Peeling for L0-Regularized Least-Squares

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L0-Regularized Least-Squares

Framework

Sparse linear models

- Linear regression with a sparse optimizer
- Applications in signal processing, machine learning, statistics, etc...

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ℓ_0-regularized least-squares

$$\min_x \ \frac{1}{2}\|\mathrm{y} - \mathrm{A}\mathrm{x}\|_2^2 + \lambda\|x\|_0$$

Ingredients

- Least-squares loss to ensure the linear model fitting
- ℓ_0-norm that counts the non-zeros in x
- Tradeoff parameter $\lambda > 0$ to control the sparsity

Solving ℓ_0-regularized problems

Initial problem

$$\min_x \ \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0$$

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Non-linear, non-convex,
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😵

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Big-M method

Reformulation

$$\begin{aligned} \min_x \ & \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0 \\ \text{s.t.} \ & -M \leq x \leq M \end{aligned}$$

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Add binary variables to linearize the ℓ_0-norm

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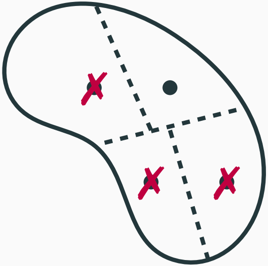
Generic Mixed-Integer Program solvers

Specialized Branch-and-Bound solvers

Branch-and-Bound

Branch-and-Bound

"Enumerate all candidate solutions and discard sub-optimal ones."



Main principles

Branching:	Divide the search space
Bounding:	Test whether a region can contain optimal solutions
Pruning:	Discard regions without optimal solutions

Tree search

$$\begin{cases} \min & \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0 \\ \text{s.t.} & -M \leq x \leq M \end{cases}$$

N^0

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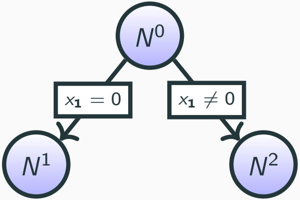
$$\begin{cases} \min & \frac{1}{2}\|\mathrm{y} - \mathrm{A}\mathrm{x}\|_2^2 + \lambda\|\mathrm{x}\|_0 \\ \text{s.t.} & -M \leq \mathrm{x} \leq M \end{cases}$$



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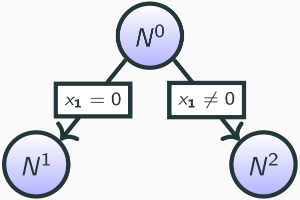

Same problem + constraint $x_1 = 0$

Same problem + constraint $x_1 \neq 0$

Tree search

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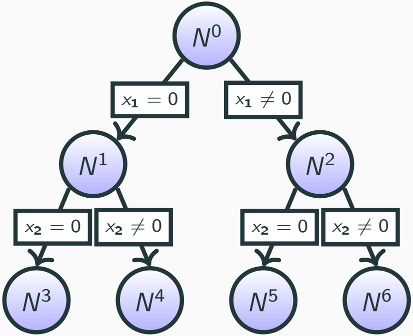

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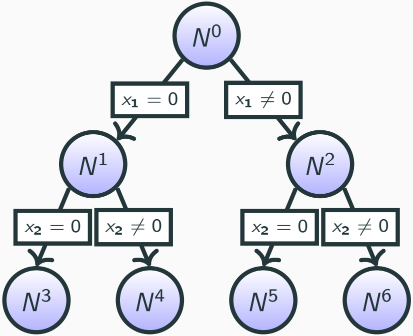

Subproblem
Still NP-hard

Tree search

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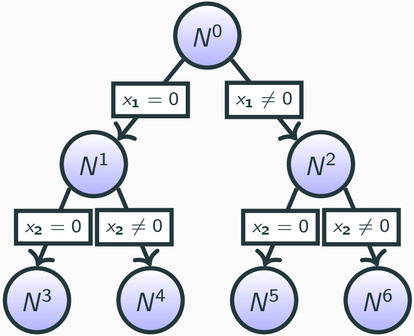

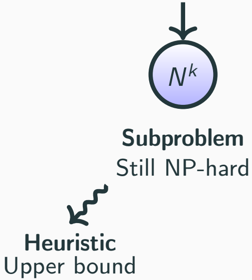

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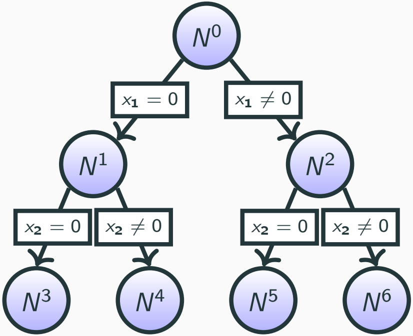

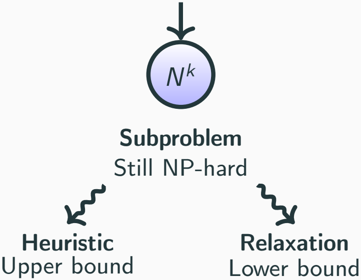

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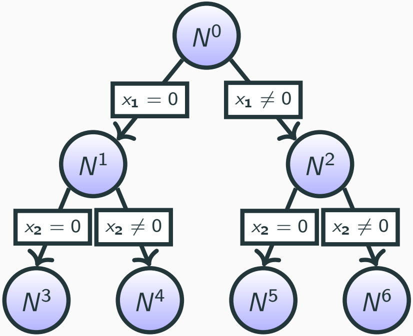

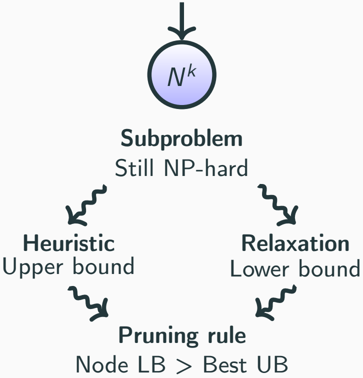

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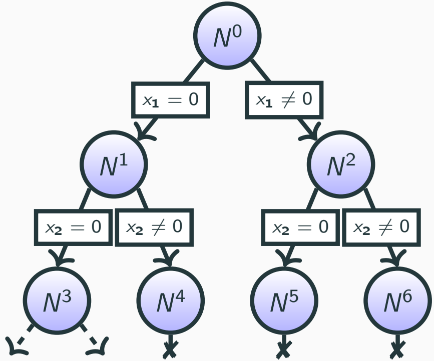

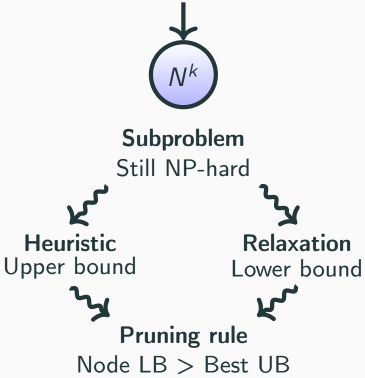

Relaxation construction

Node problem

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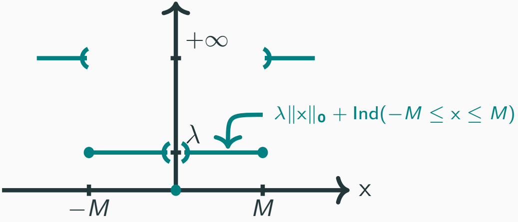

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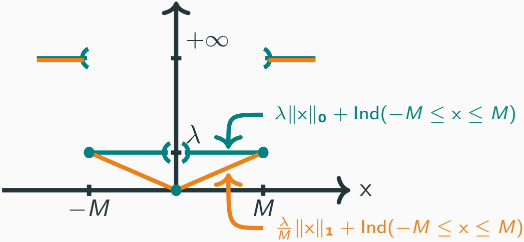

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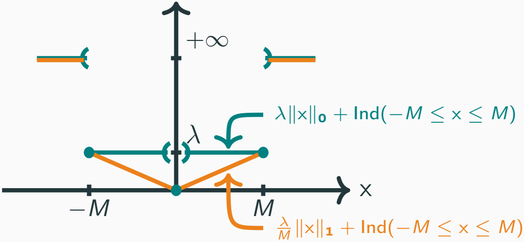
↓

Relaxation

$$\begin{cases} \min & \frac{1}{2}\|y - Ax\|_2^2 + \frac{\lambda}{M}\|x\|_1 \\ \text{s.t.} & -M \leq x \leq M \end{cases}$$



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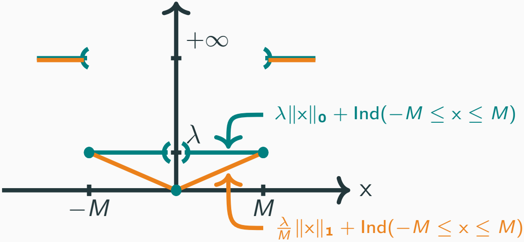


Dilemma in the choice of M

Large value of M to recover interesting solutions

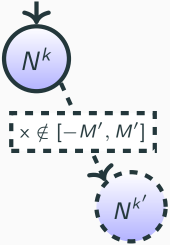
Small value of M to build strong relaxations

Our solution: Peeling

Start with M ***large and refine it locally along branches***

Peeling

Bound tightening



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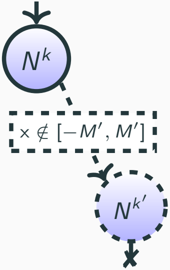
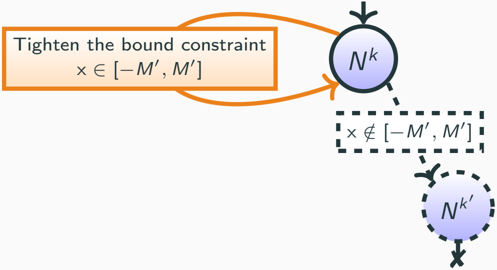


Find some
$M' \leq M$
such that $N^{k'}$ is pruned

Bound tightening



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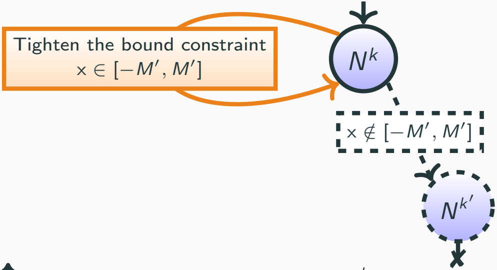

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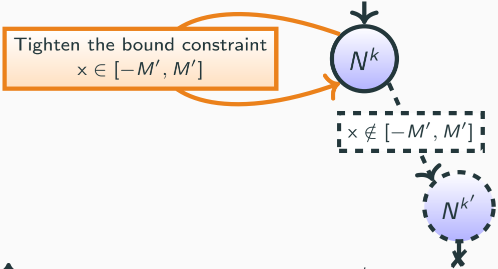

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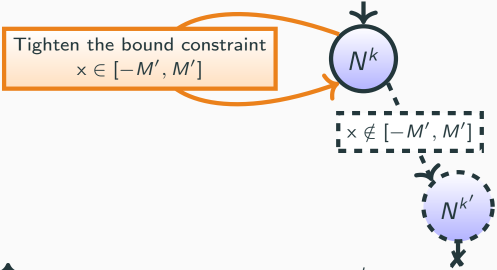

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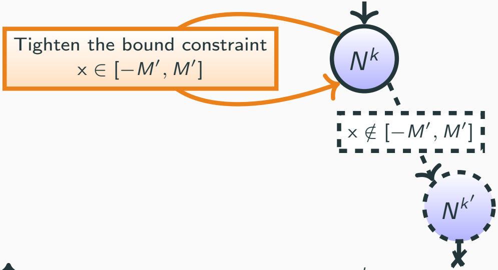
Ingredients

- Already-computed quantities
- Cost-free dual evaluation

Bound tightening



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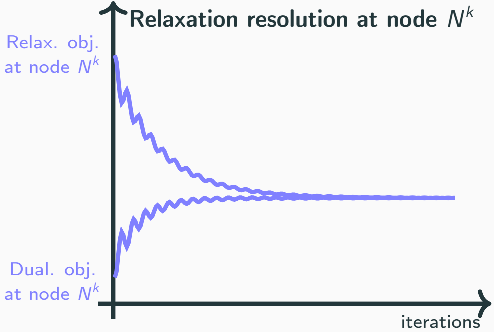

Ingredients

- Already-computed quantities
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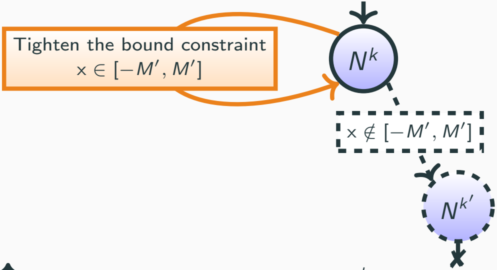
- Dual link between nodes

$$D_{N^{k'}}(u) = D_{N^k}(u) + \gamma(u)$$

Bound tightening



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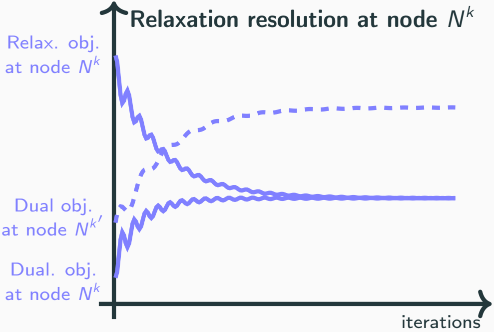

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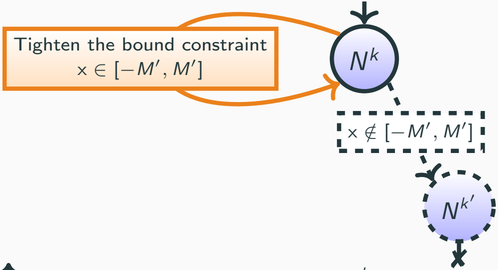

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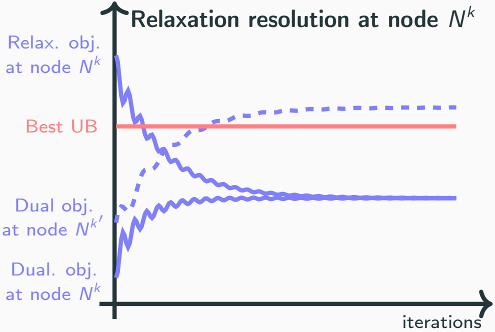

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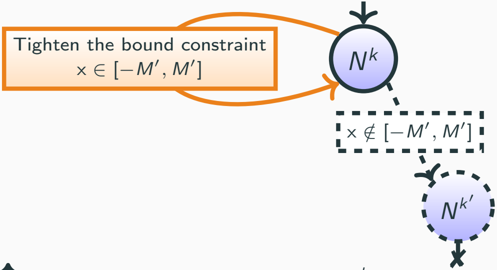

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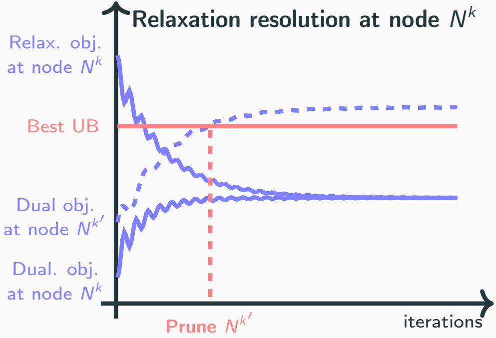

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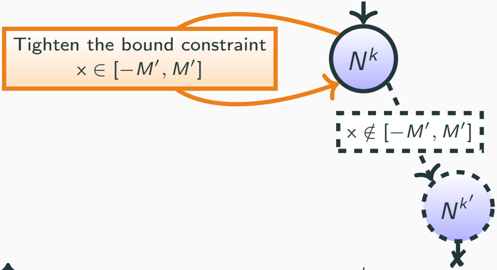

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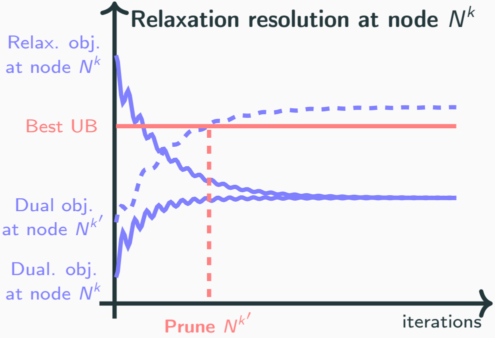
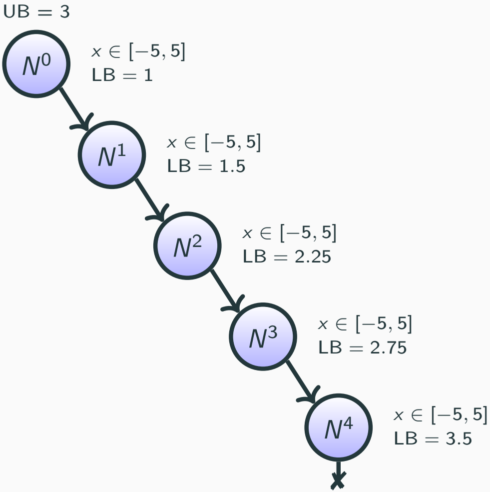


Ingredients

- Already-computed quantities
- Cost-free dual evaluation
- Dual link between nodes
- Coordinate-wise tightening
- Non-symmetric tightening

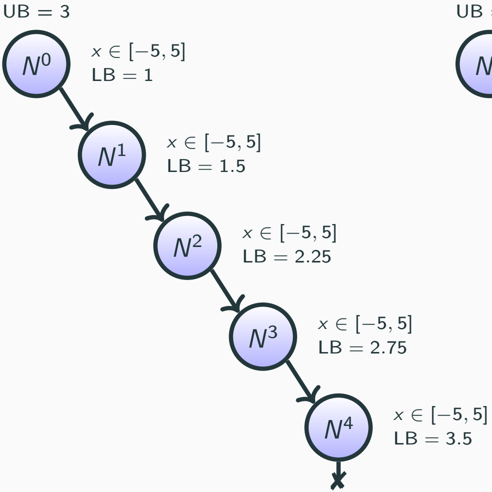
$$D_{N^{k'}}(\mathrm{u}) = D_{N^k}(\mathrm{u}) + \gamma(\mathrm{u})$$

Standard BnB

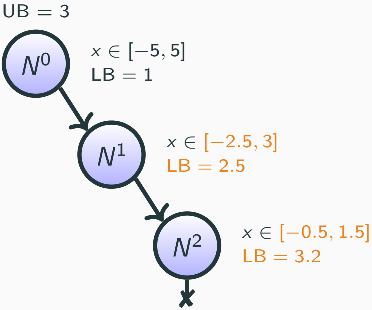


Propagation along branches

Standard BnB



BnB with peeling



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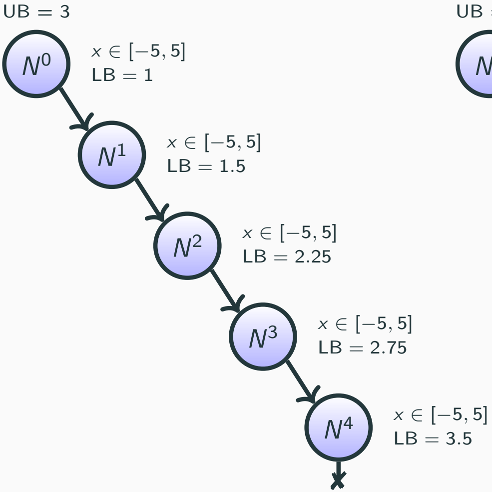
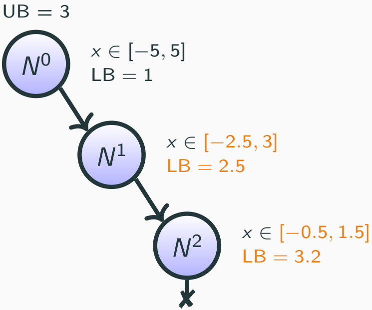
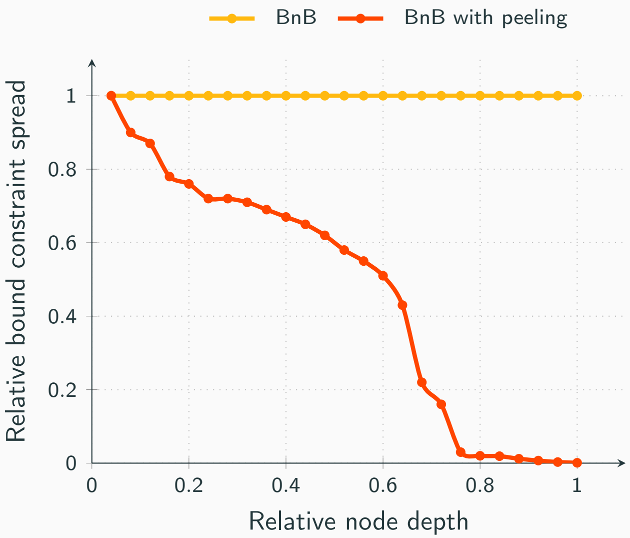


BnB with peeling



- ✔ Branches pruned early-on
- ✔ Less nodes explored
- ✔ Reduce solve time

Bound spread reduction



Numerical results

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$$\min_{\mathsf{x}\in[-M,M]} \ \frac{1}{2}\|\mathsf{y}-\mathsf{A}\mathsf{x}\|_2^2+\lambda\|\mathsf{x}\|_0$$

Data A **and** y : Sparse regression
Parameter λ : Set statistically
Bound M : $M=\gamma\|\mathsf{x}^\star\|_\infty$

Numerical results

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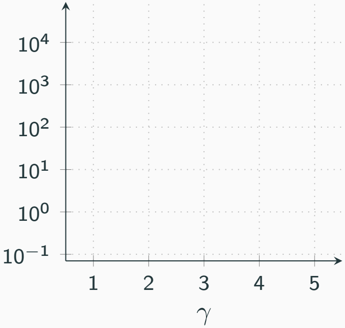
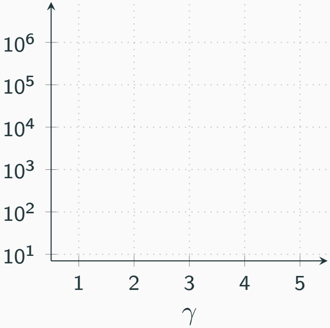
Solve time (sec.)



Node count



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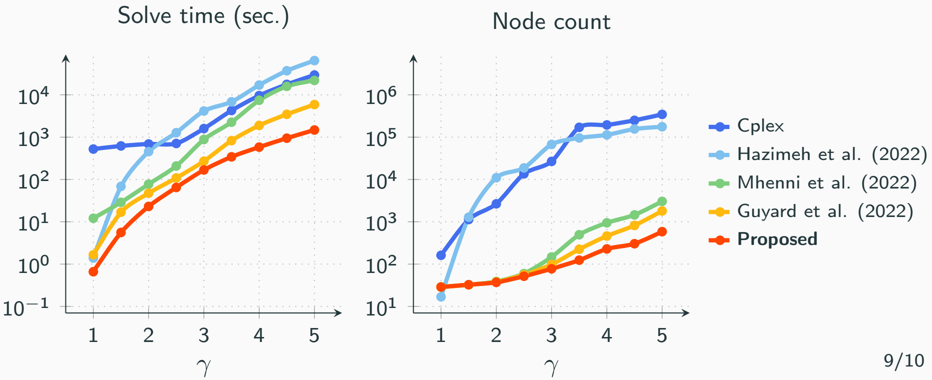

Take-home message

ℓ_0-regularized least-squares

$$\min_{x} \quad \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0$$
$$\text{s.t.} \quad -M \leq x \leq M$$

Wide interval $[-M, M]$ to recover the solution

Tight interval $[-M, M]$ to build strong relaxations

Peeling

Start with wide interval $[-M, M]$ and refine it along the BnB branches

Question time

🤓