Discrete optimization methods for sparse problems

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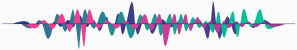
Rennes, France

Sparse problems

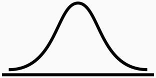
Two goals, one problem

Optimize a function $\xleftrightarrow{\textbf{Sparse problems}}$ **Sparse solution**



Signal processing

Machine learning



High-dim. statistics

And many others

Example

Sparse GLM

Features $A \in \mathbb{R}^{m \times n}$

Targets $y \in \mathbb{R}^{m}$

$\downarrow$

$\max_x \mathcal{L}(Ax, y)$

No unique solution when $m \ll n$

$\downarrow$

$\max_x \mathcal{L}(Ax, y)$ with x sparse

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Sparse PCA

Features $A \in \mathbb{R}^{m \times n}$

Covariance $\Sigma = A^{\mathrm{T}} A$

$\downarrow$

$\max_{\|x\|_2 = 1} \ x^{\mathrm{T}} \Sigma x$

Not relevant when $m \ll n$

$\downarrow$

$\max_{\|x\|_2 = 1} \ x^{\mathrm{T}} \Sigma x$ with x sparse

Example

Heart disease dataset (LIBSVM)

Age	Sex	Cholesterol	Blood pressure	...	**Disease**
31	M	50.3 mg/dl	95 mm/hg	...	**No**
35	F	54.9 mg/dl	98 mm/hg	...	**Yes**
42	F	49.8 mg/dl	92 mm/hg	...	**Yes**
37	M	59.1 mg/dl	89 mm/hg	...	**No**
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Data $\longrightarrow$ **Logistic regression** $\longrightarrow$ **Estimator** x

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Involves all the features
Explainability and robustness issues

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Data $\longrightarrow$ **Logistic regression** $\longrightarrow$ **Estimator** x

Force the use of only few features via sparsity in x

Involves all the features
Explainability and robustness issues

Objective, constraint or both ?

Optimize a function $\longleftrightarrow$ **Sparse optimization** $\longleftrightarrow$ **Sparse solution**

Loss function $F(x)$

Counting function $\|x\|_0$

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Optimize a function $\longleftrightarrow$ **Sparse optimization** $\longleftrightarrow$ **Sparse solution**

Loss function $F(\mathrm{x})$

Counting function $\|\mathrm{x}\|_0$

Constrainted version

$\min_{\mathrm{x}} F(\mathrm{x})$ s.t. $\|\mathrm{x}\|_0 \leq \kappa$

Minimized version

$\min_{\mathrm{x}} \|\mathrm{x}\|_0$ s.t. $F(\mathrm{x}) \leq \epsilon$

Penalized version

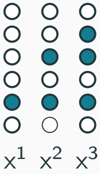
$\min_{\mathrm{x}} F(\mathrm{x}) + \lambda\|\mathrm{x}\|_0$

A bit of history

1990

Sparse heuristics

MP, OMP, IHT and co.



A bit of history

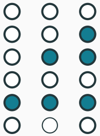
1990 **2000**

Sparse heuristics

MP, OMP, IHT and co.

Formulation with ℓ_0-norm

RIP, NSP, EkR and co.



x^1 x^2 x^3

Heuristic

$\equiv$

ℓ_0-prob.

A bit of history

1990 — Sparse heuristics

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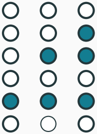


x^1 x^2 x^3

2000 — Formulation with ℓ_0-norm

RIP, NSP, EkR and co.

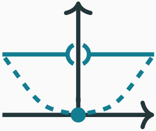
Heuristic

$|||$

ℓ_0-prob.

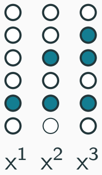
2005 — Convex approx. of ℓ_0-norm

Lasso, Elastic-Net and co.



A bit of history

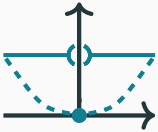
1990	2000	2005	2013
Sparse heuristics	Formulation with ℓ_0-norm	Convex approx. of ℓ_0-norm	Concave approx. of ℓ_0-norm
MP, OMP, IHT and co.	RIP, NSP, EkR and co.	Lasso, Elastic-Net and co.	SCAD, MCP CEL0 and co.

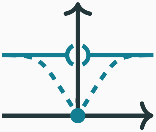


Heuristic

|||

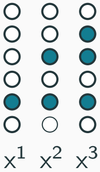
ℓ_0-prob.





A bit of history

1990	2000	2005	2013	2016
Sparse heuristics	Formulation with ℓ_0-norm	Convex approx. of ℓ_0-norm	Concave approx. of ℓ_0-norm	Exact methods for ℓ_0-problems
MP, OMP, IHT and co.	RIP, NSP, EkR and co.	Lasso, Elastic-Net and co.	SCAD, MCP CEL0 and co.	MIP, BnB CP and co.



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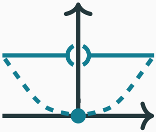

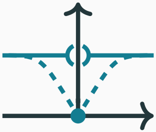

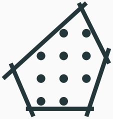

Mixed-Integer Optimization

Handeling the L0-norm with MIO tools

Sparse problem

$\min_x F(x) + \lambda\|x\|_0$

$\downarrow$ Solver

Solution

Handeling the L0-norm with MIO tools

Sparse problem

$\min_{x} F(x) + \lambda \|x\|_0$

Solver

Solution

ℓ_0-norm is non-linear, non-convex, non-smooth, NP-hard, ...

Handeling the L0-norm with MIO tools

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ℓ_0-norm is non-linear, non-convex, non-smooth, NP-hard, ...

😵

The ℓ_0-norm counts the number of non-zeros in x

Encode the nullity in some binary vector z

We know how to deal with this in MIO !

Fitting the MIO formalism

Linearizing the ℓ_0-norm

Real vector $\mathrm{x} \in \mathrm{R}^n$ and binary vector $\mathrm{z} \in \mathrm{B}^n$:

$$\|\mathrm{x}\|_0 = 1^{\mathrm{T}}\mathrm{z} \quad \text{if} \quad \mathrm{x} \odot (1 - \mathrm{z}) = 0$$

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$$\downarrow$$

$$\boxed{\min_{x,z} F(x) + \lambda 1^T z + H(x, z)}$$

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Generic MIO solvers
Branch-and-bound
Cutting planes
...

Overview of solution methods

Branch-and-Bound

"Explore regions in the feasible space and discard those that cannot contain solutions."

Cutting Planes

"Add valid cuts to restrict the feasible space until the optimum is isolated."

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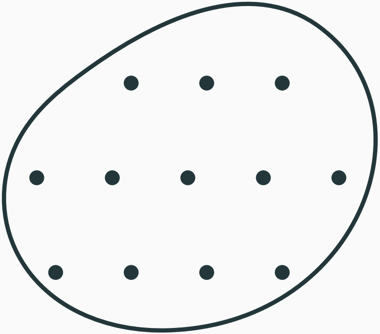

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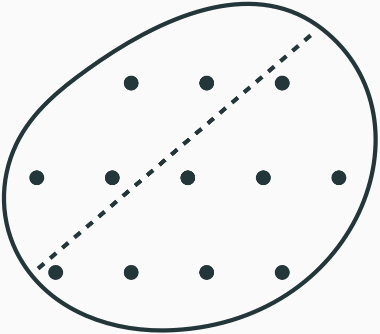

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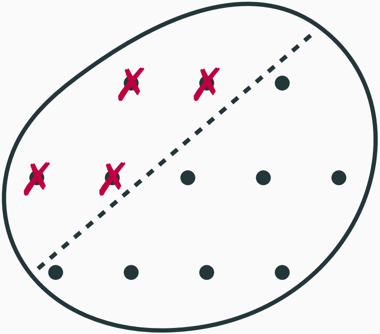

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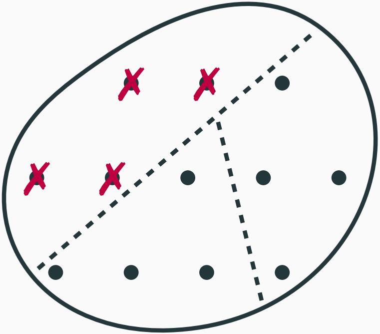

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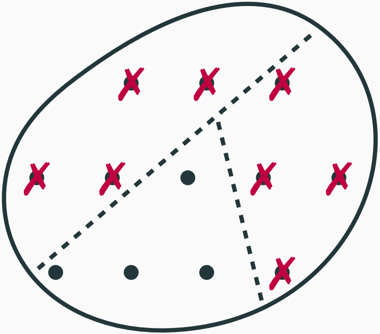

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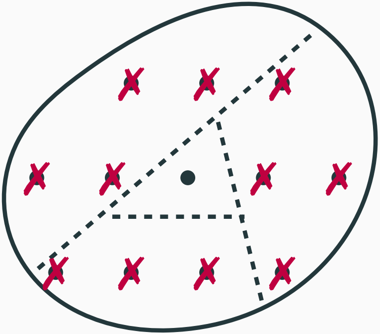

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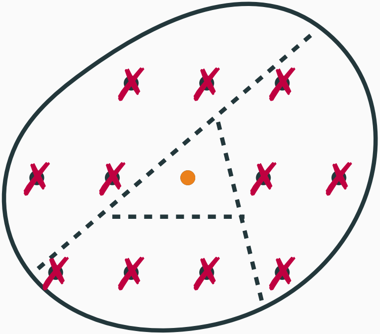

Ongoing Research Directions

Non-exhaustive list

Inria / Insa / Ensai / CentraleSupélec
T. Guyard, C. Herzet, C. Elvira, A. Arslan
Generalization, acceleration

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Lund University
M. Carlsson, C. Olsson...
Quadratic envelope

Frankfurt / Wurzburg Universities
C. Kanzow, A. Tillmann, ...
Optimality conditions

London Business School
J. Pauphilet, R. Cory-Wright, ...
Healthcare applications

Ponts ParisTech
M. De Lara, P. Chancelier, A. Parmentier, ...
Non-convex analysis for ℓ_0-norm, ML appli.

Centrale Nantes / ENSTA Bretagne
S. Bourguignon, J. Ninin, ...
Branch-and-Bound for ℓ_0-problems

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Generalization, acceleration

IRIT / I3S
E. Soubies, L. Blanc-Féraud, ...
Tight relax. of ℓ_0-norm

Ongoing Research Directions

Non-exhaustive list

MIT
D. Bertsimas, R. Mazmuder, ...
MIO theory for ℓ_0-problems

Lund University
M. Carlsson, C. Olsson...
Quadratic envelope

Google Deep Mind
H. Hazimeh, A. Dedieu, ...
MIO-based heuristics

Frankfurt / Wurzburg Universities
C. Kanzow, A. Tillmann, ...
Optimality conditions

London Business School
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Healthcare applications

Berkley
A. Atamtürk, A. Gomès, ...
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Take-home message

- ℓ_0-norm problems arise in many applied mathematical fields
- Mixed-integer optimization tools to address them
- Structure exploitation is the key to achieve competitive performances
- Active research area
  - $\rightarrow$ Theoretical results
  - $\rightarrow$ Efficiency, flexibility and accessibility of solution methods
  - $\rightarrow$ Software development
  - $\rightarrow$ Diffusion to other communities

Question time

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