Screen-and-Relax for Sparse Support Identification

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Sparse problems

Framework

Sparse optimization

- Minimize a loss with a sparse optimizer
- Applications in signal processing, machine learning, statistics, etc...

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$$x^\star \in \operatorname{argmin}_{x \in \mathbb{R}^n} f(Ax) + \underbrace{\lambda\|x\|_1 + h(x)}_{g(x)}$$

Working hypotheses

- f and h are proper, closed and convex functions
- f and h are differentiable with Lipschitz-continuous gradient
- h is separable
  - + h minimized at $x = 0$
  - + non-degeneracy assumption

Solving sparse problems

$x^\star \in \operatorname{argmin}_x \; f(Ax) + g(x)$

Solution methods

- Composite objective: smooth + non-smooth-separable
- First-order methods accessing ∇f, ∂g, prox_g, ...
    - Proximal gradient descent
    - Coordinate descent
    - Alternating direction method of multipliers
    - ...

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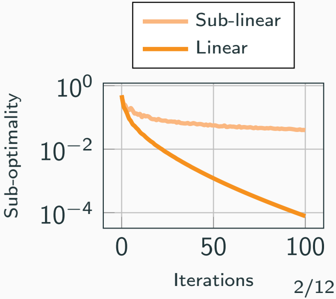
Convergence rates

- Sub-linear: $P(\mathrm{x}^{(k)}) - P(\mathrm{x}^\star) \leq C/k^\gamma$
- Linear: $P(\mathrm{x}^{(k)}) - P(\mathrm{x}^\star) \leq Ce^{-\gamma k}$
  - → Asymptotically (Peyré et al., 2015)
  - → Strong convexity (Aujol et al., 2023)
- Super-linear $P(\mathrm{x}^{(k)}) - P(\mathrm{x}^\star) \leq Ce^{-\gamma k^2}$
  - → Prox-Newton (Bareilles et al., 2022)
  - → Jérôme Malick's talk



Sparse structure exploitation

Variable with many useless and few informative entries

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Screening tests

- Identify zeros in $x^\star$
- Dimensionality shrinking
- Computational savings

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- Objective smoothing
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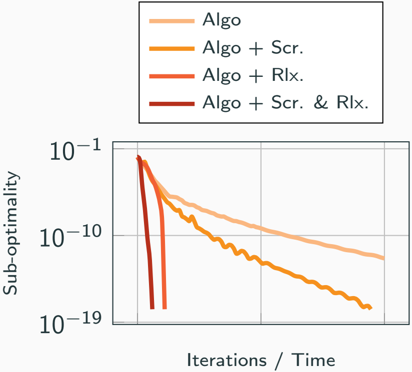

Screening and Relaxing Tests

Characterize the nullity in $\mathrm{x}^\star$ from $\partial g(\mathrm{x}^\star)$

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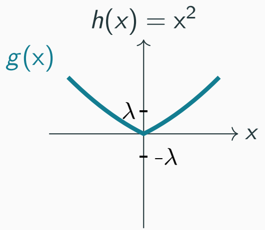
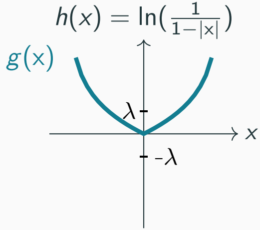
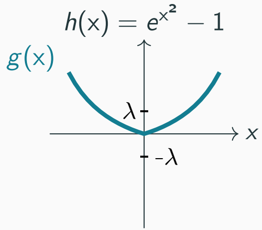
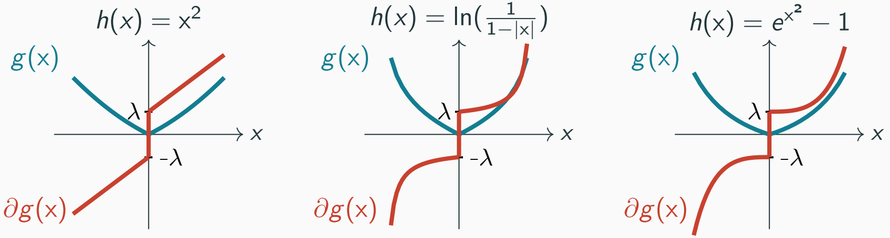






Geometrical idea

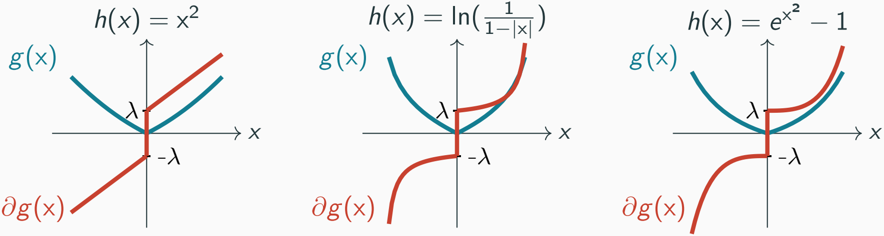
Characterize the nullity in $x^\star$ **from** $\partial g(x^\star)$



Geometrical idea

$g(\mathrm{x}) = \lambda\|\mathrm{x}\|_1 + h(\mathrm{x})$

Characterize the nullity in $\mathrm{x}^\star$ from $\partial g(\mathrm{x}^\star)$



Geometrical screening and relaxing test

$$\partial_i g(\mathrm{x}^\star) \subset [-\lambda, \lambda] \implies x_i^\star = 0$$
$$\partial_i g(\mathrm{x}^\star) \not\subset [-\lambda, \lambda] \implies x_i^\star \neq 0$$

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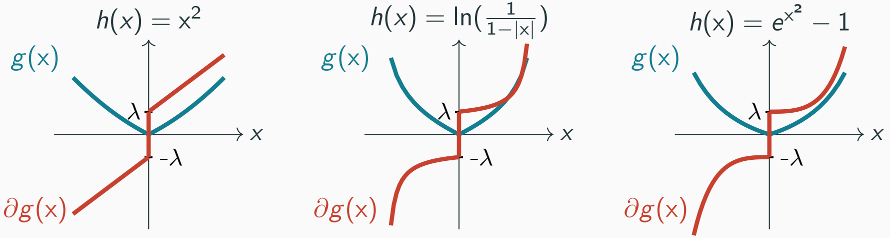

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$\partial g(\mathrm{x}^\star)$ **is not available**

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$$\boxed{a_i^\top u^\star \in \partial_i g(x^\star)}$$

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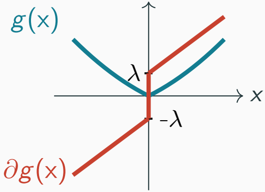

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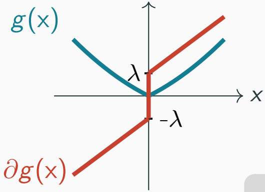

Screening and relaxing test

$$|a_i^\top u^\star| < \lambda \implies x_i^\star = 0$$
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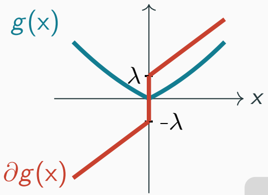

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$u^\star$ **is not available**

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$$\boxed{\text{Safe region: } u^\star \in \mathcal{R}}$$

Safe screening and relaxing test

$$\max_{u \in \mathcal{R}} |a_i^T u| < \lambda \quad \Longrightarrow \quad x_i^\star = 0$$
$$\min_{u \in \mathcal{R}} |a_i^T u| > \lambda \quad \Longrightarrow \quad x_i^\star \neq 0$$

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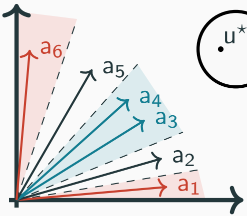

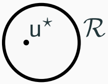

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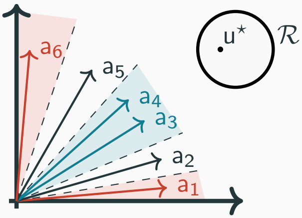
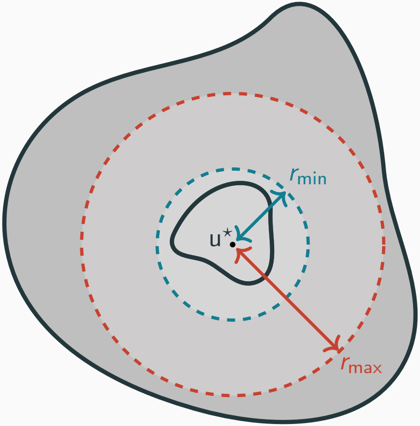


Identifiability of the nullity in $x^\star$
If h is strictly convex at $x = 0$, all zero and non-zero entries can be identified if $\mathcal{R}$ is sufficently tight.

Working regimes



$\mathcal{R} \subset \mathcal{S}(u^\star, r_{\text{min}}) \implies$ all tests passed

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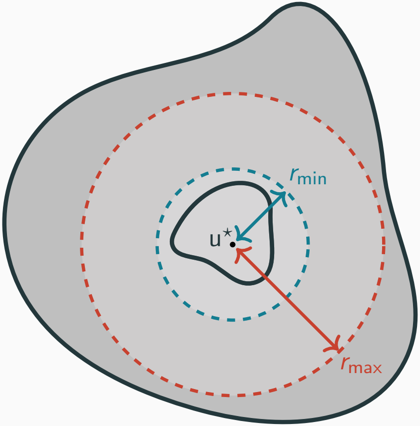
Working regimes



$\mathcal{R} \subset \mathcal{S}(u^\star, r_{\min}) \implies$ all tests passed
$\mathcal{R} \supset \mathcal{S}(u^\star, r_{\max}) \implies$ no tests passed

$\boxed{r_{\min} > 0}$

We know how to construct safe regions with a radius proportional to the (square root of) the duality gap

↓

Guaranty to identify all zeros and non-zeros in $x^\star$ in finite time

Toward new solution methods

Reformulation

Initial problem

$\min_x f(Ax) + \lambda \|x\|_1 + h(x)$

Reformulation

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$$\min_x f(\mathrm{A}\mathrm{x}) + \lambda\|\mathrm{x}\|_1 + h(\mathrm{x})$$

Perfect identif.
$\mathcal{S}_{\mathbf{0}}^{\star} = \{i \mid x_i^{\star} = 0\}$
$\mathcal{S}_{\mathbf{1}}^{\star} = \{i \mid x_i^{\star} \neq 0\}$

Ideal reformulation

$$\begin{cases} \min_x & f(\mathrm{A}_{\mathcal{S}_{\mathbf{1}}^{\star}}\mathrm{x}_{\mathcal{S}_{\mathbf{1}}^{\star}}) + \lambda\boldsymbol{\theta}_{\mathcal{S}_{\mathbf{1}}^{\star}}^{\top}\mathrm{x}_{\mathcal{S}_{\mathbf{1}}^{\star}} + h(\mathrm{x}_{\mathcal{S}_{\mathbf{1}}^{\star}}) \\ \text{s.t.} & \mathrm{x}_{\mathcal{S}_{\mathbf{0}}^{\star}} = 0 \end{cases}$$

Reformulation

Initial problem

$$\min_x f(\mathrm{Ax}) + \lambda\|\mathrm{x}\|_1 + h(\mathrm{x})$$

Screen & Relax
$\mathcal{S}_0 \subseteq \mathcal{S}_0^\star$
$\mathcal{S}_1 \subseteq \mathcal{S}_1^\star$
$\mathcal{S}_\bullet = (\mathcal{S}_0 \cup \mathcal{S}_1)^\complement$

Perfect identif.
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Ideal reformulation

$$\begin{cases} \min_x & f(\mathrm{A}_{\mathcal{S}_1^\star}\mathrm{x}_{\mathcal{S}_1^\star}) + \lambda\boldsymbol{\theta}_{\mathcal{S}_1^\star}^\top \mathrm{x}_{\mathcal{S}_1^\star} + h(\mathrm{x}_{\mathcal{S}_1^\star}) \\ \text{s.t.} & \mathrm{x}_{\mathcal{S}_0^\star} = 0 \end{cases}$$

Partial reformulation

$$\begin{cases} \min & f(\mathrm{A}_{\mathcal{S}_1\cup\mathcal{S}_\bullet}\mathrm{x}_{\mathcal{S}_1\cup\mathcal{S}_\bullet}) + \lambda\boldsymbol{\theta}_{\mathcal{S}_1}^\top \mathrm{x}_{\mathcal{S}_1} + h(\mathrm{x}_{\mathcal{S}_1}) \\ & \qquad + \lambda\|\mathrm{x}_{\mathcal{S}_\bullet}\|_1 + h(\mathrm{x}_{\mathcal{S}_\bullet}) \\ \text{s.t.} & \mathrm{x}_{\mathcal{S}_0} = 0 \end{cases}$$

Algorithm embedding

Algorithm 1 Screen & Relax

```
initialize (S_0, S_1, S_•) = (∅, ∅, [[1, n]])
repeat
    // Iterate update
    x^k_{S_•} ← 1^{st}OrderIteration(x^{k-1})
    x^k_{S_1} ← 2^{nd}OrderIteration(x^{k-1})
    x^k_{S_0} ← 0
    // Problem update
    Construct a new safe region R^k from x^k
    Update (S_0, S_1, S_•) using R^k
until convergence criterion is met
```

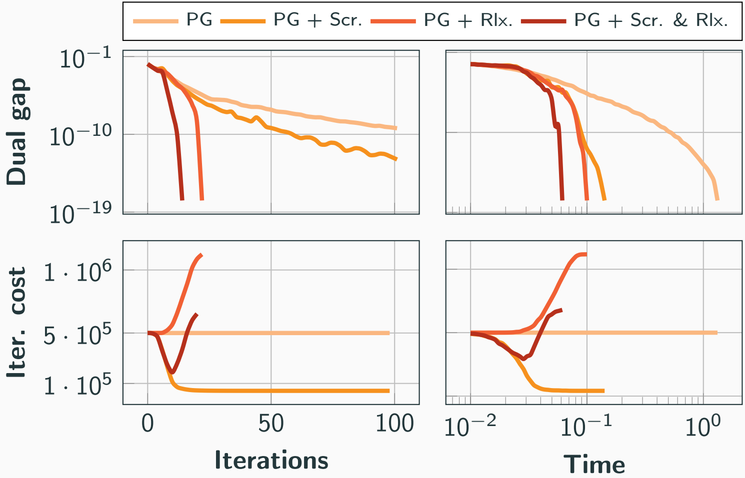
Iteration cost reduction: $n \to n - |\mathcal{S}_0|$

Faster convergence rate: (sub-)linear $\to$ super-linear

Numerics

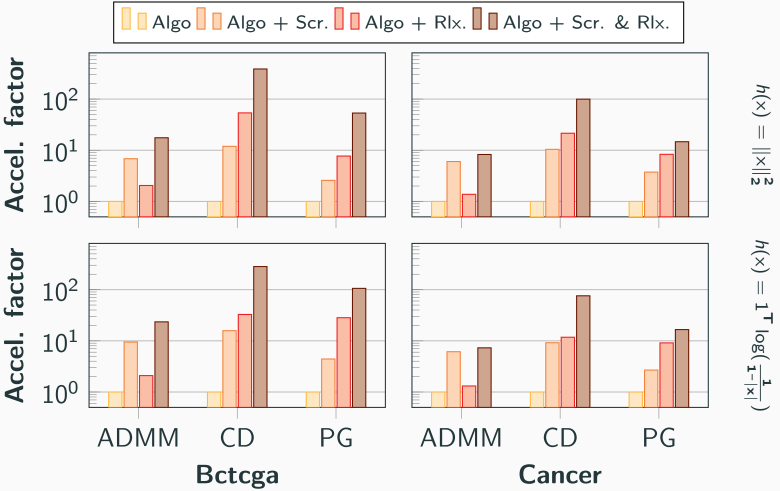
Screen & Relax effects

$$\min_{\mathrm{x}} f(\mathrm{Ax}) + \lambda \|\mathrm{x}\|_1 + h(\mathrm{x})$$



Regularization path fitting

$$\min_x f(Ax) + \lambda\|x\|_1 + h(x)$$



Regression - 356 × 17,322

Classification - 100 × 12,600

Take-home message

- Structure matters in sparse problems
- Identification of zeros
  - Screening tests
  - Allows for dimensionality reduction
  - Computational savings
- Identification of non-zeros
  - Relaxing tests
  - Allows for objective smoothing
  - Faster convergence rate
- Screen & Relax strategy to benefit from
  - Computational savings
  - Accelerated convergence

Question time