Towards Stronger Relaxations for ℓ_0-Regularized Problems

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Workshop on ℓ_0-based minimization
Institut Henri Poincaré

Joint work with Cédric Herzet, Clément Elvira, Ayşe-Nur Arslan
and discussions with Emmanuel Soubies

Problem of Interest

ℓ_0-Regularized Problems

$$p^\star = \min_{\mathbf{x} \in \mathbf{R}^n} \; f(\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$

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$$\boxed{\text{How to construct a generic Branch-and-Bound algorithm?}}$$

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Framework:

- f is proper, closed, convex
- h is proper, closed, convex and separable as $h = \sum_i h_i$
- h_i are supercoercive, even, and verify $h_i \geq h_i(0) = 0$
- $\mathbf{0} \in \operatorname{int}\operatorname{dom}(f) \cap \operatorname{int}\operatorname{dom}(h)$

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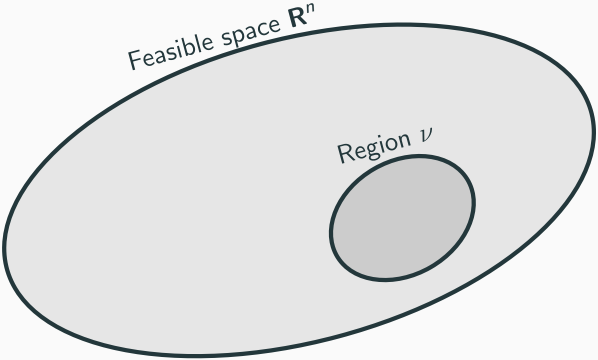
Principle: Explore regions in the feasible space and prune those that cannot contain any optimal solution.



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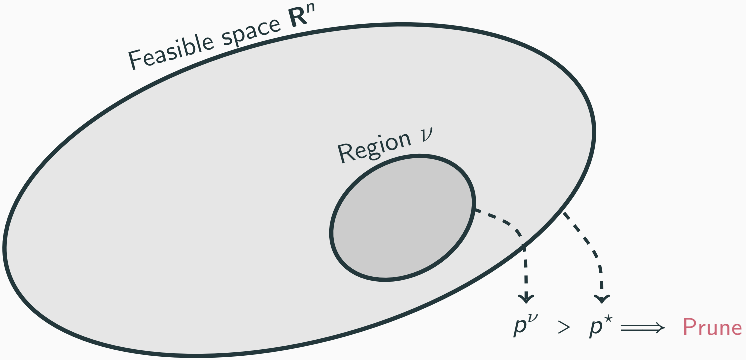

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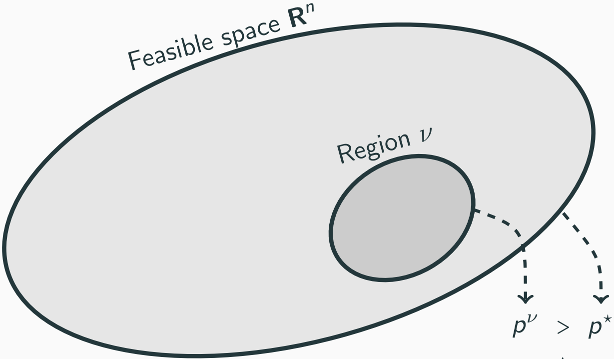

$$p^\nu \quad > \quad p^\star \implies \text{Prune}$$

$$\Uparrow$$

$$p^\nu \geq p^\nu_{\mathrm{lb}} > p^\star_{\mathrm{ub}} \geq p^\star$$

Pruning test

Branch-and-Bound Ingredients

Region: $\nu = (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_\bullet) \equiv$ partition of $\{1, \ldots, n\}$ driving sparsity:

- $x_i = 0$ for all $i \in \mathcal{S}_0$
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$p^\nu_{\mathrm{lb}} \quad =$ optimal value of a relaxation

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Convex Relaxations

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Idea: Convexify the objective function

$$p^{\nu} \quad = \min_{\mathbf{x}\in\mathbf{R}^n} \; f(\mathbf{x}) + g^{\nu}(\mathbf{x})$$

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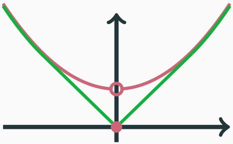

Plot: Functions g_i^{ν} and $g^{\nu}_{\mathrm{cvx},i}$ when $i \in \mathcal{S}_\bullet$ for $h_i(x_i) = x_i^2$.

Spotlight Result 1

The bi-conjugate admits a generic closed-form expression.

Let $g_i(x) = h_i(x) + \lambda\|x\|_0$, we have

$$g_i^{**}(x) = \begin{cases} \tau_i|x| & \text{if } |x| \leq \mu_i \\ h_i(x) + \lambda & \text{otherwise} \end{cases}$$

where (τ_i, μ_i) are some "easy-to-compute" quantities.

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Efficient solution methods for the convex relaxation

Proximal gradient

Coordinate descent

Splitting methods

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Idea: Dualize the problem

$$p^\nu = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x}) + g^\nu(\mathbf{x})$$

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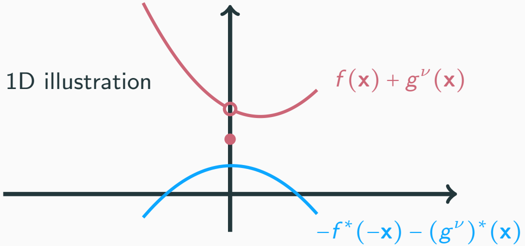

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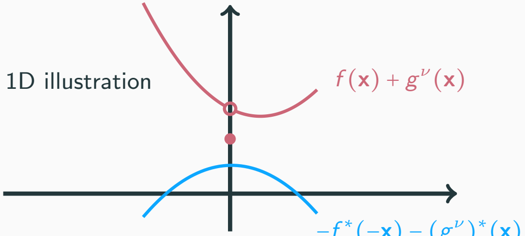
1D illustration



Dual Relaxations

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Spotlight Result 3

Convex relaxations are stronger than dual relaxations.

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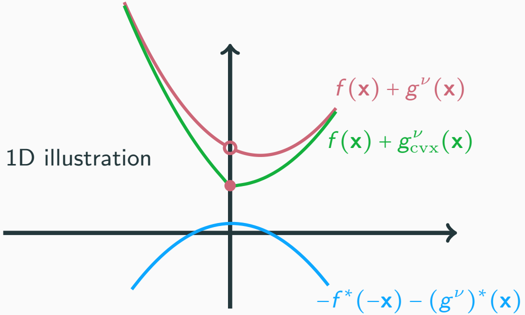

Dual Relaxations

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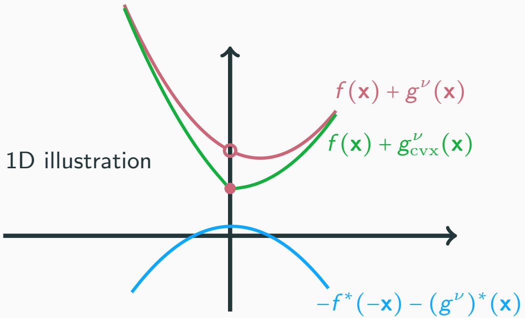

But...

- Evaluating the dual objective always gives a valid lower-bound
- No need to solve the dual relaxation to optimality
- Simple dual link between regions: **simultaneous pruning**

Can we do better?

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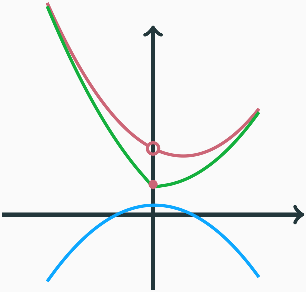

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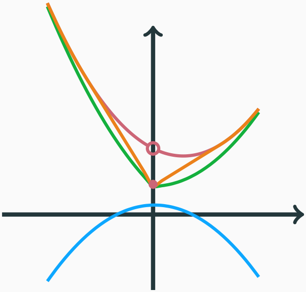

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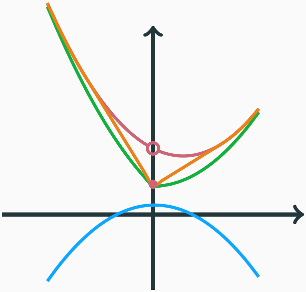

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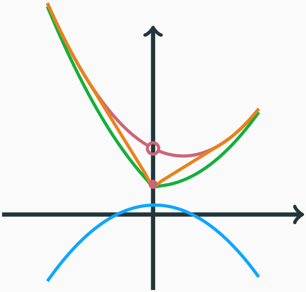

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The relaxation needs to be valid, tight and tractable

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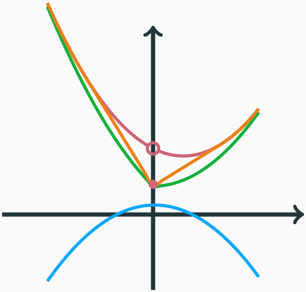

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1. Validity : $g^{\nu}_{\mathrm{ncvx}} \leq g^{\nu}$
2. Tightness : $g^{\nu}_{\mathrm{ncvx}} \geq g^{\nu}_{\mathrm{cvx}}$
3. Tractability: $f + g^{\nu}_{\mathrm{ncvx}}$ is convex

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Least-squares loss and ℓ_2-norm penalty

$$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} \tfrac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_0 + \tfrac{\gamma}{2}\|\mathbf{x}\|_2^2$$

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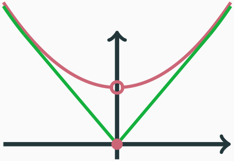

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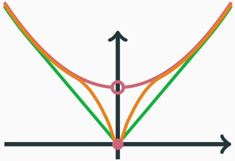

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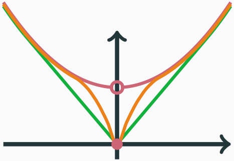

Sufficient conditions:

- $\boldsymbol{\alpha} = \sqrt{2\lambda\boldsymbol{\beta}}$ $\rightarrow$ validity and tightness ✔
- $\mathbf{A}^{\mathrm{T}}\mathbf{A} + \gamma\mathbf{I} - \mathrm{diag}(\boldsymbol{\beta}) \succeq \mathbf{0}$ $\rightarrow$ tractability ✔

Takeways

- Generalizing BnB ≡ Generic relaxation in a given region
    - Generic convex relaxation
    - Generic dual relaxation
    - Both having their own advantages
- Can we do better?
    - Non-convex relaxation of the penalty
    - Tune parameters to ensure validity, tightness and tractability
    - Case with least-squares loss and ℓ_2-norm penalty
    - We'll see next if it works in practice!

Question time!

We are building a benchmark for sparse problems with

BenchOpt

We are looking for contributors, datasets, solvers, ...

Feel free to join!