Branch-and-Bound Algorithms for ℓ_0-Regularized Problems

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Inria and Insa Rennes

PhD defense - November 27th, 2024

ℓ_0-regularized problems

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

ℓ_0-regularized problems

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

1) Loss $f(\mathbf{Ax})$

Models the quantity to minimize in the application at hand

ℓ_0-regularized problems

Problem

$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

1) Loss $f(\mathbf{A}\mathbf{x})$

Models the quantity to minimize in the application at hand

2) ℓ_0-norm $\|\mathbf{x}\|_0$

Counts non-zeros in its input to promote sparse solutions

ℓ_0-regularized problems

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$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

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Models the quantity to minimize in the application at hand

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Counts non-zeros in its input to promote sparse solutions

3) Penalty $h(\mathbf{x})$

Linked to the application or artificially introduced to enable solution methods

ℓ_0-regularized problems

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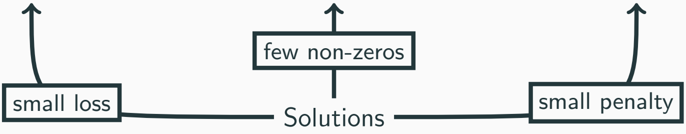


ℓ_0-regularized problems

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

tradeoff parameter $\lambda > 0$

1) Loss $f(\mathbf{A}\mathbf{x})$

Models the quantity to minimize in the application at hand

2) ℓ_0-norm $\|\mathbf{x}\|_0$

Counts non-zeros in its input to promote sparse solutions

3) Penalty $h(\mathbf{x})$

Linked to the application or artificially introduced to enable solution methods

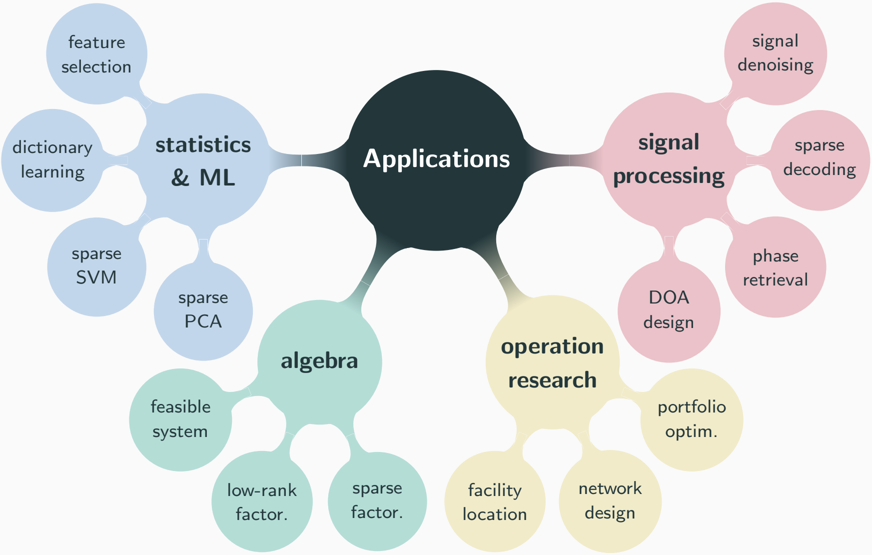
small loss

few non-zeros

small penalty

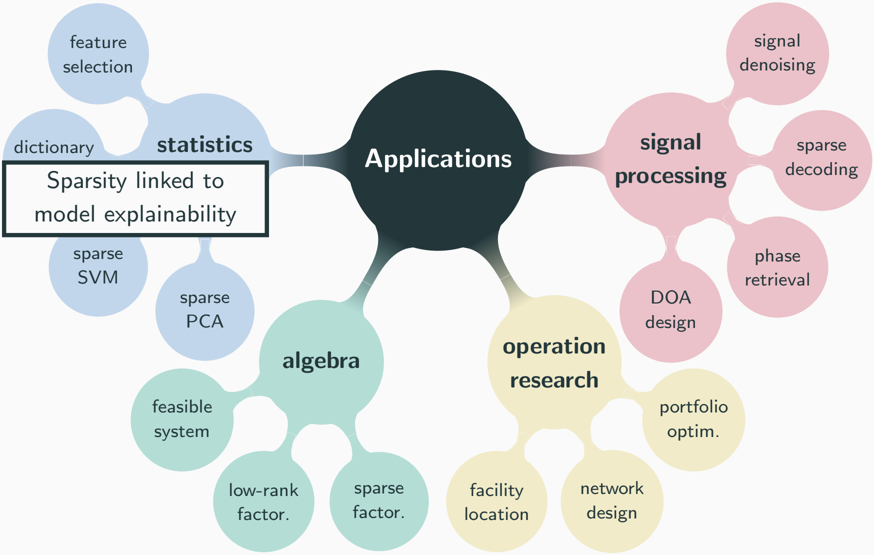
Solutions

One problem to rule them all



A. Tillmann *et. al* (2024)

One problem to rule them all



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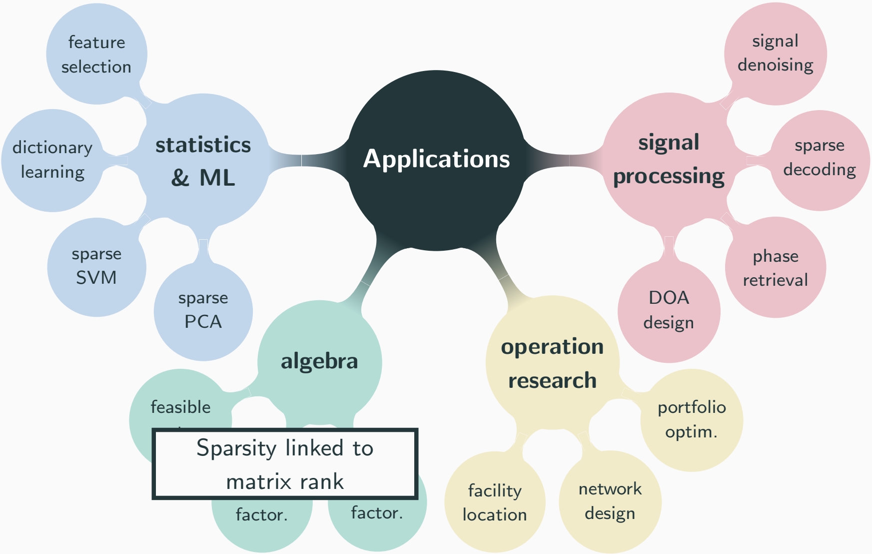

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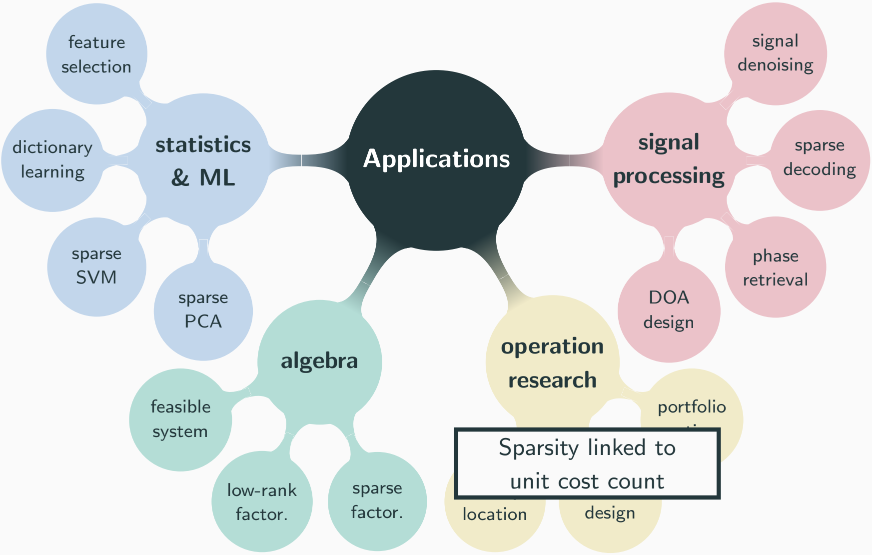
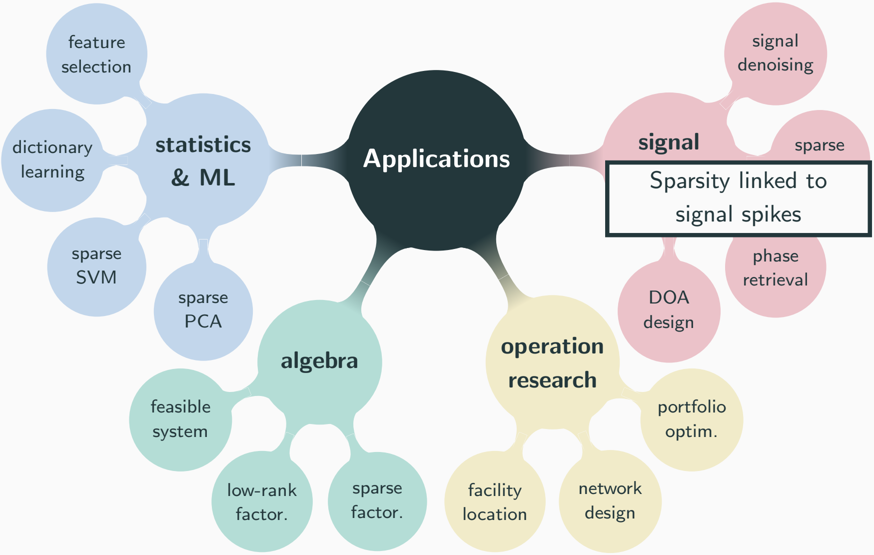


Sparsity linked to unit cost count

A. Tillmann *et. al* (2024)

One problem to rule them all



A. Tillmann *et. al* (2024)

Solution methods

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

Question

$\boxed{\text{How to design generic and efficient solution methods ?}}$

Solution methods

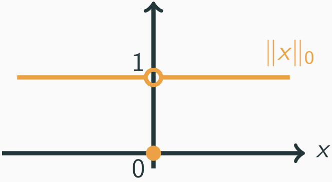
Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

Question

How to design generic and efficient solution methods ?

ℓ_0-norm in 1d



Problem

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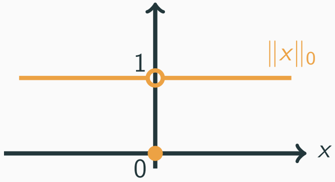


Properties

- non-convex
- non-smooth
- non-continuous
- ...

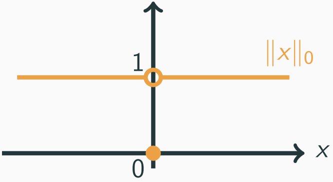
Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

Question

How to design generic and efficient solution methods ?

ℓ_0-norm in 1d



Properties

- non-convex
- non-smooth
- non-continuous
- ...

$\downarrow$

Can be tackled using generic MIP solvers

D. Bertsimas *et. al* (2016)

Numerics

Problem

$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{100 \times 300}$ / f : Quadratic / h : Bound constraint

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$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

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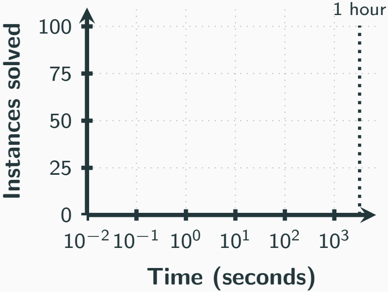

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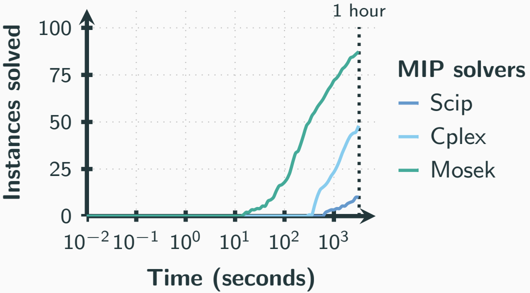

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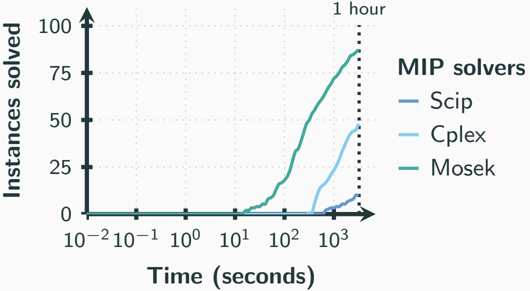

Can we do better ?

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

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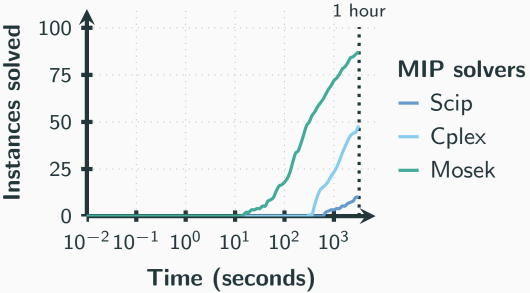


Can we do better ?

↓

BnB solvers

Specialized for the problem at hand

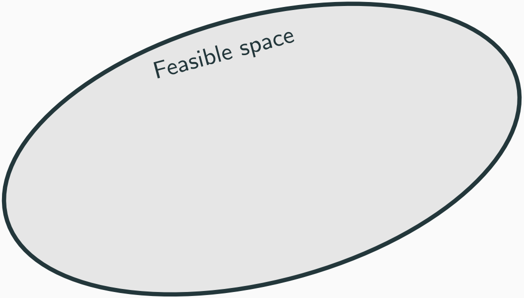
Branch-and-Bound Algorithms

BnB – Algorithmic principle

Explore regions in the feasible space and prune those that cannot contain any optimal solution.

BnB – Algorithmic principle

Explore regions in the feasible space and prune those that cannot contain any optimal solution.



BnB – Algorithmic principle

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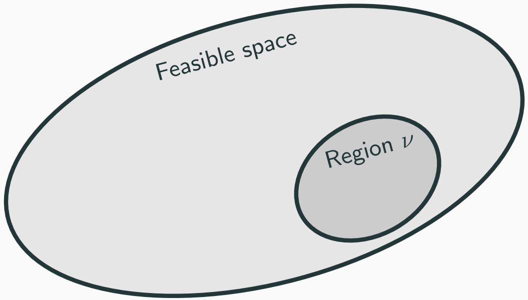

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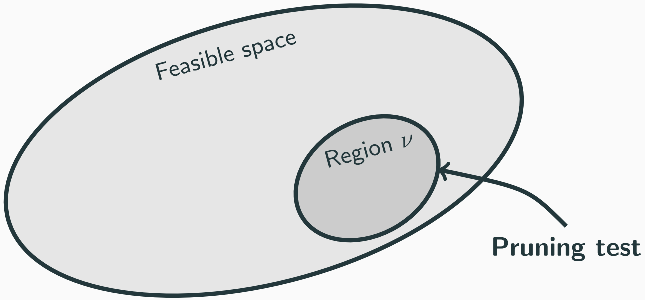

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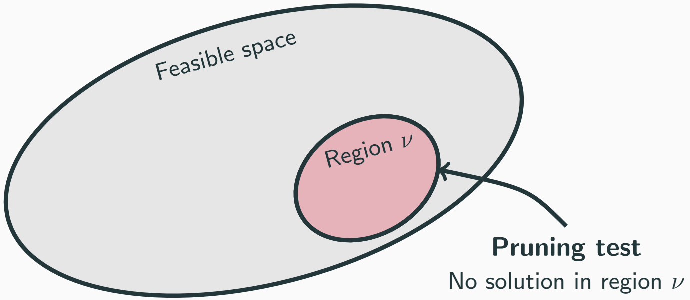

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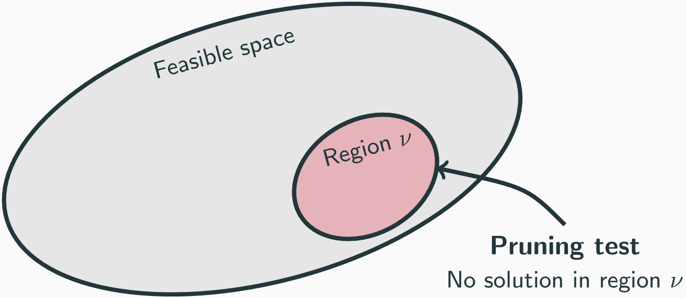

Branching – Region management
Bounding – Pruning test evaluation

BnB – How to construct regions in the feasible space ?

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

BnB – How to construct regions in the feasible space ?

Problem

$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

Observation

Solutions expected to be sparse

BnB – How to construct regions in the feasible space ?

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$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

Observation

Solutions expected to be sparse

Method

Drive the sparsity of the optimization variable

BnB – How to construct regions in the feasible space ?

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Observation

Solutions expected to be sparse

$\mathbf{R}^n$

$\downarrow$

ν_0

Method

Drive the sparsity of the optimization variable

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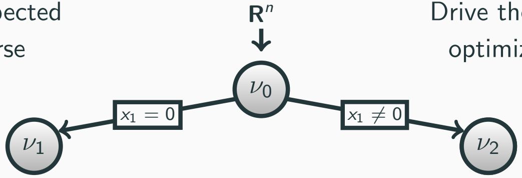

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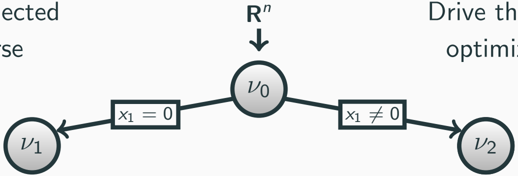


Pruning test
Solutions in ν_1 ?

Pruning test
Solutions in ν_2 ?

Problem

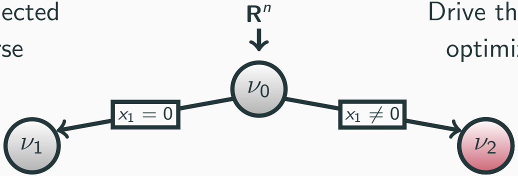
$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

Observation

Solutions expected to be sparse

Method

Drive the sparsity of the optimization variable



Pruning test
Solutions in ν_1 ?

Pruning test
Solutions in ν_2 ?
No

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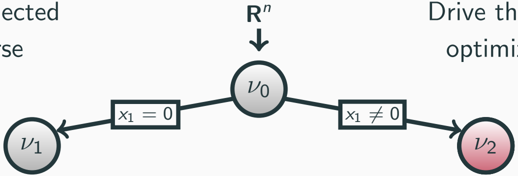

Pruning test
Solutions in ν_1 ?
Maybe

Pruning test
Solutions in ν_2 ?
No

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

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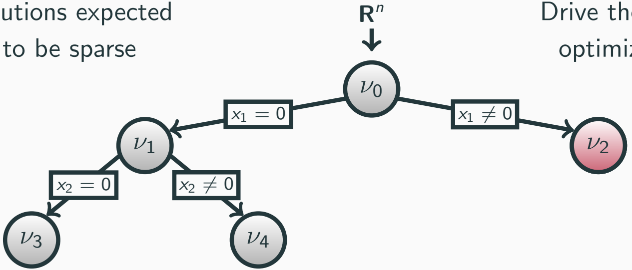

Problem

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Observation

Solutions expected to be sparse

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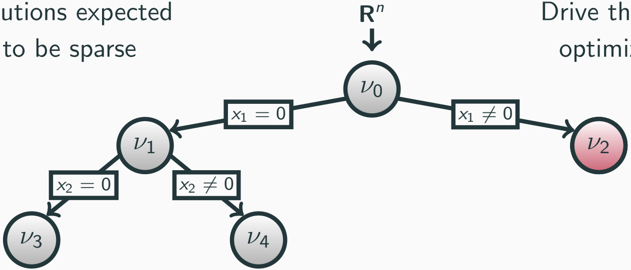


Pruning test
Solutions in ν_3 ?

Pruning test
Solutions in ν_4 ?

Problem

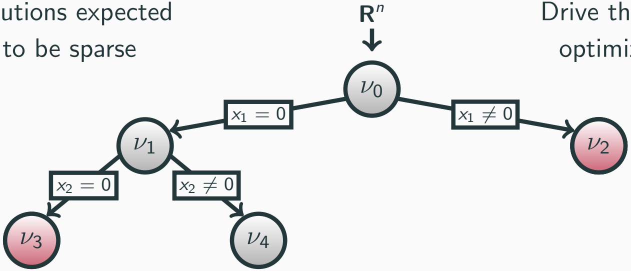
$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

Observation

Solutions expected to be sparse

Method

Drive the sparsity of the optimization variable



Pruning test
Solutions in ν_3 ?
No

Pruning test
Solutions in ν_4 ?
Maybe

BnB – How to construct regions in the feasible space ?

Problem

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Solutions expected to be sparse

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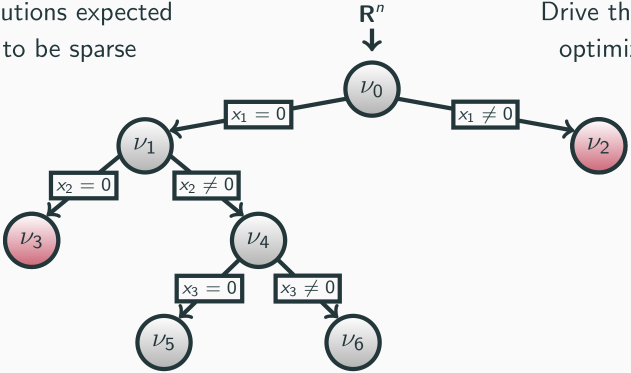
Drive the sparsity of the optimization variable



BnB – How to construct regions in the feasible space ?

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

Observation

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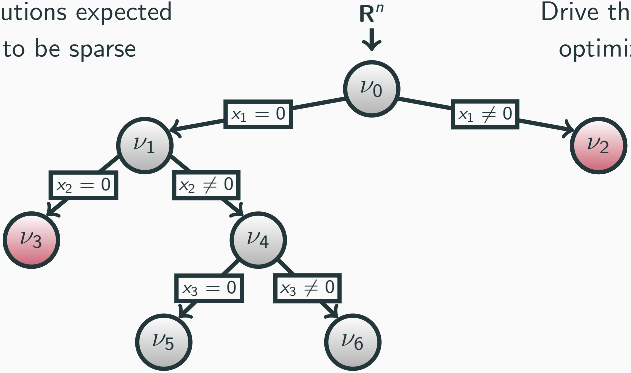
Drive the sparsity of the optimization variable



Pruning test
Solutions in ν_5 ?

Pruning test
Solutions in ν_6 ?

BnB – How to construct regions in the feasible space ?

Problem

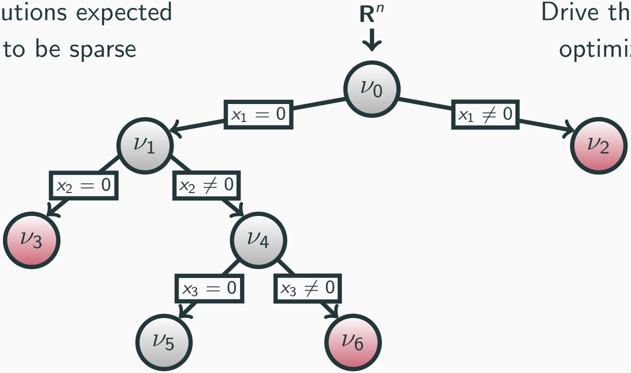
$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

Observation

Solutions expected to be sparse

Method

Drive the sparsity of the optimization variable



Pruning test
Solutions in ν_5 ?
Maybe

Pruning test
Solutions in ν_6 ?
No

BnB – How to construct regions in the feasible space ?

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$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

Observation

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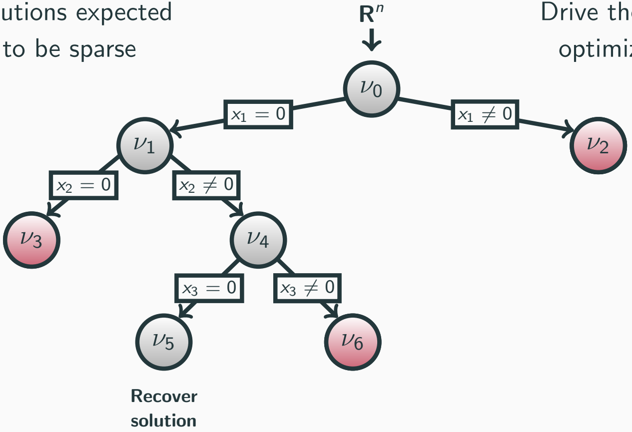
Method

Drive the sparsity of the optimization variable



Problem

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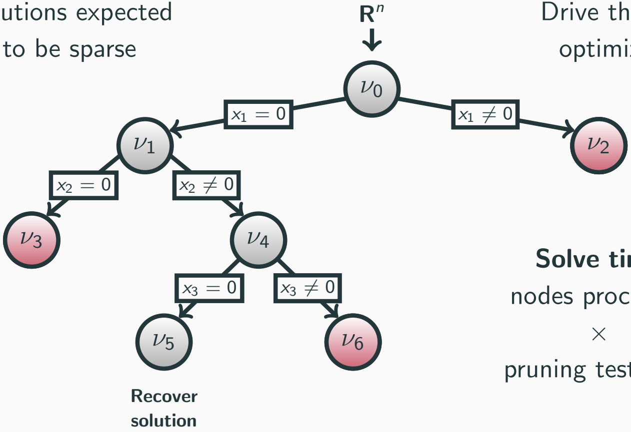
Observation

Solutions expected to be sparse



Method

Drive the sparsity of the optimization variable

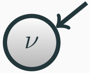
Solve time

nodes processed

$\times$

pruning test time

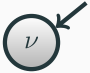
BnB – How to evaluate a pruning test ?



Pruning test

Solutions in region ν ?

BnB – How to evaluate a pruning test ?



Pruning test

Solutions in region ν ?

Problem

$p^\star = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

BnB – How to evaluate a pruning test ?

ν

Pruning test

Solutions in region ν ?

Problem

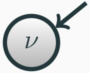
$$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

restrict to ν

Restriction to region ν

$$p^\nu = \min_{\mathbf{x}\in\nu} \; f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

BnB – How to evaluate a pruning test ?



Pruning test

Solutions in region ν ?

Problem

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restrict to ν

Restriction to region ν

$$p^\nu = \min_{\mathbf{x}\in\nu} \; f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

compare

Pruning test

$$p^\nu > p^\star$$

ν

Pruning test

Solutions in region ν ?

Problem

$$p^\star = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{Ax}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\downarrow$ restrict to ν

Restriction to region ν

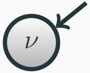
$$p^\nu = \min_{\mathbf{x} \in \nu} \ f(\mathbf{Ax}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\downarrow$ compare

Pruning test

$$p^\nu > p^\star$$

$\longrightarrow$ prune ν



Pruning test

Solutions in region ν ?

Problem

$$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\downarrow$ $\boxed{\text{restrict to } \nu}$

Restriction to region ν

$$p^\nu = \min_{\mathbf{x}\in\nu} \; f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\downarrow$ $\boxed{\text{compare}}$

Pruning test

$$p^\nu_{\mathrm{lb}} > p^\star_{\mathrm{ub}}$$

$\longrightarrow$ prune ν

BnB – How to evaluate a pruning test ?

ν

Pruning test

Solutions in region ν ?

Problem

$p^\star = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

restrict to ν

Restriction to region ν

$p^\nu = \min_{\mathbf{x} \in \nu} \; f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

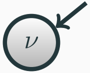
compare

Pruning test

$p^\nu_{\mathrm{lb}} > p^\star_{\mathrm{ub}}$ $\longrightarrow$ prune ν

Easy task

$\boxed{\text{Compute an upper bound on } p^\star}$



Pruning test

Solutions in region ν ?

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$\downarrow$ restrict to ν

Restriction to region ν

$$p^\nu = \min_{\mathbf{x}\in\nu} \; f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\downarrow$ compare

Pruning test

$$p^\nu_{\mathrm{lb}} > p^\star_{\mathrm{ub}} \longrightarrow \text{prune } \nu$$

Easy task

$\boxed{\text{Compute an upper bound on } p^\star}$

Construct and evaluate
a feasible vector in each
region explored to refine $p^\star_{\mathrm{ub}}$

BnB – How to evaluate a pruning test ?

ν

Pruning test

Solutions in region ν ?

Problem

$p^\star = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

restrict to ν

Restriction to region ν

$p^\nu = \min_{\mathbf{x} \in \nu} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

compare

Pruning test

$p^\nu_{\mathrm{lb}} > p^\star_{\mathrm{ub}}$ $\longrightarrow$ prune ν

Easy task

Compute an upper bound on $p^\star$

Construct and evaluate a feasible vector in each region explored to refine $p^\star_{\mathrm{ub}}$

Main challenge

Compute a lower bound on p^ν

BnB – Standard lower bounding strategy

Restriction to region ν

$$p^{\nu} = \min_{\mathbf{x} \in \nu} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$

BnB – Standard lower bounding strategy

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$$p^\nu = \min_{\mathbf{x} \in \nu} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$

Requirement 1 – Tight lower bound on p^ν
Requirement 2 – Tractable lower bound on p^ν

BnB – Standard lower bounding strategy

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Requirement 2 – Tractable lower bound on p^{ν}

$\downarrow$

Standard strategy

Construct and solve a relaxation

BnB – Standard lower bounding strategy

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$$p^\nu = \min_{\mathbf{x} \in \nu} f(\mathbf{A}\mathbf{x}) + \underbrace{\lambda \|\mathbf{x}\|_0 + h(\mathbf{x})}_{g(\mathbf{x})}$$

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Relaxation for region ν

$$p^{\nu}_{\mathrm{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\mathrm{lb}}(\mathbf{x})$$

BnB – Standard lower bounding strategy

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$g_{\text{lb}} \leq g$

g_{lb} convex

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Lower bound

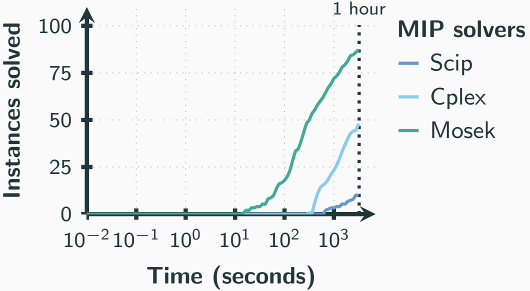
$$\boxed{p^\nu_{\text{lb}} \leq p^\nu}$$

BnB – Numerics

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Quadratic / h : Bound constraint



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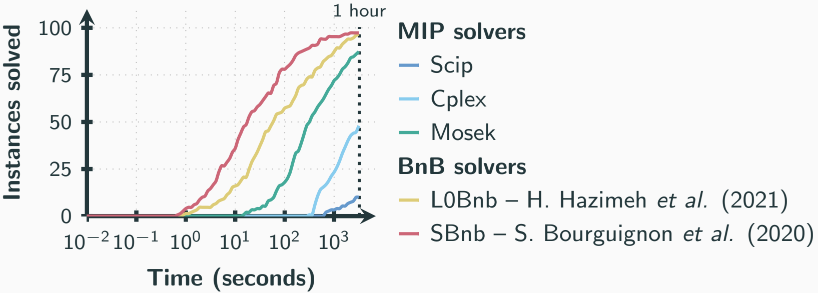

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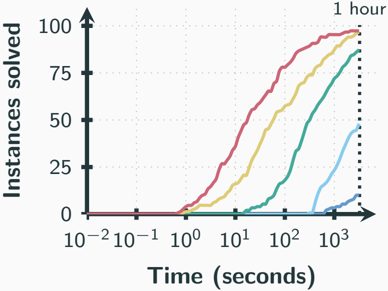

MIP solvers

- Scip
- Cplex
- Mosek

BnB solvers

- L0Bnb – H. Hazimeh *et al.* (2021)
- SBnb – S. Bourguignon *et al.* (2020)

Observation

Better performance with BnB solvers

BnB – Numerics

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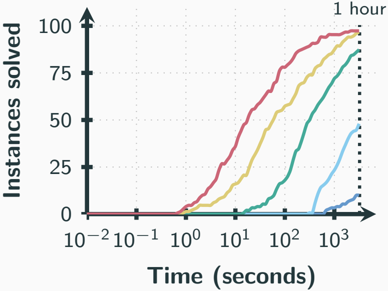

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Better performance with BnB solvers ... but they are instance-specific

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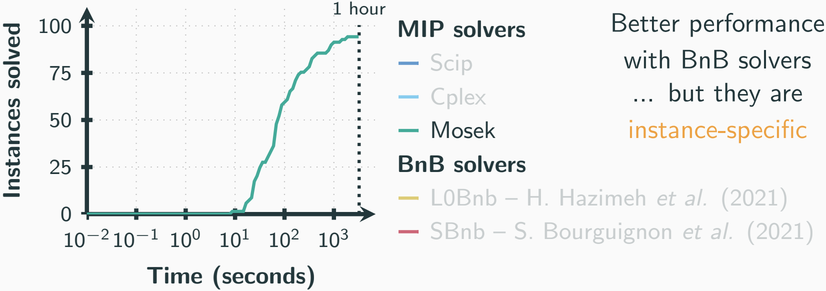

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Problem

$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$

$\rightarrow g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

$\rightarrow$ many applications

Let's recap

Problem

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BnB algorithm

Restriction to region ν

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→ instance-specific construction of g_{lb}

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Axis 1

How to construct relaxations generically ?

Manuscript – Chap. 3

→ Journal paper (202x)

Axis 1 – How to construct relaxations generically ?

Axis 1 – Generic relaxation construction

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Axis 1 – Generic relaxation construction

Restriction to region ν

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M. Pilanci *et al.* (2016)	S. Bourguignon *et al.* (2020)	H. Hazimeh *et al.* (2021)

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Generalization by setting $g_{\mathrm{lb}} = g_{\mathrm{cvx}}$

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Generalization by setting $g_{\mathrm{lb}} = g_{\mathrm{cvx}}$

$\rightarrow$ h proper, closed, convex

$\rightarrow$ h separable, even, supercoercive, $h \geq h(\mathbf{0}) = 0$

Axis 1 – Convex envelope characterization

Spotlight result

The convex envelope of $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$ admits a closed-form expression.

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Spotlight result

The convex envelope of $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$ admits a closed-form expression.

Theorem (1d version) – Let $g(x) = \lambda\|x\|_0 + h(x)$, one has

$$g_{\mathsf{cvx}}(x) = \begin{cases} \tau|x| & \text{if } |x| \leq \mu \\ h(x) + \lambda & \text{otherwise} \end{cases}$$

where (τ, μ) are some "easy-to-compute" quantities.

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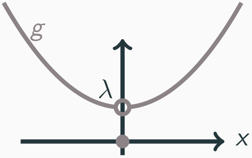

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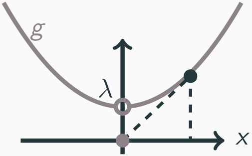

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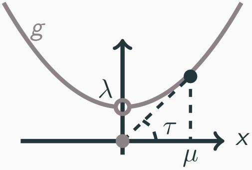

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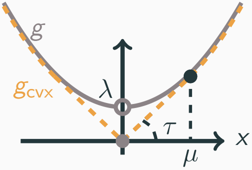

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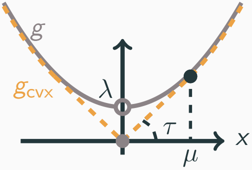

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↓

Generic relaxation construction

Characterize $g_{\mathsf{lb}} = g_{\mathsf{cvx}}$

Encompasses prior contributions

Axis 1 – Convex envelope characterization

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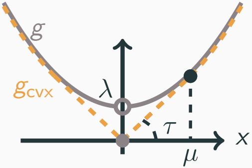

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Generic relaxation construction

Characterize $g_{\mathsf{lb}} = g_{\mathsf{cvx}}$

Encompasses prior contributions

↓

Practical relaxation construction

Closed-form for $\partial g_{\mathsf{cvx}}$ and $\mathrm{prox}_{g_{\mathsf{cvx}}}$

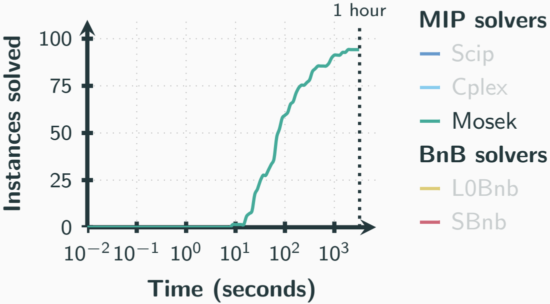
Enables standard solution methods

Axis 1 – Numerics

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / h : Bound cstr. + ℓ_1-norm



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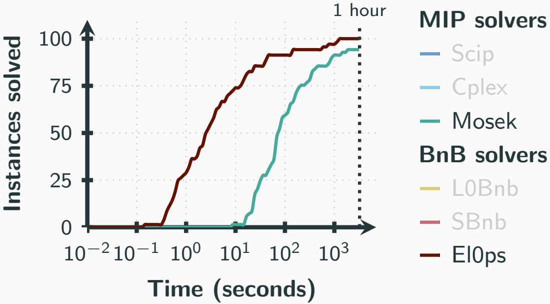

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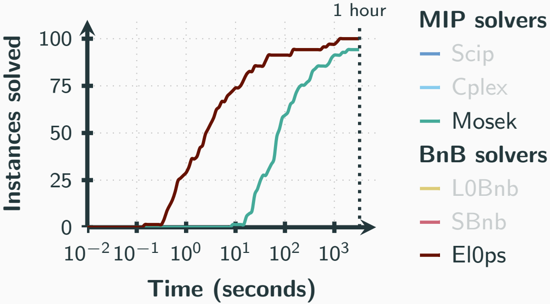

El0ps is a generic BnB solver with state-of-the-art performance

Let's recap

Problem

$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$

→ $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

→ many applications

↓ BnB algorithm

Restriction to region ν

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→ pruning test

→ lower bound on p^ν

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→ instance-specific construction of g_{lb}

Axis 1

How to construct relaxations generically ?

1) Set $g_{\mathrm{lb}} = g_{\mathrm{cvx}}$
2) Closed-form expression
3) Generalize BnB method

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Axis 2

How to solve relaxations efficiently ?

Manuscript – Chap. 6

→ ICASSP (2022)

→ Journal paper (202x)

Axis 2 – How to solve relaxations efficiently ?

Axis 2 – Convex optimization

Relaxation for region ν

$$\min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\mathsf{cvx}}(\mathbf{x})$$

Relaxation for region ν

$$\min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\mathrm{cvx}}(\mathbf{x})$$

$\updownarrow$ $\boxed{\text{reformulation}}$

Convex problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

Axis 2 – Convex optimization

Relaxation for region ν

$\min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\text{cvx}}(\mathbf{x})$

$\updownarrow$ $\boxed{\text{reformulation}}$

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \tilde{g}(\mathbf{x})$

- $\tilde{g}$ proper, closed, convex
- $\tilde{g}$ separable
- $\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

Axis 2 – Convex optimization

Relaxation for region ν

$\min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\text{cvx}}(\mathbf{x})$

remind g_{cvx} acts as an ℓ_1-norm near $\mathbf{x} = \mathbf{0}$

$\updownarrow$ $\boxed{\text{reformulation}}$

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$\uparrow$ $\boxed{\text{reformulation}}$ $\downarrow$

Applications beyond the BnB scope

Convex problem

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$\Big\updownarrow$ $\boxed{\text{reformulation}}$

Applications beyond the BnB scope $\longrightarrow$

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$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

1st-order methods

Proximal gradient

Coordinate descent

...

iteration

$\mathbf{x}^0 \longrightarrow \boxed{\mathbf{x}^k \to \mathbf{x}^{k+1}} \longrightarrow \mathbf{x}^\star$

Axis 2 – Convex optimization

Relaxation for region ν

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$\updownarrow$ reformulation

Applications beyond the BnB scope $\rightarrow$

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$\tilde{g}$ separable

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

1^{st}-order methods

Proximal gradient

Coordinate descent

...

iteration

$\mathbf{x}^0 \longrightarrow \boxed{\mathbf{x}^k \rightarrow \mathbf{x}^{k+1}} \longrightarrow \mathbf{x}^\star$

Solving cost

cost per iteration

$\times$

number of iterations

Axis 2 – Convex optimization

Relaxation for region ν

$\min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\text{cvx}}(\mathbf{x})$

remind g_{cvx} acts as an ℓ_1-norm near $\mathbf{x} = \mathbf{0}$

$\boxed{\text{reformulation}}$

Applications beyond the BnB scope

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\tilde{g}$ proper, closed, convex

$\tilde{g}$ separable

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

1st-order methods

Proximal gradient

Coordinate descent

...

iteration

$\mathbf{x}^0 \longrightarrow \boxed{\mathbf{x}^k \to \mathbf{x}^{k+1}} \longrightarrow \mathbf{x}^\star$

Solving cost

cost per iteration

$\times$

number of iterations

dimension of $\mathbf{x}$

Relaxation for region ν

$\min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\text{cvx}}(\mathbf{x})$

remind g_{cvx} acts as an ℓ_1-norm near $\mathbf{x} = \mathbf{0}$

$\uparrow$ reformulation $\downarrow$

Applications beyond the BnB scope $\rightarrow$

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

- $\tilde{g}$ proper, closed, convex
- $\tilde{g}$ separable
- $\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

1st-order methods

Proximal gradient

Coordinate descent

...

iteration

$\mathbf{x}^0 \longrightarrow \boxed{\mathbf{x}^k \to \mathbf{x}^{k+1}} \longrightarrow \mathbf{x}^\star$

Solving cost

cost per iteration

$\times$

number of iterations

dimension of $\mathbf{x}$

regularity of $f/\tilde{g}$

Axis 2 – Sparse structure exploitation

Convex problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

Task

$\boxed{\text{Reduce the solving cost}}$

Axis 2 – Sparse structure exploitation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Task

Reduce the solving cost

Option 1

Reduce the dimension of $\mathbf{x}$

Axis 2 – Sparse structure exploitation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

$\Downarrow$

sparse solution $\mathbf{x}^\star$

Task

Reduce the solving cost

Option 1

Reduce the dimension of $\mathbf{x}$

Axis 2 – Sparse structure exploitation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

$\downarrow$

sparse solution $\mathbf{x}^\star$

Task

Reduce the solving cost

$\downarrow$

Option 1

Reduce the dimension of $\mathbf{x}$

$\downarrow$

Screening tests

Identify zeros in $\mathbf{x}^\star$

L. El Ghaoui *et al.* (2011)

E. Ndiaye *et al.* (2020)

Axis 2 – Sparse structure exploitation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

$\downarrow$

sparse solution $\mathbf{x}^\star$

Task

Reduce the solving cost

Option 1

Reduce the dimension of $\mathbf{x}$

$\downarrow$

Screening tests

Identify zeros in $\mathbf{x}^\star$

L. El Ghaoui *et al.* (2011)

E. Ndiaye *et al.* (2020)

Option 2

Improve the regularity of $f/\tilde{g}$

Axis 2 – Sparse structure exploitation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

$\downarrow$

sparse solution $\mathbf{x}^\star$

Task

Reduce the solving cost

Option 1

Reduce the dimension of $\mathbf{x}$

$\downarrow$

Screening tests

Identify zeros in $\mathbf{x}^\star$

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E. Ndiaye *et al.* (2020)

Option 2

Improve the regularity of $f/\tilde{g}$

$\downarrow$

Smoothing tests

Identify non-zeros in $\mathbf{x}^\star$

Axis 2 – Sparse structure exploitation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\tilde{g}$ non-smooth at $\mathbf{x} = \mathbf{0}$

$\downarrow$

sparse solution $\mathbf{x}^\star$

Task

Reduce the solving cost

Option 1

Reduce the dimension of $\mathbf{x}$

$\downarrow$

Screening tests

Identify zeros in $\mathbf{x}^\star$

L. El Ghaoui *et al.* (2011)

E. Ndiaye *et al.* (2020)

Option 2

Improve the regularity of $f/\tilde{g}$

$\downarrow$

Smoothing tests

Identify non-zeros in $\mathbf{x}^\star$

Axis 2 – Screening and smoothing tests

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Axis 2 – Screening and smoothing tests

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Axis 2 – Screening and smoothing tests

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

solution $\mathbf{x}^\star$

if strong duality holds

$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$

solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\downarrow$ solution $\mathbf{x}^\star$

if strong duality holds

$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$

$\uparrow$ solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Intermediate result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a dual solution $\mathbf{u}^\star$.

Axis 2 – Screening and smoothing tests

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

solution $\mathbf{x}^\star$

if strong duality holds

$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$

solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Intermediate result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a dual solution $\mathbf{u}^\star$.

Safe region

$\mathcal{R} \subseteq \mathbf{R}^m$ with $\mathbf{u}^\star \in \mathcal{R}$

Axis 2 – Screening and smoothing tests

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

solution $\mathbf{x}^\star$

if strong duality holds

$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$

solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Intermediate result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a dual solution $\mathbf{u}^\star$.

Safe region

$\mathcal{R} \subseteq \mathbf{R}^m$ with $\mathbf{u}^\star \in \mathcal{R}$

Spotlight result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a safe region $\mathcal{R}$.

Axis 2 – Screening and smoothing tests

remind $\tilde{g}(\mathbf{x}) = \sum_{i=1}^{n} \tilde{g}_i(x_i)$

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

solution $\mathbf{x}^\star$

if strong duality holds

$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$

solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Intermediate result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a dual solution $\mathbf{u}^\star$.

Safe region

$\mathcal{R} \subseteq \mathbf{R}^m$ with $\mathbf{u}^\star \in \mathcal{R}$

Spotlight result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a safe region $\mathcal{R}$.

remind $\tilde{g}(\mathbf{x}) = \sum_{i=1}^{n} \tilde{g}_i(x_i)$

Convex problem

$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\downarrow$ solution $\mathbf{x}^\star$

if strong duality holds

$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$

$\uparrow$ solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u} \in \mathbf{R}^m} D(\mathbf{u})$

Intermediate result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a dual solution $\mathbf{u}^\star$.

Safe region

$\mathcal{R} \subseteq \mathbf{R}^m$ with $\mathbf{u}^\star \in \mathcal{R}$

Spotlight result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a safe region $\mathcal{R}$.

Theorem – Given a safe region $\mathcal{R}$, note $\mathbf{a}^{\mathrm{T}}\mathcal{R} = \left\{\mathbf{a}^{\mathrm{T}}\mathbf{u} \mid \mathbf{u} \in \mathcal{R}\right\}$, one has

$$\text{Screening test:} \quad \mathbf{a}_i^{\mathrm{T}}\mathcal{R} \subseteq \mathrm{int}(\partial\tilde{g}_i(0)) \implies x_i^\star = 0$$

$$\text{Smoothing test:} \quad \mathbf{a}_i^{\mathrm{T}}\mathcal{R} \subseteq \mathrm{cmpl}(\partial\tilde{g}_i(0)) \implies x_i^\star \neq 0$$

remind $\tilde{g}(\mathbf{x}) = \sum_{i=1}^{n} \tilde{g}_i(x_i)$

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

solution $\mathbf{x}^\star$

if strong duality holds

$$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$$

solution $\mathbf{u}^\star$

Dual problem

$\max_{\mathbf{u}\in\mathbf{R}^m} D(\mathbf{u})$

Intermediate result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a dual solution $\mathbf{u}^\star$.

Safe region

$\mathcal{R} \subseteq \mathbf{R}^m$ with $\mathbf{u}^\star \in \mathcal{R}$

Spotlight result

Some zeros and non-zeros in $\mathbf{x}^\star$ can be identified from a safe region $\mathcal{R}$.

Theorem – Given a safe region $\mathcal{R}$, note $\mathbf{a}^{\mathrm{T}}\mathcal{R} = \{\mathbf{a}^{\mathrm{T}}\mathbf{u} \mid \mathbf{u} \in \mathcal{R}\}$, one has

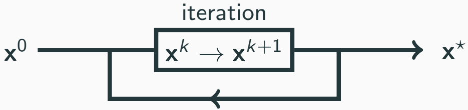
$$\text{Screening test:} \quad \mathbf{a}_i^{\mathrm{T}}\mathcal{R} \subseteq \operatorname{int}(\partial\tilde{g}_i(0)) \implies x_i^\star = 0$$

$$\text{Smoothing test:} \quad \underbrace{\mathbf{a}_i^{\mathrm{T}}\mathcal{R} \subseteq \operatorname{cmpl}(\partial\tilde{g}_i(0))}_{\text{easy to evaluate if } \mathcal{R} \text{ has a simple shape}} \implies x_i^\star \neq 0$$

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

1^{st}-order methods



Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

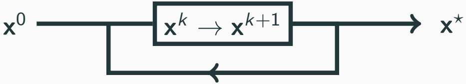
cost per iteration

$\times$

number of iterations

1st-order methods

iteration



Convex problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

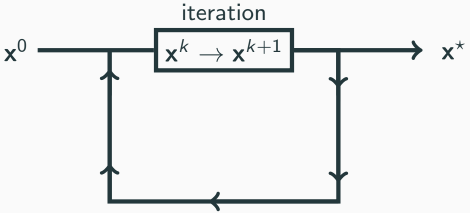
Solving cost

cost per iteration

$\times$

number of iterations

1^{st}-order methods



Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

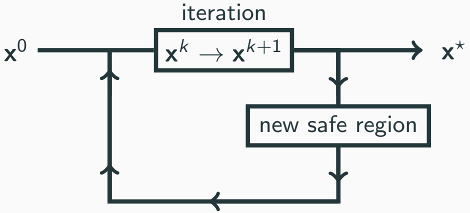
Solving cost

cost per iteration

$\times$

number of iterations

1^{st}-order methods



Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

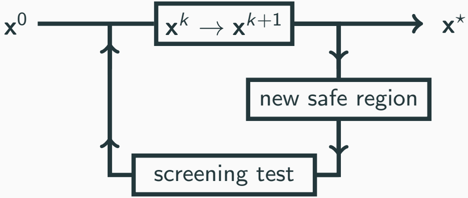
cost per iteration

$\times$

number of iterations

1^{st}-order methods

iteration



Identify some $x_i^\star = 0$

Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

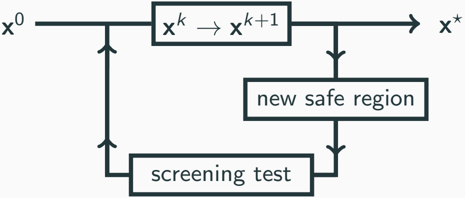
cost per iteration

$\times$

number of iterations

1^{st}-order methods

iteration



Identify some $x_i^\star = 0$

$\downarrow$

Set $x_i = 0$

Reduce cost

per iteration

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

cost per iteration

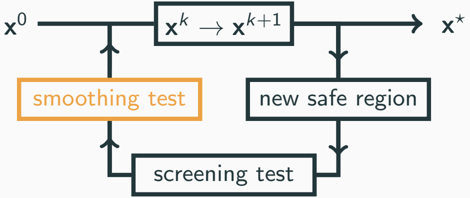
$\times$

number of iterations

1^{st}-order methods

iteration

Identify some $x_i^\star \neq 0$



Identify some $x_i^\star = 0$

$\downarrow$

Set $x_i = 0$

Reduce cost

per iteration

Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

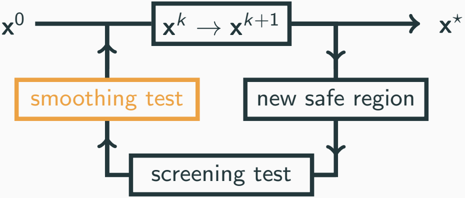
cost per iteration

$\times$

number of iterations

1^{st}-order methods

iteration



Identify some $x_i^\star \neq 0$

$\downarrow$

Smooth $\tilde{g}_i$, use

2^{nd}-order iterations

Reduce number

of iterations

Identify some $x_i^\star = 0$

$\downarrow$

Set $x_i = 0$

Reduce cost

per iteration

Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

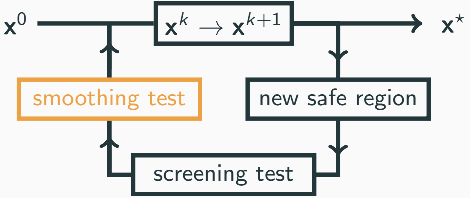
cost per iteration

$\times$

number of iterations

1^{st}-order methods

iteration



Identify some $x_i^\star \neq 0$

$\downarrow$

Smooth $\tilde{g}_i$, use

2nd-order iterations

Reduce number

of iterations

$\tilde{g}(\mathbf{x}) = \|\mathbf{x}\|_1 + \|\mathbf{x}\|_2^2$

$\downarrow$

$\tilde{g}(\mathbf{x}) = \mathbf{s}^{\mathrm{T}}\mathbf{x} + \|\mathbf{x}\|_2^2$

Identify some $x_i^\star = 0$

$\downarrow$

Set $x_i \ = \ 0$

Reduce cost

per iteration

Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

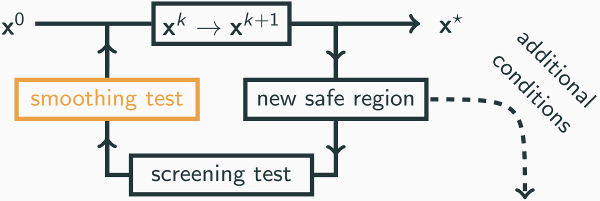
cost per iteration

$\times$

number of iterations

1^{st}-order methods

iteration



Identify some $x_i^\star \neq 0$

$\downarrow$

Smooth $\tilde{g}_i$, use

2^{nd}-order iterations

Reduce number

of iterations

$\tilde{g}(\mathbf{x}) = \|\mathbf{x}\|_1 + \|\mathbf{x}\|_2^2$

$\downarrow$

$\tilde{g}(\mathbf{x}) = \mathbf{s}^{\mathrm{T}}\mathbf{x} + \|\mathbf{x}\|_2^2$

Identify some $x_i^\star = 0$

$\downarrow$

Set $x_i = 0$

Reduce cost

per iteration

All zeros and non-zeros

identified after a finite

number of iterations

Axis 2 – Dynamic identification

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

Solving cost

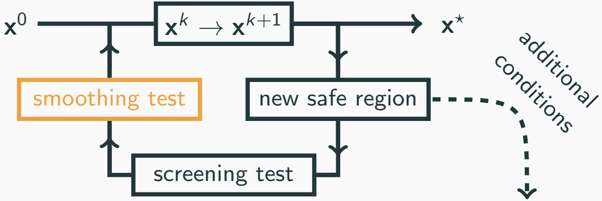
cost per iteration

$\times$

number of iterations

1^{st}-order methods

iteration



Identify some $x_i^\star \neq 0$

$\downarrow$

Smooth $\tilde{g}_i$, use

2^{nd}-order iterations

Reduce number

of iterations

$\tilde{g}(\mathbf{x}) = \|\mathbf{x}\|_1 + \|\mathbf{x}\|_2^2$

$\downarrow$

$\tilde{g}(\mathbf{x}) = \mathbf{s}^{\mathrm{T}}\mathbf{x} + \|\mathbf{x}\|_2^2$

Identify some $x_i^\star = 0$

$\downarrow$

Set $x_i = 0$

Reduce cost

per iteration

All zeros and non-zeros

identified after a finite

number of iterations

$\downarrow$

Smooth problem of

reduced dimension

Axis 2 – Numerics

Convex problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net

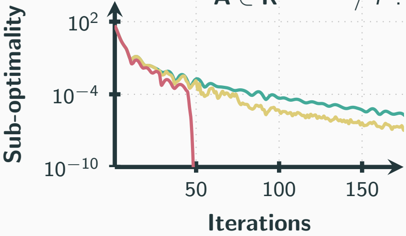
Accelerated proximal gradient

- Vanilla method
- With screening
- With screening and smoothing

Convex problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net



Accelerated proximal gradient

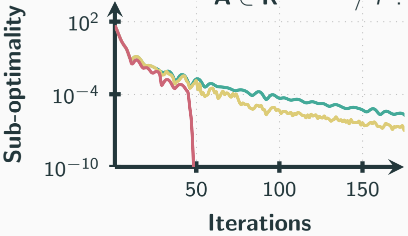
- Vanilla method
- With screening
- With screening and smoothing

Axis 2 – Numerics

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net



Accelerated proximal gradient

- Vanilla method
- With screening
- With screening and smoothing

1^{st}**-order** $\rightarrow$ 2^{nd}**-order**

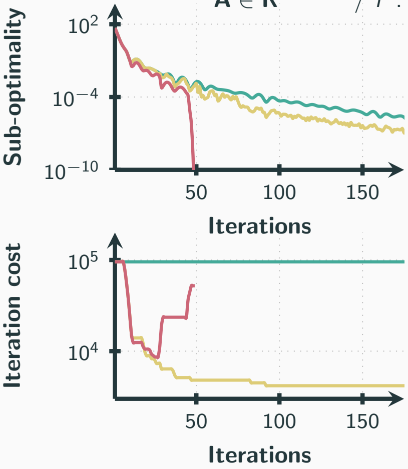
Faster convergence

Axis 2 – Numerics

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net



Accelerated proximal gradient

- Vanilla method
- With screening
- With screening and smoothing

1st-order → 2nd-order

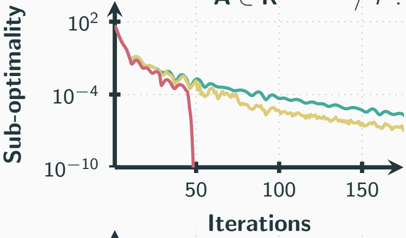
Faster convergence

More expensive iterations

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net



Accelerated proximal gradient

- Vanilla method
- With screening
- With screening and smoothing

1^{st}**-order** $\rightarrow$ 2^{nd}**-order**

Faster convergence

More expensive iterations

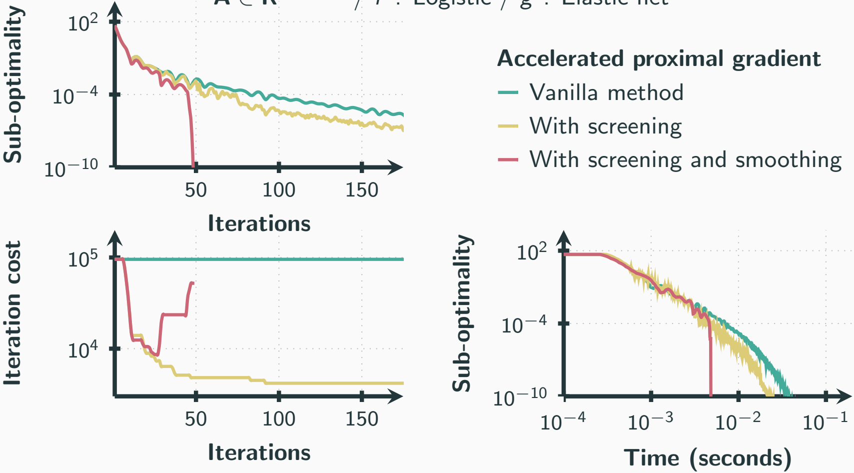
What about the solving time ?

Axis 2 – Numerics

Convex problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Logistic / $\tilde{g}$: Elastic-net



Let's recap

Problem

$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$

→ $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

→ many applications

BnB algorithm

Restriction to region ν

$p^\nu = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$

→ pruning test

→ lower bound on p^ν

standard strategy

Relaxation for region ν

$p^\nu_{\mathrm{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\mathrm{lb}}(\mathbf{x})$

→ instance-specific construction of g_{lb}

Axis 1

How to construct relaxations generically ?

1) Set $g_{\mathrm{lb}} = g_{\mathrm{cvx}}$
2) Closed-form expression
3) Generalize BnB method

Axis 2

How to solve relaxations efficiently ?

1) Cast as convex problem
2) Screening/smoothing
3) Reduce solving cost

Let's recap

Problem

$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$

→ $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

→ many applications

BnB algorithm

Restriction to region ν

$p^\nu = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$

→ pruning test

→ lower bound on p^ν

standard strategy

Relaxation for region ν

$p^\nu_{\mathrm{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\mathrm{lb}}(\mathbf{x})$

→ instance-specific construction of g_{lb}

Axis 1

How to construct relaxations generically ?

1) Set $g_{\mathrm{lb}} = g_{\mathrm{cvx}}$
2) Closed-form expression
3) Generalize BnB method

Axis 2

How to solve relaxations efficiently ?

1) Cast as convex problem
2) Screening/smoothing
3) Reduce solving cost

Axis 3

How to improve the standard strategy ?

Manuscript – Chap. 4-5

→ ICASSP (2022)

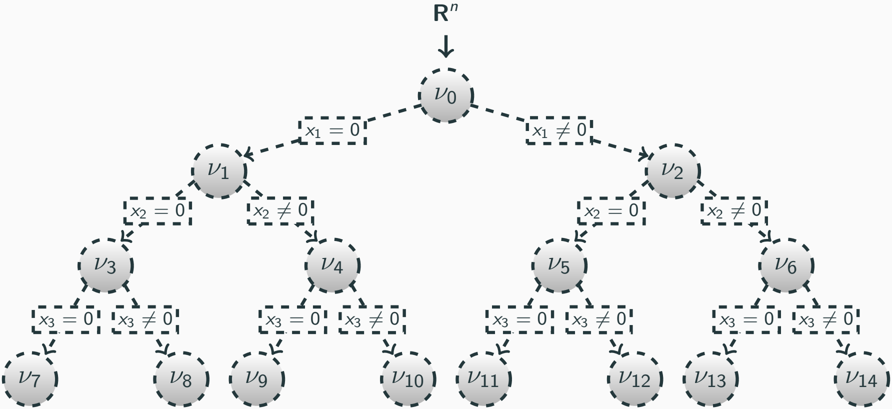
→ EUSIPCO (2023)

→ ICML (2024)

Axis 3 – How to improve the standard strategy ?

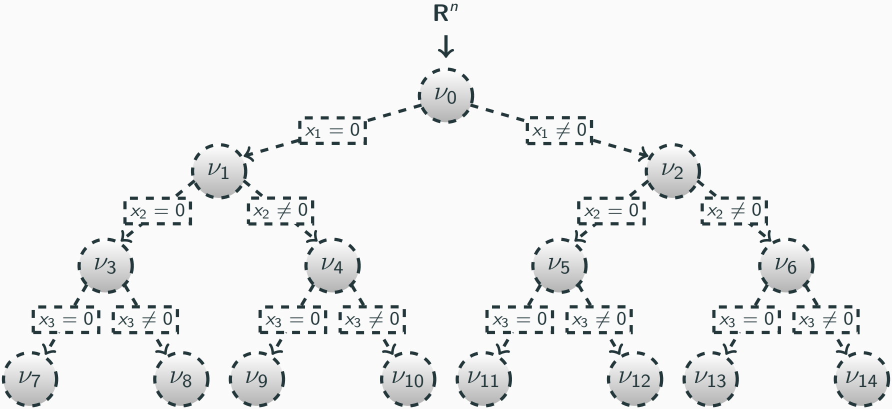
Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$



Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

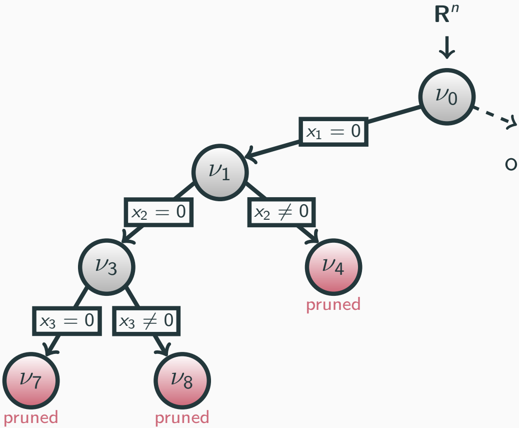


Solving time

$\boxed{\text{number of nodes explored} \times \text{pruning test time}}$

Problem

$$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$



Standard strategy

one relaxation solved per node

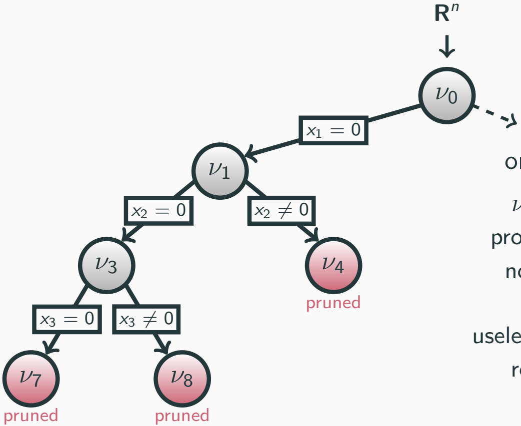
Solving time

$$\boxed{\text{number of nodes explored} \times \text{pruning test time}}$$

Axis 3 – Algorithmic complexity

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$



Standard strategy

one relaxation solved per node

ν_1 and ν_3 processed but not pruned

↓

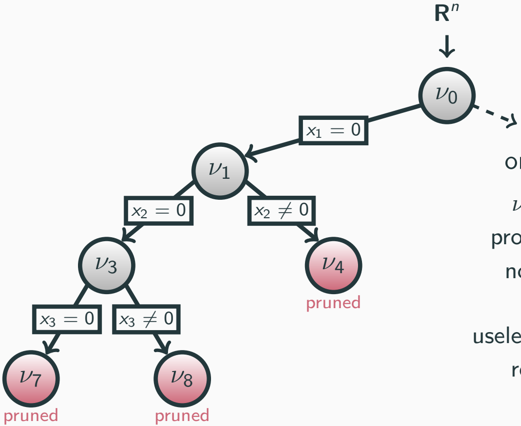
useless relaxation resolution

Solving time

$$\boxed{\text{number of nodes explored} \times \text{pruning test time}}$$

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$



Standard strategy

one relaxation solved per node

ν_1 and ν_3 processed but not pruned

↓

useless relaxation resolution

ν_4, ν_7 and ν_8 pruned with rough lower bounds

↓

unnecessarily tight lower bounds

Solving time

$\boxed{\text{number of nodes explored} \times \text{pruning test time}}$

Axis 3 – Dual bounds

Question

Can we balance the complexity/tightness tradeoff when computing the lower bound on p^ν ?

Restriction to region ν

$$p^\nu = \min_{\mathbf{x} \in \nu} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$$

$\rightarrow g(\mathbf{x}) = \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$

Axis 3 – Dual bounds

Question

Can we balance the complexity/tightness tradeoff
when computing the lower bound on p^ν ?

standard strat.

Restriction to region ν

$p^\nu = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g(\mathbf{x})$

→ $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

$\vee|$

Relaxation for region ν

$p^\nu_{\mathrm{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\mathrm{lb}}(\mathbf{x})$

→ $p^\nu_{\mathrm{lb}} \leq p^\nu$

Axis 3 – Dual bounds

Question

Can we balance the complexity/tightness tradeoff when computing the lower bound on p^ν ?

standard strat.

dual problem

Restriction to region ν

$$p^\nu = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g(\mathbf{x})$$

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$\geq$

Relaxation for region ν

$$p^\nu_{\text{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\text{lb}}(\mathbf{x})$$

$\rightarrow p^\nu_{\text{lb}} \leq p^\nu$

$\geq$

Dual relaxation for region ν

$$p^\nu_{\text{dual}} = \max_{\mathbf{u}\in\mathbf{R}^m} \; D^\nu(\mathbf{u})$$

Axis 3 – Dual bounds

Question

Can we balance the complexity/tightness tradeoff when computing the lower bound on p^ν ?

standard strat.

dual problem

Restriction to region ν

$$p^\nu = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x})$$

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$$p^\nu_{\text{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g_{\text{lb}}(\mathbf{x})$$

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→ $p^\nu_{\text{dual}} \leq p^\nu_{\text{lb}} \leq p^\nu$

Axis 3 – Dual bounds

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Can we balance the complexity/tightness tradeoff when computing the lower bound on p^ν ?

Restriction to region ν

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standard strat.

$\vee\!|$

Relaxation for region ν

$$p^\nu_{\mathrm{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\mathrm{lb}}(\mathbf{x})$$

$\rightarrow p^\nu_{\mathrm{lb}} \leq p^\nu$

dual problem

$\vee\!|$

Dual relaxation for region ν

$$p^\nu_{\mathrm{dual}} = \max_{\mathbf{u}\in\mathbf{R}^m} \; D^\nu(\mathbf{u})$$

$\rightarrow p^\nu_{\mathrm{dual}} \leq p^\nu_{\mathrm{lb}} \leq p^\nu$

Any evaluation of D^ν gives a lower-bound on p^ν

Axis 3 – Successor link

Standard lower bound

$$p^{\nu} \geq p^{\nu}_{\mathrm{lb}}$$

Tight but expensive

Dual lower bound

$$p^{\nu} \geq D^{\nu}(\mathbf{u})$$

Rough but economical

Axis 3 – Successor link

Standard lower bound	**Dual lower bound**
$p^\nu \geq p^\nu_{\text{lb}}$	$p^\nu \geq D^\nu(\mathbf{u})$
Tight but expensive	Rough but economical

A. Atamtürk *et al.* (2020) / G. Samain *et al.* (2023)

Axis 3 – Successor link

Standard lower bound	Dual lower bound
$p^\nu \geq p^\nu_{\text{lb}}$	$p^\nu \geq D^\nu(\mathbf{u})$
Tight but expensive	Rough but economical

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Spotlight result

Successor nodes in the BnB tree share similar dual lower bounds.

Axis 3 – Successor link

Standard lower bound

$p^\nu \geq p^\nu_{\mathrm{lb}}$

Tight but expensive

Dual lower bound

$p^\nu \geq D^\nu(\mathbf{u})$

Rough but economical

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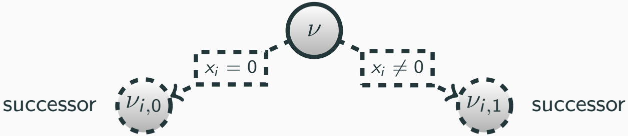

Axis 3 – Successor link

Standard lower bound	Dual lower bound
$p^{\nu} \geq p^{\nu}_{\mathsf{lb}}$	$p^{\nu} \geq D^{\nu}(\mathbf{u})$

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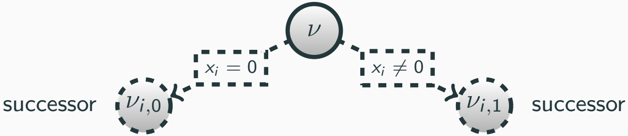


Theorem – Let $\nu_{i,b}$ with $b \in \{0, 1\}$ be a successor of node ν, then

$$D^{\nu_{i,b}}(\mathbf{u}) = D^{\nu}(\mathbf{u}) + \Delta^{i,b}(\mathbf{u})$$

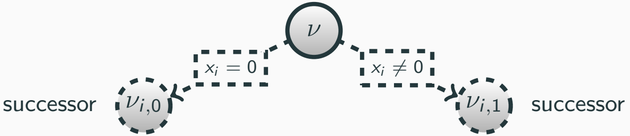
Axis 3 – Successor link

Standard lower bound	**Dual lower bound**
$p^\nu \geq p^\nu_{\mathrm{lb}}$	$p^\nu \geq D^\nu(\mathbf{u})$
Tight but expensive	Rough but economical

A. Atamtürk *et al.* (2020) / G. Samain *et al.* (2023)

Spotlight result

$\boxed{\text{Successor nodes in the BnB tree share similar dual lower bounds.}}$



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$$D^{\nu_{i,b}}(\mathbf{u}) = \underbrace{D^\nu(\mathbf{u})}_{\text{Independent of } \nu_{i,b}} + \underbrace{\Delta^{i,b}(\mathbf{u})}_{\text{Virtually cost-free}}$$

Standard lower bound

$p^{\nu} \geq p^{\nu}_{\text{lb}}$

Tight but expensive

Dual lower bound

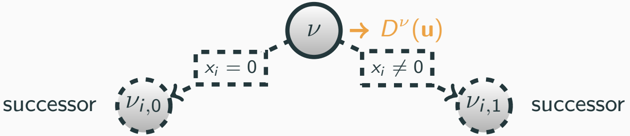
$p^{\nu} \geq D^{\nu}(\mathbf{u})$

Rough but economical

A. Atamtürk *et al.* (2020) / G. Samain *et al.* (2023)

Spotlight result

Successor nodes in the BnB tree share similar dual lower bounds.



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Axis 3 – Successor link

Standard lower bound	**Dual lower bound**
$p^\nu \geq p^\nu_{\mathsf{lb}}$	$p^\nu \geq D^\nu(\mathbf{u})$

Tight but expensive

Rough but economical

A. Atamtürk *et al.* (2020) / G. Samain *et al.* (2023)

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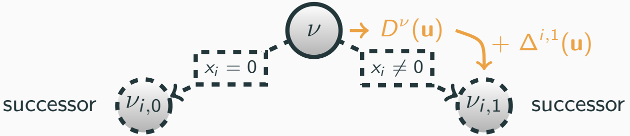
Successor nodes in the BnB tree share similar dual lower bounds.



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Standard pruning

Solve one relaxation per node

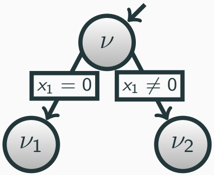
Select two successors to test next

Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

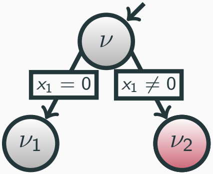
Select two successors to test next



Standard pruning

Solve one relaxation per node

Select two successors to test next



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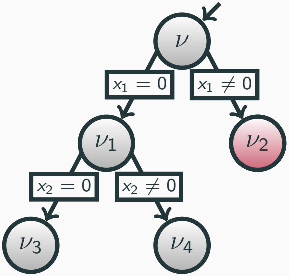

Standard pruning

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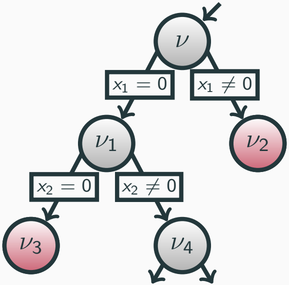
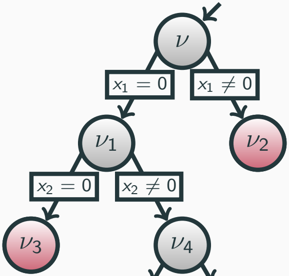


Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next



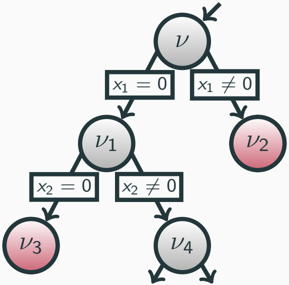
Slow and costly
tree expansion

Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next



Slow and costly

tree expansion

Simultaneous pruning

Obtain $D^\nu(\mathbf{u})$ during node processing

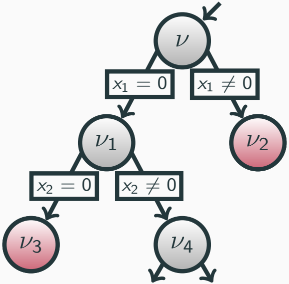
Test all successors with dual bounds

Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next



Slow and costly
tree expansion

Simultaneous pruning

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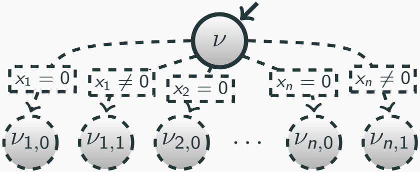
Test all successors with dual bounds



Standard pruning

Solve one relaxation per node

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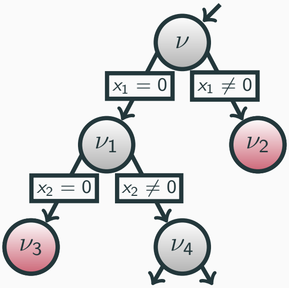

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tree expansion

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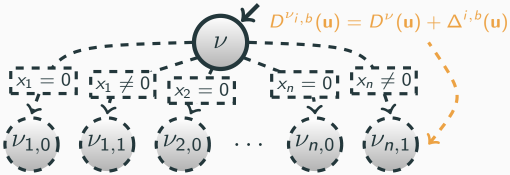

Standard pruning

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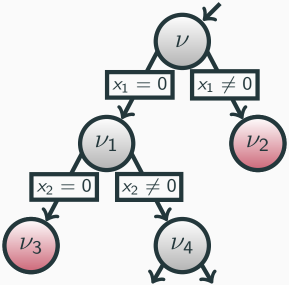

Slow and costly
tree expansion

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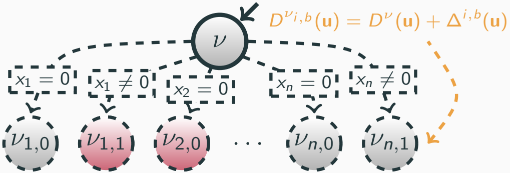
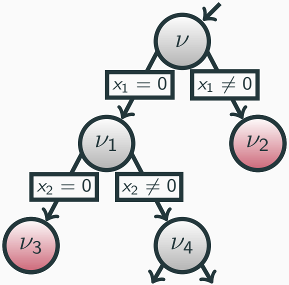


Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next

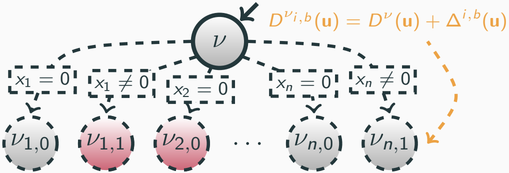


Slow and costly
tree expansion

Simultaneous pruning

Obtain $D^{\nu}(\mathbf{u})$ during node processing

Test all successors with dual bounds



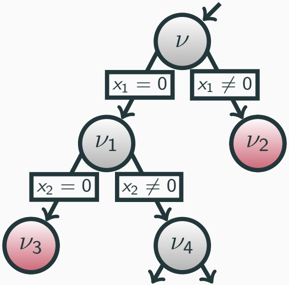
Tree expansion

Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next

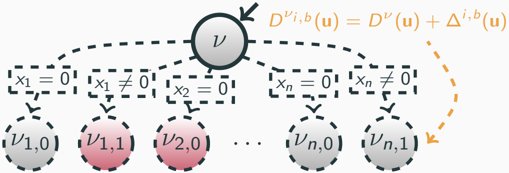


Slow and costly tree expansion

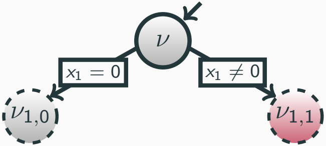
Simultaneous pruning

Obtain $D^{\nu}(\mathbf{u})$ during node processing

Test all successors with dual bounds



Tree expansion



Standard pruning

Solve one relaxation per node

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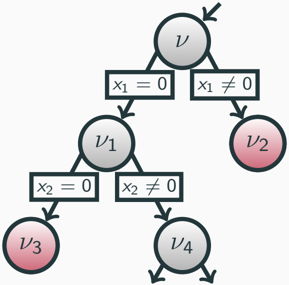
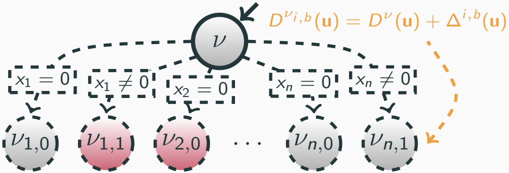


Slow and costly tree expansion

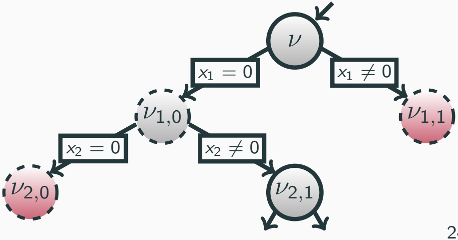
Simultaneous pruning

Obtain $D^{\nu}(\mathbf{u})$ during node processing

Test all successors with dual bounds



Tree expansion

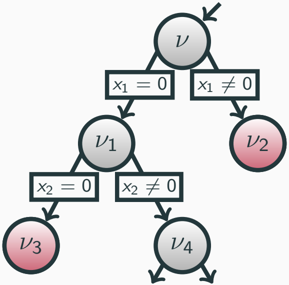


Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next

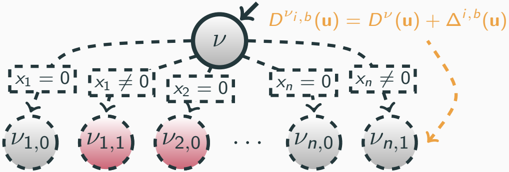


Slow and costly
tree expansion

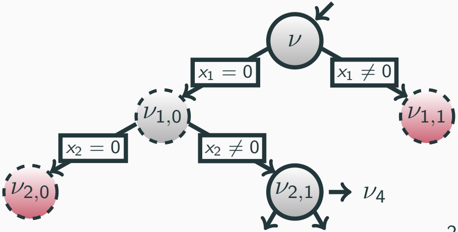
Simultaneous pruning

Obtain $D^{\nu}(\mathbf{u})$ during node processing

Test all successors with dual bounds



Tree expansion

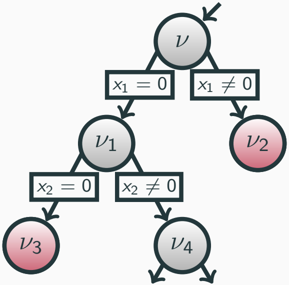


Axis 3 – Paradigm shift

Standard pruning

Solve one relaxation per node

Select two successors to test next



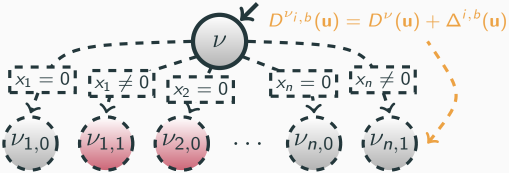
Slow and costly

tree expansion

Simultaneous pruning

Obtain $D^{\nu}(\mathbf{u})$ during node processing

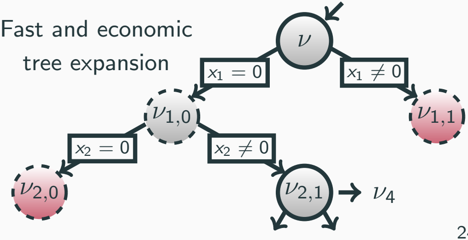
Test all successors with dual bounds



Tree expansion

Fast and economic

tree expansion



Axis 3 – Numerics

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

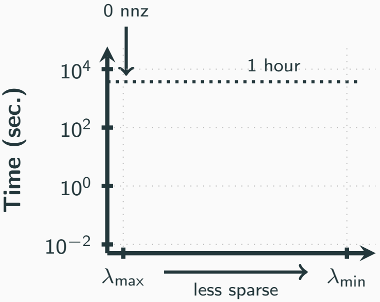
$\mathbf{A} \in \mathbf{R}^{62\times 2000}$ from ML dataset / f : Logistic / h : ℓ_2-norm

Axis 3 – Numerics

Problem

$$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{62 \times 2000}$ from ML dataset / f : Logistic / h : ℓ_2-norm



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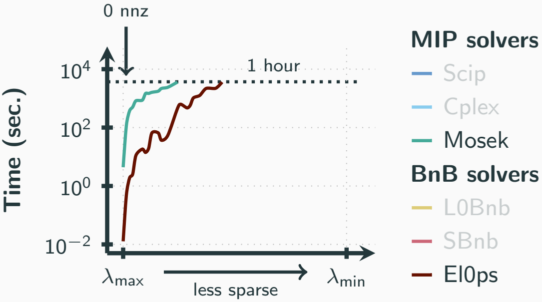

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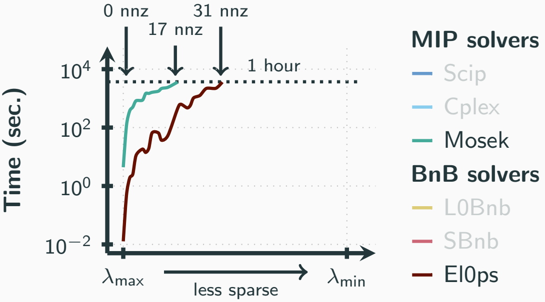
$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{62\times 2000}$ from ML dataset / f : Logistic / h : ℓ_2-norm

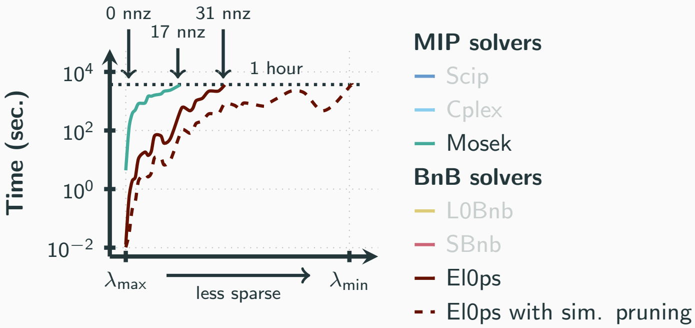


Axis 3 – Numerics

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

$\mathbf{A} \in \mathbf{R}^{62\times 2000}$ from ML dataset / f : Logistic / h : ℓ_2-norm

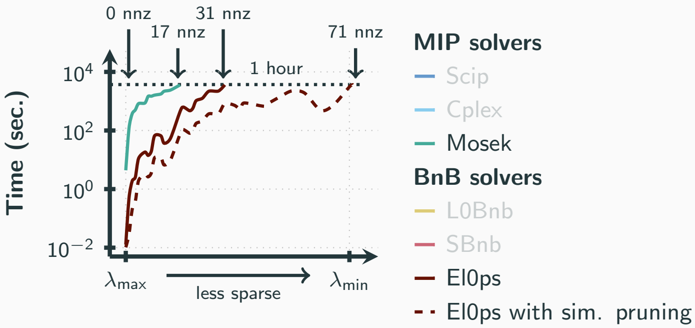


Axis 3 – Numerics

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

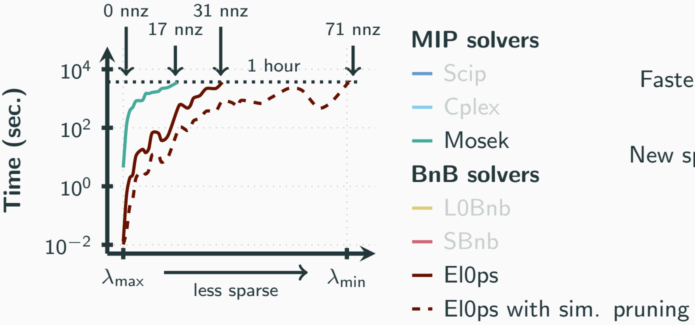
$\mathbf{A} \in \mathbf{R}^{62\times 2000}$ from ML dataset / f : Logistic / h : ℓ_2-norm



Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

$\mathbf{A} \in \mathbf{R}^{62\times 2000}$ from ML dataset / f : Logistic / h : ℓ_2-norm



Faster solving time

↓

New sparsity regimes

Let's recap

Problem

$p^\star = \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + g(\mathbf{x})$

→ $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

→ many applications

↓ BnB algorithm

Restriction to region ν

$p^\nu = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g(\mathbf{x})$

→ pruning test

→ lower bound on p^ν

↓ standard strategy

Relaxation for region ν

$p^\nu_{\mathrm{lb}} = \min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\mathrm{lb}}(\mathbf{x})$

→ instance-specific construction of g_{lb}

Axis 1

How to construct relaxations generically ?

1) Set $g_{\mathrm{lb}} = g_{\mathrm{cvx}}$
2) Closed-form expression
3) Generalize BnB method

Axis 2

How to solve relaxations efficiently ?

1) Cast as convex problem
2) Screening/smoothing
3) Accelerate solution

Axis 3

How to improve the standard strategy ?

1) Dual bound $D^\nu(\mathbf{u}) \leq p^\nu$
2) Link between successors
3) Change of paradigm

Conclusion

Contributions

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

minimize loss / sparse solutions

Contributions

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

minimize loss / sparse solutions

Question

How to design generic and efficient solution methods ?

Contributions

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

minimize loss / sparse solutions

Question

How to design generic and efficient solution methods ?

1) Generic solver

Axis 1

BnB solver with
generic framework

Contributions

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

minimize loss / sparse solutions

Question

How to design generic and efficient solution methods ?

1) Generic solver

Axis 1

BnB solver with generic framework

2) Efficient solver

Axis 2 & 3

Efficient relaxation solution, simultaneous pruning

Contributions

Problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$

minimize loss / sparse solutions

Question

How to design generic and efficient solution methods ?

1) Generic solver

Axis 1

BnB solver with generic framework

2) Efficient solver

Axis 2 & 3

Efficient relaxation solution, simultaneous pruning

3) Practical solver

El0ps

Flexible with state-of-the-art performance

Perspectives

Perspectives

Extension to other formulations

Minimize loss $f(\mathbf{Ax})$

Force sparsity with $\|\mathbf{x}\|_0$

$\downarrow$ $\boxed{\text{contributions}}$

Regularized version

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \lambda\|\mathbf{x}\|_0$

Perspectives

Extension to other formulations

Minimize loss $f(\mathbf{A}\mathbf{x})$

Force sparsity with $\|\mathbf{x}\|_0$

contributions

Regularized version

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0$

connections

Constrained version

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x})$ s.t. $\|\mathbf{x}\|_0 \leq k$

Extension to other formulations

Minimize loss $f(\mathbf{A}\mathbf{x})$

Force sparsity with $\|\mathbf{x}\|_0$

$\downarrow$ contributions $\downarrow$

Regularized version

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0$

$\uparrow$ connections $\downarrow$

Constrained version

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x})$ s.t. $\|\mathbf{x}\|_0 \leq k$

$\rightarrow$ non-separability of the ℓ_0-norm constraint

Perspectives

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$\rightarrow$ non-separability of the ℓ_0-norm constraint

Towards stronger relaxations

Restriction to region ν

$\min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g(\mathbf{x})$

$\boxed{\text{lower bound}}$

Convex relaxation

$\min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\text{cvx}}(\mathbf{x})$

Perspectives

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$\downarrow$ improve

Non-cvx relaxation

$\min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\text{non-cvx}}(\mathbf{x})$

Perspectives

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Force sparsity with $\|\mathbf{x}\|_0$

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Non-cvx relaxation

$\min_{\mathbf{x}\in\nu} f(\mathbf{Ax}) + g_{\text{non-cvx}}(\mathbf{x})$

$\rightarrow$ tune $g_{\text{non-cvx}}$ to preserve the overall convexity

Perspectives

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Broader exploitation of smoothing tests

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{Ax}) + \tilde{g}(\mathbf{x})$

accelerate

Screen/smooth tests

$x_i^\star = 0$ or $x_i^\star \neq 0$

Perspectives

Extension to other formulations

Minimize loss $f(\mathbf{Ax})$

Force sparsity with $\|\mathbf{x}\|_0$

contributions ↓

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↑ connections ↓

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accelerate ↓

Screen/smooth tests

$x_i^\star = 0$ or $x_i^\star \neq 0$

↓

Set $x_i = 0$

Tailored to any instance

Perspectives

Extension to other formulations

Minimize loss $f(\mathbf{A}\mathbf{x})$
Force sparsity with $\|\mathbf{x}\|_0$

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Set $x_i = 0$
Tailored to any instance

Smooth $\tilde{g}_i$
Depends on the instance

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Screen/smooth tests

$x_i^\star = 0$ or $x_i^\star \neq 0$

Set $x_i = 0$ — Tailored to any instance

Smooth $\tilde{g}_i$ — Depends on the instance

$\rightarrow$ Newton accel. for proximal identification

G. Bareilles *et al.* (2022)

Question time !

🤓

Context – Machine learning application

Tabular ML dataset

	Feature 1	Feature 2	...	Feature n	Target
Sample 1	$a_{1,1}$	$a_{1,2}$	$\cdots$	$a_{1,n}$	y_1
Sample 2	$a_{2,1}$	$a_{2,2}$	$\cdots$	$a_{2,n}$	y_2
Sample 3	$a_{3,1}$	$\boxed{\mathbf{A} \in \mathbf{R}^{m\times n}}$	$\cdots$	$a_{3,n}$	$\boxed{\mathbf{y} \in \mathbf{R}^{m}}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Sample m	$a_{m,1}$	$a_{m,2}$	$\cdots$	$a_{m,n}$	y_m

Features $\mathbf{A} \in \mathbf{R}^{m\times n}$ $\xleftrightarrow[\text{weights } \mathbf{x} \in \mathbf{R}^n]{}$ Target $\mathbf{y} = \phi(\mathbf{A}\mathbf{x})$

Model accuracy

Loss $\mathcal{L}_\phi(\mathbf{A}\mathbf{x}, \mathbf{y})$

Model explicability

Sparsity-inducing $\mathcal{R}(\mathbf{x})$

Optimization problem

$\min_{\mathbf{x}\in\mathbf{R}^n} \mathcal{L}_\phi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \lambda\mathcal{R}(\mathbf{x})$

Context – Algebra application

Sparse Component Analysis

Goal

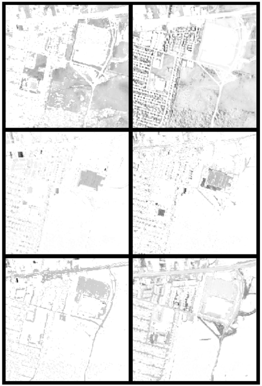
Given $\mathbf{M} \in \mathbf{R}^{m\times n}$, find $\mathbf{D} \in \mathbf{R}^{m\times r}$ and $\mathbf{B} \in \mathbf{R}^{r\times n}$ such that $\mathbf{M} \simeq \mathbf{DB}$ with sparse columns in $\mathbf{B}$.

$\downarrow$

Optimization problem

$$\min_{\mathbf{D}\in\mathbf{R}^{m\times r},\mathbf{B}\in\mathbf{R}^{r\times n}} \frac{1}{2}\|\mathbf{M}-\mathbf{DB}\|_F^2 + \lambda \sum_{i=1}^{n} \|\mathbf{b}_i\|_0$$

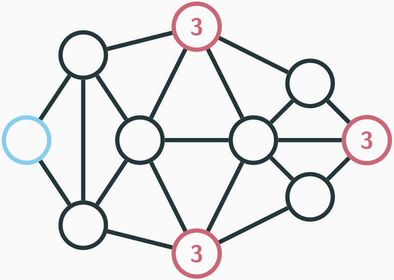
J. Cohen, N. Gillis (2019)



Extract material abundance map from hyperspectral image

Context – Operation research application

Max. capacity per edge: 10
Edge construction cost: 5



Which edges to build to transport flows from source to sink nodes ?

Network design

$$\begin{cases} \min\ Q(\mathbf{x}) + \lambda\|\mathbf{x}\|_0 \\ \text{s.t. } \mathbf{D}\mathbf{x} \leq \mathbf{d},\ \ \mathbf{x} \leq \mathbf{c} \\ \mathbf{x} \in \mathbf{R}_+^{\mathrm{card}(E)} \end{cases}$$

Q : transportation cost

λ : unit construction cost

$\mathbf{D}\mathbf{x} \leq \mathbf{d}$: flow conservation

$\mathbf{x} \leq \mathbf{c}$: capacity constraint

Context – Signal processing application

Compressed sensing

Sparse signal

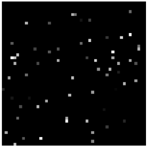
$\mathbf{x} \in \mathbf{R}^n$

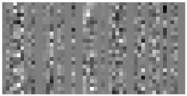
Linear measurement

Recover $\mathbf{x}$ from $\mathbf{A}$ and $\mathbf{y}$

Observation

$\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon \in \mathbf{R}^m$





Goal

Find $\mathbf{x}$ such that $\mathbf{y} \simeq \mathbf{A}\mathbf{x}$

$m \ll n$: no unique solution

Goal (with sparse prior)

Find $\mathbf{x}$ sparse such that $\mathbf{y} \simeq \mathbf{A}\mathbf{x}$

Optimization problem

$\min_{\mathbf{x}\in\mathbf{R}^n} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2$

sparsity-inducing function

Sparse optimization problem

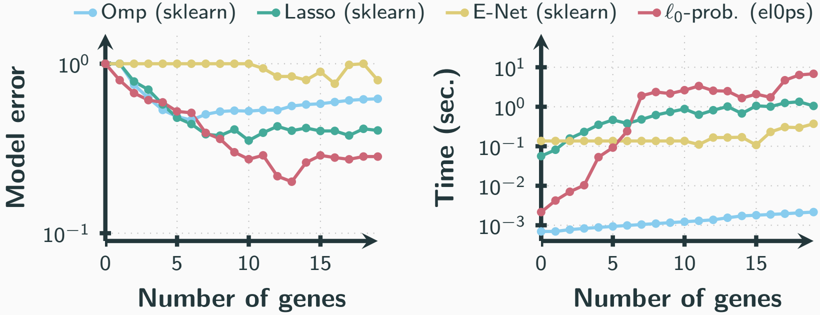
$\min_{\mathbf{x}\in\mathbf{R}^n} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_0$

Context – Balancing solution quality and problem hardness

Riboflavin dataset - P. Bühlmann *et al.* (2014)

Colony	AADK	AAPA	ABFA	ABH	...	ZUR	**B2 prod.**
#1	8.49	8.11	8.32	10.28	...	7.42	**-6.64**
#2	7.29	6.39	11.32	9.42	...	6.99	**-5.43**
...	...	...	...	...	...	...	...
#71	6.85	8.27	7.98	8.04	...	6.65	**-7.58**

4,088 genes



Context – Balancing solution quality and problem hardness

Sparse signal

$\mathbf{x} \in \mathbf{R}^n$

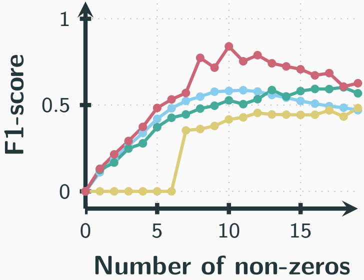
Observation

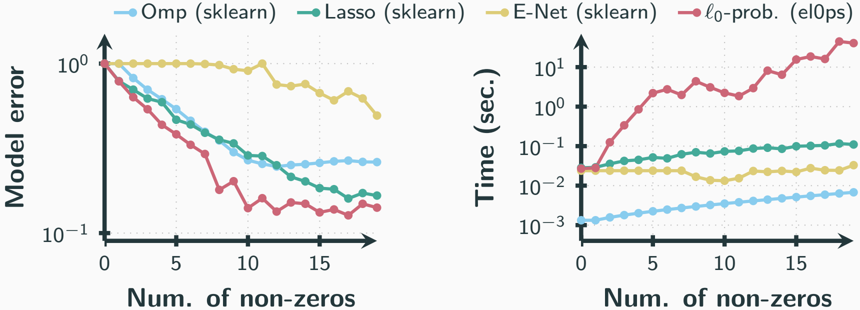
$\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\epsilon} \in \mathbf{R}^m$

Setup

$\mathbf{A} \in \mathbf{R}^{100\times 200}$ highly correlated, $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma\mathbf{I})$,

$\mathbf{x}$ with 10 unit spikes of random location





Context – Balancing solution quality and problem hardness

Sparse signal

$\mathbf{x} \in \mathbf{R}^n$

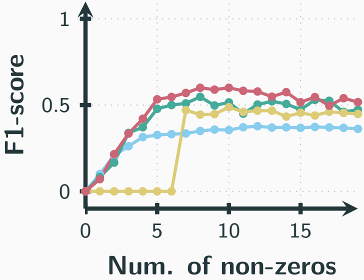
Observation

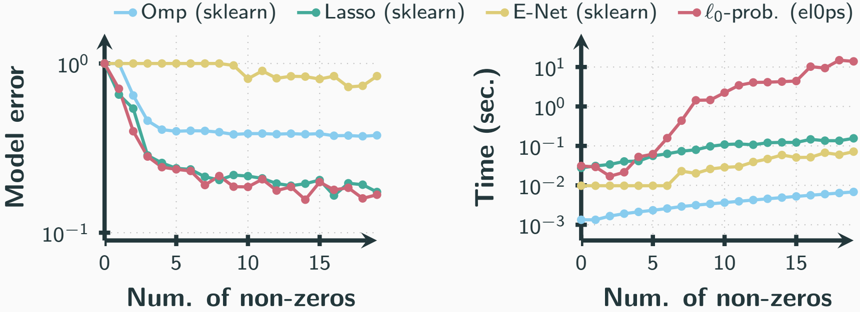
$\mathbf{y} = \mathbf{A}\mathbf{x} + \boldsymbol{\epsilon} \in \mathbf{R}^m$

Setup

$\mathbf{A} \in \mathbf{R}^{100 \times 200}$ highly correlated, $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma\mathbf{I})$,

$\mathbf{x}$ with 10 unit and evenly-spaced spikes



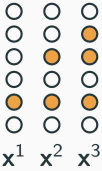


Context – A bit of history

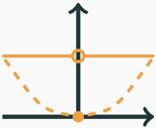
Problem

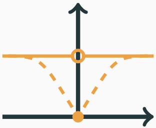
$$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0$$

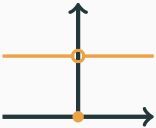
1995	2000	2005	2010	2015
Heuristics	Recovery cond.	Convex approx.	Concave approx.	Exact methods
MP, OMP, ...	RIP, NSP, ...	Lasso, Elastic-Net, ...	SCAD, MCP, ...	MIP, BnB, ...
S. Mallat (1993)	E. Candes (2004)	R. Tibshirani (2005)	C. Zhang (2010)	D. Bertsimas (2016)



OMP solves ℓ_0-problem under RIP







Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

linearize ℓ_0-norm

MIP formulation

$$\begin{cases} \min & f(\mathbf{A}\mathbf{x}) + \lambda\mathbf{1}^{\mathrm{T}}\mathbf{z} + h(\mathbf{x}) \\ \text{s.t.} & x_i = 0 \implies z_i = 0, \ \forall i \\ & \mathbf{x}\in\mathbf{R}^n, \ \mathbf{z}\in\{0,1\}^n \end{cases}$$

avoid logical cstr.

Practical MIP formulation

$$\begin{cases} \min & f(\mathbf{A}\mathbf{x}) + \lambda\mathbf{1}^{\mathrm{T}}\mathbf{z} + h_{\mathrm{mip}}(\mathbf{x},\mathbf{z}) \\ \text{s.t.} & \mathbf{x}\in\mathbf{R}^n, \ \mathbf{z}\in\{0,1\}^n \end{cases}$$

Use generic MIP solvers

Need standardized expressions

linear/quadratic/conic/...

Lifted formulation

$$\|\mathbf{x}\|_0 = \mathbf{1}^{\mathrm{T}}\mathbf{z}$$

for all $\mathbf{x}\in\mathbf{R}^n$ and $\mathbf{z}\in\{0,1\}^n$

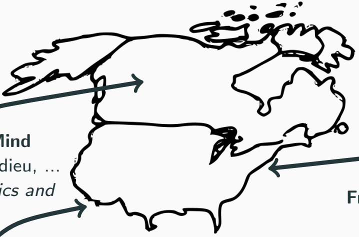
if $x_i = 0 \implies z_i = 0, \ \forall i$

Construct h_{mip} depending on h

$h(\mathbf{x})$	$h_{\mathrm{mip}}(\mathbf{x},\mathbf{z})$
$\eta(\|\mathbf{x}\|_\infty \leq M)$	$\eta(-M\mathbf{z} \leq \mathbf{x} \leq M\mathbf{x})$
$\alpha\|\mathbf{x}\|_2^2$	$\sum_{i=1}^n \alpha\frac{x_i^2}{z_i}$

Context – Research community

Non-exhaustive list



MIT
D. Bertsimas, R. Mazmuder, ...
MIO tools for ℓ_0-problems

Google Deep Mind
H. Hazimeh, A. Dedieu, ...
MIO-based heuristics and softwares

Lund University
M. Carlsson, C. Olsson...
Quadratic envelope

Frankfurt / Wurzburg Universities
C. Kanzow, A. Tillmann, ...
Optimality conditions

Berkley
A. Atamtürk, A. Gomès, ...
Convex-based acceleration

London Business School
J. Pauphilet, R. Cory-Wright, ...
Healthcare applications

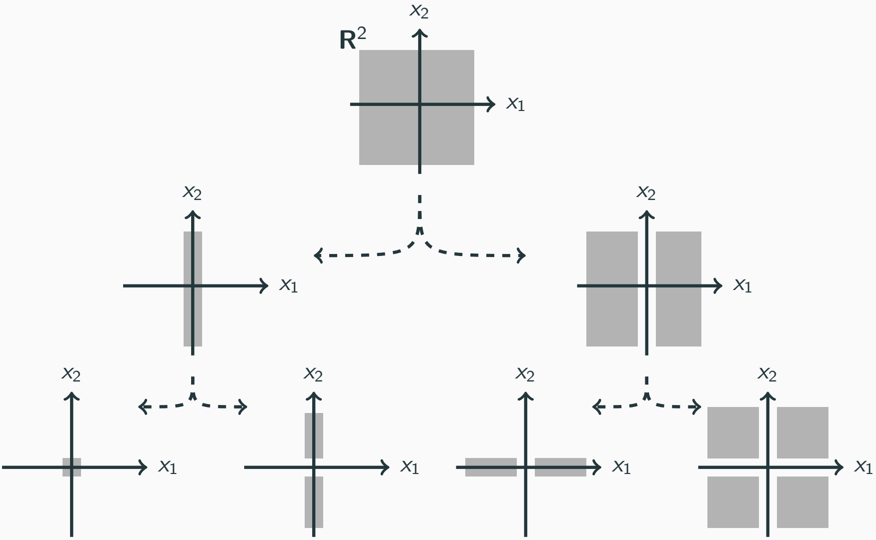
Ponts ParisTech
M. De Lara, P. Chancelier, A. Parmentier, ...
Non-convex analysis for ℓ_0-norm, ML appli.

Centrale Nantes / ENSTA Bretagne
S. Bourguignon, J. Ninin, ...
Branch-and-Bound for ℓ_0-problems

Inria / CentraleSupélec
C. Herzet, C. Elvira, A. Arslan, ...
Generalization, acceleration

IRIT / I3S
E. Soubies, L. Blanc-Féraud, ...
Strong relax. of ℓ_0-norm

BnB – Region separation



Axis 1 – Relaxation construction

Region $\nu \equiv (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_\bullet)$ with $\begin{cases} x_i = 0 & \text{if } i \in \mathcal{S}_0 \\ x_i \neq 0 & \text{if } i \in \mathcal{S}_1 \\ x_i \in \mathbf{R} & \text{if } i \in \mathcal{S}_\bullet \end{cases}$

Restriction to region ν

$$p^\nu = \min_{\mathbf{x} \in \nu} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x}) \quad \text{with} \quad g_i(x_i) = \lambda \|x_i\|_0 + h_i(x_i)$$

$\boxed{\text{reformulation}}$

Restriction to region ν

$$p^\nu = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g^\nu(\mathbf{x}) \quad \text{with} \quad g_i^\nu(x_i) = \begin{cases} g_i(x_i) + \eta(x_i = 0) & \text{if } i \in \mathcal{S}_0 \\ g_i(x_i) + \eta(x_i \neq 0) & \text{if } i \in \mathcal{S}_1 \\ g_i(x_i) & \text{if } i \in \mathcal{S}_\bullet \end{cases}$$

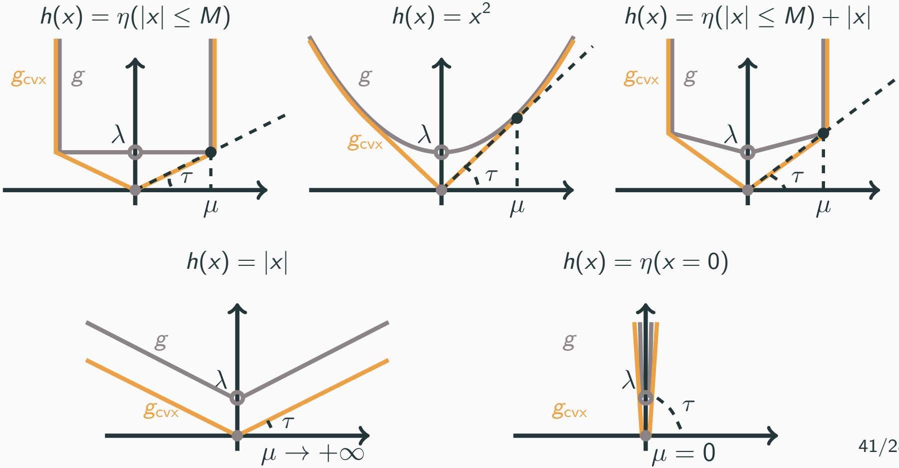
$\boxed{g_{\text{lb}}^\nu \leq g,\ g_{\text{lb}}^\nu \text{ convex}}$

Relaxation for region ν

$$p_{\text{lb}}^\nu = \min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g_{\text{lb}}^\nu(\mathbf{x}) \quad \text{with} \quad g_{i,\text{lb}}^\nu(x_i) = \begin{cases} \eta(x_i = 0) & \text{if } i \in \mathcal{S}_0 \\ h_i(x_i) + \lambda & \text{if } i \in \mathcal{S}_1 \\ g_{i,\text{cvx}}(x_i) & \text{if } i \in \mathcal{S}_\bullet \end{cases}$$

Axis 1 – Graphical interpretation

$$g(x) = \lambda\|x\|_0 + h(x) \xrightarrow{\text{convexify}} g_{\text{cvx}}(x) = \begin{cases} \tau|x| & \text{if } |x| \leq \mu \\ \lambda + h(x) & \text{otherwise} \end{cases}$$



Axis 2 – Reduced and smoothed formulation

Convex problem

$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_1 + h(\mathbf{x})$

$\downarrow$

Knowledge of zeros/positive/negative entries in the solutions

$\downarrow$

Reduced/smoothed formulation

$\min_{\tilde{\mathbf{x}}\in\mathbf{R}^{\tilde{n}}} f(\tilde{\mathbf{A}}\tilde{\mathbf{x}}) + \lambda\boldsymbol{\theta}^{\mathrm{T}}\tilde{\mathbf{x}} + h(\tilde{\mathbf{x}})$

Reduced dimension $\tilde{n} \ll n$

Smooth objective if f/h smooth

1st-order methods

Proximal gradient

Coordinate descent

$\downarrow$

Sub-linear/linear convergence rate

Cost $\mathcal{O}(nm)$ per iteration

2nd-order methods

Newton's method

LBFGS

$\downarrow$

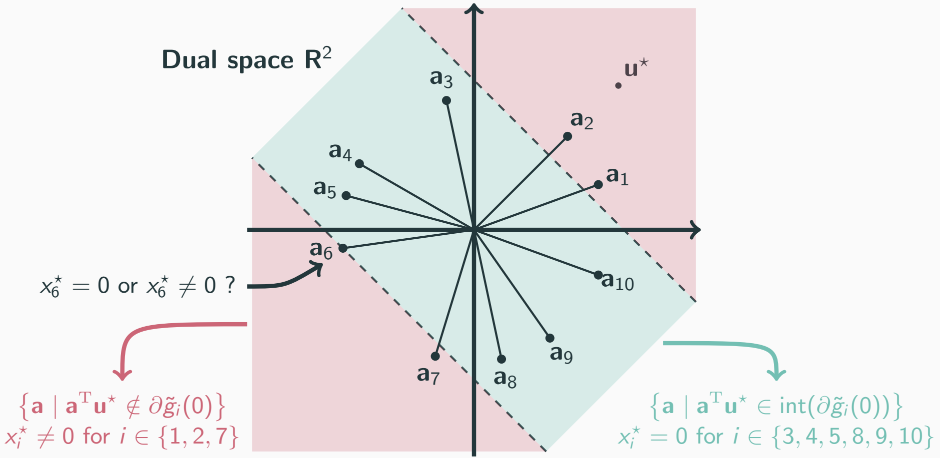
Super-linear convergence rate

Cost $\mathcal{O}(\tilde{n}m)$ per iteration

Axis 2 – Graphical interpretation (dual)

$$\text{Screening:} \quad \mathbf{a}_i^{\mathrm{T}}\mathbf{u}^\star \in \mathrm{int}(\partial\tilde{g}_i(0)) \implies x_i^\star = 0$$

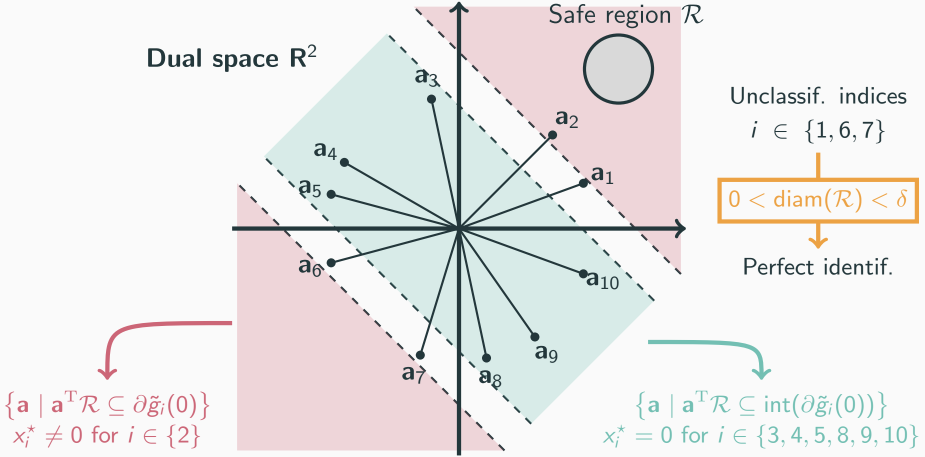
$$\text{Smoothing:} \quad \mathbf{a}_i^{\mathrm{T}}\mathbf{u}^\star \in \mathrm{cmpl}(\partial\tilde{g}_i(0)) \implies x_i^\star \neq 0$$



Axis 2 – Graphical interpretation (safe)

$$\text{Safe screening:} \quad \mathbf{a}_i^{\mathrm{T}}\mathcal{R} \subseteq \mathrm{int}(\partial \tilde{g}_i(0)) \implies x_i^\star = 0$$

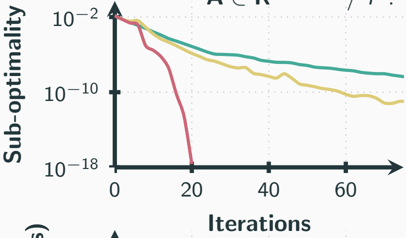
$$\text{Safe smoothing:} \quad \mathbf{a}_i^{\mathrm{T}}\mathcal{R} \subseteq \mathrm{cmpl}(\partial \tilde{g}_i(0)) \implies x_i^\star \neq 0$$



Convex problem

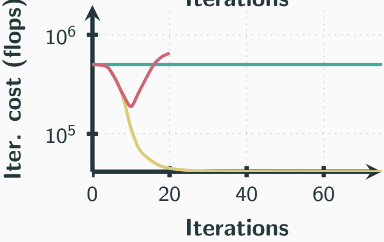
$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

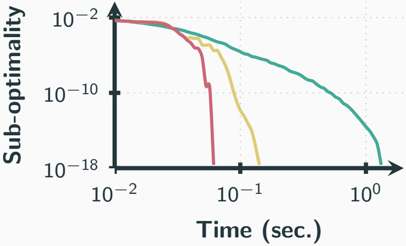
$\mathbf{A} \in \mathbf{R}^{100\times 300}$ / f : Quadratic / $\tilde{g}$: $\ell_1\ell_2$-norm



Accelerated proximal gradient

- Vanilla method
- With screening
- With screening and smoothing





ν Region $\nu \equiv (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_\bullet)$ with $\begin{cases} x_i = 0 & \text{if } i \in \mathcal{S}_0 \\ x_i \neq 0 & \text{if } i \in \mathcal{S}_1 \\ x_i \in \mathbf{R} & \text{if } i \in \mathcal{S}_\bullet \end{cases}$

Restriction to region ν

$$p^\nu \;=\; \min_{\mathbf{x}\in\nu} f(\mathbf{A}\mathbf{x}) + g(\mathbf{x}) \qquad \text{with} \quad g_i(x_i) = \lambda\|x_i\|_0 + h_i(x_i)$$

$\boxed{g_{\text{lb}}^\nu \leq g,\ g_{\text{lb}}^\nu \text{ convex}}$

Relaxation for region ν

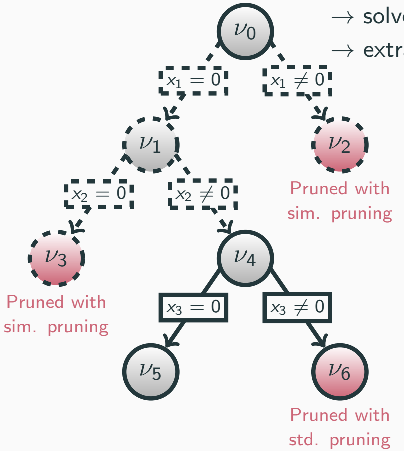
$$p_{\text{lb}}^\nu \;=\; \min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + g_{\text{lb}}^\nu(\mathbf{x}) \qquad \text{with} \quad g_{i,\text{lb}}^\nu(x_i) = \begin{cases} \eta(x_i = 0) & \text{if } i \in \mathcal{S}_0 \\ h_i(x_i) + \lambda & \text{if } i \in \mathcal{S}_1 \\ g_{i,\text{cvx}}(x_i) & \text{if } i \in \mathcal{S}_\bullet \end{cases}$$

$\boxed{\text{dualize}}$

Dual relaxation for region ν

$$p_{\text{dual}}^\nu = \max_{\mathbf{u}\in\mathbf{R}^m} -f^*(-\mathbf{u}) - (g_{\text{lb}}^\nu)^*(\mathbf{A}^{\mathrm{T}}\mathbf{u}) \quad \text{with} \quad (g_{i,\text{lb}}^\nu)^*(\mathbf{a}_i^{\mathrm{T}}\mathbf{u}) = \begin{cases} 0 & \text{if } i \in \mathcal{S}_0 \\ h_i^*(\mathbf{a}_i^{\mathrm{T}}\mathbf{u}) - \lambda & \text{if } i \in \mathcal{S}_1 \\ [h_i^*(\mathbf{a}_i^{\mathrm{T}}\mathbf{u}) - \lambda]_+ & \text{if } i \in \mathcal{S}_\bullet \end{cases}$$

Axis 3 – Combining both paradigms



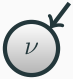
→ solve standard relaxation
→ extract dual lower bound

Solve relaxations in nodes ν_0, ν_4, ν_5 and ν_6

Bypass nodes ν_1, ν_2 and ν_3 using simultaneous pruning

Problem

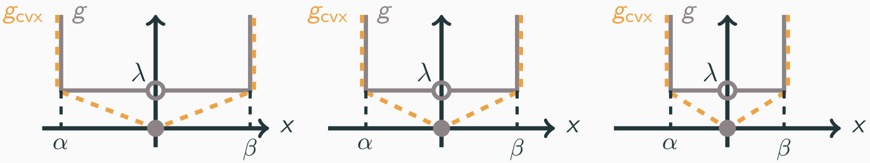
$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + \eta(\boldsymbol{\alpha} \leq \mathbf{x} \leq \boldsymbol{\beta})$



Pruning test

Construct and solve a relaxation

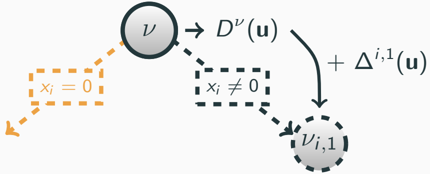
$g(\mathbf{x}) = \lambda\|\mathbf{x}\|_0 + \eta(\boldsymbol{\alpha} \leq \mathbf{x} \leq \boldsymbol{\beta}) \to g_{\mathrm{cvx}}$



Practical side – Large interval $[\alpha, \beta]$ to obtain relevant solutions.
Numerical side – Small interval $[\alpha, \beta]$ to obtain strong relaxations.

Axis 4 – Peeling tests

Simultaneous pruning test



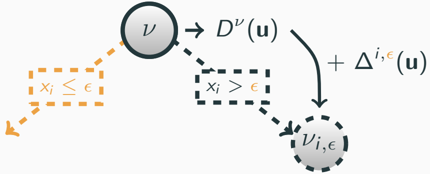
$$D^{\nu_{i,1}}(\mathbf{u}) = D(\mathbf{u}) + \Delta^{i,1}(\mathbf{u})$$

Result 1

$\rightarrow$ peeling test gives weaker conclusions but is easier to pass

$$D^{\nu_{i,\epsilon}}(\mathbf{u}) \ \geq \ D^{\nu_{i,1}}(\mathbf{u})$$

Peeling test



$$D^{\nu_{i,\epsilon}}(\mathbf{u}) = D(\mathbf{u}) + \Delta^{i,\epsilon}(\mathbf{u})$$

Result 2

$\rightarrow$ we can find the smallest $\epsilon > 0$ such that the peeling test is passed

$$p^{\star}_{\mathrm{ub}} \ < \ D^{\nu_{i,\epsilon}}(\mathbf{u})$$

Numerics – El0ps

Problem

$$\min_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \lambda\|\mathbf{x}\|_0 + h(\mathbf{x})$$

El0ps

Problem management

Builtin instances of f/h

User-defined instances of f/h

- $\rightarrow$ function value
- $\rightarrow$ convex conjugate
- $\rightarrow$ subdifferential
- $\rightarrow$ proximal operator

Branch-and-Bound solver

Generic backbone (Axis 1)

$\rightarrow$ generic convex relaxation

Relaxation solution (Axis 2)

- $\rightarrow$ convex optim. solver
- $\rightarrow$ accel. with screen/smooth

Reduced complexity (Axis 3)

- $\rightarrow$ dual bounds
- $\rightarrow$ simultaneous pruning

Perspectives – Stronger relaxations

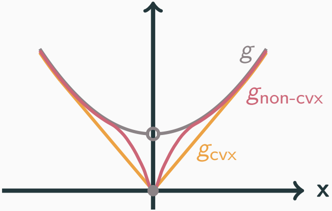
$$
\begin{aligned}
p^{\nu} &= \min_{\mathbf{x} \in \nu} \; f(\mathbf{A}\mathbf{x}) + g(\mathbf{x}) \\
&\vee\!| \\
p^{\nu}_{\text{non-cvx}} &= \min_{\mathbf{x} \in \nu} \; f(\mathbf{A}\mathbf{x}) + g_{\text{non-cvx}}(\mathbf{x}) \\
&\vee\!| \\
p^{\nu}_{\text{cvx}} &= \min_{\mathbf{x} \in \nu} \; f(\mathbf{A}\mathbf{x}) + g_{\text{cvx}}(\mathbf{x})
\end{aligned}
$$

→ Need valid and tractable relaxation

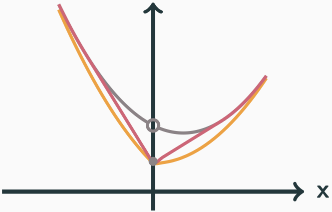
$$
\begin{aligned}
g(\mathbf{x}) &= \lambda \|\mathbf{x}\|_0 + \tfrac{\gamma}{2}\|\mathbf{x}\|_2^2 \\
&\vee\!| \\
g_{\text{non-cvx}}(\mathbf{x}) &= \mathrm{Mcp}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{x}) + \tfrac{\gamma}{2}\|\mathbf{x}\|_2^2 \\
&\vee\!| \\
g_{\text{cvx}}(\mathbf{x}) &= \mathrm{Berhu}_{\lambda,\gamma}(\mathbf{x})
\end{aligned}
$$

→ Tune $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ depending on f and $\mathbf{A}$ to ensure the overall objective convexity

Regularization functions



Objective functions



Perspectives – Proximal identification

Convex problem

$$\mathbf{x}^\star \in \operatorname{argmin}_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{A}\mathbf{x}) + \tilde{g}(\mathbf{x})$$

Screening/smoothing

Optimality conditions

$$\mathbf{A}^{\mathrm{T}}\mathbf{u}^\star \in \partial\tilde{g}(\mathbf{x}^\star)$$

$$\boxed{\tilde{g}(\mathbf{x}^\star) = \textstyle\sum_{i=1}^n \tilde{g}_i(x_i^\star)}$$

Identif. from dual solution

$$\mathbf{a}_i^{\mathrm{T}}\mathbf{u}^\star \to x_i^\star = 0 \text{ or } x_i^\star \neq 0$$

$$\boxed{\mathcal{R} \subseteq \mathbf{R}^m \text{ with } \mathbf{u}^\star \in \mathcal{R}}$$

Identif. from safe region

$$\mathbf{a}_i^{\mathrm{T}}\mathcal{R} \to x_i^\star = 0 \text{ or } x_i^\star \neq 0$$

Safe but can miss indices

$$\mathbf{w} = \operatorname{prox}_{\tilde{g}}(\mathbf{x}) \longleftrightarrow \mathbf{x} - \mathbf{w} \in \partial\tilde{g}(\mathbf{w})$$

unsafe exploit.

make safe

Proximal identification

Optimality conditions

$$\mathbf{x}^\star = \operatorname{prox}_{\tilde{g}}(\mathbf{x}^\star)$$

$$\boxed{\tilde{g}(\mathbf{x}^\star) = \textstyle\sum_{i=1}^n \tilde{g}_i(x_i^\star)}$$

Identif. from prox. operator

$$\operatorname{prox}_{\tilde{g}_i}(x_i^\star) \to x_i^\star = 0 \text{ or } x_i^\star \neq 0$$

$$\boxed{\mathbf{x} \in \mathbf{R}^n \text{ near } \mathbf{x}^\star}$$

Identif. from arbitrary point

$$\operatorname{prox}_{\tilde{g}_i}(x_i) \to x_i^\star = 0 \text{ or } x_i^\star \neq 0$$

Unsafe but classify all indices

Perspectives – Newton acceleration

Algorithm 1: Our approach

Input: $\mathbf{x}^0 \in \mathbf{R}^n$

Initialize $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_\bullet) = (\emptyset, \emptyset, [1, n])$
for $k = 1, 2, \dots, k_{\max}$ **do**
  `// Update iterate`
  $\mathbf{x}^k_{\mathcal{S}_0} \leftarrow \mathbf{0}$
  $\mathbf{x}^k_{\mathcal{S}_1} \leftarrow 2^{\text{nd}}\text{OrderIteration}(\mathbf{x}^{k-1}_{\mathcal{S}_1})$
  $\mathbf{x}^k_{\mathcal{S}_\bullet} \leftarrow 1^{\text{st}}\text{OrderIteration}(\mathbf{x}^{k-1}_{\mathcal{S}_\bullet})$
  `// Update structure knowledge`
  $\mathcal{R} \leftarrow \text{SafeRegion}(\mathbf{x}^k)$
  $\mathcal{S}_0 \leftarrow \mathcal{S}_0 \cup \text{ScreeningTest}(\mathcal{R})$
  $\mathcal{S}_1 \leftarrow \mathcal{S}_1 \cup \text{SmoothingTest}(\mathcal{R})$
  $\mathcal{S}_\bullet \leftarrow [1, n] \setminus (\mathcal{S}_0 \cup \mathcal{S}_1)$
end

- $\mathcal{S}_0 \subseteq \mathcal{S}_0^\star$ and $\mathcal{S}_1 \subseteq \mathcal{S}_1^\star$ at any iter.
- $\mathcal{S}_\bullet \neq \emptyset$ until $k \geq k_0$

$\rightarrow$ Safe but uncomplete identification

$$\boxed{\begin{aligned} \mathcal{S}_0^\star &= \{i \mid x_i^\star = 0\} \\ \mathcal{S}_1^\star &= \{i \mid x_i^\star \neq 0\} \end{aligned}}$$

Algorithm 2: G. Bareilles *et al.*

Input: $\mathbf{x}^0 \in \mathbf{R}^n$

for $k = 1, 2, \dots, k_{\max}$ **do**
  `// Update iterate`
  $\tilde{\mathbf{x}}^k \leftarrow \text{ProxIteration}(\mathbf{x}^{k-1})$
  `// Get local structure`
  $(\mathcal{S}_0, \mathcal{S}_1) \leftarrow \text{ProxIdentification}(\tilde{\mathbf{x}}^k)$
  `// Follow local structure`
  $\mathbf{x}^k \leftarrow \text{StructureUpdate}_{(\mathcal{S}_0, \mathcal{S}_1)}(\tilde{\mathbf{x}}^k)$
end

- $\mathcal{S}_\bullet = \emptyset$ at any iter.
- $\mathcal{S}_0 \neq \mathcal{S}_0^\star$ and $\mathcal{S}_1 \neq \mathcal{S}_1^\star$ until $k \geq k_0$

$\rightarrow$ Unsafe but complete identification