New directions for tensor Netorks

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Table des matières

1 Introduction

During my participation in the **Quant25** summer program, I had the opportunity to attend a series of advanced lectures and conferences on modern topics in quantum physics and quantum information. Among them, the talk given by **Dr. Miles Stoudenmire** at the Max Planck Institute for the Physics of Complex Systems (MPIPKS) particularly caught my attention.

The presentation, entitled *"New directions for tensor Netorks"*, was a lecture on tensor networks, which are powerful mathematical tools for efficiently representing and manipulating very high-dimensional objects. They were initially developed to describe quantum wavefunctions of many-body systems, whose Hilbert space dimension grows exponentially with the number of degrees of freedom.

For a quantum system of N spin-1/2 particles, the most general state can be written as :

$$|\psi\rangle = \sum_{n_1,\ldots,n_N} T^{n_1\ldots n_N} |n_1 \ldots n_N\rangle,$$

where the tensor $T^{n_1\ldots n_N}$ has 2^N components. This exponential growth makes any naïve approach rapidly infeasible.

The central idea of tensor networks is that, for many physically relevant systems, the useful information is highly structured, in particular by quantum entanglement. Tensor networks exploit this structure to provide compressed, yet controlled, representations of these high-dimensional objects.

2 Tensors and diagrammatic notation

A tensor can be viewed as a generalization of scalars, vectors, and matrices. A tensor of order N is denoted :

$$T^{n_1 n_2 \ldots n_N},$$

and can be represented graphically as a node with N outgoing legs.

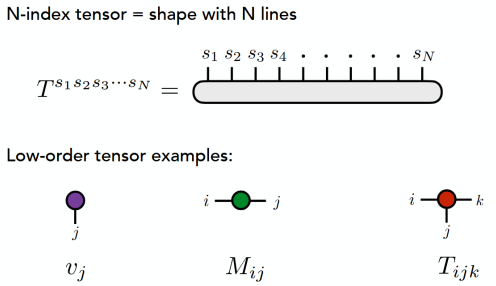


FIGURE 1 – Tensors and diagrammatic notation

Diagrammatic notation is particularly useful :

- A leg corresponds to an index.
- A leg connecting two tensors corresponds to an index contraction.
- A free leg corresponds to an unsummed index.

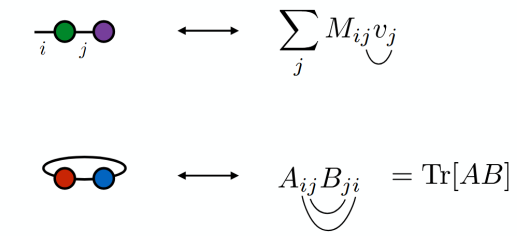


FIGURE 2 – Diagrammatic notation of the inner product and the trace

For example :

- The inner product $\sum_j v_j w_j$ is represented by two nodes connected by a single leg.
- The trace $\sum_n M_{nn}$ is represented by a leg leaving a tensor and reconnecting to itself.
- The contraction $\omega_i = \sum_j M_{ij} v_j$ corresponds to joining one leg between M and v.

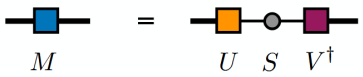
This graphical representation allows one to reason about complex algebraic calculations in a visual and intuitive way.

3 Singular Value Decomposition (SVD)

The **Singular Value Decomposition (SVD)** is a central mathematical tool underlying most tensor network algorithms. It provides both a canonical factorization of matrices and a precise way to quantify and control information loss.

Given a complex matrix $M \in \mathbb{C}^{m\times n}$, the SVD states that M can always be decomposed as :

$$M = USV^\dagger,$$



where :
— $U \in \mathbb{C}^{m\times m}$ is unitary and contains the left singular vectors,
— $V \in \mathbb{C}^{n\times n}$ is unitary and contains the right singular vectors,
— $S \in \mathbb{R}^{m\times n}$ is diagonal (rectangular if $m \neq n$) with non-negative entries

$$S = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r),$$

where $r = \mathrm{rank}(M)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the **singular values**.

The singular values quantify how much each corresponding singular vector contributes to the matrix. Large singular values encode the dominant structure of M, while small ones represent fine details or noise.

Optimal low-rank approximation A crucial property of the SVD is that it provides the *best possible* low-rank approximation of a matrix. If one truncates the decomposition by keeping only the first k singular values,

$$M \approx M_k = \sum_{i=1}^{k} \sigma_i \, |u_i\rangle\langle v_i|,$$

then M_k is the rank-k matrix that minimizes the Frobenius norm of the error :

$$\|M - M_k\|_F^2 = \sum_{i>k} \sigma_i^2.$$

For example, if

$$S = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad a \geq b \geq c,$$

and only the largest singular value a is kept, the discarded information is precisely quantified by :

$$\|M - M_1\|_F^2 = b^2 + c^2.$$

This result is not heuristic : it is mathematically optimal. In tensor networks, this guarantees that truncations introduce the smallest possible error for a given bond dimension.

Physical interpretation In quantum physics, matrices often arise from bipartitions of a quantum state. In this context :

— singular values correspond to Schmidt coefficients,

— their squares represent probabilities,

— rapid decay of σ_i indicates weak entanglement.

This direct link between linear algebra and entanglement is the reason SVD is indispensable in tensor network methods.

4 Generalization of SVD to tensors

The SVD naturally generalizes to higher-order tensors by reshaping them into matrices. This procedure is the backbone of tensor network decompositions such as MPS.

Consider a tensor $T^{n_1 n_2 \dots n_N}$ of order N. Choose a bipartition of its indices into two disjoint sets :

$$\{n_1, \dots, n_k\} \quad \text{and} \quad \{n_{k+1}, \dots, n_N\}.$$

By grouping each set into a composite index, the tensor is reshaped into an effective matrix :

$$T^{(n_1 \dots n_k)(n_{k+1} \dots n_N)} \equiv M_{ij}.$$

An SVD can then be applied :

$$M_{ij} = \sum_{\alpha=1}^{\chi} U_{i\alpha} S_\alpha V^\dagger_{\alpha j}.$$

This decomposition splits the original tensor into two tensors connected by an internal index α of dimension χ, called the **bond dimension**.

Controlled compression If the singular values S_α decay rapidly, the sum can be truncated to $\chi_{\max} \ll \dim(i), \dim(j)$ without significant loss of accuracy. The truncation error is again controlled by :

$$\varepsilon^2 = \sum_{\alpha > \chi_{\max}} S_\alpha^2.$$

This leads to an exponential reduction in storage cost :

$$\underbrace{d^N}_{\text{full tensor}} \quad \longrightarrow \quad \underbrace{N d \chi^2}_{\text{tensor network}} \quad ,$$

where d is the local physical dimension.

Connection to entanglement When the tensor represents a quantum state, the bond dimension χ has a clear physical meaning :

— χ equals the Schmidt rank across the chosen bipartition,

— larger χ is required for highly entangled states,

— weakly entangled states admit small χ and strong compression.

In one-dimensional gapped systems, the area law for entanglement entropy ensures that χ remains moderate, explaining the success of MPS methods.

Iterative construction of tensor networks By repeatedly applying this tensor-SVD procedure to successive bipartitions, a high-order tensor can be factorized into a chain of low-rank tensors. This systematic decomposition is what transforms an exponentially large object into a tractable tensor network with controlled accuracy.

5 Product states, entanglement, and Schmidt rank

To understand why tensor networks are efficient for some quantum states and not for others, it is essential to analyze the structure of quantum entanglement.

Consider a bipartite quantum system $A \cup B$, with Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

A general pure state can be written as :

$$|\psi\rangle = \sum_{i,j} C_{ij} \, |i\rangle_A \otimes |j\rangle_B,$$

where C_{ij} is a complex coefficient matrix.

Product states A state is called a **product state** if it can be factorized as :

$$|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B.$$

In this case, the coefficient matrix factorizes as $C_{ij} = a_i b_j$ and has rank one. Such states contain no quantum correlations between subsystems.

For example :

$$|\uparrow\rangle \otimes |\downarrow\rangle$$

is a product state and can be represented with minimal complexity.

Schmidt decomposition Any bipartite pure state admits a **Schmidt decomposition**, obtained by performing an SVD of the coefficient matrix C_{ij} :

$$|\psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_\alpha \, |\alpha\rangle_A \otimes |\alpha\rangle_B,$$

where :

— $\lambda_\alpha \geq 0$ are the Schmidt coefficients,

— $\{|\alpha\rangle_A\}$ and $\{|\alpha\rangle_B\}$ are orthonormal bases,

— χ is the **Schmidt rank**.

The Schmidt rank χ is the number of non-zero Schmidt coefficients and provides a direct measure of bipartite entanglement.

Entangled states A state is **entangled** if and only if $\chi > 1$. For instance, the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle \right)$$

has Schmidt rank $\chi = 2$ and cannot be written as a product state.

Entanglement entropy The Schmidt coefficients define the reduced density matrix of subsystem A :

$$\rho_A = \sum_{\alpha=1}^{\chi} \lambda_\alpha^2 |\alpha\rangle_A \langle\alpha|.$$

The von Neumann entanglement entropy is then :

$$S_A = -\sum_{\alpha=1}^{\chi} \lambda_\alpha^2 \log \lambda_\alpha^2.$$

A rapid decay of λ_α implies low entanglement and allows an efficient tensor network representation with small bond dimension.

6 Matrix Product States (MPS)

Matrix Product States (MPS) provide a systematic way to generalize the Schmidt decomposition from two subsystems to a one-dimensional chain of many quantum degrees of freedom.

Consider a chain of N sites, each with local Hilbert space dimension d. A general quantum state reads :

$$|\psi\rangle = \sum_{n_1,\ldots,n_N} C_{n_1 n_2 \ldots n_N} |n_1 n_2 \ldots n_N\rangle,$$

with d^N complex coefficients.

MPS representation An MPS rewrites these coefficients as a product of local tensors :

$$C_{n_1 n_2 \ldots n_N} = \sum_{\alpha_1,\ldots,\alpha_{N-1}} A^{n_1}_{1,\alpha_1} A^{n_2}_{\alpha_1,\alpha_2} \ldots A^{n_N}_{\alpha_{N-1}},$$

or equivalently :

$$|\psi\rangle = \sum_{n_1,\ldots,n_N} A_1^{n_1} A_2^{n_2} \ldots A_N^{n_N} |n_1 \ldots n_N\rangle.$$

Each tensor $A_j^{n_j}$ carries :

— one physical index n_j of dimension d,

— two virtual indices of dimensions χ_{j-1} and χ_j (except at the boundaries).

Bond dimension and entanglement The virtual index dimension χ_j equals the Schmidt rank across the bipartition

$$\{1, \ldots, j\} \,|\, \{j+1, \ldots, N\}.$$

Thus :

— product states correspond to $\chi_j = 1$ everywhere,

— entangled states require $\chi_j > 1$,

— highly entangled states demand large χ_j.

The MPS ansatz efficiently captures states whose entanglement entropy grows slowly with system size, as guaranteed by area laws in one dimension.

Canonical form Using successive SVDs, an MPS can be brought into a **canonical form** where tensors satisfy orthonormality conditions :

$$\sum_{n_j} (A_j^{n_j})^\dagger A_j^{n_j} = I \quad \text{or} \quad \sum_{n_j} A_j^{n_j} (A_j^{n_j})^\dagger = I.$$

In this form, Schmidt coefficients appear explicitly on the bonds, making truncation and normalization straightforward.

Computational efficiency Thanks to their structure, MPS allow efficient computation of :

— norms and inner products,

— local expectation values,

— correlation functions.

The computational cost typically scales as $\mathcal{O}(N\chi^3)$, which is polynomial rather than exponential in system size.

Physical relevance MPS accurately describe ground states of gapped one-dimensional Hamiltonians, such as spin chains and fermionic wires. They also form the variational class underlying powerful algorithms like DMRG and TEBD, making them a cornerstone of modern numerical quantum many-body physics.

7 Other tensor network architectures

There exist many generalizations :

— MPS / Tensor Train (TT),

— Tree Tensor Networks,

— PEPS (Projected Entangled Pair States) for 2D lattices,

— Tensor grids for two-dimensional systems.

Each architecture is adapted to a particular geometry and entanglement structure.

8 Tensor networks and quantum circuits

A quantum circuit can be interpreted as a tensor network :

— Quantum gates are tensors.

— Wires represent qubits.

— The final state is obtained by contracting all tensors.

This viewpoint allows one to classically simulate quantum circuits as long as the entanglement structure remains moderate.

The Quantum Fourier Transform (QFT) can always be represented and simulated as a tensor network.

9 Time evolution and numerical algorithms

Local operations can temporarily destroy the MPS structure, but it can be restored using SVD and controlled truncation.

Two key algorithms are :

— **TEBD** (Time-Evolving Block Decimation) for time evolution $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$,

— **DMRG** (Density Matrix Renormalization Group), used to find the ground state of Hamiltonians such as the Heisenberg chain :

$$H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1}.$$

DMRG relies on a local alternating optimization procedure, equivalent to solving a sequence of small diagonalization problems.

10 New directions : machine learning

A recent and promising direction consists in using tensor networks as machine learning models :

— Tensors play the role of parameters or weights.

— Contractions replace classical neural networks.

A major advantage is that the same models can be implemented on classical or quantum hardware. This supports the idea of the quantum computer as a machine specialized in tensor network contraction.

11 Conclusion

This lecture shows that tensor networks are no longer only tools for condensed matter physics or one-dimensional quantum systems, but rather constitute a universal language connecting :

— many-body quantum physics,

— quantum circuit simulation,
— machine learning,
— and quantum computation.

They offer a conceptually elegant and numerically efficient route toward understanding complex quantum systems in two and three dimensions.

Attending Dr. Miles's talk was a highly enriching experience that deepened my understanding of how tensor network formalisms can provide a powerful framework for describing quantum many-body systems. Beyond the specific results on efficient state representation, this work illustrates how ideas from entanglement geometry, renormalization groups, and computational complexity are becoming essential for simulating and characterizing complex quantum phases.

This conference also gave me valuable insight into the broader landscape of quantum information science, where theoretical physics and high-performance computing now interact in powerful ways to overcome the curse of dimensionality. It strengthened my interest in the intersection between **many-body physics, topological order, and tensor network algorithms**, and highlighted how such approaches are fundamental to the theoretical underpinning of future quantum technologies.

Remerciements

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references

https://indico.flatironinstitute.org/event/889/attachments/542/773/Miles%20Stoudenmire%20-%20TensorNetworks.pdf