MATRIX COMPUTATIONS: HOMEWORK 1, 20 September 2023

In this homework, we study the factorization of a matrix $A \in \mathbb{R}^{n\times n}$ in the form $A = LU$ where $L \in \mathbb{R}^{n\times n}$ is a *unit* lower-triangular matrix, i.e., with ones on the diagonal, and $U \in \mathbb{R}^{n\times n}$ is an upper-triangular matrix.

Definition Let $A = [a_{ij}]_{i=1,j=1}^{n,n} \in \mathbb{R}^{n\times n}$ and let $k \in \{1, \ldots, n\}$.

- A *principal submatrix* of A is obtained by selecting a subset of indices $\mathcal{S} = \{i_1, i_2, \ldots, i_k\}$ where $1 \le i_1 < i_2 < \ldots < i_k \le n$, and keeping the k rows indexed by these values along with their corresponding columns indexed by the same values. The resulting submatrix is of dimension $k \times k$.
- The determinant of a principal submatrix is called a *principal minor* of A.
- The kth *leading principal submatrix* of A is the principal submatrix of A obtained using the subset of indices $\mathcal{S} = \{1, \ldots, k\}$, i.e., it is the submatrix given by the *first* k rows and the *first* k columns of A.
- The kth *leading principal minor* of A, denoted by m_k is defined as the determinant of the kth leading principal submatrix of A, i.e., $m_k = \det([a_{ij}]_{i=1,j=1}^{k,k})$.

To answer the homework questions, you can use the major theorems from your Bac 1 course: LEPL1101-Algèbre linéaire.

Exercise A: LU factorization

(A1) Not all matrices, even symmetric ones or nonsingular ones, can be written as $A = LU$. Assuming that $\det(A) \neq 0$, prove that the following matrix has no LU factorization

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Remark 1. Note that we can prove through Gaussian elimination that any matrix $A \in \mathbb{R}^{n\times n}$ can be factorized into the following form

$$PA = LU$$

where $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, L is a unit lower-triangular matrix, and U is an upper-triangular matrix. But in this homework, we focus on the specific case without permutation, i.e., when $P = I$ (the identity matrix).

(A2) Assuming that A is nonsingular and that there exists a LU factorization $A = LU$, prove that the factorization is unique.

(A3) Assuming that the leading principal minors of $A \in \mathbb{R}^{n\times n}$ $(n \ge 2)$ are nonzero for every $k \in \{1, \ldots, n-1\}$, i.e., $m_k \neq 0$ for all $k \in \{1, \ldots, n-1\}$, prove that A admits a LU decomposition of the form

$$A = LU$$

with $U_{1,1} = m_1$ and $U_{i,i} = \frac{m_i}{m_{i-1}}$ for $i = 2, \ldots, n$. Note that A need not be nonsingular.

Hint: Proceed by induction on the order of the matrix A.

(A4) Assuming that the leading principal minors of $A \in \mathbb{R}^{n\times n}$ $(n \geq 2)$ are nonzero for every $k \in \{1, \dots, n-1\}$, i.e., $m_k \neq 0$ for all $k \in \{1, \dots, n-1\}$, prove that A admits a decomposition of the form

$$A = LDU$$

where $L \in \mathbb{R}^{n\times n}$ is a lower-triangular matrix with ones on the diagonal, $U \in \mathbb{R}^{n\times n}$ is an upper-triangular matrix with ones on the diagonal and $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix such that $D_{1,1} = m_1$ and $D_{i,i} = \frac{m_i}{m_{i-1}}$ for $i = 2, \dots, n$.

(A5) Assuming that A is nonsingular, prove that A admits a LU (or LDU) factorization if and only if all leading principal minors are nonzero.

(A6) Assuming that A is symmetric, nonsingular and admits a factorization $A = LDU$, with L a unit lower-triangular matrix, U a unit upper-triangular matrix and D a diagonal matrix, prove that the matrix U is given by $U = L^\top$ and thus that the factorization can be written as $A = LDL^\top$.

(A7) Assuming that A is symmetric, prove that the following are equivalent

1. A is positive definite ($A \succ 0$), i.e., for all $x \in \mathbb{R}^n \setminus \{0\}$, $x^\top Ax > 0$;
2. All leading principal minors of A are positive, i.e., for all $k \in \{1, \dots, n\}$, $m_k > 0$.

(A8) Prove that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ admits a factorization $A = LL^\top$ where $L \in \mathbb{R}^{n\times n}$ is a lower-triangular matrix with positive diagonal entries if and only if A is positive definite ($A \succ 0$). This factorization is known as the *Cholesky factorization*.

We are now interested in establishing necessary and sufficient conditions for the positive semidefiniteness of a matrix based on its minors, similar to the result in **(A7)** for positive definite matrices.

(A9) Give a counterexample to illustrate that the condition of all leading principal minors of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ being nonnegative, i.e., $m_k \geq 0$ for all $k \in \{1, \dots, n\}$, is not sufficient to establish that A is positive semidefinite ($A \succeq 0$).

The previous question showed that it is not enough to consider only the leading principal minors of the matrix A.

(A10) Assuming that A is symmetric, prove that the following are equivalent

1. A is positive semidefinite ($A \succeq 0$), i.e., for all $x \in \mathbb{R}^n$, $x^\top Ax \geq 0$;
2. All principal minors of A are nonnegative.

Practical information

The homework solution should be written in English.
Please, submit the pdf file containing your solution on Moodle with file name
`[LINMA2380] - Homework 1 - Group XX`.
Note, as you are in master, we strongly recommend you to write your report in latex. Deadline for turning in the homework: Monday 16 October 2023 (11:59pm). It is expected that each group makes the homework individually. If your group has problems or questions, you are welcome to contact the teaching assistants: julien.calbert@uclouvain.be, guillaume.berger@uclouvain.be.