MATRIX COMPUTATIONS: HOMEWORK 3, 27 November 2023

Theorem 1 (Schur's Theorem)**.** Every matrix $A \in \mathbb{C}^{n\times n}$ is unitarily similar to an upper-triangular matrix, i.e., there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $T = U^*AU$ is upper-triangular.

Theorem 2 (Schur's complement lemma)**.** Let $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ be sub-matrices of the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

If A is invertible:

$$\det(M) = \det(A)\det(D - CA^{-1}B).$$

If D is invertible:

$$\det(M) = \det(D)\det(A - BD^{-1}C).$$

Exercise A:

Using Schur's theorem and the QR decomposition, prove the following results concisely.

(A1) For any invertible matrix A, there exist unitary matrices U and V such that $AA^* = U\Lambda U^*$ and $A^*A = V\Lambda V^*$ where $A = U\Sigma V^*$ and Σ is positive semidefinite.

(A2) For any pair of invertible matrices $A, B \in \mathbb{C}^{n\times n}$, there exist unitary matrices $Q, Z \in \mathbb{C}^{n\times n}$ such that Q^*AZ and Q^*BZ are upper-triangular matrices. Moreover, how can we use this result to find the zeros of the polynomial $\det(sB - A)$ (we call them the generalized eigenvalues of $sB - A$)?

(A3) For any pair of invertible matrices $A, B \in \mathbb{C}^{n\times n}$, there exist unitary matrices $Q, Z \in \mathbb{C}^{n\times n}$ such that Q^*AZ and Z^*BQ are upper-triangular matrices. Moreover, how can you use this result to find the zeros of the polynomial $\det(sI - AB)$ (i.e., the eigenvalues of the matrix AB) and the zeros of the polynomial $\det\begin{pmatrix} sI & -A \\ -B & sI \end{pmatrix}$ (i.e., the eigenvalues of the matrix $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$) ?

Hint: use the Schur's complement lemma for the second polynomial.

(A4) Bonus: Prove that the previous decompositions are also valid when the invertibility conditions are not satisfied by using the following results:

- *Unitary matrices form a compact set*;
- *From any sequence contained in a compact set, we can extract a convergent sub-sequence in this set.*

Exercise B:

(B1) Let $A, B \in \mathbb{C}^{n\times n}$. If A or B is invertible, prove that the eigenvalues of AB and BA are the same (with same algebraic multiplicity).

(B2) Bonus: Prove the previous theorem without assuming that A or B is invertible.

Exercise C:

From the circuit diagram in Figure 1, we have established a system of equations which describe the circuit. In the frequency domain, the system of equations can be rewritten as the form $(sB - A)u(s) = l(s)$ where $u(s) = \begin{pmatrix} u_1(s) & \dots & u_7(s) \end{pmatrix}^\top$, $l(s) = \begin{pmatrix} 0 & \dots & 0 & -e(s) \end{pmatrix}^\top$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(C1) Give the expression for the coefficients a, b, c, d as a function of c_1, c_2, c_3 where

$$\det(sB - A) = as^3 + bs^2 + cs + d$$

using Schur's complement lemma. Then, for $c_1 = c_2 = c_3 = 1$, compute the zeros of this polynomial and verify your results with Matlab's function `eig(A,B)`[1] (the `eig(A,B)` function uses the triangularization of Q^*AZ and Q^*BZ).

(C2) Add to the matrix B a perturbation $B_{6,6} = 10 \cdot \texttt{eps}$ where `eps`[2] is the epsilon machine of Matlab. What happens to the determinant of the matrix $sB - A$? Compute the zeros of this perturbed polynomial $\det(sB - A)$ with Matlab's function `eig(A,B)`.

(C3) Bonus: Can you interpret the effect of this disturbance by adding a corresponding component to the circuit?

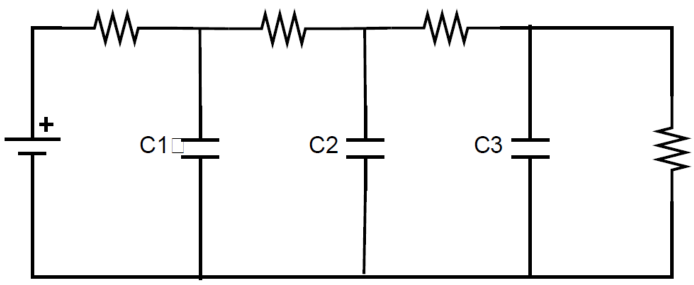

Figure 1: Electrical circuit with four resistors of 1Ω and three capacitors (c_1, c_2, c_3).

[1]Note that you can use any language other than Matlab, with the equivalent function of `eig(A,B)`.
[2]You can use the value 10^{-14} for any other programming language.

Practical information

The homework solution should be written in English.
Please, submit the `pdf` file containing your solution on Moodle with file name
`[LINMA2380] - Homework 3 - Group XX`.
As you are in master, we strongly recommend you to write your report in latex.
Deadline for turning in the homework: Monday 27 November 2023 (11:59pm).
It is expected that each group makes the homework individually.
If your group has problems or questions, you are welcome to contact the teaching assistants: julien.calbert@uclouvain.be, guillaume.berger@uclouvain.be.