# LINMA2370 Project Part I :
# Modelling the Dynamics of Tropical Rainforests

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## Introduction

Remote sensing is used to monitor tree cover distributions in different regions on the Earth. This is an important step towards building predictions about how forests might change due to various factors such as changes in rainfall, wildfires, man-made activities etc. Especially interesting data comes from the tropics (for example Amazon forest in South America, Congo forest or forests in Australasia). It is believed that the tropics exist in a delicate equilibrium and can be in one of the three states: forest with dense tree cover (Figure 1a); savannah with partial tree cover and partial grass cover (Figure 1b); treeless but grass covered arid state (Figure 1c).

One may wonder what factors determine if a given tropic will be in a forest state, savannah, or arid state. Data indicates that amount of rainfall is the key determinant on which of the 3 different states a given tropical forest exists in. Secondly, the forest fires are also important factors behind whether a given tropic is in 3 different forest states. Wildfires use grass as a fuel and keeps burning trees until they reach an equilibrium with a certain proportion of grass and tree cover. But, if the tree cover reaches a certain threshold value, then the grass cover is suppressed, which subsequently leads to lower fuel for the wildfires. Such low/minimal wildfires combined with high rainfall can lead the tropic to a forest state.

(a) Forest state.

(b) Savannah state.

(c) Arid state.

Figure 1: Different forest states based on the tree and grass cover.

We will simulate different models with increasing complexity to understand how the amount of tree cover changes as a function of the amount of rainfall, and parameters that determine the heterogeneity of the landscape and the feedback between vegetation and the regional climate.

# 1 Single-State Modelling

## Presentation of the Analytical Model

We denote by $T(t)$ the the *tree cover* (expressed as a fraction of total cover, i.e. $0 \leq T \leq 1$). The rate of change in tree cover is modelled by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = r(R)T\big(1 - \frac{T}{k}\big) - m_n T\frac{h_n}{T + h_n} - m_f T\frac{h_f^p}{T^p + h_f^p}, \tag{1}$$

$$\text{where } r(R) = r_m \frac{R}{h_R + R}. \tag{2}$$

We succinctly explain the different terms:

$$r(R)T\big(1 - \frac{T}{k}\big)$$

is the *logistic* growth function describing the influence of the total number of trees in the area on the growth.

The *Michaelis–Menten* model for the *expansion rate* $r(R)$ as a function of $R$, the *average rainfall* in mm/day was provided in (2), in which $r_m$ is the maximum tree cover expansion rate (in 1/year) while $h_R$ is the amount of rainfall (in mm/day) where $r(R)$ is reduced by half of its maximum.

The factor $T\big(1 - \frac{T}{k}\big)$ accounts for the limited resources of the soil that restrains the maximum number of trees. The maximum carrying capacity of the land is expressed as a fraction $k$ of the tree cover ($0 \leq k \leq 1$). As the net tree cover approaches this $k$, the rate of tree growth saturates.

$$-m_n T\frac{h_n}{T + h_n}$$

is a *Monod model* for the so-called *nursing effect*: if the tree cover decreases, it will cause a decrease in the protection of plants and seedlings, causing a reduction in the growth rate of tree cover (negative proportionality constant $-m_n$). Below a certain tree cover limit ($h_n$), the rate is drastically reduced since the seedlings will not have much protective cover from the nursing/protective trees.

| Param. | Description | Default value | Units |
|---|---|---|---|
| $R$ | default amount of rainfall | 2.0 | mm/day |
| $r_m$ | maximal rate of tree cover expansion rate | 0.3 | 1/year |
| $h_R$ | rainfall value where $r$ is reduced by half | 0.5 | mm/day |
| $m_n$ | maximal loss rate due to nursing effect | 0.15 | 1/year |
| $h_n$ | tree cover below which rate of loss increases steeply (nursing effect) | 0.1 | |
| $m_f$ | maxima loss rate due to fire mortality | 0.11 | 1/year |
| $h_f$ | tree cover below which rate of loss increases steeply (fire mortality) | 0.6 | |
| $p$ | Hill function exponent | 7 | |
| $k$ | Maximal carrying capacity | 0.90 | |
| $b$ | Local feedback parameter | 2 | mm/day |
| $r_R$ | Maximal rainfall rate toward equilibrium | 1 | 1/year |

Table 1: List of parameter values.

$$-m_f T \frac{h_f^p}{T^p + h_f^p}$$

represents the *fire mortality*, modelled through a *sigmoidal Hill function* (with proportionality constant $-m_f$ and speed exponent $p$). This effect determines what would be the decrease in the rate of change in tree cover if there was a wildfire. Below a certain tree cover threshold ($h_f$), the wildfires are more intensively fuelled by abundant grass.

Numerical values of the parameters of the model to be considered for the simulations are provided in Table 1.

The mathematical and biological foundations of the different terms of (1) and (2) are more extensively described in Sections 5.2, 5.3 and 5.5 of the lecture notes, as well as in the following research paper (optional reading for the interested students willing to understand more deeply the model):

van Nes, E. H., Hirota, M., Holmgren, M. and Scheffer, M. (2014). *Tipping points in tropical tree cover: linking theory to data.* Global change biology, 20(3), 1016-1021.

## Questions

1. Take the values of rainfall ($R$) in the range $[0, 5\,\mathrm{mm/day}]$ with 100 equally spaced values in between. Furthermore, consider 5 different initial tree cover values $(T_0) : 0.05, 0.25, 0.50, 0.75, 1$.

   For each of the $(R, T_0)$ pair, *solve the differential equation* (1) *numerically* for $T(t)$. during 600 years.

   *Use explicit Runge-Kutta method of order 4 for all simulations. One unit of time in this simulation represents one year. This is available in scipy library 'scipy.integrate.solve_ivp'.*

   *Parameters with units* mm/day *in Table 1 can be directly used in the differential equations without any need for conversion into* mm/year.

   *Plot* the trajectory of tree cover $T(t; R)$, in 5 separate figures (one figure each for 5 different initial tree cover values $T_0$).

   *The x-axis should represent 'time (t)', the y-axis should represent 'rainfall value (R)' and z-axis should represent the value of tree cover (T) across time and rainfall value.*

   *Alternatively, instead of a 3D plot, you can also use 2D heatmaps where the tree cover will be visualized as different colors, while the x-axis represents the simulation time (*600 years*), and y-axis represents the amount of rainfall (R between* 0 *and* 5 mm/day*).*

2. You will observe that, for at each pair $(R, T_0)$, the tree covers will reach an equilibrium value at the end of each simulation, denoted $T(600)$. These are fixed points that are derived empirically by simulating (1).

   *Write* a function that, first, identifies the fixed points $T(600)$ reached for each $R$ over different values of $T_0$.

   *Plot* $T(600)$ versus $(R, T_0)$ (one 3D plot containing five curves $T(600; R)$ for 5 different $T_0$).

   On this plot, *locate and identify* the different stable equilibrium states, defined as follows:

   | | | |
   |---|---|---|
   | arid: | $0$ | $\leq T < 5\,\%$ |
   | savannah: | $5\,\%$ | $\leq T \leq 70\,\%$ |
   | forest: | $70\,\%$ | $< T \leq 1$ |

   - Do you expect this plot recover all the fixed points of the dynamical system?

     *It is* not *required, for this part, to calculate (analytically or numerically) all of these points.*

   - If some fixed points are missing, then answer why they cannot be recovered with this simulation approach.

# 2 Two-State Modelling

## Extended Model

We first modelled the tree cover dynamics as a function of the current tree cover ($T(t)$) and used the amount of rainfall ($R$) as one of the input parameter. However, in reality there is a phenomenon called 'vegetation-rainfall feedback' that increases the amount of rainfall locally where trees are abundant. This increase in local rainfall due to the forest feedback occurs in addition to the default global level of rainfall that occurs independently. Once rainfall changes due to this local feedback, it will further influence the rate of change in tree cover. Hence, the tree cover dynamics should be modelled as a *two-dimensional dynamical system* to include the effect of the rate of change in the rainfall as a function of the tree cover. The tree covers still evolves according to (1), now with $R = R(t)$ an additional state variable of the system, whereas $R(t)$ is determined by:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = r_R\Big(\Big(R_{\text{constant}} + b\frac{T}{k}\Big) - R\Big). \tag{3}$$

Note that (1) and (3) form a system of two coupled ordinary differential equations.

In (3), $r_R$ is the maximum rate toward equilibrium for rainfall; $R_{\text{constant}}$ is the default amount of rainfall in the absence of any amount of tree cover in the tropic; $b$ determines the amount of increase in the rainfall from the tree cover ($T$) to maximum carrying capacity ($k$) ratio ($T$). In absence of local vegetation-rainfall feedback, $b = 0$. In that case, if we initialize the rainfall dynamics at $R = 0\,\text{mm/day}$, the final amount of rainfall would reach $R_{\text{constant}}$ as the time increases.

## Questions

1. *Write* code to solve the two-dimensional equations for tree cover and rainfall dynamics.

   Simulate, during 600 years the two-dimensional tree-cover $T(t)$ and rainfall $R(t)$ dynamics for different pairs of initial values $(T_0, R_0, R_{\text{constant}})$, where $T_0$ is the initial tree-cover ranging from 0 to 100 %, and $R_0$ is the initial rainfall ranging from 0 to 5 mm/day. Assume that for all simulations, the default rainfall $R_{\text{constant}}$ is the same as $R$ (Table 1).

2. *Plot* $T(600)$ versus $(R_0, T_0)$ (3D plot or 2D heatmaps where the final tree cover is visualized as different levels of color).

   *Compare* this equilibrium plot to the one obtained from the one-dimensional model. *Comment*, based on your understanding of the biophysical model.