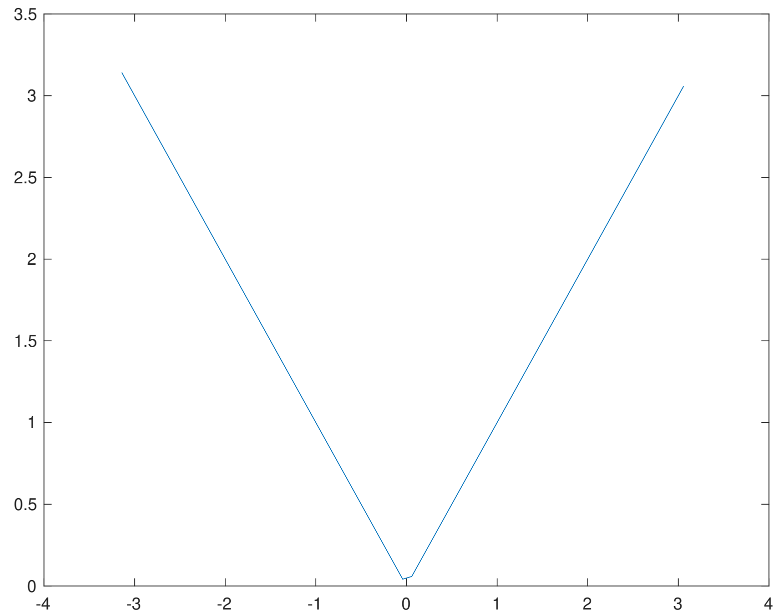
Exercise 6

Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$.

(a) The graph of f



(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \frac{\pi}{2}, & n = 0 \\ \frac{-1+(-1)^n}{n^2\pi}, & n \neq 0 \end{cases}$$

(i) Let $n = 0$. Then,

$$\begin{aligned} \hat{f}(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} \theta d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} -\theta d\theta \\ &= \frac{1}{2\pi}\left.\frac{\theta^2}{2}\right|_0^{\pi} - \frac{1}{2\pi}\left.\frac{\theta^2}{2}\right|_{-\pi}^{0} \\ &= \frac{\pi}{4} + \frac{\pi}{4} \\ &= \frac{\pi}{2} \end{aligned}$$

(ii) Let $n \neq 0$. Then,

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \theta e^{-in\theta}d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} -\theta e^{-in\theta}d\theta \\
&\vdots \quad \text{(Integration By Parts)} \\
&= \frac{1}{2\pi}\Big(\frac{-1+(-1)^n}{n^2} + \frac{-1+(-1)^n}{n^2}\Big) \\
&= \frac{-1+(-1)^n}{n^2\pi}
\end{aligned}$$

(c) What is the Fourier series of f in terms of sines and cosines?

$$\begin{aligned}
f(\theta) &\sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} \\
&\sim \frac{\pi}{2} + \sum_{n=-\infty}^{-1} \frac{-1+(-1)^n}{n^2\pi}e^{in\theta} + \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2\pi}e^{in\theta} \\
&\sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2} \cdot \frac{e^{in\theta}+e^{-in\theta}}{2} \\
&\sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2}\cos(n\theta) \\
&\sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd } \geq 1} \frac{1}{n^2}\cos(n\theta)
\end{aligned}$$

(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof. By **Corollary 2.3**, we have that

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$$

as f is continous on the circle with its Fourier series absolutely convergent. Then

$$\begin{aligned} f(0) &= 0 \\ &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd } \geq 1}^{\infty} \frac{1}{n^2} \cos(0) \\ &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd } \geq 1}^{\infty} \frac{1}{n^2} \end{aligned}$$

implying

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd } \geq 1}^{\infty} \frac{1}{n^2} \tag{1}$$

it is clear from (1) that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Furthermore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n \text{ odd } \geq 1} \frac{1}{n^2} + \sum_{n \text{ even } \geq 2} \frac{1}{n^2} \\ &= \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

resulting in the following equality,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{2}$$

and it is clear from (2) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

Exercise 10

Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that

$$\hat{f}(n) = O\left(\frac{1}{|n|^k}\right) \quad \text{as } |n| \to \infty. \tag{1}$$

Proof. To formally show (1), we perform induction on k. Let $k = 1$, and $n \neq 0$. Then

$$\hat{f}'(n) = in\hat{f}(n)$$

holds as a result of **Corollary 2.4**. Furthermore,

$$\begin{aligned}
|\hat{f}'(n)| &\leq \frac{1}{2\pi}\int_0^{2\pi} |f'(\theta)e^{-in\theta}|d\theta \\
&\leq C_1 \int_0^{2\pi} |f'(\theta)|d\theta \\
&= C
\end{aligned}$$

for some $C \in \mathbb{R}$. So

$$|\hat{f}(n)| \leq \frac{C}{|n|}$$

and the base case holds. Now suppose the statement holds for $k \geq 1$. Then, for $k + 1$

$$\begin{aligned}
|\hat{f}(n)| &\leq \frac{|\hat{f}'(n)|}{|n|} \\
&\leq \frac{1}{|n|} \cdot \frac{C}{|n|^k} \\
&= \frac{C}{|n|^{k+1}}
\end{aligned}$$

holds from our induction hypothesis, and

$$\hat{f}(n) = O\left(\frac{1}{|n|^k}\right) \quad \text{as } |n| \to \infty.$$

$\square$

Exercise 11

Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[0,1]$ such that

$$\int_0^1 |f_k(x) - f(x)|dx \to 0 \quad \text{as } k \to \infty.$$

Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$.

Proof. To show $\hat{f}_k(n) \to \hat{f}(n)$ converges uniformly in n as $k \to \infty$, we first look at the following

$$\begin{aligned}
|\hat{f}_k(n) - \hat{f}(n)| &= |\widehat{(f_k - f)}(n)| \\
&= \Big| \int_0^1 [f_k(x) - f(x)]e^{-2\pi inx}dx \Big| \\
&\leq \int_0^1 |f_k(x) - f(x)|dx \\
&\to 0
\end{aligned}$$

holds from our assumption, concluding $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$. $\square$

Exercise 12

Prove that if a series of complex numbers $\sum c_n$ converges to s, then $\sum c_n$ is Cesàro summable to s.

Proof. Define

$$s_n = \sum_{k=0}^{n} c_k$$

and assume without loss of generality, $s_n \to 0$ as $n \to \infty$. Given $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that

$$|s_n - s| = |s_n| < \epsilon, \quad \forall n > N_0$$

then $\exists N > n$ such that

$$\begin{aligned}
\sigma_n &= \frac{1}{N} \sum_{n=0}^{N-1} s_n \\
&\leq \frac{1}{N} \Big| \sum_{n=0}^{N-1} s_n \Big| \\
&\leq \frac{1}{N} \sum_{n=0}^{N-1} |s_n| \\
&< \frac{1}{N} \cdot N\epsilon \\
&= \epsilon
\end{aligned}$$

So $\sum c_n$ is Cesàro summable to s. $\square$

Exercise 15

Prove that Fejér kernel is given by

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

Proof. Let

$$NF_N(x) = D_0(x) + \cdots + D_{N-1}(x)$$

where $D_n(x)$ is the Dirichlet kernel. Therefore, if $\omega = e^{ix}$, we have

$$\begin{aligned}
NF_N(x) &= \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega} \\
&= \frac{1}{1-\omega}\Big(\sum_{n=0}^{N-1}\omega^{-n} - \sum_{n=0}^{N-1}\omega^{n+1}\Big) \\
&= \frac{1}{1-\omega}\Big(\frac{1-\omega^{-N}}{1-\omega^{-1}} - \omega\frac{1-\omega^{N}}{1-\omega}\Big) \\
&= \frac{1}{(1-\omega)^2}\omega(\omega^{\frac{N}{2}} - \omega^{\frac{-N}{2}})^2 \\
&= \frac{(\omega^{\frac{N}{2}} - \omega^{\frac{-N}{2}})^2}{(\omega^{\frac{1}{2}} - \omega^{\frac{-1}{2}})^2} \\
&= \frac{(e^{\frac{iNx}{2}} - e^{\frac{-iNx}{2}})^2}{(e^{\frac{ix}{2}} - e^{\frac{-ix}{2}})^2} \\
&= \frac{\sin^2(Nx/2)}{\sin^2(x/2)}
\end{aligned}$$

It is clear from above that the Fejér kernel is given by

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

□

Exercise 17a

Prove that if f has a jump discontinuity at θ, then

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) - f(\theta^-)}{2}, \quad \text{with } 0 \le \theta < 1$$

Proof. We will begin this proof with investingating the Poisson kernel, given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

It is should be noted that $P_r(\theta)$ is even. Furthermore, from **Lemma 5.5** we can conclude $P_r(\theta)$ is a good kernel. This will be useful as

$$\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta)d\theta = \frac{1}{2\pi}\int_{0}^{\pi} P_r(\theta)d\theta = \frac{1}{2}$$

Additionally, with $A_r(f)(\theta) = (f * P_r)(\theta)$. Let $\epsilon > 0$, $\exists \delta > 0$ with

$$|f(\theta - y) - f(\theta^-)| < \frac{\epsilon}{2}, \quad -\pi \le \theta < \delta \tag{2}$$
$$|f(\theta - y) - f(\theta^+)| < \frac{\epsilon}{2}, \quad \delta < \theta \le \pi \tag{3}$$

$$\begin{aligned}\left|A_r(f)(\theta) - \frac{f(\theta^+) - f(\theta^-)}{2}\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - y)P_r(\theta)d\theta - \frac{f(\theta^+) - f(\theta^-)}{2}\right| \\ &= \end{aligned}$$

□