Exercise 5

Let

$$f(\theta) = \begin{cases} 0, & \theta = 0 \\ \log(\frac{1}{\theta}), & 0 < \theta \leq 2\pi \end{cases}$$

and define a sequence of functions in $\mathcal{R}$ by

$$f_n(\theta) = \begin{cases} 0, & 0 \leq \theta \leq \frac{1}{n} \\ f(\theta), & \frac{1}{n} < \theta \leq 2\pi \end{cases}$$

Prove that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$. However, f does not belong in $\mathcal{R}$.

Proof. Let $m, n \in \mathbb{N}$ with $m \geq n$. Then

$$\begin{aligned}
\|f_m - f_n\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f_m(\theta) - f_n(\theta)|^2 d\theta \\
&= \frac{1}{2\pi}\int_{\frac{1}{m}}^{\frac{1}{n}} \log(\theta)^2 d\theta \\
&= \frac{1}{2\pi}\left(\theta\log(\theta)^2 - 2\theta\log(\theta) + 2\theta\right)\Big|_{\frac{1}{m}}^{\frac{1}{n}} \\
&= \frac{1}{2\pi}\left(\theta([\log(\theta) - 1]^2 + 1)\right)\Big|_{\frac{1}{m}}^{\frac{1}{n}} \\
&= \frac{1}{2\pi}\left[\frac{1}{n}([\log(n) + 1]^2 + 1) - \frac{1}{m}([\log(m) + 1]^2 + 1)\right] \\
&\to 0
\end{aligned}$$

for sufficiently large m, n. So $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with $f_n \to f$ as $n \to \infty$. However, since f is unbounded, so f does not belong in $\mathcal{R}$. $\square$

Exercise 6

Consider the sequence $\{a_k\}_{k=-\infty}^{\infty}$ defined by

$$a_k = \begin{cases} \frac{1}{k}, & k \geq 1 \\ 0, & k \leq 0 \end{cases}$$

Prove that $\{a_k\} \in \ell^2(\mathbb{Z})$, but that no Riemann integrable function has k^{th} Fourier coefficient equal to a_k for all k.

Proof. First, to show $\{a_k\} \in \ell^2(\mathbb{Z})$, we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |a_k|^2 &= \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &< \infty \end{aligned}$$

so $\{a_k\} \in \ell^2(\mathbb{Z})$. Now suppose for contradiction that $\exists f \in \mathcal{R}$. Then the Abel means of f is defined to be

$$A_r(f)(\theta) = (f * P_r)(\theta)$$

where

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

for $0 \leq r < 1$. Furthermore,

$$\sup |(f * P_r)(\theta)| \leq \sup |f(\theta)| \tag{1}$$

implying $A_r(f)(\theta)$ is bounded. However, $A_r(f)(\theta)$ can also be written as

$$\begin{aligned} A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\ &= \sum_{n=1}^{\infty} \frac{r^n}{n} e^{in\theta} \end{aligned}$$

Specifically, let $\theta = 0$. Then

$$\begin{aligned} \lim_{r \to 1^-} A_r(f)(0) &= \lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{r^n}{n} \\ &= \lim_{r \to 1^-} -\log(1-r) \\ &= \infty \end{aligned}$$

which is a contradiction to (1). $\square$

Exercise 7

Show that the trigonometric series

$$\sum_{n\geq 2} \frac{1}{\log(n)} \sin(nx)$$

converges for every x, yet it is not the Fourier series of a Riemann integrable function.

Proof. Suppose $M, N \in \mathbb{N}$ with $M \geq N$, and let

$$S_N = \sum_{n=2}^{N} \frac{1}{\log(n)} \sin(nx)$$

and

$$B_N = \sum_{n=2}^{N} \sin(nx) \leq B$$

where B is the bound for B_N. Then, using summation by parts,

$$\begin{aligned}|S_M - S_N| &= \Big|\frac{1}{\log(N)}B_N - \frac{1}{\log(M)}B_{M-1} - \sum_{n=M}^{N-1}\Big(\frac{1}{\log(n+1)} - \frac{1}{\log(n)}\Big)B_n\Big| \\ &\leq B\Big(\Big|\frac{1}{\log(N)} + \frac{1}{\log(M)}\Big| + \Big|\sum_{n=M}^{N-1}\frac{\log\left(\frac{n}{n+1}\right)}{\log(n)\log(n+1)}\Big|\Big)\end{aligned}$$

Furthermore,

$$\frac{\log\left(\frac{n}{n+1}\right)}{\log(n)\log(n+1)} \leq \frac{1}{n\log(n)^2} \tag{1}$$

implying the sum

$$\sum_{n=M}^{N-1}\frac{\log\left(\frac{n}{n+1}\right)}{\log(n)\log(n+1)}$$

converges by the integral test with results from (1). Alternatively, Dirichlet's test could be used to conclude

$$\sum_{n\geq 2} \frac{1}{\log(n)} \sin(nx)$$

converges.

Now suppose for contradiction $\exists f \in \mathcal{R}$ with such a Fourier series. Then

$$\begin{aligned}\sum_{n\geq 2} \frac{1}{\log(n)} \sin(nx) &= \sum_{n\geq 2} \frac{1}{\log(n)} \Big(\frac{e^{inx} - e^{-inx}}{2i}\Big) \\ &= \sum_{n\geq 2} \frac{1}{2i\log(n)} e^{inx} - \sum_{n\leq -2} \frac{1}{2i\log(-n)} e^{inx} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{inx}\end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{2i\log(n)}, & n \geq 2 \\ 0, & |n| \leq 1 \\ -\frac{1}{2i\log(-n)}, & n \leq -2 \end{cases}$$

are the Fourier coefficients of f. Furthermore,

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{4} \sum_{n=2}^{\infty} \frac{1}{|i\log(n)|^2}$$

Note that the series above diverges, implying $\{c_n\} \notin \ell^2(\mathbb{Z})$, which is a contradiction to the assumption $f \in \mathcal{R}$. $\square$

Exercise 8a

Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Use Parseval's identity to find the sums of the following two series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}$$

In fact, they are $\frac{\pi^4}{96}$ and $\frac{\pi^4}{90}$, respectively. $(*)$

Proof. From Exercise 6 in Chapter 2, the Fourier coefficients of f are

$$a_n = \hat{f}(n) = \begin{cases} \frac{\pi}{2}, & n = 0 \\ \frac{-1+(-1)^n}{n^2\pi}, & n \neq 0 \end{cases}$$

Then,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |a_n|^2 &= \frac{\pi^2}{4} + \sum_{n \neq 0} \left| \frac{-1+(-1)^n}{n^2\pi} \right|^2 \\ &= \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n \text{ odd } \geq 1} \frac{1}{n^4} \\ &= \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \end{aligned}$$

and

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \theta^2 d\theta \\ &= \left. \frac{\theta^3}{3\pi} \right|_0^{\pi} \\ &= \frac{\pi^2}{3} \end{aligned}$$

By Parseval's identity,

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2 \iff \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^2}{3} \tag{1}$$

it is clear from (1) that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Furthermore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^4} &= \sum_{n \text{ odd } \geq 1}\frac{1}{n^4} + \sum_{n \text{ even } \geq 2}\frac{1}{n^4} \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} + \sum_{n \text{ even } \geq 2}\frac{1}{n^4} \\ &= \frac{\pi^4}{96} + \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4}\end{aligned}$$

resulting in the following equality,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96} + \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{2}$$

and it is clear from (2) that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

where both results matches with $(*)$. $\square$

Exercise 9

Show that for α not an integer, the Fourier series of

$$\frac{\pi}{\sin(\pi\alpha)} e^{i(\pi-x)\alpha}$$

on $[0, 2\pi]$ is given by

$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}$$

Apply Parseval's formula to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin(n\alpha)^2}$$

Proof. Let $n \in \mathbb{Z}$. Then

(i) $n \neq 0$:

$$\begin{aligned}
a_n &= \hat{f}(n) \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi}{\sin(\pi\alpha)} e^{i(\pi-x)\alpha - inx} dx \\
&= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)} \int_0^{2\pi} e^{-i(n+\alpha)x} dx \\
&= \frac{-e^{i\pi\alpha}}{2\sin(\pi\alpha)} \cdot \frac{e^{-i(n+\alpha)x}}{i(n+\alpha)} \Bigg|_{x=0}^{x=2\pi} \\
&= \frac{-e^{i\pi\alpha}(1 - e^{-2\pi i(n+\alpha)})}{2i \sin(\pi\alpha)(n+\alpha)} \\
&= \frac{1}{n+\alpha}
\end{aligned}$$

(ii) $n \neq 0$:

$$\begin{aligned}
a_0 &= \hat{f}(n) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi}{\sin(\pi\alpha)} e^{i\pi\alpha} \cdot e^{-i\alpha x} dx \\
&= \frac{-e^{i\pi\alpha}}{2i\alpha\sin(\pi\alpha)} \cdot e^{-i\alpha x}\Big|_{x=0}^{x=2\pi} \\
&= \frac{-e^{i\pi\alpha}(e^{-2\pi\alpha i} - 1)}{2i\sin(\pi\alpha)} \cdot \frac{1}{\alpha} \\
&= \frac{1}{\alpha}
\end{aligned}$$

By (i) and (ii). The Fourier series is given by

$$\sum_{n=-\infty}^{\infty} a_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}$$

Furthermore, by Parseval's identity, we have

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2$$

where

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx$$

So, we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} &= \|f\|^2 \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{\pi}{|\sin(n\alpha)|}|e^{i(\pi-x)\alpha}|\Big)^2 dx \\
&= \frac{\pi}{2\sin(n\alpha)^2}\, x\Big|_{x=0}^{x=2\pi} \\
&= \frac{\pi^2}{2\sin(n\alpha)^2}
\end{aligned}$$

$\square$

Exercise 12

Prove that $\displaystyle\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$.

Proof. To start, let $D_N(x)$ be the N^{th} Dirichlet kernel. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} D_N(x)dx &= \int_{-\pi}^{\pi} \sum_{\substack{|n|\leq N\\ n\neq 0}} e^{inx} + \int_{-\pi}^{\pi} dx \\
&= \sum_{\substack{|n|\leq N\\ n\neq 0}} \int_{-\pi}^{\pi} e^{inx} + x\big|_{x=-\pi}^{\pi} \\
&= \sum_{\substack{|n|\leq N\\ n\neq 0}} \frac{1}{in}\, e^{inx}\Big|_{x=-\pi}^{\pi} + 2\pi \\
&= 2\pi
\end{aligned}$$

Furthermore, let $f : [-\pi, \pi] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{1}{\sin(x/2)} - \frac{2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

A simple use of L'Hopital rule will show that the function is continous on $[-\pi, \pi]$, and Riemann Lebesque Lemma tells us that

$$\int_{-\pi}^{\pi} \sin([N+1/2]x)\Big(\frac{1}{\sin(x/2)} - \frac{2}{x}\Big) \to 0, \qquad \text{as } N \to \infty$$

and

$$\begin{aligned}
2\pi &= \lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{2\sin([N+1/2]x)}{x}dx \\
&= 2\lim_{n\to\infty} \int_{-(N+1/2)\pi}^{(N+1/2)\pi} \frac{sin(u)}{u}du \\
&= 4\int_0^\infty \frac{sin(u)}{u}du
\end{aligned}$$

So we have $\displaystyle\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$. $\square$

Exercise 15

Let f be 2π-periodic and Riemann integrable on $[-\pi, \pi]$.

(a) Show that

$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-inx}dx$$

hence

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]e^{-inx}dx$$

(b) Now assume that f satisfies a Hölder condition of order α, namely

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

for some $0 < \alpha \leq 1$, some $C > 0$, and all x, h Use part a) to show that

$$\hat{f}(n) = O(1/|n|^{\alpha})$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x},$$

where $0 < \alpha < 1$, satisfies

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

and $\hat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$.

Proof. By Chapter 2 Exercise 1, we have that

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-in(x+\frac{\pi}{n})}dx \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-in}dx
\end{aligned}$$

So $\hat{f}(n)$ can be written as the following sum

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2}\hat{f}(n) + \frac{1}{2}\hat{f}(n) \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-in}dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}[f(x) - f(x+\pi/n)]e^{-inx}dx\end{aligned}$$

Now we assume that f satisfies a Hölder condition of order α as in part b). Then

$$\begin{aligned}|\hat{f}(n)| &= \Big|\frac{1}{4\pi}\int_{-\pi}^{\pi}[f(x) - f(x+\pi/n)]e^{-inx}dx\Big| \\ &\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x) - f(x+\pi/n)|dx \\ &\leq \frac{1}{4\pi}\cdot C\Big|\frac{\pi}{n}\Big|^{\alpha} x\Big|_{x=-\pi}^{x=\pi} \\ &= \frac{C\pi^{\alpha}}{2}\cdot\frac{1}{|n|^{\alpha}}\end{aligned}$$

So $\hat{f}(n) = O(1/|n|^{\alpha})$. Lastly, let z be the largest integer such that

$$2^{z}|h| \leq 1$$

Then

$$\begin{aligned}|f(x+h) - f(x)| &= \Big|\sum_{k=0}^{\infty} 2^{-k\alpha}(e^{i2^{k}(x+h)} - e^{i2^{k}x})\Big| \\ &\leq \sum_{k=0}^{z}|2^{-k\alpha}(e^{i2^{k}(x+h)} - e^{i2^{k}x})| + \sum_{k=z+1}^{\infty}|2^{-k\alpha}(e^{i2^{k}(x+h)} - e^{i2^{k}x})| \\ &\leq |h|\sum_{k=0}^{z} 2^{(1-\alpha)k} + 2\sum_{k=z+1}^{\infty} 2^{-k\alpha} \\ &\leq |h|\frac{1-2^{(1-\alpha)(z+1)}}{1-2^{(1-\alpha)}} + 2\frac{2^{-(z+2)\alpha}}{1-2^{-\alpha}} \\ &= \underbrace{\left(\frac{1}{2^{(1-\alpha)}-1} + \frac{2}{1-2^{-\alpha}}\right)}_{C}|h|^{\alpha}\end{aligned}$$

So f satisfies $|f(x+h)-f(x)| \leq C|h|^\alpha$, where $0<\alpha<1$, and

$$\begin{aligned}
\hat{f}(N) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-iNx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x}\cdot e^{-iNx} \\
&= \frac{1}{2\pi}\sum_{k=0}^{\infty} 2^{-k\alpha}\underbrace{\int_{-\pi}^{\pi} e^{i(2^k-N)x}dx}_{I_N}
\end{aligned}$$

where

$$I_N = \int_{-\pi}^{\pi} e^{i(2^k-N)x}dx = \begin{cases} 2\pi, & 2^k = N \\ 0, & 2^k \neq N \end{cases}$$

which concludes the final part where $\hat{f}(N) = 1/N^\alpha$ whenever $N = 2^k$. $\square$