Essay for the specialised integration theory course, VT23

The Malý-Pfeffer integral

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Introduction

Given a suitable set $A \subseteq \mathbb{R}^n$ and suitable $w \colon \mathbb{R}^n \to \mathbb{R}^n$ the divergence theorem states that

$$\int_A \operatorname{Div} w \, d\mathscr{L}^n = \int_{\partial A} w \cdot \nu_A \, d\mathscr{H}^{n-1}$$

Here $\nu_A \colon \partial A \to S^{n-1} \subseteq \mathbb{R}^n$ is the exteriour unit normal and Div denotes the divergence. A very general formulation of the divergence theorem is presented in Pfeffer, the guass green theorem where the integral on the left hand side is formulated using the Pfeffer integral, referred to in [1] as the R-integral. This integral was generalised to the R^*-integral in [1].

Definition of the integral

We begin by defining the essential interiour, exteriour and boundary.

Definition 1 (essential interiour, exteriour, boundary). *We calle the set of density points in A essential interiour* $\operatorname{int}_* A$ *of A. The essential exteriour* $\operatorname{ext}_* A = \operatorname{int}_* A^{\complement}$ *is the essential interiour of the complement of A. The essential boundary is given by*

$$\partial_* A = \mathbb{R}^n \setminus \left(\operatorname{int}_* A \cup \operatorname{ext}_* A \right).$$

Let now $x \in \operatorname{int} A$ then there exists $r > 0$ such that $B_r(x) \subseteq A$. Since

$$\frac{|A \cap B_r(x)|}{|B_r(x)|} = 1 \xrightarrow{r \to 0} 1$$

we have that x is a density point and thus $x \in \operatorname{int}_* A$. Therefore we have that $\operatorname{int} A \subseteq \operatorname{int}_* A$. It then follows further that $\operatorname{ext}_* A \subseteq \operatorname{ext} A$ and also $\partial_* A \subseteq \partial A$. We can now introduce the notion of relative perimiter of a set

Definition 2 (relative perimiter). *We define the relative perimiter of a measurable set E to be*

$$P(E, in\ A) = \mathscr{H}^{n-1}(\partial_* E \cap \operatorname{int}_* A).$$

where $\mathscr{H}^{n-1}$ *denotes the* $(n-1)$*-dimensional Hausdorff-measure. For convenience we write*

$$P(E) = P(E, in\ \mathbb{R}^n).$$

We can now proceed to define $\mathscr{BV}$-sets

Definition 3 ($\mathscr{BV}$-sets). *A measurable set* $A \subseteq \mathbb{R}^n$ *is called a* $\mathscr{BV}$*-set if* $|A| + P(A) < \infty$. *We denote by* $\mathscr{BV}$ *the set of all* $\mathscr{BV}$*-sets and by* $\mathscr{BV}_c$ *the set of all bounded* $\mathscr{BV}$*-sets.*

The reason for looking at $\mathscr{BV}$-sets is the following. Given a $\mathscr{BV}$-set A there exists an exteriour normal $\nu_A \colon \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ which is unique up to $\mathscr{H}^{n-1}$-nullsets. One can now define a topology on the $\mathscr{BV}_c$-sets.

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Definition 4 (Topology on $\mathscr{BV}_c$)**.** *We say a sequence $A\colon \mathbb{N} \to \mathscr{BV}_c$ converges to A_* if there exists a compact $K \subseteq \mathbb{R}^n$ such that $A_k \subseteq K$, $\sup_k P(A_k) < \infty$ and $|A_* \triangle A_k| \to 0$ as $k \to \infty$. Here $A \triangle B = (B \setminus A) \sqcup (A \setminus B)$ denotes the symmetric difference.*

Using this topology on $\mathscr{BV}_c$ we can proceed to define charges (not the legal sort).

Definition 5 (Charge)**.** *A function $F\colon \mathscr{BV}_c \to \mathbb{R}$ is called*

- *finitely additive if $F(A \sqcup B) = F(A) + F(B)$ for all disjoint $A, B \in \mathscr{BV}_c$.*

- *continuous if $A_k \to A_*$ implies that $F(A_k) \to F(A_*)$.*

- *a charge if it is finitely additive and continuous*

One sees from the definition that charges form a linear space. It is shown in [2, Corollary 1.10.4] that the space of dyadic cubes is dense in $\mathscr{BV}_c$. As a consequence we have that

Lemma 6 (Density of cubes)**.** *Let F be a charge such that $F(C) = 0$ for all dyadic cubes $C \subseteq \mathbb{R}^n$. Then $F = 0$.*

We now give some important examples of charges.

Claim (Indefinite Lebesgue-Integrals are charges)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$ then the indefinite integral of f*

$$F\colon A \mapsto \int_A f \, \mathrm{d}\mathscr{L}^n$$

is a charge.

Proof. F is finitely additive. Let $A_k \to A_*$ then we have by the dominated convergence theorem with majorant $\mathbb{1}_K f$ that

$$\int_{A_k} f \, \mathrm{d}\mathscr{L}^n \xrightarrow{k \to \infty} \int_{A_*} f \, \mathrm{d}\mathscr{L}^n$$

and hence F is a charge. $\qquad\square$

This example motivates why charges are used in the definition of the R^*-integral which generalises the Lebesgue-integral. Additionally it follows from this example and the Radon-Nikodym derivative (c.f. e.g. ??) that every measure which is absolutely continuous w.r.t. the Lebesgue measure is a charge. For the next example we require an alternative characterisation of charges.

Lemma 7 (Alternative characterisation of charges)**.** *A finitely additive $F\colon \mathscr{BV}_c \to \mathbb{R}$ is a charge iff for every $\varepsilon > 0$ then there exists $\theta > 0$ such that $|F(A)| < \varepsilon$ for all $A \in \mathscr{BV}_c$ such that $A \subseteq B_{1/\varepsilon}(0)$, $P(A) < 1/\varepsilon$ and $|A| \leq \theta$.*

Proof. Assume the alternative characterisation holds. Let $A_k \to A_*$ in $\mathscr{BV}_c$ then for all $\varepsilon > 0$ small such that $A_k \subseteq B = B_{1/\varepsilon}(0)$ and $P(A_k) < 1/\varepsilon$ for all k. Choose $\theta > 0$ according to the alternative characterisation. We set $B_k = A_* \setminus A_k$ then also $B_k \subseteq B$, $P(B_k) \leq 1/\varepsilon$ and $|B_k| \leq \theta$ for k large enough. Analogously for and $C_k = A_k \setminus A_*$ we

have $C_k \subseteq B$, $P(C_k) \le 1/\varepsilon$ and $|C_k| \le \theta$ for k large enough. Thus $|F(B_k)| \le \varepsilon$ for k large enough and as $\varepsilon > 0$ was arbitrarily small it follows that $F(B_k) \to 0$ as $k \to \infty$. Analogously we have $F(C_k) \to 0$ as $k \to \infty$. No2

$$F(A_k) + F(A_* \setminus A_k) = F(A_k \cup A_*) = F(A_*) + F(A_k \setminus A_*)$$

implies

$$F(A_k) = F(A_*) + F(C_k) - F(B_k) \xrightarrow{k \to \infty} F(A_*)$$

so F is a charge.

Assume the alternative characterisation does not hold. Then there exists an $\varepsilon > 0$ s.t. there exists a sequence of $A_k \in \mathscr{BV}_c$ such that $A_k \subseteq B_{1/\varepsilon}(0)$, $P(A_k) < 1/\varepsilon$, $|A_k| \le 1/k$ and $|F(A_k)| > \varepsilon$. But then $A_k \to 0$ in $\mathscr{BV}_c$ and $|F(A_k)| \not\to 0$ in $\mathbb{R}$ and F is not a charge. $\qquad\square$

A further example for a charge is given by fluxes.

Claim (Fluxes are charges). *For $E \in \mathscr{BV}$ and $w \in C(\mathrm{cl}(E); \mathbb{R}^n)$ we have that the flux of w*

$$F : A \mapsto \int_{\partial_*(A \cap E)} w \cdot \nu_{A \cap E} \, \mathrm{d}\mathcal{H}^{n-1}$$

is a charge. Here $\nu_{A \cap E} : \partial_*(A \cap E) \to S^{n-1} \subseteq \mathbb{R}^n$ denotes the outer unit normal.

Proof. We follow [2, Example 2.1.4]. We have that F is finitely additive. Let $\varepsilon > 0$ and $A \in \mathscr{BV}_c$ such that $P(A) < 1/\varepsilon$, $A \subseteq B$ and $|A| \le \theta$ where $B =_{1/\varepsilon} (0)$ and $\theta > 0$. Now choose $w \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $|v - w| < \eta$ on $B \cap \mathrm{cl} E$. It then follows that

$$\begin{aligned}
|F(A)| &\le \int_{\partial_*(A \cap E)} |v - w| \, \mathrm{d}\mathcal{H}^{n-1} + \int_{A \cap E} |\mathrm{Div}\, w| \, \mathrm{d}\mathcal{L}^n \\
&\le \eta P(A \cap E) + |A \cap E| \|\mathrm{Div}\, w\|_\infty \\
&\le \eta(P(A) + P(B \cap E)) + |A| \|\mathrm{Div}\, w\|_\infty \\
&\le \eta\left(\frac{1}{\varepsilon} + P(B \cap E)\right) + \theta \|\mathrm{Div}\, w\|_\infty.
\end{aligned}$$

Now choosing η and then θ small enough we obtain that

$$|F(A)| \le \varepsilon.$$

$\qquad\square$

We now define the regularity of $\mathscr{BV}_c$-sets.

Definition 8 (Regularity of $\mathscr{BV}_c$-sets). *For $E \in \mathscr{BV}_c$ we define the regularity*

$$r(E) = \begin{cases} \frac{|E|}{\mathrm{diam}(E)P(E)} & \text{if } |E| > 0 \\ 0 & \text{else} \end{cases}$$

And we define what an ε-isoparametric set is

Definition 9 (ε-isoparametric). *We call $E \in \mathscr{BV}_c$ ε-isoparametric if for all $T \in \mathscr{BV}$*

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{P(T, in \, E)}{\varepsilon} \, .$$

We now show that cubes are in fact an example of ε-isoparametric for some small ε which only depends on the dimension n.

Proposition 10 (Cubes are ε-isoparametric). *There exists a $c_1 > 0$ which depends only on the dimension s.t. every cube $C \subseteq \mathbb{R}^n$ is ε-isoparametric for every $\varepsilon < c_1$.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Definition 11 (Gauge). *We call a set thin if it is σ-finite w.r.t. $\mathscr{H}^{n-1}$. A mapping $\delta \colon \mathbb{R}^n \to \mathbb{R}^n_{>0}$ for which $\{\delta = 0\}$ is thin is called a gauge.*

Definition 12 (Partitions). *Let δ be a gauge and $\varepsilon > 0$. We call*

$$\mathscr{P} = \big\{(E_1, x_1), \dots, (E_p, x_p)\big\}$$

a partition of the set $\bigcup \mathscr{P} = \bigcup_i E_i$ if $E_i \in \mathscr{BV}_c$ are disjoint sets and $x_i \in \mathbb{R}^n$. A partition is called

- *dyadic if E_i is a dyadic cube and $x_i \in E_i$ for all i*

- *ε-regular if $r(E_i \cup \{x_i\}) > \varepsilon$ for all i*

- *strongly ε-regular if it is ε-regular, E_i is ε-isoperimetric and $x_i \in \mathrm{cl}_* E_i$ for all i*

- *δ-fine if $E_i \subseteq B_{\delta(x_i)}(x_i)$ for all i*

After all these definitions we are finally able to define the R^*-integral.

Definition 13 (R^*-integral). *A function $f \colon \mathbb{R}^n \to \mathbb{R}$ is called R^*-integrable with respect to a charge G if there is a charge F, s.t. for all $\varepsilon > 0$ there exists a gauge δ such that*

$$\sum_{i=1}^{p} |f(x_i) G(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\big\{(E_1, x_1), \dots, (E_p, x_p)\big\}$. We call F an indefinite integral of f with respect to G and write

$$F = (R^*) \int f \, \mathrm{d}G \, .$$

Uniqueness of the integral

We now would like to prove the uniqueness of the integral. For this we require a version of Cousin's Lemma.

Definition 14 (Nice dyadic cubes)**.** *Let* $\sigma \colon C \to \mathbb{R}_{>0}$*. A dyadic cube C is called nice if there exist a dyadic σ-fine partition of C. A dyadic cube which is not nice is called faulty.*

Lemma 15 (Cousin)**.** *All dyadic cubes are nice.*

Proof. Assume a dyadic cube $C = C^1$ is faulty and has diameter r. Then C^1 can be written as the disjoint union $C = \bigsqcup_i C_i$ of dyadic cubes C_i with diameters less than $r/2$. Since C is faulty at least one of the C_i, say $C^2 = C_i$ is also faulty. Inductively we obtain a sequence of nested faulty dyadic cubes C^j with diameters less than $r/2^j$. Thus

$$\bigcap_j C^j = \{x\}$$

for some $x \in C$. Let j be s.t. $r/2^j < \sigma(x)$. Then we have that $\operatorname{diam}(C^j) < \sigma(x)$ and $x \in C^j$ so C^j is nice. This is a contradiction. $\qquad\square$

We cite without proof the following proposition from [2, Lemma 1.3.2]

Proposition 16. *There exists a constant c_2 which depends only on the dimension n such that for all $\delta > 0$ and $T \subseteq \mathbb{R}^n$ with $t = \mathscr{H}^{n-1}(T) < \infty$ there exists a family of dyadic cubes $\mathscr{K}$ with diameters less than δ and*

$$T \subseteq \operatorname{int} \bigcup \mathscr{K} \qquad \text{and} \qquad \sum_{K \in \mathscr{K}} P(K) < c_2 t \,.$$

With this proposition we can prove the following result from [2, Lemma 2.6.4].

Lemma 17. *Let C be a cube F be a charge, $\varepsilon > 0$ and δ be a gauge. Then there exist a δ-fine dyadic partition $\mathscr{P} = \big\{(C_1, x_1), \dots, (C_q, x_q)\big\}$ such that*

$$|F|\Big(C \setminus \bigcup \mathscr{P}\Big) < \varepsilon \,.$$

Proof. We essentially follow [2, Lemma 2.6.4].

The idea is to involve Cousin's lemma. Since Cousin's lemma only applies for $\sigma > 0$ everywhere but our definition of a gauge allows for δ to vanish on a thin set we have to choose our σ wisely. Set $T = \{\delta = 0\}$. Since T is thin there exist $T_i \subseteq C$ such that $T = \bigcup_{i \geq 1} T_i$ and $t_i = \mathscr{H}^{n-1}(T_i) < \infty$. Let $\varepsilon > 0$. Now define $\varepsilon_i > 0$ such that

$$\varepsilon_i \leq \frac{\varepsilon}{2^i} \qquad \text{and} \qquad \varepsilon_i \leq \frac{1}{c_2 t_i}$$

and $C \subseteq B_{1/\varepsilon_i}(0)$ for all i. The reason for the choice of ε in this way will become clear later. By the alternative characterisation of charges there exist θ_i such that $|F(A)| < \varepsilon_i$

for each $A \in \mathscr{BV}_c$ such that $A \subseteq C$, $P(A) < 1/\varepsilon_i$ and $|A| < \theta_i$ for all i. By Proposition 16 there exist families $\mathscr{K}_i$ of dyadic cubes with diameters less than $\theta_i \varepsilon_i$,

$$T_i \subseteq \operatorname{int}\bigcup \mathscr{K}_i \qquad \text{and} \qquad \sum_{K \in \mathscr{K}} \operatorname{diam}(K)^{n-1} < c_2 t_i.$$

Let $\mathscr{K}$ be a disjoint subfamily of $\mathscr{K}_0 = \bigcup_i \mathscr{K}$ such that $\bigcup \mathscr{K} = \bigcup \mathscr{K}_0$. Define a function $\sigma \colon C \to \mathbb{R}_{>0}$ by

$$\sigma(x) = \begin{cases} \operatorname{diam}(K) & \text{if } x \in K \cap T \text{ for a } K \in \mathscr{K} \\ \delta(x) & \text{else} \end{cases}$$

By Cousin's Lemma 15 there exists a dyadic σ-fine partition $\mathscr{C} = \{(C_1, x_1), \ldots, (C_p, x_p)\}$ of C. We now do a reordering of indices such that

$$\mathscr{C} = \Big\{ \underbrace{(C_1, x_1), \ldots, (C_q, x_q)}_{C_i \nsubseteq \bigcup \mathscr{K} \text{ for } 1 \leq i \leq q} \quad, \quad \underbrace{(C_{k_1+1}, x_{k_1+1}), \ldots, (C_{k_2}, x_{k_2})}_{\substack{C_i \subseteq \bigcup \mathscr{K}_1 \text{ for all} \\ q=k_1 < i \leq k_2}}$$

$$, \quad \cdots \quad , \quad \underbrace{(C_{k_m+1}, x_{k_m+1}), \ldots, (C_{k_{m+1}}, x_{k_{m+1}})}_{\substack{C_i \subseteq \mathscr{K}_m \setminus ((\bigcup \mathscr{K}_1) \cup \cdots \cup (\bigcup \mathscr{K}_{m-1})) \\ \text{for all } k_m < i \leq k_{m+1}=p}} \Big\}.$$

We set $\mathscr{P} = \{(C_1, x_1), \ldots, (C_1, x_1)\}$. Since by construction we have that $x_j \in T$ implies that there exists a cube $K \in \mathscr{K}_i$ such that $C_i \subseteq K$ it follows that $\mathscr{P}$ is a δ-fine dyadic partition. We now have by Proposition 16 and the choice of ε_i that

$$P\left(\bigcup_{k_i < k \leq k_{i+1}} C_k \right) \leq \sum_{K \in \mathscr{K}_i} P(K) \leq \frac{1}{\varepsilon_i}$$

and it thus follows further that

$$\left| \bigcup_{k_i < k \leq k_{i+1}} C_k \right| \leq \left| \bigcup \mathscr{K}_i \right| \leq \sum_{K \in \mathscr{K}_i} \operatorname{diam}(K) P(K) < \varepsilon_i \theta_i \sum_{K \in \mathscr{K}_i} P(K) \leq \theta_i$$

so

$$|F|\left(C \setminus \bigcup_{i=1}^{q} C_i \right) = |F|\left(\bigcup_{i=q+1}^{p} C_j \right)$$

$$\leq \sum_{i=1}^{m} |F|\left(\bigcup_{k_i < k \leq k_{i+1}} C_k \right)$$

$$\leq \sum_{i=1}^{m} \varepsilon_i$$

$$\leq \sum_{i \geq 1} 2^{-i} \varepsilon = \varepsilon$$

$\square$

Claim. *The integral is unique.*

Proof. We follow [1, Proposition 3.4]. Let F_1 and F_2 be R_*-integrals of f w.r.t. G. We set $H = |F_1 - F_2|$. Now choose a cube $C \subseteq \mathbb{R}^n$ and $0 < \varepsilon < c_1$. Let δ be the corresponding gauge to ε in the definition of the integral. By Lemma 17 there exists a δ-fine dyadic partition $\mathscr{P} = \{(C_1, x_1), \ldots, (C_q, x_q)\}$ such that

$$|H|\left(C \setminus \bigcup \mathscr{P}\right) < \varepsilon.$$

It then follows that

$$|H(C)| \leq |H|\left(C \setminus \bigcup \mathscr{P}\right) + |H|\left(\bigcup \mathscr{P}\right)$$

$$\leq \varepsilon + \left|\sum_i (F_1(C_i) - F_2(C_i))\right|$$

$$\leq \varepsilon + \sum_i |F_1(C_i) - f(x_i)G(C_i)| + \sum_i |f(x_i)G(C_i) - F_2(C_i)|$$

$$\leq 3\varepsilon \xrightarrow{\varepsilon \to 0} 0$$

and thus $H(C) = 0$. Since C was an arbitrary dyadic cube it follows from Lemma 6 on the density of cubes that $H = 0$ and hence $F_1 = F_2$. $\qquad\square$

Claim (Linearity of the integral). *Let f_1, f_2 be R_*-integrable and $a \in \mathbb{R}$. Then $f + ag$ is also integrable and*

$$(R^*) \int f_1 \, dG + (R^*) \int f_2 \, dG = (R^*) \int f_1 + \alpha f_2 \, dG.$$

Proof. We write $F_i = \int f_i \, d^*G$. then we have that for all $\varepsilon > 0$ there exist gauges δ_i such that

$$\sum_j |f_i(x_j)G(E_j) - F_i(E_j)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. Since the space of charges is a linear space we have that also $F_1 + \alpha F_2$ is a charge. If we now set $\delta = \min_i \delta_i$ then we obtain that

$$\sum_j |(f_1 + \alpha f_2)(x_j)G(E_j) - (F_1 + \alpha F_2)(E_j)|$$

$$\leq \sum_j |f_1(x_j)G(E_j) - F_1(E_j)| + |\alpha| \sum_j |f_2(x_j)G(E_j) - F_2(E_j)|$$

$$\leq (1 + |\alpha|)\varepsilon$$

for every strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. Thus $f_1 + \alpha f_2$ is integrable with integral $F_1 + \alpha F_2$. $\qquad\square$

What this integral is good for?

In [1, Proposition 3.5] it is stated that integral generalises the Lebesgue integral on $\mathbb{R}^n$.

Proposition 18 (Generalisation of the Lebesgue integral on $\mathbb{R}^n$)**.** *Each Lebesgue-integrable function is also R_*-integrable and the integrals conincide.*

The integral generalises the Henstock-Kurzweil integral on $\mathbb{R}$. It is shown in [1, Proposition 3.6] that

Proposition 19 (Generalisation of the Henstock-Kurzweil integral on $\mathbb{R}$)**.** *A function $f\colon \mathbb{R} \to \mathbb{R}$ is Denjoy-Perron integrable on a compact $A \subseteq \mathbb{R}$ iff it is R_*-integrable and the two integrals coincide on A.*

We now turn back to the divergence theorem. For this we need some additional conditions on our sets.

Definition 20 (Admissable sets)**.** *We call a set admissable if $\mathrm{int}_* A \subseteq A \subseteq \mathrm{cl}_* A$ and ∂A is compact. The set of admissible $\mathscr{BV}$-sets is denoted by $\mathscr{ABV}$.*

It is shown in [1, Corollary 3.18] that for admissable sets the R_*-integral and the classical Pfeffer integral which is defined in coincide. That is

Proposition 21 (Generalisation of the Pfeffer-Integral on $\mathbb{R}^n$)**.** *Let $A \in \mathscr{ABV}$. Then each Pfeffer-integrable function is also R_*-integrable and the integrals coincide on A.*

We now turn our attention to the divergence theorem which is one of the major motivations for the Pfeffer integral and hence also the R^*-integral. The following definition is from [4]. We say that $w \in C(\mathrm{cl}(A); \mathbb{R}^n)$ such that w is pointwise lipshitz on a set $E \subseteq A$ if we have that

$$\limsup_{y \to x} \frac{|w(x) - w(y)|}{|x - y|} < \infty$$

for all $x \in E$ where w was possibly extended in a neighbourhood around x. Then by Stepanoff's theorem it follows that w is differentiable almost everywhere on E and thus it is possible to define the divergence

$$\mathrm{Div}\, w = \sum_{i=1}^{n} \partial_i w_i$$

almost everywhere on E. It is shown in [4, Lemma 5.16] that $\mathrm{Div}\, w$ is well defined, i.e. it does not depend on the extension around a neighbourhood. With this we obtain from [4, Theorem 5.19] and Proposition 21 that

Theorem 22 (Divergence theorem)**.** *Let $A \in \mathscr{ABV}$, $S \subseteq A$ a thin set and $w \in C(\mathrm{cl}(A); \mathbb{R}^n)$ a continuous vector field which is pointwise Lipshitz on $A \setminus S$. Then $\mathrm{Div}\, w$ is R_*-integrable and*

$$(R^*) \int_A \mathrm{Div}\, w \, \mathrm{d}\mathscr{L}^n = \int_{\partial_* A} w \cdot \nu_A \, \mathrm{d}\mathscr{H}^{n-1}$$

where $\nu_A \colon \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ is the unit normal to A.

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