The Malý-Pfeffer integral

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May 30, 2023

Given a suitable set $A \subseteq \mathbb{R}^n$ and suitable $w \colon \mathbb{R}^n \to \mathbb{R}^n$ the divergence theorem states that

$$\int_A \operatorname{Div} w \, \mathrm{d}\mathscr{L}^n = \int_{\partial A} w \cdot \nu_A \, \mathrm{d}\mathscr{H}^{n-1}$$

Here $\nu_A \colon \partial A \to S^{n-1} \subseteq \mathbb{R}^n$ is the exteriour unit normal and Div denotes the divergence.

Definition (essential interiour, exteriour, boundary)

We calle the set of density points in A essential interiour $\mathrm{int}_* A$ of A. The essential exteriour $\mathrm{ext}_* A = \mathrm{int}_* A^{\complement}$ is the essential interiour of the complement of A. The essential boundary is given by

$$\partial_* A = \mathbb{R}^n \setminus (\mathrm{int}_* A \cup \mathrm{ext}_* A) .$$

Definition (relative perimiter)

We define the relative perimiter of a measurable set E to be

$$P(E, \text{in } A) = \mathscr{H}^{n-1}(\partial_* E \cap \text{int}_* A)\,.$$

where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff-measure.
For convenience we write

$$P(E) = P(E, \text{in } \mathbb{R}^n)\,.$$

Definition ($\mathscr{BV}$-sets)

A measurable set $A \subseteq \mathbb{R}^n$ is called a $\mathscr{BV}$-set if $|A| + P(A) < \infty$. We denote by $\mathscr{BV}$ the set of all $\mathscr{BV}$-sets and by $\mathscr{BV}_c$ the set of all bounded $\mathscr{BV}$-sets.

Definition (Topology on $\mathscr{BV}_c$)

We say a sequence $A \colon \mathbb{N} \to \mathscr{BV}_c$ converges to A_* if there exists a compact $K \subseteq \mathbb{R}^n$ such that $A_k \subseteq K$, $\sup_k P(A_k) < \infty$ and $|A_* \triangle A_k| \to 0$ as $k \to \infty$. Here $A \triangle B = (B \setminus A) \sqcup (A \setminus B)$ denotes the symmetric difference.

Definition (Charge)

A function $F\colon \mathscr{BV}_c \to \mathbb{R}$ is called

- finitely additive if $F(A \sqcup B) = F(A) + F(B)$ for all disjoint $A, B \in \mathscr{BV}_c$.
- continuous if $A_k \to A_*$ implies that $F(A_k) \to F(A_*)$.
- a charge if it is finitely additive and continuous

Example (Indefinite Lebesgue-Integrals are charges)

Let $f \in L^1_{loc}(\mathbb{R}^n)$ then the indefinite integral of f

$$F \colon A \mapsto \int_A f \, \mathrm{d}\mathscr{L}^n$$

is a charge.

Example (Fluxes are charges)

For $E \in \mathscr{BV}$ and $w \in C(\mathrm{cl}(E); \mathbb{R}^n)$ we have that the flux of w

$$F \colon A \mapsto \int_{\partial_*(A \cap E)} w \cdot \nu_{A \cap E} \, \mathrm{d}\mathscr{H}^{n-1}$$

is a charge. Here $\nu_{A \cap E} \colon \partial_*(A \cap E) \to S^{n-1} \subseteq \mathbb{R}^n$ denotes the outer unit normal.

Definition (Regularity of $\mathscr{BV}_c$-sets)

For $E \in \mathscr{BV}_c$ we define the regularity

$$r(E) = \begin{cases} \frac{|E|}{\operatorname{diam}(E) P(E)} & \text{if } |E| > 0 \\ 0 & \text{else} \end{cases}$$

Definition (ε-isoparametric)

We call $E \in \mathscr{BV}_c$ ε-isoparametric if for all $T \in \mathscr{BV}$

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{P(T, \text{in } E)}{\varepsilon}.$$

Definition (Gauge)

We call a set thin if it is σ-finite w.r.t. $\mathscr{H}^{n-1}$. A mapping $\delta\colon \mathbb{R}^n \to \mathbb{R}^n_{>0}$ for which $\{\delta = 0\}$ is thin is called a gauge.

Definition (Partitions)

Let δ be a gauge and $\varepsilon > 0$. We call

$$\mathscr{P} = \{(E_1, x_1), \ldots, (E_p, x_p)\}$$

a partition of the set $\bigcup \mathscr{P} = \bigcup_i E_i$ if $E_i \in \mathscr{BV}_c$ are disjoint sets and $x_i \in \mathbb{R}^n$. A partition is called

- dyadic if E_i is a dyadic cube and $x_i \in E_i$ for all i
- ε-regular if $r(E_i \cup \{x_i\}) > \varepsilon$ for all i
- strongly ε-regular if it is ε-regular, E_i is ε-isoperimetric and $x_i \in \mathrm{cl}_* E_i$ for all i
- δ-fine if $E_i \subseteq B_{\delta(x_i)}(x_i)$ for all i

Definition (R^*-integral)

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called R^*-integrable with respect to a charge G if there is a charge F, s.t. for all $\varepsilon > 0$ there exists a gauge δ such that

$$\sum_{i=1}^{p} |f(x_i)G(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. We call F an indefinite integral of f with respect to G and write

$$F = (R^*) \int f \, \mathrm{d}G \, .$$

Proposition

The integral is unique.

Proposition (Generalisation of the Lebesgue integral on $\mathbb{R}^n$)

Each Lebesgue-integrable function is also R_*-integrable and the integrals conincide.

Proof.

See [5, Proposition 3.5]. □

Proposition (Generalisation of the Henstock-Kurzweil integral on $\mathbb{R}$)

A function $f : \mathbb{R} \to \mathbb{R}$ is Denjoy-Perron integrable on a compact $A \subseteq \mathbb{R}$ iff it is R_*-integrable and the two integrals coincide on A.

Proof.

See [5, Proposition 3.6]. $\square$

Definition (Admissable sets)

We call a set admissable if $\operatorname{int}_* A \subseteq A \subseteq \operatorname{cl}_* A$ and ∂A is compact. The set of admissible $\mathscr{BV}$-sets is denoted by $\mathscr{ABV}$.

Proposition (Generalisation of the Pfeffer-Integral on $\mathbb{R}^n$)

Let $A \in \mathscr{ABV}$. Then each Pfeffer-integrable function is also R_*-integrable and the integrals coincide on A.

Theorem (Divergence theorem)

Let $A \in \mathscr{A}\mathscr{B}\mathscr{V}$, $S \subseteq A$ a thin set and $w \in C(\mathrm{cl}(A); \mathbb{R}^n)$ a continuous vector field which is pointwise Lipschitz on $A \setminus S$. Then $\mathrm{Div}\, w$ is R_*-integrable and

$$(R^*) \int_A \mathrm{Div}\, w \,\mathrm{d}\mathscr{L}^n = \int_{\partial_* A} w \cdot \nu_A \,\mathrm{d}\mathscr{H}^{n-1}$$

where $\nu_A \colon \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ is the unit normal to A.

Main source

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Thank you for your attention.