Specialised Course in Integration Theory, VT23

Take Home Exam

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Problem 1

a)

Claim. *The definition for a step function given in the exam and the definition given in [1, Chapter 3.4] are equivalent.*

Proof. Let f be a step function on $[a, b]$ as given in the exam, i.e.

$$f = \sum_{k=1}^{K} c_k \mathbb{1}_{I_k} \tag{1}$$

for some intervals $I_k \subseteq [a, b]$. Let $\overline{I}_k = [s_k, t_k]$ and let

$$a\colon \{1, \dots, n\} \to \{a, b, s_1, \dots, s_K, t_1, \dots, t_K\}$$

be a monotone bijection, i.e. we have for $j < k$ that also $a_j < a_k$. It then follows that $a = a_0 < a_1 < \dots < a_n = b$. Furthermore we have that f is constant on each interval (a_j, a_{j+1}) as $I_k \cap (a_j, a_{j+1})$ is either empty or the entire interval (a_j, a_{j+1}) and as f is given by equation 1. Thus f is a step function as given in [1, Chapter 3.4].

Let on the other hand f be a step function as given in [1, Chapter 3.4]. Then there exist $a = a_0 < a_1 < \dots < a_n = b$ such that f is constant on each interval $I_j = (a_{j-1}, a_j)$. We thus have that

$$f = \sum_{j=1}^{n} c_j \mathbb{1}_{I_j} + \sum_{j=0}^{n} f(a_j) \mathbb{1}_{\{a_j\}}$$

where $c_j = f(x_k)$ for some $x_k \in (a_{j-1}, a_j)$. But this is precisely of the form 1 where we interpret $\{a_j\} = [a_j, a_j]$ to be an interval. Hence f is a step function as given in the exam. Thus the claim of the equivalence of the definitions follows. $\square$

b)

Claim. *Let μ be a finite Borel-measure on the closed bounded interval $X = [a, b]$ and let $1 \leq p < \infty$. Then the space of step functions on X is dense in $L^p = L^p(X, \mu)$.*

Proof. We follow the strategy from [1, Proposition 3.4.3]. Let $f = \mathbb{1}_A$ be a characteristic function and let $\varepsilon > 0$. We know that X is a locally compact Hausdorff space with a countable basis for its topology. Thus by [1, Proposition 7.1.5] each open subset of X is F_σ. It follows with [1, Lemma 7.2.4] that there exists a closed set $F \subseteq A$ and an open set $A \subseteq U \subseteq X$ such that

$$\mu(A) \leq \mu(F) + \varepsilon \qquad \text{and} \qquad \mu(U) \leq \mu(A) + \varepsilon$$

Since X is compact we have that F is also compact. Since U is open for every $x \in U$ there exists an open interval $I_x \subseteq U$ containing x. Then $\{I_x\}_{x \in U}$ is an open covering of F and thus there exists a finite subcover $I_1, \dots, I_K$ of F. Setting $I = \bigcup_{k=1}^{K} I_k$ we see that $g = \mathbb{1}_I$ is a step function and $F \subseteq I \subseteq U$. Thus we have that

$$\begin{aligned}
\|f - g\|_p^p &= \int_X |\mathbb{1}_A - \mathbb{1}_I|^p \,\mathrm{d}\mu \\
&= \int_X \mathbb{1}_{A \setminus I \cup I \setminus A} \,\mathrm{d}\mu \\
&= \mu(A \setminus I \cup I \setminus A) \\
&\leq \mu(A \setminus I) + \mu(I \setminus A) \\
&\leq \mu(A \setminus F) + \mu(U \setminus A) \\
&\leq 2\varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary it follows that the set of step functions is dense in the set of simple functions in the space of L^p. It follows from [1, Proposition 3.4.2] that the set of simple functions is dense in L^p and as the set of step functions is dense in the set of simple functions we thus have that the set of step function is dense in L^p. $\square$

c)

Claim. *Let $\mathcal{C}$ be as in the exam and set*

$$A = \{f \in L^p = L^p([a, b], \mu) \colon \|f\|_\infty \leq 1\}$$

then we have that $\operatorname{closure}(\mathcal{C}) = A$.

Proof. Let $f \in A$. By b) there exists a sequence of step functions $\mathcal{C} \ni g_k \to f$ in L^p as $k \to \infty$. Set $E_k = (|g_k|)^{-1}([1, \infty)) \subseteq [a, b]$ and note that E_k is a finite union of intervals as g_k is a step function. We now define

$$h_k = g_k \mathbb{1}_{E_k^\complement} + \frac{g_k}{|g_k|} \mathbb{1}_{E_k} .$$

By construction h_k fulfills $\|h_k\|_\infty \leq 1$. As E_k is a finite union of intervals and g_k and $g_k/|g_k|$ are a step functions we also have that h_k is a step function so $h_k \in \mathcal{C}$. Now one calculates

$$\begin{aligned}
\|f - h_k\|_p^p &= \int_{E_k^\complement} |f - h_k|^p \,\mathrm{d}\mu + \int_{E_k} |f - h_k|^p \,\mathrm{d}\mu \\
&= \int_{E_k^\complement} |f - g_k|^p \,\mathrm{d}\mu + \int_{E_k} \left| f - \frac{g_k}{|g_k|} \right|^p \mathrm{d}\mu
\end{aligned}$$

We now have for λ-a.e. $x \in E_k$ that $f(x) \in B_1 = B_1(0)$ is contained in the unit ball. Since $B_1(0)$ is convex and $g_k(x)/|g_k(x)|$ is the projection of $g_k(x)$ onto $B_1(0)$ it then follows that for almost every $x \in E_k$

$$\left| f(x) - \frac{g_k(x)}{|g_k(x)|} \right| \leq |f(x) - g_k(x)| \,.$$

We thus have that

$$\begin{aligned}
\|f - h_k\|_p^p &= \int_{E_k^\complement} |f - g_k|^p \,\mathrm{d}\mu + \int_{E_k} \left| f - \frac{g_k}{|g_k|} \right|^p \mathrm{d}\mu \\
&\leq \int_{E_k^\complement} |f - g_k|^p \,\mathrm{d}\mu + \int_{E_k} |f - g_k|^p \,\mathrm{d}\mu \\
&= \|f - g_k\|_p^p \xrightarrow{k\to\infty} 0
\end{aligned}$$

and hence $h_k \to f$ in L^p. Thus we have shown that $A \subseteq \operatorname{closure}(\mathcal{C})$.

Let now $f_k \to f$ in L^p such that $\|f_k\|_\infty \leq 1$. Then also $f \in L^p$. Assume now that $\|f\|_\infty > 1$. Then there exists an $\varepsilon > 0$ and a set E of positive measure such that $|f| \geq 1 + \varepsilon$ on E. But then

$$\begin{aligned}
\|f - f_k\|_p^p &\geq \int_E |f - f_k|^p \,\mathrm{d}\mu \\
&\geq \int_E (|f| - |f_k|)^p \,\mathrm{d}\mu \\
&\geq \int_E (1 + \varepsilon - 1)^p \,\mathrm{d}\mu \\
&= \int_E \varepsilon^p \,\mathrm{d}\mu \\
&= \mu(E)\varepsilon^p > 0
\end{aligned}$$

and hence $f_k \not\to f$ as $k \to \infty$, a contradiction. Thus we must also have that $\|f\|_\infty \leq 1$. We have therefore shown that A is closed. Since $\mathcal{C} \subseteq A$ it then follows that

$$\operatorname{closure}(\mathcal{C}) \subseteq \operatorname{closure}(A) = A \,.$$

□

d)

Claim. *$\mathcal{S}$ is dense in $L^p = L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ for any Borel measure μ that is finite on compact sets.*

Proof. Let $f \in L^p$ and $\varepsilon > 0$. Then there exists a finite interval $K = [a, b]$ such that

$$\|f\|^p_{L^p(K^\complement)} \leq \varepsilon$$

As K is a compact interval we have that μ is a finite Borel measure on K and it thus follows from b) that there exists a step function $g \in \mathcal{S}$ with $\operatorname{supp} g \subseteq K$ such that

$$\|f - g\|^p_{L^p(K)} \leq \varepsilon\,.$$

Putting it all together we get

$$\|f - g\|^p_{L^p} = \| \underbrace{f - g}_{\substack{=f \text{ as} \\ \operatorname{supp} g \subseteq K}} \|^p_{L^p(K^\complement)} + \|f - g\|^p_{L^p(K)} \leq 2\varepsilon$$

and since $f \in L^p$ and $\varepsilon > 0$ where arbitrary it follows that $\mathcal{S}$ is dense in L^p.

□

e)

Claim. *Let μ be as in d) and let $\mathcal{S}_{rc}$ be the set of right-continuous step functions that vanish at $\pm\infty$. Then $\mathcal{S}_{rc}$ is dense in $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ for any $1 \leq p < \infty$.*

Proof. We first show that $\mathcal{S}_{rc}$ is dense in the set of step functions of the form $f = \mathbb{1}_{\{a\}}$ for some $a \in \mathbb{R}$. It follows for $\delta > 0$ that

$$\begin{aligned}
\left\|\mathbb{1}_{[a,a+\delta)} - \mathbb{1}_{\{a\}}\right\|_p^p &= \int_{\mathbb{R}} \left|\mathbb{1}_{[a,a+\delta)} - \mathbb{1}_{\{a\}}\right|^p \mathrm{d}\mu \\
&= \int_{\mathbb{R}} \left|\mathbb{1}_{(a,a+\delta)}\right|^p \mathrm{d}\mu \\
&= \int_{\mathbb{R}} \mathbb{1}_{(a,a+\delta)} \,\mathrm{d}\mu \\
&= \mu((a, a+\delta))
\end{aligned}$$

and further with [1, Proposition 1.2.5]

$$\lim_{\delta \to 0} \left\|\mathbb{1}_{[a,a+\delta)} - \mathbb{1}_{\{a\}}\right\|_p^p = \lim_{\delta \to 0} \mu((a, a+\delta)) = \mu\left(\bigcap_{\delta \to 0} (a, a+\delta)\right) = \mu(\emptyset) = 0$$

from which it follows that $\mathcal{S}_{rc}$ is dense in the set of step functions.

We now show that $\mathcal{S}_{rc}$ is dense in the set of step functions of the form

$$\sum_{k=1}^{K} c_k \mathbb{1}_{\{a_k\}}\,. \tag{2}$$

For this let $\varepsilon > 0$ and $g_k \in \mathcal{S}_{rc}$ be such that $\left\|g_k - \mathbb{1}_{\{a_k\}}\right\|_p < \varepsilon_k$ where $\varepsilon_k = \varepsilon/(K \cdot \max_k |c_k|)$. Then it follows that also $\sum_{k=1}^{K} c_k g_k \in \mathcal{S}_{rc}$ and

$$\begin{aligned}\left\|\sum_{k=1}^{K} c_k \mathbb{1}_{\{a_k\}} - \sum_{k=1}^{K} c_k g_k\right\|_p &\leq \sum_{k=1}^{K} \left\|c_k \mathbb{1}_{\{a_k\}} - c_k g_k\right\|_p \\ &= \sum_{k=1}^{K} |c_k| \left\|\mathbb{1}_{\{a_k\}} - g_k\right\|_p \\ &\leq \sum_{k=1}^{K} \max_k |c_k| \varepsilon_k \\ &= \sum_{k=1}^{K} \frac{\varepsilon}{K} = \varepsilon\,.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary it follows that $\mathcal{S}_{rc}$ is dense in the set of functions of the form (2).

Let now $f \in L^p$ and $\varepsilon > 0$. It follows from d) that there exists $g_1 \in \mathcal{S}$ such that

$$\|f - g_1\|_p \leq \varepsilon$$

Since $g_1 \in \mathcal{S}$ there exist finite intervals $I_1, \ldots, I_K$ such that

$$g_1 = \sum_{k=1}^{K} c_k \mathbb{1}_{I_k}$$

Let $\overline{I_k} = [a_k, b_k]$. Then we can write

$$g_1 = \sum_{k=1}^{K} c_k \mathbb{1}_{[a_k, b_k)} + \sum_{k=1}^{K} d_k \mathbb{1}_{\{a_k\}} + \sum_{k=1}^{K} e_k \mathbb{1}_{\{b_k\}}$$

for some $d_k, e_k \in \mathbb{R}$. Since $\mathcal{S}_{rc}$ is dense for step functions of the form (2) there exists a function $g_2 \in \mathcal{S}_{rc}$ such that

$$\left\|\sum_{k=1}^{K} d_k \mathbb{1}_{\{a_k\}} + \sum_{k=1}^{K} e_k \mathbb{1}_{\{b_k\}} - g_2\right\|_p \leq \varepsilon$$

We now set

$$g = g_1 - \sum_{k=1}^{K} d_k \mathbb{1}_{\{a_k\}} - \sum_{k=1}^{K} e_k \mathbb{1}_{\{b_k\}} + g_2 = \sum_{k=1}^{K} c_k \mathbb{1}_{[a_k, b_k)} + g_2 \in \mathcal{S}_{rc}\,.$$

Taking everything together we obtain that

$$\begin{aligned}
\|g-f\|_p &= \left\| g_1 - \sum_{k=1}^K d_k \mathbb{1}_{\{a_k\}} - \sum_{k=1}^K e_k \mathbb{1}_{\{b_k\}} + g_2 - f \right\|_p \\
&\le \|g_1 - f\|_p + \left\| \sum_{k=1}^K d_k \mathbb{1}_{\{a_k\}} - \sum_{k=1}^K e_k \mathbb{1}_{\{b_k\}} + g_2 \right\|_p \\
&\le 2\varepsilon
\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary the claim follows. □

f)

Claim. *For $s \in \mathcal{S}_{rc}$ there exists an increasing sequence $\{t_k\}_{k=0}^K$ and numbers $\{c_k\}_{k=1}^K$ such that*

$$s = \sum_{k=1}^K c_k \mathbb{1}_{[t_{k-1},t_k)} .$$

Proof. Since s is a step function which vanishes on $\pm\infty$ there exists a finite interval $[a,b]$ such that $\operatorname{supp} s \subseteq (a,b)$. Then s is also a step function on $[a,b]$ and by a) there exists an increasing sequence $\{t_k\}_{k=0}^K$ such that $a = t_0 < t_1 < \dots < t_K = b$ and such that s is constant on each interval (t_{k-1}, t_k). Hence s is of the form

$$s = \sum_{k=1}^K c_k \mathbb{1}_{(t_{k-1},t_k)} + \sum_{k=0}^K s(t_k) \mathbb{1}_{\{t_k\}} . \tag{3}$$

for a sequence of numbers $\{c_k\}_{k=1}^K$. Now since s is right-continuous we have for $k = 0, \dots, K-1$ that

$$s(t_k) = \lim_{\substack{t\to t_k\\ t>t_k}} s(t) = c_k$$

Together with the fact that $0 = s(b) = s(t_K)$ we obtain from equation 3 that

$$s = \sum_{k=1}^K c_k \mathbb{1}_{(t_{k-1},t_k)} + \sum_{k=0}^{K-1} c_k \mathbb{1}_{\{t_k\}} = \sum_{k=1}^K c_k \mathbb{1}_{[t_{k-1},t_k)}$$

as claimed. □

Problem 2

a)

Claim. *We have for every finite signed measure μ that $V_{F_\mu} \le \|\mu\|$.*

Proof. We have in the notation of [1, chapter 4.4] that

$$\begin{aligned}
V_{F_\mu} &= \sup\left\{\sum_{j=1}^{n}|F_\mu(t_j) - F_\mu(t_{j-1})| \colon (t_j)_{j=0}^n \in \mathcal{S}\right\} \\
&= \sup\left\{\sum_{j=1}^{n}|\mu((t_{j-1}, t_j])| \colon (t_j)_{j=0}^n \in \mathcal{S}\right\} \\
&\le \sup\left\{\sum_{j=1}^{n}|\mu(B_j)| \colon (B_j)_{j=1}^n \text{ are measurable and pairwise disjoint}\right\} \\
&= \|\mu\|\,.
\end{aligned}$$

□

b)

c)

Claim. *The variation $V_F = \|F\|_{\mathcal{V}}$ is a norm that makes the space $\mathcal{V}$ of all right-continuous functions of finite variation that vanish at $-\infty$ into a Banach space. Furthermore the map $\Phi \colon M_r \to \mathcal{V}, \mu \mapsto F_\mu$ is a linear isometry between the space of all finite signed Borel measures $M_r = M_r(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and the space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$.*

Proof. We know from [1, Proposition 4.4.3] that μ is bijective. Let $f, g \in \mathcal{V}$ and $\alpha \in \mathbb{R}$ then there exist $\mu, \nu \in M_r$ such that $F_\mu = f$ and $F_\nu = g$. It follows from a) and b) that

$$\|\Phi(\mu)\|_{\mathcal{V}} = V_{F_\mu} = \|\mu\|\,. \tag{4}$$

Since Φ is linear we have

$$\begin{aligned}
\|f+g\|_{\mathcal{V}} &= \|\Phi(\mu) + \Phi(\nu)\|_{\mathcal{V}} \\
&= \|\Phi(\mu+\nu)\|_{\mathcal{V}} \\
&= \|\mu+\nu\| \\
&\le \|\mu\| + \|\nu\| \\
&= \|\Phi(\mu)\|_{\mathcal{V}} + \|\Phi(\nu)\|_{\mathcal{V}} \\
&= \|f\|_{\mathcal{V}} + \|g\|_{\mathcal{V}}
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\|\alpha f\|_{\mathcal{V}} &= \|\alpha\Phi(\mu)\|_{\mathcal{V}} \\
&= \|\Phi(\alpha\mu)\|_{\mathcal{V}} \\
&= \|\alpha\mu\| \\
&= |\alpha|\|\mu\| \\
&= |\alpha|\|\Phi(\mu)\|_{\mathcal{V}} \\
&= |\alpha|\|f\|_{\mathcal{V}}
\end{aligned} \tag{6}$$

so $\|\cdot\|_{\mathcal{V}}$ fulfills the triangle inquality and homogeneity. Since $\alpha f + g$ is also right continuous and vanishes at $-\infty$ it follows from equations 5 and 6 that $\alpha f + g \in \mathcal{V}$. Thus $\mathcal{V}$ is a linear space. We also have that

$$0 = \|f\|_{\mathcal{V}} = \|\Phi(\mu)\|_{\mathcal{V}} = \|\mu\|$$

implies that $\mu = 0$ and hence $f = 0$. Thus $\|\cdot\|_{\mathcal{V}}$ provides $\mathcal{V}$ with a norm. It now follows from equation 4 that Φ defines an isometric isomorphism. Let now $f_k = \Phi(\mu_k)$ be a Cauchy sequence in $\mathcal{V}$. As Φ is an isometry we have that μ_k is Cauchy in M_r. From [1, chapter 7.3] we know that M_r is a Banach space and thus $\mu_k \to \mu$ in M_r. But then we also have that $f_k \to f = \Phi(\mu) \in \mathcal{V}$ by continuity of Φ. Thus $\mathcal{V}$ is complete and thus a Banach space.

□

Problem 3

a)

Claim. *Let F_k be a sequence of differentiable functions such that $F = \sum_{k\in\mathbb{N}} F_k$ is pointwise absolutely convergent on $[a, b]$. If*

$$\sum_{k\in\mathbb{N}} \|F_k'\|_\infty < \infty$$

then $F' = \sum_{k\in\mathbb{N}} F_k'$.

Proof. Let $x_n \to x$ in $[a, b]$ as $n \to \infty$. Then we have

$$\frac{F(x) - F(x_n)}{x - x_n} = \frac{\sum_k F_k(x) - \sum_k F_k(x_n)}{x - x_n} = \sum_k \frac{F_k(x) - F_k(x_n)}{x - x_n}$$

where the reordering of the sums is permitted as the sums are pointwise absolutely convergent. Now we have by standard estimates for differentiable functions that

$$\frac{|F_k(x) - F_k(x_n)|}{|x - x_n|} \leq \frac{\|F_k'\|_\infty |x - x_n|}{|x - x_n|} = \|F_k'\|_\infty$$

Hence the sequence

$$\left(\frac{F_k(x) - F_k(x_n)}{x - x_n}\right)_{k\in\mathbb{N}}$$

is bounded from above by the summable sequence $(\|F_k'\|_\infty)_{k\in\mathbb{N}}$. We also have for every $k \in \mathbb{N}$ that

$$\lim_{n\to\infty} \frac{F_k(x) - F_k(x_n)}{x - x_n} = F_k'(x)\,.$$

It then follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \frac{F(x) - F(x_n)}{x - x_n} = \lim_{n\to\infty} \sum_k \frac{F_k(x) - F_k(x_n)}{x - x_n} = \sum_k F_k'(x)\,.$$

As $(x_n)_{n\in\mathbb{N}}$ was an arbitrary sequence converging to x in $[a, b]$ the claim follows. □

b)

Claim. *Let $\{q_k\}_{k\in\mathbb{N}}$ be a dense set in $(0,1)$ and $I_k = [q_k, q_k + 2^{-k}]$. Then λ-a.e. point in $(0,1)$ appears in at most finitely many I_k.*

Proof. We have with the monotonicity and σ-additivity of λ that

$$\lambda\left(\liminf_{k\in\mathbb{N}} I_k\right) = \lambda\left(\bigcap_k \bigcup_{n>k} I_n\right) \leq \lambda\left(\bigcup_{n>k} I_n\right) \leq \sum_{n>k} \lambda(I_n) = \sum_{n>k} 2^{-n} = 2^{-k} \xrightarrow{k\to\infty} 0$$

So $\lambda(\liminf_k I_k) = 0$ and thus almost every point appears only finitely often in the intervals I_k. □

c)

Claim. *Let*

$$f(x) = x^2 \sin\left(\frac{1}{x^2}\right)\mathbb{1}_{x>0}$$

then f is differentiable on $\mathbb{R}$ and we have for $s<0$ and $s<t$ that

$$\frac{|f(t)-f(s)|}{t-s} \leq t-s\,.$$

Proof. If $s < t \leq 0$ then we have that

$$\frac{|f(t)-f(s)|}{t-s}\frac{|0-0|}{t-s} = 0 \leq t-s\,. \tag{7}$$

If $s \leq 0 < t$ then we have

$$\frac{|f(t)-f(s)|}{t-s} = \frac{|t^2\sin(1/t^2)|}{t-s} \leq \frac{t^2}{t} = t \leq t-s \tag{8}$$

and the second part of the claim follows.

For differentiability we note that $f=0$ is differentiable on $\mathbb{R}_{<0}$ and $f = x^2\sin\big(1/x^2\big)$ is differentiable on $\mathbb{R}_{>0}$. Thus the differentiability of f at 0 remains to be shown. For this it follows by setting $t=0$ in equation 7 and $s=0$ in equation 8 that for $x \in \mathbb{R}\setminus\{0\}$

$$\frac{|f(0)-f(x)|}{|0-x|} \leq |x| \xrightarrow{x\to 0} 0$$

and thus f is also differentiable in 0. □

d)

Claim. *Define F by*

$$F = \sum_{k \in \mathbb{N}} F_k \tag{9}$$

where

$$F_k = c_k f(\cdot - q_k)$$

for a suitable sequence $(c_k)_k$. Then F is λ-a.e. right-differentiable on $(0,1)$.

Proof. We follow the hint. We know that f is continuously differentiable on $\mathbb{R}_{>0}$ and thus we can choose c_k such that

$$C = \sum_{k \in \mathbb{N}} c_k \|f'\|_{C^0([2^{-k},1])} < \infty$$

e.g. by setting $0 < c_k \leq 1/(\|f'\|_{C^0([2^{-k},1])} \cdot k^2)$. Additionally we choose c_k such that

$$\widetilde{C} = \sum_{k \in \mathbb{N}} c_k < \infty$$

e.g. by additionally requiring $0 < c_k \leq 1/k^2$. It then follows from $|f| \leq 1$ on $[-1,1]$ that

$$\sum_{k \in \mathbb{N}} |F_k| = \sum_{k \in \mathbb{N}} c_k |f(\cdot - q_k)| \leq \sum_{k \in \mathbb{N}} c_k < \infty\,.$$

So $\sum_k F_k$ is absolutely convergent and F is in particular well-defined. By part b) there exists a measurable set $S \subseteq (0,1)$ such that $\lambda(S) = 1$ and such that every $x \in S$ is contained in at most finitely many of the intervals $[q_k, q_k + 2^{-k}]$. Thus for K large enough we have for $s \in S$ that

$$\begin{aligned}
\sum_{\substack{q_k < s \\ k \geq K}} \|F_k'\|_{C^0([s,1])} &= \sum_{\substack{q_k < s \\ k \geq K}} c_k \|f'(\cdot - q_k)\|_{C^0([s,1])} \\
&\leq \sum_{\substack{q_k < s \\ k \geq K}} c_k \|f'\|_{C^0([s-q_k,1])} \\
&\leq \sum_{\substack{q_k < s \\ k \geq K}} c_k \|f'\|_{C^0([2^{-k},1])} \\
&\leq C\,.
\end{aligned}$$

As $\sum_k F_k$ is an absolutely convergent series it follows with a) that

$$G = \sum_{\substack{q_k < s \\ k \geq K}} F_k$$

is differentiable on $[s,1]$. As we have $\operatorname{supp} F_k \subseteq [q_k, 1]$ it follows for $s < t$ that

$$\begin{aligned}
F(s) - F(t) &= \sum_k F_k(s) - \sum_k F_k(t) \\
&= \sum_{q_k < s} F_k(s) - \left(\sum_{q_k < s} + \sum_{s \le q_k < t} \right) F_k(t) \\
&= \sum_{q_k < s} (F_k(s) - F_k(t)) - \sum_{s \le q_k < t} F_k(t) \\
&= \left(\sum_{\substack{q_k < s \\ k < K}} + \sum_{\substack{q_k < s \\ k \ge K}} \right) (F_k(s) - F_k(t)) + \sum_{s \le q_k < t} F_k(t) \\
&= \sum_{\substack{q_k < s \\ k < K}} (F_k(s) - F_k(t)) + G(s) - G(t) + \sum_{s \le q_k < t} F_k(t)
\end{aligned}$$

Division by $(s-t)$ yields

$$\frac{F(s) - F(t)}{s - t} = \underbrace{\sum_{\substack{q_k < s \\ k < K}} \frac{F_k(s) - F_k(t)}{s - t}}_{=(\mathrm{I})} + \underbrace{\frac{G(s) - G(t)}{s - t}}_{=(\mathrm{II})} + \underbrace{\sum_{s \le q_k < t} \frac{F_k(t)}{s - t}}_{=(\mathrm{III})}$$

To show that F is right-differentiable in $S \subseteq [0,1]$ it suffices to show that the terms (I), (II) and (III) converge as $t \to s$ in the expression above. We note that the first term converges as $t \to s$ since by c) the F_k are differentiable and since the sum is finite. The second term converges because G is differentiable. For the third term it follows from c) that

$$\begin{aligned}
|(\mathrm{III})| &\le \sum_{s \le q_k < t} \frac{|F_k(t)|}{|s - t|} \\
&= \sum_{s \le q_k < t} \frac{|F_k(t) - F_k(s)|}{|s - t|} \\
&= \sum_{s \le q_k < t} c_k \frac{|f(t - q_k) - f(s - q_k)|}{|t - q_k - (s - q_k)|} \\
&\le \sum_{s \le q_k < t} c_k |t - q_k - (s - q_k)| \\
&= \sum_{s \le q_k < t} c_k |t - s| \\
&\le \widetilde{C} |t - s| \xrightarrow{t \to s} 0
\end{aligned}$$

and thus the claim follows. $\square$

Problem 4

a)

Claim. *The functions $f \in C(X)$ which have a continuous extension to the one-point compactification X^* of X are precisely those for which*

$$\lim_{x \to k} f(x) = \lim_{|x| \to \infty} f(x) = c \in \mathbb{R}\,. \tag{10}$$

for all $k \in \mathbb{Z}$.

Proof. Let $f \in C(X)$ be such that f has an extension f^* to X^*. Then we have for all open neighbourhoods $U \subseteq \mathbb{R}$ of c that

$$(f^*)^{-1}(U) = f^{-1}(U) \cup \{\infty\} = V \cup \{\infty\}$$

is open in X. As ∞ is contained in this open set it follows that $K^{\complement} \subseteq V$ for some compact $K \subseteq X$. As $K \subseteq X$ and $\{k\} \subseteq \mathbb{R} \setminus X$ for every $k \in \mathbb{Z}$ are compact in $\mathbb{R}$ and as $\mathbb{R}$ is regular there exists around every $k \in \mathbb{Z}$ an open neighbourhood $V_k \subseteq \mathbb{R}$ such that $V_k \cap K = \emptyset$. Thus we have $V_k \cap X \subseteq V = f^{-1}(U)$. Since U was an arbitrary open neighbourhood of c this means that $\lim_{x \to k} f(x) = c$ for all $k \in \mathbb{Z}$. Now define open sets

$$V_\infty = (\max K, \infty) \qquad \text{and} \qquad V_{-\infty} = (-\infty, \min K)\,.$$

It then follows from $V_\infty \cap K = \emptyset = V_{-\infty} \cap K$ that

$$V_\infty \cap X \subseteq V = f^{-1}(U) \qquad \text{and} \qquad V_{-\infty} \cap X \subseteq V = f^{-1}(U)\,.$$

As U was an arbitrary open neighbourhood of c this means that $\lim_{|x| \to \infty} f(x) = c$. Thus f has to fulfill condition 10.

Let now $f \in C(X)$ be such that f fulfills condition 10. Set $f^* = f$ on X and $f^*(\infty) = c$. Let $U \subseteq \mathbb{K}$ be open. Now if $c \notin U$ then $(f^*)^{-1}(U) = f^{-1}(U)$ is open by continuity of f. Else $(f^*)^{-1}(U) = f^{-1}(U) \cup \{\infty\}$. It follows from condition 10 that there exist for all $k \in \mathbb{Z}$ open neighbourhoods $V_k \subseteq X$ of k such that $f(V_k) \subseteq U$. By the same reason there exist open intervals $V_\infty = (k_+, \infty)$ and $V_{-\infty} = (-\infty, k_-)$ such that

$$f(V_\infty \cap X) \subseteq U \qquad \text{and} \qquad f(V_{-\infty} \cap X) \subseteq U\,.$$

Now define

$$K = X \setminus \left(\left(\bigcup_{k \in \mathbb{Z}} V_k \right) \cup V_\infty \cup V_{-\infty} \right).$$

Then K is bounded and closed in $\mathbb{R}$ by construction and thus compact in $\mathbb{R}$. As $K \cap \mathbb{Z} = \emptyset$ it is also compact in X and we have that

$$K^{\complement} = \left(\left(\bigcup_{k \in \mathbb{Z}} V_k \right) \cup V_\infty \cup V_{-\infty} \right) \subseteq f^{-1}(U)$$

and hence

$$(f^*)^{-1}(U) = f^{-1}(U) \cup \{\infty\} = f^{-1}(U) \cup \left(K^{\complement} \cup \{\infty\} \right)$$

is open. Thus f^* is continuous in X^*. $\square$

b)

Claim. *The set $C_0(X)^*$ is a closed linear subspace of $C_0(\mathbb{R})$.*

Proof. Let $f \in C_0(X)$. Set $f^* = f$ on X and $f^* = 0$ on $\mathbb{Z}$. We claim that $f^* \in C_0(\mathbb{R})$.

Indeed let $U \subseteq \mathbb{R}$ be open. If $0 \notin U$ then $(f^*)^{-1}(U) = f^{-1}(U)$ which is open in X and hence also open in $\mathbb{R}$. If $0 \in U$ then $(f^*)^{-1}(U) = f^{-1}(U) \cup \mathbb{Z}$. As $f \in C_0(X)$ there exists a compact $K \subseteq X$ such that $f(X \setminus K) \subseteq U$. It then follows that

$$(f^*)^{-1}(U) = f^{-1}(U) \cup \mathbb{Z} = f^{-1}(U) \cup X \setminus K \cup \mathbb{Z} = f^{-1}(U) \cup \mathbb{R} \setminus K$$

is also open in $\mathbb{R}$. Hence $f^* \in C(\mathbb{R})$. It also follows that $f^*(\mathbb{R} \setminus K) = \{0\} \cup f(\mathbb{R} \setminus K) \subseteq U$. Since U was an arbitrary neighbourhood of 0 we in fact have $f^* \in C_0(\mathbb{R})$.

It remains to show that $C_0(X)$ embedded into $C_0(\mathbb{R})$ in this way forms a closed linear subspace. For this let $f^*, g^* \in C_0(\mathbb{R})$ such that $f^* = 0$ on $\mathbb{Z}$. Then we have for $\alpha \in \mathbb{R}$ that $\alpha f^* + g^* \in C_0(\mathbb{R})$ and $\alpha f^* + g^* = 0$ on $\mathbb{Z}$ so $C_0(X)$ is indeed a linear subspace. Furthermore let $f_k^* \in C_0(\mathbb{R})$ be a Cauchy sequence such that $f_k^* = 0$ on $\mathbb{Z}$ for every $k \in \mathbb{Z}$. Then f_k^* converges to some $f^* \in C_0(\mathbb{R})$. In particular f_k^* converges pointwise on $\mathbb{Z}$ to 0 and thus $f^* = 0$ on $\mathbb{Z}$. Thus $C_0(X)$ is a closed subspace of $C_0(\mathbb{R})$. □

Claim. *$C_0(X)^*$ can be identified with the space S of finite signed Borel measures on $\mathbb{R}$ $\mu \in M_r(\mathbb{R}) = M_r(\mathbb{R}, \mathcal{B}(R), \mathbb{R})$ such that $|\mu|(\mathbb{Z}) = 0$.*

Proof. We know from [1, Theorem 7.3.6] that $C_0(X)^*$ can be identified with the space of finite signed Borel measures on X denoted by $M_r(X)$. We now claim that $M_r(X)$ can be identified with the space S. Indeed if $\widehat{\mu} \in M_r(X)$ then $\mu = \widehat{\mu}(\cdot \cap X) \in M_r(\mathbb{R})$ and $|\mu|(\mathbb{Z}) = 0$ as for all $Z \subseteq \mathbb{Z}$ we have that

$$\mu(Z) = \widehat{\mu}(Z \cap X) = \widehat{\mu}(\emptyset) = 0\,.$$

On the other hand if $\mu \in M_r(\mathbb{R})$ such that $|\mu|(\mathbb{Z}) = 0$ then $\mu \in M_r(X)$. Additionally the mapping $\widehat{\mu} \mapsto \mu$ is injective and thus an embedding.

It remains to be shown that S is a closed linear space. For this let $\alpha \in \mathbb{R}$ and $\mu, \nu \in S$ then

$$|\alpha\mu + \nu|(\mathbb{Z}) \leq \alpha|\mu|(\mathbb{Z}) + |\nu|(\mathbb{Z}) = 0$$

and thus $\alpha\mu + \nu \in S$ and S is a linear space. Let now $\mu_k \in S$ be a Cauchy sequence in $M_r(\mathbb{R})$. By completeness μ_k converge to some $\mu \in M_r(\mathbb{R})$ as $k \to \infty$ (c.f. [1, Chapter 7.3]). It then follows that

$$|\mu|(\mathbb{Z}) = |\mu - \mu_k|(\mathbb{Z}) \leq \|\mu - \mu_k\| \xrightarrow{k\to\infty} 0$$

so also $\mu \in S$. Hence S is closed. □

c)

Claim. *The set $C_0(X)^{\perp}$ can be identified (via the identification in b)) with the subspace of measures $\mu \in M_r(\mathbb{R})$ whose support is contained in $\mathbb{Z}$.*

Proof. Let μ be a measure in $C_0(X)^{\perp}$. Then we have for all $f \in C^0(X)$ that $\langle \widehat{\mu}, f \rangle = 0$ where we write $\langle \widehat{\mu}, f \rangle = \int_X f \,\mathrm{d}\widehat{\mu}$. This implies in particular that for all measurable $A \subseteq X$ we have that

$$0 = \langle \widehat{\mu}, \mathbb{1}_A \rangle = \widehat{\mu}(A) = \mu(A)\,.$$

Hence $|\mu|(X) = 0$ and thus $\operatorname{supp}\mu \subseteq \overline{\mathbb{R} \setminus X} = \mathbb{Z}$.

On the other hand let $\mu \in M_r(\mathbb{R})$ be such that $\operatorname{supp}\mu \subseteq \mathbb{Z} = \mathbb{R} \setminus X$ then we have for all measurable $A \subseteq X$ that $\widehat{\mu}(A) = \mu(A) = 0$ so $\widehat{\mu} = 0$ on X and thus for all $f \in C_0(X)$ we have that $\langle \widehat{\mu}, f \rangle = 0$. Hence μ is a measure in $C_0(X)^{\perp}$.

It remains to show that $C_0(X)^{\perp}$ forms a linear subspace. For this let $\alpha \in \mathbb{R}$ and $\mu, \nu \in M_r(\mathbb{R})$ such that $\operatorname{supp}\mu \subseteq \mathbb{Z}$ and $\operatorname{supp}\nu \subseteq \mathbb{Z}$. Then also $\alpha\mu + \nu \in M_r(\mathbb{Z})$ and we have that $\operatorname{supp}(\alpha\mu + \nu) \subseteq \mathbb{Z}$ and the claim follows. □

d)

Claim. *The measures in $C_0(X)^{\perp}$ are singular with respect to λ.*

Proof. Let $\mu \in C_0(X)^{\perp}$ then by c) we have $\operatorname{supp}\mu \subseteq \mathbb{Z}$. Since this means that $|\mu|(\mathbb{R} \setminus \mathbb{Z}) = 0$ we have that $|\mu|$ is concentrated on $\mathbb{Z}$. Now as $\mathbb{Z}$ is a λ-nullset we have that λ is concentrated on $\mathbb{R} \setminus \mathbb{Z}$ so μ and λ are singular. □

Claim. *The measures in $C_0(X)^{\perp}$ are absolutely continuous with respect to the counting measure ν on $\mathbb{Z}$.*

Proof. Let $\mu \in C_0(X)^{\perp}$ then and let f be measurable with respect to the counting measure such that $f(k) = \mu(\{k\})$ for all $k \in \mathbb{Z}$. E.g. we could take

$$f = \mu(\{k\})\mathbb{1}_{[k-1/2,k+1/2)}\,.$$

By c) we have that $\operatorname{supp}\mu \subseteq \mathbb{Z}$ and it follows for all measurable $A \subseteq X$ using the σ-additivity of μ that

$$\mu(A) = \mu(A \cap \mathbb{Z}) = \mu\left(\bigcup_{k \in A \cap \mathbb{Z}} \{k\}\right) = \sum_{k \in A \cap \mathbb{Z}} \mu(\{k\}) = \sum_{k \in A \cap \mathbb{Z}} f(k) = \int_A f \,\mathrm{d}\nu\,.$$

Thus μ is absolutely continuous with respect to ν. □

e)

Claim. *If we combine the answers from b) and c) we get that every measure* $\mu \in M_r(\mathbb{R})$ *can be decomposed into* $\mu = \mu_1 + \mu_2$ *where* $\mu_1 \in C_0(X)^*$ *and* $\mu_2 \in C_0(X)^\perp$ *are mutually singular measures. We can write this as* $M_r(\mathbb{R}) = C_0(X)^* \oplus C_0(X)^\perp$.

Proof. For $\mu \in M_r(\mathbb{R})$ set $\mu_1 = \mu(\cdot \cap X)$ and $\mu_2 = \mu(\cdot \cap \mathbb{Z})$. We have for all measurable sets $A \subseteq \mathbb{R}$ that

$$\mu(A) = \mu(A \cap X) + \mu(A \cap \mathbb{Z}) = \mu_1(A) + \mu_2(A)\,.$$

Also μ_1 is concentrated on X and μ_2 is concantrated on $\mathbb{Z}$ so μ_1 and μ_2 are mutually singular. As μ_2 is concentrated on $\mathbb{Z}$ we have that $\operatorname{supp}\mu_2 \subseteq \mathbb{Z}$. It then follows from c) that $\mu_2 \in C_0(X)^\perp$. It also follows from $\mu_1(Z) = 0$ for all $Z \subseteq \mathbb{Z}$ that $|\mu_1|(\mathbb{Z}) = 0$. Thus we have by b) that $\mu_1 \in C_0(X)^*$. □

References

[1] D. L. Cohn, *Measure theory*, Second, ser. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2013, pp. xxi+457, ISBN: 978-1-4614-6955-1; 978-1-4614-6956-8. DOI: 10.1007/978-1-4614-6956-8. [Online]. Available: https://doi-org.ludwig.lub.lu.se/10.1007/978-1-4614-6956-8.