Essay for the Specialised Course in Integration Theory, VT23

The Malý-Pfeffer integral

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Introduction

The following essay presents the R_*-integral defined in [1] as part of the course MATP39, Specialised Course in Integration Theory. We will refer to this integral also as the Malý-Pfeffer integral. First we will briefly motivate the integral. Then we give the construction of the integral. After that we show the uniqueness of the definition and the linearity of the integral. Then we state some facts regarding this integral to motivate why the integral is potentially interesting. Finally we give a conclusion regarding. The report and the presentation of this topic can be found online under [6]

Motivation of the Malý-Pfeffer integral.

Given a suitable set $A \subseteq \mathbb{R}^n$ and suitable $w \colon \mathbb{R}^n \to \mathbb{R}^n$ the divergence theorem states that

$$\int_A \operatorname{Div} w \, \mathrm{d}\mathscr{L}^n = \int_{\partial A} w \cdot \nu_A \, \mathrm{d}\mathscr{H}^{n-1} \tag{1}$$

Here $\nu_A \colon \partial A \to S^{n-1} \subseteq \mathbb{R}^n$ is the exterior unit normal, Div denotes the divergence, $\mathscr{L}^n$ refers to the n-dimensional Lebesgue measure and $\mathscr{H}^{n-1}$ to the $(n-1)$-dimensional Hausdorff measure. A relatively general formulation of the divergence theorem was presented in [3] (1991) with few requirements on w. For this the integral on the left hand side of (1) is formulated using the Pfeffer integral, referred to in [1] as the R-integral. This integral was relatively recently generalised to the R_*-integral in [1] (2016). It was shown that the R_*-integral generalises in a certain way the Lebesgue the Pfeffer and the Henstock-Kurzweil integral in a way that the general formulation of the divergence theorem still holds.

Definition of the integral

In this section we will define $\mathscr{BV}$-sets, charges, partitions and finally the Malý-Pfeffer integral. We begin by defining the essential interior, exterior and boundary.

Definition 1 (essential interior, exterior, boundary)**.** *We call the set of density points of a set $A \subseteq \mathbb{R}^n$ essential interior $\operatorname{int}_*(A)$ of A. The essential exterior $\operatorname{ext}_*(A) = \operatorname{int}_* \left(A^{\complement} \right)$ is the essential interior of the complement of A. The essential boundary is given by*

$$\partial_*(A) = \mathbb{R}^n \setminus (\operatorname{int}_*(A) \cup \operatorname{ext}_*(A)).$$

Let now $x \in \operatorname{int}(A)$ be an interior point. Then there exists $r > 0$ such that $B_r(x) \subseteq A$. Since we have

$$\frac{|A \cap B_r(x)|}{|B_r(x)|} = 1$$

it follows that x is a density point and thus $x \in \operatorname{int}_*(A)$. Therefore we have that $\operatorname{int}(A) \subseteq \operatorname{int}_*(A)$. It then follows further that $\operatorname{ext}_*(A) \subseteq \operatorname{ext}(A)$ and also $\partial_*(A) \subseteq \partial(A)$. We can now introduce the notion of relative perimiter of a set

Definition 2 (relative perimiter). *We define the relative perimiter of a measurable set $E \subseteq \mathbb{R}^n$ with regard to a measurable set $A \subseteq \mathbb{R}^n$ to be*

$$P(E, \text{in } A) = \mathcal{H}^{n-1}(\partial_*(E) \cap \text{int}_*(A)).$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff-measure. For convenience we write

$$P(E) = P(E, \text{in } \mathbb{R}^n).$$

To illustrate this definition one can take as $A = H = \mathbb{R}_{>0} \times \mathbb{R}$ and $E = B_1(0)$ as shown in Figure 1. Then one can see that $\partial_*(E) = \partial(E)$ and $\text{int}_*(A) = \text{int}(A)$ so we have in this particular case that $P(E, \text{in } A) = \pi$ and $P(E) = 2\pi$.

We proceed to define $\mathcal{BV}$-sets where the $\mathcal{BV}$ stands for bounded variation

Definition 3 ($\mathcal{BV}$-sets). *A measurable set $A \subseteq \mathbb{R}^n$ is called a $\mathcal{BV}$-set if $|A| + P(A) < \infty$ where $|\cdot| = \mathcal{L}^n$ denotes the Lebesgue measure of a set. We denote by $\mathcal{BV}$ the set of all $\mathcal{BV}$-sets and by $\mathcal{BV}_c$ the set of all bounded $\mathcal{BV}$-sets.*
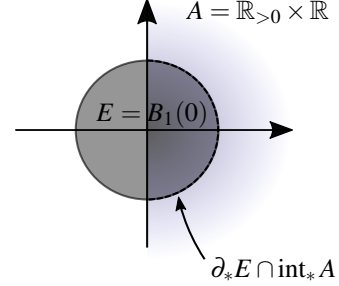

Figure 1: An illustration of the relative perimeter.

The reason for looking at $\mathcal{BV}$-sets is the following. Given a $\mathcal{BV}$-set A there exists an exterior normal $\nu_A \colon \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ which is unique up to $\mathcal{H}^{n-1}$-nullsets (see also [1]). One can now define a topology on the $\mathcal{BV}_c$-sets.

Definition 4 (Topology on $\mathcal{BV}_c$). *We say a sequence $A \colon \mathbb{N} \to \mathcal{BV}_c$ converges to A_* if there exists a compact $K \subseteq \mathbb{R}^n$ such that $A_k \subseteq K$, $\sup_k P(A_k) < \infty$ and $|A_* \triangle A_k| \to 0$ as $k \to \infty$. Here $A \triangle B = (B \setminus A) \sqcup (A \setminus B)$ denotes the symmetric difference.*

Using this topology on $\mathcal{BV}_c$ we can proceed to define charges (not the legal sort).

Definition 5 (Charge). *A function $F \colon \mathcal{BV}_c \to \mathbb{R}$ is called*

- *finitely additive if $F(A \sqcup B) = F(A) + F(B)$ for all disjoint $A, B \in \mathcal{BV}_c$.*

- *continuous if $A_k \to A_*$ implies that $F(A_k) \to F(A_*)$.*

- *a charge if it is finitely additive and continuous*

One sees from the definition that charges form a linear space. In the following we call a cube C *dyadic* if there exist $l \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$ such that

$$C = 2^{-k} \prod_{i=1}^{n} [l_i, l_i + 1] \subseteq \mathbb{R}^n.$$

As a consequence of the density of dyadic cubes in $\mathcal{BV}_c$ we have the following proposition.

Lemma 6 (Density of cubes). *Let F be a charge such that $F(C) = 0$ for all dyadic cubes $C \subseteq \mathbb{R}^n$. Then $F = 0$.*

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Proof. Since [2, Corollary 1.10.4] states that the space of dyadic cubes is dense in $\mathscr{BV}_c$ there exists for given $A \in \mathscr{BV}_c$ a sequence $A \colon \mathbb{N} \to \mathscr{BV}_c$ of dyadic cubes for which $A_k \to A$ as $k \to \infty$. The claim then follows from

$$0 = F(A_k) \xrightarrow{k \to \infty} F(A_*).$$

$\square$

We now give some important examples of charges.

Claim (Indefinite Lebesgue-Integrals are charges)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$ then the indefinite integral of f*

$$F \colon A \mapsto \int_A f \, \mathrm{d}\mathscr{L}^n$$

is a charge.

Proof. F is finitely additive. Let $A_k \to A_*$ then we have by the dominated convergence theorem with majorant $\mathbb{1}_K f$ that

$$\int_{A_k} f \, \mathrm{d}\mathscr{L}^n \xrightarrow{k \to \infty} \int_{A_*} f \, \mathrm{d}\mathscr{L}^n$$

and hence F is a charge. $\square$

This example motivates why charges are used in the definition of the R_*-integral to extend the Lebesgue-integral. Additionally it follows from this example and the Radon-Nikodym derivative (see also [4, Chapter 4.2]) that every measure which is absolutely continuous with respect to the Lebesgue measure is a charge. For the next example we require an alternative characterisation of charges.

Lemma 7 (Alternative characterisation of charges)**.** *Let $F \colon \mathscr{BV}_c \to \mathbb{R}$, then the following are equivalent*

  (i) F is a charge

 (ii) for every $\varepsilon > 0$ then there exists a $\theta > 0$ such that for all $A \in \mathscr{BV}_c$ with $A \subseteq B_{1/\varepsilon}(0)$, $P(A) < 1/\varepsilon$ and $|A| \leq \theta$ we have that $|F(A)| < \varepsilon$.

Proof. Assume the alternative characterisation (ii) holds. Let $A_k \to A_*$ in $\mathscr{BV}_c$ and let $\varepsilon > 0$ small such that $A_k \subseteq B = B_{1/\varepsilon}(0)$ and $P(A_k) < 1/\varepsilon$ for all k. Choose $\theta > 0$ according to the alternative characterisation. We set $B_k = A_* \setminus A_k$ then also $B_k \subseteq B$, $P(B_k) \leq 1/\varepsilon$ and $|B_k| \leq \theta$ for k large enough. Analogously for and $C_k = A_k \setminus A_*$ we have $C_k \subseteq B$, $P(C_k) \leq 1/\varepsilon$ and $|C_k| \leq \theta$ for k large enough. Thus $|F(B_k)| \leq \varepsilon$ for k large enough and as $\varepsilon > 0$ was arbitrarily small it follows that $F(B_k) \to 0$ as $k \to \infty$. Analogously we have $F(C_k) \to 0$ as $k \to \infty$. Now

$$F(A_k) + F(A_* \setminus A_k) = F(A_k \cup A_*) = F(A_*) + F(A_k \setminus A_*)$$

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implies

$$F(A_k) = F(A_*) + F(C_k) - F(B_k) \xrightarrow{k \to \infty} F(A_*)$$

so F is a charge and (i) holds.

Assume the alternative characterisation (ii) does not hold. Then there exists an $\varepsilon > 0$ and a sequence of $A_k \in \mathscr{BV}_c$ such that $A_k \subseteq B_{1/\varepsilon}(0)$, $P(A_k) < 1/\varepsilon$, $|A_k| \leq 1/k$ and $|F(A_k)| > \varepsilon$. But then $A_k \to 0$ in $\mathscr{BV}_c$ and $|F(A_k)| \not\to 0$ in $\mathbb{R}$ and F is not a charge. $\quad\square$

A further example for a charge is given by fluxes.

Claim (Fluxes are charges). *For a set $E \in \mathscr{BV}$ and $w \in C(\mathrm{cl}(E); \mathbb{R}^n)$ we have that the flux of w*

$$F : A \mapsto \int_{\partial_*(A \cap E)} w \cdot \nu_{A \cap E} \, \mathrm{d}\mathscr{H}^{n-1}$$

is a charge. Here $\nu_{A \cap E} : \partial_*(A \cap E) \to S^{n-1} \subseteq \mathbb{R}^n$ denotes the outer unit normal.

Proof. We follow [2, Example 2.1.4]. We have that F is finitely additive. Let $\varepsilon > 0$ and $A \in \mathscr{BV}_c$ such that $P(A) < 1/\varepsilon$, $A \subseteq B$ and $|A| \leq \theta$ where $B = B_{1/\varepsilon}(0)$ and $\theta > 0$. Let $\eta > 0$ and choose $v \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $|v - w| < \eta$ on $B \cap \mathrm{cl}\, E$. It then follows that

$$
\begin{aligned}
|F(A)| &= \left| \int_{\partial_*(A \cap E)} (v - w) \cdot \nu_{A \cap E} \, \mathrm{d}\mathscr{H}^{n-1} + \int_{A \cap E} \mathrm{Div}\, v \, \mathrm{d}\mathscr{L}^n \right| \\
&\leq \int_{\partial_*(A \cap E)} |v - w| \, \mathrm{d}\mathscr{H}^{n-1} + \int_{A \cap E} |\mathrm{Div}\, v| \, \mathrm{d}\mathscr{L}^n \\
&\leq \eta P(A \cap E) + |A \cap E| \|\mathrm{Div}\, w\|_\infty \\
&\leq \eta (P(A) + P(B \cap E)) + |A| \|\mathrm{Div}\, w\|_\infty \\
&\leq \eta \left(\frac{1}{\varepsilon} + P(B \cap E) \right) + \theta \|\mathrm{Div}\, w\|_\infty .
\end{aligned}
$$

Now choosing η and then θ small enough we obtain that

$$|F(A)| \leq \varepsilon .$$

The claim now follows from Lemma 7 with the alternative characterisation of charges.

$\quad\square$

We now define the regularity of $\mathscr{BV}_c$-sets.

Definition 8 (Regularity of $\mathscr{BV}_c$-sets). *For $E \in \mathscr{BV}_c$ we define the regularity*

$$
r(E) = \begin{cases} \frac{|E|}{\mathrm{diam}(E) P(E)} & \text{if } |E| > 0 \\ 0 & \text{else} \end{cases}
$$

And we define what an ε-isoparametric set is

Definition 9 (ε-isoparametric)**.** *We call $E \in \mathscr{BV}_c$ ε-isoparametric if for all $T \in \mathscr{BV}$*

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{P(T, in\ E)}{\varepsilon}\ .$$

This property is rather technical and is used for example in [1, Definition 3.11] for extending the integral to functions which are defined on arbtirary sets $A \subseteq \mathbb{R}^n$. For cubes one can obtain the following result

Proposition 10 (Cubes are ε-isoparametric)**.** *There exists a constant $c_1 > 0$ which depends only on the dimension such that every cube $C \subseteq \mathbb{R}^n$ is ε-isoparametric for every $\varepsilon < c_1$.*

Proof. See [1, Lemma 3.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We call a set thin if it is σ-finite with respect to $\mathscr{H}^{n-1}$. A mapping $\delta \colon \mathbb{R}^n \to \mathbb{R}^n_{>0}$ for which $\{\delta = 0\}$ is thin is called a gauge. Note that this definition of a gauge differs from the typical definition of a gauge used for the Henstock-Kurzweil integral in that it allows for δ to vanish. Now we can give a definition of partitions.

Definition 11 (Partitions)**.** *Let δ be a gauge and $\varepsilon > 0$. We call*

$$\mathscr{P} = \big\{(E_1, x_1), \ldots, (E_p, x_p)\big\}$$

a partition of the set $\bigsqcup \mathscr{P} = \bigsqcup_i E_i$ if for all i the $E_i \in \mathscr{BV}_c$ are disjoint sets and $x_i \in \mathbb{R}^n$. A partition is called

- *dyadic if E_i is a dyadic cube and $x_i \in \mathrm{cl}(E_i)$ for all i*

- *ε-regular if $r(E_i \cup \{x_i\}) > \varepsilon$ for all i*

- *strongly ε-regular if it is ε-regular, E_i is ε-isoperimetric and $x_i \in \mathrm{cl}_* E_i$ for all i*

- *δ-fine if $E_i \subseteq B_{\delta(x_i)}(x_i)$ for all i*

In Figure 2 one can see an example of a δ-fine dyadic partition of the interval $[a, b]$. Note that the intervals get smaller as δ gets smaller. Also note that δ can arbitrary as long as it is a map from $\mathbb{R}^n$ to $\mathbb{R}_{\geq 0}$ which only vanishes on a thin set.

After all these definitions we are finally able to define the R_*-integral.

Definition 12 (R_*-integral)**.** *A function $f \colon \mathbb{R}^n \to \mathbb{R}$ is called R_*-integrable with respect to a charge G if there is a charge F, such that for all $\varepsilon > 0$ there exists a gauge δ such that*

$$\sum_{i=1}^{p} |f(x_i)G(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\big\{(E_1, x_1), \ldots, (E_p, x_p)\big\}$. We call F an indefinite integral of f with respect to G and write

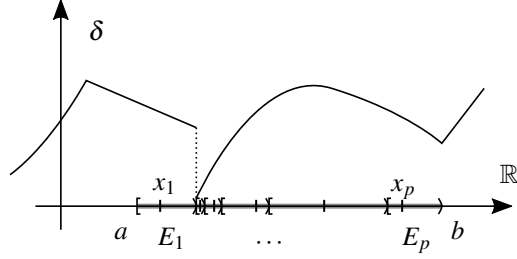$$F = (R_*) \int f \, \mathrm{d}G.$$

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Figure 2: Example of a partition.

Uniqueness of the integral

We now would like to prove the uniqueness of the integral. For this we require a version of Cousin's Lemma.

Definition 13 (Nice dyadic cubes). *Let $\sigma\colon C \to \mathbb{R}_{>0}$. A dyadic cube C is called nice if there exist a dyadic σ-fine partition of C. A dyadic cube which is not nice is called faulty.*

Lemma 14 (Cousin). *All dyadic cubes are nice.*

Proof. Assume a dyadic cube $C = C^1$ is faulty and has diameter r. Then C^1 can be written as the disjoint union $C = \bigsqcup_i C_i$ of dyadic cubes C_i with diameters less than $r/2$. Since C is faulty at least one of the C_i, say $C^2 = C_i$ is also faulty. Inductively we obtain a sequence of nested faulty dyadic cubes C^j with diameters less than $r/2^j$. Thus

$$\bigcap_j \mathrm{cl}\big(C^j\big) = \{x\}$$

for some $x \in \mathrm{cl}(C)$. Let j be such that $r/2^j < \sigma(x)$. Then we have that $\mathrm{diam}(C^j) < \sigma(x)$ and $x \in \mathrm{cl}\big(C^j\big)$ so C^j is nice. This is a contradiction. □

We cite without proof the following proposition from [2, Lemma 1.3.2]

Proposition 15. *There exists a constant $c_2 > 0$ which depends only on the dimension n such that for all $\delta > 0$ and $T \subseteq \mathbb{R}^n$ with $t = \mathcal{H}^{n-1}(T) < \infty$ there exists a family of dyadic cubes $\mathcal{K}$ with diameters less than δ and*

$$T \subseteq \mathrm{int}\bigsqcup \mathcal{K} \qquad \text{and} \qquad \sum_{K \in \mathcal{K}} P(K) < c_2 t \,.$$

With this proposition we can prove the following result from [2, Lemma 2.6.4].

Lemma 16. *Let C be a cube F be a charge, $\varepsilon > 0$ and δ be a gauge. Then there exist a δ-fine dyadic partition $\mathcal{P} = \big\{(C_1, x_1), \dots, (C_q, x_q)\big\}$ such that*

$$\left| F\big(C \setminus \bigsqcup \mathcal{P}\big) \right| < \varepsilon \,.$$

Proof. We essentially follow [2, Lemma 2.6.4].

The idea is to involve Cousin's lemma. Since Cousin's lemma only applies for $\sigma > 0$ everywhere but our definition of a gauge allows for δ to vanish on a thin set we have to choose our σ wisely. Set $T = \{\delta = 0\}$. Since T is thin there exist $T_i \subseteq C$ such that $T = \bigcup_{i \geq 1} T_i$ and $t_i = \mathcal{H}^{n-1}(T_i) < \infty$. Let $\varepsilon > 0$. Now define $\varepsilon_i > 0$ such that

$$\varepsilon_i \leq \frac{\varepsilon}{2^i} \qquad \text{and} \qquad \varepsilon_i \leq \frac{1}{c_2 t_i}$$

and $C \subseteq B_{1/\varepsilon_i}(0)$ for all i. The reason for the choice of ε in this way will become clear later. By the alternative characterisation of charges there exist θ_i such that $|F(A)| < \varepsilon_i$ for each $A \in \mathscr{BV}_c$ such that $A \subseteq C$, $P(A) < 1/\varepsilon_i$ and $|A| < \theta_i$ for all i. By Proposition 15 there exist families $\mathscr{K}_i$ of dyadic cubes with diameters less than $\theta_i \varepsilon_i$,

$$T_i \subseteq \text{int}\left(\bigsqcup \mathscr{K}_i\right) \qquad \text{and} \qquad \sum_{K \in \mathscr{K}} \text{diam}(K)^{n-1} < c_2 t_i.$$

Let $\mathscr{K}$ be a disjoint subfamily of $\mathscr{K}_0 = \bigcup_i \mathscr{K}$ such that $\bigsqcup \mathscr{K} = \bigcup \mathscr{K}_0$. Define a function $\sigma \colon C \to \mathbb{R}_{>0}$ by

$$\sigma(x) = \begin{cases} \text{diam}(K) & \text{if } x \in K \cap T \text{ for a } K \in \mathscr{K} \\ \delta(x) & \text{else} \end{cases}$$

By Cousin's Lemma 14 there exists a dyadic σ-fine partition $\mathscr{C} = \{(C_1, x_1), \ldots, (C_p, x_p)\}$ of C. We now do a reordering of indices such that

$$\mathscr{C} = \Big\{ \underbrace{(C_1, x_1), \ldots, (C_q, x_q)}_{C_i \nsubseteq \bigsqcup \mathscr{K} \text{ for } 1 \leq i \leq q} \quad, \quad \underbrace{\big(C_{k_1+1}, x_{k_1+1}\big), \ldots, \big(C_{k_2}, x_{k_2}\big)}_{\substack{C_i \subseteq \bigsqcup \mathscr{K}_1 \text{ for all} \\ q = k_1 < i \leq k_2}}$$

$$, \quad \ldots \quad, \quad \underbrace{\big(C_{k_m+1}, x_{k_m+1}\big), \ldots, \big(C_{k_{m+1}}, x_{k_{m+1}}\big)}_{\substack{C_i \subseteq \mathscr{K}_m \backslash ((\bigsqcup \mathscr{K}_1) \cup \cdots \cup (\bigsqcup \mathscr{K}_{m-1})) \\ \text{for all } k_m < i \leq k_{m+1} = p}} \Big\}.$$

We set $\mathscr{P} = \{(C_1, x_1), \ldots, (C_1, x_1)\}$. Since by construction we have that $x_j \in T$ implies that there exists a cube $K \in \mathscr{K}_i$ such that $C_i \subseteq K$ it follows that $\mathscr{P}$ is a δ-fine dyadic partition. We now have by Proposition 15 and the choice of ε_i that

$$P\left(\bigsqcup_{k_i < k \leq k_{i+1}} C_k\right) \leq \sum_{K \in \mathscr{K}_i} P(K) \leq \frac{1}{\varepsilon_i}$$

and it thus follows further that

$$\left|\bigsqcup_{k_i < k \leq k_{i+1}} C_k\right| \leq \left|\bigsqcup \mathscr{K}_i\right| \leq \sum_{K \in \mathscr{K}_i} \text{diam}(K) P(K) < \varepsilon_i \theta_i \sum_{K \in \mathscr{K}_i} P(K) \leq \theta_i$$

so

$$\left| F\left(C \setminus \bigsqcup_{i=1}^{q} C_i \right) \right| = \left| F\left(\bigsqcup_{i=q+1}^{p} C_j \right) \right|$$

$$\leq \sum_{i=1}^{m} \left| F\left(\bigsqcup_{k_i < k \leq k_{i+1}} C_k \right) \right|$$

$$\leq \sum_{i=1}^{m} \varepsilon_i$$

$$\leq \sum_{i \geq 1} 2^{-i} \varepsilon = \varepsilon$$

$\square$

Claim. *The integral is unique.*

Proof. We follow [1, Proposition 3.4]. Let F_1 and F_2 be R_*-integrals of f with respect to G. We set $H = |F_1 - F_2|$. Now choose a cube $C \subseteq \mathbb{R}^n$ and $0 < \varepsilon < c_1$. Let δ be the corresponding gauge to ε in the definition of the integral. By Lemma 16 there exists a δ-fine dyadic partition $\mathscr{P} = \{(C_1, x_1), \ldots, (C_q, x_q)\}$ such that

$$\left| H\left(C \setminus \bigsqcup \mathscr{P} \right) \right| < \varepsilon.$$

It then follows that

$$|H(C)| \leq \left| H\left(C \setminus \bigsqcup \mathscr{P} \right) \right| + \left| H\left(\bigsqcup \mathscr{P} \right) \right|$$

$$\leq \varepsilon + \left| \sum_i (F_1(C_i) - F_2(C_i)) \right|$$

$$\leq \varepsilon + \sum_i |F_1(C_i) - f(x_i)G(C_i)| + \sum_i |f(x_i)G(C_i) - F_2(C_i)|$$

$$\leq 3\varepsilon \xrightarrow{\varepsilon \to 0} 0$$

and thus $H(C) = 0$. Since C was an arbitrary dyadic cube it follows from Lemma 6 on the density of cubes that $H = 0$ and hence $F_1 = F_2$.

$\square$

Claim (Linearity of the integral)**.** *Let f_1, f_2 be R_*-integrable and $a \in \mathbb{R}$. Then $f_1 + af_2$ is also integrable and*

$$(R_*) \int f_1 \, dG + (R_*) \int f_2 \, dG = (R_*) \int f_1 + \alpha f_2 \, dG.$$

Proof. We write $F_i = \int f_i \, d^*G$. then we have that for all $\varepsilon > 0$ there exist gauges δ_i such that

$$\sum_j |f_i(x_j)G(E_j) - F_i(E_j)| < \varepsilon$$

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for each strongly ε-regular δ-fine partition $\{(E_1,x_1),\ldots,(E_p,x_p)\}$. Since the space of charges is a linear space we have that also $F_1 + \alpha F_2$ is a charge. If we now set $\delta = \min_i \delta_i$ then we obtain that

$$\sum_j \left| (f_1 + \alpha f_2)(x_j)G(E_j) - (F_1 + \alpha F_2)(E_j) \right|$$
$$\leq \sum_j \left| f_1(x_j)G(E_j) - F_1(E_j) \right| + |\alpha| \sum_j \left| f_2(x_j)G(E_j) - F_2(E_j) \right|$$
$$\leq (1 + |\alpha|)\varepsilon$$

for every strongly ε-regular δ-fine partition $\{(E_1,x_1),\ldots,(E_p,x_p)\}$. Thus $f_1 + \alpha f_2$ is integrable with integral $F_1 + \alpha F_2$. $\qquad\square$

What this integral is good for?

In this section we cite some interesting results for the Malý-Pfeffer without giving much deeper motivation or proof. The first result shows that the R_*-integral is an extension of the Lebesgue integral on $\mathbb{R}^n$.

Proposition 17 (Generalisation of the Lebesgue integral on $\mathbb{R}^n$)**.** *Each Lebesgue-integrable function is also R_*-integrable and the integrals conincide.*

Proof. See [1, Proposition 3.5]. $\qquad\square$

From this it follows that the Malý-Pfeffer integral does not depend on $\mathscr{L}^n$-nullsets. We now define the according to [4, Appendix H] the Henstock-Kurzweil integral.

Definition 18 (Henstock-Kurzweil integral)**.** *Given a bounded interval $[a,b] \subseteq \mathbb{R}$ and a function $f\colon [a,b] \to \mathbb{R}$ we call f Henstock-Kurzweil integrable with integral $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a nowhere vanishing gauge δ such that*

$$\left| \sum_i F|E_i| - L \right| < \varepsilon$$

for all δ-fine dyadic partitions $\{(E_1,x_1),\ldots,(E_p,x_p)\}$ of $[a,b]$.

It can also be shown that the integral generalises the Henstock-Kurzweil integral on $\mathbb{R}$.

Proposition 19 (Generalisation of the Henstock-Kurzweil integral on $\mathbb{R}$)**.** *A function $f\colon \mathbb{R} \to \mathbb{R}$ is Henstock-Kurzweil integrable on a compact $A \subseteq \mathbb{R}$ iff it is R_*-integrable and the two integrals coincide on A.*

Proof. See [1, Proposition 3.6]. $\qquad\square$

We now turn back to the divergence theorem. For this we need some additional conditions on our sets.

Definition 20 (Admissable sets). *We call a set admissable if* $\text{int}_* A \subseteq A \subseteq \text{cl}_* A$ *and* ∂A *is compact. The set of admissible* $\mathcal{BV}$*-sets is denoted by* $\mathcal{ABV}$.

One can then show that for admissable sets the R_*-integral and the classical Pfeffer integral which is defined in coincide

Proposition 21 (Generalisation of the Pfeffer-Integral on $\mathbb{R}^n$). *Let* $A \in \mathcal{ABV}$. *Then each Pfeffer-integrable function is also* R_**-integrable and the integrals coincide on A.*

Proof. See [1, Corollary 3.18]. □

We now turn our attention to the divergence theorem which is one of the major motivations for the Pfeffer integral and hence also the R_*-integral. The following definition is from [3]. We say that $w \in C(\text{cl}(A); \mathbb{R}^n)$ such that w is pointwise Lipschitz on a set $E \subseteq A$ if we have that

$$\limsup_{y \to x} \frac{|w(x) - w(y)|}{|x - y|} < \infty$$

for all $x \in E$ where w was possibly extended in a neighbourhood around x. Then by Stepanoff's theorem it follows that w is differentiable almost everywhere on E. Hence it is possible to define the divergence

$$\text{Div}\, w = \sum_{i=1}^{n} \partial_i w_i$$

almost everywhere on E. It is shown in [3, Lemma 5.16] that $\text{Div}\, w$ is well defined, that is it does not depend on the extension around a neighbourhood. With this one can obtain

Theorem 22 (Divergence theorem). *Let* $A \in \mathcal{ABV}$, $S \subseteq A$ *a thin set and* $w \in C(\text{cl}(A); \mathbb{R}^n)$ *a continuous vector field which is pointwise Lipschitz on* $A \setminus S$. *Then* $\text{Div}\, w$ *is* R_**-integrable and*

$$(R_*) \int_A \text{Div}\, w \, d\mathcal{L}^n = \int_{\partial_* A} w \cdot \nu_A \, d\mathcal{H}^{n-1}$$

where $\nu_A : \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ *is the unit normal to A.*

Proof. Follows from [3, Theorem 5.19] and Proposition 21. □

Summary

In summary the Malý-Pfeffer integral is an interesting extension of the Henstock-Kurzweil integral. Similarly, it is defined using gauges albeit with slight differences to the gauges used when defining the Henstock-Kurzweil integral. On the one hand these gauges are required to be strongly ε-regular. On the other they are allowed to vanish on thin sets. These additional requirements guarantee that a relatively general version of the divergence theorem holds for this integral. A further interesting concept is the concept of charges. This approach to integration however also has visible limitations. For one, the space over which one integrates is not particularly general. Another problem is that the construction is quite technical in many parts to fix some shortcomings of Henstock-Kurzweil integration.

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