Essay for the specialised integration theory course, VT23

The Pfeffer integral

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Introduction

Given a suitable set $A \subseteq \mathbb{R}^n$ the divergence theorem states that

$$\int_A \operatorname{Div} w \, \mathrm{d}\mathscr{L}^n = \int_{\partial A} w \cdot \nu_A \, \mathrm{d}\mathscr{H}^{n-1}$$

holds for each $w \in C^1(\mathbb{R}^n;\mathbb{R}^n)$. Here $\nu_A \colon \partial A \to S^{n-1} \subseteq \mathbb{R}^n$ is the exteriour unit normal and

$$\operatorname{Div} w = \sum_i \partial_i w_i$$

is the divergence. One of the major motivations of the Pfeffer integral is to generalise the integral on the Lebesgue-integral on the left-hand-side in such a way that the divergence theorem holds for w with fewer regularity conditions.

Definition of the integral

Definition 1 (essential interiour, exteriour, boundary)**.** *We define the essential interiour* $\operatorname{int}_* A$ *to be the set of density points of A. The essential exteriour* $\operatorname{ext}_* A = \operatorname{int}_* A^{\complement}$ *is the essential interiour of the complement of A. The essential boundary is given by*

$$\partial_* A = \mathbb{R}^n \setminus (\operatorname{int}_* A \cup \operatorname{ext}_* A)\,.$$

Definition 2 (relative perimiter)**.** *We define the relative perimiter of a measurable set E to be*

$$P(E, in\ A) = \mathscr{H}^{n-1}(\partial_* E \cap \operatorname{int}_* A)\,.$$

where $\mathscr{H}^{n-1}$ *denotes the* $(n-1)$*-dimensional Hausdorff-measure. We write* $P(E) = P(E, in\ \mathbb{R}^n)$*.*

Definition 3 (BV-sets)**.** *A measurable set* $A \subseteq \mathbb{R}^n$ *is called a BV-set if* $|A| + P(A) < \infty$*. We denote by* $\mathscr{BV}$ *the set of all BV-sets and by* $\mathscr{BV}_c$ *the set of all bounded BV-sets.*

The reason for looking at BV-sets is the following. Given a BV-set A there exists an exteriour normal $\nu_A \colon \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ which is unique up to $\mathscr{H}^{n-1}$-nullsets.

Definition 4 (Regularity of BVc-sets)**.** *For* $E \in \mathscr{BV}_c$ *we define the regularity*

$$r(E) = \begin{cases} \frac{|E|}{\operatorname{diam}(E) P(E)} & \text{if } |E| > 0 \\ 0 & \text{else} \end{cases}$$

Definition 5 (isoparametric)**.** *We call* $E \in \mathscr{BV}_c$ ε*-isoparametric if for all* $T \in \mathscr{BV}$

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{P(T, in\ E)}{\varepsilon}\,.$$

We now show that cubes are in fact ε-isoparametric for some small ε which only depends on the dimension n.

Proposition 6 (Cubes are ε-isoparametric)**.** *Every cube $C \subseteq \mathbb{R}^n$ is ε-isoparametric for $\varepsilon < \kappa$ for some κ which depends only on the dimension.*

Proof. Let $T \in \mathscr{BV}$ □

Definition 7 (Gauge)**.** *We call a set thin if it is σ-finite w.r.t. $\mathscr{H}^{n-1}$. A mapping $\delta \colon \mathbb{R}^n \to \mathbb{R}^n_{>0}$ for which $\{\delta = 0\}$ is called a gauge.*

Definition 8 (Partitions)**.** *Let δ be a gauge and $\varepsilon > 0$. We call a partition*

$$\big\{(E_1, x_1), \ldots, (E_p, x_p)\big\}$$

with disjoint sets $E_i \in \mathscr{BV}_c$ and points $x_i \in \mathbb{R}^n$

- *ε-regular if $r(E_i \cup \{x_i\}) > \varepsilon$ for all i*
- *strongly ε-regular if it is ε-regular, E_i is ε-isoperimetric and $x_i \in \mathrm{cl}_* E_i$ for all i*
- *δ-fine if $E_i \subseteq B_{\delta(x_i)}(x_i)$ for all i*

Definition 9 (Topology on $\mathscr{BV}_c$)**.** *We say a sequence $A \colon \mathbb{N} \to \mathscr{BV}_c$ converges to A_* if there exists a compact $K \subseteq \mathbb{R}^n$ such that $A_k \subseteq K$, $\sup_k P(A_k) < \infty$ and $|A_* \triangle A_k| \to 0$ where Δ denotes the symmetric difference.*

Definition 10 (Charge)**.** *A function $F \colon \mathscr{BV}_c \to \mathbb{R}$ is called*

- *finitely additive if $F(A \sqcup B) = F(A) + F(B)$ for all $A, B \in \mathscr{BV}_c$ disjoint.*
- *continuous if $A_k \to A_*$ implies that $F(A_k) \to F(A_*)$.*
- *a Charge if it is finitely additive and continuous*

One sees from the definition that charges form a linear space. We now give some examples of charges.

Claim (Volume integrals as charges)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$ then the indefinite integral of f*

$$F \colon A \mapsto \int_A f \, \mathrm{d}\mathscr{L}^n$$

is a charge.

Proof. F is finitely additive. Let $A_k \to A_*$ then we have by the dominated convergence theorem with majorant $\mathbb{1}_K f$ that

$$\int_{A_k} f \, \mathrm{d}\mathscr{L}^n \xrightarrow{k \to \infty} \int_{A_*} f \, \mathrm{d}\mathscr{L}^n$$

and hence F is a charge. □

In particular it follows from this example that every measure which is absolutely continuous w.r.t. the Lebesgue measure is a charge. A further example is given by

Claim (Fluxes are charges). *For $E \in \mathscr{BV}$ and $w \in C(\mathrm{cl}(E);\mathbb{R}^n)$ we have that the flux of w*

$$F\colon A \mapsto \int_{\partial_*(A\cap E)} w \cdot \nu_{A\cap E}\,\mathrm{d}\mathscr{H}^{n-1}$$

is a charge. Here $\nu_{A\cap E}\colon \partial_*(A\cap E) \to S^{n-1} \subseteq \mathbb{R}^n$ denotes the outer unit normal.

Proof. We have that F is finitely additive. □

Definition 11. *A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called R^*-integrable with respect to a charge G if there is a charge F, s.t. for all $\varepsilon > 0$ there exists a gauge δ such that*

$$\sum_{i=1}^{p} |f(x_i)G(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\{(E_1,x_1),\dots,(E_p,x_p)\}$. We call F an indefinite integral of f with respect to G and write

$$F = \int^* f\,\mathrm{d}G\,.$$

We now would like to prove the uniqueness of the integral. For this we require

Lemma 12 (Density of cubes). *Let F be a charge such that $F(C) = 0$ for all cubes $C \subseteq \mathbb{R}^n$. Then $F = 0$.*

Proof. See [1, Lemma 2.4]. □

Lemma 13. *Let C be a cube $F \geq 0$ be a charge, $\varepsilon > 0$ and δ be a gauge. Then there exist disjoint dyadic cubes C_i and points $x_i \in C_i$ such that* $\mathrm{diam}(C_i) < \delta(x_i)$ *for all i and*

$$f\left(A \setminus \bigcup_i C_i\right) < \varepsilon$$

Proof. See [3, Lemma 2.6.4]. □

Claim. *The integral is unique.*

Proof. We follow [1, Proposition 3.4]. Let F_1 and F_2 be R_*-integrals of f w.r.t. G. We set $H = |F_1 - F_2|$. Now choose a cube $C \subseteq \mathbb{R}^n$ and $0 < \varepsilon$ s.t. C is ε-isoparametric. By Lemma 13 there exist pairwise disjoint dyadic cubes $E_i \subseteq C$ such that

$$H\left(C \setminus \bigcup_i E_i\right) < \varepsilon$$

and $\operatorname{diam}(E_i) < \delta(x_i)$ for $x_i \in E_i$ and $i \in \{1,\dots,q\}$. It then follows that

$$\begin{aligned} H(C) &\le H\Big(C \setminus \bigcup \mathscr{P}\Big) + H\Big(\bigcup \mathscr{P}\Big) \\ &\le \varepsilon + \left|\sum_i (F_1(E_i) - F_2(E_i))\right| \\ &\le \varepsilon + \sum_i |F_1(E_i) - f(x_i)G(E_i)| + \sum_i |f(x_i)G(E_i) - F_2(E_i)| \\ &\le 3\varepsilon \xrightarrow{\varepsilon \to 0} 0 \end{aligned}$$

and thus $H(C) = 0$. Since C was an arbitrary cube it follows from Lemma **??** on the density of cubes that $H = 0$ and hence $F_1 = F_2$.

$\square$

Claim (Linearity of the integral)**.** *Let f_1, f_2 be R^*-integrable and $a \in \mathbb{R}$. Then $f + ag$ is also integrable and*

$$\int f_1 \, \mathrm{d}^* G + \int f_2 \, \mathrm{d}^* G = \int f_1 + \alpha f_2 \, \mathrm{d}^* G.$$

Proof. We write $F_i = \int f_i \, \mathrm{d}^* G$. then we have that for all $\varepsilon > 0$ there exist gauges δ_i such that

$$\sum_j \big| f_i(x_j) G(E_j) - F_i(E_j) \big| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\{(E_1, x_1), \dots, (E_p, x_p)\}$. Since the space of charges is a linear space we have that also $F_1 + \alpha F_2$ is a charge. If we now set $\delta = \min_i \delta_i$ then we obtain that

$$\begin{aligned} &\sum_j \big|(f_1 + \alpha f_2)(x_j) G(E_j) - (F_1 + \alpha F_2)(E_j)\big| \\ &\le \sum_j \big| f_1(x_j) G(E_j) - F_1(E_j) \big| + |\alpha| \sum_j \big| f_2(x_j) G(E_j) - F_2(E_j) \big| \\ &\le (1 + |\alpha|)\varepsilon \end{aligned}$$

for every strongly ε-regular δ-fine partition $\big\{(E_1, x_1), \dots, (E_p, x_p)\big\}$. Thus $f_1 + \alpha f_2$ is integrable with integral $F_1 + \alpha F_2$. $\square$

What this integral is good for?

In [1, Proposition 3.5] it is stated that integral generalises the Lebesgue integral on $\mathbb{R}^n$.

Proposition 14 (Generalisation of the Lebesgue integral on $\mathbb{R}^n$)**.** *Each Lebesgue-integrable function is also R_*-integrable and the integrals conincide.*

The integral generalises the Henstock-Kurzweil integral on $\mathbb{R}$. It is shown in [1, Proposition 3.6] that

Proposition 15 (Generalisation of the Henstock-Kurzweil integral on $\mathbb{R}$)**.** *A function $f\colon \mathbb{R} \to \mathbb{R}$ is Denjoy-Perron integrable on a compact $A \subseteq \mathbb{R}$ if it is R^*-integrable and the two integrals coincide on A.*

We now turn back to the divergence theorem. For this we need some additional conditions on our sets.

Definition 16 (Admissable sets)**.** *We call a set admissable if $\operatorname{int}_* A \subseteq A \subseteq \operatorname{cl}_* A$ and ∂A is compact. The set of admissible BV-sets is denoted by $\mathscr{A}\mathscr{B}\mathscr{V}$.*

It is shown in [1, Corollary 3.18] that for admissable sets the R_*-integral and the classical Pfeffer integral which is defined in **c**oincide. That is

Proposition 17 (Generalisation of the Pfeffer-Integral on $\mathbb{R}^n$)**.** *Let $A \in \mathscr{A}\mathscr{B}\mathscr{V}$. Then each Pfeffer-integrable function is also R_*-integrable and the integrals coincide on A.*

Since a very general version of the divergence theorem holds for Pfeffer-integrals one can obtain that

Theorem 18 (Divergence theorem)**.** *Let $A \in \mathscr{A}\mathscr{B}\mathscr{V}$, $S \subseteq A$ a thin set and $w \in C(\operatorname{cl}(A); \mathbb{R}^n)$ a continuous vector field which is pointwise Lipshitz on $A \setminus S$. Then $\operatorname{Div} w$ is R_*-integrable and*

$$\int_A^* \operatorname{Div} w \, \mathrm{d}\mathscr{L}^n = \int_{\partial_* A} w \cdot \nu_A \, \mathrm{d}\mathscr{H}^{n-1}$$

where $\nu_A\colon \partial_* A \to S^{n-1} \subseteq \mathbb{R}^n$ is the unit normal to A.

Bibliography

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