# Some relations between equilibria of harmonic vector fields and the domain topology.

**Master Thesis**

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# Contents

# Todo list



General TODOs

- Check for typos.
- use =: for definitions.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [26] and [23]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- $C$ is used once for critical points, once for level sets.
- Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

Some questions

- Should I state Hopf's Lemma?

# 1 Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\Omega = \mathrm{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. Unless otherwise stated we denote by

$$f: X \to \mathbb{R}$$

a $C^2$ function on $X$. Often $f$ will be assumed to be harmonic. We also denote by

$$u: X \to \mathbb{R}^d$$

a vector field of class $C^1$. In the following we often assume that $u$ is in fact *harmonic*, that is $u$ fulfils $\mathrm{Div}\, u = 0$ and $\mathrm{curl}\, u = 0$. Often but not always we assume that in fact $u = \nabla f$ is a gradient field. One question we seek to answer in this thesis is the following:

**Question 1.1** (Flowthrough with stagnation point)**.** Does there exist a region $X \subseteq \mathbb{R}^3$ homeomorphic to a ball with flow $u$ through the region such that

1. $u$ is a harmonic vector field
2. $u$ has an interior stagnation point
3. the boundary on which $u$ enters the region is simply connected?

The answer for this will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Another question we will consider is of the type:

**Question 1.2** (stagnation points of harmonic vector fields without inflow or outflow)**.** Let $u$ be a harmonic vector field in a domain $X$ such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary behaviour.

## General definitions

We start by requiring some regularity for the boundary of $X$. More precisely, we require $X$ to be a compact Riemannian manifold with corners:

**Definition 1.3** (Manifolds with corners, [10])**.** We introduce the notation

$$H_j^d = \mathbb{R}_{\geq 0}^j \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d .$$

where $j \in \{0,\dots,d\}$. A *manifold with (convex) corners* is a topological space $X$ together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exists an open neighbourhood $U_x$ of $x$, a number $j = j(x)$ and a diffeomorphism $\phi \colon U_x \to H_j^d$ in $\mathscr{A}$ with $\phi(x) = 0$. We further define for $k \in \{0,\dots,d\}$ a collection of sets

$$X_k = \{x \in X \colon j(x) = d-k\}, \tag{1.1}$$

which form a stratification of $X$.

More generally we give the definition of a stratification as

**Definition 1.4** (Stratified space, [10])**.** Let $X$ be a topological space. A *stratum* is a subspace $X_j \subseteq X$, $j \in \mathscr{J}$, indexed by a partially ordered set $\mathscr{J}$ such that

1. each $X_j$ is a manifold (without boundary) of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$ iff $j \prec k$.

The pair of $X$ and the collection of strata is called a *stratified space*. In the case that $X_j \subseteq \overline{X}_k$ and additionally $n(k) = 0$ or $n(k) = n(j)+1$ we will write $X_j \precsim X_k$ or, abusing notation, we will write $X_k = X_{j+1}$.

In the case that the stratification arises through relation (1.1) we have precisely $X_j \precsim X_{j+1}$ for $j \in \{1,\dots,d\}$ and $X_0 \precsim X_0$. Note that in general for a given stratum $X_j$ the stratum $X_{j+1}$ such that $X_j \precsim X_{j+1}$ need not be unique. In the following we assume, unless otherwise stated, that the stratification is finite, that is $\#\mathscr{J} < \infty$ and that the interior $\Omega$ corresponds to a single stratum.

For completeness we also give the definition of the contingent cone for a stratification $X_j$ of $X$:

**Definition 1.5** (contingent cone, [16, Def. 4.6])**.** We denote the *(Bouligand) contingent cone* for a set $Y \subseteq X$ at $x \in \overline{Y}$ by $C_x Y$. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exists sequences $\lambda_n \to 0$ and $x_n \to x$ in $Y$ such that

$$\lim_n \lambda_n (x_n - x) = v .$$

For clarification we give an example

**Example 1.6** (Cubical domain)**.** Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$. Then we have a stratification given by

$$\begin{aligned}
X_0 &= \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\
X_1 &= I \times \{-1\} \cup I \times \{1\} \cup \{-1\} \times I \cup \{1\} \times I \\
X_2 &= I \times I
\end{aligned}$$

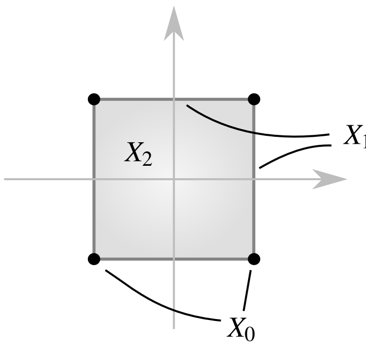



Figure 1.1: A stratification of $X$.

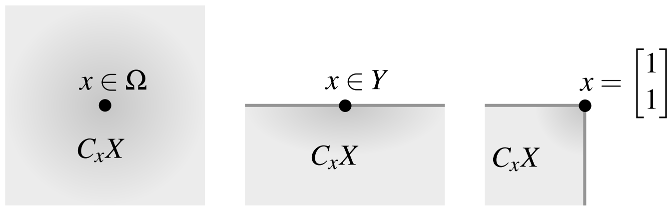



Figure 1.2: The contingent cones for various $x \in X$.

where $I = (-1, 1) \subseteq \mathbb{R}$. The stratification is depicted in figure 1.1. For an interior point $x \in X_2$ we have the contingent cone $C_x X = T_x \mathbb{R}^d$. For a boundary point $x \in Y = I \times \{1\} \subset X_1$ we have the contingent cone

$$C_x X = \left\{ v \in T_x \mathbb{R}^2 \colon v \cdot n \leq 0 \right\}$$

where the basis vector $n = e_2$ is the outer unit normal. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top \in X_0$ we have

$$C_x X = \left\{ v \in T_x \mathbb{R}^2 \colon v_1 \leq 0 \text{ and } v_2 \leq 0 \right\}.$$

The situation is depicted in figure 1.2. The contingent cone on the other parts of the square $\Sigma = \partial X$ is given by similar formulas.

In the following we define the emergent and the entrant boundary in a way that generalises [23, p.282] for stratified manifolds.

**Definition 1.7** (Emergent and entrant boundary)**.** We call a vector $v \in T_x \mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ if

1. $v$ points into $\Omega$ or
2. $v$ lies in the dual cone of the contingent cone $C_x X$, that is

$$v \in (C_x X)^* = \left\{ w \in T_x^* X \colon \left\langle w, w' \right\rangle \geq 0 \text{ for all } w' \in C_x X \right\}.$$

We call $v$ *strictly entrant* if in addition $v$ is not tangential to $\Sigma$ and if $v \in (C_x X)^*$ then $v$ lies in the relative interior $\operatorname{rel\,int}(C_x X)^*$. Analogously $v$ is *(strictly) emergent* if $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which $u$ is entrant. We define the *strictly entrant boundary* $\Sigma^-$ to be the set of strictly entrant boundary points of $u$. In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* $\Sigma^+$. Further define the *tangential boundary* $\Sigma^0$ to be

$$\Sigma^0 = \Sigma^{\leq 0} \cup \Sigma^{\geq 0} \setminus \left( \Sigma^+ \cup \Sigma^- \right) \subseteq \Sigma. \tag{1.2}$$

We would now like to illustrate the preceding definitions.

**Example 1.8.** Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$ and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 - x_2^2 \,. \end{aligned} \tag{1.3}$$

This induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u\colon \Omega &\to \mathbb{R}^3 \\ x &\mapsto 2\begin{bmatrix} x_1 & -x_2 \end{bmatrix}^\top . \end{aligned} \tag{1.4}$$

For a boundary point $x \in I \times \{1\}$ the dual cone of the contingent cone $C_xX$ is given by

$$(C_xX)^* = \{-re_2\colon r \geq 0\} \,.$$

Now we have that $x \in \Sigma^{\leq 0}$ if

$$0 \geq n \cdot u = -2x_2$$

which is always fulfilled and thus $I \times \{1\} \subseteq \Sigma^{\leq 0}$. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ the dual of the contingent cone is given by

$$(C_xX)^* = C_xX \,.$$

Since $v = u(x) = 2\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ we have that $v \notin (C_xX)^*$ and $-v \notin (C_xX)^*$ and thus $x \notin \Sigma^{\geq 0} \cup \Sigma^{\leq 0}$. By analogous argumentation on the other sides of the square $\Sigma = \partial X$ one obtains that

$$\begin{aligned} \Sigma^{\leq 0} &= I \times \{1\} \cup I \times \{-1\} \\ \Sigma^{\geq 0} &= \{1\} \times I \cup \{-1\} \times I \,. \end{aligned}$$

A plot of the sets can be seen in figure 1.3.

Given a vector field $u\colon X \to \mathbb{R}^d$ and a stratification $X_j$ of $X$ we can construct for every $j \in \mathscr{J}$ a vector field

$$u_j\colon X_j \to T^*X_j \,.$$

Here $T^*X_j$ denotes the cotangent space of the manifold $X_j$ which is defined for instance in [14, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j\big|_x\colon \mathbb{R}^d \cong T_x^*\mathbb{R}^d \to T_x^*X_j \tag{1.5}$$

denote the orthogonal projection of a vector at $x$ onto the cotangent space of the stratum $X_j$ at $x$. Now let

$$u_j = \pi_j \circ u\big|_{X_j} \in C^1(T^*X_j) \tag{1.6}$$

be the projection of $u$ onto the cotangent bundle $T^*X_j$.

The following are slight generalisation of definitions given in [26, p.138f], [24, §5] and [23, p.282f] to include harmonic vector fields.

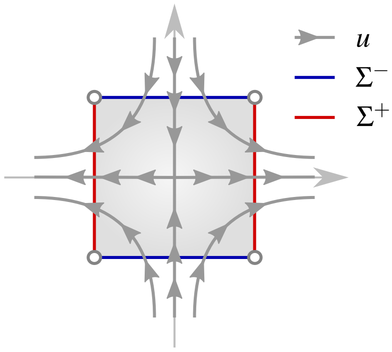



Figure 1.3: Depiction of the entrant and emergent boundaries for the function $f$ given by equation (1.3)

**Definition 1.9** (Stagnation points)**.** Let $u_j\colon X_j \to T^*X_j$ be a $C^1$ vector field on a stratum $X_j$ of $X$. We call the zeroes $x \in X_j$ of $u_j$ *stagnation points of $u_j$ on $X_j$*. If $x \in \Omega$ then we call $x$ an *interior stagnation point*. If $u(x) \in (C_xX)^*$ we call $x$ an *essential stagnation point*. The set of all essential stagnation points of $u_j$ is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$ and the essential stagnation points of $u$ and $-u$ on $X_j$ are called *stagnation points of $u$ on $X_j$*. A stagnation point $x$ of $u_j$ is called *non-degenerate* if it is contained in the relative interior $\operatorname{rel\,int} X_j$ and the derivative

$$Du_j(x) = Du_j\big|_x \in T_xT^*X_j \cong \mathbb{R}^{n(j)\times n(j)}$$

is bijective. In addition we say that $x$ has *index $k$* if $Du_j(x)$ has exactly $k$ negative eigenvalues. $u_j$ is called *(essentially) non-degenerate* if all its (essential) stagnation points are non-degenerate. Finally, we call a non-degenerate essential stagnation point of $u_j$ such that additionally $u(x) \in \operatorname{rel\,int}(C_xX)^*$ *regular*. Boundary points which are non-regular essential stagnation points are called *irregular boundary points*. We call the set of all irregular boundary points the irregular boundary $\Sigma^{\mathrm{irr}}$. $u_j$ is called *regular* if it has no irregular boundary points. We can define the *k-th type number* $\mathrm{Ind}_{j,k}(u)$ of the stratum $X_j$ to be the number of regular stagnation points of $u_j$ of index $k$, that is

$$\mathrm{Ind}_{j,k}(u) = \#\big\{x \in \mathrm{Cr}_j(u)\colon x \text{ has index } k\big\}. \tag{1.7}$$

To illustrate the preceding definitions we return to our previous example.

**Example 1.10.** Let $X$, $f$ and $u$ be as in example 1.8. We have that $u_2 = u$ and thus one sees from equation (1.4) that the origin 0 is the sole stagnation point of $u$ on the stratum $X_2$. Since we have that

$$Du(x) = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}$$

for all $x \in \Omega$ we see that $Du(0)$ is bijective and thus the origin is a non-degenerate interior stagnation point. Since $Du(0)$ has exactly one negative eigenvalue we see that the origin has index

1. Since an interior stagnation point is also an essential stagnation point we have $\mathrm{Ind}_{2,k} = \delta_{k1}$ where $\delta$ denotes the Kronecker delta. For $x \in I \times \{1\} = Y$ we calculate

$$u_1(x) = \pi_1 \circ u(x) = (u - n \cdot un)(x) = 2x_1 e_1$$

and thus we have that $x = e_2$ is the unique stagnation point of $u$ on $I \times \{1\}$. Consider the curve

$$\begin{aligned} \gamma \colon I &\to Y \\ t &\mapsto te_1 + e_2 \end{aligned}$$

then $\gamma(0) = e_2$ and we have

$$Du_1(e_1)\big(\gamma'(0)\big) = (u_1 \circ \gamma)'(0) = (2te_1)'(0) = 2e_1 = 2\gamma'(0)$$

and thus $e_1$ is an eigenvector of $Du_1(e_2)$ to eigenvalue 2. Since $e_1$ spans the eigenspace $T_{e_2}Y$ it follows that $e_2$ is a non-degenerate stagnation point of $u_1$ with index 0. Now since $u(e_2) \in \operatorname{rel\,int}(C_xX)^*$ we have that $e_2$ is in fact a regular point. Proceeding in this manner for the other segments of the square $\Sigma$ we obtain that $\mathrm{Ind}\, 1, k = 2\delta_{0k}$. If we now consider the point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ then we have that $u_0(x) = 0$ and thus $x$ is a stagnation point. Now the derivative $Du_0 = 0 \in T_xT^*X_0 = 0$ is bijective and thus we have that $x$ has index 0. Since however $u(x) \notin (C_xX)^*$ we have that $x$ is not an essential stagnation point. Analogous argumentation on the other three corners yields that $\mathrm{Ind}_{0,0} = 0$.

The following characterisation of the irregular boundary will come in handy for showing the density of Morse functions later on:

**Proposition 1.11** (Characterisation of the irregular boundary)**.** *The condition that the stagnation point $x \in X_j$ lies in $\Sigma^{irr}$ is equivalent to that $x$ is stagnation point of $u_{j+1}$ for a stratum $X_j \precsim X_{j+1}$.*

*Proof.* We calculate for some boundary point $x \in \Sigma$:

$$(C_xX)^* \setminus \operatorname{rel\,int}(C_xX)^* = \left\{ w \in T_xX \;\middle|\; \begin{array}{l} \langle w, w' \rangle \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ \langle w, w' \rangle = 0 \text{ for some } w' \in \partial C_xX \setminus \{0\} \end{array} \right\}.$$

Now $w' \in \partial C_xX \setminus \{0\}$ iff $w' \in T_xX_{j+1}$ is orthogonal to $T_xX_j$ at $x$. Thus we have that

$$u(x) \in (C_xX)^* \setminus \operatorname{rel\,int}(C_xX)^*$$

iff $u(x) \in (C_xX)^*$ and there exists a normal $n \in T_xX_{j+1}$ such that

$$0 = (n \cdot u\, n)(x) = (n \cdot u_{j+1}\, n)(x) = u_{j+1}(x) - u_j(x)$$

from which the claim follows. □

**Definition 1.12** (Morse functions)**.** We call $u$ *Morse* if for all $j \in \mathscr{J}$ we have that $u_j$ is regular. If both $u$ and $-u$ are Morse we call $u$ *strongly Morse*. For a Morse function $u$ we define the *interior type numbers $M_k$* to be the number of essential interior stagnation points of $u$ of index $k$, that is

$$M_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \mathrm{Ind}_{j,k}(u) = \#\left\{ x \in \bigcup_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \mathrm{Cr}_j(u) \colon x \text{ has index } k \right\}. \tag{1.8}$$

The total number $M$ of interior stagnation points of $u$ is then given by

$$M = \sum_k M_k \,. \tag{1.9}$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of $u$ of index $k$, that is

$$\mu_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) < d}} \mathrm{Ind}_{j,k}(u) \tag{1.10}$$

We further write $\nu_k$ for the $k$-th boundary type number of $-u$. We define the *type number* to be the number of essential stagnation points of $u$ of index $k$, that is

$$\mathrm{Ind}_k(u) = \sum_{j \in \mathscr{J}} \mathrm{Ind}_{j,k}(u) = M_k + \mu_k \,. \tag{1.11}$$

We return to our example:

**Example 1.13.** Let $X$, $f$ and $u$ be as in example 1.8. By the calculations of the previous example 1.10 we have that $u$ is Morse and we can calculate the interior type numbers

$$M_k = \mathrm{Ind}_{2,k} = \delta_{2k}$$

and the boundary type numbers

$$\mu_k = \mathrm{Ind}_{0,k}(u) + \mathrm{Ind}_{1,k}(u) = 2\delta_{0k}$$

This then yields the type numbers

$$\mathrm{Ind}_k(u) = M_k + \mu_k = \delta_{2k} + 2\delta_{0k} \,.$$

The previous definitions translate naturally to $f$. That is, we call $f$ Morse, non-degenerate, et cetera if $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call $x$ a *critical point* of $f$ if it is a stagnation point of $u$. Note that most authors refer to regular and essential stagnation points as simply non-degenerate stagnation points and that this naming was introduced simply to distinguish between these different concepts.

## Density of Morse functions

In the following section we argue that $u$ and $f$ being Morse is not a great restriction. Given $u$ we define the modification

$$u^\varepsilon = u + \varepsilon \tag{1.12}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that the set $E$ of $\varepsilon$ for which $u^\varepsilon$ is Morse is residual in $\mathbb{R}$. Recall that a *residual* set is a set whose complement is *meagre*, that is whose complement is the countable union of nowhere dense subsets. Since residual sets are dense in a Baire space by the Baire category theorem we can use $u^\varepsilon$ to approximate a degenerate $u$. Our approach is to use Thom's theorem which is inspired by the approach in [14, Chapter 6].

**Definition 1.14** (Transversality, [14, §3.2])**.** We call a function $g\colon Y_1 \to Y_2$ between two manifolds $Y_1$ and $Y_2$ (without boundary) *transverse* to a submanifold $A \subseteq Y_2$ if for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\mathrm{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y_2\,.$$

As an application of this definition we make the following observation:

**Proposition 1.15** (Transversal characterisation of non-degeneracy)**.** *Let $u_j\colon X_j \to T^*X_j$ be a differentiable vector field. Then $u_j$ is non-degenerate iff $u_j$ is transverse to the zero section $A_j$ of the cotangent space $T^*X_j$.*

*Proof.* First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A) = C$ is the set of stagnation points. Unravelling the definition of transversality we get that $u_j$ is transverse to the zero section iff for all $x \in C = u_j^{-1}(A)$ we have that

$$\mathrm{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}TX\,. \tag{1.13}$$

As $A$ is the zero section we have $T_{u_j(x)}A = 0$ and equation (1.13) is equivalent to stating that $Du_j$ is of full rank at $x$. But $Du_j$ being of full rank at all stagnation points is equivalent to $u_j$ being non-degenerate. $\square$

The alternative characterisation of non-degeneracy given in proposition 1.15 is sometimes used as a definition of non-degeneracy. We can now state a weakened version of the Thom's transversality theorem from [14, Theorem 2.7]:

**Theorem 1.16** (Parametric transversality theorem, [14, §3 Theorem 2.7])**.** *Let $\mathscr{E}, Y_1, Y_2$ be $C^r$-manifolds (without boundary) and $A \subseteq Y_2$ a $C^r$ submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A\,.$$

*Let further $F\colon \mathscr{E} \to C^r(Y_1, Y_2)$ be such that the evaluation map*

$$\begin{aligned} F^{ev}\colon \mathscr{E} \times Y_1 &\to Y_2 \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

*is $C^r$ and transverse to $A$. Then the set*

$$E = \{\varepsilon \in \mathscr{E} \colon F_\varepsilon \text{ is transverse to } A\}$$

*is residual in $\mathscr{E}$.*

*Proof.* See [14, Theorem 2.7] for details. □

From this we obtain a generalisation of the results in [23, §2] which will prove useful later:

**Corollary 1.17** (Density of boundary generic functions)**.** *Let $u\colon X \to T^*X$ be a harmonic vector field on $X$ and let $X_j$ be a stratification of $X$. Assume that $u$ has no irregular stagnation points. Then there exists a $\delta > 0$ and a residual (and thus dense) set $E \subseteq B_\delta \subseteq \mathbb{R}^d$ such that for every $\varepsilon \in E$ the following statements hold:*

1. *$u_j^\varepsilon \to u_j$ converge uniformly on all strata $X_j$ as $\varepsilon \to 0$.*
2. *If $x_\varepsilon \to x$ is a convergent sequence of stagnation points of $u_j^\varepsilon$ as $\varepsilon \to 0$ then $x$ is a stagnation point of $u_j$.*
3. *$u^\varepsilon$ is strongly Morse.*
4. *Additionally we can find for every $\eta > 0$ a $\delta > 0$ such that all stagnation points of $u^\varepsilon$ are contained in an $\eta$-neighbourhood of the set of stagnation points of $u$.*
5. *the property of being entrant or emergent of stagnation points of $u^\varepsilon$ is preserved, that is a stagnation point $x^\varepsilon$ of $u^\varepsilon$ lies in $\Sigma^\pm(u^\varepsilon)$ iff it lies in $\Sigma^\pm(u)$.*
6. *If $u_j$ is non-degenerate on the stratum $X_j$ we have for all $k$ that*

$$\mathrm{Ind}_{X_j,k}(u^\varepsilon) = \mathrm{Ind}_{X_j,k}(u) \qquad \textit{and} \qquad \mathrm{Ind}_{X_j,k}(-u^\varepsilon) = \mathrm{Ind}_{X_j,k}(-u)\,.$$

*Proof. Part 1.* Follows from compactness of $\overline{X}_j \subseteq X$ and the continuity of $\pi_j$.

*Part 2.* Let $x_\varepsilon \to x$ be a convergent sequence of stagnation points on the stratum $X_j$. By part 1 $u_j^\varepsilon \to u_j$ as $\varepsilon \to 0$ uniformly which implies

$$0 = \lim_{\varepsilon} u_j^\varepsilon(x_\varepsilon) = u_j(x) \tag{1.14}$$

and thus $x$ is a stagnation point of $u_j$.

*Part 3.* The following is essentially an adaptation of a proof given in [23, §2]. We first show that we can choose a $\delta > 0$ such that for all $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ the function $u^\varepsilon$ has no irregular stagnation points. Assume not. Then there exists a sequence $\varepsilon_k \to 0$ and irregular stagnation points $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ of $u^{\varepsilon_k}$. By compactness of $X$ we can assume that $x_k \to x$ for some $x \in X$ after taking a sub-sequence. After taking a further sub-sequence we can also assume that all $x_k$ lie in a stratum $X_j$. The condition that the $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ are stagnation points means that after taking a further sub-sequence there exists a stratum $X_{j-1}$ such that $x_k$ is also stagnation point of this stratum by proposition 1.11. But then $x \in \overline{X}_j$ is also a stagnation point of $X_{j-1}$ by part 2. Analogously $x$ is also stagnation point of $X_j$. Thus $x \in \Sigma^{\mathrm{irr}}$ is an irregular stagnation point. A contradiction.

The next part of the proof is inspired by the use of transversality in [14, §6 Theorem 1.2] to show a similar statement. Set $r = 2$, $\mathscr{E} = B_\delta$ and $Y_2 = T^*X_j$ in the previous theorem. We initially set

$Y_1 = X_j = \Omega$. We would like to apply the parametric transversality theorem to the function

$$
\begin{aligned}
F\colon \mathscr{E} &\to C^\infty(X_j, T^*X_j)\\
\varepsilon &\mapsto u^\varepsilon
\end{aligned}
$$

and note that $F^{\mathrm{ev}}$ is sufficiently smooth. We need to show that $F^{\mathrm{ev}}$ is transverse to the zero section $A \subseteq T^*X_j$. Then the parametric transversality theorem yields a residual $E_j \subseteq \mathscr{E}$ on which $F_\varepsilon = u^\varepsilon$ is transverse to $A$. For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$
\mathrm{Image}\Big(DF^{\mathrm{ev}}_{(\varepsilon,x)}\Big) = T_xT^*X_j \tag{1.15}
$$

since

$$
DF^{\mathrm{ev}}_{(\varepsilon,x)} = \big[\mathrm{Id}_{d\times d} \mid Du_x\big]
$$

is surjective. Proposition 1.15 now implies that $u^\varepsilon$ is non-degenerate on $X_j$ for $\varepsilon \in E_j$.

Analogously we set $Y_1 = X_j$ to be an arbitrary strata in the previous proof and replace $u^\varepsilon$ with the projection $u^\varepsilon_j$. To show that equation (1.15) holds we resort to the fact that

$$
DF^{\mathrm{ev}}_{(\varepsilon,x)} = D\big(u^\varepsilon_j(x)\big)_{(\varepsilon,x)} = D\pi_j \circ (Du^\varepsilon(x))_{(\varepsilon,x)}
$$

is surjective as a concatenation of surjective functions. Thus there also exists a residual set $E_j \subseteq \mathscr{E}$ on which $u^\varepsilon_j$ is non-degenerate on $X_j$.

Now the intersection

$$
E = \bigcap_j E_j \subseteq \mathscr{E} = B_\delta
$$

is residual and for every $\varepsilon \in E$ the function $u^\varepsilon$ fulfils condition 3.

*Part 4.* Let $C_\eta$ denote the open $\eta$-neighbourhood of the set of stagnation points of $u$. Since $u$ has no irregular stagnation points we have for any stratum $X_j$ that $u_j \neq 0$ on the compact set $\overline{X}_j \setminus C_\eta$ which implies that we can choose $\delta > 0$ so small that $|u_j| > 2\delta$ on $\overline{X}_j \setminus C_\eta$ for all strata $X_j$. For any $\varepsilon \in B_\delta$ it then follows that $u^\varepsilon$ has no stagnation points on the set $\overline{X}_j \setminus C_\eta$ which yields the claim.

*Part 5.* Now consider a stratum $X_j$ and the continuous mapping

$$
\begin{aligned}
\Phi\colon X_j &\to \mathbb{R}_{\geq 0}\\
x &\mapsto \mathrm{dist}(u(x), \partial C_x X)
\end{aligned}
$$

on $X_j$. $\Phi$ is positive on the set of stagnation points $C$ of $u_j$ on $X_j$ and thus we can choose $\eta > 0$ such that $\Phi$ is also positive in the neighbourhood $C_{2\eta}$. Choose $\delta > 0$ smaller than in part 4. Now the mapping $\Phi$ attains a positive minimum on the compact set $\overline{C_\eta}$. We can assume that $\delta > 0$ is less than this minimum. The choice of $\delta$ in this way ensures that emergent stagnation points of $u_j$ are also emergent stagnation points of $u^\varepsilon_j$ on $X_j$. Analogous argumentation with $-u$ then ensures that entrant stagnation points of $u_j$ are also entrant stagnation points of $u^\varepsilon_j$ on $X_j$. Since there are finitely many strata $X_j$ we can choose $\delta > 0$ such that part 5 follows.

*Part 6*. Pick $\delta > 0$ as in part 5 and such that If $x$ is a non-degenerate stagnation point of $u$ on the stratum $X_j$ it follows from the inverse function theorem that there exists for sufficiently small $\delta$ a neighbourhood around $x$ on which there is a one-to-one correspondence between the stagnation points of $u$ and $u^\varepsilon$. Since there are by proposition 2.1 at most finitely many non-degenerate stagnation points of $u$ we can choose $\delta$ to be minimal over all these stagnation points. The equality of the indexes then follows from $Du^\varepsilon = Du$. $\square$

# 2 Some general remarks

Bring order into this section.

The finiteness of the number of critical points is a known fact which is mentioned for example in [23]. For completeness we give the following proposition:

**Proposition 2.1.** *The number of non-degenerate stagnation points of $u_j$ on $X_j$ is finite.*

*Proof.* Let $x \in X_j$ be a non-degenerate stagnation point of $u_j$. Since $Du_j(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq X_j$ of $x$ on which $u_j$ is bijective. Hence $x$ is the only stagnation point in $U_x$. Let $C$ denote the set of all non-degenerate stagnation points of $u_j$. Then the sets $U_x$ for $x \in C$ together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \tag{2.1}$$

form an open cover of $\overline{X}_j$. But $\overline{X}_j$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that $U_x$ is contained the finite subcover. Thus it follows that $\#C < \infty$ is finite. $\square$

As a consequence we obtain the following observation:

**Corollary 2.2.** *For a Morse $u$ the type numbers $M_0, \dots, M_d$ and the boundary type numbers $\mu_0, \dots, \mu_{d-1}$ are finite.*

State the theorem of Sard

We state Morse's lemma according to [14, §6, Lemma 1.1]

**Lemma 2.3.** *Let $f\colon X \to \mathbb{R}$ be $C^{2+r}$ and $x$ be a non-degenerate critical point of index $k$. Then there exists a $C^r$ chart $(\varphi, U)$ at $x$ such that we have*

$$f\colon \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{d} y_j^2 .$$

*Proof.* See for example [14, §6]. $\square$

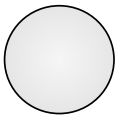
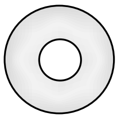
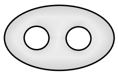
| Domain | Picture | $b_0$ | $b_1$ | $b_k$, $k \geq 2$ |
|---|---|---|---|---|
| Disk $D$ |  | 1 | 0 | 0 |
| Annulus $2D \setminus D$ |  | 1 | 1 | 0 |
| Two holed button |  | 1 | 2 | 0 |

Table 2.1: Betti numbers for selected domains in $\mathbb{R}^2$.

## Betti numbers

Let $H_k(X;\mathbb{R})$ denote the $k$-th homology space of $X$. For an introduction and definition of these we refer the reader to [13, Chapter 2]. We define the $k$-th Betti number as the dimension

$$b_k = \dim_{\mathbb{R}} H_k(X;\mathbb{R})\,. \tag{2.2}$$

We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

**Example 2.4** (In flatland)**.** In $d = 2$ dimensions the 0-th Betti number counts the number of connected components of $\Omega$ and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely we give the Betti numbers for selected domains in table 2.1.

**Example 2.5** (In spaceland)**.** In $d = 3$ dimensions the 0-th Betti number counts the number of connected components of $\Omega$, the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 2.2.

Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.

## The Morse inequalities

We state the Morse inequalities.

**Theorem 2.6** (Strong Morse inequalities, [1, Theorem 2.4])**.** *Let $X$ be a manifold with corners*

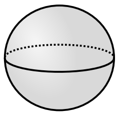
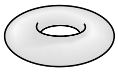
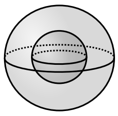
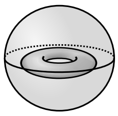
| Domain | Picture | $b_0$ | $b_1$ | $b_2$ | $b_k, k \geq 3$ |
|---|---|---|---|---|---|
| Ball $B$ |  | 1 | 0 | 0 | 0 |
| Solid torus $S^1 \times D$ |  | 1 | 1 | 0 | 0 |
| Ball with bubble $2B \setminus B$ |  | 1 | 0 | 1 | 0 |
| Ball with bubble in shape of torus |  | 1 | 1 | 1 | 0 |

Table 2.2: Betti numbers for selected domains in $\mathbb{R}^3$.

*and $f\colon X \to \mathbb{R}$ be Morse. Then we have for $l \in \{0, \ldots, d\}$ the inequalities*

$$\sum_{k=0}^{l} (-1)^{k+l} \operatorname{Ind}_j(f) \geq \sum_{k=0}^{l} (-1)^{k+l} b_j(X)\,.$$

*For $l = d$ we in fact have equality*

$$\sum_{k=0}^{d} (-1)^k \operatorname{Ind}_k(f) = \chi(X)$$

*where the Euler characteristic*

$$\chi(X) = \sum_{k=0}^{d} (-1)^k b_k(X)$$

*is the alternating sum of the Betti numbers.*

*Proof.* A for manifolds with $C^1$ boundary is given for instance in [24, Theorem 10.2']. The definition of regular critical points of $f$ and their index given in definition 1.9 coincides with the definition of a critical point and its co-index of $-f$ given in [1]. The result then follows from [1, Theorem 2.4]. Nonetheless we will give an idea of the proof. □

Give outline of proof idea.

Define for $a \in \mathbb{R}$ the manifold $X^a := \{x \in X\colon\ f(x) \leq a\}$. The main idea is to inspect what happens to the manifold $X^a$ as we vary $a$. This is the approach used in [20], [1] and [10]. The proof involves two main steps. The first is to show the following

**Lemma 2.7.** *Assume that the interval $[a,b]$ contains no critical value of $f$. Then $X^a \simeq X^b$ are homotopic.*

*Proof.* See for instance [1, p.7]. $\square$

The second step involves showing that

**Lemma 2.8.** *If the interval $[a,b]$ contains one critical value $c \in \mathbb{R}$ with non-degenerate essential critical points $x_1, \ldots, x_r$ of $f$ with coindices $\lambda_1, \ldots, \lambda_r$ then $X^b$ is obtained from $X^a$ by attaching the $\lambda$-cells*

The final part of the proof is then given in all details in **??**§I.5]Milnor1963. We shall however sketch for the equality in equation (**??**). For this define a generalisation of the Euler characteristic by

$$\chi(X,Y) := \sum_k \dim_{\mathbb{R}} H_k(X,Y)\,. \tag{2.3}$$

One can show that this is additive, in the sense that given suitable $X \subseteq Y \subseteq Z$ we have that

$$\chi(X,Z) = \chi(X,Y) + \chi(Y,Z) \tag{2.4}$$

**Corollary 2.9** (Weak Morse inequalities)**.** *Let $X$ be a manifold with corners and $f\colon X \to \mathbb{R}$ Morse. Then we have for $k \in \{0,\ldots,d\}$ the inequalities*

$$\mathrm{Ind}_k(f) \geq b_k(X)\,.$$

*Proof.*

Write some proof.

$\square$

If we now assume that $f$ is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [24, Corollary 10.1].

**Corollary 2.10** (Morse inequalities for $f$ harmonic, $d = 2$)**.** *Let $d = 2$, $\Omega$ and $f$ be regular and assume that $f$ is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M + \mu_1 - \mu_0 &= b_1 - b_0\,. \end{aligned}$$

In dimensions $d = 3$ we obtain [24, Corollary 10.2]

**Corollary 2.11** (Morse inequalities for $f$ harmonic, $d = 3$)**.** *Let $d = 3$, $\Omega$ and $f$ be regular and assume that $f$ is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0 \\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= b_2 - b_1 + b_0 \,. \end{aligned}$$

Give a classical example of a Morse function to determine the Betti numbers.

## On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2,3\}$.

**Proposition 2.12** (Harmonic vector fields on simply connected domains)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and $u$ be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f\colon \Omega \to \mathbb{R}$.*
2. *$f$ is harmonic.*
3. *$u$ is in fact $C^\infty$.*
4. *The components $u_i = \partial_i f$ are harmonic.*

*Proof.*
1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.
2. This follows from $\Delta f = \operatorname{Div} u = 0$.
3. This follows from the fact that $f$ is harmonic
4. This follows from $u_i = \partial_i f$.

□

If one considers not necessarily simply connected domains $\Omega$ then we obtain the properties of proposition 2.12 at least locally.

# 3 The case of connected entrant boundaries on simply connected domains in $d = 2$ dimensions

We will start this section by giving an essentially negative answer to question 1.1 in $d = 2$ dimensions. Thus it is not possible to have a harmonic function with interior critical point on a simply connected planar domain with connected entrant boundary. The proof of this will involve Morse theory on manifolds with corners which was introduced in the previous chapter. We shall then state a result from [2] which more generally relates the number of entrant components with the number of critical points in the plane using tools from complex analysis. After that we give examples showing that dropping the condition of simply connectedness on the domain yields a harmonic function with interior stagnation point. We will also consider the analogous problem in $d = 4$ dimensions and show that there exists a function and domain with the desired properties.

## A negative result using Morse theory

We now give a result which is essentially a negative answer to question 1.1. It was possible to prove this statement using level sets of critical points or using invariant manifolds. The following proof involves Morse theory since the techniques of the proof generalise to the three dimensional case.

**Proposition 3.1** (Negative answer to question 1.1 in $d = 2$ dimensions)**.** *Let $d = 2$ and $X$ be a simply connected manifold with corners and let $f\colon X \to \mathbb{R}$ have no irregular critical points. Assume further that $\Sigma^-$ contains at least two points and is simply connected. Then $f$ has no non-degenerate interior critical point.*

*Proof.* Let $\gamma = \{x_1, x_2\} = \partial\Sigma^-$. Then we can cut the domain along a curve $\Gamma$ such that the endpoints $\gamma = \partial\Gamma$ of the cut coincide with $x_1$ and $x_2$, that is $\partial\Gamma = \{x_1, x_2\}$. Now we obtain two new domains $X^+$ and $X^-$ such that $\partial X^+ \cap \Sigma^- \subseteq \gamma$ and $\partial X^- \cap \Sigma^+ \subseteq \gamma$. We can assume that $\Gamma$ is a smooth manifold and corresponds to the stratum $X_{\Gamma^+}$ for $X^+$ and $X_{\Gamma^-}$ for $X^-$. Analogously $\gamma$ corresponds to strata $X_{\gamma^\pm}$ on $X^\pm$. Locally around the corner point $x_1$ we have a situation depicted as in figure 3.1. We assume that we chose $\Gamma$ in such a way that it forms an acute angle with $u = \nabla f$ at the boundary points $\gamma$. For the following argumentation we require that $u$ is strongly Morse on both $X^+$ and $X^-$, so assume for a moment that this is the case. Since each point of $\gamma$ is an essential critical point for either $f$ or $-f$ on precisely one of the domains $X^+$ or $X^-$ we have

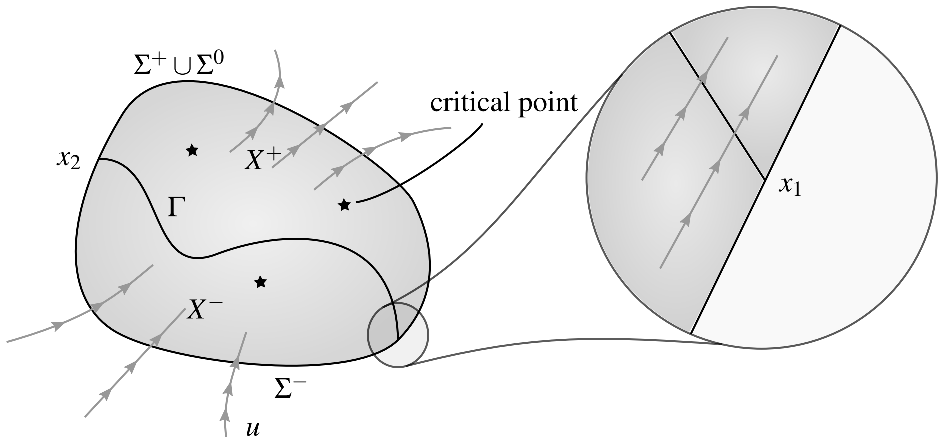



Figure 3.1: The situation at hand.

the relation

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^+,0}(-f) + \mathrm{Ind}_{\gamma^-,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) = 2\,. \tag{3.1}$$

We now focus our attention on $X^+$. Since no essential critical points of $f$ lie on $\Sigma^+ \setminus \gamma$ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f) \tag{3.2}$$

where $\delta_{ij}$ denotes the Kronecker delta. Analogously we have on $X^-$ that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{3.3}$$

In addition we have on $\Gamma$ that the emergent critical points of $f$ on $X^+$ are the entrant critical points of $-f$ on $X^-$, that is

$$\mathrm{Ind}_{\Gamma^+,0}(f) = \mathrm{Ind}_{\Gamma^-,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma^+,1}(f) = \mathrm{Ind}_{\Gamma^-,0}(-f)\,. \tag{3.4}$$

Using equations (3.2), (3.3) and (3.4) we obtain

$$\mu_0^+ - \mathrm{Ind}_{\gamma^+,0}(f) = \nu_1^- \qquad \text{and} \qquad \mu_1^+ = \nu_0^- - \mathrm{Ind}_{\gamma^-,0}(-f)\,. \tag{3.5}$$

Consider the Morse inequality for $f$ on $X^+$

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) \tag{3.6}$$

and the Morse inequality for $-f$ on $X^-$

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X)\,. \tag{3.7}$$

We now add equations (3.6) and (3.7) and insert relations (3.5) to obtain

$$M^- + M^+ - \mathrm{Ind}_{\gamma^+,0}(f) - \mathrm{Ind}_{\gamma^-,0}(-f) = -2\chi(X) = -2\,.$$

Since $\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \le 2$ by equation (3.1) and $M^\pm \ge 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

The claim remains to be shown in the case that $f$ is not strongly Morse on $X^+$ and $X^-$. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 1.17 applied separately to the domains $X^+$ and $X^-$. Since $E^\pm$ are residual in $B_\delta$ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. It follows from the slanted angles at which $\Gamma$ approaches $\gamma$ that if the points $x_1, x_2$ are essential critical points of $f$ that they then are in fact regular. Hence we obtain that

$$\mathrm{Ind}_{\gamma^+,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma^+,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma^-,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{3.8}$$

By the same corollary $u^\varepsilon$ has no essential stagnation points on $\Sigma^+(u)$ and $-u$ has no essential stagnation points on $\Sigma^-(u)$. The claim then follows by the calculations above where we replace $f$ with $f^\varepsilon$ and then note that $M^\varepsilon = M$. □

## A generalisation of this result

We could also have used tools from complex analysis to show proposition 3.1. In fact, [2] gives a more refined result which we will state in the following.

Let $J^\pm$ denote the number of connected components of $\Sigma^\pm$ which are proper subsets of a component of $\Sigma$. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\ge 0} \sqcup \Sigma_{\le 0}$ such that $\Sigma_{\ge 0} \subseteq \Sigma^{\ge 0}$ and $\Sigma_{\le 0} \subseteq \Sigma^{\le 0}$. Let now $J^{\ge 0}$ denote the minimal number of connected components of $\Sigma_{\ge 0}$ which are proper subsets of a component of $\Sigma$. We state a consequence of a result from [2, Theorem 2.1 and 2.2]:

**Proposition 3.2** (Weakening of [2, Theorem 2.1 and 2.2].). *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u\colon X \to \mathbb{R}$ be a harmonic vector field. Then we have*

$$M \le b_1 - b_0 + \frac{J^+ + J^-}{2}\,.$$

*If in addition we assume that there are no stagnation points on the boundary then we have*

$$M \le b_1 - b_0 + J^{\ge 0}\,.$$

*Proof.* See [2, Theorem 2.1] and [2, Theorem 2.2] where we set $\underline{\alpha} = n$ to be the outer unit normal and $D = b_0 - b_1$. □

If we set $X$ to be homeomorphic to the ball, that is $b_1 - b_0 = -1$ and require the entrant boundary to be simply connected we then obtain from proposition 3.2 that $M \le 0$, that is $f$ has no non-degenerate critical point.

incorporate this result in the following examples.

## If we allow for holes in the domain

In the following we will give examples of harmonic vector fields on domains with holes with simply connected entrant boundaries and interior stagnation points. For this define two differential operators in $d = 2$ dimensions by

$$\nabla^\perp f = \operatorname{Curl} f = \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix} \tag{3.9}$$

and

$$\operatorname{curl} u = -\partial_1 u_2 + \partial_2 u_1 .$$

The next proposition gives us a recipe to generate harmonic vector fields.

**Proposition 3.3.** *Let $\psi \colon \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}$ be harmonic then $u = \nabla^\perp \psi$ is a harmonic vector field.*

*Proof.* Since $\operatorname{Div} \nabla^\perp = 0$ we have

$$\operatorname{Div} u = \operatorname{Div} \nabla^\perp \psi = 0$$

and one calculates

$$\operatorname{curl} u = \operatorname{curl} \nabla^\perp \psi = -\Delta \psi = 0 .$$

□

The function $\psi$ is also called a *stream function*. The next example is our first example of a harmonic vector fields in $d = 2$ dimensions with an interior stagnation points for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component.

**Example 3.4** (Flow through tube with hole and stagnation point)**.** Consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{3.10}$$

where

$$\Phi_2 = \log(|\cdot|) \tag{3.11}$$

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and $e_i$ is the $i$-th unit vector. A plot of the streamlines in figure 3.2 indicates that $u = \nabla^\perp \psi$ fulfils the requirements on the domain

$$X = \psi^{-1}([-0.5, 2]) \cap (\mathbb{R} \times [-2, 2]) .$$

Indeed, an elementary calculation reveals that the origin is a stagnation point of $u$.

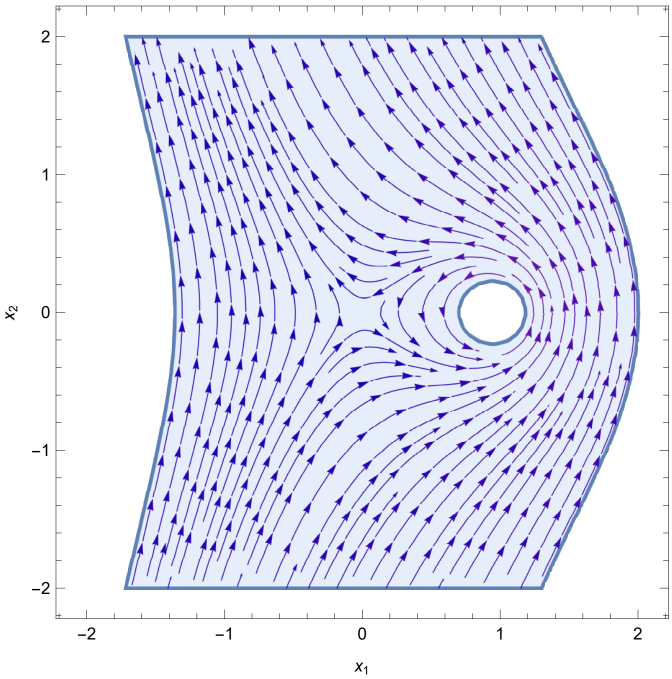



Figure 3.2: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap (\mathbb{R}\times[-2,2])$. Here $\psi$ is given by equation (3.10).

Example 3.4 highlights the importance of the requirement in proposition 3.1 that the domain be simply connected. We now give a similar example, this time with two holes in the domain.

**Example 3.5** (Flow through tube with two holes and stagnation points)**.** We consider the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x-e_1) - \Phi_2(x+e_1) + x_1 \end{aligned} \tag{3.12}$$

A plot of the streamlines in figure 3.3 indicates that $u = \nabla^{\perp}\psi$ on the domain

$$X = \psi^{-1}([-0.7,0.7]) \cap (\mathbb{R}\times[-2,2])$$

has the desired properties.

## The case of $d = 4$ dimensions

We ask whether there exists a harmonic vector field on a simply connected domain with connected entrant boundary and with interior stagnation point in a dimension higher than $d = 2$. Indeed, in $d = 4$ dimensions we can readily give an example of such a harmonic vector field.

**Example 3.6** (Connected entrant boundary in $d = 4$ dimensions)**.** Consider as domain $X = B_1 \subseteq \mathbb{R}^4$

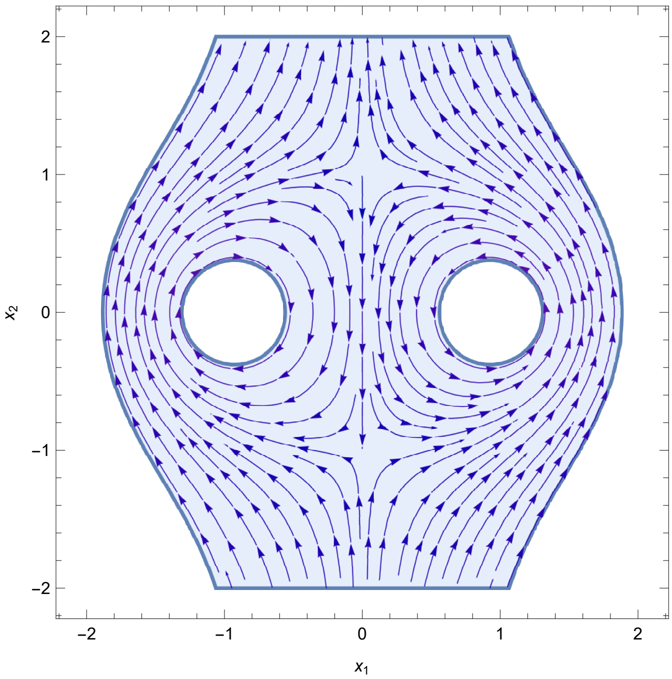



Figure 3.3: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2, 2])$. Here $\psi$ is given by equation (3.12).

the unit ball and the harmonic function

$$\begin{aligned} f : X &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned} \tag{3.13}$$

This has a critical point at the origin. We will show in proposition 3.7 that the entrant boundary $\Sigma^-$ is in fact connected.

**Proposition 3.7** (Simply connected entry boundary in example 3.6)**.** *The harmonic function given by equation* (3.13) *has simply connected entrant boundary.*

*Proof.* First observe that the boundary $\Sigma = S^3$ can be away from the equator locally parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{3.14}$$

on the boundary $\Sigma = S^3$ we have that $x_1$ is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_\pm : \mathbb{R}^3 \supseteq B_1 &\to \Sigma \subseteq \mathbb{R}^4 \\ \bar{x} &\mapsto x = (x_1, \bar{x}) \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{3.15}$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of $f$

$$\nabla f = 2\begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the outer unit normal to the boundary $\Sigma$ is given by

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2\big(x_1^2 + x_2^2 - x_3^2 - x_4^2\big) .$$

Using condition (3.14) we obtain the equivalent condition

$$0 < \pm 1 - 2\big(x_3^2 + x_4^2\big)$$

Define the cylinder

$$C = \big\{\bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2\big\} = \mathbb{R} \times B_{1/\sqrt{2}} \subseteq \mathbb{R}^3$$

If we return to our parametrisation (3.15) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(\bar{x}) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm\big(\Sigma^+\big)$$

and

$$B_1 \setminus C = \psi_\pm\big(\Sigma^-\big) .$$

Since $\phi_+$ is a homeomorphism onto the northern hemisphere and $\phi_-$ a homeomorphism onto the southern hemisphere and the surface $S^2$ corresponds to the equatorial sphere the claim then follows. The situation is depicted in figure 3.4. $\square$

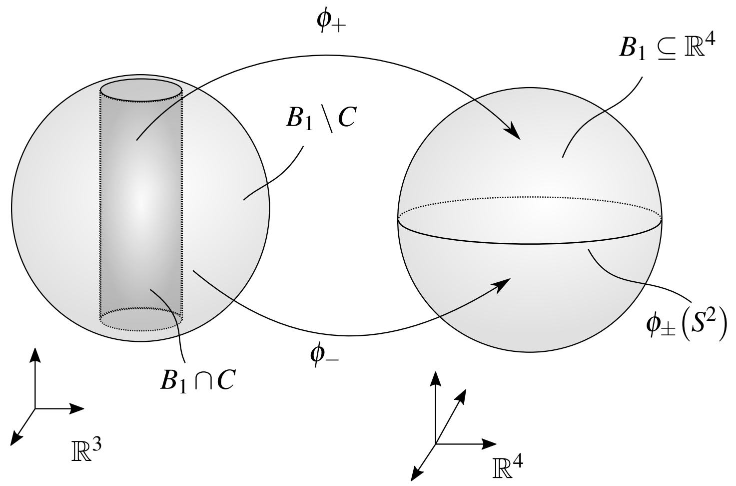



Figure 3.4: Visualisation of the situation in the proof of proposition 3.7.

# 4 The case of simply connected entrant boundaries in $d = 3$ dimensions.

In the following chapter we will discuss question 1.1 in the case of $d = 3$ dimensions. That is, we are looking for a harmonic function with interior stagnation point on a simply connected domain and connected entrant boundary. We will first give a negative answer in the case of a cylinder. After that we will use Morse theory to essentially argue that the number of interior stagnation points of such an example must be an even number. Finally we will give an example of a function and a domain with the desired properties.

## A negative result for the cylinder

The following proof comes from [28] and is a negative answer to question 1.1 for the cylinder.

**Proposition 4.1.** *Let $\Omega = (0,1) \times U \subseteq \mathbb{R}^3$ be an open cylinder where $U \subseteq \mathbb{R}^2$ is an open set. Let further $f\colon X = \overline{\Omega} \to \mathbb{R}$ be harmonic such that the sides $[0,1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^+$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^-$ is the emergent boundary. Then $f$ cannot have an interior critical point.*

*Proof.* Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary $\Sigma$. Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times \partial U$ such that $\partial_1 f(x)$ is minimal on $X$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. □

## A condition on the interior type numbers

Mimicking the proof of proposition 3.1 in two dimensions we obtain a condition on the type numbers for a harmonic function with interior stagnation point and simply connected entrant boundary.

**Proposition 4.2.** *Let $X \subset \mathbb{R}^3$ be a compact manifold with corners. Let $f\colon X \to \mathbb{R}$ be a Morse harmonic function. Assume that the strictly entrant boundary $\Sigma^-$ is non-empty and simply connected and that $\gamma = \partial\Sigma^-$ is a one-dimensional manifold with corners homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Then we have that*

$$M_1 - M_2 = 0\,.$$

*Proof.* As in the two dimensional case we split the domain $X$ with a surface $\Gamma$ such that $\partial\Gamma = \gamma = \partial\Sigma^-$. Denote the two arising domains by $X^+$ and $X^-$ where $\partial X^- \cap \Sigma^+ \subseteq \gamma$ and $\partial X^+ \cap \Sigma^- \subseteq \gamma$. Since by proposition 2.1 there are finitely many interior critical points in $X$ we can also assume that no interior critical points lie on $\Gamma$. Furthermore we assume that $\Gamma$ approaches $\gamma$ at a slanted angle. For the following argumentation we require that $f$ is strongly Morse on both $X^+$ and $X^-$ so assume for a moment that this is the case. By assumption we have that $\gamma$ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Since $f$ is non-degenerate the the number of maxima and minima of $f$ on $\gamma$ must be equal and thus

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) = \mathrm{Ind}_{\gamma^+,1}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \tag{4.1}$$

We now turn our attention to $X^+$. Since no essential critical points lie on $\Sigma^+$ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f)\,. \tag{4.2}$$

Analogously we have on $X^-$ that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{4.3}$$

In addition we have that the emergent critical points on $\Gamma = \Gamma^+$ of $f$ on $X^+$ are the entrant critical points on $\Gamma = \Gamma^-$ of $-f$ on $X^-$, that is

$$\begin{aligned} \mathrm{Ind}_{\Gamma^+,0}(f) &= \mathrm{Ind}_{\Gamma^-,2}(-f) \\ \mathrm{Ind}_{\Gamma^+,1}(f) &= \mathrm{Ind}_{\Gamma^-,1}(-f)\,. \\ \mathrm{Ind}_{\Gamma^+,2}(f) &= \mathrm{Ind}_{\Gamma^-,0}(-f) \end{aligned} \tag{4.4}$$

Using equations (4.2), (4.3) and (4.4) we obtain

$$\begin{aligned} \mu_0^+ - \nu_2^- &= \mathrm{Ind}_{\gamma^+,0}(f) \\ \mu_1^+ - \nu_1^- &= \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,1}(-f)\,. \\ \mu_2^+ - \nu_0^- &= -\mathrm{Ind}_{\gamma^-,0}(-f) \end{aligned} \tag{4.5}$$

We observe the Morse inequalities for $f$

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X)\,. \tag{4.6}$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \tag{4.7}$$

where the $M_k$ continue to denote the interior type numbers of $f$. We now subtract equation (4.7) from (4.6) and insert relations (4.5) to obtain then with equation (4.1)

$$\begin{aligned}0 &= M_1^- - M_2^- + M_1^+ - M_2^+ + \mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) - \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,0}(-f)\\ &= M_1 - M_2\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that $f$ is not strongly Morse on $X^+$ and $X^-$. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 1.17 applied separately to the domains $X^+$ and $X^-$. Since $E^\pm$ are residual in $B_\delta$ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. Since $x_1, x_2$ are non-degenerate critical points of $f$ due to the slanted angle at which $\Gamma$ approaches $\gamma$ we obtain that

$$\mathrm{Ind}_{\gamma,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-f) \tag{4.8}$$

By the same corollary we can assume that $f^\varepsilon$ has no essential critical points on $\Sigma^+(f)$ and $-f^\varepsilon$ has no essential critical points on $\Sigma^-(f)$. The claim then follows by the calculations above where we replace $f$ with $f^\varepsilon$ and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. □

## A harmonic function with interior critical point and simply connected entrant boundary in $d = 3$ dimensions

Based on example 3.6 in $d = 4$ dimensions of a harmonic vector field with interior stagnation point the author argued that it would be simplest to construct such a vector field in $d = 3$ dimensions in a similar manner. More specifically we choose as domain the ball and polynomials as our harmonic function. In choosing the ball as domain we also take into account the result from proposition 4.1 which is a negative result to question 1.1 for the cylinder. Based on the result from proposition 4.2 we see that the number of stagnation points must be at least two. The author then implemented a Mathematica routine to generate harmonic polynomials with two stagnation points and a plotting function to inspect what occurs. Indeed, this approach yielded a function with the desired properties as we shall discuss in this section.

**Example 4.3** (A harmonic function with interior critical point and simply connected entrant boundary)**.** Consider the domain $X = \overline{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large, and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R}\\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1x_2^2 + x_2x_3\end{aligned} \tag{4.9}$$

This induces the harmonic vector field

$$\begin{aligned} u\colon X &\to \mathbb{R}^3\\ x &\mapsto \nabla f(x) = \begin{bmatrix} x_1(1-x_1) + x_2^2 \\ x_2(2x_1 - 1) + x_3 \\ x_2 \end{bmatrix}\end{aligned}$$

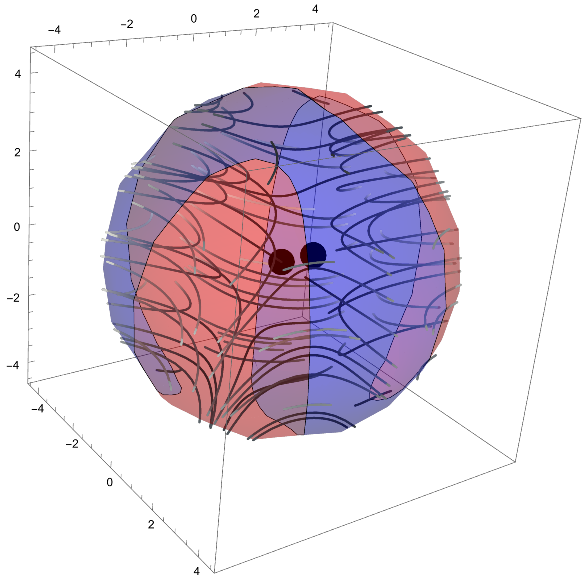



Figure 4.1: A stream plot of the function $u$. The interior stagnation points are highlighted in black. $\Sigma^+$ is shaded red, $\Sigma^-$ blue.

It follows from setting $u(x) = 0$ that $x_2 = 0$ and then $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of $u$. Conversely one sees that these are zeroes of $u$. Hence $f$ has two interior critical points for $r > 1$. Figure 4.1 shows a stream plot of $u$ with the two interior stagnation points highlighted as black dots. The boundary of the domain is shaded in blue for $\Sigma^-$ and in red for $\Sigma^+$. The stereographic projection of the boundary is plotted in figure 4.2. This plot indicates that the entrant boundary and the emergent boundary are simply connected. Indeed, we will show this in theorem 4.4.

The remainder of this section will be devoted to the proof of the following theorem:

**Theorem 4.4.** *The harmonic function given by equation* (4.9) *on the sphere has interior stagnation points and connected emergent and entrant boundaries for sufficiently large $r > 0$.*

Before we proceed to the proof we require some definitions from algebraic geometry. For an introduction this subject we refer the reader to [9], [11] or [12]. For polynomials $p_1, \dots, p_k \in \mathbb{R}[x_1, \dots, x_d]$ and a set $U \subseteq \mathbb{R}^d$ we denote the variety generated by these polynomials on $U$ by $V_U(p_1, \dots, p_k)$. In the case that $U = \mathbb{R}^d$ we write $V(\dots) = V_U(\dots)$.

**Definition 4.5** (Smoothness, [12, §5])**.** We call a $k$ dimensional algebraic variety $V_U(p_1, \dots, p_n)$ *smooth* or *non-singular* if the the jacobian

$$\begin{bmatrix} \nabla p_1 & \cdots & \nabla p_n \end{bmatrix}^\top$$

is of rank $d - k$ on $V_U(p_1, \dots, p_n)$. This criterion of smoothness is also called the *Jacobi criterion*.

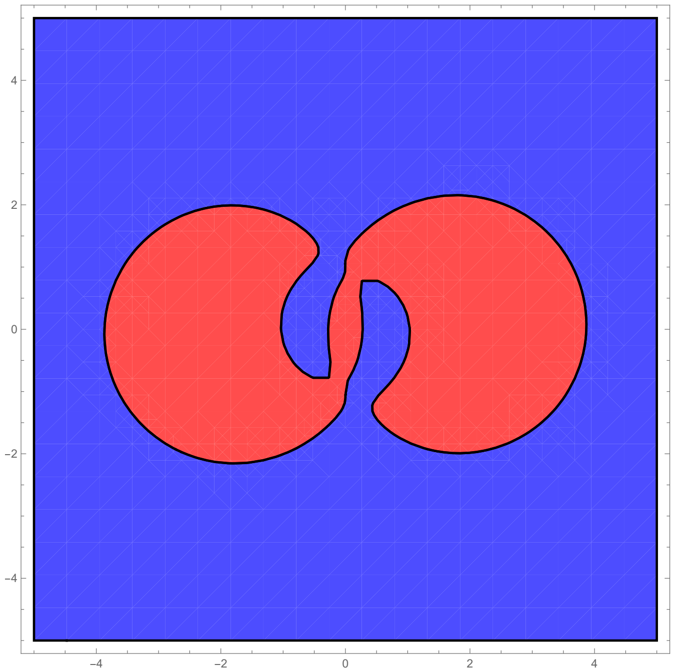



Figure 4.2: Stereographic projection of the surface $\Sigma$. $\Sigma^+$ is shaded red, $\Sigma^-$ blue.

By the implicit function theorem this means that the variety $V_U(p_1,\ldots,p_n)$ in fact defines a $d-k$ dimensional submanifold of $\mathbb{R}^d$.

The proof of theorem 4.4 requires lemmas 4.6 and 4.9 which we will show later on.

*Proof of theorem 4.4.* It was already discussed in example 4.3 that $f$ has an interior critical point at the origin and is harmonic. For the connectedness of the entrant and emergent boundaries we calculate

$$rn \cdot u(x) = x_1^2(1-x_1) + x_2^2(3x_1-1) + 2x_2x_3 =: p_1(x) \tag{4.10}$$

and define further

$$P_2(r,x) := x_1^2 + x_2^2 + x_3^2 - r^2 . \tag{4.11}$$

Thus we have that the tangential boundary $\Sigma^0 = V(p_1, P_2(r,\cdot))$ is precisely the variety generated by the polynomials $p_1$ and $P_2(r,\cdot)$ for a fixed radius $r > 0$. In lemma 4.6 we will show that the variety $V(p_1, P_2(r,\cdot))$ is in fact smooth and in lemma 4.9 we will then show that it is in fact connected. Thus $\Sigma^0$ then defines a simple closed curve on $\Sigma$ and the stereographic projection of $\Sigma^0$ defines a simple closed planar curve. This is indicated by the black curve in figure 4.2. By the Jordan curve theorem this curve then splits the plane into two connected regions, one of which is simply connected. The preimage of these connected regions under the stereographic projections then corresponds precisely to the entrant and emergent boundaries. From this it follows that the entrant and emergent boundaries are simply connected which proves the theorem. $\square$

From now onwards we assume that $p_1$ and $P_2$ are given by equations 4.10 and 4.11. We first show the smoothness which was required in the proof of theorem 4.4.

**Lemma 4.6** (Smoothness)**.** *There exists $R > 0$ such that for every $r > R$ the variety $V(p_1, P_2(r,\cdot))$ is smooth.*

*Proof.* One calculates

$$T := \begin{bmatrix} \nabla p_1(x) & \frac{1}{2}\nabla_x P_2(r,x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 + 3x_2^2 + 2x_1 & x_1 \\ 6x_1x_2 - 2x_2 + 2x_3 & x_2 \\ 2x_2 & x_3 \end{bmatrix}$$

By the Jacobi criterion it is sufficient to show that this matrix is of full rank on $V(p_1, P_2(r,\cdot))$. This is equivalent to showing that

$$0 \neq \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = -9x_1^2x_2 + 3x_2^3 + 4x_1x_2 - 2x_1x_3 =: h_1(x)\,,$$

$$0 \neq \det \begin{bmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} = 6x_1x_2x_3 - 2x_2x_3 + 2x_3^2 - 2x_2^2 =: h_2(x) \text{ or}$$

$$0 \neq \det \begin{bmatrix} T_{31} & T_{32} \\ T_{11} & T_{12} \end{bmatrix} = 2x_1x_2 + 3x_1^2x_3 - 3x_2^2x_3 - 2x_1x_3 =: h_3(x)$$

for any $x \in V(p_1, P_2(r,\cdot))$. This in turn is equivalent to showing that

$$V(p_1, P_2(r,\cdot), h_1, h_2, h_3) = \emptyset\,. \tag{4.12}$$

Indeed, consider the variety

$$V(p_1, h_1, h_2, h_3)\,. \tag{4.13}$$

Maple calculates the Gröbner basis with lexicographic order $x_1 < x_2 < x_3$

$$72x_1^8 - 198x_1^7 + 228x_1^6 - 153x_1^5 + 56x_1^4 - 5x_1^3\,, \tag{4.14}$$

$$72x_1^5x_2 - 126x_1^4x_2 + 102x_1^3x_2 - 51x_1^2x_2 + 5x_1x_2\,, \tag{4.15}$$

$$-24x_1^7 + 42x_1^6 - 2x_1^5 - 23x_1^4 + 7x_1^3 + 10x_1x_2^2\,, \tag{4.16}$$

$$48x_1^4x_2 - 60x_1^3x_2 + 13x_1^2x_2 + 15x_2^3\,, \tag{4.17}$$

$$24x_1^4x_2 - 30x_1^3x_2 + 29x_1^2x_2 - 10x_1x_2 + 5x_1x_3\,, \tag{4.18}$$

$$72x_1^7 - 126x_1^6 + 6x_1^5 + 69x_1^4 - 31x_1^3 + 10x_1^2 - 10x_2^2 + 20x_2x_3\,, \tag{4.19}$$

$$-72x_1^7 + 414x_1^6 - 654x_1^5 + 399x_1^4 - 97x_1^3 + 10x_1^2 - 30x_2^2 + 20x_3^2\,. \tag{4.20}$$

For an introduction to Gröbner bases we refer the reader to for example [6]. We will however only use the fact that the polynomials (4.14)-(4.20) generate the variety (4.13). We see from the basis vector (4.14) that for $x \in V(p_1, h_1, h_2, h_3)$ the coordinate $x_1$ can take only finitely many values. It then follows with (4.15) that also $x_2$ can take only finitely many values and finally with (4.16) that $x_3$ can also take only finitely many values. So the variety (4.13) contains finitely many points. Thus if we choose $R$ so large that all of these points are contained in the ball $B_R$ then we have that (4.12) holds for all $r > R$. $\square$

We now define $p_2$ to be the dehomogenisation of $P_2$, that is

$$p_2 := P_2(1,\cdot)\,.$$

Analogously let $P_1$ denote the homogenisation of $p_1$, that is

$$P_1(\varepsilon,x) := \varepsilon^3 p_1(x/\varepsilon)\,.$$

By rescaling the variety $V(p_1,P_2(r,\cdot))$ we obtain

$$V(p_1,P_2(r,\cdot)) = rV(x\mapsto p_1(rx),p_2) = rV(P_1(\varepsilon,\cdot),p_2) = r\mathscr{V}_\varepsilon \tag{4.21}$$

where we set $\varepsilon = 1/r$ and $\mathscr{V}_\varepsilon := V(P_1(\varepsilon,\cdot),p_2)$. Motivated by taking the limit $r\to\infty$ we inspect the variety $\mathscr{V}_0$ closer. The next proposition is required in the proof of lemma 4.9. It essentially states that the varieties $\mathscr{V}_\varepsilon \to \mathscr{V}_0$ converge as $\varepsilon\to 0$ outside of singular points. We will thus also need a notion of convergence of subsets on a metric space.

**Definition 4.7** (Hausdorff metric, tubular neighbourhood)**.** The *Hausdorff metric* for two sets $A,B\subseteq X$ is given by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\operatorname{dist}(x,B),\,\sup_{y\in B}\operatorname{dist}(y,A)\right\} \tag{4.22}$$

where

$$\operatorname{dist}(x,B) = \inf_{y\in B} d(x,y) \tag{4.23}$$

is the smallest distance from $x$ to $B$. For a given $\delta>0$ and a subset $A\subseteq\mathbb{R}^d$ we call the union of $\delta$ balls

$$\operatorname{Tub}_\delta(A) = \bigcup_{x\in A} B_\delta(x) \tag{4.24}$$

a *tubular neighbourhood* of $A$.

**Proposition 4.8** (Convergence of $\mathscr{V}_\varepsilon$ at smooth points)**.** *Let $U\subseteq\mathbb{R}^3$ be an open set such that $\mathscr{V}_0$ is smooth in an open neighbourhood of $\overline{U}$. Let further $\eta>0$. Then there exists a $\delta>0$ such that for all $\varepsilon<\delta$ we have that the Hausdorff distance satisfies*

$$d_H(\mathscr{V}_\varepsilon\cap U,\mathscr{V}_0\cap U) < \eta$$

*and additionally $\mathscr{V}_\varepsilon\cap U$ is isotopic to $\mathscr{V}_0\cap U$.*

*Proof.* Consider the mapping

$$F = \begin{bmatrix} P_1 \\ p_2 \end{bmatrix} : \mathbb{R}^4 \to \mathbb{R}^2\,.$$

Since $V(P_1(0,\cdot),p_2)$ is smooth on an open neighbourhood of $\overline{U}$ we have by the Jacobi criterion that

$$DF(0,\cdot) = \left[\begin{array}{c|c} D_\varepsilon P_1(0,\cdot) & D_x P_1(0,\cdot) \\ 0 & Dp_2 \end{array}\right] \tag{4.25}$$

is of full rank on $\overline{U}$. By the implicit function theorem there exists at every point $x \in \overline{U}$ open neighbourhoods $\Omega_x \subseteq \mathbb{R}^4$ of $(0,x)$ and $\omega_x \subseteq \mathbb{R}^3$, a coordinate permutation $I \in O(4)$ and a continuously differentiable mapping $g_x \colon \omega_x \to \mathbb{R}$ such that

$$V(P_1,p_2) \cap \Omega_x = \{I(y,g_x(y)) \colon y \in \omega_x\}\,.$$

Since $D_x F(0,x)$ is of full rank we can assume (possibly after shrinking the open sets) that $I$ does not permute the $\varepsilon$-coordinate. Thus we can write $y = (\varepsilon, y_1, y_2) \in \omega_x$. We can also assume that $\Omega_x = B_{\delta_x} \times W_x \subseteq \mathbb{R}^4$ for some open $W_x \subseteq \mathbb{R}^3$ and some $\delta_x > 0$. Hence we also obtain that $\omega_x = B_{\delta_x} \times w_x$ for some open $w_x \subseteq \mathbb{R}^2$ and we can define our isotopy on $\Omega_x$ as

$$\begin{aligned} \varphi_x \colon B_{\delta_x} \times w_x &\to W_x \\ y &\mapsto \operatorname{proj}_{\mathbb{R}^4 \to W_x} I(y, g_x(y))\,. \end{aligned}$$

Note that $\varphi_x(\{\varepsilon\} \times w_x) = \mathscr{V}_\varepsilon \cap W_x$. From this it also follows that we can choose $\delta_x$ such that

$$d_H(\mathscr{V}_\varepsilon \cap W_x, \mathscr{V}_0 \cap W_x) < \eta\,.$$

Now for $x \in \overline{U}$ the $\Omega_x$ form an open cover of $\overline{U}$. By compactness there exists a finite subcover. Set $\delta > 0$ to be the minimum of all $\delta_x$ for the $\Omega_x$ in this finite subcover and the claim follows. $\square$

The next lemma shows the connectedness of the variety $V(p_1, P_2(r,\cdot))$. Because the proof is quite lengthy, a part of the proof has been split off into proposition 4.10 which will be proved later on.

**Lemma 4.9** (Connectedness)**.** *There exists an $r > 0$ such that the planar variety $V(p_1, P_2(r,\cdot))$ has one connectivity component.*

*Proof.* By lemma 4.6 there exists a $R > 0$ such that for all $r > R$ we have that the variety $V(p_1, P_2(r,\cdot))$ is smooth and by equation (4.21) $\mathscr{V}_\varepsilon$ is also smooth for $\varepsilon < 1/R$. We inspect $\mathscr{V}_0$ closer. Observe that

$$P_1(0,x) = -x_1^3 + 3x_1x_2^2$$

which is the monkey saddle embedded into $\mathbb{R}^3$. We thus define curves

$$\widetilde{\alpha}^{\pm} := \left\{ t \begin{bmatrix} \pm\sqrt{3} & 1 \end{bmatrix}^\top : t \in \mathbb{R} \right\}$$

and $\widetilde{\alpha}^0 := \{0\} \times \mathbb{R}$. We then define $\alpha^\bullet := (\widetilde{\alpha}^\bullet \times \mathbb{R}) \cap S^2$. Setting $A := \alpha^- \cup \alpha^+ \cup \alpha^0$ we have the relation

$$\mathscr{V}_0 = V(P_1(\varepsilon,\cdot),p_2) = A\,.$$

Thus $\mathscr{V}_0$ consists of six smooth arcs originating at the singularity $e_3$ and ending at the singularity $-e_3$. Similar to the classical beach ball. Now consider for $\rho > 0$ the open sets $W_\rho := B_\rho \times \mathbb{R} \subseteq \mathbb{R}^3$ and $U_\rho := \mathbb{R}^3 \setminus W_\rho$. Since $\mathscr{V}_0$ is smooth in a neighbourhood of $\overline{U}_\rho$ we obtain from proposition 4.8 that in a certain sense $\mathscr{V}_\varepsilon$ is obtained from $\mathscr{V}_0$ by a small deformation on $U_\rho$. Thus in order to show connectedness of $\mathscr{V}_\varepsilon$ for sufficiently small $\varepsilon > 0$ we have to inspect what happens around the points $\pm e_3$. Now observe that we have the symmetry

$$p_1(x_1, -x_2, -x_3) = p_1(x) \tag{4.26}$$

and thus it suffices to inspect what happens around the point $e_3$. For this parametrise the neighbourhood $S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)$ of $e_3$ by the diffeomorphism

$$\begin{aligned} \psi\colon B_{1/2} &\to S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right) \\ \tilde{x} &\mapsto x = \begin{bmatrix} \tilde{x} & \sqrt{1-|\tilde{x}|^2} \end{bmatrix}^\top . \end{aligned}$$

We set

$$\widetilde{\mathscr{V}_\varepsilon} := \psi^{-1}\left(\mathscr{V}_\varepsilon \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)\right) = V_{B_{1/2}}\left(x \mapsto P_1\left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right)\right) = V_{B_{1/2}}\left(\widetilde{P}_1(\varepsilon, \cdot)\right)$$

where we defined

$$\widetilde{P}_1(\varepsilon, x) := P_1\left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right) = -x_1^3 + 3x_1x_2^2 + \varepsilon\left(x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2}\right).$$

In a similar manner we define $\widetilde{\alpha}^\bullet := \psi^{-1}(\alpha^\bullet)$, $\widetilde{U}_\rho := \psi^{-1}\left(U_\rho\right)$, $\widetilde{W}_\rho := \psi^{-1}\left(W_\rho\right)$ and $\widetilde{A} := \psi^{-1}(A)$. Now let the sets $W$ and $C \subseteq W$ be as in proposition 4.10. Pick $\rho > 0$ so small that $\widetilde{W}_{2\rho} \subseteq W$. Now we can pick $\eta$ smaller than the minimum distance between two arcs of $\widetilde{\mathscr{V}_0}$ on $\widetilde{U}_\rho$. We also assume that $\eta$ is smaller than the Hausdorff distance between $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ and $C$. Now choose $\delta$ as in proposition 4.8. We can assume that $0 < \varepsilon < \delta$. We thus have a situation as in figure 4.3. Number the arcs lying close to $A \cap U_\rho$ as in figure 4.4. Since $\widetilde{\gamma}$ by proposition 4.10 splits $\widetilde{W}_{2\rho}$ and lies in $C \cap \widetilde{\mathscr{V}_\varepsilon}$ we must have that $\widetilde{\gamma}$ in fact connects arcs 3 and 6. As the variety is smooth we must also have that arcs 1 and 2 are connected and analogously that arcs 4 and 5 are connected. By equation (4.26) the situation around $-e_3$ is analogous but mirrored. We thus obtain analogous connections between the six arcs at $-e_3$ which is summarised in figure 4.5. Thus we in fact have that $\mathscr{V}_\varepsilon$ is connected. The claim then follows from relation (4.21). $\square$

The following proposition is part of the proof of lemma 4.9:

**Proposition 4.10.** *Let $\widetilde{U}_\rho$, $\widetilde{\alpha}^\bullet$, $\widetilde{A}$, $\widetilde{\mathscr{V}_\varepsilon}$ and $\widetilde{P}_1$ be as in the proof of lemma 4.9. There exists an open cube $W \subseteq \mathbb{R}^2$ containing the origin and a set $C \subseteq W$ such that $C$ has positive Hausdorff distance to the set $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that for any sufficiently small $\varepsilon > 0$ there is an arc $\widetilde{\gamma}$ entirely contained in $C \cap \widetilde{\mathscr{V}_\varepsilon}$ splitting $W$ into two parts.*

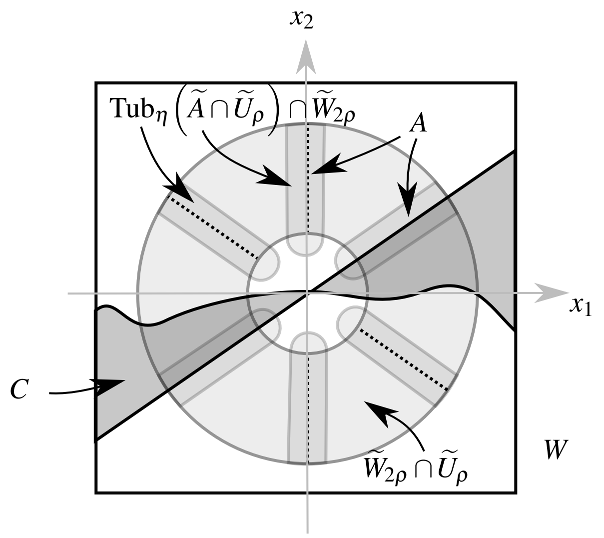



Figure 4.3: An overview of the situation around $e_3$.

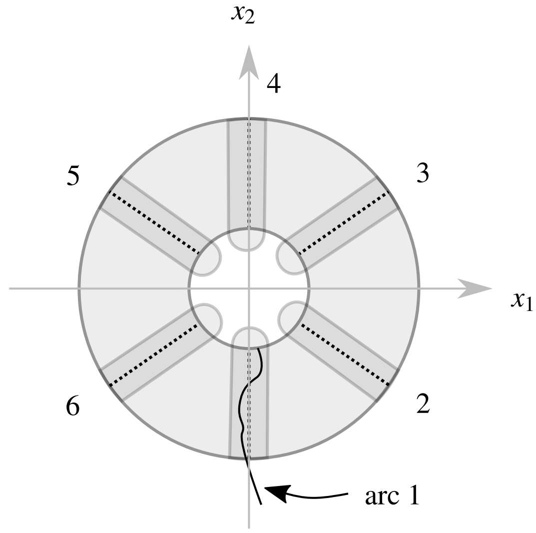



Figure 4.4: The numbering of the arcs around $e_3$.

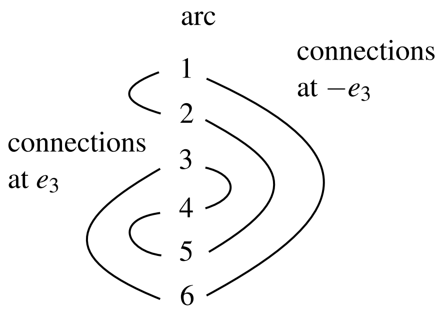



Figure 4.5: The connection of the arcs at $\pm e_3$.

*Proof.* We borrow notation from the proof of lemma 4.9. Let $W \subseteq \mathbb{R}^2$ be an open cube around the origin which we fix later. Define polynomials

$$\tilde{q}_1(x) := -x_1^3 + 3x_1x_2^2 \tag{4.27}$$

$$\tilde{q}_2(x) := x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2} \tag{4.28}$$

and observe that

$$\widetilde{P}_1(\varepsilon, \cdot) = \tilde{q}_1 + \varepsilon \tilde{q}_2 \,. \tag{4.29}$$

By equation (4.27) $\tilde{q}_1$ is a monkey saddle and $V_W(\tilde{q}_1) = \widetilde{A}$ is similar to figure 4.6a. The signs in the figure indicate the sign of $\tilde{q}_1$ in a given region. From equation (4.28) we obtain that $\nabla\tilde{q}_2(0) = e_2$ and $\tilde{q}_2(0) = 0$. Hence we observe that the $V_W(\tilde{q}_2)$ looks similar to figure 4.6b for a sufficiently small neighbourhood $W$. More concretely we choose $W$ so small that the arc $\widetilde{\beta} = V_W(\tilde{q}_2)$ has positive distance to $\widetilde{A} \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that a given vertical line in $W$ intersects $\widetilde{\beta}$ in precisely one point. Now we claim that the set $C$ consisting of the vertical lines between $\widetilde{\alpha}^+$ and $\widetilde{\beta}$ fulfills the claim.

We first show that there exists a curve $\widetilde{\gamma}$ through the origin which is entirely contained in $C \cap \widetilde{\mathscr{V}}_\varepsilon$. For this note that we have $\widetilde{P}_1(\varepsilon, 0) = 0$ and $\nabla_x \widetilde{P}_1(\varepsilon, x)\big|_{x=0} = \varepsilon e_2$ and thus there exists locally around the origin a parametrisation $\widetilde{\gamma}(t) = (t, \widetilde{\gamma}_2(t))$. By a similar argument there also exists locally a parametrisation $\widetilde{\beta}(t) = \big(t, \widetilde{\beta}_2(t)\big)$. We see that $\widetilde{\gamma}$ lies locally below $\widetilde{\alpha}^+$ and thus we need to show that $\widetilde{\gamma}_2$ also lies locally above $\widetilde{\beta}_2$ on the right half plane and locally below $\widetilde{\beta}_2$ on the left half plane. We calculate derivatives

$$\begin{aligned} D\tilde{q}_1\big|_{x=0} &= 0 \\ D^2\tilde{q}_1\big|_{x=0} &= 0 \\ D^3\tilde{q}_1\big|_{x=0}(v,v,v) &= -6v_1^3 + 18v_1v_2^2 \end{aligned}$$

and

$$\begin{aligned} D\tilde{q}_2\big|_{x=0} &= e_2^\top \\ D^2\tilde{q}_2\big|_{x=0} &= \begin{bmatrix} 2 & \\ & -2 \end{bmatrix} \\ D^3\tilde{q}_2\big|_{x=0}(v,v,v) &= 3v_1^2v_2 + v_2^3 \end{aligned}$$

and now by relation (4.29)

$$\begin{aligned} Dh\big|_{x=0} &= \varepsilon D_x\tilde{q}_2 \\ D^2h\big|_{x=0} &= \varepsilon D_x^2\tilde{q}_2 \\ D^3h\big|_{x=0}(v,v,v) &= -6v_1^3 + 18v_1v_2^2 + \varepsilon\big(3v_1^2v_2 + v_2^3\big)\,. \end{aligned} \tag{4.30}$$

where we wrote $h(x) = \widetilde{P}_1(\varepsilon, x)$ for brevity. We have $0 = h \circ \widetilde{\gamma}$ and thus

$$0 = \partial_t(h \circ \widetilde{\gamma})\big|_{t=0} = \Big(Dh\big|_{\widetilde{\gamma}}\widetilde{\gamma}'\Big)_{t=0} = \varepsilon\widetilde{\gamma}_2'(0) \tag{4.31}$$

so $\widetilde{\gamma}_2(0) = 0$ and in particular $\widetilde{\gamma}'(0) = e_1$. Taking another derivative we get

$$\begin{aligned}
0 &= \partial_t^2 (h \circ \widetilde{\gamma})\big|_{t=0} \\
&= \Big( D^2 h\big|_{\widetilde{\gamma}} (\widetilde{\gamma}', \widetilde{\gamma}') + Dh\big|_{\widetilde{\gamma}} \widetilde{\gamma}'' \Big)_{t=0} \\
&= \big( D^2 h\big|_{x=0} (e_1, e_1) + Dh\big|_{x=0} \widetilde{\gamma}'' \big)_{t=0} \\
&= \varepsilon \big( 2 + \widetilde{\gamma}_2'' \big)(0)
\end{aligned} \tag{4.32}$$

so $\widetilde{\gamma}''(0) = -2e_2$. The third derivative then yields

$$\begin{aligned}
0 &= \partial_t^3 (h \circ \widetilde{\gamma})\big|_{t=0} \\
&= \Big( D^3 h\big|_{\widetilde{\gamma}} (\widetilde{\gamma}', \widetilde{\gamma}', \widetilde{\gamma}') + 3D^2 h\big|_{\widetilde{\gamma}} (\widetilde{\gamma}', \widetilde{\gamma}'') + Dh\big|_{\widetilde{\gamma}} \widetilde{\gamma}''' \Big)_{t=0} \\
&= \big( D^3 h\big|_{x=0} (e_1, e_1, e_1) - 6D^2 h\big|_{x=0} (e_1, e_2) + Dh\big|_{x=0} \widetilde{\gamma}''' \big)_{t=0} \\
&= \big( -6 + \varepsilon (\widetilde{\gamma}_2''') \big)(0)
\end{aligned}$$

so $\widetilde{\gamma}_2'''(0) = 6/\varepsilon$. Analogously we observe that it follows from $0 = \tilde{q}_2 \circ \widetilde{\beta}$ that

$$0 = \varepsilon \partial_t \Big( \tilde{q}_2 \circ \widetilde{\beta} \Big)\big|_{x=0} = \varepsilon \Big( D\tilde{q}_2\big|_{\widetilde{\gamma}} \widetilde{\beta}' \Big)_{t=0} \tag{4.33}$$

and

$$0 = \varepsilon \partial_t^2 \Big( \tilde{q}_2 \circ \widetilde{\beta} \Big)\big|_{x=0} = \varepsilon \Big( D^2 \tilde{q}_2\big|_{\widetilde{\gamma}} \Big( \widetilde{\beta}', \widetilde{\beta}' \Big) + D\tilde{q}_2\big|_{\widetilde{\gamma}} \widetilde{\beta}'' \Big)_{t=0}. \tag{4.34}$$

By the relations (4.30), equation (4.33) is identical to equation (4.31) and equation (4.34) is identical to equation (4.32) with $\widetilde{\gamma}$ replaced by $\widetilde{\beta}$. Now equations (4.31) and (4.32) determined $\widetilde{\gamma}'(0)$ and $\widetilde{\gamma}''(0)$ uniquely and thus we have that $\widetilde{\gamma}(0) = \widetilde{\beta}(0)$, $\widetilde{\gamma}'(0) = \widetilde{\beta}'(0)$ and $\widetilde{\gamma}''(0) = \widetilde{\beta}''(0)$. For the third derivative we observe that

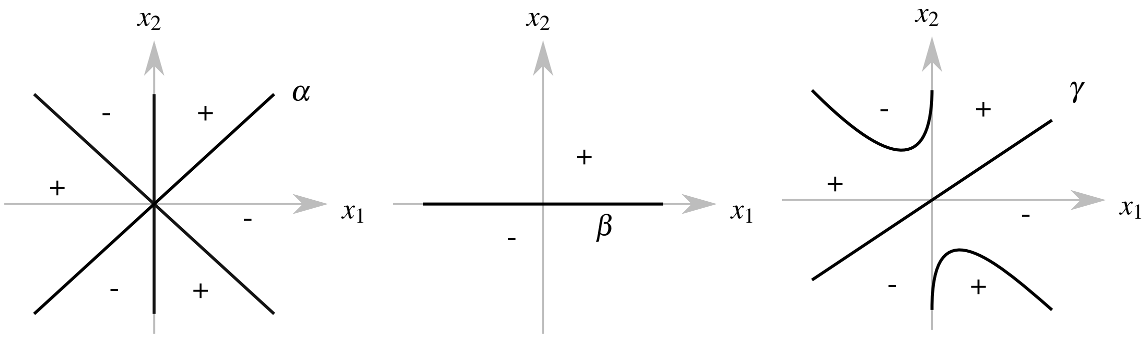
$$\begin{aligned}
0 &= \partial_t^3 \Big( \tilde{q}_2 \circ \widetilde{\beta} \Big)\big|_{t=0} \\
&= \Big( D^3 \tilde{q}_2\big|_{\widetilde{\gamma}} \Big( \widetilde{\beta}', \widetilde{\beta}', \widetilde{\beta}' \Big) + 3D^2 \tilde{q}_2\big|_{\widetilde{\gamma}} \Big( \widetilde{\beta}', \widetilde{\beta}'' \Big) + D\tilde{q}_2\big|_{\widetilde{\gamma}} \widetilde{\beta}''' \Big)_{t=0} \\
&= \Big( D^3 \tilde{q}_2\big|_{x=0} (e_1, e_1, e_1) - 6D^2 \tilde{q}_2\big|_{x=0} (e_1, e_2) + D\tilde{q}_2\big|_{x=0} \widetilde{\beta}''' \Big)_{t=0} \\
&= \widetilde{\beta}_2'''(0)
\end{aligned}$$

so $\widetilde{\beta}'''(0) = 0$. Thus we obtain

$$\partial_t^k \Big( \widetilde{\gamma}_2 - \widetilde{\beta}_2 \Big)\big|_{t=0} = 0$$

for $k \leq 2$ and

$$\partial_t^3 \Big( \widetilde{\gamma}_2 - \widetilde{\beta}_2 \Big)\big|_{t=0} = 6/\varepsilon > 0.$$




(a) Sketch of the set $V(\tilde{q}_1)$. (b) Sketch of the set $V(\tilde{q}_2)$. (c) Sketch of the set $\widetilde{\mathscr{V}}_\varepsilon$.

Figure 4.6: Sketches of varieties.

Hence we indeed have that $\widetilde{\gamma}_2 > \widetilde{\beta}_2$ in a sufficiently small neighbourhood of the origin for $t > 0$ and also that $\widetilde{\gamma}_2 < \widetilde{\beta}_2$ in a sufficiently small neighbourhood for $t < 0$.

Now that we have established that $\widetilde{\gamma}$ lies locally in $C$ it remains to be shown that $\widetilde{\gamma}$ indeed reaches the boundary $\partial W$. For this let $\widetilde{\gamma}\colon [0,b) \to C$ be a maximally extended parametrisation of $\widetilde{\gamma}$. Since $C$ is compact there exists a sequence $t_k \to b$ such that $x_k = \widetilde{\gamma}(t_k) \to x$ is convergent. Then we have by continuity that also $h(x) = 0$ so also $x \in \mathscr{V}_\varepsilon$. As $\mathscr{V}_\varepsilon$ is smooth we have that $Dh(x)$ is of full rank. Thus by the implicit function theorem we must in fact have that $b < \infty$ and the parametrisation of $\widetilde{\gamma}$ can be extended beyond the point $b$. But this means that $x \in \partial C$. Again by smoothness $x$ cannot lie in the origin. Now note that $\widetilde{P}_1(\varepsilon,\cdot) = \tilde{q}_1 < 0$ on the arc $\widetilde{\beta}$ and that $\widetilde{P}_1(\varepsilon,\cdot) = \tilde{q}_2 > 0$ on the arc $\widetilde{\alpha}^+$. So $\widetilde{\gamma}$ cannot intersect $\partial C$ on $\widetilde{\beta}$ or on $\widetilde{\alpha}$. Thus we must have that $x \in \partial W$ and hence $\widetilde{\gamma}$ splits $W$ in the right half plane into two parts. On the left half plane the argumentation is analogous. Then $\widetilde{\gamma}$ divides the plane into two parts which yields the claim. For clarity the idea of the proof is also depicted in figure 4.7. Also note that as a consequence $\widetilde{\mathscr{V}}_\varepsilon$ looks similar to figure 4.6c. $\square$

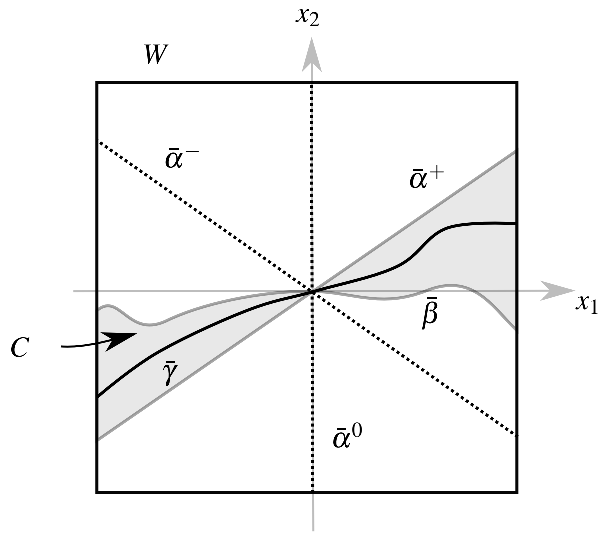



Figure 4.7: An overview of the situation in proposition 4.10.

# 5 No in- or outflow in $d = 2$ dimensions

In the second part of this thesis we will discuss harmonic vector fields without inflow or outflow through the boundary. More generally we will first consider harmonic vector fields without boundary critical points. Here we will show in proposition 5.1 that in $d = 2$ dimensions there is a strong relation between the number of critical points and the domain topology. We will then discuss briefly how this relates to the Poincaré-Hopf index theorem. After that we will give examples which illustrate these results.

## Relations between the domain topology and the number of stagnation points

Our first result relates the number of critical points with the domain topology. We shall give two proofs of this result, one involving Morse theory and the other involving the argument principle. The reason for this is that the result follows quickly from the argument principle. However, the techniques of the proof involving Morse theory generalise more readily to $d = 3$ dimensions.

**Proposition 5.1.** *Let $X \subset \mathbb{R}^2$ be a compact manifold with corners and Betti numbers $b_0 = 1$, and $b_1$ and let $u\colon X \to \mathbb{R}^2$ be a strongly Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where $M$ denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of $X$.*

*Proof.* We slit the domain $X$ such that it is homeomorphic to the disk as is depicted in figure 5.1. Denote the slit by $\Gamma$. Since the number of interior stagnation points is finite by proposition 2.1, we can choose $\Gamma$ in such a way that it does not contain any interior stagnation points. We write denote the boundary of $\Gamma$ by $\gamma = \partial\Gamma = \Gamma \cap \Sigma$ and define points $\{x_1, \ldots, x_{2b_1}\} = \gamma$. Note that there are $2b_1$ many such points. Without loss of generality we can assume that $\Gamma$ and $u$ form an acute angle at each point of $\gamma$. The situation is depicted in figure 5.1. For the following argumentation we require that $u$ is strongly Morse on the new domain $\tilde{X}$ so assume for a moment that this is the case. Since each $x_j$ is either an essential critical point of $u$ or of $-u$ on the slit domain $\tilde{X}$ we have that

$$\mathrm{Ind}_{\gamma,0}(u) + \mathrm{Ind}_{\gamma,0}(-u) = 2b_1\,. \tag{5.1}$$

Since there are no stagnation points on $\Sigma$ we have the relations

$$\mu_k = \mathrm{Ind}_{\Gamma,k}(u) + \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \nu_k = \mathrm{Ind}_{\Gamma,k}(-u) + \mathrm{Ind}_{\gamma,k}(-u) \tag{5.2}$$

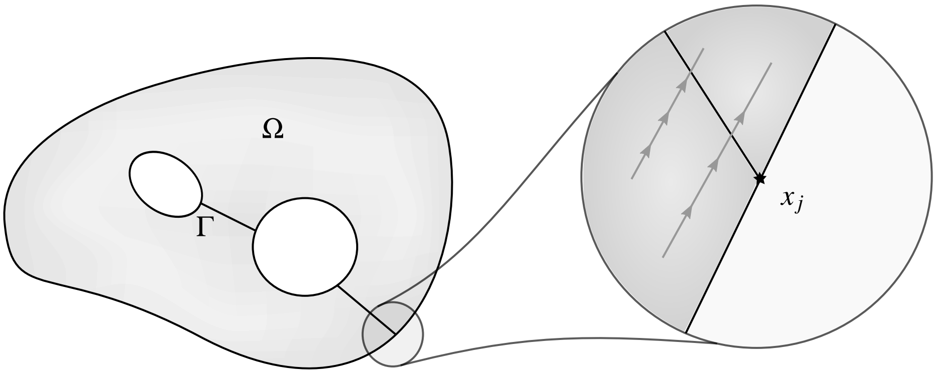



Figure 5.1: How we slit the domain.

for all $k \in \{0,1\}$. All entrant stagnation points of $u$ on $\Gamma$ are also emergent stagnation points of $-u$ on $\Gamma$ (and vice versa) and hence we have the relations

$$\mathrm{Ind}_{\Gamma,0}(u) = \mathrm{Ind}_{\Gamma,1}(-u) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(u) = \mathrm{Ind}_{\Gamma,0}(-u)\,. \tag{5.3}$$

Equations (5.2) and (5.3) yield

$$\mu_0 - \mathrm{Ind}_{\gamma,0}(u) = \nu_1 \qquad \text{and} \qquad \mu_1 = \nu_0 - \mathrm{Ind}_{\gamma,0}(-u)\,. \tag{5.4}$$

Since $\tilde{X}$ is now simply connected $u$ is by proposition 2.12 the gradient of a harmonic function $f$ on this new domain. For this $f$ we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \tag{5.5}$$

and for $-f$ the Morse inequalities

$$M + \nu_1 - \nu_0 = -\chi(\tilde{X}) = -1\,. \tag{5.6}$$

Adding equations (5.5) and (5.6) and using the relation (5.4) and then (5.1) we obtain

$$-2 = 2M - \mathrm{Ind}_{\gamma,0}(u) - \mathrm{Ind}_{\gamma,0}(u) = 2M - 2b_1$$

from which the claim follows.

The claim remains to be shown in the case that $u$ is not strongly Morse on $\tilde{X}$. In this case let $u^{\varepsilon}$ for $\varepsilon \in E$ be a strongly Morse function as in corollary 1.17. Since the $x_1,\dots,x_{2b_1} \in \gamma$ are non-degenerate stagnation points of $u$ due to the slanted angle at which $\Gamma$ approaches $\gamma$ we obtain that

$$\mathrm{Ind}_{\gamma,k}(u^{\varepsilon}) = \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-u^{\varepsilon}) = \mathrm{Ind}_{\gamma,k}(-u) \tag{5.7}$$

By the same corollary $u^{\varepsilon}$ has no stagnation points on $\Sigma$. The claim then follows by the calculations above where we replace $u$ with $u^{\varepsilon}$ and then note that $M^{\varepsilon} = M$. $\square$

We now give an alternative and simpler proof of proposition 5.1 using the argument principle.

*Alternative proof.* As before we slit the domain such that it is homeomorphic to a disk. By proposition 2.12 $u$ is the gradient of a harmonic function $f$ on this new domain $\tilde{X}$. Let $h \in \mathrm{Hol}(\tilde{X})$ be the holomorphic function given by $h = \nabla f$. Let $\gamma$ traverse the boundary of the slit domain such that the domain lies to the left of $\gamma$. We now determine the change of argument $\arg h$ along $\gamma$. For this consider first the the slits. Since $\nabla f$ is continuously differentiable along the slit and $\gamma$ traverses the slit once in one direction and once in the other, the contribution to the change of $\arg h$ from the slits vanishes. On the other hand as $\gamma$ traverses the boundary $\Sigma$ the contribution to the change in argument of $\arg h$ is $2\pi$ for every hole in the domain since $h \cdot \gamma' = u \cdot \gamma'$ does not change sign as $\gamma$ traverses a hole in clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is $-2\pi$ for the outer boundary component which is traversed counterclockwise. Since we have $b_1$ holes in the domain the total change of $\arg h$ as $\gamma$ traverses $\Sigma$ is $2\pi(b_1 - 1)$. Since $h$ has no poles it follows from the argument principle (see for example [8, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M$$

from which the claim follows. □

We say that $u$ has no *inflow* on a boundary subset $S \subseteq \Sigma$ if $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* if $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can give the following corollary.

**Corollary 5.2.** *Let $X$ be a compact manifold with corners and $u \colon X \to \mathbb{R}^2$ a strongly Morse harmonic vector field without inflow or outflow on $\Sigma$. Then we have the relation $M = -\chi(X)$ where $M$ is the number of stagnation points and $\chi(X)$ is the Euler characteristic of $X$.*

This corollary could also have been proved using the Poincaré-Hopf index theorem. For this define:

**Definition 5.3** ([5, Definition 1.1.1]). The *Poincaré-Hopf index* $\mathrm{Ind}_{\mathrm{PH},x}(f)$ of an isolated interior stagnation point $x$ of $u$ is the degree of the Gauss map $u/|u| \colon S_\varepsilon^{d-1}(x) \to S^{d-1}$ for sufficiently small $\varepsilon > 0$. The *total index* $\mathrm{Ind}_{\mathrm{PH}}(u)$ is the sum of indexes $\mathrm{Ind}_{\mathrm{PH},x}(f)$ for every interior stagnation point $x$ of $u$.

Note that in $d = 2$ dimensions a point $x$ with Morse index $k$ has Poincaré-Hopf index $(-1)^k$. Thus we have for a harmonic vector field in $d = 2$ dimensions that $M = -\mathrm{Ind}_{\mathrm{PH}}(u)$ and thus corollary 5.2 also follows from the Poincaré-Hopf index theorem:

**Theorem 5.4** (Poincaré-Hopf index theorem, [21, §6]). *Let $u \colon X \to \mathbb{R}^d$ be a vector field on a manifold with corners without inflow. Then we have that the total index $\mathrm{Ind}_{PH}(u)$ equals the Euler characteristic $\chi(X)$.*

*Proof.* See for example [21, §6]. □

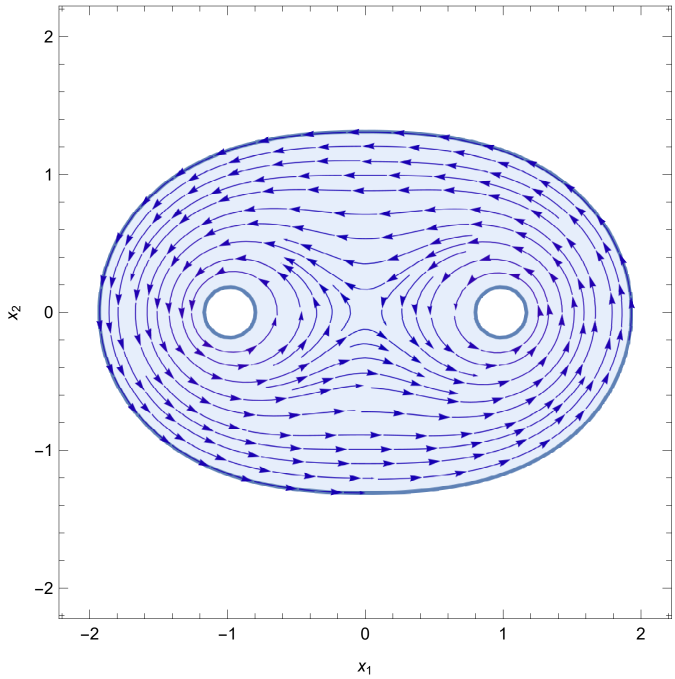



Figure 5.2: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here $\psi$ is given by equation (5.8).

## Examples of harmonic vector fields

We would like to illustrate the previous results with examples. We first give an example of a harmonic vector field in $d = 2$ dimensions without inflow or outflow and with one stagnation point.

**Example 5.5** (No in- or outflow)**.** Consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + \Phi_2(x + e_1) \end{aligned} \tag{5.8}$$

where $\Phi_2$ is as in example 3.4. A plot of the streamlines in figure 5.2 indicates that $u = \nabla^{\perp}\psi$ in the domain $X = \psi^{-1}([-1,1])$ has the desired properties. Indeed, since $\psi$ is constant on each component of $\partial X$ the function $u$ has neither inflow nor outflow. It follows from $\psi(-x) = \psi(x)$ that $u(-x) = -u(x)$ and thus the origin $x = 0$ is a stagnation point. By proposition 5.1 it is in fact the sole stagnation point of $u$ on $X$.

In a second example given by [28] we fix the domain rather than the function.

**Example 5.6** (No in- or outflow)**.** Set $X = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We let the stream function $\psi$ be determined by the system

$$\begin{aligned} \Delta\psi &= 0 && \text{, on } \Omega \\ \psi &= 0 && \text{, on the outer ring } 4S^1 \\ \psi &= 1 && \text{, on the inner rings } S^1(-2e_1) \cup S^1(2e_1) \end{aligned} \tag{5.9}$$

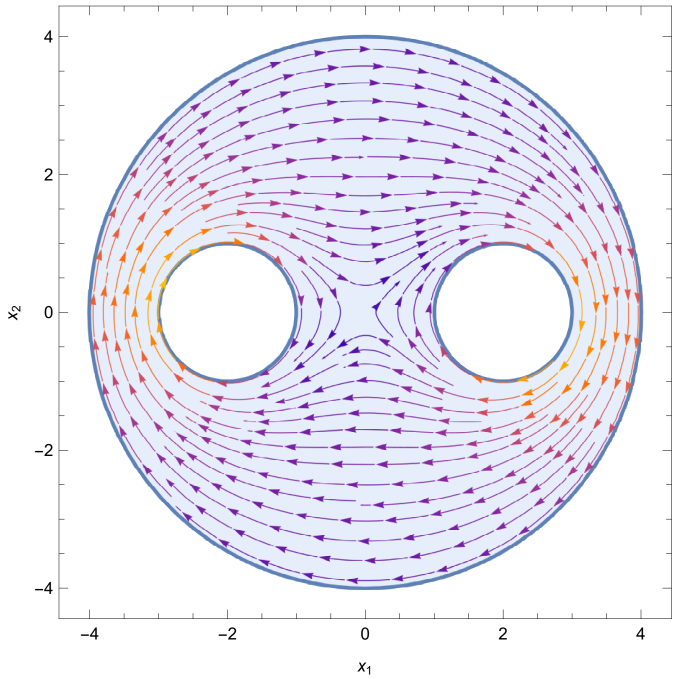



Figure 5.3: A plot of $u = \nabla^{\perp}\psi$ where $\psi$ is the numerical solution to (5.9).

and set $u = \nabla^{\perp}\psi$. A numerical solution to this system is plotted in figure 5.3. Again, it follows from symmetry that the origin is a stagnation point and from proposition 5.1 that it is in fact the sole stagnation point of $u$.

Our third example highlights the importance of assuming that $u$ be Morse in corollary 5.2.

**Example 5.7** (Stagnation points on the boundary)**.** In this example given by [28] we again start by fixing the domain. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We let the stream function $\psi$ be determined by the system

$$\begin{aligned} \Delta\psi &= 0 && \text{, on } \Omega \\ \psi &= 0 && \text{, on the outer ring } 4S^1 \\ \psi &= -1 && \text{, on the left inner ring } S^1(-2e_1) \\ \psi &= 1 && \text{, on the right inner ring } S^1(2e_1) \end{aligned} \tag{5.10}$$

and then set $u = \nabla^{\perp}\psi$. The numerical solution to this system is plotted in figure 5.4. Here we obtain from the symmetry $\psi(-x_1, x_2) = \psi(x)$ that $\psi = 0$ on the $x_2$-axis. Since also $\psi = 0$ on $4S^1$ we have two stagnation points at $\pm 4e_2$. This function again has no in- or outflow through the boundary. The domain contains two holes so $\chi(X) = -1$. Now $u$ has no interior stagnation point, seemingly contradicting corollary 5.2. But since the two stagnation points at $\pm 4e_2$ lie on the boundary $u$ is in fact not Morse and thus we cannot apply corollary 5.2. This shows the importance of the assumption that $u$ is Morse. Proposition 5.8 essentially states that in this case

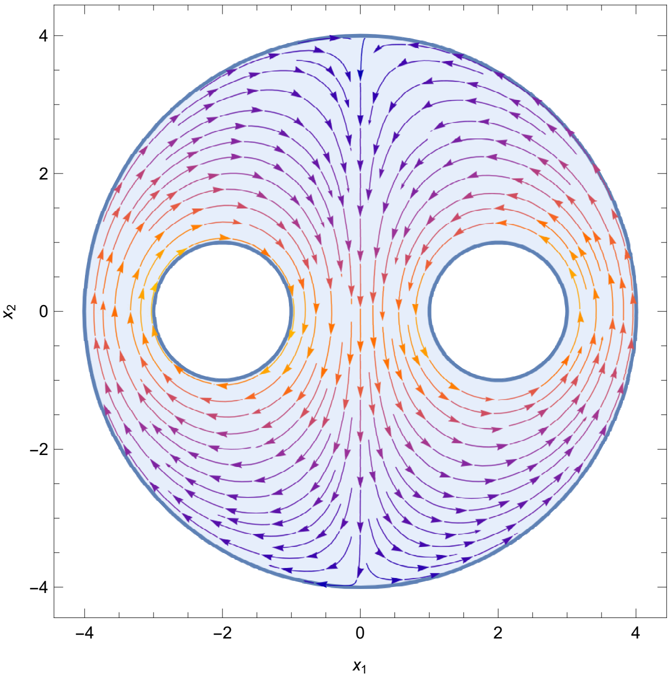



Figure 5.4: A plot of $u = \nabla^{\perp}\psi$ where $\psi$ is the numerical solution to (5.10).

the stagnation points of $u$ on the boundary count half as much as stagnation points in the interior. This explains why there are two critical points in this example.

**Proposition 5.8** (Critical points on the boundary, [2, Theorem 1.1])**.** *Let $\Omega \subseteq \mathbb{R}^2$ be bounded, open, with piecewise $C^{1,\alpha}$ boundary. Let further $f : X \to \mathbb{R}$ be harmonic and constant on each boundary component, say $f = a_j$ on a boundary component $\Gamma_j$. Then there exist finitely many critical points $x_1, \dots, x_k \in X$ of $u$ with multiplicities $m_1, \dots, m_k$ and we have that*

$$\sum_{x_j \in \Omega} m_j + \frac{1}{2} \sum_{x_j \in \Sigma} m_j = -\chi(X)\,.$$

*Proof.* The proof uses complex analysis techniques and can be found in [2, Theorem 1.1]. $\square$

# 6 No in- or outflow in $d = 3$ dimensions

This question is in part inspired by a question posed in **Lortz1970**.

We obtain as a quick consequence of the hairy ball theorem

**Proposition 6.1.** *Let $X$ have Betti numbers $b_0$, $b_1$ and $b_2$. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without boundary stagnation points. Then we have*

$$b_2 \leq b_1 \,.$$

*Proof.* Assume not. Since $\Omega$ has $b_2$ bubbles and $b_1$ holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since $u$ has no boundary stagnation points on $\Gamma$ we the restriction $u|_\Gamma$ does not vanish. But $\Gamma$ is homeomorphic to the Ball in contradiction to the hairy ball theorem. $\square$

We also obtain the following result:

**Proposition 6.2.** *Let $X \subseteq \mathbb{R}^3$ be a compact manifold with corners and Betti numbers $b_0$, $b_1$ and $b_2$. Let $u\colon X \to \mathbb{R}$ be a strongly Morse harmonic vector field without boundary critical points. Then the following relation for the interior type numbers of $u$ holds:*

$$M_1 = M_2 \,.$$

*Proof.* As in the two dimensional case we cut the domain $X$ with a surface $\Gamma$ such that the slit domain is homeomorphic to a ball with bubbles. Since the number of interior stagnation points of $u$ is finite by proposition 2.1, we can choose $\Gamma$ in such a way that it does not contain any stagnation points. We also denote the arcs at which $\Gamma$ meets $\Sigma$ by $\gamma_1, \ldots, \gamma_{b_1} \subseteq \partial\Gamma$. Note that there are $b_1$ many such curves. We can assume that $\Gamma$ and the $\gamma_j$ are manifolds with corners and that $\Gamma$ approaches each $\gamma_j$ at a slanted angle. The cut now yields a new domain $\tilde{X}$ which is a covering space of $X$. On this covering space we denote the cover of the cut $\Gamma$ and the sets $\gamma_j$ by $\Gamma^i$ and $\gamma_j^i$ with $i \in (1,2)$. Since this new domain $\tilde{X}$ is homeomorphic to a ball with bubbles the vector field $u = \nabla f$ is the gradient of a harmonic function $f$ by proposition 2.12. For the following argumentation we require that $u$ is strongly Morse on $\tilde{X}$, so assume for a moment that this is the case. Now we have that each $\gamma_j$ is homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Since $f$ is non-degenerate the number of maxima and minima of $f$ on $\gamma_j^1 \cup \gamma_j^2$ must be equal and thus

$$\sum_i \Big(\mathrm{Ind}_{\gamma_j^i,0}(f) + \mathrm{Ind}_{\gamma_j^i,1}(-f)\Big) = \sum_i \Big(\mathrm{Ind}_{\gamma_j^i,1}(f) + \mathrm{Ind}_{\gamma_j^i,0}(-f)\Big) \,. \tag{6.1}$$

Since on $\Gamma$ all entrant stagnation points of $u$ are also emergent stagnation points of $-u$ (and vice versa) we have the relations

$$\begin{aligned}\operatorname{Ind}_{\Gamma^1,0}(\pm u) &= \operatorname{Ind}_{\Gamma^2,2}(\mp u)\\ \operatorname{Ind}_{\Gamma^1,1}(\pm u) &= \operatorname{Ind}_{\Gamma^2,1}(\mp u)\\ \operatorname{Ind}_{\Gamma^1,2}(\pm u) &= \operatorname{Ind}_{\Gamma^2,0}(\mp u)\,.\end{aligned} \tag{6.2}$$

As there are no boundary critical points on $\Sigma$ it follows for the boundary type numbers that

$$\begin{aligned}\mu_k &= \sum_i \left(\operatorname{Ind}_{\Gamma^i,k} + \sum_j \operatorname{Ind}_{\gamma^i_j,k}\right)(f)\\ \nu_k &= \sum_i \left(\operatorname{Ind}_{\Gamma^i,k} + \sum_j \operatorname{Ind}_{\gamma^i_j,k}\right)(-f)\,.\end{aligned} \tag{6.3}$$

Equations (6.3) and (6.2) yield

$$\begin{aligned}\mu_0 - \nu_2 &= \sum_{i,j} \operatorname{Ind}_{\gamma^i_j,0}(f)\\ \mu_1 - \nu_1 &= \sum_{i,j} \Big(\operatorname{Ind}_{\gamma^i_j,1}(f) - \operatorname{Ind}_{\gamma^i_j,1}(-f)\Big)\\ \mu_2 - \nu_0 &= -\sum_{i,j} \operatorname{Ind}_{\gamma^i_j,0}(-f)\end{aligned} \tag{6.4}$$

For $f$ we have the Morse inequalities

$$M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 = -\chi\big(\tilde{X}\big) \tag{6.5}$$

and for $-f$ the Morse inequalities

$$M_1 + \nu_2 - M_2 - \nu_1 + \nu_0 = -\chi\big(\tilde{X}\big)\,. \tag{6.6}$$

Subtracting equation (6.6) from (6.5) and using the relation (6.4) we obtain together with equation (6.1) that

$$\begin{aligned}0 &= 2(M_2 - M_1) + \sum_{i,j} \Big(\operatorname{Ind}_{\gamma^i_j,0}(f) - \operatorname{Ind}_{\gamma^i_j,1}(f) + \operatorname{Ind}_{\gamma^i_j,1}(-f) - \operatorname{Ind}_{\gamma^i_j,0}(-f)\Big)\\ &= 2(M_2 - M_1)\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that $f$ is not strongly Morse on $X^+$ and $X^-$. In this case let $f^\varepsilon$ for $\varepsilon \in E$ be a strongly Morse function as in corollary 1.17. Since $x_1, x_2$ are non-degenerate critical points of $f$ due to the slanted angle at which $\Gamma$ approaches each $\gamma_j$ we obtain that

$$\operatorname{Ind}_{\gamma_j,k}(f^\varepsilon) = \operatorname{Ind}_{\gamma_j,k}(f) \qquad \text{and} \qquad \operatorname{Ind}_{\gamma_j,k}(-f^\varepsilon) = \operatorname{Ind}_{\gamma_j,k}(-f) \tag{6.7}$$

By the same corollary we can assume that $f^\varepsilon$ has no critical points on $\Sigma$. The claim then follows by the calculations above where we replace $f$ with $f^\varepsilon$ and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. $\square$

As a consequence we obtain the following:

**Corollary 6.3.** *Let $X$ be a compact manifold with corners and $u\colon X \to \mathbb{R}^d$ a strongly Morse harmonic vector field without inflow or outflow on the boundary $\Sigma$. The we have that*

$$M_1 = M_2 \,.$$

# Symbols

| | |
|---|---|
| $d$ | Dimensions $d = 2$ or $d = 3$ |
| $X$ | Compact domain in $\mathbb{R}^d$, often assumed to be a manifold with corners. |
| $\Omega$ | Interior $\Omega = \text{int}(X)$ |
| $f \colon X \to \mathbb{R}$ | A harmonic function. |
| $u \colon X \to \mathbb{R}^d$ or $T^*X$ | A harmonic vector field. |
| $X_j$ | A stratification of $X$ as given in definition 1.4. Often but not always assumed to be given by equation (1.1) |
| $\Sigma$ | Boundary $\Sigma = \partial X$ |
| $\Sigma^-$, $\Sigma^{\leq 0}$ | (Strictly) entrant boundary. See definition 1.7. |
| $\Sigma^+$, $\Sigma^{\geq 0}$ | (Strictly) emergent boundary. See definition 1.7. |
| $\Sigma^0$ | Tangential boundary. See definition 1.7. |
| $\Sigma^{\text{irr}}$ | Irregular boundary. See definition 1.9. |
| $B_r(x)$, $B_r$ | Ball of radius $r$ around the point $x$ / the origin. |
| $S^{d-1}(x)$, $S^{d-1}$ | $(d-1)$-dimensional sphere around $x$ / the origin. |
| $e_j$ | $j$-th unit vector in $\mathbb{R}^d$. |
| $u_j$ | Projection of $u$ to the cotangent bundle $T^*X_j$. See equation (1.6). |
| $\pi_j$ | Orthogonal projection onto the cotangent bundle $T^*X_j$. See equation (1.5). |
| $\text{Cr}_j$ | Set of essential stagnation points. See definition 1.9. |
| $\text{Ind}_{j,k}$ | $k$-th type number on the stratum $X_j$. See equation (1.7). |
| $\text{Ind}_k$ | $k$-th type number. See equation (1.11). |
| $M_k$ | $k$-th interior type numbers. See equation (1.8). |
| $M$ | Total number of stagnation points. See equation (1.9). |
| $\mu_k$ | $k$-th boundary type numbers of $f$. See equation (1.10). |
| $\nu_k$ | $k$-th boundary type numbers of $-f$. See definition 1.9. |
| $u^\varepsilon$ | modification to $u$ as in equation (1.12) |
| $A$ | submanifold, can be thought of as the zero section of $T^*X$ |
| $b_k$ | Betti number as defined in equation (2.2) |
| $\chi(X)$ | Euler characteristic. |
| $\nabla^\perp$ | Orthogonal gradient. Given by equation (3.9). |
| $\Phi_2$ | Multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$. Given by equation (3.11). |
| $V_U(\dots)$, $V(\dots)$ | Algebraic variety. |
| $d_H(A,B)$ | Hausdorff metric. See equation (4.22). |

| | |
|---|---|
| $\mathrm{dist}(x,A)$ | Distance between $x$ and $A$. See equation (4.23) |
| $\mathrm{Tub}_\delta(A)$ | Tubular neighbourhood of $A$. See equation (4.24). |

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