

Some title

Master Thesis

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Contents

Introduction	4
General definitions	4
On assuming non-degeneracy	5
Some general remarks	7
Betti numbers	8
The Morse inequalities	8
On harmonic vector fields	9
Harmonic functions, $d = 2$	10
A proof involving level-sets	10
A proof involving invariant manifolds	12
Harmonic vector fields, $d = 2$	14
No inflow or outflow	14
An example of inflow on one side and outflow on the other	17
Harmonic functions, $d = 3$	20
The cylinder	20
Harmonic vector fields, $d = 3$	21
Harmonic functions, $d = 4$	22
Bibliography	24

Todo list

2do	2
Some amazing introduction	4
of which class?	5
This was stated somewhere in Morse1969. Also, what is with the boundary stagnation points	8
Define Betti numbers.	8
complete this section	8
2do	8
More citations.	8
Give outline of proof idea.	8
Give a classical example of a Morse function to determine the betti numbers.	9
Give an outline of the proof.	9
write omega-limit.	10
This figure is not quite accurate as it does not contain loops on the boundary. Also, add E to the figure.	10
use argument with ∇f here to show that extrema can be assumed to be alternating.	12
More precise.	12
elaborate	13
elaborate	13
Check with A	13
One could use the argument principle for Riemann surfaces.	15
Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.	15
Check the signs of this example. Give explanation for why it works.	19
A little more rigour would not harm.	21
Introduce Morse inequalities for $-f$	21
Check that the transition at the boundary is legal.	22

General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Generalise argument principle proof to one inflow, outflow
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.

Some questions

- Should I state Hopf's Lemma?
- Weak formulation - a distraction? \rightarrow Hartman, Wintner

Introduction

Some amazing introduction

Unless otherwise stated we denote by $\Omega \subseteq \mathbb{R}^d$ an open bounded subset of \mathbb{R}^d with boundary $\Sigma = \partial\Omega$. In the following we will work in dimensions $d \in \{2, 3\}$. We denote with

$$f: \overline{\Omega} \rightarrow \mathbb{R}$$

a scalar function of class C^2 . We also denote by

$$u: \overline{\Omega} \rightarrow \mathbb{R}^d$$

a vector field of class C^1 . Often but not always u can be thought of as a *harmonic vector field*, that is u is of type C^1 and fulfils

$$\text{Div } u = 0 \quad \text{and} \quad \text{curl } u = 0.$$

Also often but not always we assume that globally $u = \nabla f$ is a gradient field, implying that f is harmonic. One question we seek to answer during this thesis is the following.

Question 1 (Flowthrough with stagnation point). Does there exist a tube $\Omega \subseteq \mathbb{R}^3$ with flow u through the tube such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. u enters the tube on the one end and exits the tube on the other?

To make the formulation more precise we begin with some general definitions regarding stagnation points and the boundary conditions.

General definitions

In the following we define the emergent and the entrant boundary as in [2, p.282]

Definition 2 (Emergent and entrant boundary). We call a vector $v \in T_x \mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ iff v is not tangent to Σ and directed into the interior of Ω . Analogously if v is not tangent and directed to the exterior we call v *emergent*. We define the *entrant boundary* Σ^- to be the set of boundary points at which u is entrant. Analogously define the *emergent boundary* Σ^+ to be the set of boundary points at which u is emergent. Further define Σ^0 to contain all other boundary points such that we have a decomposition of the boundary

$$\Sigma = \Sigma^- \sqcup \Sigma^0 \sqcup \Sigma^+.$$

Rewrite: Use differential topology language in the following section.

The following are slight generalisation of the definitions given in [1, p.138f], [3, §5] and [2, p.282f] to include harmonic vector fields. We call the zeroes of u *stagnation points*. A stagnation point $x \in \Omega$ is called *non-degenerate* if the derivative $Du(x)$ is invertible. We say that x has *index* k if $Du(x)$ has exactly k negative eigenvalues. u is called *non-degenerate* if all its stagnation points are non-degenerate.

We now define the *interiour type numbers* M_k to be the number of stagnation points of u of index k . The total number of stagnation points is thus given by

$$M = \sum_k M_k.$$

We call Ω a *regular domain* if Σ is a manifold of class C^2 . In the following definition we require Ω to be regular.

Definition 3 (Boundary type numbers). For a boundary point x let

$$\pi_x: \mathbb{R}^d \cong T_x \mathbb{R}^d \rightarrow T_x \Sigma$$

denote the orthogonal projection of a vector at x onto the tangent space of Σ at x . Let

$$\tilde{u} = \pi \circ u|_{\Sigma^-} \in C^1(T\Sigma^-) \quad (1)$$

be the restriction and orthogonal projection of u onto the tangent bundle of Σ^- . We define the *boundary type numbers* μ_k to be the number of stagnation points of u_{Σ^-} on the entrant boundary Σ^- of index k . We further write v_k for the k -th boundary type number of $-u$.

We now call u *regular* iff u and \tilde{u} are non-degenerate and all stagnation points lie in Ω .

The previous definitions translate naturally to f . We call f regular, non-degenerate, et cetera iff $u = \nabla f$ is regular, non-degenerate, et cetera. Similarly we call x a critical point of index k if it is a stagnation point of u of index k .

On assuming non-degeneracy

In the following section we argue that assuming non-degeneracy of u and f is not a great restriction. Given u we define the modification

$$u_\varepsilon = u + \varepsilon \quad (2)$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that u_ε is for almost all choices of ε non-degenerate and can thus be used to approximate a degenerate u . Our approach is to use Thom's theorem which is inspired by the approach in [4, Chapter 6]. In this section we refer to X and Y as generic manifolds.

Definition 4 (Transversality). We call a function $g: X \rightarrow Y$ transverse to a submanifold $A \subseteq Y$ iff for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\text{Image}(dg_x) + T_{g(x)}A = T_{g(x)}Y.$$

of which class?

As an application we make the following observation.

Proposition 5 (Transversal characterisation of non-degeneracy). *Let $u: X \rightarrow TX$ be a differentiable vector field. Then u non-degenerate iff u is transverse to the zero section A of TX .*

Proof. First note that we have that $x \in u^{-1}(A)$ iff $u(x) = 0$ and thus $u^{-1}(A) = C$. Unraveling the definition of transversality we get that u is transverse to the zero section iff for all $x \in C = u^{-1}(A)$ we have that

$$\text{Image}(du_x) + T_{u(x)}A = T_{u(x)}TX. \quad (3)$$

As A is the zero section we have $T_{u(x)}A = 0$ and equation (3) is equivalent to stating that du is of full rank. But du being of full rank at all points in C is equivalent to u being non-degenerate. \square

The following version of Thom's transversality theorem is an adaption (i.e. weakening) of [4, Theorem 2.7] to our needs.

Theorem 6 (Parametric transversality theorem.). *Let E, X, Y be C^r -manifolds (without boundary) and $A \subseteq Y$ a C^r submanifold such that*

$$r > \dim X - \dim Y + \dim A.$$

Let further $F: E \rightarrow C^r(X, Y)$ be such that the evaluation map

$$\begin{aligned} F^{ev}: E \times X &\rightarrow Y \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

is C^r and transverse to A . Then the set

$$\cap (F; A) = \{\varepsilon \in E : F_\varepsilon \text{ is transverse to } A\}$$

is dense.

Proof. See [4, Theorem 2.7] for details. \square

Using proposition 5 we obtain the corollary

Corollary 7. *Let u be a harmonic vector field on the regular domain $\overline{\Omega}$. Then for almost every $\varepsilon \in \mathbb{R}^d$ we have that*

- u_ε given by equation (2) is non-degenerate on Ω and
- \tilde{u}_ε as in (1) is non-degenerate on Σ .

Proof. Set $r = 2$, $E = \mathbb{R}^d$ and $Y = TX$ where we initially assume that $X = \Omega$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F: E &\rightarrow C^\infty(X, TX) \\ \varepsilon &\mapsto u_\varepsilon \end{aligned}$$

We note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq TX$. Then the parametric transversality theorem yields a dense $E_\Omega \subseteq E$ on which F is transverse to A . For this note that $E \times C = F^{-1}(A)$. It then follows for all $(\varepsilon, x) \in F^{-1}(A)$ that

$$\text{Image}\left(dF_{(\varepsilon, x)}^{\text{ev}}\right) = T_x TX \quad (4)$$

since we have that

$$DF_{(\varepsilon, x)}^{\text{ev}} = [Du_x \mid \text{Id}_{d \times d}]$$

is surjective. Proposition 5 now yields that u_ε non-degenerate on E_Ω .

Analogously we set $X = \Sigma$ in the previous proof and replace u_ε with the restriction \tilde{u}_ε . To show that equation (4) holds we resort to the fact that

$$dF_{(\varepsilon, x)}^{\text{ev}} = d(\tilde{u}_\varepsilon(x))_{(\varepsilon, x)} = d\pi \circ (du_\varepsilon(x))_{(\varepsilon, x)}$$

is surjective as a concatenation of surjective functions. Thus there also exists a dense set $E_\Sigma \subseteq \mathbb{R}$ on which \tilde{u}_ε is non-degenerate on Σ . Now the set $E_\Omega \cap E_\Sigma \subseteq \mathbb{R}$ is dense and for every ε in this set has the desired properties. \square

As a consequence we get a version of the results in [2, §2].

We call a boundary point $x \in \Sigma$ *ordinary* iff $u(x)$ is not tangent to Σ .

Proposition 8. *Assume all stagnation points of u lie in Ω and that the stagnation points on Ω are ordinary. Then there exists a positive $\delta > 0$ such that for (Lebesgue) almost every $\varepsilon \in B_\delta$ we have that*

- u_ε is regular
- there is a one to one relation of the stagnation points in Ω and Σ preserving index and the property of being entrant or emergent.

Proof. We follow [2, text]. First choose δ so small that for all stagnation points of u_ε lie in Ω . If there did not exist such a δ we could choose a sequence $\varepsilon_n \rightarrow 0$ in \mathbb{R}^d such that u_{ε_n} had a stagnation point x_n on Σ . Since Σ is compact we can assume that x_n converges to a stagnation point x . But then x is also a stagnation point of u . A contradiction.

We note that since $\overline{\Omega}$ is compact we have that $\nabla u_\varepsilon \rightarrow \nabla u$ uniformly as $|\varepsilon| \rightarrow 0$. \square

Some general remarks

We make the following remarks

Proposition 9. *Let u be non-degenerate. Then the number of stagnation points is finite.*

Proof. Let x be a non-degenerate stagnation point. Since $Du(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq \Omega$ of x on which

u is bijective. Hence x is the only stagnation point in U_x . Let C denote the set of all stagnation points of u . Then the sets U_x together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \quad (5)$$

form an open cover of $\overline{\Omega}$. But $\overline{\Omega}$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is in the finite subcover. Thus it follows that $\#C < \infty$ is finite. \square

As a consequence we obtain the following.

Corollary 10. *For a non-degenerate u the type numbers M_0, \dots, M_d and the boundary type numbers μ_0, \dots, μ_{d-1} are finite.*

Betti numbers

For a formal definition of Betti numbers we refer to ??.

Define Betti numbers.

We do however give some examples of Betti numbers for some domains.

complete this section

- comment on that betti numbers are finite

This was stated somewhere in Morse1969. Also, what is with the boundary stagnation points

The Morse inequalities

We state the Morse inequalities.

Theorem 11 (Morse inequalities). *Let Ω and f be regular. Then we have the inequalities*

$$\begin{aligned} 0 &\leq M_0 + \mu_0 - R_0 \\ 0 &\leq M_1 + \mu_1 - R_1 - (M_0 + \mu_0 - R_0) \\ &\vdots \\ 0 &\leq M_{d-1} + \mu_{d-1} - R_{d-1} - \dots + (-1)^{d-1} (M_0 + \mu_0 - R_0) \\ 0 &= M_d - R_d - (M_{d-1} + \mu_{d-1} - R_{d-1}) + \dots + (-1)^d (M_0 + \mu_0 - R_0). \end{aligned}$$

Proof. See [3, Theorem 10.2]. \square

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [3, Corollary 10.1].

Corollary 12 (Morse inequalities for f harmonic, $d = 2$). *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} 0 &\leq \mu_0 - R_0 \\ 0 &= M + \mu_1 - R_1 - \mu_0 + R_0. \end{aligned}$$

More citations.

Give outline of proof idea.

In dimensions $d = 3$ we obtain [3, Corollary 10.2]

Corollary 13 (Morse inequalities for f harmonic, $d = 3$). *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} 0 &\leq \mu_0 - R_0 \\ 0 &\leq M_1 + \mu_1 - R_1 - \mu_0 + R_0 \\ 0 &= M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 - \mu_0 + R_0. \end{aligned}$$

Give a classical example of a Morse function to determine the betti numbers.

Give an outline of the proof.

On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2, 3\}$.

Proposition 14 (Harmonic vector fields on simply connected domains). *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. $u = \nabla f$ is the gradient field of some function $f: \Omega \rightarrow \mathbb{R}$.
2. f is harmonic.
3. u is in fact C^∞ .
4. The components $u_i = \partial_i f$ are harmonic.

Proof. 1. Since $\text{curl } u = 0$ this is a direct consequence of Stokes theorem.

2. This follows from $\Delta f = \text{Div } u = 0$.
3. This follows from the fact that f is harmonic
4. This follows from $u_i = \partial_i f$.

□

If one considers not necessarily simply connected domains Ω then we obtain the previous properties at least locally.

Harmonic functions, $d = 2$

The following result is essentially a negative to question 1 in $d = 2$ dimensions.

Proposition 15. *Let Ω be homeomorphic to $B_1 \subseteq \mathbb{R}^2$. Let further $f: \overline{\Omega} \rightarrow \mathbb{R}$ be regular harmonic with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \Sigma$ is not connected.*

We shall give two different proofs of this result. One involving level-sets and the other involving invariant manifolds

A proof involving level-sets

Sketch of Proof. Let $y_c = f(x_1)$ and x_1, \dots, x_M be all the critical points such that $f(x_i) = y_c$. We claim that the level set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

can be represented by a multigraph G which divides the boundary Σ into 4 components. To show this let $\gamma_i: (a_i, b_i) \rightarrow C$ for $i \in \{1, \dots, 4\}$ parametrise the curves in C intersecting at x_1 . These can be constructed with the initial value problem

$$\begin{aligned} \gamma' &= (\nabla f)^\perp|_\gamma \\ \gamma(0) &= \gamma_0 \end{aligned}$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1 . We assume that the intervals on which the γ_i are defined are maximal. We thus have for

$$\begin{aligned} \gamma_i^- &= \lim_{t \rightarrow a_i} \gamma(t) \\ \gamma_i^+ &= \lim_{t \rightarrow b_i} \gamma(t) \end{aligned}$$

that $\gamma_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $\nabla f^\perp = 0$. This argument can be applied to all of the x_1, \dots, x_M . We therefore have a situation similar to the one depicted in figure 1.

Thus C can be represented by a multigraph G with vertices v_1, \dots, v_K and edges $e_1, \dots, e_L \subseteq C$. In the following we identify the graph G with its planar embedding in $\overline{\Omega}$. Assume G contains a cycle with vertex sequence v_{i_1}, \dots, v_{i_j} and edges e_{i_1}, \dots, e_{i_j} . Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E . By the maximum principle $f = y_c$ on E and thus $f = y_c$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with the boundary Σ is at least four and thus the boundary Σ is divided into at least four components.

Now choose four neighbouring components $\omega_1, \dots, \omega_4$ as depicted in figure 2 Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle

write omega-limit.

This figure is not quite accurate as it does not contain loops on the boundary. Also, add E to the figure.

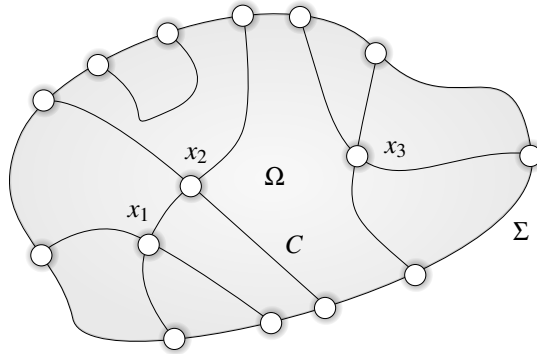


Figure 1: The situation at hand: The edges represent level curves and the interior vertices critical points.

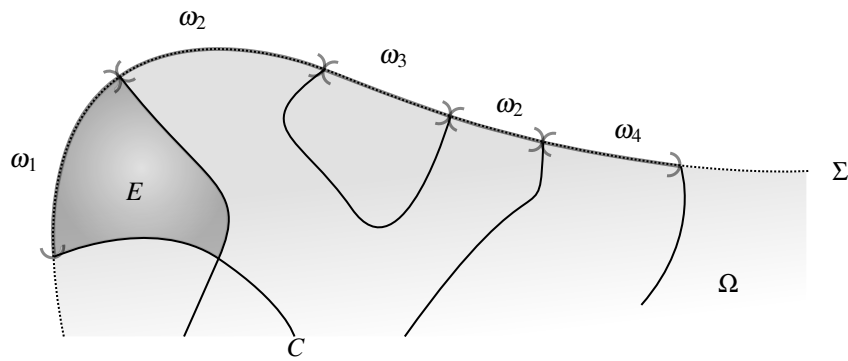


Figure 2: The choice of $\omega_1, \dots, \omega_4$.

yields that ω_1 contains a local maximum or minimum of f since $f = y_c$ is constant on the other boundaries $\partial A \setminus \omega_1$. By the same argument $\omega_2, \dots, \omega_4$ also contain local extrema. Since the $\partial \omega_i$ cannot be extremal points on Σ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. \square

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let x_1, \dots, x_M denote the critical points of f . Let $\lambda_i: (a_i, b_i) \rightarrow \bar{\Omega}$ for $i \in \{1, 2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i: (a_i, b_i) \rightarrow \bar{\Omega}$ for $i \in \{3, 4\}$ be chosen to parametrise the stable manifolds of x_1 . As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\begin{aligned} \lambda_i^- &= \lim_{t \rightarrow a_i} \lambda(t) \\ \lambda_i^+ &= \lim_{t \rightarrow b_i} \lambda(t) \end{aligned}$$

that $\lambda_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\bar{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices v_1, \dots, v_K and edges $e_1, \dots, e_L \subseteq \bar{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Once again we identify the graph with its planar embedding. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle A with vertices x_{i_1}, \dots, x_{i_j} and edges e_{i_1}, \dots, e_{i_j} . The set of cycles forms a partial ordering with respect to the property 'contains another cycle'. We can assume that our chosen cycle A contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to one another. We also note that the edges cannot cross. We can thus describe the trail x_{i_1}, \dots, x_{i_j} by a set of directives of the type

$$(d_1, \dots, d_K) \in \{l, r, s\}^J.$$

Here l, r and s stand for 'left', 'right' and 'straight' respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 3.

An example of the trail 'srsr' is given in figure 4. We now note that cycles of the type r, \dots, r or l, \dots, l cannot occur as we otherwise would have a directed cycle. Thus there

use argument with ∇f here to show that extrema can be assumed to be alternating.

More precise.

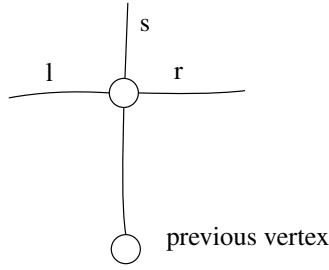


Figure 3: Explanation of the directives 'l', 'r' and 's'.

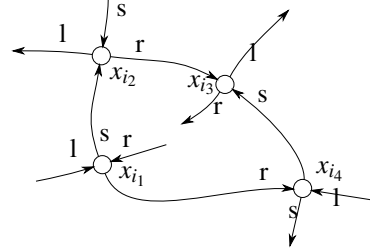


Figure 4: An example for a cycle.

exists a vertex where the chosen direction is s . Without loss of generality this vertex is x_{i_1} . Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1} . Now consider new cycle B starting at x_{i_1} with directives r, \dots, r . Since all vertices of B lie within the cycle A we must at some step reach a vertex on the cycle A. But then cycle B is a new distinct cycle contained in cycle A, a contradiction to the minimality of A. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

Now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur. We now pick a tree \tilde{G} out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each 'neighbouring' leaf of this tree has opposite signage and the claim follows. \square

elaborate

elaborate

The strategy in the above proofs can be generalised to show the following

Conjecture 16. Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with Betti numbers $R_0 = 1$ and R_1 . Let further $f: \Omega \rightarrow \mathbb{R}$ be regular harmonic with M critical points. Assume that $\bar{\Sigma}^- \subseteq \Sigma$ on a given connected component of the boundary Σ consists of at most one connected component. Then we have

$$\frac{4}{3}M \leq R_1 + 1.$$

This inequality can probably be improved considerably.

Check with
A

Harmonic vector fields, $d = 2$

No inflow or outflow

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ iff $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* iff $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can state the following result.

Proposition 17 (Upper bound on M). *Let $d = 2$ and Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1 . Let further $u: \bar{\Omega} \rightarrow \mathbb{R}^2$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$M + 1 \leq R_1.$$

Sketch of proof. As in the second proof of proposition 15 the critical manifolds form a directed multigraph. Since no critical manifold can intersect with the boundary each vertex of the graph has degree 4 and we thus have $2M$ edges. Now we obtain with Euler's polyhedron formula for a planar graph with multiple components

$$\begin{aligned} \# \text{ minimal cycles} &= \# \text{ faces} - 1 \\ &= 1 + \# \text{ components} - \# \text{ vertices} + \# \text{ edges} - 1 \\ &\geq 1 + 1 - M + 2M - 1 = M + 1 \end{aligned}$$

Here we use the term ‘minimal’ as in the second proof of proposition 15. Note that each minimal cycle must contain a hole of the domain since else we could restrict u to a simply connected region containing this cycle. Then by proposition 14 u would correspond to the gradient of a harmonic function in this region and we would obtain a contradiction as in the proof of proposition 15. Hence the number of minimal cycles is a lower bound on the number of holes R_1 of the domain. \square

In fact using the Morse inequalities we can obtain the stronger result.

Proposition 18. *Let Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1 and let $u: \bar{\Omega} \rightarrow \mathbb{R}^2$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$M + 1 = R_1$$

Sketch of proof. We slit the domain such that it is homeomorphic to the disk. By proposition 14 u is the gradient of a harmonic function f on this new domain. f has no critical points on the boundary Σ and fulfils on the cuts the conditions

$$\mu_0 = v_1 \quad \mu_1 = v_0 \tag{6}$$

since every entrant critical point is also an emergent critical point on the other side of the cut of shifted index. We have for this new cut domain the Morse inequalities for f and $-f$

$$M + \mu_1 - R_1 - \mu_0 + R_0 = 0 \tag{7}$$

$$M + v_1 - R_1 - v_0 + R_0 = 0. \tag{8}$$

Adding equations (7) and (8) and using the relation (6) we obtain

$$2(M - R_1 + R_0) = 0$$

from which the claim follows. \square

We now give an alternative proof using the argument principle.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition ?? u is the gradient of a harmonic function f on this new domain. Let $h \in \text{Hol}(\mathbb{C})$ be a holomorphic function such that $h' = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ . We now determine the change of argument $\arg h'$ along γ . For this consider first the parts of γ traversing the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other the contribution in the change of $\arg h'$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h'$ is 2π for every hole in the domain since $h' = u$ is tangent to Σ and traverses the holes clockwise direction. Similarly the contribution to the change in argument of $\arg h'$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have R_1 holes in the domain the total change of $\arg h'$ as γ traverses Σ is $2\pi(R_1 - 1)$. Since h has no poles it follows from the argument principle (see for example [5, Chapter VIII]) that

$$2\pi(R_1 - 1) = \int_{\gamma} d \arg(h'(z)) = 2\pi M \quad (9)$$

From this the claim follows. \square

In the following we would like to give examples for harmonic vector fields. In order to do this we define two differential operators for $d = 2$ by

$$\nabla^{\perp} f = \text{Curl } f = \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix}$$

and

$$\text{curl } u = -\partial_1 u_2 + \partial_2 u_1$$

Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

The following proposition gives us a recipe to generate harmonic vector fields in $d = 2$ dimensions.

Proposition 19. *Let $\psi: \Omega \rightarrow \mathbb{R}$ be harmonic then $\nabla^{\perp} \psi$ is a harmonic vector field.*

Proof. Since $\text{Div } \nabla^{\perp} \psi = 0$ we have

$$\text{Div } u = \text{Div } \nabla^{\perp} \psi = 0$$

One could use the argument principle for Riemann surfaces.

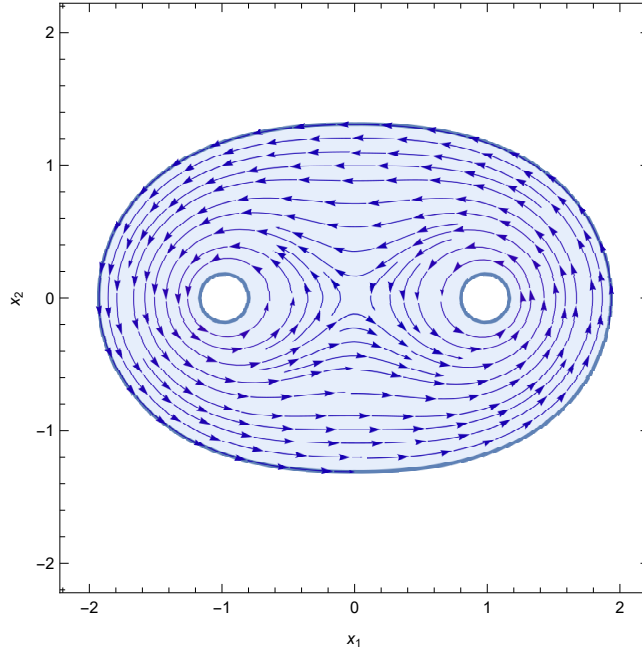


Figure 5: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-1, 1])$. Here ψ is given by equation (10).

and one calculates

$$\operatorname{curl} u = \operatorname{curl} \nabla^\perp \psi = -\Delta \psi = 0.$$

□

The function ψ is also called a stream function.

We now give an example of a harmonic vector field without inflow or outflow and with one critical point. For this consider the stream function

$$\begin{aligned} \psi: \mathbb{R}^2 \setminus (\{-e_1, e_1\}) &\rightarrow \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + \Phi_2(x + e_1) \end{aligned} \quad (10)$$

where

$$\Phi_2 = \log(|\cdot|)$$

is a multiple of the fundamental solution of the laplace equation on \mathbb{R}^2 and $e_i = \delta_i$ are the unit vectors. Figure 5 indicates that $u = \nabla^\perp \psi$ has the desired properties.

In a second example given by [6] we fix the domain rather than the function. For

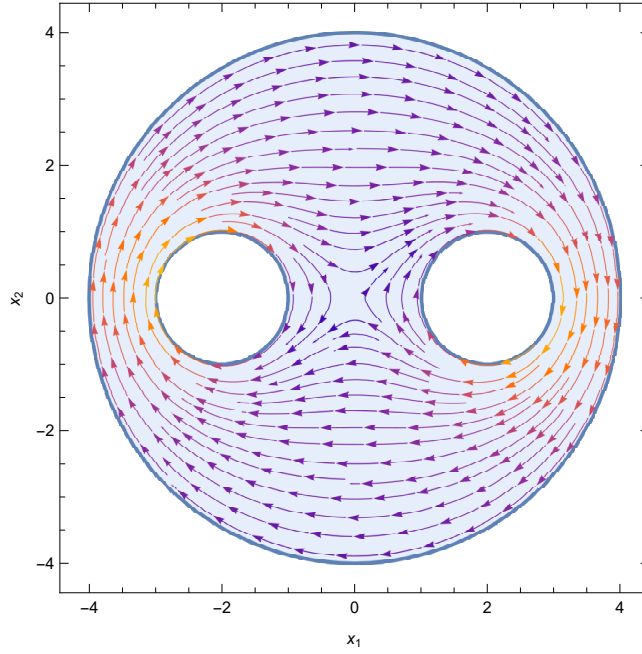


Figure 6: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (11).

this set $\overline{\Omega} = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We then have the system

$$\begin{aligned} \Delta \psi &= 0 \quad , \text{ on } \Omega \\ \psi &= 0 \quad , \text{ on the outer ring } 4S^1 \\ \psi &= 1 \quad , \text{ on the inner rings } S^1(-2e_1) \cup S^1(2e_1) \end{aligned} \tag{11}$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 6.

An example of inflow on one side and outflow on the other

In the following we aim to give examples of domains in $d = 2$ dimensions for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component. For this consider first the stream function

$$\begin{aligned} \psi: \mathbb{R}^2 \setminus (\{-e_1, e_1\}) &\rightarrow \mathbb{R}^2 \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{12}$$

Figure 7 indicates that $u = \nabla^\perp \psi$ fulfills the requirements.

Now we would like to have a harmonic vector field similar to the example with two holes with inflow on the one side and outflow on the other. For this consider the streamline

$$\begin{aligned} u: \mathbb{R}^2 \setminus (\{-e_1, e_1\}) &\rightarrow \mathbb{R}^2 \\ x &\mapsto \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1 \end{aligned} \tag{13}$$

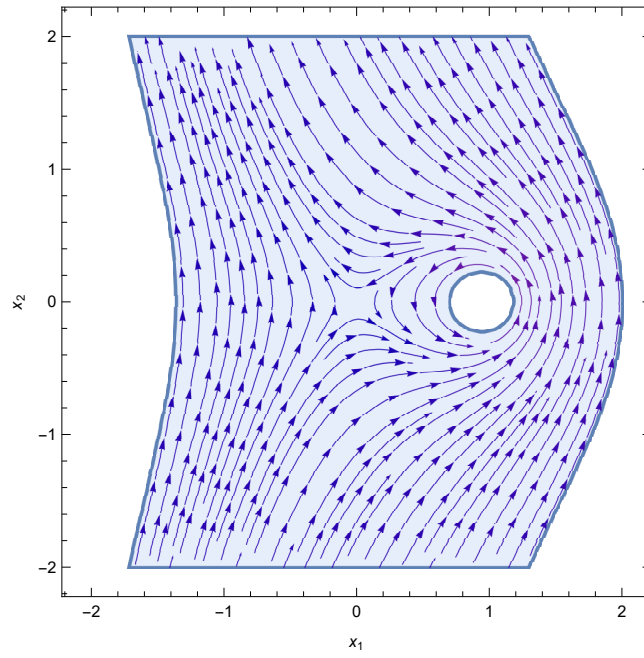


Figure 7: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.5, 2]) \cap \mathbb{R} \times [-2, 2]$. Here ψ is given by equation (12).

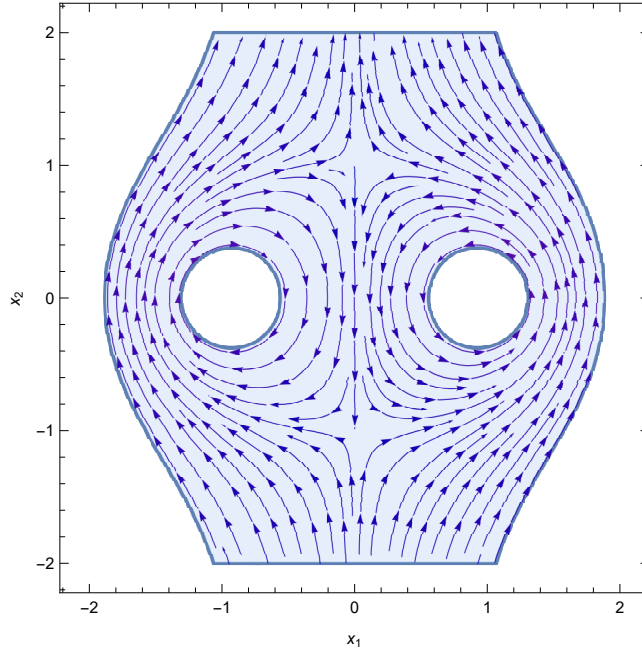


Figure 8: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap \mathbb{R} \times [-2, 2]$. Here ψ is given by equation (13).

Figure 8 indicates that $u = \nabla^\perp \psi$ is the function we are looking for.

In another example given by [6] we once again fix the domain rather than the function. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We now have the system

$$\begin{aligned}
 \Delta \psi &= 0 && , \text{ on } \Omega \\
 \psi &= 0 && , \text{ on the outer ring } 4S^1 \\
 \psi &= -1 && , \text{ on the left inner ring } S^1(-2e_1) \\
 \psi &= 1 && , \text{ on the right inner ring } S^1(2e_1)
 \end{aligned} \tag{14}$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 9.

Check the signs of this example. Give explanation for why it works.

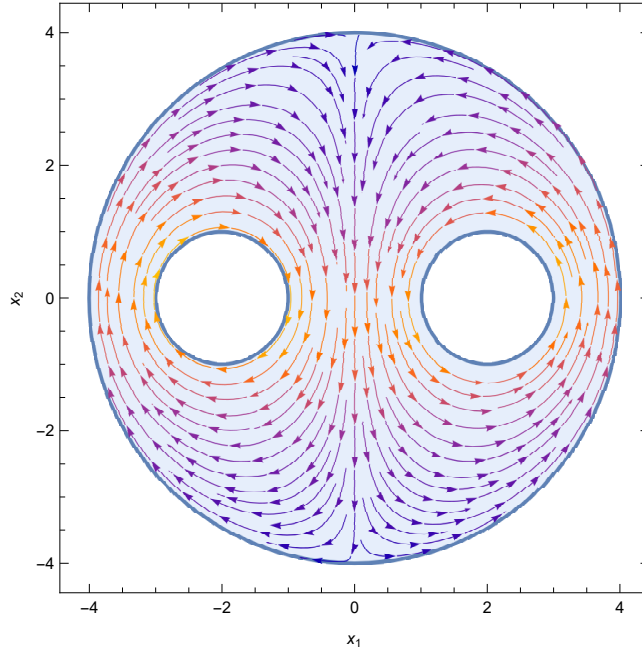


Figure 9: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (14).

Harmonic functions, $d = 3$

The cylinder

The following proof comes from [6]

Proposition 20. *Let $\Omega = (0, 1) \times B_1 \subseteq \mathbb{R}^3$ be the cylinder. Let further $f: \overline{\Omega} \rightarrow \mathbb{R}$ be regular harmonic with no inflow or outflow on the sides $\partial(0, 1) \times B_1$, no outflow on $\{0\} \times B_1$ and no inflow on $\{1\} \times B_1$. Then f cannot have a critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ . Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0, 1) \times S^1$ such that $\partial_1 f(x)$ is minimal on $\overline{\Omega}$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0,$$

a contradiction. □

Harmonic vector fields, $d = 3$

We obtain as a quick consequence of the hairy ball theorem

Proposition 21. *Let Ω have Betti numbers R_0 , R_1 and R_2 . Let $u: \overline{\Omega} \rightarrow \mathbb{R}$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$R_2 \leq R_1.$$

Proof. Assume not. Since Ω has R_2 bubbles and R_1 holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no inflow or outflow on Γ we have that the restriction $u|_{\Gamma} \in T\Gamma$ is a vector field on Γ . Since u is regular $u|_{\Gamma}$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. \square

A little more rigour would not harm.

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 22. *Let Ω have Betti numbers R_0 , R_1 and R_2 . Let $u: \overline{\Omega} \rightarrow \mathbb{R}$ be a regular harmonic vector field without inflow or outflow. Then we have the following relation for critical points of u*

$$M_2 = M_1$$

Attempt at proof. As in the two-dimensional case we begin by cutting up the domain such that the slit domain is homeomorphic to the ball with bubbles. Once again by proposition 14 u is the gradient of a harmonic function u on this new domain. f has no critical points on the boundary Σ and on the cut boundary it fulfils the conditions

$$\mu_0 = \nu_2 \quad \mu_1 = \nu_1 \quad \mu_2 = \nu_0 \quad (15)$$

by the same reasoning. We now have the Morse inequalities for f and $-f$

$$M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 + \mu_0 - R_0 = 0 \quad (16)$$

$$M_1 + \nu_2 - R_2 - M_2 - \nu_1 + R_1 + \nu_0 - R_0 = 0 \quad (17)$$

It then follows by subtracting equation (17) from (16) and using relations (15) that

$$2(M_2 - M_1) = 0.$$

\square

Introduce Morse inequalities for $-f$.

Harmonic functions, $d = 4$

Define the harmonic function

$$\begin{aligned} f: B_1 \subseteq \mathbb{R}^4 &\rightarrow \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2. \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in \mathbb{R}^4 with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \quad (18)$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_{\pm}: B_1 \subseteq \mathbb{R}^3 &\rightarrow \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \quad (19)$$

with inverses $\psi_{\pm} = (\phi_{\pm})^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^{\top}$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^{\top}.$$

Thus we have $x \in \Sigma^{\pm}$ iff

$$0 < \pm \nabla f \cdot n = \pm 2(x_1^2 + x_2^2 - x_3^2 - x_4^2)$$

Using condition (18) we obtain the equivalent condition

$$0 < \pm 1 - 2(x_3^2 + x_4^2)$$

Define the cylinder

$$C = \{\bar{x} \in \mathbb{R}^3: x_3^2 + x_4^2 < 1/2\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (19) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_{\pm}(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_{\pm}(\Sigma^+).$$

Analogously we have

$$B_1 \setminus C = \psi_{\pm}(\Sigma^-).$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 10.

Check that the transition at the boundary is legal.

□

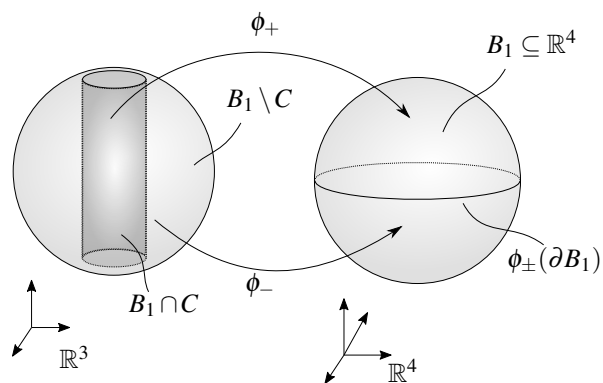


Figure 10: Visualisation of the situation.

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