Stagnation Points of Harmonic Vector Fields and the Domain Topology

Some Applications of Morse Theory

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Definition (Harmonic vector field)

Let $X \subset \mathbb{R}^d$ be a suitable domain. A function $f\colon X \to \mathbb{R}$ is *harmonic* if it satisfies the equation

$$0 = \Delta f = \sum_j \partial_j^2 f\,.$$

A vector field $u\colon X \to \mathbb{R}^d$ is *harmonic* if it is locally the gradient of a harmonic function, that is locally $u = \nabla f$.

Why harmonic vector fields?

- gravitational field in classical mechanics
- steady state heat flow
- irrotational flow of an inviscid incompressible medium
- electrostatic field in vacuum
- magnetostatic field in vacuum

Definition (Stagnation point)

We call the zeros of a vector field $u\colon X \to \mathbb{R}^d$ *stagnation points*. The stagnation points of $u = \nabla f$ are called *critical points* of $f\colon X \to \mathbb{R}$.

Why stagnation points?

$$\text{vector field topology} \xleftrightarrow{\text{Morse theory}} \text{stagnation points}$$

The following question is inspired by (Alber 1992, "Existence of threedimensional, steady, inviscid, incompressible flows with nonvanishing vorticity"):

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

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Answer

- For $d = 2$ dimensions: Not possible, see (Koppenhöfer 2024, Prop. 4.1). The proof uses Morse theory but one can also see this with the argument principle from complex analysis.

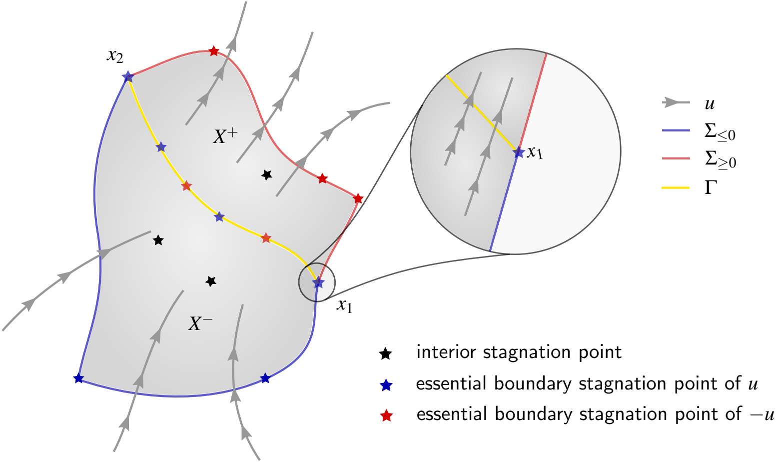

Figure: Idea of the proof involving Morse theory.

But if one allows for holes in $d=2$ dimensions it becomes possible.

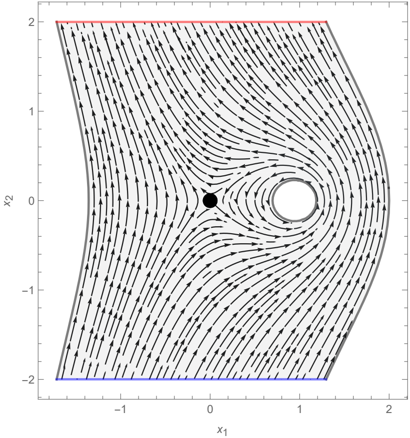

Figure: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap (\mathbb{R}\times[-2,2])$.
Here $\psi := \Phi_2(x-e_1)+x_1$.

Question (Flowthrough with stagnation point)

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1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- For $d = 2$ dimensions: No.
- For $d = 2$ dimensions for a domain with holes: Possible.

Proposition (Negative answer for cylinders, (Koppenhöfer 2024, Prop. 5.1))

Let $X = [0,1] \times \overline{U} \subset \mathbb{R}^d$ *be a cylinder where* $U \subset \mathbb{R}^{d-1}$ *is a bounded open set with* C^1 *boundary. Let further* $f\colon X \to \mathbb{R}$ *be non-constant and harmonic such that the sides* $[0,1] \times \partial U = \Sigma^0$ *are the tangential boundary, the lid* $\{0\} \times U = \Sigma^{\leq 0}$ *is the entrant boundary and the lid* $\{1\} \times U = \Sigma^{\geq 0}$ *is the emergent boundary. Then* f *cannot have an interior critical point.*

Proof.

See blackboard or (Koppenhöfer 2024). □

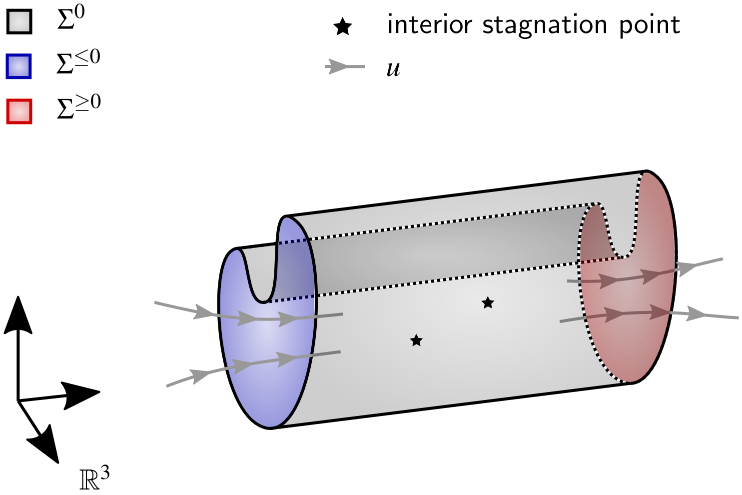

Figure: This kind of situation is not possible.

Question (Flowthrough with stagnation point)

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1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- For $d = 2$ dimensions: No.
- For $d = 2$ dimensions for a domain with holes: Possible.
- For the cylinder: No.

Example (Connected entrant boundary in $d = 4$ dimensions, (Koppenhöfer 2024, Ex. 4.7))

Consider as domain $X = B_1 \subset \mathbb{R}^4$ the unit ball and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned}$$

This has a critical point at the origin. It is shown in (Koppenhöfer 2024, Prop. 4.8) that the entrant and emergent boundaries are in fact connected.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- For $d = 2$ dimensions: No.
- For $d = 2$ dimensions for a domain with holes: Possible.
- For the cylinder: No.
- In $d = 4$ dimensions: Yes.

So what about $d = 3$ dimensions?

Definition (Interior type numbers)

Let $u\colon X \to \mathbb{R}^d$ be a vector field and x an interior stagnation point of u. We say that x is *non-degenerate* if the derivative $Du(x)$ is bijective. If in addition u is irrotational we say that x has *Morse index* k if $Du(x)$ has exactly k negative eigenvalues. The *interior type number* M_k denotes the number of interior stagnation points of index k.

Example (Interior type numbers)

Consider the harmonic function $f = x_1^2 + x_2^2 - x_3^2 - x_4^2$. One calculates

$$u = \nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

which has the sole stagnation point $x = 0$. Furthermore

$$Du = 2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

which implies that $x = 0$ is non-degenerate and has Morse index 2. This implies $M_2 = 1$ and $M_k = 0$ for $k \neq 2$.

Proposition (Condition on the interior type numbers, (Koppenhöfer 2024, Prop. 5.2))

Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with corners and let $f: X \to \mathbb{R}$ be a harmonic function without irregular stagnation points and such that f is non-degenerate on the interior $\mathrm{int}(X)$. Let $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ be a disjoint decomposition of the boundary into simply connected nonempty sets such that we have for the strictly entrant boundary $\Sigma^- \subseteq \Sigma_{\leq 0}$ and for the strictly emergent boundary $\Sigma^+ \subseteq \Sigma_{\geq 0}$. Additionally we require that $\gamma := \partial \Sigma_{\leq 0}$ is a one-dimensional manifold diffeomorphic to the circle S^1. Then we have the relation $M_1 = M_2$ between the interior type numbers.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- For $d = 2$ dimensions: No.
- For $d = 2$ dimensions for a domain with holes: Possible.
- For the cylinder: No.
- In $d = 4$ dimensions: Yes.
- In $d = 3$ dimensions: Has to have an even number of stagnation points.

Example (A harmonic function with interior critical points and connected entrant and emergent boundaries, (Koppenhöfer 2024, Ex. 5.3))

For $d = 3$ dimensions we have for r sufficiently large the example $X = B_r$, $u = \nabla f$ with

$$f = \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_1 x_2^2}{2} + x_1 x_2^2 + x_2 x_3 \,.$$

This is

- harmonic
- has critical points at 0 and e_1
- connected entrant and emergent boundaries.

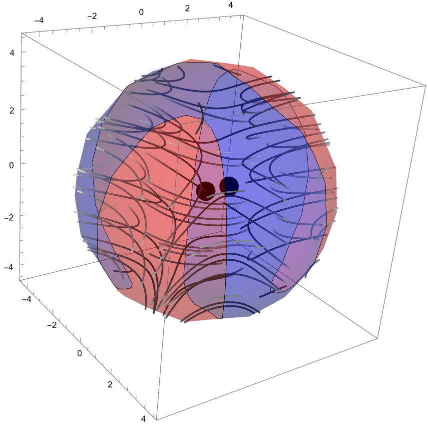


Figure: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

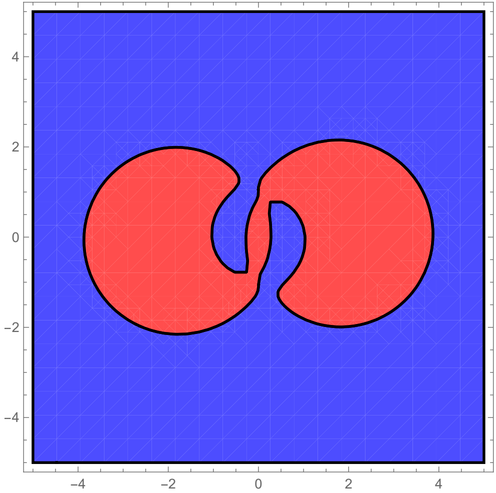
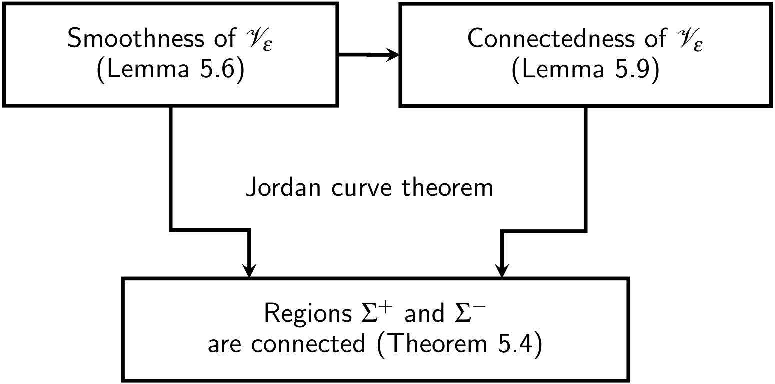


Figure: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

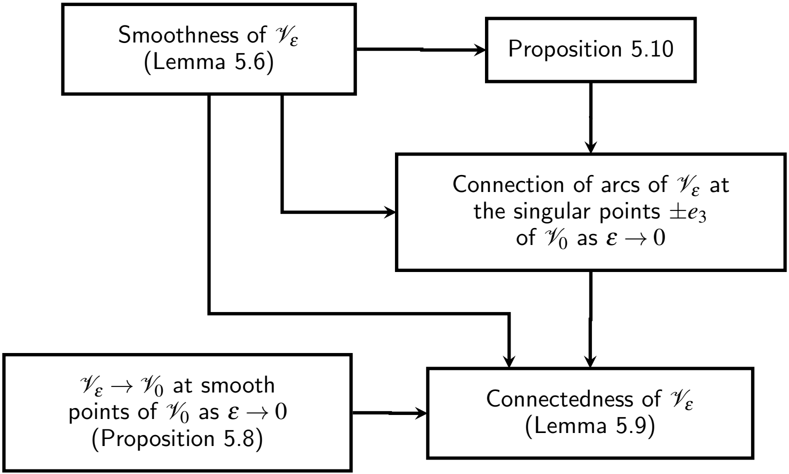
Let $\varepsilon = 1/r$ and let $r\mathscr{V}_\varepsilon$ be the curve separating Σ^+ and Σ^-.



Proof of smoothness (Lemma 5.6)

- Here smoothness means that $\mathscr{V}_\varepsilon$ is a differentiable manifold $\rightarrow$Jacobi criterion
- $\mathscr{V}_\varepsilon$ is an algebraic variety $\rightarrow$use Gröbner bases to show that Jacobi criterion is fulfilled for sufficiently large $r > 0$

Proof of connectedness (Lemma 5.9)



Remark (Thickening Σ^0)

One can perturb this solution to show that there exists a harmonic vector field on B_r with interior stagnation point such that Σ^+ and Σ^- have positive distance from one another and are connected. This is done in (Koppenhöfer 2024, Ex. 5.12).

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Answer

- For $d = 2$ dimensions: No.
- For $d = 2$ dimensions for a domain with holes: Possible.
- For the cylinder: No.
- In $d = 3$ dimensions: Yes.
- In $d = 3$ dimensions: Has to have an even number of stagnation points.

The following question is inspired by (Lortz 1970):

Question (Harmonic vector fields without inflow or outflow)

Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

The following answers the question in $d = 2$ dimensions:

Proposition (Condition on the number of stagnation points, (Koppenhöfer 2024, Prop. 6.5))

Let $X \subset \mathbb{R}^2$ be a compact planar manifold with smooth boundary and let $u \colon X \to TX$ be a harmonic vector field without boundary stagnation points and such that the interior stagnation points are isolated. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points counting multiplicities and $\chi(X)$ is the Euler characteristic of X.

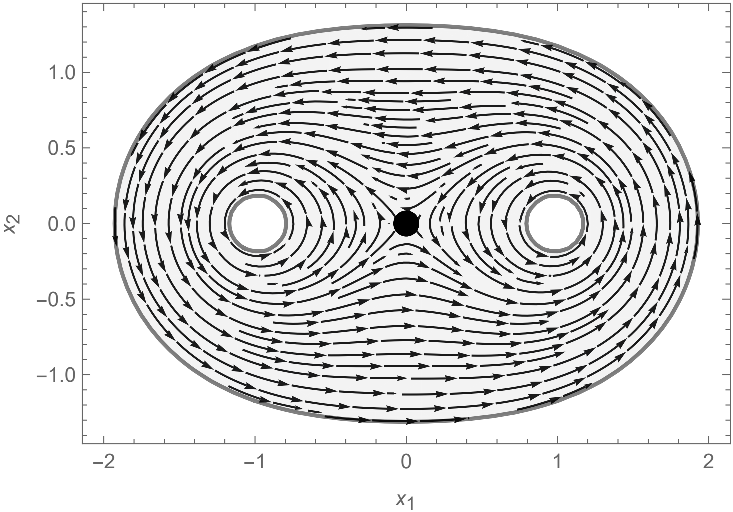

Figure: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here $\psi := \Phi_2(x-e_1)+\Phi_2(x+e_1)$.

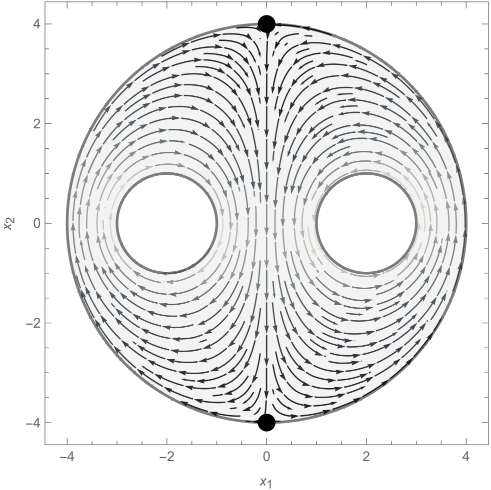

Figure: A plot of $u = \nabla^{\perp}\psi$ in the domain X given as in (Koppenhöfer 2024, Ex. 6.9).

Proposition (Condition on the domain topology, (Koppenhöfer 2024, Prop. 6.11))

Let X be a compact orientable odd dimensional manifold with smooth boundary. Let further $u\colon X \to TX$ be a smooth vector field with isolated stagnation points on the interior and without boundary stagnation points. Then the Euler characteristic of the domain $\chi(X) = 0$ has to vanish.

Corollary (Condition on the type numbers and domain, (Koppenhöfer 2024, Cor. 6.12))

Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with smooth boundary and let further $u \colon X \to TX$ be a Morse harmonic vector field with no inflow or outflow through the boundary. Then we have the condition $M_1 = M_2$ between the interior type numbers and the Euler characteristic of the domain $\chi(X) = 0$ vanishes.

Summary

- Is it possible to have an interior stagnation point and connected entrant and emergent boundaries?
  - $d = 2$ dimensions: No, unless one allows for holes in the domain
  - $d = 3$ dimensions: Yes and $M_1 = M_2$, but not possible for cylinders
  - $d = 4$ dimensions: Yes by a simple example
  - techniques: Morse theory for manifolds with corners
- What is the relation between interior stagnation points and the domain topology for harmonic vector fields without inflow or outflow through the boundary?
  - $d = 2$ dimensions: $M = -\chi(X)$
  - $d = 3$ dimensions: $\chi(X) = 0$ and $M_1 = M_2$.
  - techniques: Morse index theorem

Sources I

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Thank you for your attention.