Some title

Master Thesis

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Some amazing introduction

Some general remarks

On assuming non-degeneracy

The case $n = 2$

Claim. *Let Ω be homoemorph to $B_1 \subseteq \mathbb{R}^2$. Let further $f\colon \overline{\Omega} \to \mathbb{R}$ be harmonic and admissable as in Morse with critical point $x_0 \in \Omega$. Then $\Sigma^- \subseteq \partial\Omega$ consists fo at least 2 components.*

A proof involving level-sets

Proof. Let $y_c = f(x_0)$ and $x_0, \ldots, x_N$ be all the critical points s.t. $f(x_i) = y_c$. We claim that the set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

divides $\partial\Omega$ into 4 components. To show this let $\gamma_i\colon (a_i, b_i) \to C$ parametrise the curves in C intersecting at x_0. These can be constructed with the initial value problem

$$\gamma' = (Df)^{\perp}\big|_{\gamma}$$
$$\gamma(0) = \gamma_0$$

where $\gamma_0 \in C$ is chosen sufficiently near x_0. Without loss of generality the intervals on which the γ_i are defined are maximal. We thus have for

$$\gamma_i^- = \lim_{t \to a_i} \gamma(t)$$
$$\gamma_i^+ = \lim_{t \to b_i} \gamma(t)$$

that $\gamma_i^{\pm} \in ([)c]x_0, \ldots, x_N, \partial\Omega$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $Df^{\perp} = 0$. We therefore have a situation similar to the one depicted in [TODO: make figure]. One sees that C can thus be represented by a graph G with vertices $v_0, \ldots, v_M$ and edges $e_0, \ldots, e_L \subseteq C$. Assume G contains a cycle with vertex sequence $v_{i_1}, \ldots, v_{i_K}$ and edges $e_{i_1}, \ldots, v_{i_K}$. Then

$$\partial E = \bigcup_k e_{i_k} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E.By the maximum principle $f = 0$ on E and thus $f = 0$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with $\partial\Omega$ is at least 4 and thus $\partial\Omega$ is divided into 4 components. Now choose 4 neighbouring components as depicted in figure [TODO: insert figure]. Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle yields that ω_1 contains a local maximum or minimum of f. Analogously $\omega_2, \ldots, \omega_4$ also contain local extrema. Since the $\partial\omega_i$ cannot be extremal points on $\partial\Omega$ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. $\square$

A proof involving invariant manifolds

Proof. Let $x_0, \dots, x_N$ denote the critical points of f. Let $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1, 2\}$ parametrise the unstable manifolds of the critical point x_0 and $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1, 2\}$ be chosen to parametrise the stable manifolds of x_0. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^{\pm} \in ([)c]x_0, \dots, x_N, \partial\Omega$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $Df \perp = 0$. Thus all invariant manifolds of all critical points form an directed graph G with vertices $v_1, \dots, v_M$ and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Our graph is in fact acyclic directed. TODO: continue proof $\square$

The case $n = 3$

The case of a single critical point

The case of dimensions $n = 4$

Define the harmonic function

$$
\begin{aligned}
f \colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\
x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,.
\end{aligned}
$$

This has the origin as a stagnation point. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$
\sum_i x_i^2 = 1 \tag{1}
$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$
\begin{aligned}
\Sigma^\pm \colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\
\bar{x} &\mapsto x, \pm x_1 \geq 0
\end{aligned} \tag{2}
$$

We now calculate the derivative of f

$$
Df = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top
$$

and the normal to ∂B_1

$$
n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .
$$

Thus we have $x \in \Sigma^\pm$ iff

$$
0 < \pm Df \cdot n = \pm 2\big(x_1^2 + x_2^2 - x_3^2 - x_4^2\big)
$$

Using the condition (1) we obtain the equivalent condition

$$
0 < \pm 1 - 2\big(x_3^2 + x_4^2\big)
$$

Define the cylinder

$$
C = \left\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \right\} = \mathbb{R} \times B_{1/\sqrt{2}}
$$

If we return to our parametrisation (2) we see that we have $\bar{x} \in B_1 \cap C$ iff $\Sigma^\pm(x) \in \Sigma^+$. Analogously we have that $\bar{x} \in B_1 \setminus C$ iff $\Sigma^\pm(x) \in \Sigma^-$. Taking into account that $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$ the claim follows from a picture.

(TODO: Elaborate here with some argument with homeormophisms) $\square$

Bibliography

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