Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

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Contents

Todo list

General TODOs
- Check for typos.
- use $=:$ for definitions.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [23] and [20]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.
- Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

Some questions
- Should I state Hopf's Lemma?

1 Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\Omega = \text{int}(X)$. In the following we will work in dimensions $d \in \{2, 3\}$. Unless otherwise stated we denote by

$$f \colon X \to \mathbb{R}$$

a C^2 function on X. Often f will be assumed to be harmonic. We also denote by

$$u \colon X \to \mathbb{R}^d$$

a vector field of class C^1. In the following we often assume that u is in fact *harmonic*, that is u fulfils $\text{Div}\, u = 0$ and $\text{curl}\, u = 0$. Often but not always we assume that in fact $u = \nabla f$ is a gradient field. One question we seek to answer in this thesis is the following:

Question 1.1 (Flowthrough with stagnation point)**.** Does there exist a region $X \subseteq \mathbb{R}^3$ homeomorphic to a ball with flow u through the region such that

1. u is a harmonic vector field

2. u has an interior stagnation point

3. the boundary on which u enters the region is simply connected?

The answer for this will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Another question we will consider is of the type:

Question 1.2 (stagnation points of harmonic vector fields without inflow or outflow)**.** Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary behaviour.

General definitions

We start by requiring some regularity for the boundary of X. More precisely, we require X to be a compact Riemannian manifold with corners:

Definition 1.3 (Manifolds with corners, [9])**.** We introduce the notation

$$H_j^d = \mathbb{R}_{\geq 0}^j \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d \,.$$

where $j \in \{0,\dots,d\}$. A *manifold with (convex) corners* is a topological space X together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exists an open neighbourhood U_x of x, a number $j = j(x)$ and a diffeomorphism $\phi : U_x \to H_j^d$ in $\mathscr{A}$ with $\phi(x) = 0$. We further define for $k \in \{0,\dots,d\}$ a collection of sets

$$X_k = \{x \in X : \; j(x) = d - k\}, \tag{1.1}$$

which form a stratification of X.

More generally we give the definition of a stratification as

Definition 1.4 (Stratified space, [9])**.** Let X be a topological space. A *stratum* is a subspace $X_j \subseteq X$, $j \in \mathscr{J}$, indexed by a partially ordered set $\mathscr{J}$ such that

1. each X_j is a manifold (without boundary) of dimension $n = n(j)$

2. $X = \bigcup_j X_j$

3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$ iff $j \prec k$.

The pair of X and the collection of strata is called a *stratified space*. In the case that $X_j \subseteq \overline{X}_k$ and additionally $n(k) = 0$ or $n(k) = n(j) + 1$ we will write $X_j \precsim X_k$ or, abusing notation, we will write $X_k = X_{j+1}$.

In the case that the stratification arises through relation (1.1) we have precisely $X_j \precsim X_{j+1}$ for $j \in \{1,\dots,d\}$ and $X_0 \precsim X_0$. Note that in general for a given stratum X_j the stratum X_{j+1} such that $X_j \precsim X_{j+1}$ need not be unique. In the following we assume, unless otherwise stated, that the stratification is finite, that is $\# \mathscr{J} < \infty$ and that the interior Ω corresponds to a single stratum.

For completeness we also give the definition of the contingent cone for a stratification X_j of X:

Definition 1.5 (contingent cone, [13, Def. 4.6])**.** We denote the *(Bouligand) contingent cone* for a set $Y \subseteq X$ at $x \in \overline{Y}$ by $C_x Y$. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exists sequences $\lambda_n \to 0$ and $x_n \to x$ in Y such that

$$\lim_n \lambda_n (x_n - x) = v \,.$$

For clarification we give an example

Example 1.6 (Cubical domain)**.** Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$. Then we have a stratification given by

$$X_0 = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$
$$X_1 = I \times \{-1\} \cup I \times \{1\} \cup \{-1\} \times I \cup \{1\} \times I$$
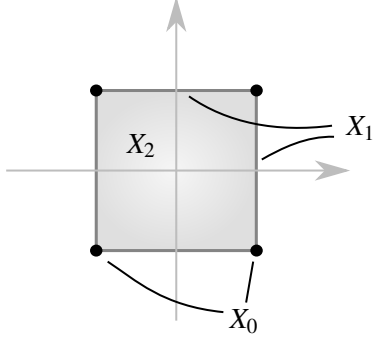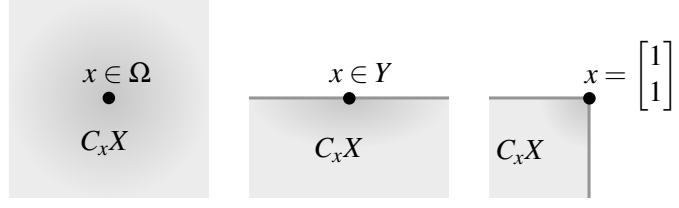$$X_2 = I \times I$$

Figure 1.1: A stratification of X.

Figure 1.2: The contingent cones for various $x \in X$.

where $I = (-1,1) \subseteq \mathbb{R}$. The stratification is depicted in figure 1.1. For an interior point $x \in X_2$ we have the contingent cone $C_x X = T_x \mathbb{R}^d$. For a boundary point $x \in Y = I \times \{1\} \subset X_1$ we have the contingent cone

$$C_x X = \left\{ v \in T_x \mathbb{R}^2 : v \cdot n \leq 0 \right\}$$

where the basis vector $n = e_2$ is the outer unit normal. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top \in X_0$ we have

$$C_x X = \left\{ v \in T_x \mathbb{R}^2 : v_1 \leq 0 \text{ and } v_2 \leq 0 \right\}.$$

The situation is depicted in figure 1.2. The contingent cone on the other parts of the square $\Sigma = \partial X$ is given by similar formulas.

In the following we define the emergent and the entrant boundary in a way that generalises [20, p.282] for stratified manifolds.

Definition 1.7 (Emergent and entrant boundary)**.** We call a vector $v \in T_x \mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ if

1. v points into Ω or

2. v lies in the dual cone of the contingent cone $C_x X$, that is

$$v \in (C_x X)^* = \left\{ w \in T_x^* X : \langle w, w' \rangle \geq 0 \text{ for all } w' \in C_x X \right\}.$$

We call v *strictly entrant* if in addition v is not tangential to Σ and if $v \in (C_x X)^*$ then v lies in the relative interior $\operatorname{rel int}(C_x X)^*$. Analogously v is *(strictly) emergent* if $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which u is entrant. We define the *strictly entrant boundary* Σ^- to be the set of strictly entrant boundary points of u. In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* Σ^+. Further define the *tangential boundary* Σ^0 to be

$$\Sigma^0 = \Sigma^{\leq 0} \cup \Sigma^{\geq 0} \setminus \left(\Sigma^+ \cup \Sigma^- \right) \subseteq \Sigma. \tag{1.2}$$

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We would now like to illustrate the preceding definitions.

Example 1.8. Consider the domain to be the cube $X = [-1, 1]^2 \subseteq \mathbb{R}^2$ and the harmonic function

$$f \colon X \to \mathbb{R}$$
$$x \mapsto x_1^2 - x_2^2. \tag{1.3}$$

This induces the harmonic vector field $u = \nabla f$, or more precisely

$$u \colon \Omega \to \mathbb{R}^3$$
$$x \mapsto 2 \begin{bmatrix} x_1 & -x_2 \end{bmatrix}^\top. \tag{1.4}$$

For a boundary point $x \in I \times \{1\}$ the dual cone of the contingent cone $C_x X$ is given by

$$(C_x X)^* = \{-r e_2 \colon r \geq 0\}.$$

Now we have that $x \in \Sigma^{\leq 0}$ if

$$0 \geq n \cdot u = -2x_2$$

which is always fulfilled and thus $I \times \{1\} \subseteq \Sigma^{\leq 0}$. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ the dual of the contingent cone is given by

$$(C_x X)^* = C_x X.$$

Since $v = u(x) = 2 \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ we have that $v \notin (C_x X)^*$ and $-v \notin (C_x X)^*$ and thus $x \notin \Sigma^{\geq 0} \cup \Sigma^{\leq 0}$. By analogous argumentation on the other sides of the square $\Sigma = \partial X$ one obtains that

$$\Sigma^{\leq 0} = I \times \{1\} \cup I \times \{-1\}$$
$$\Sigma^{\geq 0} = \{1\} \times I \cup \{-1\} \times I.$$

A plot of the sets can be seen in figure 1.3.

Given a vector field $u \colon X \to \mathbb{R}^d$ and a stratification X_j of X we can construct for every $j \in \mathcal{J}$ a vector field

$$u_j \colon X_j \to T^* X_j.$$

Here $T^* X_j$ denotes the cotangent space of the manifold X_j which is defined for instance in [11, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j\big|_x \colon \mathbb{R}^d \cong T_x^* \mathbb{R}^d \to T_x^* X_j \tag{1.5}$$

denote the orthogonal projection of a vector at x onto the cotangent space of the stratum X_j at x. Now let

$$u_j = \pi_j \circ u\big|_{X_j} \in C^1(T^* X_j) \tag{1.6}$$

be the projection of u onto the cotangent bundle $T^* X_j$.

The following are slight generalisation of definitions given in [23, p.138f], [21, §5] and [20, p.282f] to include harmonic vector fields.
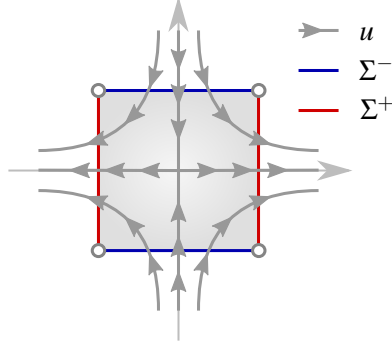
Figure 1.3: Depiction of the entrant and emergent boundaries for the function f given by equation (1.3)

Definition 1.9 (Stagnation points). Let $u_j\colon X_j \to T^*X_j$ be a C^1 vector field on a stratum X_j of X. We call the zeroes $x \in X_j$ of u_j *stagnation points* of u_j on X_j. If $x \in \Omega$ then we call x an *interior stagnation point*. If $u(x) \in (C_xX)^*$ we call x an *essential stagnation point*. The set of all essential stagnation points of u_j is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$ and the essential stagnation points of u and $-u$ on X_j are called *stagnation points of u on X_j*. A stagnation point x of u_j is called *non-degenerate* if it is contained in the relative interior $\mathrm{rel\,int}\,X_j$ and the derivative

$$Du_j(x) = Du_j\big|_x \in T_x T^* X_j \cong \mathbb{R}^{n(j) \times n(j)}$$

is bijective. In addition we say that x has *index k* if $Du_j(x)$ has exactly k negative eigenvalues. u_j is called *(essentially) non-degenerate* if all its (essential) stagnation points are non-degenerate. Finally, we call a non-degenerate essential stagnation point of u_j such that additionally $u(x) \in \mathrm{rel\,int}(C_xX)^*$ *regular*. Boundary points which are non-regular essential stagnation points are called *irregular boundary points*. We call the set of all irregular boundary points the irregular boundary Σ^{irr}. u_j is called *regular* if it has no irregular boundary points. We can define the *k-th type number* $\mathrm{Ind}_{j,k}(u)$ of the stratum X_j to be the number of regular stagnation points of u_j of index k, that is

$$\mathrm{Ind}_{j,k}(u) = \#\big\{x \in \mathrm{Cr}_j(u)\colon x \text{ has index } k\big\}. \tag{1.7}$$

To illustrate the preceding definitions we return to our previous example.

Example 1.10. Let X, f and u be as in example 1.8. We have that $u_2 = u$ and thus one sees from equation (1.4) that the origin 0 is the sole stagnation point of u on the stratum X_2. Since we have that

$$Du(x) = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}$$

for all $x \in \Omega$ we see that $Du(0)$ is bijective and thus the origin is a non-degenerate interior stagnation point. Since $Du(0)$ has exactly one negative eigenvalue we see that the origin has index

1. Since an interior stagnation point is also an essential stagnation point we have $\text{Ind}_{2,k} = \delta_{k1}$ where δ denotes the Kronecker delta. For $x \in I \times \{1\} = Y$ we calculate

$$u_1(x) = \pi_1 \circ u(x) = (u - n \cdot un)(x) = 2x_1 e_1$$

and thus we have that $x = e_2$ is the unique stagnation point of u on $I \times \{1\}$. Consider the curve

$$\gamma \colon I \to Y$$
$$t \mapsto te_1 + e_2$$

then $\gamma(0) = e_2$ and we have

$$Du_1(e_1)\big(\gamma'(0)\big) = (u_1 \circ \gamma)'(0) = (2te_1)'(0) = 2e_1 = 2\gamma'(0)$$

and thus e_1 is an eigenvector of $Du_1(e_2)$ to eigenvalue 2. Since e_1 spans the eigenspace $T_{e_2}Y$ it follows that e_2 is a non-degenerate stagnation point of u_1 with index 0. Now since $u(e_2) \in \text{rel int}(C_xX)^*$ we have that e_2 is in fact a regular point. Proceeding in this manner for the other segments of the square Σ we obtain that $\text{Ind}\,1, k = 2\delta_{0k}$. If we now consider the point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ then we have that $u_0(x) = 0$ and thus x is a stagnation point. Now the derivative $Du_0 = 0 \in T_xT^*X_0 = 0$ is bijective and thus we have that x has index 0. Since however $u(x) \notin (C_xX)^*$ we have that x is not an essential stagnation point. Analogous argumentation on the other three corners yields that $\text{Ind}_{0,0} = 0$.

The following characterisation of the irregular boundary will come in handy for showing the density of Morse functions later on:

Proposition 1.11 (Characterisation of the irregular boundary). *The condition that the stagnation point $x \in X_j$ lies in Σ^{irr} is equivalent to that x is stagnation point of u_{j+1} for a stratum $X_j \precsim X_{j+1}$.*

Proof. We calculate for some boundary point $x \in \Sigma$:

$$(C_xX)^* \setminus \text{rel int}(C_xX)^* = \left\{ w \in T_xX \,\middle|\, \begin{array}{l} \langle w, w' \rangle \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ \langle w, w' \rangle = 0 \text{ for some } w' \in \partial C_xX \setminus \{0\} \end{array} \right\}.$$

Now $w' \in \partial C_xX \setminus \{0\}$ iff $w' \in T_xX_{j+1}$ is orthogonal to T_xX_j at x. Thus we have that

$$u(x) \in (C_xX)^* \setminus \text{rel int}(C_xX)^*$$

iff $u(x) \in (C_xX)^*$ and there exists a normal $n \in T_xX_{j+1}$ such that

$$0 = (n \cdot u\, n)(x) = (n \cdot u_{j+1}\, n)(x) = u_{j+1}(x) - u_j(x)$$

from which the claim follows. $\square$

Definition 1.12 (Morse functions). We call u *Morse* if for all $j \in \mathscr{J}$ we have that u_j is regular. If both u and $-u$ are Morse we call u *strongly Morse*. For a Morse function u we define the *interior type numbers* M_k to be the number of essential interior stagnation points of u of index k, that is

$$M_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \text{Ind}_{j,k}(u) = \#\left\{ x \in \bigcup_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \text{Cr}_j(u) : x \text{ has index } k \right\}. \tag{1.8}$$

The total number M of interior stagnation points of u is then given by

$$M = \sum_k M_k . \tag{1.9}$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of u of index k, that is

$$\mu_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) < d}} \mathrm{Ind}_{j,k}(u) \tag{1.10}$$

We further write ν_k for the k-th boundary type number of $-u$. We define the *type number* to be the number of essential stagnation points of u of index k, that is

$$\mathrm{Ind}_k(u) = \sum_{j \in \mathscr{J}} \mathrm{Ind}_{j,k}(u) = M_k + \mu_k . \tag{1.11}$$

We return to our example:

Example 1.13. Let X, f and u be as in example 1.8. By the calculations of the previous example 1.10 we have that u is Morse and we can calculate the interior type numbers

$$M_k = \mathrm{Ind}_{2,k} = \delta_{2k}$$

and the boundary type numbers

$$\mu_k = \mathrm{Ind}_{0,k}(u) + \mathrm{Ind}_{1,k}(u) = 2\delta_{0k}$$

This then yields the type numbers

$$\mathrm{Ind}_k(u) = M_k + \mu_k = \delta_{2k} + 2\delta_{0k} .$$

The previous definitions translate naturally to f. That is, we call f Morse, non-degenerate, et cetera if $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call x a *critical point* of f if it is a stagnation point of u. Note that most authors refer to regular and essential stagnation points as simply non-degenerate stagnation points and that this naming was introduced simply to distinguish between these different concepts.

Density of Morse functions

In the following section we argue that u and f being Morse is not a great restriction. Given u we define the modification

$$u^{\varepsilon} = u + \varepsilon \tag{1.12}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that the set E of ε for which u^ε is Morse is residual in $\mathbb{R}$. Recall that a *residual* set is a set whose complement is *meagre*, that is whose complement is the countable union of nowhere dense subsets. Since residual sets are dense in a Baire space by the Baire category theorem we can use u^ε to approximate a degenerate u. Our approach is to use Thom's theorem which is inspired by the approach in [11, Chapter 6].

Definition 1.14 (Transversality, [11, §3.2]). We call a function $g\colon Y_1 \to Y_2$ between two manifolds Y_1 and Y_2 (without boundary) *transverse* to a submanifold $A \subseteq Y_2$ if for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\text{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y_2 \,.$$

As an application of this definition we make the following observation:

Proposition 1.15 (Transversal characterisation of non-degeneracy). *Let $u_j\colon X_j \to T^*X_j$ be a differentiable vector field. Then u_j is non-degenerate iff u_j is transverse to the zero section A_j of the cotangent space T^*X_j.*

Proof. First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A) = C$ is the set of stagnation points. Unravelling the definition of transversality we get that u_j is transverse to the zero section iff for all $x \in C = u_j^{-1}(A)$ we have that

$$\text{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}TX \,. \tag{1.13}$$

As A is the zero section we have $T_{u_j(x)}A = 0$ and equation (1.13) is equivalent to stating that Du_j is of full rank at x. But Du_j being of full rank at all stagnation points is equivalent to u_j being non-degenerate. $\qquad\square$

The alternative characterisation of non-degeneracy given in proposition 1.15 is sometimes used as a definition of non-degeneracy. We can now state a weakened version of the Thom's transversality theorem from [11, Theorem 2.7]:

Theorem 1.16 (Parametric transversality theorem, [11, §3 Theorem 2.7]). *Let $\mathscr{E}, Y_1, Y_2$ be C^r-manifolds (without boundary) and $A \subseteq Y_2$ a C^r submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A \,.$$

Let further $F\colon \mathscr{E} \to C^r(Y_1, Y_2)$ be such that the evaluation map

$$F^{ev}\colon \mathscr{E} \times Y_1 \to Y_2$$
$$(\varepsilon, x) \mapsto F_\varepsilon(x)$$

is C^r and transverse to A. Then the set

$$E = \{\varepsilon \in \mathscr{E}\colon F_\varepsilon \text{ is transverse to } A\}$$

is residual in $\mathscr{E}$.

Proof. See [11, Theorem 2.7] for details. $\qquad\square$

From this we obtain a generalisation of the results in [20, §2] which will prove useful later:

Corollary 1.17 (Density of boundary generic functions). *Let $u\colon X \to T^*X$ be a harmonic vector field on X and let X_j be a stratification of X. Assume that u has no irregular stagnation points. Then there exists a $\delta > 0$ and a residual (and thus dense) set $E \subseteq B_\delta \subseteq \mathbb{R}^d$ such that for every $\varepsilon \in E$ the following statements hold:*

1. *$u_j^\varepsilon \to u_j$ converge uniformly on all strata X_j as $\varepsilon \to 0$.*

2. *If $x_\varepsilon \to x$ is a convergent sequence of stagnation points of u_j^ε as $\varepsilon \to 0$ then x is a stagnation point of u_j.*

3. *u^ε is strongly Morse.*

4. *Additionally we can find for every $\eta > 0$ a $\delta > 0$ such that all stagnation points of u^ε are contained in an η-neighbourhood of the set of stagnation points of u.*

5. *the property of being entrant or emergent of stagnation points of u^ε is preserved, that is a stagnation point x^ε of u^ε lies in $\Sigma^\pm(u^\varepsilon)$ iff it lies in $\Sigma^\pm(u)$.*

6. *If u_j is non-degenerate on the stratum X_j we have for all k that*

$$\mathrm{Ind}_{X_j,k}(u^\varepsilon) = \mathrm{Ind}_{X_j,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{X_j,k}(-u^\varepsilon) = \mathrm{Ind}_{X_j,k}(-u).$$

Proof. Part 1. Follows from compactness of $\overline{X}_j \subseteq X$ and the continuity of π_j.

Part 2. Let $x_\varepsilon \to x$ be a convergent sequence of stagnation points on the stratum X_j. By part 1 $u_j^\varepsilon \to u_j$ as $\varepsilon \to 0$ uniformly which implies

$$0 = \lim_\varepsilon u_j^\varepsilon(x_\varepsilon) = u_j(x) \tag{1.14}$$

and thus x is a stagnation point of u_j.

Part 3. The following is essentially an adaptation of a proof given in [20, §2]. We first show that we can choose a $\delta > 0$ such that for all $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ the function u^ε has no irregular stagnation points. Assume not. Then there exists a sequence $\varepsilon_k \to 0$ and irregular stagnation points $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ of u^{ε_k}. By compactness of X we can assume that $x_k \to x$ for some $x \in X$ after taking a sub-sequence. After taking a further sub-sequence we can also assume that all x_k lie in a stratum X_j. The condition that the $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ are stagnation points means that after taking a further sub-sequence there exists a stratum X_{j-1} such that x_k is also stagnation point of this stratum by proposition 1.11. But then $x \in \overline{X}_j$ is also a stagnation point of X_{j-1} by part 2. Analogously x is also stagnation point of X_j. Thus $x \in \Sigma^{\mathrm{irr}}$ is an irregular stagnation point. A contradiction.

The next part of the proof is inspired by the use of transversality in [11, §6 Theorem 1.2] to show a similar statement. Set $r = 2$, $\mathscr{E} = B_\delta$ and $Y_2 = T^*X_j$ in the previous theorem. We initially set

$Y_1 = X_j = \Omega$. We would like to apply the parametric transversality theorem to the function

$$F\colon \mathscr{E} \to C^\infty(X_j, T^*X_j)$$
$$\varepsilon \mapsto u^\varepsilon$$

and note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq T^*X_j$. Then the parametric transversality theorem yields a residual $E_j \subseteq \mathscr{E}$ on which $F_\varepsilon = u^\varepsilon$ is transverse to A. For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$\text{Image}\left(DF^{\text{ev}}_{(\varepsilon,x)}\right) = T_x T^*X_j \tag{1.15}$$

since

$$DF^{\text{ev}}_{(\varepsilon,x)} = \left[\text{Id}_{d\times d} \mid Du_x\right]$$

is surjective. Proposition 1.15 now implies that u^ε is non-degenerate on X_j for $\varepsilon \in E_j$.

Analogously we set $Y_1 = X_j$ to be an arbitrary strata in the previous proof and replace u^ε with the projection u_j^ε. To show that equation (1.15) holds we resort to the fact that

$$DF^{\text{ev}}_{(\varepsilon,x)} = D\left(u_j^\varepsilon(x)\right)_{(\varepsilon,x)} = D\pi_j \circ \left(Du^\varepsilon(x)\right)_{(\varepsilon,x)}$$

is surjective as a concatenation of surjective functions. Thus there also exists a residual set $E_j \subseteq \mathscr{E}$ on which u_j^ε is non-degenerate on X_j.

Now the intersection

$$E = \bigcap_j E_j \subseteq \mathscr{E} = B_\delta$$

is residual and for every $\varepsilon \in E$ the function u^ε fulfils condition 3.

Part 4. Let C_η denote the open η-neighbourhood of the set of stagnation points of u. Since u has no irregular stagnation points we have for any stratum X_j that $u_j \neq 0$ on the compact set $\overline{X}_j \setminus C_\eta$ which implies that we can choose $\delta > 0$ so small that $|u_j| > 2\delta$ on $\overline{X}_j \setminus C_\eta$ for all strata X_j. For any $\varepsilon \in B_\delta$ it then follows that u^ε has no stagnation points on the set $\overline{X}_j \setminus C_\eta$ which yields the claim.

Part 5. Now consider a stratum X_j and the continuous mapping

$$\Phi\colon X_j \to \mathbb{R}_{\geq 0}$$
$$x \mapsto \text{dist}(u(x), \partial C_x X)$$

on X_j. Φ is positive on the set of stagnation points C of u_j on X_j and thus we can choose $\eta > 0$ such that Φ is also positive in the neighbourhood $C_{2\eta}$. Choose $\delta > 0$ smaller than in part 4. Now the mapping Φ attains a positive minimum on the compact set $\overline{C_\eta}$. We can assume that $\delta > 0$ is less than this minimum. The choice of δ in this way ensures that emergent stagnation points of u_j are also emergent stagnation points of u_j^ε on X_j. Analogous argumentation with $-u$ then

ensures that entrant stagnation points of u_j are also entrant stagnation points of u_j^ε on X_j. Since there are finitely many strata X_j we can choose $\delta > 0$ such that part 5 follows.

Part 6. Pick $\delta > 0$ as in part 5 and such that If x is a non-degenerate stagnation point of u on the stratum X_j it follows from the inverse function theorem that there exists for sufficiently small δ a neighbourhood around x on which there is a one-to-one correspondence between the stagnation points of u and u^ε. Since there are by proposition 2.1 at most finitely many non-degenerate stagnation points of u we can choose δ to be minimal over all these stagnation points. The equality of the indexes then follows from $Du^\varepsilon = Du$. $\qquad\square$

2 Some general remarks

Bring order into this section.

The finiteness of the number of critical points is a known fact which is mentioned for example in [20]. For completeness we give the following proposition:

Proposition 2.1. *The number of non-degenerate stagnation points of u_j on X_j is finite.*

Proof. Let $x \in X_j$ be a non-degenerate stagnation point of u_j. Since $Du_j(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq X_j$ of x on which u_j is bijective. Hence x is the only stagnation point in U_x. Let C denote the set of all non-degenerate stagnation points of u_j. Then the sets U_x for $x \in C$ together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \tag{2.1}$$

form an open cover of $\overline{X}_j$. But $\overline{X}_j$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is contained the finite subcover. Thus it follows that $\#C < \infty$ is finite. □

As a consequence we obtain the following observation:

Corollary 2.2. *For a Morse u the type numbers $M_0, \ldots, M_d$ and the boundary type numbers $\mu_0, \ldots, \mu_{d-1}$ are finite.*

State the theorem of Sard

We state Morse's lemma according to [11, §6, Lemma 1.1]

Lemma 2.3. *Let $f \colon X \to \mathbb{R}$ be C^{2+r} and x be a non-degenerate critical point of index k. Then there exists a C^r chart (φ, U) at x such that we have*

$$f \colon \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{d} y_j^2.$$
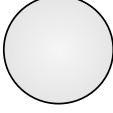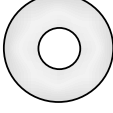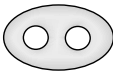
Proof. See for example [11, §6]. □

Domain	Picture	b_0	b_1	$b_k, k \geq 2$
Disk D		1	0	0
Annulus $2D \setminus D$		1	1	0
Two holed button		1	2	0

Table 2.1: Betti numbers for selected domains in $\mathbb{R}^2$.

Betti numbers

Let $H_k(X;\mathbb{R})$ denote the k-th homology space of X. For an introduction and definition of these we refer the reader to [10, Chapter 2]. We define the k-th Betti number as the dimension

$$b_k = \dim_{\mathbb{R}} H_k(X;\mathbb{R}). \tag{2.2}$$

We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

Example 2.4 (In flatland). In $d = 2$ dimensions the 0-th Betti number counts the number of connected components of Ω and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely we give the Betti numbers for selected domains in table 2.1.

Example 2.5 (In spaceland). In $d = 3$ dimensions the 0-th Betti number counts the number of connected components of Ω, the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 2.2.

Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.

The Morse inequalities

We state the Morse inequalities.

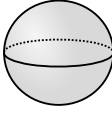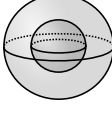**Theorem 2.6** (Strong Morse inequalities, [1, Theorem 2.4]). *Let X be a manifold with corners*

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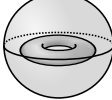
Domain	Picture	b_0	b_1	b_2	$b_k, k \geq 3$
Ball B		1	0	0	0
Solid torus $S^1 \times D$		1	1	0	0
Ball with bubble $2B \setminus B$		1	0	1	0
Ball with bubble in shape of torus		1	1	1	0

Table 2.2: Betti numbers for selected domains in $\mathbb{R}^3$.

and $f \colon X \to \mathbb{R}$ be Morse. Then we have for $l \in \{0,\ldots,d\}$ the inequalities

$$\sum_{k=0}^{l}(-1)^{k+l}\operatorname{Ind}_j(f) \geq \sum_{k=0}^{l}(-1)^{k+l}b_j(X).$$

For $l = d$ we in fact have equality

$$\sum_{k=0}^{d}(-1)^{k}\operatorname{Ind}_k(f) = \chi(X)$$

where the Euler characteristic

$$\chi(X) = \sum_{k=0}^{d}(-1)^{k}b_k(X)$$

is the alternating sum of the Betti numbers.

Proof. A for manifolds with C^1 boundary is given for instance in [21, Theorem 10.2']. The definition of regular critical points of f and their index given in definition 1.9 coincides with the definition of a critical point and its co-index of $-f$ given in [1]. The result then follows from [1, Theorem 2.4].

Give outline of proof idea.

$\square$

Corollary 2.7 (Weak Morse inequalities). *Let X be a manifold with corners and $f \colon X \to \mathbb{R}$ Morse. Then we have for $k \in \{0, \ldots, d\}$ the inequalities*

$$\mathrm{Ind}_k(f) \geq b_k(X).$$

Proof.

Write some proof.

$\square$

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [21, Corollary 10.1].

Corollary 2.8 (Morse inequalities for f harmonic, $d = 2$). *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\mu_0 \geq b_0$$
$$M + \mu_1 - \mu_0 = b_1 - b_0.$$

In dimensions $d = 3$ we obtain [21, Corollary 10.2]

Corollary 2.9 (Morse inequalities for f harmonic, $d = 3$). *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\mu_0 \geq b_0$$
$$M_1 + \mu_1 - \mu_0 \geq b_1 - b_0$$
$$M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 = b_2 - b_1 + b_0.$$

Give a classical example of a Morse function to determine the Betti numbers.

On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2, 3\}$.

Proposition 2.10 (Harmonic vector fields on simply connected domains). *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f \colon \Omega \to \mathbb{R}$.*

2. *f is harmonic.*

3. *u is in fact C^∞.*

4. *The components $u_i = \partial_i f$ are harmonic.*

Proof.   1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.

2. This follows from $\Delta f = \operatorname{Div} u = 0$.

3. This follows from the fact that f is harmonic

4. This follows from $u_i = \partial_i f$.

$\square$

If one considers not necessarily simply connected domains Ω then we obtain the properties of proposition 2.10 at least locally.

3 The case of connected entrant boundaries on simply connected domains in $d = 2$ dimensions

We will start this section by giving an essentially negative answer to question 1.1 in $d = 2$ dimensions. Thus it is not possible to have a harmonic function with interior critical point on a simply connected planar domain with connected entrant boundary. The proof of this will involve Morse theory on manifolds with corners which was introduced in the previous chapter. We shall then state a result from [2] which more generally relates the number of entrant components with the number of critical points in the plane using tools from complex analysis. After that we give examples showing that dropping the condition of simply connectedness on the domain yields a harmonic function with interior stagnation point. We will also consider the analogous problem in $d = 4$ dimensions and show that there exists a function and domain with the desired properties.

A negative result using Morse theory

We now give a proof involving Morse theory since the techniques of the proof generalise to the three dimensional case.

Proposition 3.1 (Negative answer to question **??** in $d = 2$ dimensions)**.** *Let $d = 2$ and X be a simply connected manifold with corners and let $f \colon X \to \mathbb{R}$ have no irregular critical points. Assume further that Σ^- contains at least two points and is simply connected. Then f has no non-degenerate interior critical point.*

Proof. Let $\gamma = \{x_1, x_2\} = \partial \Sigma^-$. Then we can cut the domain along a curve Γ such that the endpoints $\gamma = \partial \Gamma$ of the cut coincide with x_1 and x_2, that is $\partial \Gamma = \{x_1, x_2\}$. Now we obtain two new domains X^+ and X^- such that $\partial X^+ \cap \Sigma^- \subseteq \gamma$ and $\partial X^- \cap \Sigma^+ \subseteq \gamma$. We can assume that Γ is a smooth manifold and corresponds to the stratum X_{Γ^+} for X^+ and X_{Γ^-} for X^-. Analogously γ corresponds to strata $X_{\gamma^\pm}$ on $X^\pm$. Locally around the corner point x_1 we have a situation depicted as in figure 3.1. We assume that we chose Γ in such a way that it forms an acute angle with $u = \nabla f$ at the boundary points γ. For the following argumentation we require that u is strongly Morse on both X^+ and X^-, so assume for a moment that this is the case. Since each point of γ is an essential critical point for either f or $-f$ on precisely one of the domains X^+ or X^- we have the relation

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^+,0}(-f) + \mathrm{Ind}_{\gamma^-,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) = 2 \,. \tag{3.1}$$
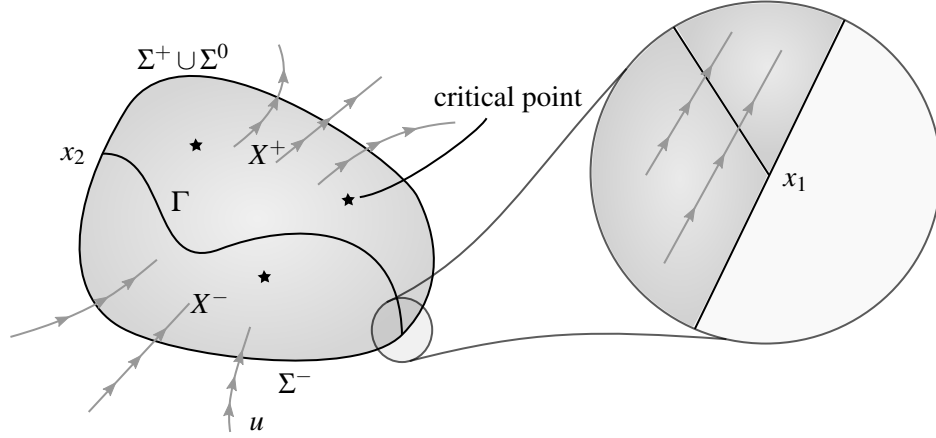
Figure 3.1: The situation at hand.

We now focus our attention on X^+. Since no essential critical points of f lie on $\Sigma^+ \setminus \gamma$ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f) \tag{3.2}$$

where δ_{ij} denotes the Kronecker delta. Analogously we have on X^- that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f). \tag{3.3}$$

In addition we have on Γ that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^-, that is

$$\mathrm{Ind}_{\Gamma^+,0}(f) = \mathrm{Ind}_{\Gamma^-,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma^+,1}(f) = \mathrm{Ind}_{\Gamma^-,0}(-f). \tag{3.4}$$

Using equations (3.2), (3.3) and (3.4) we obtain

$$\mu_0^+ - \mathrm{Ind}_{\gamma^+,0}(f) = \nu_1^- \qquad \text{and} \qquad \mu_1^+ = \nu_0^- - \mathrm{Ind}_{\gamma^-,0}(-f). \tag{3.5}$$

Consider the Morse inequality for f on X^+

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) \tag{3.6}$$

and the Morse inequality for $-f$ on X^-

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X). \tag{3.7}$$

We now add equations (3.6) and (3.7) and insert relations (3.5) to obtain

$$M^- + M^+ - \mathrm{Ind}_{\gamma^+,0}(f) - \mathrm{Ind}_{\gamma^-,0}(-f) = -2\chi(X) = -2.$$

Since $\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \le 2$ by equation (3.1) and $M^\pm \ge 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

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The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 1.17 applied separately to the domains X^+ and X^-. Since $E^\pm$ are residual in B_δ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. It follows from the slanted angles at which Γ approaches γ that if the points x_1, x_2 are essential critical points of f that they then are in fact regular. Hence we obtain that

$$\text{Ind}_{\gamma^+, k}(f^\varepsilon) = \text{Ind}_{\gamma^+, k}(f) \qquad \text{and} \qquad \text{Ind}_{\gamma^-, k}(-f^\varepsilon) = \text{Ind}_{\gamma^-, k}(-f). \tag{3.8}$$

By the same corollary u^ε has no essential stagnation points on $\Sigma^+(u)$ and $-u$ has no essential stagnation points on $\Sigma^-(u)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M^\varepsilon = M$. $\qquad \square$

A generalisation of this result

We could also have used tools from complex analysis to show proposition **??**. In fact, [2] gives a more refined result which we will state in the following.

Let $J^\pm$ denote the number of connected components of $\Sigma^\pm$ which are proper subsets of a component of Σ. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\geq 0} \sqcup \Sigma_{\leq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma_{\geq 0}$ which are proper subsets of a component of Σ. We state a consequence of a result from [2, Theorem 2.1 and 2.2]:

Proposition 3.2 (Weakening of [2, Theorem 2.1 and 2.2].)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u \colon X \to \mathbb{R}$ be a harmonic vector field. Then we have*

$$M \leq b_1 - b_0 + \frac{J^+ + J^-}{2}.$$

If in addition we assume that there are no stagnation points on the boundary then we have

$$M \leq b_1 - b_0 + J^{\geq 0}.$$

Proof. See [2, Theorem 2.1] and [2, Theorem 2.2] where we set $\underline{\alpha} = n$ to be the outer unit normal and $D = b_0 - b_1$. $\qquad \square$

If we set X to be homeomorphic to the ball, that is $b_1 - b_0 = -1$ and require the entrant boundary to be simply connected we then obtain from proposition 3.2 that $M \leq 0$, that is f has no non-degenerate critical point.

> incorporate this result in the following examples.

If we allow for holes in the domain

We would like to give examples for harmonic vector fields. For this define two differential operators for $d = 2$ dimensions by

$$\nabla^\perp f = \mathrm{Curl}\, f = \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix} \tag{3.9}$$

and

$$\mathrm{curl}\, u = -\partial_1 u_2 + \partial_2 u_1 \,.$$

The next proposition gives us a recipe to generate harmonic vector fields in $d = 2$ dimensions.

Proposition 3.3. *Let* $\psi \colon \Omega \to \mathbb{R}$ *be harmonic then* $u = \nabla^\perp \psi$ *is a harmonic vector field.*

Proof. Since $\mathrm{Div}\, \nabla^\perp = 0$ we have

$$\mathrm{Div}\, u = \mathrm{Div}\, \nabla^\perp \psi = 0$$

and one calculates

$$\mathrm{curl}\, u = \mathrm{curl}\, \nabla^\perp \psi = -\Delta \psi = 0 \,.$$

$\square$

The function ψ is also called a *stream function*. In the following we aim to give examples of harmonic vector fields on domains in $d = 2$ dimensions with interior stagnation points for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component.

Example 3.4 (Flow through tube with hole and stagnation point)**.** Consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{3.10}$$

where

$$\Phi_2 = \log(|\cdot|) \tag{3.11}$$

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and e_i is the i-th unit vector. A plot of the streamlines in figure 3.2 indicates that $u = \nabla^\perp \psi$ fulfils the requirements on the domain

$$X = \psi^{-1}([-0.5, 2]) \cap (\mathbb{R} \times [-2, 2]) \,.$$

Indeed, an elementary calculation reveals that the origin is a stagnation point of u.
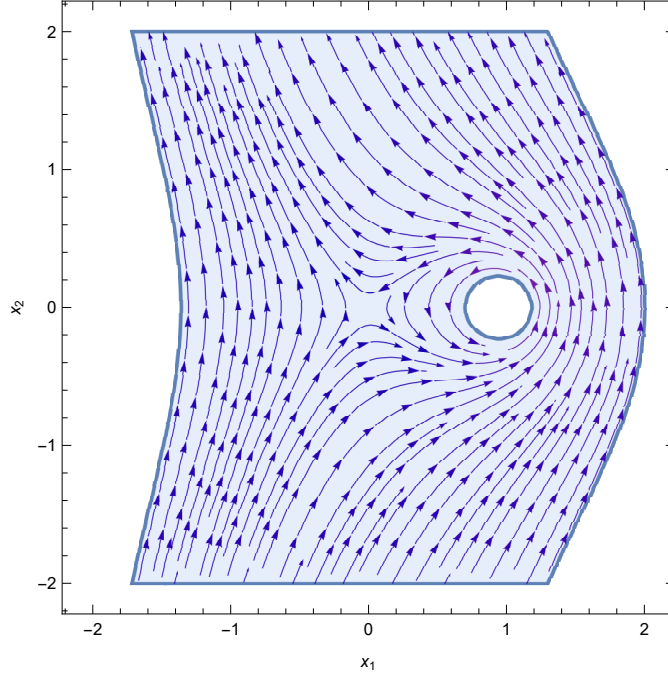
Figure 3.2: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.5, 2]) \cap (\mathbb{R} \times [-2, 2])$. Here ψ is given by equation (3.10).

Example 3.4 highlights the importance of the requirement in proposition 3.1 that the domain be simply connected. We now give a similar example, this time with two holes in the domain.

Example 3.5 (Flow through tube with two holes and stagnation points)**.** We consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1 \end{aligned} \tag{3.12}$$

A plot of the streamlines in figure 3.3 indicates that $u = \nabla^\perp \psi$ on the domain

$$X = \psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2, 2])$$

has the desired properties.

The case of $d = 4$ dimensions

We ask whether there exists a harmonic vector field on a simply connected domain with connected entrant boundary and with interior stagnation point in a dimension higher than $d = 2$. Indeed, in $d = 4$ dimensions we can readily give an example of such a harmonic vector field.

Example 3.6 (Connected entrant boundary in $d = 4$ dimensions)**.** Consider as domain $X = B_1 \subseteq \mathbb{R}^4$
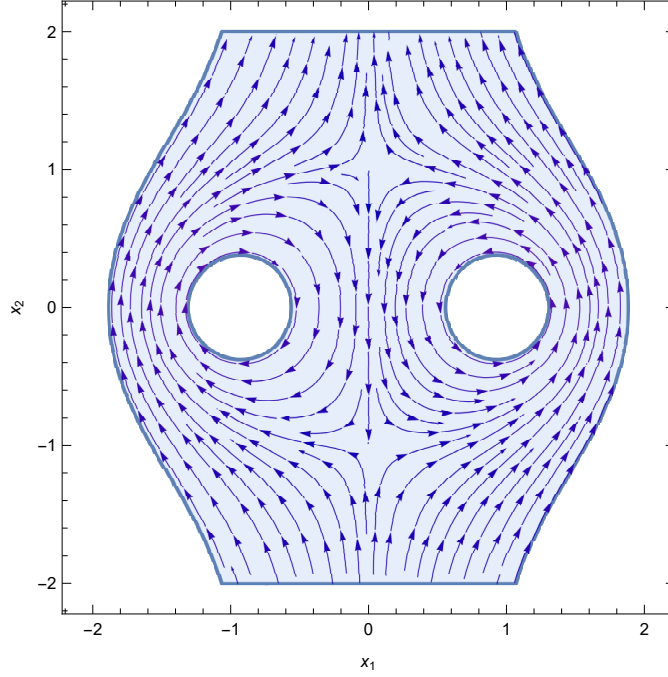
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Figure 3.3: A plot of $u = \nabla^{\perp} \psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2, 2])$. Here ψ is given by equation (3.12).

the unit ball and the harmonic function

$$
\begin{aligned}
f \colon X &\to \mathbb{R} \\
x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2.
\end{aligned}
\tag{3.13}
$$

This has a critical point at the origin. We will show in proposition 3.7 that the entrant boundary Σ^- is in fact connected.

Proposition 3.7 (Simply connected entry boundary in example 3.6)**.** *The harmonic function given by equation* (3.13) *has simply connected entrant boundary.*

Proof. First observe that the boundary $\Sigma = S^3$ can be away from the equator locally parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$
\sum_i x_i^2 = 1
\tag{3.14}
$$

on the boundary $\Sigma = S^3$ we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$
\begin{aligned}
\phi_{\pm} \colon \mathbb{R}^3 \supseteq B_1 &\to \Sigma \subseteq \mathbb{R}^4 \\
\bar{x} &\mapsto x = (x_1, \bar{x}) \text{ such that } \pm x_1 \geq 0
\end{aligned}
\tag{3.15}
$$

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with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the outer unit normal to the boundary Σ is given by

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top.$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2 \left(x_1^2 + x_2^2 - x_3^2 - x_4^2 \right).$$

Using condition (3.14) we obtain the equivalent condition

$$0 < \pm 1 - 2 \left(x_3^2 + x_4^2 \right)$$

Define the cylinder

$$C = \left\{ \bar{x} \in \mathbb{R}^3 : x_3^2 + x_4^2 < 1/2 \right\} = \mathbb{R} \times B_{1/\sqrt{2}} \subseteq \mathbb{R}^3$$

If we return to our parametrisation (3.15) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(\bar{x}) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm \left(\Sigma^+ \right)$$

and

$$B_1 \setminus C = \psi_\pm \left(\Sigma^- \right).$$

Since ϕ_+ is a homeomorphism onto the northern hemisphere and ϕ_- a homeomorphism onto the southern hemisphere and the surface S^2 corresponds to the equatorial sphere the claim then follows. The situation is depicted in figure 3.4. $\qquad\square$
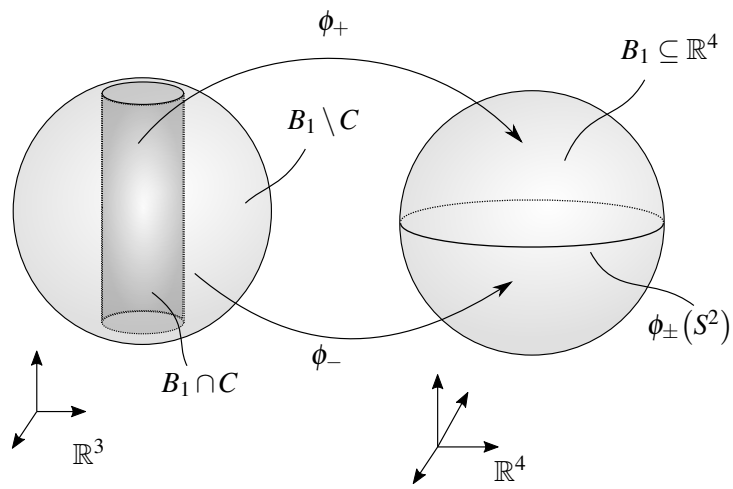
Figure 3.4: Visualisation of the situation in the proof of proposition 3.7.

4 The case of simply connected entrant boundaries in $d = 3$ dimensions.

In the following chapter we will discuss Question **??** in the case of $d = 3$ dimensions. We will first give a negative answer in the case of a cylinder. After that we will use Morse theory to essentially argue that the number of interior stagnation points of such an example with simply connected entrant boundary must be an even number. Finally we will give an example of a function and a domain with the desired properties.

A negative result for the cylinder

The following proof comes from [25] and is a negative answer to quesion **??** for the cylinder.

Proposition 4.1. *Let $\Omega = (0, 1) \times U \subseteq \mathbb{R}^3$ be an open cylinder where $U \subseteq \mathbb{R}^2$ is an open set. Let further $f \colon X = \overline{\Omega} \to \mathbb{R}$ be harmonic such that the sides $[0, 1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^+$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^-$ is the emergent boundary. Then f cannot have an interior critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ. Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0, 1) \times \partial U$ such that $\partial_1 f(x)$ is minimal on X. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0,$$

a contradiction. $\qquad\square$

A condition on the interior type numbers

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 4.2. *Let $X \subset \mathbb{R}^3$ be a compact manifold with corners. Let $f \colon X \to \mathbb{R}$ be a Morse harmonic function. Assume that the strictly entrant boundary Σ^- is non-empty and simply*

connected and that $\gamma = \partial \Sigma^-$ is a one-dimensional manifold with corners homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Then we have that

$$M_1 - M_2 = 0.$$

Proof. As in the two dimensional case we split the domain X with a surface Γ such that $\partial \Gamma = \gamma = \partial \Sigma^-$. Denote the two arising domains by X^+ and X^- where $\partial X^- \cap \Sigma^+ \subseteq \gamma$ and $\partial X^+ \cap \Sigma^- \subseteq \gamma$. Since by proposition 2.1 there are finitely many interior critical points in X we can also assume that no interior critical points lie on Γ. Furthermore we assume that Γ approaches γ at a slanted angle. For the following argumentation we require that f is strongly Morse on both X^+ and X^- so assume for a moment that this is the case. By assumption we have that γ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Since f is non-degenerate the the number of maxima and minima of f on γ must be equal and thus

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) = \mathrm{Ind}_{\gamma^+,1}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \tag{4.1}$$

We now turn our attention to X^+. Since no essential critical points lie on Σ^+ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f). \tag{4.2}$$

Analogously we have on X^- that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f). \tag{4.3}$$

In addition we have that the emergent critical points on $\Gamma = \Gamma^+$ of f on X^+ are the entrant critical points on $\Gamma = \Gamma^-$ of $-f$ on X^-, that is

$$\begin{aligned}
\mathrm{Ind}_{\Gamma^+,0}(f) &= \mathrm{Ind}_{\Gamma^-,2}(-f) \\
\mathrm{Ind}_{\Gamma^+,1}(f) &= \mathrm{Ind}_{\Gamma^-,1}(-f). \\
\mathrm{Ind}_{\Gamma^+,2}(f) &= \mathrm{Ind}_{\Gamma^-,0}(-f)
\end{aligned} \tag{4.4}$$

Using equations (4.2), (4.3) and (4.4) we obtain

$$\begin{aligned}
\mu_0^+ - \nu_2^- &= \mathrm{Ind}_{\gamma^+,0}(f) \\
\mu_1^+ - \nu_1^- &= \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,1}(-f). \\
\mu_2^+ - \nu_0^- &= -\mathrm{Ind}_{\gamma^-,0}(-f)
\end{aligned} \tag{4.5}$$

We observe the Morse inequalities for f

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X). \tag{4.6}$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \tag{4.7}$$

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where the M_k continue to denote the interior type numbers of f. We now subtract equation (4.7) from (4.6) and insert relations (4.5) to obtain then with equation (4.1)

$$0 = M_1^- - M_2^- + M_1^+ - M_2^+ + \mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) - \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,0}(-f)$$
$$= M_1 - M_2$$

from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 1.17 applied separately to the domains X^+ and X^-. Since $E^\pm$ are residual in B_δ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. Since x_1, x_2 are non-degenerate critical points of f due to the slanted angle at which Γ approaches γ we obtain that

$$\mathrm{Ind}_{\gamma,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-f) \qquad (4.8)$$

By the same corollary we can assume that f^ε has no essential critical points on $\Sigma^+(f)$ and $-f^\varepsilon$ has no essential critical points on $\Sigma^-(f)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. $\qquad\square$

A harmonic function with interior critical point and simply connected entrant boundary in $d = 3$ dimensions

Based on example 3.6 in $d = 4$ dimensions of a harmonic vector field with interior stagnation point the author argued that it would be simplest to construct such a vector field in $d = 3$ dimensions in a similar manner. More specifically we choose as domain the ball and polynomials as our harmonic function. In choosing the ball as domain we also take into account the result from proposition 4.1 which is a negative result to question **??** for the cylinder. Based on the result from proposition 4.2 we see that the number of stagnation points must be at least two. The author then implemented a Mathematica routine to generate harmonic polynomials with two stagnation points and a plotting function to inspect what occurs. Indeed, this approach yielded a function with the desired properties as we shall discuss in the following example.

Example 4.3 (A harmonic function with interior critical point and simply connected entrant boundary). Consider the domain $X = \overline{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large, and the harmonic function

$$f \colon X \to \mathbb{R}$$
$$x \mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1 x_2^2 + x_2 x_3 \qquad (4.9)$$

This induces the harmonic vector field

$$u \colon X \to \mathbb{R}^3$$
$$x \mapsto \nabla f(x) = \begin{bmatrix} x_1(1-x_1) + x_2^2 \\ x_2(2x_1 - 1) + x_3 \\ x_2 \end{bmatrix}$$
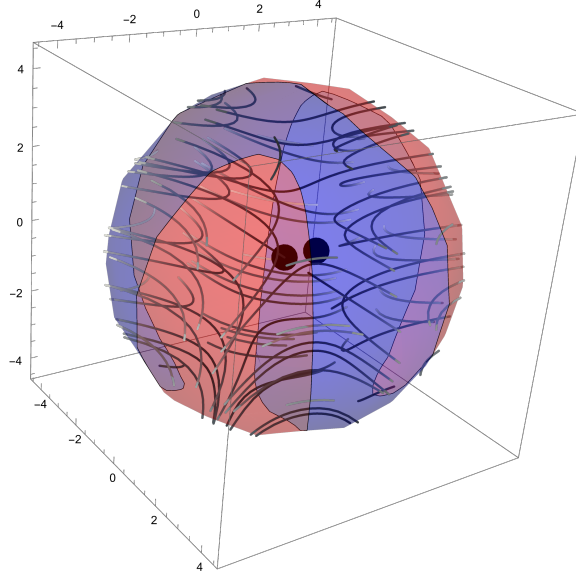
30

Figure 4.1: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

It follows from setting $u(x) = 0$ that $x_2 = 0$ and then $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of u. Conversely one sees that these are zeroes of u. Hence f has two interior critical points for $r > 1$. Figure 4.1 shows a stream plot of u with the two interior stagnation points highlighted as black dots. The boundary of the domain is shaded in blue for Σ^- and in red for Σ^+. The stereographic projection of the boundary is plotted in figure 4.2. This plot indicates that the entrant boundary and the emergent boundary are simply connected. We will give a formal proof of this in proposition 4.4.

> Bring order into this section. Rename propositions to lemmas / etc.

We start by stating some required definitions. For polynomials $p_1, \ldots, p_k \in \mathbb{R}[x_1, \ldots, x_d]$ and a set $U \subseteq \mathbb{R}^d$ we denote the variety generated by these polynomials on U by $V_U(p_1, \ldots, p_k)$. In the case that $U = \mathbb{R}^d$ we write $V(\ldots) = V_U(\ldots)$. We will also need a notion of convergence of subsets on a metric space. We define the *Hausdorff metric* for two sets $A, B \subseteq X$ to be given by

$$d_H(A, B) = \max\left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \tag{4.10}$$

where
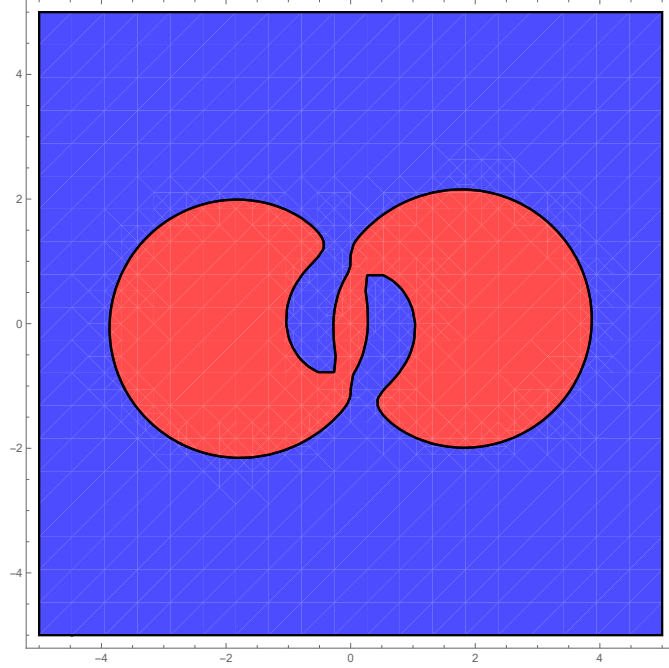
$$\text{dist}(x, B) = \inf_{y \in B} d(x, y) \tag{4.11}$$

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Figure 4.2: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

is the smallest distance from x to B. For a given $\delta > 0$ and a subset $A \subseteq \mathbb{R}^d$ we call the union of δ balls

$$\text{Tub}_\delta(A) = \bigcup_{x \in A} B_\delta(x) \tag{4.12}$$

a *tubular neighbourhood* of A. We have the following proposition, the proof of which will span several propositions.

Proposition 4.4. *The harmonic function given by equation* (4.9) *has simply connected emergent and entrant boundaries.*

Proof. We calculate

$$rn \cdot u(x) = x_1^2(1 - x_1) + x_2^2(3x_1 - 1) + 2x_2x_3 = p_1(x)$$

and define further

$$P_2(r, x) = x_1^2 + x_2^2 + x_3^2 - r^2.$$

Thus we have that the tangential boundary $\Sigma^0 = V(p_1, P_2(r, \cdot))$ is precisely the variety generated by the polynomials p_1 and $P_2(r, \cdot)$ for a fixed radius $r > 0$. It will be shown in proposition 4.5 that there exists $r > 0$ such that Σ^0 is smooth and connected. The stereographic projection of Σ^0 then defines a simple closed planar curve. This is indicated by the black curve in figure 4.2. By the Jordan curve theorem this curve then splits the plane into two connected regions, one of

which is simply connected. The preimage of these connected regions under the stereographic projections then corresponds precisely to the entrant and emergent boundaries from which the claim follows. $\square$

The following definitions will play a central role in the upcoming propositions. We now define p_2 to be the dehomogenisation of P_2, that is

$$p_2 = P_2(1,\cdot).$$

Analogously let P_1 denote the homogenisation of p_1, that is

$$P_1(\varepsilon,x) = \varepsilon^3 p_1(x/\varepsilon).$$

Proposition 4.5. *There exists an $r > 0$ such that the planar variety $V(p_1, P_2(r,\cdot))$ is smooth and has one connectivity component.*

Proof. By proposition 4.6 there exists a $R > 0$ such that for all $r > R$ we have that the variety $V(p_1, P_2(r,\cdot))$ is smooth. By rescaling we obtain

$$V(p_1, P_2(r,\cdot)) = rV(p_1(r\cdot), p_2) = rV(P_1(\varepsilon,\cdot), p_2) = r\mathcal{V}_\varepsilon \qquad (4.13)$$

where we set $\varepsilon = 1/r$ and $\mathcal{V}_\varepsilon = V(P_1(\varepsilon,\cdot), p_2)$. Motivated by taking the limit $r \to \infty$ we inspect the variety $\mathcal{V}_0$ closer. We observe that

$$P_1(0,x) = -x_1^3 + 3x_1 x_2^2$$

which is the monkey saddle embedded into $\mathbb{R}^3$. We thus define curves

$$\widetilde{\alpha}^\pm = \left\{ t \begin{bmatrix} \pm\sqrt{3} & 1 \end{bmatrix}^\top : t \in \mathbb{R} \right\}$$

and $\widetilde{\alpha}^0 = \{0\} \times \mathbb{R}$. We then define $\alpha^\bullet = (\widetilde{\alpha}^\bullet \times \mathbb{R}) \cap S^2$. Setting $A = \alpha^- \cup \alpha^+ \cup \alpha^0$ we have the relation

$$\mathcal{V}_0 = V(P_1(\varepsilon,\cdot), p_2) = A.$$

Thus $\mathcal{V}_0$ consists of six smooth arcs originating at the singularity e_3 and ending at the singularity $-e_3$. Similar to the classical beach ball. Now consider for $\rho > 0$ the open sets $W_\rho = B_\rho \times \mathbb{R} \subseteq \mathbb{R}^3$ and $U_\rho = \mathbb{R}^3 \setminus W_\rho$. Since $\mathcal{V}_0$ is smooth in a neighbourhood of $\overline{U}_\rho$ we obtain from proposition 4.6 that in a certain sense $\mathcal{V}_\varepsilon$ is obtained from $\mathcal{V}_0$ by a small deformation on U_ρ. Thus in order to show connectedness of $\mathcal{V}_\varepsilon$ for sufficiently small $\varepsilon > 0$ we have to inspect what happens around the points $\pm e_3$. Now observe that we have the symmetry

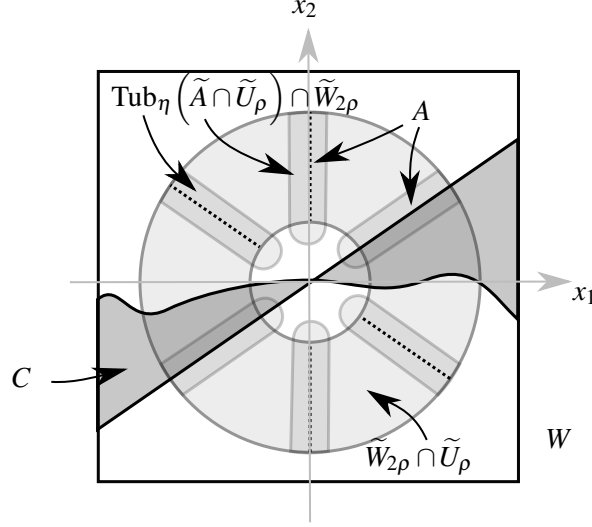$$p_1(x_1, -x_2, -x_3) = p_1(x) \qquad (4.14)$$

Figure 4.3: An overview of the situation around e_3.

and thus it suffices to inspect what happens around the point e_3. For this parametrise the neighbourhood $S^2 \cap (B_{1/2} \times \mathbb{R}_{\geq 0})$ of e_3 by the diffeomorphism

$$\psi \colon B_{1/2} \to S^2 \cap (B_{1/2} \times \mathbb{R}_{\geq 0})$$
$$\tilde{x} \mapsto x = \begin{bmatrix} \tilde{x} & \sqrt{1 - |\tilde{x}|^2} \end{bmatrix}^\top.$$

We set

$$\widetilde{\mathcal{V}_\varepsilon} = \psi^{-1} \left(\mathcal{V}_\varepsilon \cap (B_{1/2} \times \mathbb{R}_{\geq 0}) \right) = V_{B_{1/2}} \left(x \mapsto P_1 \left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2} \right) \right) = V_{B_{1/2}} \left(\widetilde{P_1}(\varepsilon, \cdot) \right)$$

where we defined

$$\widetilde{P_1}(\varepsilon, x) = P_1 \left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2} \right) = -x_1^3 + 3x_1 x_2^2 + \varepsilon \left(x_1^2 - x_2^2 + x_2 \sqrt{1 - x_1^2 - x_2^2} \right).$$

In a similar manner we define $\widetilde{\alpha}^\bullet = \psi^{-1}(\alpha^\bullet)$, $\widetilde{U}_\rho = \psi^{-1}(U_\rho)$, $\widetilde{W}_\rho = \psi^{-1}(W_\rho)$ and $\widetilde{A} = \psi^{-1}(A)$. Now let the sets W and $C \subseteq W$ be as in proposition 4.8. Pick $\rho > 0$ so small that $\widetilde{W}_{2\rho} \subseteq W$. Now we can pick η smaller than the minimum distance between two arcs of $\widetilde{\mathcal{V}_0}$ on $\widetilde{U}_\rho$. We also assume that η is smaller than the Hausdorff distance between $(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-) \cap \widetilde{U}_\rho$ and C. Now choose δ as in proposition 4.7. We can assume that $R > 1/\delta$. We thus have a situation as in figure 4.3. Number the arcs lying close to $A \cap U_\rho$ as in figure 4.4. Since $\widetilde{\gamma}$ by proposition 4.8 splits $\widetilde{W}_{2\rho}$ and lies in C we must have that $\widetilde{\gamma}$ in fact connects arcs 3 and 6. As the variety is smooth we must also have that arcs 1 and 2 are connected and analogously that arcs 4 and 5 are connected. By equation (4.14) the situation around $-e_3$ is analogous but mirrored. We thus obtain analogous connections between the six arcs at $-e_3$ which is summarised in figure 4.5. Thus we in fact have that $\mathcal{V}_\varepsilon$ is connected. The claim then follows from relation (4.13). $\qquad\square$
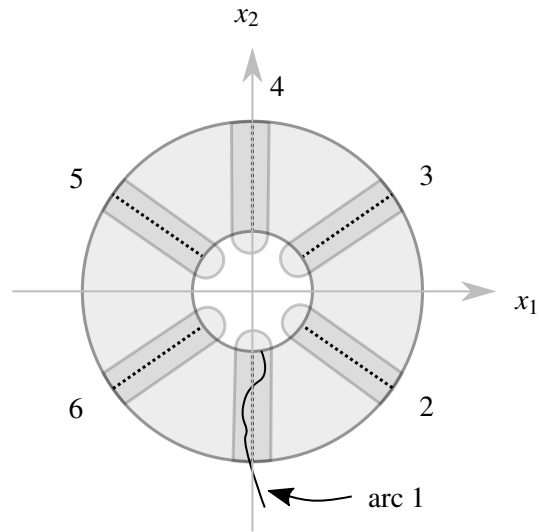
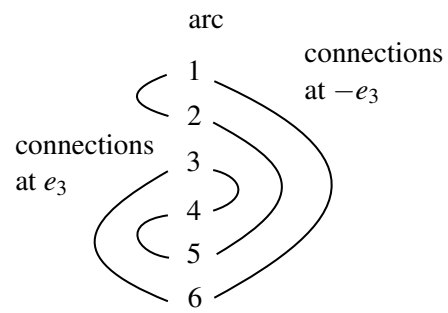Figure 4.4: The numbering of the arcs around e_3.

Figure 4.5: The connection of the arcs at $\pm e_3$.

Proposition 4.6 (Smoothness). *There exists $R > 0$ such that for every $r > R$ the variety $V(p_1, P_2(r, \cdot))$ is smooth.*

Proof. One calculates

$$T = \begin{bmatrix} \nabla p_1(x) & \frac{1}{2}\nabla_x P_2(r,x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 + 3x_2^2 + 2x_1 & x_1 \\ 6x_1 x_2 - 2x_2 + 2x_3 & x_2 \\ 2x_2 & x_3 \end{bmatrix}$$

By the Jacobi criterion it is sufficient to show that this matrix is of full rank on $V(p_1, P_2(r, \cdot))$. This is equivalent to showing that

$$0 \neq \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = -9x_1^2 x_2 + 3x_2^3 + 4x_1 x_2 - 2x_1 x_3 = h_1(x),$$

$$0 \neq \det \begin{bmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} = 6x_1 x_2 x_3 - 2x_2 x_3 + 2x_3^2 - 2x_2^2 = h_2(x) \text{ or}$$

$$0 \neq \det \begin{bmatrix} T_{31} & T_{32} \\ T_{11} & T_{12} \end{bmatrix} = 2x_1 x_2 + 3x_1^2 x_3 - 3x_2^2 x_3 - 2x_1 x_3 = h_3(x)$$

for any $x \in V(p_1, P_2(r, \cdot))$. This in turn is equivalent to showing that

$$V(p_1, P_2(r, \cdot), h_1, h_2, h_3) = \emptyset. \tag{4.15}$$

Indeed, consider the variety

$$V(p_1, h_1, h_2, h_3) \tag{4.16}$$

Maple calculates the Gröbner basis with lexicographic order $x_1 < x_2 < x_3$

$$72x_1^8 - 198x_1^7 + 228x_1^6 - 153x_1^5 + 56x_1^4 - 5x_1^3,$$
$$72x_1^5 x_2 - 126x_1^4 x_2 + 102x_1^3 x_2 - 51x_1^2 x_2 + 5x_1 x_2,$$
$$-24x_1^7 + 42x_1^6 - 2x_1^5 - 23x_1^4 + 7x_1^3 + 10x_1 x_2^2,$$
$$48x_1^4 x_2 - 60x_1^3 x_2 + 13x_1^2 x_2 + 15x_2^3,$$
$$24x_1^4 x_2 - 30x_1^3 x_2 + 29x_1^2 x_2 - 10x_1 x_2 + 5x_1 x_3,$$
$$72x_1^7 - 126x_1^6 + 6x_1^5 + 69x_1^4 - 31x_1^3 + 10x_1^2 - 10x_2^2 + 20x_2 x_3,$$
$$-72x_1^7 + 414x_1^6 - 654x_1^5 + 399x_1^4 - 97x_1^3 + 10x_1^2 - 30x_2^2 + 20x_3^2,$$

from which we see that the variety (4.16) contains finitely many points. Thus if we choose R so large that all of these points are contained in the ball B_R then we have that (4.15) holds for all $r > R$. $\qquad\square$

Proposition 4.7 (Convergence at smooth points). *For $\varepsilon \in \mathbb{R}$ and an open set $W \subseteq \mathbb{R}^3$ define the variety $\mathcal{V}_\varepsilon = V_W(P_1(\varepsilon, \cdot), p_2)$. Let $U \subseteq \mathbb{R}^3$ be an open set such that $\mathcal{V}_0$ is smooth in an open neighbourhood of $\overline{U}$. Let further $\eta > 0$. Then there exists a $\delta > 0$ such that for all $\varepsilon < \delta$ we have that the Hausdorff distance satisfies*

$$d_H(\mathcal{V}_\varepsilon, \mathcal{V}_0) < \eta$$

and additionally $\mathcal{V}_\varepsilon$ is isotopic to $\mathcal{V}_0$.

Proof. Consider the mapping

$$F = \begin{bmatrix} P_1 \\ p_2 \end{bmatrix} : \mathbb{R}^4 \to \mathbb{R}^2.$$

Since $V(P_1(0, \cdot), p_2)$ is smooth on an open neighbourhood of $\overline{U}$ we have by the Jacobi criterion that

$$DF = \begin{bmatrix} * & \Big| & \begin{matrix} D_x P_1(0, \cdot) \\ Dp_2 \end{matrix} \end{bmatrix} \tag{4.17}$$

is of full rank on $\overline{U}$. By the implicit function theorem there exists at every point $x \in \overline{U}$ open neighbourhoods $\Omega_x \subseteq \mathbb{R}^4$ of $(x, 0)$ and $\omega_x \subseteq \mathbb{R}^3$, a coordinate permutation $I \in O(4)$ and a mapping $g_x \in C^1(\omega_x; \mathbb{R})$ such that

$$V(P_1, p_2) \cap \Omega_x = \{I(y, g_x(y)) : y \in \omega_x\}.$$

By equation (4.17) we can assume (possibly after shrinking the open sets) that I does not permute the ε-coordinate. We can also assume that $\Omega_x = B_{\delta_x} \times W_x \subseteq \mathbb{R}^4$ for some open $W_x \subseteq \mathbb{R}^3$ and some $\delta_x > 0$. Thus we also obtain that $\omega_x = B_{\delta_x} \times w_x$ for some open $w_x \subseteq \mathbb{R}^2$ and we can define our isotopy on Ω_x as

$$\varphi \colon B_{\delta_x} \times w_x \to W_x$$
$$y \mapsto \mathrm{proj}_{\mathbb{R}^4 \to W} I(y, g_x(y))$$

where $y = (\varepsilon, y_1, y_2)$. Note that $\varphi(\{\varepsilon\} \times w_x) = \mathcal{V}_\varepsilon \cap W_x$. From this it also follows that we can choose δ_x such that

$$d_H(\mathcal{V}_\varepsilon \cap W_x, \mathcal{V}_0 \cap W_x) < \eta.$$

Now for $x \in \overline{U}$ the Ω_x form an open cover of $\overline{U}$. By compactness there exists a finite subcover. Set $\delta > 0$ to be the minimum of all δ_x for the Ω_x in this finite subcover and the claim follows. $\square$

Proposition 4.8. *Let $\widetilde{U}_\rho$, $\widetilde{\alpha}$, ε, et cetera be as in the proof of proposition 4.5. There exists an open cube $W \subseteq \mathbb{R}^2$ containing the origin and a set $C \subseteq W$ such that C has positive Hausdorff distance to the set $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that for any sufficiently small $\varepsilon > 0$ there is an arc $\widetilde{\gamma}$ entirely contained in $C \cap \widetilde{\mathcal{V}}_\varepsilon$ splitting W into two parts.*

Proof. We borrow notation from the proof of proposition 4.5. Let $W \subseteq \mathbb{R}^2$ be an open cube around the origin which we fix later. Define polynomials

$$\tilde{q}_1(x) = -x_1^3 + 3x_1 x_2^2 \tag{4.18}$$

$$\tilde{q}_2(x) = x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2} \tag{4.19}$$

and observe that

$$\widetilde{P}_1(\varepsilon, \cdot) = \tilde{q}_1 + \varepsilon\tilde{q}_2. \tag{4.20}$$

By equation (4.18) $\tilde{q}_1$ is a monkey saddle and $V_W(\tilde{q}_1) = \widetilde{A}$ is similar to figure 4.6a. The signs in the figure indicate the sign of $\tilde{q}_1$ in a given region. From equation (4.19) we obtain that $\nabla\tilde{q}_2(0) = e_2$ and $\tilde{q}_2(0) = 0$. Hence we observe that the $V_W(\tilde{q}_2)$ looks similar to figure 4.6b for a sufficiently small neighbourhood W. More concretely we choose W so small that the arc $\widetilde{\beta} = V_W(\tilde{q}_2)$ has positive distance to $\widetilde{A} \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that a given vertical line in W intersects $\widetilde{\beta}$ in precisely one point. Now we claim that the set C consisting of the vertical lines between $\widetilde{\alpha}^+$ and $\widetilde{\beta}$ fulfills the claim.

Indeed, consider the right half plane. Here we have that $\widetilde{P}_1(\varepsilon, \cdot) = \tilde{q}_1 \leq 0$ on the arc $\widetilde{\beta}$ and that $\widetilde{P}_1(\varepsilon, \cdot) = \tilde{q}_2 \geq 0$ on the arc $\widetilde{\alpha}^+$. By the intermediate value theorem thus there must be for each $(x_1, 0) \in W$ a corresponding point $(x_1, \widetilde{\gamma}(x_1)) \in V_W\left(\widetilde{P}_1(\varepsilon, \cdot)\right) = \widetilde{\mathcal{V}}_\varepsilon$ sandwiched between $\widetilde{\beta}$ and $\widetilde{\alpha}^+$. Since $\widetilde{\mathcal{V}}_\varepsilon$ is smooth $\widetilde{\gamma}$ in fact defines an arc which we also denote by $\widetilde{\gamma}$. An analogous argumentation using signage extends $\widetilde{\gamma}$ to the third quadrant. Then $\widetilde{\gamma}$ divides the plane into two parts and it follows that $\widetilde{\mathcal{V}}_\varepsilon$ which yields the claim. For clarity the idea of the proof is also depicted in figure 4.7. Also note that as a consequence $\widetilde{\mathcal{V}}_\varepsilon$ looks similar to figure 4.6c. $\qquad\square$

(a) Sketch of the set $V(\tilde{q}_1)$. (b) Sketch of the set $V(\tilde{q}_2)$. (c) Sketch of the set $\widetilde{\mathcal{V}_\varepsilon}$.
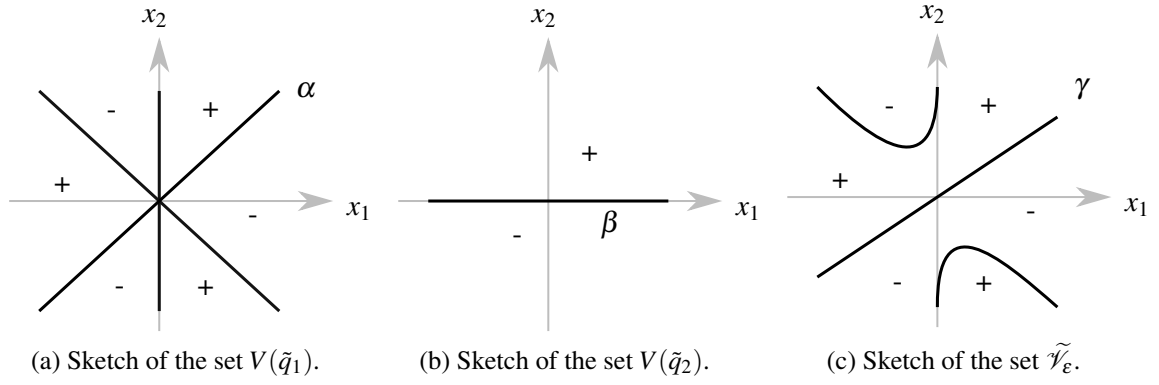
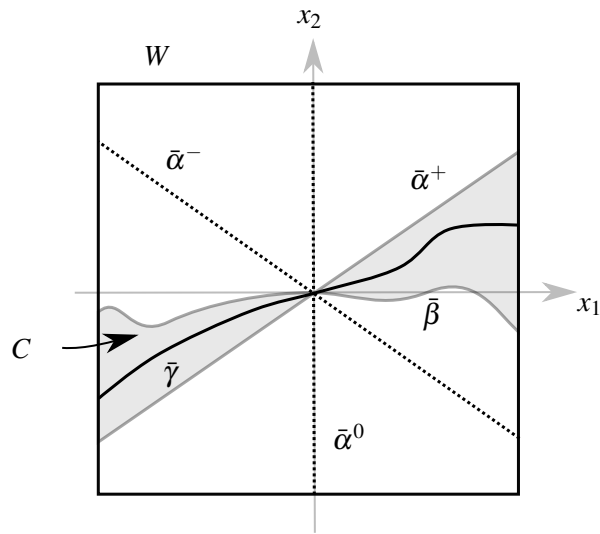Figure 4.6: Sketches of varieties.

Figure 4.7: An overview of the situation in proposition 4.8.

5 No in- or outflow in $d = 2$ dimensions

Proposition 5.1. *Let $X \subset \mathbb{R}^2$ be a compact manifold with corners and Betti numbers $b_0 = 1$, and b_1 and let $u \colon X \to \mathbb{R}^2$ be a strongly Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of X.*

Proof. We slit the domain X such that it is homeomorphic to the disk as is depicted in figure 5.1. Denote the slit by Γ. Since the number of interior stagnation points is finite by proposition 2.1, we can choose Γ in such a way that it does not contain any interior stagnation points. We write denote the boundary of Γ by $\gamma = \partial\Gamma = \Gamma \cap \Sigma$ and define points $\{x_1, \ldots, x_{2b_1}\} = \gamma$. Note that there are $2b_1$ many such points. Without loss of generality we can assume that Γ and u form an acute angle at each point of γ. The situation is depicted in figure 5.1. For the following argumentation we require that u is strongly Morse on the new domain $\tilde{X}$ so assume for a moment that this is the case. Since each x_j is either an essential critical point of u or of $-u$ on the slit domain $\tilde{X}$ we have that

$$\mathrm{Ind}_{\gamma,0}(u) + \mathrm{Ind}_{\gamma,0}(-u) = 2b_1 \,. \tag{5.1}$$

Since there are no stagnation points on Σ we have the relations

$$\mu_k = \mathrm{Ind}_{\Gamma,k}(u) + \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \nu_k = \mathrm{Ind}_{\Gamma,k}(-u) + \mathrm{Ind}_{\gamma,k}(-u) \tag{5.2}$$

for all $k \in \{0,1\}$. All entrant stagnation points of u on Γ are also emergent stagnation points of $-u$ on Γ (and vice versa) and hence we have the relations

$$\mathrm{Ind}_{\Gamma,0}(u) = \mathrm{Ind}_{\Gamma,1}(-u) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(u) = \mathrm{Ind}_{\Gamma,0}(-u) \,. \tag{5.3}$$
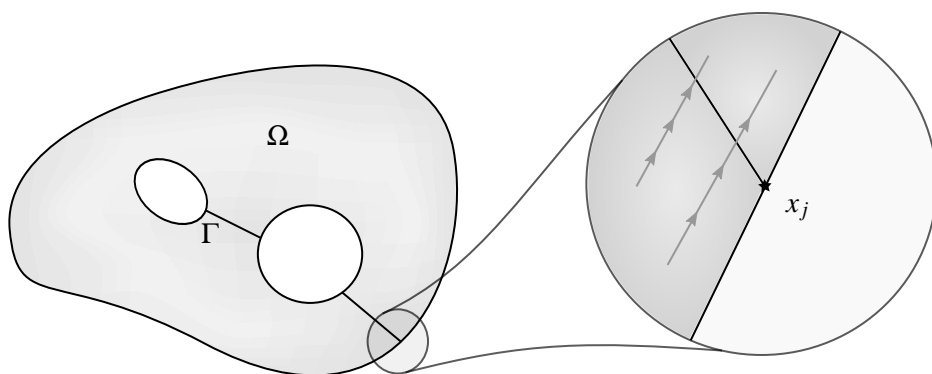

Figure 5.1: How we slit the domain.

Equations (5.2) and (5.3) yield

$$\mu_0 - \mathrm{Ind}_{\gamma,0}(u) = v_1 \qquad \text{and} \qquad \mu_1 = v_0 - \mathrm{Ind}_{\gamma,0}(-u). \qquad (5.4)$$

Since $\tilde{X}$ is now simply connected u is by proposition 2.10 the gradient of a harmonic function f on this new domain. For this f we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \qquad (5.5)$$

and for $-f$ the Morse inequalities

$$M + v_1 - v_0 = -\chi(\tilde{X}) = -1. \qquad (5.6)$$

Adding equations (5.5) and (5.6) and using the relation (5.4) and then (5.1) we obtain

$$-2 = 2M - \mathrm{Ind}_{\gamma,0}(u) - \mathrm{Ind}_{\gamma,0}(u) = 2M - 2b_1$$

from which the claim follows.

The claim remains to be shown in the case that u is not strongly Morse on $\tilde{X}$. In this case let u^ε for $\varepsilon \in E$ be a strongly Morse function as in corollary 1.17. Since the $x_1, \ldots, x_{2b_1} \in \gamma$ are non-degenerate stagnation points of u due to the slanted angle at which Γ approaches γ we obtain that

$$\mathrm{Ind}_{\gamma,k}(u^\varepsilon) = \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-u^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-u) \qquad (5.7)$$

By the same corollary u^ε has no stagnation points on Σ. The claim then follows by the calculations above where we replace u with u^ε and then note that $M^\varepsilon = M$. $\qquad \square$

We now give an alternative proof using the argument principle.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition 2.10 u is the gradient of a harmonic function f on this new domain $\tilde{X}$. Let $h \in \mathrm{Hol}(\tilde{X})$ be the holomorphic function given by $h = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ. We now determine the change of argument $\arg h$ along γ. For this consider first the the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other, the contribution to the change of $\arg h$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h$ is 2π for every hole in the domain since $h \cdot \gamma' = u \cdot \gamma'$ does not change sign as γ traverses a hole in clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have b_1 holes in the domain the total change of $\arg h$ as γ traverses Σ is $2\pi(b_1 - 1)$. Since h has no poles it follows from the argument principle (see for example [7, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M$$

from which the claim follows. $\qquad \square$

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ if $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* if $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can give the following corollary.

Corollary 5.2. *Let X be a compact manifold with corners and $u \colon X \to \mathbb{R}^2$ a strongly Morse harmonic vector field without inflow or outflow on Σ. Then we have the relation $M = -\chi(X)$ where M is the number of stagnation points and $\chi(X)$ is the Euler characteristic of X.*

This result could also have been proved using the Poincaré-Hopf index theorem. For this define:

Definition 5.3 ([5, Definition 1.1.1])**.** The *Poincaré-Hopf index* $\mathrm{Ind}_{\mathrm{PH},x}(f)$ of an isolated interior stagnation point x of u is the degree of the Gauss map $u/|u| \colon S_\varepsilon^{d-1}(x) \to S^{d-1}$ for sufficiently small $\varepsilon > 0$. The *total index* $\mathrm{Ind}_{\mathrm{PH}}(u)$ is the sum of indexes $\mathrm{Ind}_{\mathrm{PH},x}(f)$ for every interior stagnation point x of u.

Note that in $d = 2$ dimensions a point x with Morse index k has Poincaré-Hopf index $(-1)^k$. Thus we have for a harmonic vector field in $d = 2$ dimensions that $M = -\mathrm{Ind}_{\mathrm{PH}}(u)$ and thus corollary 5.2 also follows from the Poincaré-Hopf index theorem:

Theorem 5.4 (Poincaré-Hopf index theorem, [18, §6])**.** *Let $u \colon X \to \mathbb{R}^d$ be a vector field on a manifold with corners without inflow. Then we have that the total index $\mathrm{Ind}_{PH}(u)$ equals the Euler characteristic $\chi(X)$.*

Proof. See for example [18, §6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now give an example of a harmonic vector field in $d = 2$ dimensions without inflow or outflow and with one stagnation point.

Example 5.5. Consider the stream function

$$
\begin{aligned}
\psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\
x &\mapsto \Phi_2(x - e_1) + \Phi_2(x + e_1)
\end{aligned}
\tag{5.8}
$$

where Φ_2 is as in example 3.4. A plot of the streamlines in figure 5.2 indicates that $u = \nabla^\perp \psi$ in the domain $X = \psi^{-1}([-1,1])$ has the desired properties. Indeed, since ψ is constant on each component of ∂X the function u has neither inflow nor outflow. It follows from $\psi(-x) = \psi(x)$ that $u(-x) = -u(x)$ and thus the origin $x = 0$ is a stagnation point. By proposition 5.1 it is in fact the sole stagnation point of u on X.

In a second example given by [25] we fix the domain rather than the function.

Example 5.6. Set $X = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We then have the system

$$
\begin{aligned}
\Delta \psi &= 0 \quad \text{, on } \Omega \\
\psi &= 0 \quad \text{, on the outer ring } 4S^1 \\
\psi &= 1 \quad \text{, on the inner rings } S^1(-2e_1) \cup S^1(2e_1)
\end{aligned}
\tag{5.9}
$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 5.3. Again it follows from symmetry that the origin is a stagnation point and from proposition 5.1 that it is in fact the sole stagnation point of u.
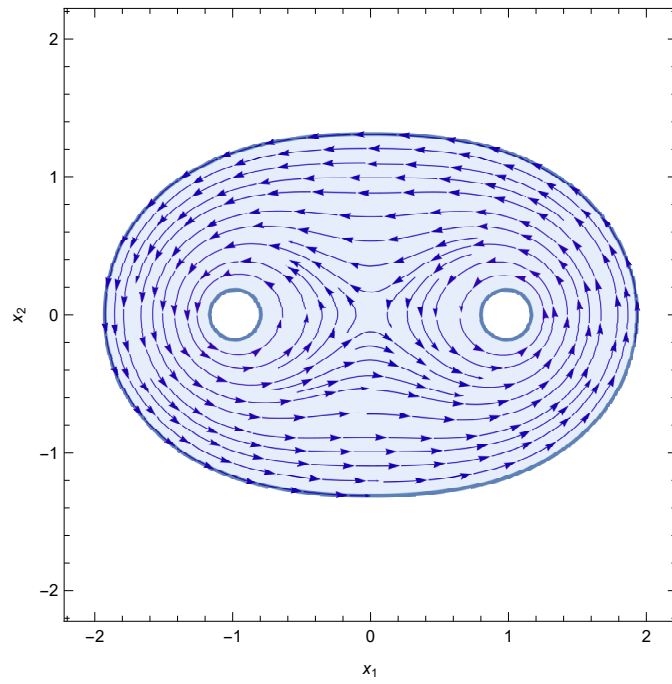
Figure 5.2: A plot of $u = \nabla^\perp \psi$ in the domain $\psi^{-1}([-1,1])$. Here ψ is given by equation (5.8).
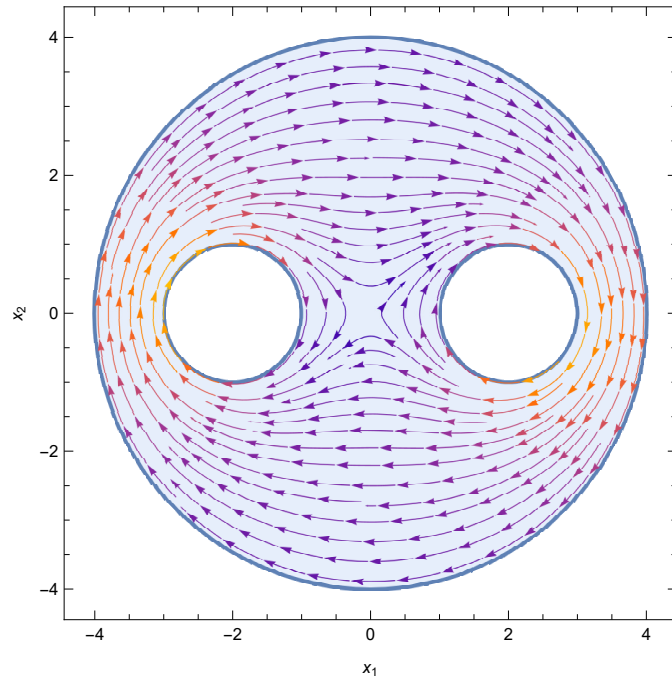

Figure 5.3: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (5.9).
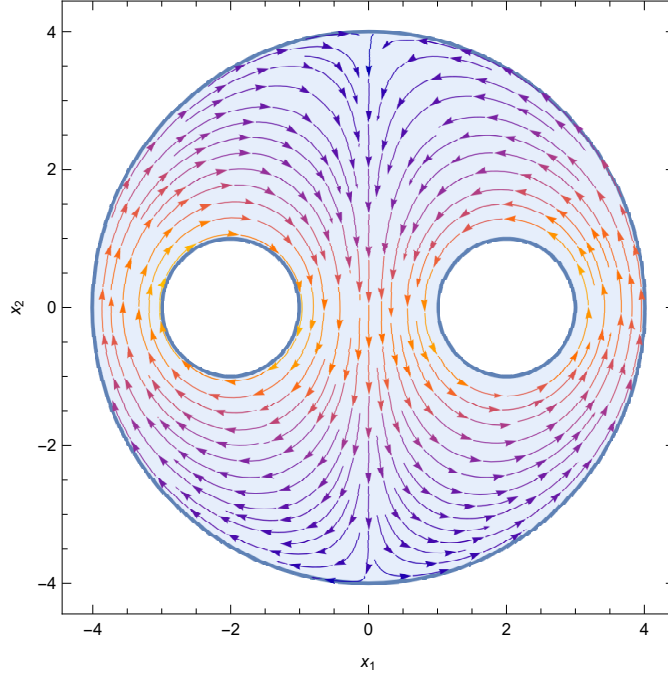
Figure 5.4: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (5.10).

Example 5.7 (Stagnation points on the boundary). In another example given by [25] we fix the domain rather than the function. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We now have the system

$$
\begin{aligned}
\Delta \psi &= 0 && \text{, on } \Omega \\
\psi &= 0 && \text{, on the outer ring } 4S^1 \\
\psi &= -1 && \text{, on the left inner ring } S^1(-2e_1) \\
\psi &= 1 && \text{, on the right inner ring } S^1(2e_1)
\end{aligned}
\tag{5.10}
$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 5.4. Here we obtain from the symmetry $\psi(-x_1, x_2) = \psi(x)$ that $\psi = 0$ on the x_2-axis. Since also $\psi = 0$ on $4S^1$ we have two stagnation points at $\pm 4e_2$.

6 No in- or outflow in $d = 3$ dimensions

We obtain as a quick consequence of the hairy ball theorem

Proposition 6.1. *Let X have Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without boundary stagnation points. Then we have*

$$b_2 \leq b_1\,.$$

Proof. Assume not. Since Ω has b_2 bubbles and b_1 holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no boundary stagnation points on Γ we the restriction $u|_\Gamma$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. $\qquad\square$

We also obtain the following result:

Proposition 6.2. *Let $X \subseteq \mathbb{R}^3$ be a compact manifold with corners and Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a strongly Morse harmonic vector field without boundary critical points. Then the following relation for the interior type numbers of u holds:*

$$M_1 = M_2\,.$$

Proof. As in the two dimensional case we cut the domain X with a surface Γ such that the slit domain is homeomorphic to a ball with bubbles. Since the number of interior stagnation points of u is finite by proposition 2.1, we can choose Γ in such a way that it does not contain any stagnation points. We also denote the arcs at which Γ meets Σ by $\gamma_1, \ldots, \gamma_{b_1} \subseteq \partial\Gamma$. Note that there are b_1 many such curves. We can assume that Γ and the γ_j are manifolds with corners and that Γ approaches each γ_j at a slanted angle. The cut now yields a new domain $\tilde{X}$ which is a covering space of X. On this covering space we denote the cover of the cut Γ and the sets γ_j by Γ^i and γ_j^i with $i \in (1,2)$. Since this new domain $\tilde{X}$ is homeomorphic to a ball with bubbles the vector field $u = \nabla f$ is the gradient of a harmonic function f by proposition 2.10. For the following argumentation we require that u is strongly Morse on $\tilde{X}$, so assume for a moment that this is the case. Now we have that each γ_j is homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Since f is non-degenerate the number of maxima and minima of f on $\gamma_j^1 \cup \gamma_j^2$ must be equal and thus

$$\sum_i \left(\mathrm{Ind}_{\gamma_j^i,0}(f) + \mathrm{Ind}_{\gamma_j^i,1}(-f) \right) = \sum_i \left(\mathrm{Ind}_{\gamma_j^i,1}(f) + \mathrm{Ind}_{\gamma_j^i,0}(-f) \right). \qquad (6.1)$$

Since on Γ all entrant stagnation points of u are also emergent stagnation points of $-u$ (and vice versa) we have the relations

$$\mathrm{Ind}_{\Gamma^1,0}(\pm u) = \mathrm{Ind}_{\Gamma^2,2}(\mp u)$$
$$\mathrm{Ind}_{\Gamma^1,1}(\pm u) = \mathrm{Ind}_{\Gamma^2,1}(\mp u) \tag{6.2}$$
$$\mathrm{Ind}_{\Gamma^1,2}(\pm u) = \mathrm{Ind}_{\Gamma^2,0}(\mp u)\,.$$

As there are no boundary critical points on Σ it follows for the boundary type numbers that

$$\mu_k = \sum_i \left(\mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma_j,k} \right)(f)$$
$$\nu_k = \sum_i \left(\mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma_j,k} \right)(-f)\,. \tag{6.3}$$

Equations (6.3) and (6.2) yield

$$\mu_0 - \nu_2 = \sum_{i,j} \mathrm{Ind}_{\gamma_j,0}(f)$$
$$\mu_1 - \nu_1 = \sum_{i,j} \left(\mathrm{Ind}_{\gamma_j,1}(f) - \mathrm{Ind}_{\gamma_j,1}(-f) \right) \tag{6.4}$$
$$\mu_2 - \nu_0 = -\sum_{i,j} \mathrm{Ind}_{\gamma_j,0}(-f)$$

For f we have the Morse inequalities

$$M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 = -\chi\left(\tilde{X}\right) \tag{6.5}$$

and for $-f$ the Morse inequalities

$$M_1 + \nu_2 - M_2 - \nu_1 + \nu_0 = -\chi\left(\tilde{X}\right)\,. \tag{6.6}$$

Subtracting equation (6.6) from (6.5) and using the relation (6.4) we obtain together with equation (6.1) that

$$0 = 2(M_2 - M_1) + \sum_{i,j} \left(\mathrm{Ind}_{\gamma_j,0}(f) - \mathrm{Ind}_{\gamma_j,1}(f) + \mathrm{Ind}_{\gamma_j,1}(-f) - \mathrm{Ind}_{\gamma_j,0}(-f) \right)$$
$$= 2(M_2 - M_1)$$

from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case let f^ε for $\varepsilon \in E$ be a strongly Morse function as in corollary 1.17. Since x_1, x_2 are non-degenerate critical points of f due to the slanted angle at which Γ approaches each γ_j we obtain that

$$\mathrm{Ind}_{\gamma_j,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma_j,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(-f) \tag{6.7}$$

By the same corollary we can assume that f^ε has no critical points on Σ. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. $\square$

As a consequence we obtain the following:

Corollary 6.3. *Let X be a compact manifold with corners and $u \colon X \to \mathbb{R}^d$ a strongly Morse harmonic vector field without inflow or outflow on the boundary Σ. The we have that*

$$M_1 = M_2 \,.$$

Symbols

d	Dimensions $d = 2$ or $d = 3$
X	Compact domain in $\mathbb{R}^d$, often assumed to be a manifold with corners.
Ω	Interior $\Omega = \text{int}(X)$
$f\colon X \to \mathbb{R}$	A harmonic function.
$u\colon X \to \mathbb{R}^d$ or T^*X	A harmonic vector field.
X_j	A stratification of X as given in definition 1.4. Often but not always assumed to be given by equation (1.1)
Σ	Boundary $\Sigma = \partial X$
$\Sigma^-, \Sigma^{\leq 0}$	(Strictly) entrant boundary. See definition 1.7.
$\Sigma^+, \Sigma^{\geq 0}$	(Strictly) emergent boundary. See definition 1.7.
Σ^0	Tangential boundary. See definition 1.7.
Σ^{irr}	Irregular boundary. See definition 1.9.
$B_r(x), B_r$	Ball of radius r around the point x / the origin.
$S^{d-1}(x), S^{d-1}$	$(d-1)$-dimensional sphere around x / the origin.
e_j	j-th unit vector in $\mathbb{R}^d$.
u_j	Projection of u to the cotangent bundle T^*X_j. See equation (1.6).
π_j	Orthogonal projection onto the cotangent bundle T^*X_j. See equation (1.5).
Cr_j	Set of essential stagnation points. See definition 1.9.
$\text{Ind}_{j,k}$	k-th type number on the stratum X_j. See equation (1.7).
Ind_k	k-th type number. See equation (1.11).
M_k	k-th interior type numbers. See equation (1.8).
M	Total number of stagnation points. See equation (1.9).
μ_k	k-th boundary type numbers of f. See equation (1.10).
ν_k	k-th boundary type numbers of $-f$. See definition 1.9.
u^ε	modification to u as in equation (1.12)
A	submanifold, can be thought of as the zero section of T^*X
b_k	Betti number as defined in equation (2.2)
$\chi(X)$	Euler characteristic.
$\nabla^\perp$	Orthogonal gradient. Given by equation (3.9).
Φ_2	Multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$. Given by equation (3.11).
$V_U(\dots), V(\dots)$	Algebraic variety.
$d_H(A, B)$	Hausdorff metric. See equation (4.10).

$\text{dist}(x, A)$ Distance between x and A. See equation (4.11)

$\text{Tub}_\delta(A)$ Tubular neighbourhood of A. See equation (4.12).

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