Stagnation Points of Harmonic Vector Fields and the Domain Topology

Some Applications of Morse Theory

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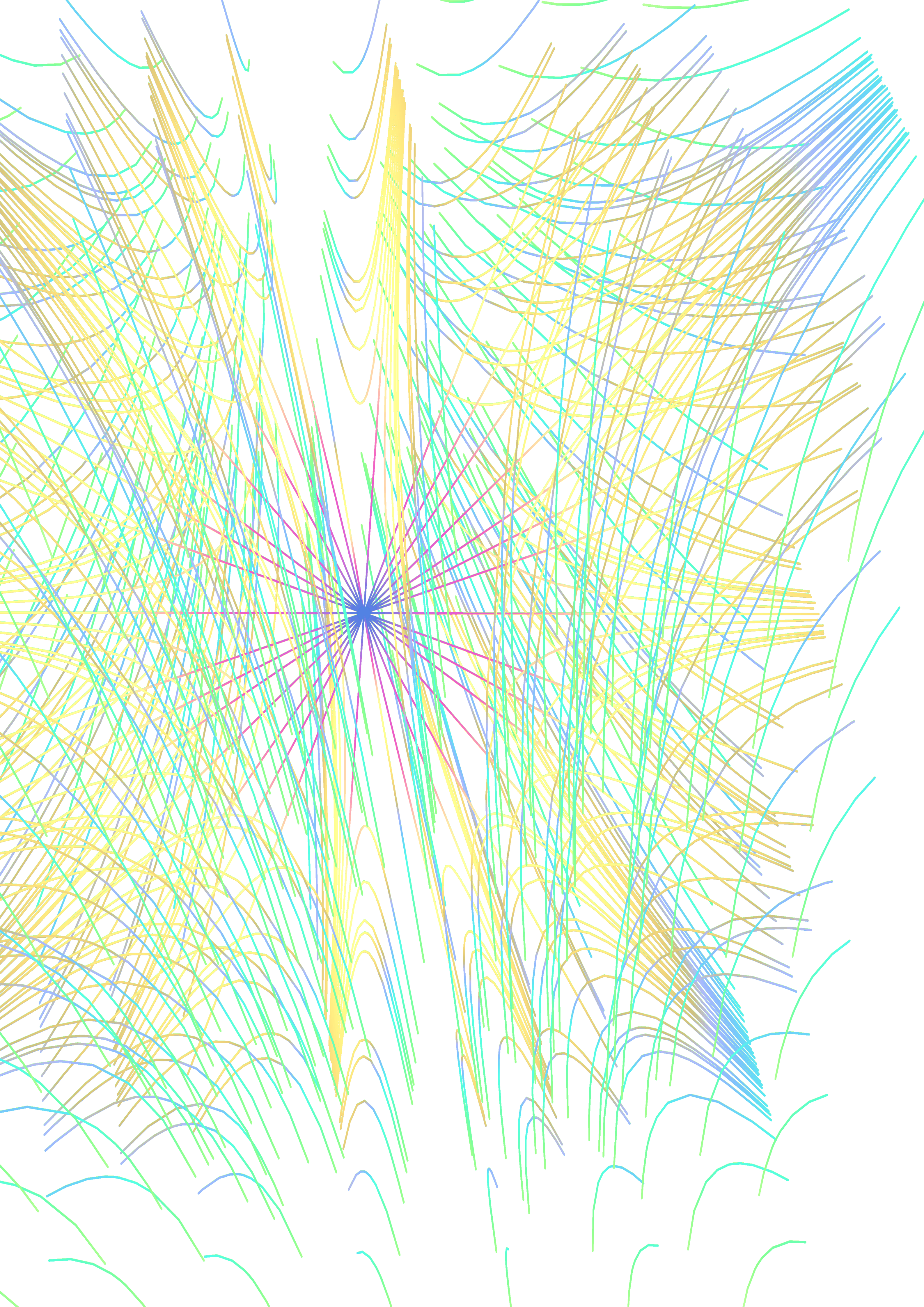


Abstract

Given a harmonic vector field u on a bounded domain we call the set on which u enters the domain entrant boundary and the set on which u exits the domain emergent boundary. In the first part of this thesis we ask the question of when it is possible to have interior stagnation points and connected entrant and emergent boundaries. The answer in two dimensions for a domain homeomorphic to a disc is 'no'. If one allows for holes in the domain the answer becomes a 'yes' for which we give explicit examples. In four dimensions a simple example shows that this is possible with the ball as the domain. In three dimensions we use Morse theory to argue that the number of stagnation points is even. With the help of this result and numerical methods we found a harmonic polynomial on the ball which has interior stagnation points and simply connected entrant and emergent sets.

The second part revolves around harmonic vector fields with interior stagnation points and without boundary stagnation points. This question yields in two dimensions an elegant relation between the domain topology and the number of interior stagnation points. As a special case we obtain statements about harmonic vector fields which are tangential and nonvanishing at the boundary. We also give examples of vector fields illustrating this point. In three dimensions we do not find an explicit example but the Morse index theorem implies that for such an example the number of stagnation points has to be even and the Euler characteristic of the domain has to vanish.

As part of the thesis we formulate a version of Morse theory by [7] and [1] for manifolds with corners. This relates the domain topology with the number of stagnation points. Additionally we give a proof for the density of Morse functions for this type of Morse theory.

Populärvetenskaplig sammanfattning

Givet ett harmoniskt vektorfält u på en begränsad domän kallar vi den mängd där u träder in i domänen för inträdesrand och den mängd där u lämnar domänen utträdesrand. I den första delen av detta examensarbete ställer vi frågan om det är möjligt att ha inre stagnationspunkter och sammanhängande inträdes- och utträdesrand. Svaret i två dimensioner är 'nej' för en domän som är homeomorf till skivan. Om man tillåter hål i domänen blir svaret 'ja', vilket vi ger explicita exempel på. I fyra dimensioner visar ett enkelt exempel att detta är möjligt med klotet som domän. I tre dimensioner använder vi Morse-teorin för att visa att antalet stagnationspunkter är jämnt. Med hjälp av detta resultat och numeriska metoder fann vi ett harmoniskt polynom i klotet som har inre stagnationspunkter och enkelt sammanhängande inträdes- och utträdesrand.

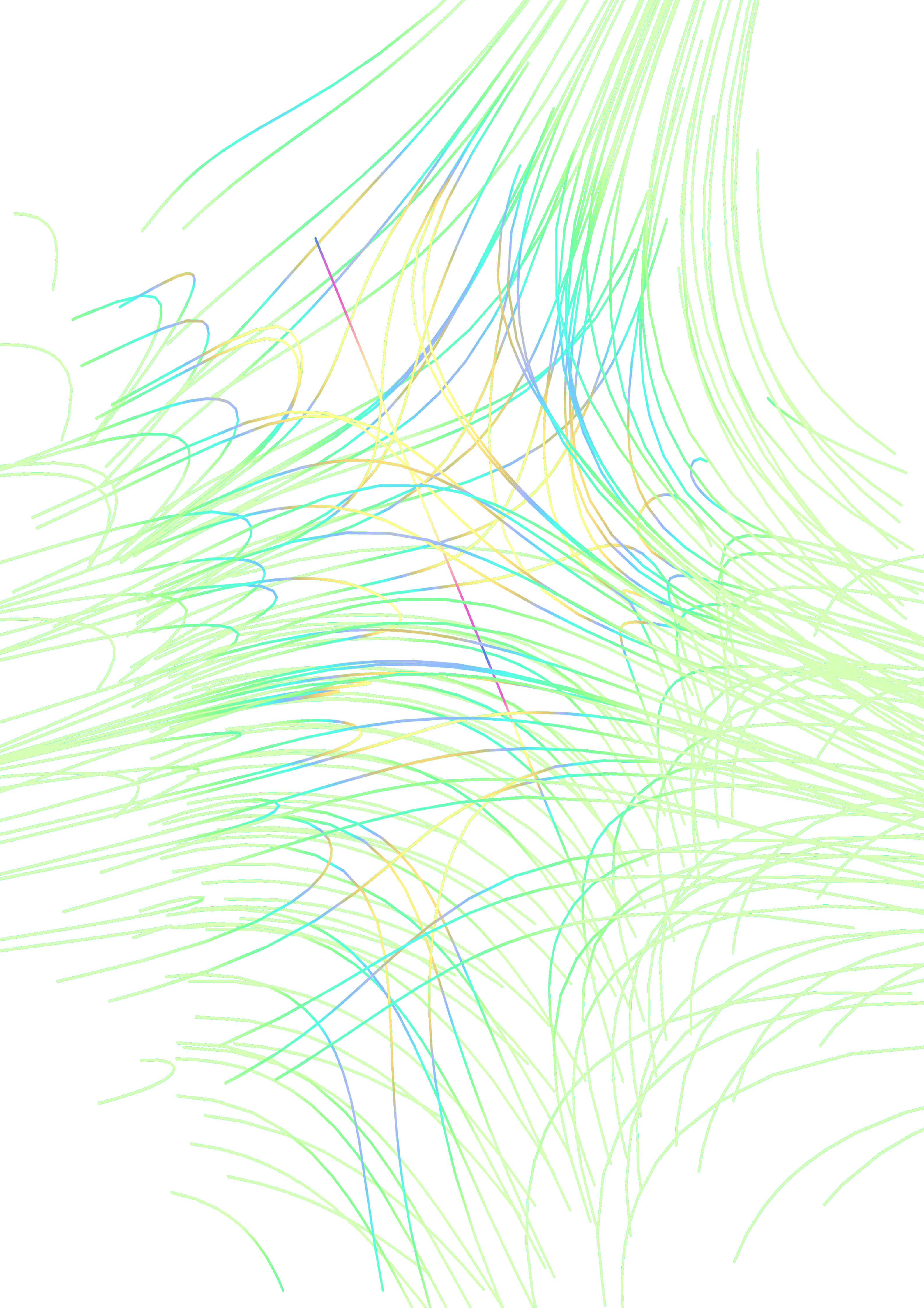
Den andra delen handlar om harmoniska vektorfält med inre stagnationspunkter och utan stagnationspunkter på randen. Denna fråga i två dimensioner ger ett mycket elegant samband mellan domänens topologi och antalet inre stagnationspunkter. Som ett specialfall får vi uttalanden om harmoniska vektorfält som är tangentiella och icke försvinnande vid randen. Vi ger också exempel på vektorfält som illustrerar denna punkt. I tre dimensioner hittar vi inget sådant exempel, men Morse indexsats implicerar att för ett sådant exempel måste antalet stagnationspunkter vara jämnt och Euler-karakteristiken för domänen måste försvinna.

Som en del av examensarbetet formulerar vi en version av Morse-teorin enligt [7] och [1] för mångfalder med kantiga hörn. Denna relaterar domäntopologin till antalet stagnationspunkter. Dessutom ger vi ett bevis för tätheten av Morsefunktioner för denna typ av Morse-teori.

Acknowledgements

I would like to express my gratitude to my supervisor Erik Wahlén for having posed many interesting questions which have led to this thesis and who has encouraged me throughout. The stimulating discussions and suggestions were very helpful. Additionally I would like to thank Martin Löfström, Anna-Lisa Rathsmann and Thomas Renström for the Master thesis discussions and Ludvig Sundell for having pointed out Gröbner bases to me. Finally I would like to thank my family, friends and teachers for having supported me throughout my studies.

Contents



1 Introduction

Harmonic vector fields arise naturally within certain areas of mathematics and physics. Classical electrostatic, magnetostatic and gravitational fields in vacuum are examples of harmonic vector fields. Other examples are the heat flow in a system that has reached steady state or an irrotational flow of an inviscid incompressible medium. The null points of for instance the electrostatic or magnetostatic fields are particularly interesting. These are examples of stagnation points of harmonic vector fields. If one considers the underlying potentials which exist at least locally then these stagnation points are critical points of a harmonic function. The study of these points has a long tradition. Walsh gives in [36] a comprehensive overview of the state of knowledge regarding critical points of harmonic functions in 1950. In particular complex analysis by that time had clarified the situation in two dimensions significantly though far less was (and is) known about the situation in three dimensions. 20 years later Morse applies his theory in [30] and [29] to the critical points of harmonic functions. Morse gives in both works a set of inequalities for two and three dimensions relating the number and type of critical points on the boundary and the interior with the domain topology. He then poses in [29] the question: To a given set of numbers of critical points and a domain topology fulfilling these inequalities does there exists a harmonic function in $\mathbb{R}^3$ with precisely these numbers of critical points? This question is answered affirmatively in 1980 in [32] with a construction using line charges. Inspired by this we will apply Morse theory to harmonic functions in an attempt to answer the following question:

Question 1.1 (Flowthrough with stagnation point, [35])**.** Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

This question is in part inspired by the requirement that u be free of stagnation points for an existence result of steady, inviscid and incompressible flows with nonvanishing vorticity in $\mathbb{R}^3$ given in [2]. The answer will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Somewhat related, [23, p.198] poses the question whether it is possible to have a harmonic vector field without inflow or outflow through the boundary with interior stagnation points in $\mathbb{R}^3$. Inspired by this we will also consider the following question:

Question 1.2 (Harmonic vector fields without inflow or outflow, [35])**.** Let $u\colon X \to \mathbb{R}^d$ be a harmonic vector field in a domain $X \subset \mathbb{R}^d$ such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

In fact we will use the Morse index theorem to deal with the following more general question:

Question 1.3 (Harmonic vector fields without boundary stagnation points)**.** Let $u\colon X \to \mathbb{R}^d$ be a harmonic vector field on a domain $X \subset \mathbb{R}^d$ without boundary stagnation points. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. More concretely the Morse index theorem implies that the number of stagnation points equals the negative Euler characteristic of the domain. In $d = 3$ dimensions we show that the number of stagnation points has to be even and the Euler characteristic of the domain has to vanish.

The thesis starts out with some mathematical preliminaries. Here we discuss some important definitions to make the formulation of these questions more precise. We also show the density of Morse functions. In the next chapter we state the Morse inequalities for manifolds with corners and give a motivation as to why they hold. The following chapter gives a negative answer to question 1.1 in the case of two dimensions. We then give examples of planar harmonic vector fields which give an affirmative answer to this question if we allow for holes in the domain. We also give an example in four dimensions which answers question 1.1 affirmatively. Chapter 5 then deals with the three-dimensional case where we give an explicit example which answers question 1.1 affirmatively in three dimensions. In the final chapter we deal with question 1.3. Here we state and apply the Morse index theorem in two and three dimensions and give explicit examples of harmonic vector fields without inflow or outflow through the boundary in $d = 2$ dimensions.

This thesis is directed to an audience with basic knowledge of harmonic functions such as the maximum principle, the mean value property and the existence and uniqueness of solutions. For a comprehensive introduction to the theory of partial differential equations more generally we refer the reader to [12]. We also assume the reader has had exposure to basic topology and some form of differential topology or differential geometry. In some parts knowledge of complex analysis as in for example [13] is required though it is not essential to understand the thesis. Elementary knowledge of convex optimisation is helpful but not a prerequisite. Despite Morse theory playing a central role in this thesis we do not assume that the reader has had prior exposure to this topic. To simplify things we avoid the language of homology theory where possible and instead give the theory in the classical sense as a set of inequalities. Thus we do not assume prior knowledge of algebraic topology.

2 Mathematical preliminaries

In this chapter we give some important definitions. We start of by defining harmonic vector fields. After that we define manifolds with corners, stratified spaces, tangent spaces and the entrant and emergent boundaries. We then give a definition of stagnation points, type numbers and Morse functions. Throughout we give examples and results to illustrate the definitions. In the final part of this chapter we prove that harmonic vector fields can essentially be approximated by harmonic Morse vector fields whilst preserving certain properties.

Harmonic vector fields

Unless otherwise stated we denote by $X \subset \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\operatorname{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. Throughout the thesis we denote by

$$f \colon X \to \mathbb{R}$$

a C^2 function on X. Often f will be assumed to be *harmonic*, that is $\Delta f = 0$ on $\operatorname{int}(X)$ where Δ denotes the Laplace operator. We also denote by

$$u \colon X \to \mathbb{R}^d$$

a vector field of class C^1. In the following we often assume that u is in fact a *harmonic vector field*, that is u fulfils $\operatorname{div} u = 0$ and $\operatorname{curl} u = 0$ on $\operatorname{int}(X)$. Note that a harmonic function f gives rise to a harmonic vector field via its gradient $u = \nabla f$. In general a harmonic vector field u is not globally the gradient of a harmonic function. This is illustrated for instance in example 4.5. On simply connected domains this implication is however true:

Proposition 2.1 (Harmonic vector fields on simply connected domains)**.** *Let $d \in \{2,3\}$ and $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and $u \colon \Omega \to \mathbb{R}^d$ be a harmonic vector field. Then*

1. *f is conservative, that is $u = \nabla f$ is the gradient field of some function $f \colon \Omega \to \mathbb{R}$.*
2. *f is harmonic.*
3. *u is in fact smooth.*
4. *The components u_i are harmonic.*

Proof. 1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.

2. This follows from $\Delta f = \operatorname{div} u = 0$.

3. This follows from the fact that f is harmonic.

4. This follows from $u_i = \partial_i f$. $\square$

If one considers not necessarily simply connected domains Ω then we obtain the properties of proposition 2.1 at least locally. More generally, we call a vector field u *irrotational* if it is at every point locally the gradient of a function f. In the following we assume that our vector fields are irrotational, unless otherwise stated.

Stratified spaces and the entrant and emergent boundary

We start by requiring some regularity for the boundary of X. For one, we require X to be a compact manifold with corners:

Definition 2.2 (Manifolds with corners, [15])**.** We introduce the notation

$$H_j^d := \mathbb{R}_{\geq 0}^j \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d$$

where $j \in \{0, \dots, d\}$. A *manifold with (convex) corners* is a topological space X together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exist open neighbourhood $U_x \subseteq X$ of x and $V_0 \subseteq H_j^d$ of the origin, a number $j = j(x)$ and a diffeomorphism $\phi_x \colon U_x \to V_0$ in $\mathscr{A}$ with $\phi_x(x) = 0$. Here diffeomorphism means that for any two charts ϕ_x and ψ_x around x there exist locally extensions of $\psi_x \circ \phi_x^{-1}$ to open sets in $\mathbb{R}^d$ which are diffeomorphic onto their image. We further define for $k \in \{0, \dots, d\}$ a collection of sets

$$X_k := \{x \in X \colon \ j(x) = d - k\} \tag{2.1}$$

which form a stratification of X.

More generally we give the definition of a stratification as follows:

Definition 2.3 (Stratified space, [15])**.** Let X be a topological space. A *stratum* is a subspace $X_j \subseteq X$, $j \in \mathscr{J}$, indexed by a partially ordered set $\mathscr{J}$ such that

1. each X_j is a manifold without boundary of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$ iff $j \prec k$.

The pair of X and the collection of strata is called a *stratified space* and we call n the *dimension of the stratum* X_j. In the case that $X_j \subseteq \overline{X}_k$ we will write $X_j \prec X_k$ using the notation of [15]. If additionally $n(k) = n(j) + 1$ we will write $X_j \precsim X_k$ and, indicating that the strata differ by one in their dimension, we may write $X_k = X_{j+1}$.

In the case that the stratification arises through relation (2.1) we have precisely $X_j \precsim X_{j+1}$ for $j \in \{0,\dots,d-1\}$. Note that in general for a given stratum X_j the stratum X_{j+1} such that $X_j \precsim X_{j+1}$ need not be unique. Unless otherwise stated we assume in the following that the stratification is finite, that is $\#\mathscr{J} < \infty$, and that the interior $\mathrm{int}(X)$ corresponds to a single stratum. The next definition generalises the definition of tangent space in [21, §1.2] to manifolds with boundary.

Definition 2.4 (Tangent space, [21, §1.2])**.** Let X be a differentiable manifold with corners and $x \in X$. Let $\mathfrak{TX}$ denote the set of all pairs (ϕ, v) of charts ϕ around x and tangent vectors $v \in T_{\phi(x)}\mathbb{R}^d \cong \mathbb{R}^d$. We say two such pairs $(\phi, v) \sim (\psi, w)$ are equivalent if we have the relation $w = D\big(\psi \circ \phi^{-1}\big)(v)$. The *tangent space* T_xX to X at x is the quotient $\mathfrak{TX}/_\sim$ with respect to this equivalence relation. The *tangent bundle* TX is the union of tangent spaces T_xX for each $x \in X$ where the differentiable structure on TX is defined in the following way: Define the projection $\pi(v) = x$ for a tangent vector $v \in T_xX$. For a given chart $\phi \colon U \to V$ we define a chart

$$\begin{aligned} D\phi \colon TU &\to TV \cong V \times \mathbb{R}^d \\ v &\mapsto D\phi\big|_{\pi(v)}(v) \end{aligned}$$

on the tangent bundle TX.

We also give the definition of the contingent cone:

Definition 2.5 (Contingent cone, [20, Def. 4.6])**.** Denote the *(Bouligand) contingent cone* for a set $A \subseteq \mathbb{R}^d$ at the point $x \in \overline{A}$ by C_xA. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exist sequences $\lambda \colon \mathbb{N} \to \mathbb{R}_{>0}$ and $x_n \to x$ in A such that

$$\lim_n \lambda_n(x_n - x) = v\,.$$

This forms a subset of the tangent space $T_x\mathbb{R}^d$. Let X be a manifold with corners and $A \subseteq X$ a subset. At a point $x \in X$ with a chart $\phi \colon U \to V$ at x we can define the contingent cone to be given by

$$C_xA := D\phi^{-1}\big|_y(C_y\phi(U \cap A))$$

where $y := \phi(x)$.

In general we will also require X to be a Riemannian manifold in order to be able to talk about gradients. Since we will in the following assume that $X \subset \mathbb{R}^d$ is embedded we will assume that we chose the metric induced by the standard metric in $\mathbb{R}^d$. To clarify the previous definitions we give an example:

Example 2.6 (Cubical domain)**.** Consider the domain to be the cube $X = [-1,1]^2 \subset \mathbb{R}^2$. This is a manifold with corners. The stratification according to equation (2.1) is given by

$$\begin{aligned} X_0 &= \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\ X_1 &= (I \times \{-1,1\}) \cup (\{-1,1\} \times I) \\ X_2 &= I \times I \end{aligned}$$

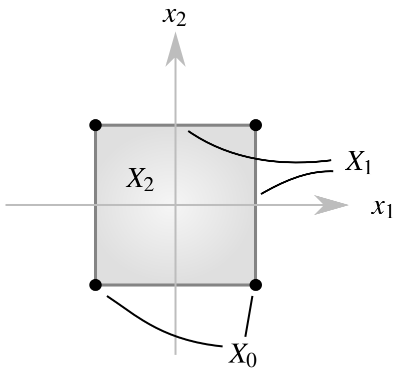


Figure 2.1: A stratification of X.

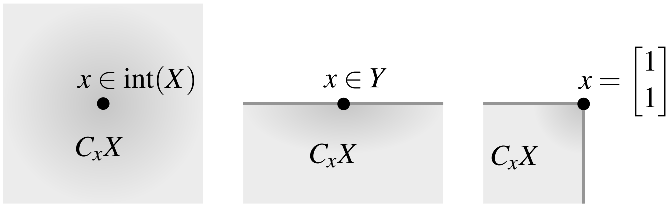


Figure 2.2: The contingent cones for various $x \in X$.

where we set $I := (-1,1) \subset \mathbb{R}$. The stratification is depicted in figure 2.1. For an interior point $x \in X_2$ we have the contingent cone $C_xX = \mathbb{R}^2$ and the tangent space $T_xX = \mathbb{R}^2$. For a boundary point $x \in Y := I \times \{1\} \subset X_1$ on the upper edge we have the contingent cone

$$C_xX = \{v \in T_xX \colon v \cdot n \leq 0\} \tag{2.2}$$

where the outer unit normal $n = e_2$ is the second basis vector and the tangent space is $T_xX = \mathbb{R}^2$. At the boundary point $x = e_1 + e_2 \in X_0$ we have

$$C_xX = \{v \in T_xX \colon v_1 \leq 0 \text{ and } v_2 \leq 0\} \tag{2.3}$$

where again $T_xX = \mathbb{R}^2$. The situation is depicted in figure 2.2. The contingent cone on the other parts of the square $\Sigma = \partial X$ is given by similar formulas.

We remind the reader of the concept of relative interior.

Definition 2.7 (Relative interior, [6, Chapter 2.1.3])**.** For a subset A of a vector space V denote by $\mathrm{aff}A$ the affine hull of A in V. Then we define the relative interior $\mathrm{rel\,int}A$ to be

$$\mathrm{rel\,int}A = \{x \in A \colon \text{There exists } r > 0 \text{ such that } \mathrm{aff}A \cap B_r(x) \subseteq A\}.$$

We can now define the emergent and the entrant boundary in a way that generalises [29, p.282] for manifolds with corners.

Definition 2.8 (Emergent and entrant boundary)**.** We call a vector $v \in T_xX$ *entrant* at a boundary point $x \in \Sigma$ if

1. v lies in the contingent cone C_xX or
2. v lies in $(C_xX)^* \oplus \left((C_xX)^*\right)^\perp$.

Here the set

$$(C_xX)^* = \left\{w \in T_xX \colon w \cdot w' \geq 0 \text{ for all } w' \in C_xX\right\} \tag{2.4}$$

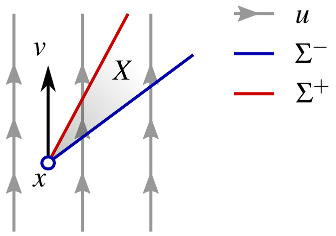


Figure 2.3: The vector v is entrant.

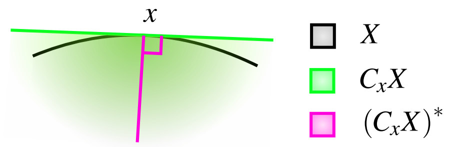


Figure 2.4: Depiction of $(C_xX)^*$ for a manifold with C^1 boundary.

denotes the dual cone of the contingent cone C_xX and $\left((C_xX)^*\right)^\perp$ denotes its orthogonal complement in T_xX. We call v *strictly entrant* if in addition v lies in the interior $\operatorname{int}(C_xX)$ or if v lies in the interior

$$\operatorname{int}\Big((C_xX)^* \oplus \left((C_xX)^*\right)^\perp\Big) = \operatorname{rel\,int}(C_xX)^* \oplus \left((C_xX)^*\right)^\perp.$$

Analogously v is *(strictly) emergent* if $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which u is entrant. We define the *strictly entrant boundary* Σ^- to be the set of strictly entrant boundary points of u. In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* Σ^+. Further define the *tangential boundary* Σ^0 to be

$$\Sigma^0 := \left(\Sigma^{\leq 0} \cup \Sigma^{\geq 0}\right) \setminus \left(\Sigma^+ \cup \Sigma^-\right) \subseteq \Sigma.$$

Although the condition that we call a vector entrant if it lies in the contingent cone C_xX is more or less intuitive the second criteria requires a little motivation. For this consider a vector field u and a tangent vector v at the corner point x as in figure 2.3. The domain X forms an acute angle at x and the v almost points into C_xX. Since v however does not point into X it does not satisfy the first condition. Now assume we place a test particle in a neighbourhood of x which is confined to X and u describes a force acting on this particle. This test particle will then move away from the point x, possibly along the boundary Σ^+. It is in this manner that we would like to call u at x entrant. This type of reasoning is made precise in the construction of a flow on X in [15]. Since this flow is generated by a discontinuous vector field several technicalities arise which is why proofs of the Morse inequalities tend to take other approaches. It motivates however why we added a second criteria for calling v entrant.

We would also like to motivate the naming of the tangential boundary. Assume that X is a manifold with C^1 boundary. Let $x \in \Sigma$ be a boundary point and let n denote the outer unit normal at x. Then we have that

$$(C_xX)^* = \left\{w \in T_xX \colon w \cdot w' \geq 0 \text{ for all } w' \in C_xX\right\} = \{-rn \colon r \geq 0\}.$$

This situation is depicted in figure 2.4. If the vector v lies in $(C_xX)^*$ but not in the relative interior $\operatorname{rel\,int}(C_xX)^*$ we in fact have that $v = 0$. Thus the tangential boundary Σ^0 consists in this case of precisely those boundary points at which u is tangential to the boundary or vanishes. We would now like to illustrate definition 2.8 with a more concrete example.

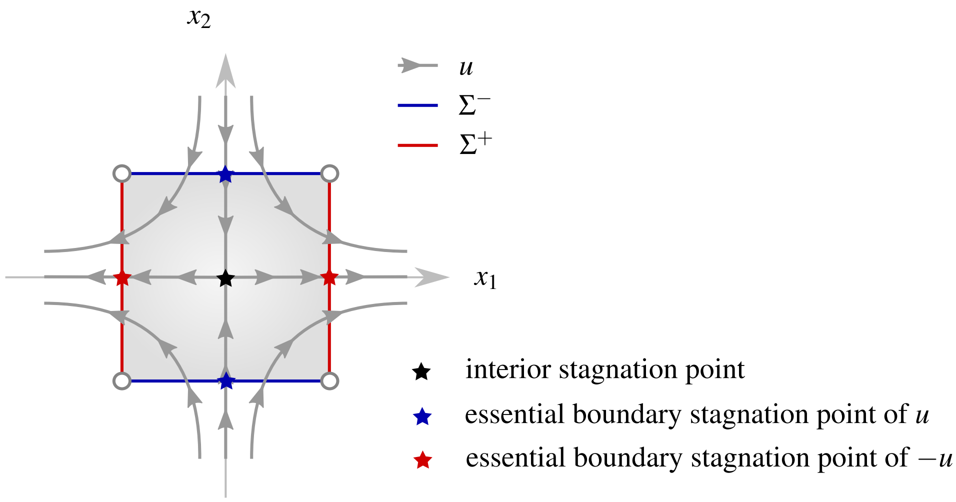


Figure 2.5: The entrant and emergent boundaries in example (2.9)

Example 2.9 (A vector field on the cube)**.** Consider the domain to be the cube $X = [-1,1]^2 \subset \mathbb{R}^2$ and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 - x_2^2 . \end{aligned} \tag{2.5}$$

This induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u\colon X &\to \mathbb{R}^2 \\ x &\mapsto 2\begin{bmatrix} x_1 & -x_2 \end{bmatrix}^\top . \end{aligned} \tag{2.6}$$

For a boundary point $x \in I \times \{1\}$ on the upper edge we have that

$$0 > -2x_2 = n \cdot u$$

and thus by relation (2.2) it follows that $x \in \Sigma^-$ so $I \times \{1\} \subseteq \Sigma^-$. At the corner point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ the dual of the contingent cone is given by

$$(C_xX)^* = (C_xX)^* \oplus \left((C_xX)^*\right)^\perp = C_xX \tag{2.7}$$

where we used the characterisation of equation (2.3). Since $v = u(x) = 2\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ we have that $v \notin (C_xX)^*$ and $-v \notin (C_xX)^*$ and thus $x \notin \Sigma^{\geq 0} \cup \Sigma^{\leq 0}$. By analogous argumentation on the other sides of the square $\Sigma = \partial X$ one obtains that

$$\begin{aligned} \Sigma^- &= \Sigma^{\leq 0} = I \times \{-1,1\} \\ \Sigma^+ &= \Sigma^{\geq 0} = \{-1,1\} \times I . \end{aligned}$$

A sketch of the sets can be seen in figure 2.5.

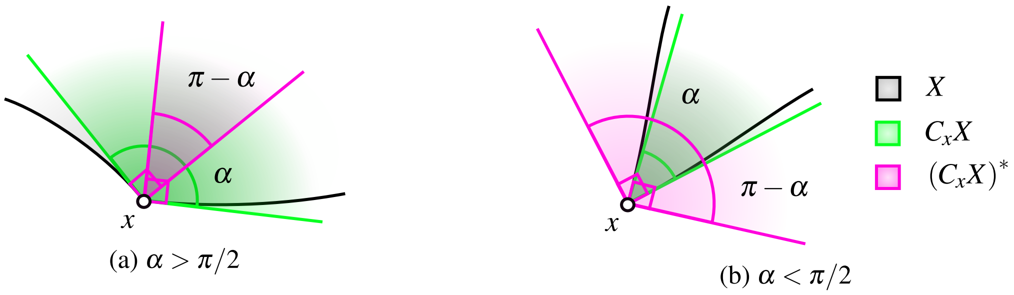


Figure 2.6: Illustration of a planar contingent cone and its dual cone at a corner.

Since the dual of the contingent cone $(C_xX)^*$ will also play a role later on we make a further remark to illustrate the concept.

Remark (The contingent cone and its dual). If the angle of the cone C_xX at a corner point x in the plane is α the angle formed by its dual cone $(C_xX)^*$ is $\pi - \alpha$. Thus equation (2.7) in the previous example is a special case for $\alpha = \pi/2$. The more general case is illustrated in figure 2.6.

Stagnation points

Given a vector field $u\colon X \to TX$ and a stratification X_j of X we can construct for every $j \in \mathscr{J}$ a vector field

$$u_j\colon X_j \to TX_j\,.$$

More precisely, for $x \in X_j$ let

$$\pi_j\big|_x\colon T_xX \to T_xX_j \tag{2.8}$$

denote the orthogonal projection of a tangent vector at x onto the tangent space of the stratum X_j at x. Let further

$$u_j := \pi_j \circ u\big|_{X_j} \in C^1(TX_j) \tag{2.9}$$

be the projection of u onto the tangent bundle TX_j. We can now give a definition of stagnation points and their index. In the following we generalise definitions given in [32, p.138f], [30, §5] and [29, p.282f] to include vector fields which are not necessarily gradient fields.

Definition 2.10 (Stagnation points)**.** Let $u_j\colon X_j \to TX_j$ be the C^1 vector field on a stratum X_j of X given by equation (2.9). We call the zeroes $x \in X_j$ of u_j *stagnation points of u_j on X_j*. If additionally $x \in \operatorname{int}(X)$ is contained in the interior then we call x an *interior stagnation point* else x is a *boundary stagnation point*. If $u(x) \in (C_xX)^*$, that is the stagnation point x lies in the entrant boundary $\Sigma^{\leq 0}$ or the interior $\operatorname{int}(X)$, we call x an *essential stagnation point*. The set of all essential stagnation points of u_j is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$ and the essential stagnation points of u and $-u$ on X_j are called *stagnation points of u on X_j*.

The reason for only considering essential stagnation points has to do with the Morse inequalities. In many proofs a flow is constructed along which f increases. Since this flow would leave the domain at emergent but not at entrant stagnation points of $u = \nabla f$ one sees that entrant and emergent stagnation points are different in character. For more details the reader is referred to chapter 3.

Definition 2.11 (Non-degeneracy and the Morse index)**.** A stagnation point x of the vector field $u_j \colon X_j \to TX_j$ is called *non-degenerate* if the derivative

$$Du_j(x) = Du_j\big|_x \colon T_xX_j \to T_xTX_j$$

is bijective. If in addition u_j is irrotational we say that x has *index* k if $Du_j(x)$ has exactly k negative eigenvalues. u_j is called *(essentially) non-degenerate* if all its (essential) stagnation points are non-degenerate. Finally, we call essential stagnation points x of u_j *regular* if additionally $u(x) \in \operatorname{rel\,int}(C_xX)^*$, that is x lies in the strictly entrant boundary Σ^- or is an interior stagnation point. u_j is called *regular* if all essential stagnation points of u_j are regular. Essential boundary stagnation points of u and $-u$ which are not regular are called *irregular stagnation points*. We can define the *k-th type number* $\operatorname{Ind}^{\mathrm{M}}_{j,k}(u)$ of the stratum X_j to be the number of regular non-degenerate stagnation points of u_j of Morse index k, that is

$$\operatorname{Ind}^{\mathrm{M}}_{j,k}(u) := \#\big\{x \in \operatorname{Cr}_j(u) \colon x \text{ has index } k\big\}. \tag{2.10}$$

Note that most authors refer to regular and essential non-degenerate stagnation points as non-degenerate stagnation points and that the definition of regular stagnation points was introduced to distinguish between these different concepts. We also emphasize that the Morse index and the type numbers are only defined for irrotational vector fields. To illustrate the preceding definitions we return to our previous example.

Example 2.12 (Stagnation points)**.** Let X, f and u be as in example 2.9. We have that $u_2 = u$ and thus one sees from equation (2.6) that the origin is the sole stagnation point of u on the stratum X_2. Since we have that

$$Du(x) = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}$$

for all $x \in \operatorname{int}(X)$ we see that $Du(0)$ is bijective and thus the origin is a non-degenerate interior stagnation point. As $Du(0)$ has exactly one negative eigenvalue we see that the origin has index 1. Since an interior stagnation point is also an essential stagnation point we have $\operatorname{Ind}^{\mathrm{M}}_{2,k} = \delta_{k1}$ where δ denotes the Kronecker delta. For $x \in I \times \{1\} =: Y$ on the upper edge we calculate

$$u_1(x) = \pi_1 \circ u(x) = (u - (n \cdot u)n)(x) = 2x_1e_1$$

with n the outer unit normal. Thus we have that $x = e_2$ is the unique stagnation point of u on Y. Consider the curve

$$\begin{aligned} \gamma \colon I &\to Y \\ t &\mapsto \begin{bmatrix} t & 1 \end{bmatrix}^\top \end{aligned}$$

then $\gamma(0) = e_2$ and we have

$$Du_1\big|_{e_2}(e_1) = Du_1\big|_{e_2}\big(\gamma'(0)\big) = (u_1 \circ \gamma)'(0) = (2te_1)'(0) = 2e_1$$

and thus e_1 is an eigenvector of $Du_1(e_2)$ to the eigenvalue 2. As e_1 spans the tangent space $T_{e_2}Y$ it follows that e_2 is a non-degenerate stagnation point of u_1 with index 0. Now since in fact $u(e_2) \in \operatorname{rel\,int}(C_xX)^*$ we have that e_2 is an essential non-degenerate regular stagnation point. Proceeding in this manner for the other segments of the square Σ we obtain that $\operatorname{Ind}^{\mathrm{M}}_{1,k} = 2\delta_{0k}$. The essential stagnation points of u and $-u$ are also depicted in figure 2.5. If we now consider the corner point $x = e_1 + e_2$ then we have that $u_0(x) = 0$ and thus x is a stagnation point. Now the derivative

$$0 = Du_0 \colon 0 = T_xX_0 \to T_xTX_0 = 0$$

is bijective and thus we have that x is a non-degenerate stagnation point of index 0. Since however $u(x) \notin (C_xX)^*$ we have that x is not an essential stagnation point. Analogous argumentation on the other three corners yields that $\operatorname{Ind}^{\mathrm{M}}_{0,0} = 0$ vanishes.

Notice that it follows from the definition that a stagnation point of u_j is irregular iff it lies in the tangential boundary Σ^0. We would like to clarify the definition of irregular stagnation points further with the following characterisation:

Proposition 2.13 (Characterisation of irregular points). *The condition that the stagnation point $x \in X_j$ is irregular is equivalent to that x is a stagnation point of u_{j+1} for a stratum $X_j \precsim X_{j+1}$.*

We first remark that x being a stagnation point of u_{j+1} makes at first sight no sense since $x \notin X_{j+1}$. We have however that $x \in \overline{X}_{j+1}$. The concept of tangent space T_xX_{j+1} and the projection π_{j+1} can be extended to the closure $\overline{X}_{j+1}$ by continuity and so we can reasonably speak of evaluating u_{j+1} at the point x.

Proof. We first claim that for some boundary point $x \in \Sigma \cap X_j$ the relations

$$\begin{aligned}
\partial(C_xX)^* &:= (C_xX)^* \setminus \operatorname{rel\,int}(C_xX)^* \\
&\overset{(+)}{=} \left\{ w \in T_xX \;\middle|\; \begin{array}{l} w \cdot w' \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ w \cdot w' = 0 \text{ for some } w' \in C_xX \setminus T_xX_j \end{array} \right\} \\
&\overset{(*)}{=} \left\{ w \in T_xX \;\middle|\; \begin{array}{l} w \cdot w' \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ w \cdot w' = 0 \text{ for some } w' \in T_xX_{j+1} \setminus T_xX_j \\ \text{where } X_j \precsim X_{j+1} \end{array} \right\}
\end{aligned} \tag{2.11}$$

hold. To see the equality $(+)$ let $w \in \partial(C_xX)^*$ and $w_k \in T_xX \setminus (C_xX)^*$ be a sequence converging to w. Then there exists for each w_k a $w'_k \in C_xX$ such that $w_k \cdot w'_k < 0$. Since the spaces $T_xX_j \perp (C_xX)^*$ are perpendicular we can assume that the component of w'_k projected onto T_xX_j vanishes. After rescaling we can assume that $|w'_k| = 1$ and after taking a subsequence that the w'_k converge to a $w' \in C_xX$. But then we have by continuity that $w \cdot w' \leq 0$ and $w' \in C_xX \setminus T_xX_j$ since $|w'| = 1$ and the component of w' projected onto T_xX_j vanishes. Thus the inclusion '$\subseteq$' follows. On the other hand if w is contained in the right but not in the left hand side of $(+)$, then $w \in \operatorname{rel\,int}(C_xX)^*$ and $w \cdot w' = 0$ for some $w' \in C_xX \setminus T_xX_j$. By definition there then exists a

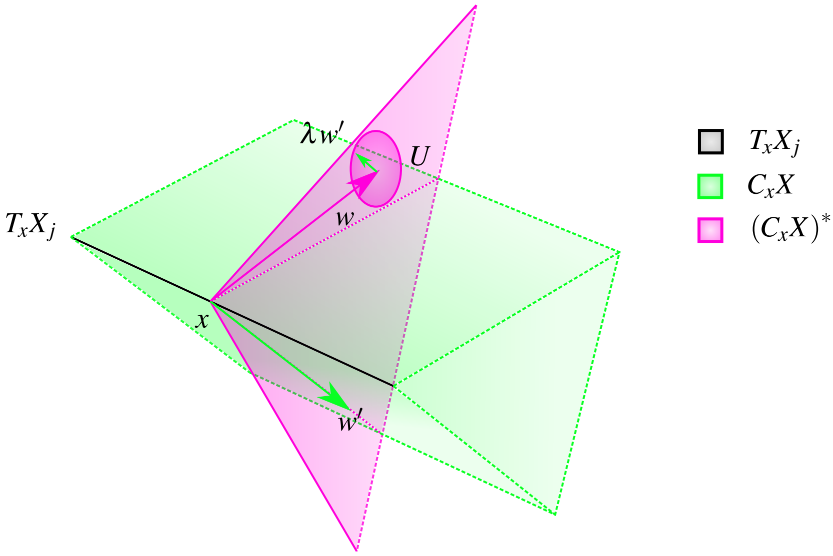


Figure 2.7: An overview of the proof of $(+)$ in equation (2.11).

relatively open neighbourhood $U \subseteq (C_xX)^*$ of w as indicated in figure 2.7. Since we have that $T_xX_j \perp (C_xX)^*$ are perpendicular we can assume that the component of w' projected onto the space T_xX_j vanishes. Then we have for $\lambda < 0$ small enough that $w + \lambda w' \in U$ and it follows that $(w + \lambda w') \cdot w' = \lambda |w'|^2 < 0$, a contradiction and '$\supseteq$' follows.

In equation $(*)$ the inclusion '$\supseteq$' follows from $T_xX_{j+1} \subseteq C_xX$. We now show the inclusion '$\subseteq$'. Let $w \in \partial(C_xX)^*$ and $w' \in (T_xX_k \cap C_xX) \setminus T_xX_j$ for some stratum $X_j \prec X_k$ with $w \cdot w' = 0$. We assume that X_k is chosen to be of minimal dimension. Now if $X_j \precsim X_k$ we are finished. Else there exist strata $X^1_{k-1}, X^2_{k-1} \precsim X_k$ adjacent to X_j as is indicated in figure 2.8. Then there further exist $w'_1 \in \left(T_xX^1_{k-1} \cap C_xX\right) \setminus T_xX_j$ and $w'_2 \in \left(T_xX^2_{k-1} \cap C_xX\right) \setminus T_xX_j$ such that $w' = w'_1 + w'_2$. But then it follows from

$$0 = w \cdot w' = \underbrace{w \cdot w'_1}_{\geq 0} + \underbrace{w \cdot w'_2}_{\geq 0}$$

that also $0 = w \cdot w'_1$ which contradicts the choice of X_k having minimal dimension. Hence the inclusion '$\subseteq$' follows.

Now let x be an irregular essential stagnation point of u_j then $v \in \partial(C_xX)^*$ where $v := u(x)$. Hence there exists by relation (2.11) a $w' \in T_xX_{j+1} \setminus T_xX_j$ such that $v \cdot w' = 0$. Additionally we have that $v \cdot w = 0$ for all $w \in T_xX_j$ since x is stagnation point of u_j. Since we have that $\operatorname{Span}(w') + T_xX_j = T_xX_{j+1}$ by dimension reasons it then follows that $v \cdot w = 0$ for all $w \in T_xX_{j+1}$ which means that $u_{j+1}(x) = 0$.

Conversely assume that $x \in X_j$ is an essential stagnation point of u for some stratum X_{j+1}. Then we have that $u(x) \cdot w = 0$ for all $w \in T_xX_{j+1}$. Since $\operatorname{Span}(w') + T_xX_j = T_xX_{j+1}$ for some $w' \in T_xX_{j+1} \setminus T_xX_j$ it then follows from equation (2.11) that $u(x) \in \partial(C_xX)^*$ and x is an irregular stagnation point. $\square$

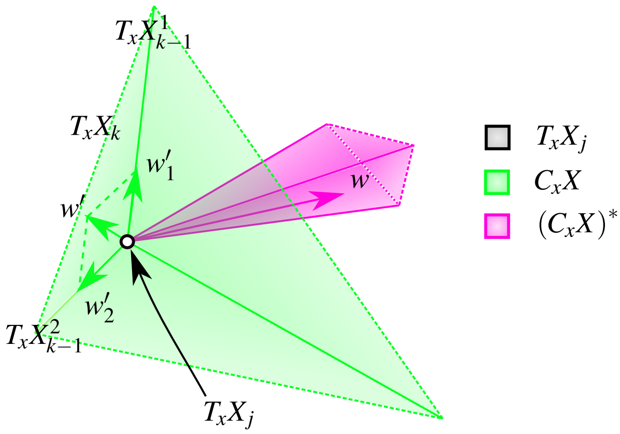


Figure 2.8: An overview of the proof of $(*)$ in equation (2.11).

Morse functions

We are now in a position to define interior and boundary type numbers which is inspired by definitions in [30, Definition 10.3]:

Definition 2.14 (Morse functions, [30, Definition 10.3]). We call u *Morse* if for all $j \in \mathscr{J}$ we have that u_j is regular and essentially non-degenerate. If both u and $-u$ are Morse we call u *strongly Morse*. For a Morse function u we define the *interior type numbers* M_k to be the number of essential interior stagnation points of u of Morse index k, that is

$$M_k := \sum_{\substack{j\in\mathscr{J} \text{ s.t.}\\ n(j)=d}} \mathrm{Ind}^{\mathrm{M}}_{j,k}(u) = \#\left\{x \in \bigcup_{\substack{j\in\mathscr{J} \text{ s.t.}\\ n(j)=d}} \mathrm{Cr}_j(u) \colon x \text{ has index } k\right\}. \tag{2.12}$$

The total number M of interior stagnation points of u is then given by

$$M := \sum_k M_k\,. \tag{2.13}$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of u of Morse index k, that is

$$\mu_k := \sum_{\substack{j\in\mathscr{J} \text{ s.t.}\\ n(j)<d}} \mathrm{Ind}^{\mathrm{M}}_{j,k}(u)\,. \tag{2.14}$$

We further write ν_k for the k-th boundary type number of $-u$. We define the *type number* to be the number of essential stagnation points of u of Morse index k on X, that is

$$\mathrm{Ind}^{\mathrm{M}}_k(u) = \mathrm{Ind}^{\mathrm{M}}_{X,k}(u) := \sum_{j\in\mathscr{J}} \mathrm{Ind}^{\mathrm{M}}_{j,k}(u) = M_k + \mu_k\,. \tag{2.15}$$

We illustrate this with our example:

Example 2.15 (Type numbers)**.** Let X, f and u be as in example 2.9. By the calculations of the previous example 2.12 we have that u is Morse and we can calculate the interior type numbers

$$M_k = \mathrm{Ind}^{\mathrm{M}}_{2,k} = \delta_{1k}$$

and the boundary type numbers

$$\mu_k = \mathrm{Ind}^{\mathrm{M}}_{0,k}(u) + \mathrm{Ind}^{\mathrm{M}}_{1,k}(u) = 2\delta_{0k}\,.$$

This then yields the type numbers

$$\mathrm{Ind}^{\mathrm{M}}_{k}(u) = M_k + \mu_k = \delta_{1k} + 2\delta_{0k}\,.$$

The finiteness of the number of critical points for Morse functions is a known fact which is mentioned for example in [29]. For completeness we give the following proposition:

Proposition 2.16. *The number of non-degenerate stagnation points of u_j on X_j is finite.*

Proof. Let $x \in X_j$ be a non-degenerate stagnation point of u_j. Since $Du_j(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq X_j$ of x on which u_j is bijective. Hence x is the only stagnation point in U_x. Let C denote the set of all non-degenerate stagnation points of u_j. Then the sets U_x for $x \in C$ together with

$$U_C = \overline{X}_j \setminus \overline{C} \tag{2.16}$$

form an open cover of $\overline{X}_j$. But $\overline{X}_j$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is contained in the finite subcover. Thus it follows that $\#C < \infty$ is finite. □

As a consequence we obtain the following observation:

Corollary 2.17. *For a Morse vector field $u\colon X \to TX$ the type numbers $M_0, \dots, M_d$ and the boundary type numbers $\mu_0, \dots, \mu_{d-1}$ are finite.*

The previous definitions translate naturally to a scalar function $f\colon X \to \mathbb{R}$. That is, we call f Morse, non-degenerate, et cetera if $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call a point x a *critical point* of f if it is a stagnation point of u.

Density of Morse functions

In the following section we argue that u and f being Morse is not a great restriction. Given u we define the modification

$$u^{\varepsilon} := u + \varepsilon \tag{2.17}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that the set E of ε for which u^{ε} is Morse is residual in $\mathbb{R}^d$. Recall that a *residual* set is a set whose complement is *meagre*, that is whose complement is the countable union of nowhere dense subsets. Since residual sets are dense in a Baire space by the Baire category theorem we can use u^{ε} to approximate a degenerate u. Our approach is to use Thom's transversality theorem which is inspired by the approach in [19, Chapter 6].

Definition 2.18 (Transversality, [19, §3.2])**.** Let X and Y be manifolds without boundary. We call a function $g\colon X \to Y$ *transverse* to a submanifold $A \subseteq Y$ without boundary if for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\mathrm{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y\,. \tag{2.18}$$

As an application of this definition we make the following observation:

Proposition 2.19 (Transversal characterisation of non-degeneracy)**.** *Let $u_j\colon X_j \to TX_j$ be a differentiable vector field. Then u_j is non-degenerate iff u_j is transverse to the zero section A of the tangent space TX_j.*

Proof. First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A)$ is the set of stagnation points. Unravelling the definition of transversality we get that u_j is transverse to the zero section iff for all $x \in u_j^{-1}(A)$ we have that

$$\mathrm{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}TX_j\,. \tag{2.19}$$

As A is the zero section we have $T_{u_j(x)}A = 0$ and equation (2.19) is equivalent to stating that Du_j is of full rank at x. But Du_j being of full rank at all stagnation points is equivalent to u_j being non-degenerate. ☐

The alternative characterisation of non-degeneracy given in proposition 2.19 is sometimes used as a definition of non-degeneracy. We can now state a weakened version of Thom's transversality theorem which is proven in [19, §3 Theorem 2.7]:

Theorem 2.20 (Parametric transversality theorem, [19, §3 Theorem 2.7])**.** *Let $\mathscr{E}, Y_1, Y_2$ be C^r-manifolds (without boundary) and $A \subseteq Y_2$ a C^r submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A\,.$$

Let further $F\colon \mathscr{E} \to C^r(Y_1, Y_2)$ be such that the evaluation map

$$\begin{aligned} F^{ev}\colon \mathscr{E} \times Y_1 &\to Y_2 \\ (\varepsilon, x) &\mapsto F_{\varepsilon}(x) \end{aligned}$$

is C^r and transverse to A. Then the set

$$E = \{\varepsilon \in \mathscr{E}\colon F_{\varepsilon} \textit{ is transverse to } A\}$$

is residual in $\mathscr{E}$.

From this we obtain a generalisation of the results in [29, §2] which will be useful later:

Corollary 2.21 (Density of harmonic Morse functions)**.** *Let $X \subset \mathbb{R}^d$ be a compact d-dimensional manifold with corners with stratification X_j and let $u\colon X \to TX$ be a smooth vector field. Assume that u has no irregular stagnation points. Then there exists a $\delta > 0$ and a residual (and thus dense) set $E \subseteq B_\delta \subset \mathbb{R}^d$ such that for every $\varepsilon \in E$ the following statements hold:*

1. *For every $\eta > 0$ we can find a $\delta > 0$ such that all stagnation points of u^ε are contained in an η-neighbourhood of the set of stagnation points of u.*
2. *A stagnation point x_ε of u^ε lies in $\Sigma^+(u)$ if it lies in $\Sigma^{\geq 0}(u^\varepsilon)$ and analogously it lies in $\Sigma^-(u)$ if it lies in $\Sigma^{\leq 0}(u^\varepsilon)$. Additionally u^ε has no irregular stagnation points.*
3. *u^ε is strongly Morse.*
4. *If u_j is non-degenerate on the stratum X_j we have for all k that*
$$\mathrm{Ind}^M_{X_j,k}(u^\varepsilon) = \mathrm{Ind}^M_{X_j,k}(u) \qquad \text{and} \qquad \mathrm{Ind}^M_{X_j,k}(-u^\varepsilon) = \mathrm{Ind}^M_{X_j,k}(-u)\,.$$

In addition note that if u is harmonic then by construction u^ε is also harmonic.

Here we defined the *η-neighbourhood* or *tubular neighbourhood* of a subset $A \subseteq \mathbb{R}^d$ by
$$A_\eta = \mathrm{Tub}_\eta(A) := \bigcup_{x\in A} B_\eta(x)\,. \tag{2.20}$$

We remark that the fact that u has no irregular stagnation points is essential to the point 4 of this corollary. Assume for instance u_d has a non-degenerate stagnation point x on the boundary Σ, which means by proposition 2.13 that x is an irregular critical point. It may then follow that for ε contained in a cone at the origin the corresponding critical point of u^ε moves outside of the domain X which in turn changes the index. Example 6.9 gives a concrete illustration of this point.

Proof. Part 1. Let C_η denote the open η-neighbourhood of the set of stagnation points of u. We have for any stratum X_j that u_j is continuous and nonvanishing on the compact set $\overline{X}_j \setminus C_\eta$ which implies that we can choose $\delta > 0$ so small that $|u_j| > 2\delta$ on $\overline{X}_j \setminus C_\eta$ for all strata X_j. For any $\varepsilon \in B_\delta$ it then follows that u^ε has no stagnation points on the set $\overline{X}_j \setminus C_\eta$ which yields the claim.

Part 2. Now consider a stratum X_j and the continuous mapping
$$\begin{aligned} \Phi\colon \overline{X}_j &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \mathrm{dist}\big(u(x), \partial(C_xX)^* \oplus T_xX_j\big) \end{aligned}$$
on $\overline{X}_j$. Since u has no irregular stagnation points Φ is positive on the compact set of stagnation points C of u_j on $\overline{X}_j$. Thus we can choose $\eta > 0$ such that Φ is also positive in the neighbourhood $C_{2\eta}$. Choose $\delta > 0$ smaller than in part 1 such that all stagnation points of u_j on $\overline{X}_j$ lie in C_η. Now the mapping Φ attains a positive minimum on the compact set $\overline{C_\eta}$. We can assume that $\delta > 0$ is less than this minimum. The choice of δ in this way ensures that entrant boundary stagnation points of u_j^ε are also strictly entrant points of u_j on X_j. Additionally it insures that all essential stagnation points of u^ε on X_j are regular. Analogous argumentation with $-u$ then ensures that all essential stagnation points of $-u^\varepsilon$ on X_j are regular and that emergent boundary

stagnation points of u_j are also strictly emergent stagnation points of u_j^ε on X_j. Since there are finitely many strata X_j we can choose $\delta > 0$ such that part 2 follows.

Part 3. It follows from part 2 that u^ε has no irregular stagnation points. Thus it remains to be shown that u_j is non-degenerate for all strata $j \in \mathscr{J}$. This part of the proof is inspired by the use of transversality in [19, §6 Theorem 1.2] to show a similar statement. Set $r = 2$, $\mathscr{E} = B_\delta$ and $Y_2 = TX_j$ in the previous theorem. We initially set $Y_1 = X_j = \text{int}(X)$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F\colon \mathscr{E} &\to C^\infty(X_j, TX_j) \\ \varepsilon &\mapsto u^\varepsilon \end{aligned}$$

for which we note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq TX_j$. Then the parametric transversality theorem yields a residual $E_j \subseteq \mathscr{E}$ on which $F_\varepsilon = u^\varepsilon$ is transverse to A. For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$\text{Image}\Big(DF^{\text{ev}}\big|_{(\varepsilon,x)}\Big) = T_x TX_j \tag{2.21}$$

since

$$DF^{\text{ev}}\big|_{(\varepsilon,x)} = \big[\text{Id}_{d\times d} \mid Du(x)\big]$$

is surjective. Proposition 2.19 now implies that u^ε is non-degenerate on X_j for $\varepsilon \in E_j$. Analogously we set $Y_1 = X_j$ to be an arbitrary stratum in the previous proof and replace u^ε with the projection u_j^ε. To show that equation (2.21) holds we resort to the fact that

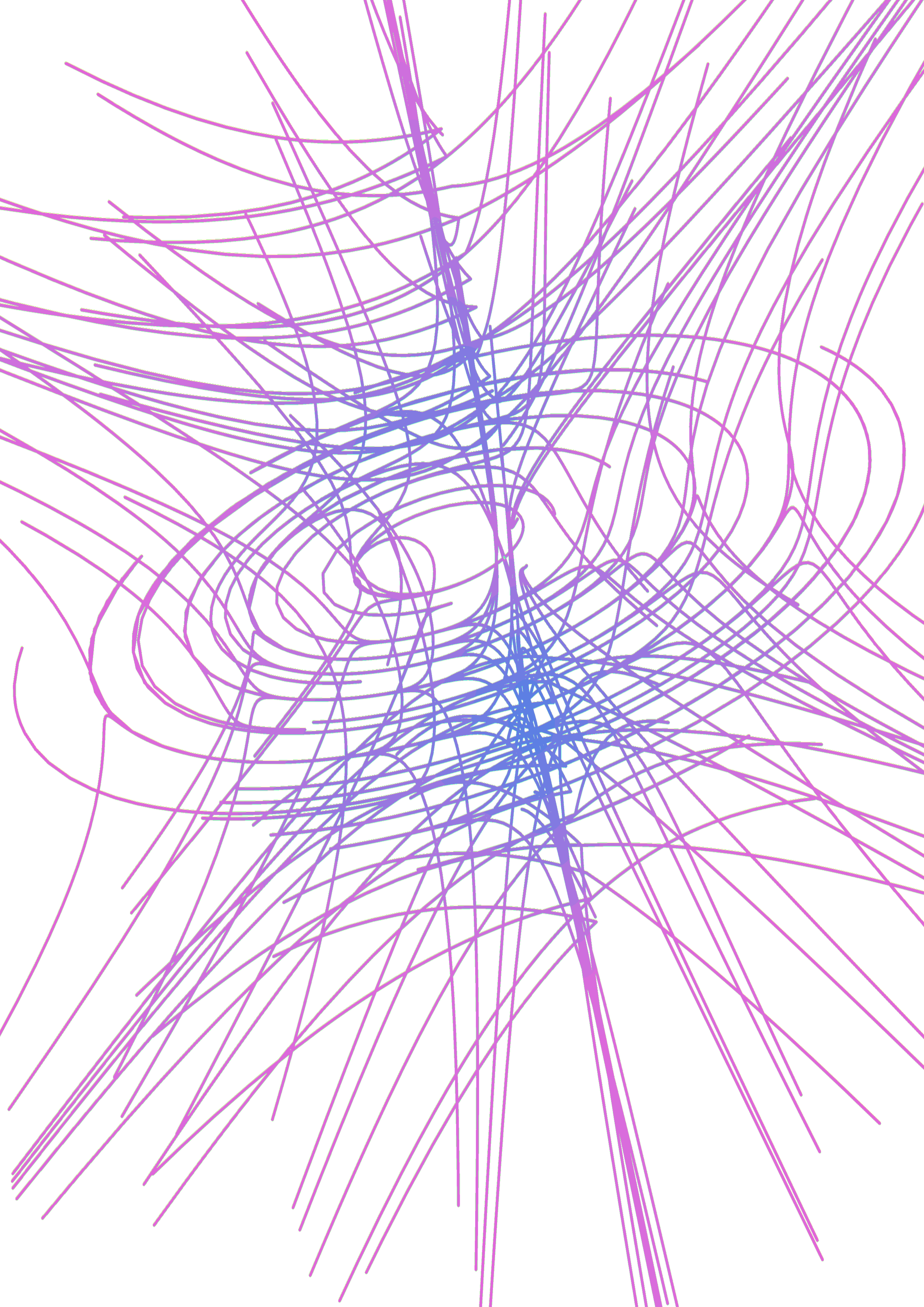
$$DF^{\text{ev}}\big|_{(\varepsilon,x)} = D\big(u_j^\varepsilon(x)\big)\big|_{(\varepsilon,x)} = D\pi_j \circ \big(Du^\varepsilon(x)\big)\big|_{(\varepsilon,x)}$$

is surjective as a composition of surjective functions. Thus there also exists a residual set $E_j \subseteq \mathscr{E}$ on which u_j^ε is non-degenerate on X_j. Now the intersection

$$E = \bigcap_j E_j \subseteq \mathscr{E} = B_\delta$$

is residual and for every $\varepsilon \in E$ the function u^ε fulfils condition 3.

Part 4. Pick $\delta > 0$ as in part 2. If x is a non-degenerate stagnation point of u on the stratum X_j it follows from the inverse function theorem that there exists for sufficiently small δ a neighbourhood around x on which there is a one-to-one correspondence between the stagnation points of u and u^ε. Since there are by proposition 2.16 at most finitely many non-degenerate stagnation points of u we can choose δ to be minimal over all these stagnation points such that this one-to-one correspondence holds. The equality of the indexes then follows from $Du^\varepsilon = Du$. $\square$



3 The Morse inequalities

The aim of this chapter is to state and motivate the Morse inequalities for manifolds with corners. For this we first introduce the Betti numbers of the domain X, then we state the Morse inequalities and finally we give a motivation as to why these inequalities hold. After that we state the Morse inequalities in the special case of $d \in \{2,3\}$ dimensions for a harmonic Morse function f.

Betti numbers

Let $H_k(X)$ denote the *k-th homology group* of X with coefficients in the field $\mathbb{K}$. For a comprehensive introduction and definition of these we refer the reader to [18, Chapter 2]. We nonetheless make remarks along the lines of [9]. The space $H_0(X)$ can be thought of as the free space generated by points where two points are considered equivalent if they are path connected. That is, two 0-*cycles* (oriented points) are considered equivalent if they are the boundary of a 1-*chain* (oriented paths). $H_1(X)$ can be thought of as generated by 1-cycles (oriented loops) which are identified if they differ by the oriented boundary of a 2-chain (oriented surface). Analogously $H_2(X)$ can be thought of as generated by 2-cycles (oriented closed surfaces) which are identified if they differ by the boundary of a 3-chain (oriented volume). In our setting with $X \subset \mathbb{R}^d$ embedded and $d \leq 3$ we have for $k \geq 3$ that $H_k(X)$ is trivial since there exist no k-cycles. We define the *k-th Betti number* as the dimension

$$b_k = b_k(X) := \dim_{\mathbb{K}} H_k(X)\,. \tag{3.1}$$

Now we can define the *Euler characteristic* to be

$$\chi = \chi(X) := \sum_k (-1)^k b_k(X) \tag{3.2}$$

the alternating sum of the Betti numbers. We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

Example 3.1 (In flatland)**.** In $d = 2$ dimensions the zeroth Betti number counts the number of connected components of X and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely, we give the Betti numbers and the Euler characteristic for selected planar domains in table 3.1.

Example 3.2 (In spaceland, [9, Table 1])**.** In $d = 3$ dimensions the zeroth Betti number counts the number of connected components of X, the first Betti number equals the total genus of Σ and the second Betti number equals the number of connected components of the boundary Σ minus the number of connected components of X. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 3.2.

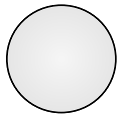
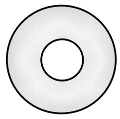
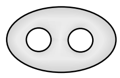
Domain	Picture	b_0	b_1	b_k, $k \geq 2$	$\chi(X)$
Disk D		1	0	0	1
Annulus $2D \setminus D$		1	1	0	0
Two holed button		1	2	0	-1

Table 3.1: Betti numbers and the Euler characteristic for selected domains in $\mathbb{R}^2$.

The Morse inequalities

We state the Morse inequalities for manifolds with corners:

Theorem 3.3 (Morse inequalities, [1, Theorem 2.4]). *Let X be a compact d-dimensional manifold with corners and $f\colon X \to \mathbb{R}$ be smooth and Morse. Then we have for $l \in \{0, \ldots, d\}$ the inequalities*

$$\sum_{k=0}^{l} (-1)^{k+l} \operatorname{Ind}_k^M(f) \geq \sum_{k=0}^{l} (-1)^{k+l} b_k(X)\,. \tag{3.3}$$

For $l = d$ we in fact have equality

$$\sum_{k=0}^{d} (-1)^{k} \operatorname{Ind}_k^M(f) = \chi(X) \tag{3.4}$$

where $\chi(X)$ denotes the Euler characteristic.

The definition of essential non-degenerate regular critical points of f and their index given in definition 2.11 coincides with the definition of a critical point and its co-index of $-f$ given in [1]. Theorem 3.3 then follows from [1, Theorem 2.4]. Nonetheless we will give an idea of the proof of this theorem.

Define for $a \in \mathbb{R}$ the manifold $X^a := \{x \in X\colon\, f(x) \leq a\}$. The main idea in the proof of the Morse inequalities for manifolds with corners is to inspect how the topology of the manifold X^a changes as we vary a. This was the idea in Morse's original proof of these inequalities for manifolds without corners in [27]. Later proofs of the Morse inequalities for manifolds without boundary using this approach added more terminology from algebraic topology and differential geometry. Proofs of the Morse inequalities can be found in [25, §5] for manifolds without boundary and in [30, Theorem 10.2'] for manifolds with C^1 boundary. There also exists proofs using dynamical system techniques which can be found for instance in [34]. The reader is referred to [5] for

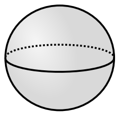
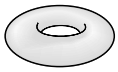
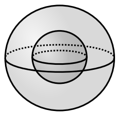
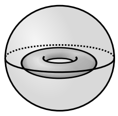
Domain	Picture	b_0	b_1	b_2	$b_k,\, k \geq 3$	$\chi(X)$
Ball B		1	0	0	0	1
Solid torus $S^1 \times D$		1	1	0	0	0
Ball with bubble $2B \setminus B$		1	0	1	0	2
Ball with bubble in shape of torus		1	1	1	0	1

Table 3.2: Betti numbers and the Euler characteristic for selected domains in $\mathbb{R}^3$.

a comprehensive overview of Morse homology on compact manifolds without boundary. We require however the Morse inequalities on manifolds with corners. This is a special case of stratified Morse theory by Goresky and McPherson which is outlined for instance in [10, §5.5]. We will try to avoid the fancy tools that are used in stratified Morse theory and instead resort to [7], [1] and [15] where the Morse inequalities are shown for manifolds with corners. In the following we will outline the main ideas of the proof to convince the reader as to why equation (3.4) is reasonable as this will be the only part of the Morse inequalities we will use.

We state a result from [7, Satz 4.1]:

Lemma 3.4 (Interval without critical value, [7, Satz 4.1])**.** *Let X be a compact manifold with corners and $f\colon X \to \mathbb{R}$ a smooth function. Assume that the interval $[a,b]$ contains no essential critical value of f. Then $X^a \simeq X^b$ are homotopic.*

Idea of proof. The proof consists of constructing a flow on X^b which fixes X^a and moves each point in $f^{-1}([a,b])$ onto $f^{-1}(\{a\})$. For this one first constructs a smooth vector field v locally around a point $x \in f^{-1}([a,b])$ such that locally $u \cdot v = -1$ and v does not point out of X. A global vector field with this property is then constructed with a smooth partition of unity. This vector field is then used to generate a flow which in turn defines a deformation retraction. □

A similar result is given in [15, Theorem 7] and [1]. The idea in [15] is to modify the vector field u on the boundary such that the modified vector field does not point out of the domain X. Since the modified vector field is however discontinuous the proof to show that this indeed induces a continuous flow becomes very technical. On the other hand the proof in [1] is based on the result [1, Theorem 2.1]. The proof and statement of this result seem however flawed to the author. We state Morse's lemma according to [15, Lemma 5].

Proposition 3.5 (Morse's lemma, [15, Lemma 5])**.** *Let $f\colon X \to \mathbb{R}$ be C^2 and x be an essential regular non-degenerate critical point of Morse index k on the stratum X_j of dimension n. Then there exists a local chart $\varphi\colon U \to V \subseteq H^d_{d-n}$ such that*

$$f \circ \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{n} y_j^2 + \sum_{j=n+1}^{d} y_j\,. \tag{3.5}$$

Idea of proof. The case that $n = d$ is the classical version of the Morse lemma and can be found for instance in [19, §6, Lemma 1.1]. Note that the definition of an essential critical point in [15] fulfilling property (3) also encompasses our definition of a non-degenerate regular essential critical point. Here the regularity condition ensures that property (3) in [15] is fulfilled. Hence this statement follows from [15, Lemma 5]. The idea of the proof involves first applying the Morse lemma on the stratum of the critical point. Then this coordinate chart is extended with Taylor in a suitable manner to a full coordinate chart on X such that equation (3.5) holds. □

Using the Morse lemma one can then show the following result:

Lemma 3.6 (Interval with critical value, [15, Theorem 8] and [7, Satz 7.1])**.** *If the interval $[a,b]$ contains one critical value $a < c < b$ with corresponding essential regular non-degenerate critical point x of Morse index k then X^b is homotopic to X^a by attaching the k-cell D^k.*

Idea of proof. This is proven for instance in [15, Theorem 8]. Here we note that we do not require what [15] calls property (3) since this was only used to show the analogue of lemma 3.4 and in the formulation of the Morse lemma. The proof is in structure very similar to [25]. One first uses the Morse lemma to choose a suitable coordinate system in a neighbourhood of the critical point. Then one chooses $\varepsilon > 0$ so small such that $B_{2\varepsilon} \subseteq V$ where V is as in proposition 3.5. The proof then proceeds in three steps. First one defines a handle H as in figure 3.1. Then one shows that $X^{c-\varepsilon}$ with H attached is homotopic to $X^{c+\varepsilon}$. Finally one shows that $X^{c-\varepsilon}$ with the k-cell D^k attached is homotopic to $X^{c-\varepsilon}$ with H attached. The claim then follows from

$$X^a \cup D^k \simeq X^{c-\varepsilon} \cup D^k \simeq X^{c-\varepsilon} \cup H \simeq X^{c+\varepsilon} \simeq X^b$$

where we also used the result of lemma 3.4. □

In the original proof [27] of the Morse inequalities Morse inspected in the analogues to lemmas 3.4 and 3.6 directly how the Betti numbers of the manifold changed from X^a to X^b. Most proofs of the Morse inequalities however use the language of homology which is why we now need to argue how lemmas 3.4 and 3.6 imply the Morse inequalities. This final part of the proof is given in all details in [25, §I.5]. As we only use the equality in equation (3.4) we shall outline its proof.

Denote the *k-th relative homology group* by $H_k(X,A)$ for spaces $A \subseteq X$. Once again we refer the reader to [18, Chapter 2] for details regarding their definition. We also give an interpretation along the lines of [9]. The homology group $H_0(X,A)$ can be thought of as being freely generated by 0-cycles in X (oriented points) which are identified if they are the boundary of a 1-chain

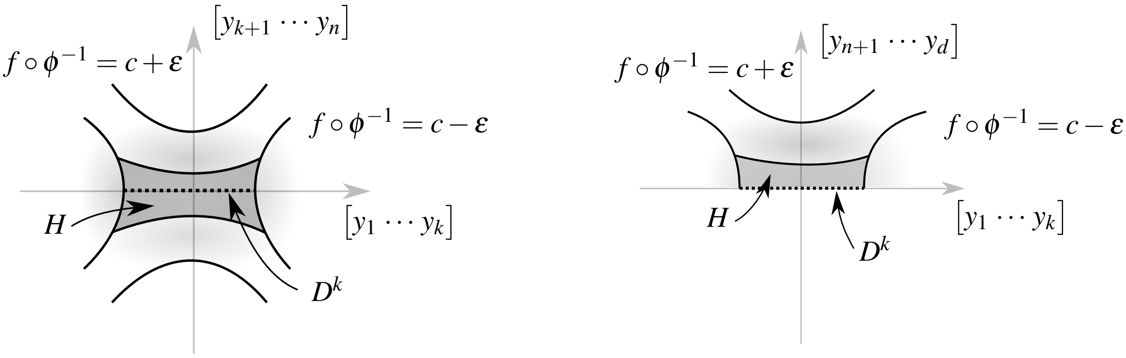


Figure 3.1: The main idea in the proof of Lemma 3.6.

(oriented paths). Here a 0-cycle is trivial if it is the boundary of a path with endpoint in A. Thus the dimension of $H_0(X,A)$ is the number of path components of X not containing a point of A. Similarly $H_1(X,A)$ is generated by 1-cycles in X which are identified if they, possibly augmented by portions of A, are the boundary of a 2-chain (oriented surface) in X. Here 1-cycles are loops and paths with endpoints in A and they are considered trivial if they, possibly augmented by 1-chains in A are the boundary of a 2-chain in X. A comparably technical description exists of $H_2(X,A)$. We note that $H_3(X,A)$ unlike $H_3(X)$ may be nontrivial in our setting and is generated by 3-cycles. Here a 3-cycle is an oriented volume with boundary contained in A and they are considered trivial if they are entirely contained in A. Thus the dimension of $H_3(X,A)$ equals the number of oriented compartments surrounded by A. We require a result from algebraic topology proven in [18, Corollary 2.24]:

Proposition 3.7 (Corollary of the excision theorem, [18, Corollary 2.24])**.** *Let $X = A \cup B$ be a CW-complex. Then we have for every k that $H_k(X,A) \cong H_k(B, A \cap B)$ are isomorphic.*

Now generalise the Euler characteristic for spaces $Y \subseteq X$ by

$$\chi(X,Y) := \sum_k (-1)^k \dim_{\mathbb{K}} H_k(X,Y). \tag{3.6}$$

It is shown in [25, §5] that this is additive, in the sense that given suitable $Z \subseteq Y \subseteq X$ we have that

$$\chi(X,Z) = \chi(X,Y) + \chi(Y,Z). \tag{3.7}$$

Let $a_0 < \cdots < a_L$ be such that $[a_l, a_{l+1}]$ contains precisely one critical value in its interior with index k_l and such that $X^{a_0} = \emptyset$ and $X = X^{a_L}$. It now follows from lemma 3.6 and proposition 3.7 that

$$\begin{aligned} H_k(X^{a_{l+1}}, X^{a_l}) &= H_k\left(X^{a_l} \cup D^{k_l}, X^{a_l}\right) \\ &= H_k\left(D^{k_l}, S^{k_l}\right) \\ &= \begin{cases} \mathbb{K} & \text{if } k = k_l, \\ 0 & \text{else} \end{cases} \end{aligned}$$

so

$$\chi(X^{a_{l+1}}, X^{a_l}) = (-1)^{k_l} . \tag{3.8}$$

Hence we obtain by inductively applying equation (3.7) and then (3.8) that

$$\chi(X) = \chi(X^{a_L}, X^{a_0}) = \sum_l \chi(X^{a_{l+1}}, X^{a_l}) = \sum_k (-1)^k \operatorname{Ind}_k^{\mathrm{M}}(f)$$

so equation (3.4) of the Morse inequalities follows for the case that no two critical points lie in the same level set. In case there are multiple critical points on the same level set the notation becomes a little more involved but the idea of the proof is the same. The proof of the inequalities (3.3) involves defining the Betti numbers for relative homology groups and then making a similar argumentation as above with these Betti numbers instead of the Euler characteristic. For details we refer the reader to [25, §5].

In the final part of this chapter we inspect the Morse inequalities in the special case that f is harmonic. If we assume that f is harmonic the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain from the Morse inequalities a result from [30, Corollary 10.1].

Corollary 3.8 (Morse inequalities for harmonic f in $\mathbb{R}^2$, [30, Corollary 10.1])**.** *Let $X \subset \mathbb{R}^2$ be a manifold with corners and $f \colon X \to \mathbb{R}$ be harmonic and Morse. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M + \mu_1 - \mu_0 &= -\chi(X) . \end{aligned}$$

In dimensions $d = 3$ we obtain from theorem 3.3 a result from [30, Corollary 10.2].

Corollary 3.9 (Morse inequalities for harmonic f in $\mathbb{R}^3$, [30, Corollary 10.2])**.** *Let $X \subset \mathbb{R}^3$ be a manifold with corners and $f \colon X \to \mathbb{R}$ be harmonic and Morse. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0 \\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= \chi(X) . \end{aligned}$$

4 Connected entrant and emergent boundaries in $\mathbb{R}^2$

We will start this section by giving an essentially negative answer to question 1.1 in $d = 2$ dimensions. Thus it is not possible to have a harmonic function with interior critical point on a simply connected planar domain with connected entrant and connected emergent boundaries. The proof of this will involve Morse theory on manifolds with corners which was introduced in the previous chapter. We shall then state a result from [3] which more generally relates the number of emergent components with the number of stagnation points in the plane using tools from complex analysis. After that we give examples of harmonic vector fields with interior stagnation points where we drop the condition of simple connectedness of the domain and only require the strictly entrant and strictly emergent boundaries to be connected. We will also consider the analogous problem in $d = 4$ dimensions and show that there exists a harmonic vector field on the unit ball with the desired properties.

A negative result for simply connected domains

We now give a result which is essentially a negative answer to question 1.1. It was possible to prove this statement using complex analysis, level sets of critical points or alternatively using invariant manifolds. The following proof involves Morse theory since the techniques of the proof generalise more readily to the three dimensional case. We remark that it was also possible to prove this with the Morse index theorem for manifolds with boundaries. Note that in the following the notation $\Sigma_{\leq 0}$ is not to be confused with the notation $\Sigma^{\leq 0}$ used for the entrant boundary.

Proposition 4.1 (Negative answer to question 1.1 in $d = 2$ dimensions)**.** *Let $X \subset \mathbb{R}^2$ be a simply connected planar compact manifold with corners and let $f\colon X \to \mathbb{R}$ be harmonic, non-degenerate on the interior* $\mathrm{int}(X)$ *and without irregular critical points. Let $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ be a disjoint decomposition of the boundary into simply connected nonempty sets such that we have for the strictly entrant boundary $\Sigma^- \subseteq \Sigma_{\leq 0}$ and for the strictly emergent boundary $\Sigma^+ \subseteq \Sigma_{\geq 0}$. Then f has no interior critical point.*

Proof. Let $\gamma = \{x_1, x_2\} = \partial\Sigma_{\leq 0}$. Then we can cut the domain along a smooth curve $\Gamma \subset \mathrm{int}(X)$ such that the endpoints $\gamma = \partial\Gamma$ of the cut coincide with x_1 and x_2, that is $\partial\Gamma = \{x_1, x_2\}$. Now we obtain two new domains X^+ and X^- such that $\partial X^+ \cap \Sigma_{\leq 0} \subseteq \gamma$ and $\partial X^- \cap \Sigma_{\geq 0} \subseteq \gamma$. Since by proposition 2.16 the number of critical points of f on $\mathrm{int}(X)$ is finite we can assume that none of them lie on Γ and further that Γ corresponds to the stratum X_{Γ^+} for X^+ and X_{Γ^-} for X^-. Analogously γ corresponds to strata $X_{\gamma^\pm}$ on $X^\pm$. Locally around the corner point x_1 we have a

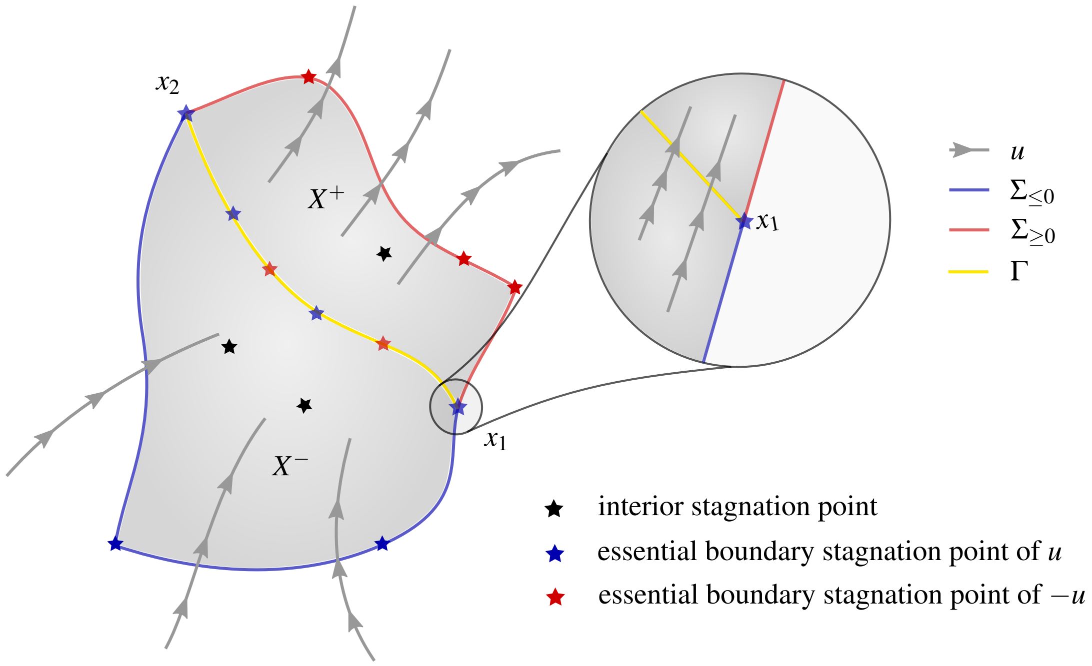


Figure 4.1: The situation at hand in proposition 4.1.

situation depicted as in figure 4.1. We assume that we chose Γ in such a way that it forms an acute angle with $u = \nabla f$ at the boundary points γ. For the following argumentation we require that u is strongly Morse on both X^+ and X^-, so assume for a moment that this is the case. Since each point of γ is an essential critical point for either f or $-f$ on precisely one of the domains X^+ or X^- we have the relation

$$\mathrm{Ind}^{\mathrm{M}}_{\gamma^+,0}(f) + \mathrm{Ind}^{\mathrm{M}}_{\gamma^+,0}(-f) + \mathrm{Ind}^{\mathrm{M}}_{\gamma^-,0}(f) + \mathrm{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f) = 2\,. \tag{4.1}$$

We now focus our attention on X^+. Since no essential critical points of f lie on $\Sigma_{\geq 0} \setminus \gamma$ it follows for the boundary type numbers that

$$\mu^+_k = \mathrm{Ind}^{\mathrm{M}}_{\Gamma^+,k}(f) + \mathrm{Ind}^{\mathrm{M}}_{\gamma^+,k}(f)\,. \tag{4.2}$$

Analogously we have on X^- that

$$\nu^-_k = \mathrm{Ind}^{\mathrm{M}}_{\Gamma^-,k}(-f) + \mathrm{Ind}^{\mathrm{M}}_{\gamma^-,k}(-f)\,. \tag{4.3}$$

In addition we have on the cut Γ that the essential critical points of f on X^+ are the essential critical points of $-f$ on X^- under a shift of index, that is

$$\mathrm{Ind}^{\mathrm{M}}_{\Gamma^+,0}(f) = \mathrm{Ind}^{\mathrm{M}}_{\Gamma^-,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}^{\mathrm{M}}_{\Gamma^+,1}(f) = \mathrm{Ind}^{\mathrm{M}}_{\Gamma^-,0}(-f)\,. \tag{4.4}$$

Using equations (4.2), (4.3) and (4.4) we obtain

$$\mu^+_0 - \mathrm{Ind}^{\mathrm{M}}_{\gamma^+,0}(f) = \nu^-_1 \qquad \text{and} \qquad \mu^+_1 = \nu^-_0 - \mathrm{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f)\,. \tag{4.5}$$

Consider the Morse inequality for f on X^+

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) \tag{4.6}$$

and the Morse inequality for $-f$ on X^-

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X) \tag{4.7}$$

where $M^\pm$ denote the number of interior critical points on the sets $X^\pm$ respectively. We now add equations (4.6) and (4.7) and insert relations (4.5) to obtain

$$M^- + M^+ - \mathrm{Ind}^{\mathrm{M}}_{\gamma^+,0}(f) - \mathrm{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f) = -2\chi(X) = -2\,.$$

Since $\mathrm{Ind}^{\mathrm{M}}_{\gamma^+,0}(f) + \mathrm{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f) \leq 2$ by equation (4.1) and $M^\pm \geq 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case observe that since Γ contains no interior stagnation points it follows from proposition 2.13 that $u = \nabla f$ has no irregular stagnation points on Γ. It also follows from the acute angle that Γ forms with u on γ that there are no irregular stagnation points on γ. Thus u has no irregular stagnation points on X^+ or X^-. Let $E^+, E^- \subseteq B_\delta$ be as in corollary 2.21 applied separately to the domains X^+ and X^-. Since E^+ and E^- are residual in B_δ we can in particular pick an $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. By the same corollary u^ε has no essential stagnation points on $\Sigma^{\geq 0}(u)$ and $-u$ has no essential stagnation points on $\Sigma^{\leq 0}(u)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M^\varepsilon = M$ as f is non-degenerate on $\mathrm{int}(X)$. $\square$

A generalisation of this result

We could also have used tools from complex analysis to show proposition 4.1. In fact, using complex analysis [3] gives a more refined result for which we require the next definition.

Definition 4.2 (Number of entrant / emergent boundary components, [3])**.** Let $J^\pm$ denote the number of connected components of $\Sigma^\pm$ which are proper subsets of a component of Σ. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma_{\geq 0}$ which are proper subsets of a component of Σ.

We state a consequence of a result from [3, Theorem 2.1 and 2.2]:

Proposition 4.3 (Special case of [3, Theorem 2.1 and 2.2])**.** *Let $X \subset \mathbb{R}^2$ be a bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u\colon X \to \mathbb{R}^2$ be a harmonic vector field, nonzero on each component. Then we have the relation*

$$M \leq -\chi(X) + \frac{J^+ + J^-}{2} \tag{4.8}$$

where M denotes the number of stagnation points of u, counting multiplicities. If in addition we assume that there are no irregular stagnation points then we have

$$M \leq -\chi(X) + J^{\geq 0}\,. \tag{4.9}$$

For a proof we refer the reader to [3, Theorem 2.1] and [3, Theorem 2.2]. Here one sets $\underline{\alpha} = n$ to be the outer unit normal and $D = \chi(X)$ to be the Euler characteristic. If we set X to be homeomorphic to the disk such that $\chi(X) = 1$ and have a decomposition of the boundary Σ as in proposition 4.1, that is $J^{\pm} = J^{\geq 0} = 1$, we then obtain from proposition 4.3 that $M \leq 0$ and f has no interior critical point.

The case of holes in the domain

If we set $J^{\pm} = J^{\geq 0} = 1$ in relations (4.8) or (4.9) we obtain the condition on the number of interior stagnation points

$$M \leq -\chi(X) + 1\,. \tag{4.10}$$

This indicates that if we allow for holes in the domain X it is possible to have a vector field with simply connected strictly entrant and strictly emergent boundaries and interior stagnation points. In fact we will give two examples where we have equality in equation (4.10) for the cases $\chi(X) = 0$ and $\chi(X) = -1$. For this define two differential operators in $d = 2$ dimensions by

$$\nabla^{\perp} f := \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix} \tag{4.11}$$

and

$$\operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1\,. \tag{4.12}$$

The next proposition gives us a recipe to generate harmonic vector fields.

Proposition 4.4. *Let $\psi\colon \mathbb{R}^2 \supseteq X \to \mathbb{R}$ be harmonic then $u = \nabla^{\perp}\psi$ is a harmonic vector field.*

Proof. We have

$$\operatorname{div} u = \operatorname{div} \nabla^{\perp} \psi = 0$$

and one calculates

$$\operatorname{curl} u = \operatorname{curl} \nabla^{\perp} \psi = \Delta \psi = 0$$

from which the claim follows. □

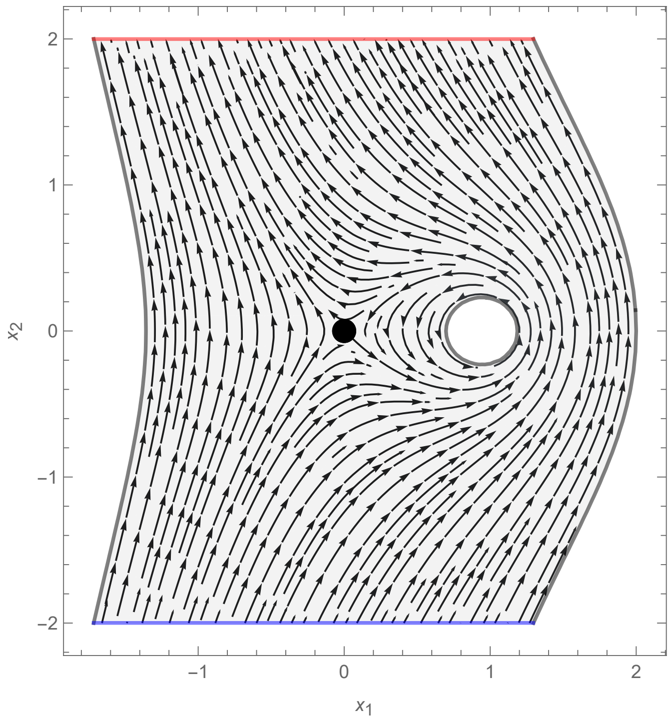


Figure 4.2: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.5,2]) \cap (\mathbb{R} \times [-2,2])$. Here ψ is given by equation (4.13).

The function ψ is also called a *stream function*. The next example is our first example of a harmonic vector fields in $d = 2$ dimensions with an interior stagnation point for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component.

Example 4.5 (Flow through tube with hole and stagnation point)**.** Consider the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{4.13}$$

where

$$\Phi_2 = \log(|\cdot|) \tag{4.14}$$

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and e_i is the i-th unit vector. A plot of the streamlines in figure 4.2 indicates that $u = \nabla^\perp \psi$ fulfils the requirements on the domain

$$X = \psi^{-1}([-0.5,2]) \cap (\mathbb{R} \times [-2,2]) .$$

Indeed, an elementary calculation reveals that the origin is a stagnation point of u.

Example 4.5 highlights the importance of the requirement in proposition 4.1 that the domain be simply connected. We now give a similar example, this time with two holes in the domain and two interior stagnation points.

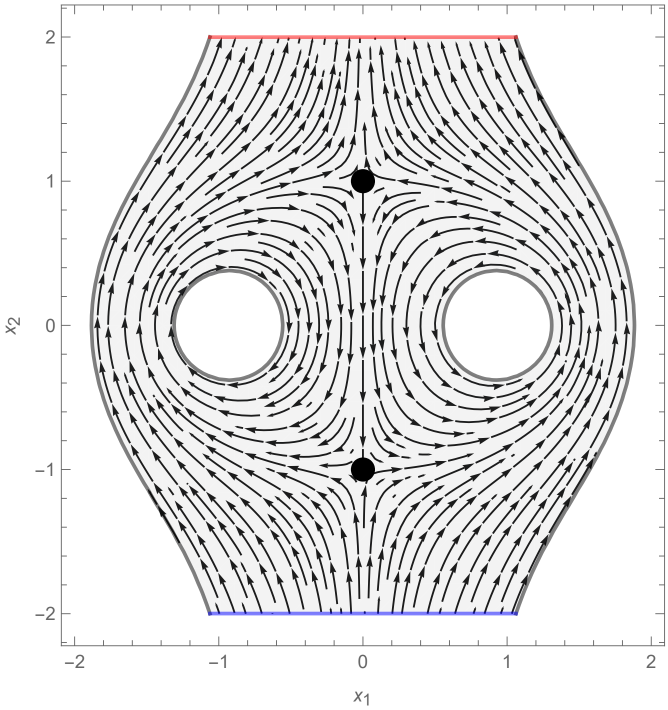


Figure 4.3: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2, 2])$. Here ψ is given by equation (4.15).

Example 4.6 (Flow through tube with two holes and stagnation points)**.** We consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1 \,. \end{aligned} \tag{4.15}$$

A plot of the streamlines in figure 4.3 indicates that $u = \nabla^{\perp}\psi$ on the domain

$$X = \psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2, 2])$$

has the desired properties.

The case of $d = 4$ dimensions

We ask whether there exists a harmonic vector field on a simply connected domain with simply connected entrant and emergent boundaries and with an interior stagnation point in a dimension higher than $d = 2$. Indeed, in $d = 4$ dimensions we can readily give an example of such a harmonic vector field.

Example 4.7 (Connected entrant boundary in $d = 4$ dimensions)**.** Consider as domain $X = B_1 \subset \mathbb{R}^4$

the unit ball and the harmonic function

$$
\begin{aligned}
f\colon X &\to \mathbb{R} \\
x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,.
\end{aligned}
\tag{4.16}
$$

This has a critical point at the origin. We will show in proposition 4.8 that the entrant and emergent boundaries are in fact connected.

Proposition 4.8 (Simply connected entry boundary in example 4.7)**.** *The harmonic function given by equation* (4.16) *has connected entrant and emergent boundaries.*

Proof. First observe that the boundary $\Sigma = S^3$ can be away from the equator locally parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$
\sum_i x_i^2 = 1 \tag{4.17}
$$

on the boundary $\Sigma = S^3$ we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$
\begin{aligned}
\phi_\pm \colon \mathbb{R}^3 \supset B_1 &\to \Sigma \subset \mathbb{R}^4 \\
\bar{x} &\mapsto x = (x_1, \bar{x}) \text{ such that } \pm x_1 \geq 0
\end{aligned}
\tag{4.18}
$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$
\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top
$$

and the outer unit normal to the boundary Σ is given by

$$
n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .
$$

Thus we have $x \in \Sigma^\pm$ iff

$$
0 < \pm \nabla f \cdot n = \pm 2 \big(x_1^2 + x_2^2 - x_3^2 - x_4^2 \big) \,.
$$

Using condition (4.17) we obtain the equivalent condition

$$
0 < \pm \big(1 - 2 \big(x_3^2 + x_4^2 \big) \big) \,. \tag{4.19}
$$

Define the cylinder

$$
C = \big\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \big\} = \mathbb{R} \times B_{1/\sqrt{2}} \subset \mathbb{R}^3 \,.
$$

If we return to our parametrisation (4.18) and take into account condition (4.19) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(\bar{x}) \in \Sigma^+$ and hence

$$
B_1 \cap C = \psi_\pm \big(\Sigma^+ \big)
$$

and

$$
B_1 \setminus \overline{C} = \psi_\pm \big(\Sigma^- \big) \,.
$$

Since ϕ_+ is a homeomorphism onto the northern hemisphere and ϕ_- a homeomorphism onto the southern hemisphere and the surface S^2 corresponds to the equatorial sphere the claim then follows. The situation is depicted in figure 4.4. $\square$

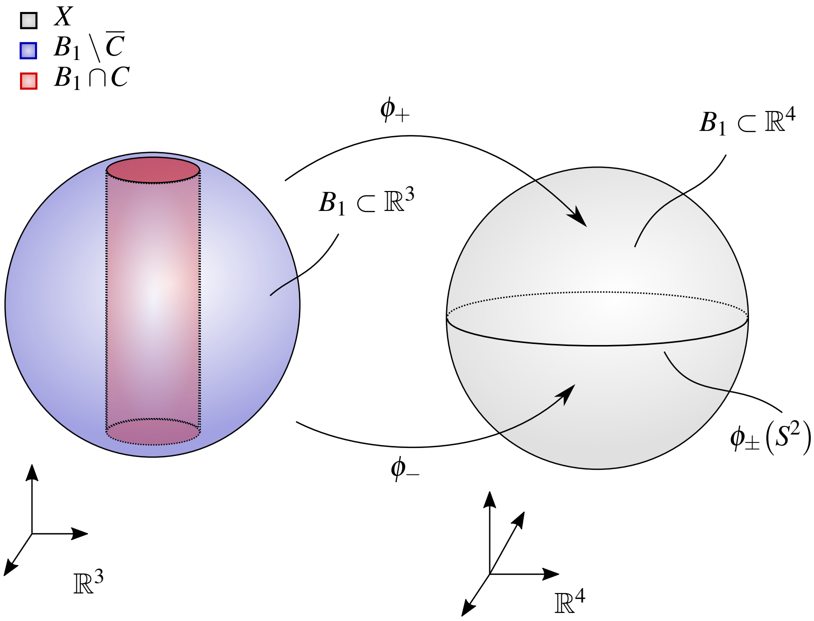


Figure 4.4: Visualisation of the situation in the proof of proposition 4.8.

5 Connected entrant and emergent boundaries in $\mathbb{R}^3$

In the following chapter we will discuss question 1.1 in the case of $d = 3$ dimensions. That is, we are looking for a harmonic function with interior stagnation point on a simply connected domain and connected entrant and emergent boundaries. We will first give a negative answer in the case that the domain is a cylinder where one lid is the entrant and the other lid is the emergent boundary. After that we will use Morse theory to essentially argue that the number of interior stagnation points of such an example must be an even number. We then give an example of a function and a domain with the desired properties. Finally we show that we can modify this example such that the entrant and emergent boundaries have positive distance from one another whilst preserving the interior stagnation points.

A negative result for the cylinder

The following proof comes from [35] and is a negative answer to question 1.1 for the cylinder where one lid is the entrant and the other lid is the emergent boundary.

Proposition 5.1 (Negative answer for cylinders, [35])**.** *Let $X = [0,1] \times \overline{U} \subset \mathbb{R}^d$ be a cylinder where $U \subset \mathbb{R}^{d-1}$ is a bounded open set with C^1 boundary. Let further $f\colon X \to \mathbb{R}$ be non-constant and harmonic such that the sides $[0,1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^{\leq 0}$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^{\geq 0}$ is the emergent boundary. Then f cannot have an interior critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the strong maximum principle that $\partial_1 f$ attains its strict minimum on the boundary Σ. As $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f \geq 0$ on the lids $\{0,1\} \times U$ there exists a point $x \in (0,1) \times \partial U$ such that $\partial_1 f(x) < 0$ is minimal on X. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. □

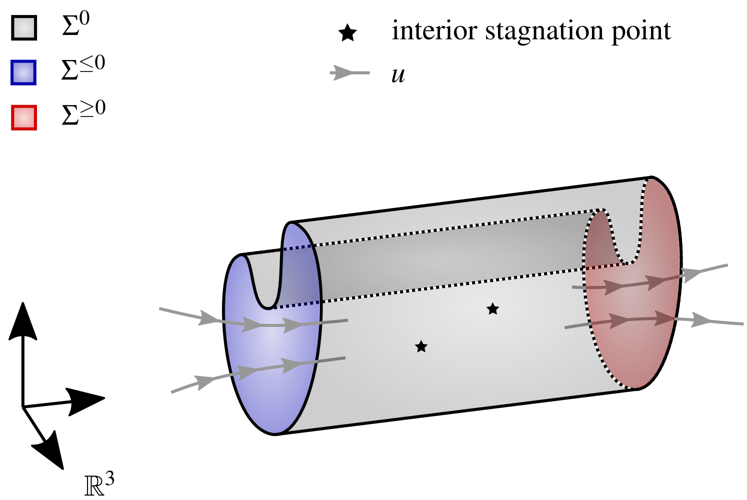


Figure 5.1: Proposition 5.1 states that this kind of situation is not possible.

This result means that a situation as depicted in figure 5.1 is not possible. We will however see in example 5.12 that this result does not hold in a topological sense. In particular, it is possible to have a harmonic vector field with interior stagnation point on a simply connected domain such that the entrant and the emergent boundaries are simply connected and are separated by a strip of tangential boundary with positive thickness.

A condition on the interior type numbers

Mimicking the proof of proposition 4.1 in two dimensions we obtain a condition on the type numbers for a harmonic function with interior stagnation point and simply connected entrant and emergent boundaries. We remark that it was also possible to prove this result using the Morse index theorem for manifolds with corners which generalises it to harmonic vector fields.

Proposition 5.2 (Condition on the interior type numbers)**.** *Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with corners and let $f\colon X \to \mathbb{R}$ be a harmonic function without irregular stagnation points and such that f is non-degenerate on the interior* $\operatorname{int}(X)$*. Let $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ be a disjoint decomposition of the boundary into simply connected nonempty sets such that we have for the strictly entrant boundary $\Sigma^- \subseteq \Sigma_{\leq 0}$ and for the strictly emergent boundary $\Sigma^+ \subseteq \Sigma_{\geq 0}$. Additionally we require that $\gamma := \partial\Sigma_{\leq 0}$ is a one-dimensional manifold diffeomorphic to the circle S^1. Then we have the relation $M_1 = M_2$ between the interior type numbers.*

Proof. As in the two dimensional case we split the domain X with a surface Γ such that $\partial\Gamma = \gamma = \partial\Sigma_{\leq 0}$. Denote the two arising domains by X^+ and X^- where $\partial X^- \cap \Sigma_{\geq 0} \subseteq \gamma$ and $\partial X^+ \cap \Sigma_{\leq 0} \subseteq \gamma$. Since by proposition 2.16 there are finitely many interior critical points in X we can also assume that no interior critical points lie on Γ. Furthermore we assume that Γ approaches γ such that $u = \nabla f$ and Γ form an acute angle at all points on γ. For the following argumentation we require that f is strongly Morse on both X^+ and X^- so assume for a moment that this is the case. By assumption we have that γ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Since f is non-degenerate on γ the number of maxima and minima of f on γ must be equal and thus

$$\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,0}(f)+\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,1}(-f)=\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,1}(f)+\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f)\,. \tag{5.1}$$

We now turn our attention to X^+. Since no essential critical points lie on $\Sigma_{\geq 0}\setminus\gamma$ it follows for the boundary type numbers that

$$\mu_k^+=\operatorname{Ind}^{\mathrm{M}}_{\Gamma^+,k}(f)+\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,k}(f)\,. \tag{5.2}$$

Analogously we have on X^- that

$$\nu_k^-=\operatorname{Ind}^{\mathrm{M}}_{\Gamma^-,k}(-f)+\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,k}(-f)\,. \tag{5.3}$$

In addition we have that the essential critical points on $\Gamma=\Gamma^+$ of $\pm f$ on X^+ are the essential critical points on $\Gamma=\Gamma^-$ of $\mp f$ on X^- under a shift of index, that is

$$\begin{aligned}
&\operatorname{Ind}^{\mathrm{M}}_{\Gamma^+,0}(f)=\operatorname{Ind}^{\mathrm{M}}_{\Gamma^-,2}(-f)\,,\\
&\operatorname{Ind}^{\mathrm{M}}_{\Gamma^+,1}(f)=\operatorname{Ind}^{\mathrm{M}}_{\Gamma^-,1}(-f)\text{ and}\\
&\operatorname{Ind}^{\mathrm{M}}_{\Gamma^+,2}(f)=\operatorname{Ind}^{\mathrm{M}}_{\Gamma^-,0}(-f)\,.
\end{aligned} \tag{5.4}$$

Using equations (5.2), (5.3) and (5.4) we obtain

$$\begin{aligned}
\mu_0^+-\nu_2^- &=\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,0}(f)\,,\\
\mu_1^+-\nu_1^- &=\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,1}(f)-\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,1}(-f)\text{ and}\\
\mu_2^+-\nu_0^- &=-\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f)\,.
\end{aligned} \tag{5.5}$$

We observe the Morse inequality for f

$$M_2^++\mu_2^+-M_1^+-\mu_1^++\mu_0^+=\chi(X^+)=\chi(X) \tag{5.6}$$

and the Morse inequality for $-f$

$$M_1^-+\nu_2^--M_2^--\nu_1^-+\nu_0^-=\chi(X^-)=\chi(X) \tag{5.7}$$

where the $M_k^\pm$ denote the interior type numbers of f on the respective domains $X^\pm$. We now subtract equation (5.7) from (5.6) and insert relations (5.5) to obtain then with equation (5.1)

$$\begin{aligned}
0&=M_2^+-M_1^+-M_1^-+M_2^-+\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,0}(f)+\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,1}(-f)-\operatorname{Ind}^{\mathrm{M}}_{\gamma^+,1}(f)-\operatorname{Ind}^{\mathrm{M}}_{\gamma^-,0}(-f)\\
&=M_2-M_1
\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case observe that since Γ contains no interior stagnation points of $u=\nabla f$ it follows from proposition 2.13 that u has no irregular points on Γ. It also follows from the slanted angle that Γ forms with u on γ and the fact that γ is a C^1 manifold that there are no irregular stagnation points on γ. Thus f

has no irregular stagnation points on X^+ or X^-. Let $E^+, E^- \subseteq B_\delta$ be as in corollary 2.21 applied separately to the domains X^+ and X^-. Since $E^\pm$ are residual in B_δ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. By the same corollary we can assume that f^ε has no essential critical points on $\Sigma^+(f)$ and $-f^\varepsilon$ has no essential critical points on $\Sigma^-(f)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$ as f is non-degenerate on $\mathrm{int}(X)$. $\square$

THE example

Based on example 4.7 in $d = 4$ dimensions of a harmonic vector field with interior stagnation point the author argued that it would be simplest to construct such a vector field in $d = 3$ dimensions in a similar manner. More specifically we choose as domain the ball and polynomials as our harmonic function. In choosing the ball as domain we also take into account the result from proposition 5.1 which is a negative result to question 1.1 for the cylinder. Based on the result from proposition 5.2 we see that the number of stagnation points must be at least two. The author then implemented a Mathematica routine to generate harmonic polynomials with two stagnation points and a plotting function to inspect what occurs. Indeed, this approach yielded a function with the desired properties as we shall discuss in this section.

Example 5.3 (A harmonic function with interior critical point and simply connected entrant and emergent boundaries)**.** Consider the domain $X = \overline{B}_r \subset \mathbb{R}^3$ with $r > 0$ sufficiently large, and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1 x_2^2 + x_2 x_3 \,. \end{aligned} \tag{5.8}$$

This induces the harmonic vector field

$$\begin{aligned} u\colon X &\to \mathbb{R}^3 \\ x &\mapsto \nabla f(x) = \begin{bmatrix} x_1(1-x_1) + x_2^2 \\ x_2(2x_1 - 1) + x_3 \\ x_2 \end{bmatrix} . \end{aligned}$$

It follows from setting $u(x) = 0$ that $x_2 = 0$ and then $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of u. Conversely one sees that these are zeroes of u. Hence f has two interior critical points for $r > 1$. Figure 5.2 shows a stream plot of u with the two interior stagnation points highlighted as black dots. The boundary of the domain is shaded in blue for the strictly entrant boundary Σ^- and in red for the strictly emergent boundary Σ^+. The stereographic projection of the boundary is plotted in figure 5.3. This plot indicates that Σ^- and Σ^+ are simply connected. Indeed, we will show this in theorem 5.4.

The remainder of this section will be devoted to the proof of the following theorem:

Theorem 5.4 (Positive answer to question 1.1 in $\mathbb{R}^3$)**.** *The harmonic function given by equation* (5.8) *on the ball $X = \overline{B}_r$ has interior stagnation points and connected strictly emergent and strictly entrant boundaries for sufficiently large $r > 0$.*

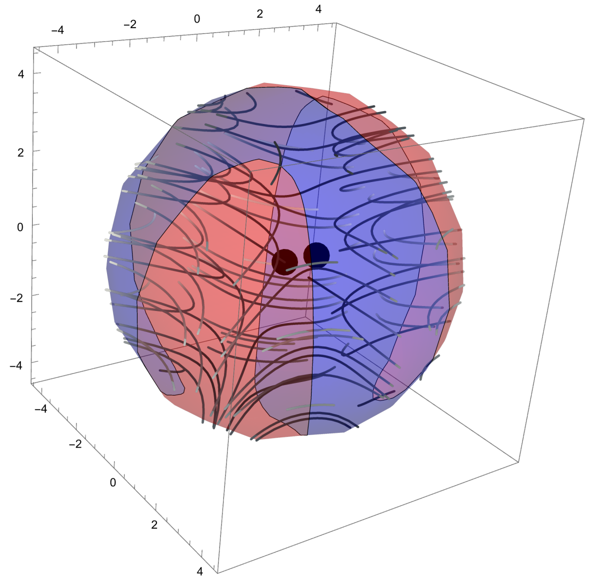


Figure 5.2: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

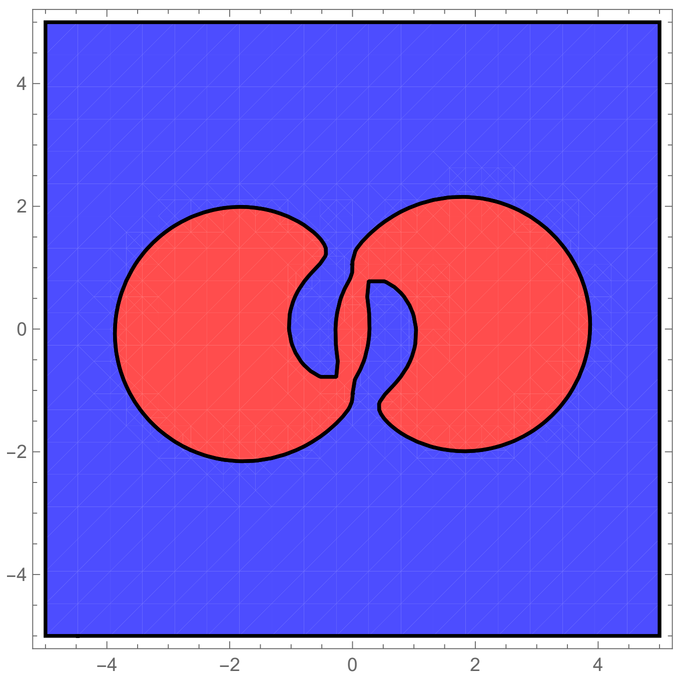


Figure 5.3: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

Before we proceed to the proof we require some definitions from algebraic geometry. For an introduction this subject we refer the reader to [14], [16] or [17]. For a given set of polynomials $p_1, \dots, p_k \in \mathbb{R}[x_1, \dots, x_d]$ and a set $U \subseteq \mathbb{R}^d$ we denote the variety generated by these polynomials on U by $V_U(p_1, \dots, p_k)$. In the case that $U = \mathbb{R}^d$ we write $V(\dots) = V_U(\dots)$.

Definition 5.5 (Smoothness, [17, §5])**.** We call an algebraic variety $V_U(p_1, \dots, p_k)$ *smooth* or *non-singular* if the Jacobian

$$\begin{bmatrix} Dp_1 & \cdots & Dp_k \end{bmatrix}$$

is of full rank on $V_U(p_1, \dots, p_k)$. This criterion of smoothness is also called the *Jacobi criterion*. By the implicit function theorem this means that the variety $V_U(p_1, \dots, p_k)$ in fact defines a $d-k$ dimensional submanifold of $\mathbb{R}^d$.

The proof of theorem 5.4 requires lemmas 5.6 and 5.9 which we will show later on.

Proof of theorem 5.4. It was already discussed in example 5.3 that f has an interior critical point at the B origin and is harmonic. For the connectedness of the entrant and emergent boundaries we calculate

$$rn \cdot u(x) = x_1^2(1 - x_1) + x_2^2(3x_1 - 1) + 2x_2x_3 =: p_1(x) \tag{5.9}$$

and define further

$$P_2(r,x) := x_1^2 + x_2^2 + x_3^2 - r^2\,. \tag{5.10}$$

Thus we have that the tangential boundary $\Sigma^0 = V(p_1, P_2(r,\cdot))$ is precisely the variety generated by the polynomials p_1 and $P_2(r,\cdot)$ for a fixed radius $r > 0$. In lemma 5.6 we will show that the variety $V(p_1, P_2(r,\cdot))$ is in fact smooth and in lemma 5.9 we will then show that it is in fact connected. Thus Σ^0 then defines a simple closed curve on Σ and the stereographic projection of Σ^0 defines a simple closed planar curve. This is indicated by the black curve in figure 5.3. By the Jordan curve theorem this curve splits the plane into two connected regions, one of which is simply connected. The preimage of these connected regions under the stereographic projection then corresponds precisely to the strictly entrant and strictly emergent boundaries. From this it follows that the strictly entrant and strictly emergent boundaries are simply connected which proves the theorem. $\square$

From now onwards we assume that p_1 and P_2 are given by equations (5.9) and (5.10). We first show the smoothness which was required in the proof of theorem 5.4.

Lemma 5.6 (Smoothness)**.** *There exists $R > 0$ such that for every $r > R$ the variety $V(p_1, P_2(r,\cdot))$ is smooth.*

Proof. One calculates

$$T := \begin{bmatrix} \nabla p_1(x) & \tfrac{1}{2}\nabla_x P_2(r,x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 + 3x_2^2 + 2x_1 & x_1 \\ 6x_1x_2 - 2x_2 + 2x_3 & x_2 \\ 2x_2 & x_3 \end{bmatrix}.$$

By the Jacobi criterion it is sufficient to show that this matrix is of full rank on $V(p_1, P_2(r,\cdot))$. This is equivalent to showing that

$$
\begin{aligned}
0 \neq \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} &= -9x_1^2x_2 + 3x_2^3 + 4x_1x_2 - 2x_1x_3 =: h_1(x)\,, \\
0 \neq \det \begin{bmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} &= 6x_1x_2x_3 - 2x_2x_3 + 2x_3^2 - 2x_2^2 =: h_2(x) \text{ or} \\
0 \neq \det \begin{bmatrix} T_{31} & T_{32} \\ T_{11} & T_{12} \end{bmatrix} &= 2x_1x_2 + 3x_1^2x_3 - 3x_2^2x_3 - 2x_1x_3 =: h_3(x)
\end{aligned}
$$

for any $x \in V(p_1, P_2(r,\cdot))$. This in turn is equivalent to showing that

$$V(p_1, P_2(r,\cdot), h_1, h_2, h_3) = \emptyset\,. \tag{5.11}$$

Indeed, consider the variety

$$V(p_1, h_1, h_2, h_3)\,. \tag{5.12}$$

Maple calculates the Gröbner basis with lexicographic order $x_1 < x_2 < x_3$

$$72x_1^8 - 198x_1^7 + 228x_1^6 - 153x_1^5 + 56x_1^4 - 5x_1^3\,, \tag{5.13}$$
$$72x_1^5x_2 - 126x_1^4x_2 + 102x_1^3x_2 - 51x_1^2x_2 + 5x_1x_2\,, \tag{5.14}$$
$$-24x_1^7 + 42x_1^6 - 2x_1^5 - 23x_1^4 + 7x_1^3 + 10x_1x_2^2\,, \tag{5.15}$$
$$48x_1^4x_2 - 60x_1^3x_2 + 13x_1^2x_2 + 15x_2^3\,, \tag{5.16}$$
$$24x_1^4x_2 - 30x_1^3x_2 + 29x_1^2x_2 - 10x_1x_2 + 5x_1x_3\,, \tag{5.17}$$
$$72x_1^7 - 126x_1^6 + 6x_1^5 + 69x_1^4 - 31x_1^3 + 10x_1^2 - 10x_2^2 + 20x_2x_3\,, \tag{5.18}$$
$$-72x_1^7 + 414x_1^6 - 654x_1^5 + 399x_1^4 - 97x_1^3 + 10x_1^2 - 30x_2^2 + 20x_3^2\,. \tag{5.19}$$

For an introduction to Gröbner bases we refer the reader for example to [11]. We will however only use the fact that the polynomials (5.13)-(5.19) generate the variety (5.12). We see from the basis vector (5.13) that for $x \in V(p_1, h_1, h_2, h_3)$ the coordinate x_1 can take only finitely many values. It then follows with (5.14) that also x_2 can take only finitely many values and finally with (5.15) that x_3 can also take only finitely many values. So the variety (5.12) contains finitely many points. Thus if we choose R so large that all of these points are contained in the ball B_R then we have that (5.11) holds for all $r > R$. □

We now define p_2 to be the dehomogenisation of P_2, that is

$$p_2 := P_2(1,\cdot)\,. \tag{5.20}$$

Analogously let P_1 denote the homogenisation of p_1, that is

$$P_1(\varepsilon, x) := \varepsilon^3 p_1(x/\varepsilon)\,. \tag{5.21}$$

By rescaling the variety $V(p_1, P_2(r,\cdot))$ we obtain

$$V(p_1, P_2(r,\cdot)) = rV(x \mapsto p_1(rx), p_2) = rV(P_1(\varepsilon,\cdot), p_2) = r\mathscr{V}_\varepsilon \tag{5.22}$$

where we set $\varepsilon = 1/r$ and $\mathscr{V}_\varepsilon := V(P_1(\varepsilon,\cdot), p_2)$. Motivated by taking the limit $r \to \infty$ we inspect the variety $\mathscr{V}_0$ closer. The next proposition is required in the proof of lemma 5.9. It essentially states that the varieties $\mathscr{V}_\varepsilon \to \mathscr{V}_0$ converge as $\varepsilon \to 0$ outside of singular points. We will thus also need a notion of convergence of subsets on a metric space.

Definition 5.7 (Hausdorff metric)**.** The *Hausdorff metric* for two sets $A, B \subseteq X$ is given by

$$d_H(A,B) := \max\left\{ \sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A) \right\} \tag{5.23}$$

where

$$\operatorname{dist}(x,B) := \inf_{y\in B} d(x,y) \tag{5.24}$$

is the smallest distance from x to B.

Proposition 5.8 (Convergence of $\mathscr{V}_\varepsilon$ at smooth points)**.** *Let $U \subset \mathbb{R}^3$ be an open bounded set such that $\mathscr{V}_0$ is smooth in an open neighbourhood of $\overline{U}$. Let further $\eta > 0$. Then there exists a $\delta > 0$ such that for all $\varepsilon < \delta$ we have that the Hausdorff distance satisfies*

$$d_H(\mathscr{V}_\varepsilon \cap U, \mathscr{V}_0 \cap U) < \eta$$

and additionally $\mathscr{V}_\varepsilon \cap U$ is isotopic to $\mathscr{V}_0 \cap U$.

Proof. The idea is to use the implicit function theorem. For this consider the mapping

$$F = \begin{bmatrix} P_1 \\ p_2 \end{bmatrix} : \mathbb{R}^4 \to \mathbb{R}^2 .$$

Since $V(P_1(0,\cdot), p_2)$ is smooth on an open neighbourhood of $\overline{U}$ we have by the Jacobi criterion that the derivative

$$DF(0,\cdot) = \left[\begin{array}{c|c} D_\varepsilon P_1(0,\cdot) & D_x P_1(0,\cdot) \\ 0 & Dp_2 \end{array}\right] \tag{5.25}$$

is of full rank on $\overline{U}$. By the implicit function theorem there exists at every point $x \in \overline{U}$ open neighbourhoods $\Omega_x \subseteq \mathbb{R}^4$ of $(0,x)$ and $\omega_x \subset \mathbb{R}^3$, a coordinate permutation $I \in O(4)$ and a continuously differentiable mapping $g_x \colon \omega_x \to \mathbb{R}$ such that

$$V(P_1, p_2) \cap \Omega_x = \{I(y, g_x(y)) \colon y \in \omega_x\} .$$

Since $D_x F(0,x)$ is of full rank we can assume (possibly after shrinking the open sets) that I does not permute the ε-coordinate. Thus we can write $y = (\varepsilon, y_1, y_2) \in \omega_x$. We can also assume that $\Omega_x = B_{\delta_x} \times W_x \subset \mathbb{R}^4$ for some open $W_x \subset \mathbb{R}^3$ and some $\delta_x > 0$. Hence we also obtain that $\omega_x = B_{\delta_x} \times w_x$ for some open $w_x \subset \mathbb{R}^2$ and we can define our isotopy on Ω_x as

$$\begin{aligned} \varphi_x \colon B_{\delta_x} \times w_x &\to W_x \\ y &\mapsto \mathrm{proj}_{\mathbb{R}^4 \to W_x} I(y, g_x(y)) \,. \end{aligned}$$

Note that $\varphi_x(\{\varepsilon\} \times w_x) = \mathscr{V}_\varepsilon \cap W_x$. From this it also follows that we can choose δ_x such that

$$d_H(\mathscr{V}_\varepsilon \cap W_x, \mathscr{V}_0 \cap W_x) < \eta \,.$$

Now for $x \in \overline{U}$ the Ω_x form an open cover of $\overline{U}$. By compactness there exists a finite subcover. Set $\delta > 0$ to be the minimum of all δ_x for the Ω_x in this finite subcover and the claim follows. □

The next lemma shows the connectedness of the variety $V(p_1, P_2(r,\cdot))$. Because the proof is quite lengthy, a part of the proof has been split off into proposition 5.10 which will be proved later on.

Lemma 5.9 (Connectedness)**.** *There exists an $r > 0$ such that the planar variety $V(p_1, P_2(r,\cdot))$ has one connectivity component.*

Proof. By lemma 5.6 there exists a $R > 0$ such that for all $r > R$ we have that the variety $V(p_1, P_2(r,\cdot))$ is smooth and by equation (5.22) $\mathscr{V}_\varepsilon$ is also smooth for $\varepsilon < 1/R$. We inspect $\mathscr{V}_0$ closer. Observe that

$$P_1(0,x) = -x_1^3 + 3x_1x_2^2$$

which is the monkey saddle embedded into $\mathbb{R}^3$. We thus define arcs

$$\widetilde{\alpha}^{\pm} := \left\{ t \begin{bmatrix} \pm\sqrt{3} & 1 \end{bmatrix}^\top : t \in \mathbb{R} \right\}$$

and $\widetilde{\alpha}^0 := \{0\} \times \mathbb{R} \subset \mathbb{R}^2$. We then define arcs $\alpha^\bullet := (\widetilde{\alpha}^\bullet \times \mathbb{R}) \cap S^2 \subset \mathbb{R}^3$. Setting $A := \alpha^- \cup \alpha^+ \cup \alpha^0$ we have the relation

$$\mathscr{V}_0 = V(P_1(\varepsilon,\cdot), p_2) = A \,.$$

Thus $\mathscr{V}_0$ consists of six smooth arcs originating at the singularity e_3 and ending at the singularity $-e_3$. Similar to the classical beach ball. Now consider for $\rho > 0$ the open sets $W_\rho := B_\rho \times \mathbb{R} \subseteq \mathbb{R}^3$ and $U_\rho := B_2 \setminus \overline{W}_\rho \subseteq \mathbb{R}^3$. Since $\mathscr{V}_0$ is smooth in a neighbourhood of $\overline{U}_\rho$ we obtain from proposition 5.8 that in a certain sense $\mathscr{V}_\varepsilon$ is obtained from $\mathscr{V}_0$ by a small deformation on U_ρ. Thus in order to show connectedness of $\mathscr{V}_\varepsilon$ for sufficiently small $\varepsilon > 0$ we have to inspect what happens around the points $\pm e_3$. Now observe that we have the symmetry

$$p_1(x_1, -x_2, -x_3) = p_1(x) \tag{5.26}$$

and thus it suffices to inspect what happens around the point e_3. For this parametrise the neighbourhood $S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)$ of e_3 by the diffeomorphism

$$\begin{aligned} \psi \colon \mathbb{R}^2 \supset B_{1/2} &\to S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right) \subset \mathbb{R}^3 \\ \tilde{x} &\mapsto x = \begin{bmatrix} \tilde{x} & \sqrt{1 - |\tilde{x}|^2} \end{bmatrix}^\top . \end{aligned}$$

We set

$$\widetilde{\mathscr{V}}_\varepsilon := \psi^{-1}\left(\mathscr{V}_\varepsilon \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)\right) = V_{B_{1/2}}\left(x \mapsto P_1\left(\varepsilon, x_1, x_2, \sqrt{1-x_1^2-x_2^2}\right)\right) = V_{B_{1/2}}\left(\widetilde{P}_1(\varepsilon,\cdot)\right)$$

where we defined

$$\widetilde{P}_1(\varepsilon,x) := P_1\left(\varepsilon, x_1, x_2, \sqrt{1-x_1^2-x_2^2}\right) = -x_1^3 + 3x_1x_2^2 + \varepsilon\left(x_1^2 - x_2^2 + 2x_2\sqrt{1-x_1^2-x_2^2}\right).$$

In a similar manner we define

$$\begin{aligned}\widetilde{U}_\rho &:= \psi^{-1}\left(U_\rho\right) = B_{1/2} \setminus \overline{B}_\rho\,,\\ \widetilde{W}_\rho &:= \psi^{-1}\left(W_\rho\right) = B_{1/2} \cap B_\rho \text{ and}\\ \widetilde{A} &:= \psi^{-1}(A) \quad = B_{1/2} \cap \left(\widetilde{\alpha}^- \cup \widetilde{\alpha}^+ \cup \widetilde{\alpha}^0\right)\end{aligned}$$

to be subsets of $\mathbb{R}^2$. Also note that $\widetilde{\alpha}^\bullet \cap B_{1/2} = \psi^{-1}(\alpha^\bullet)$. We require the following proposition which will be proved later:

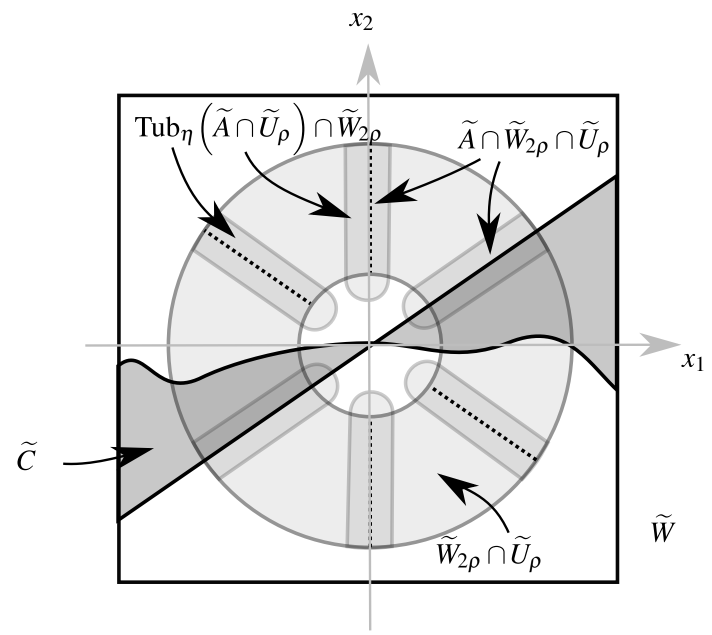


Figure 5.4: An overview of the situation around e_3.

Proposition 5.10 (Construction of $\widetilde{C}$ and $\widetilde{\gamma}$)**.** *Let $\widetilde{U}_\rho$, $\widetilde{\alpha}^\bullet$, $\widetilde{A}$, $\widetilde{\mathscr{V}}_\varepsilon$ and $\widetilde{P}_1$ be as in the proof of lemma 5.9. There exists an open cube $\widetilde{W} \subset \mathbb{R}^2$ containing the origin and a set $\widetilde{C} \subseteq \widetilde{W}$ such that $\widetilde{C}$ has positive Hausdorff distance to the set $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that for any sufficiently small $\varepsilon > 0$ there is an arc $\widetilde{\gamma}$ entirely contained in $\widetilde{C} \cap \widetilde{\mathscr{V}}_\varepsilon$ splitting $\widetilde{W}$ into two parts.*

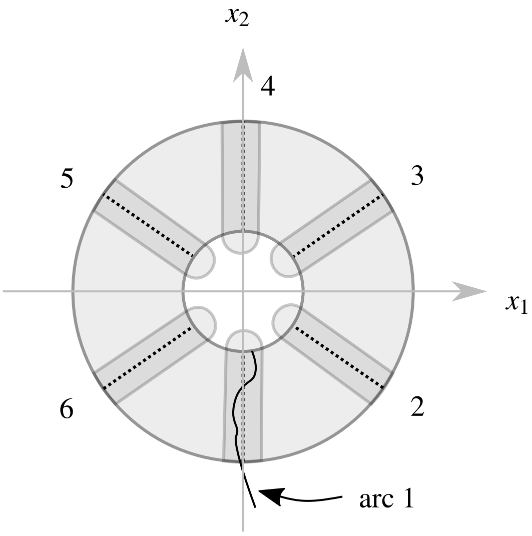


Figure 5.5: The numbering of the arcs around e_3 as viewed from above.

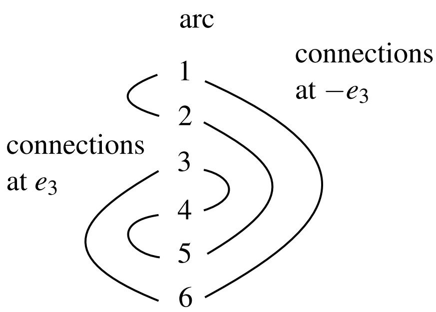
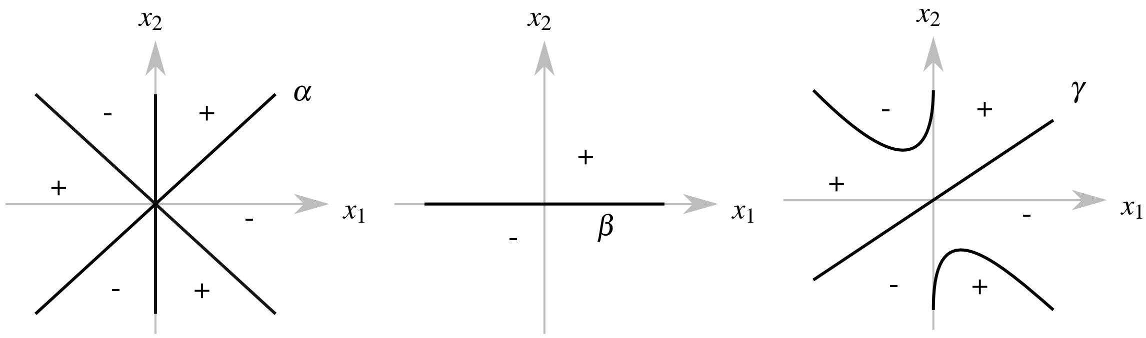


Figure 5.6: The connection of the arcs at $\pm e_3$.

Now let the sets $\widetilde{W} \subset \mathbb{R}^2$ and $\widetilde{C} \subseteq \widetilde{W}$ be as in this proposition. Pick $\rho > 0$ so small that $\widetilde{W}_{2\rho} \subseteq \widetilde{W}$. We can pick η smaller than half the minimum distance between two arcs of $\widetilde{\mathscr{V}_0}$ on $\widetilde{U}_\rho$. We also assume that η is smaller than half the Hausdorff distance between $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ and $\widetilde{C}$. Now choose δ as in proposition 5.8. We can assume that $0 < \varepsilon < \delta$. We thus have a situation as in figure 5.4.

Number the arcs of $\mathscr{V}_\varepsilon \cap U_\rho$ lying close to $A \cap U_\rho$ as in figure 5.5. Since the curve $\widetilde{\gamma}$ by proposition 5.10 splits $\widetilde{W}_{2\rho}$ and lies in $\widetilde{C} \cap \widetilde{\mathscr{V}_\varepsilon}$ we must have that $\psi \circ \widetilde{\gamma}$ in fact connects arcs 3 and 6. As the variety is smooth we must also have that arcs 1 and 2 are connected and analogously that arcs 4 and 5 are connected. By equation (5.26) the situation around $-e_3$ is analogous but mirrored. We thus obtain corresponding connections between the six arcs at $-e_3$ which is summarised in figure 5.6. Thus we in fact have that the variety $\mathscr{V}_\varepsilon$ is connected. The claim then follows with the relation (5.22). $\square$

It now only remains to show proposition 5.10.



(a) Sketch of the set $V(\tilde{q}_1)$. (b) Sketch of the set $V(\tilde{q}_2)$. (c) Sketch of the set $\widetilde{\mathscr{V}}_\varepsilon$.

Figure 5.7: Sketches of varieties.

Proof of proposition 5.10. We borrow notation from the proof of lemma 5.9. Let $\widetilde{W} \subset \mathbb{R}^2$ be an open cube around the origin which we fix later. Define polynomials

$$\tilde{q}_1(x) := -x_1^3 + 3x_1x_2^2 \tag{5.27}$$

$$\tilde{q}_2(x) := x_1^2 - x_2^2 + 2x_2\sqrt{1 - x_1^2 - x_2^2} \tag{5.28}$$

and observe that

$$\widetilde{P}_1(\varepsilon, \cdot) = \tilde{q}_1 + \varepsilon \tilde{q}_2\,. \tag{5.29}$$

By equation (5.27) $\tilde{q}_1$ is a monkey saddle and $V_{\widetilde{W}}(\tilde{q}_1) = \widetilde{A} \cap \widetilde{W}$ is similar to figure 5.7a. The signs in the figure indicate the sign of $\tilde{q}_1$ in a given region. From equation (5.28) we obtain that $\tilde{q}_2(0) = 0$ and $\nabla \tilde{q}_2(0) = 2e_2$. Hence we observe that the $V_{\widetilde{W}}(\tilde{q}_2)$ looks similar to figure 5.7b for a sufficiently small neighbourhood $\widetilde{W}$. More concretely we choose $\widetilde{W}$ so small that the arc $\widetilde{\beta} := V_{\widetilde{W}}(\tilde{q}_2)$ has positive distance to $\widetilde{A} \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that a given vertical line in $\widetilde{W}$ intersects $\widetilde{\beta}$ in precisely one point. Now we claim that the set $\widetilde{C}$ consisting of the vertical lines between $\widetilde{\alpha}^+$ and $\widetilde{\beta}$ fulfils the claim. For clarity the situation is depicted in figure 5.8.

We first show that there exists a curve $\widetilde{\gamma}$ through the origin which is entirely contained in $\widetilde{C} \cap \widetilde{\mathscr{V}}_\varepsilon$. For this note that we have $\widetilde{P}_1(\varepsilon, 0) = 0$ and $\nabla_x \widetilde{P}_1(\varepsilon, x)\big|_{x=0} = \varepsilon e_2$ and thus there exists locally around the origin a parametrisation $\widetilde{\gamma}(t) = (t, \widetilde{\gamma}_2(t))$ of the variety $\widetilde{\mathscr{V}}_\varepsilon$. By a similar argument there also exists locally around the origin a parametrisation $\widetilde{\beta}(t) = \big(t, \widetilde{\beta}_2(t)\big)$ of the variety $\widetilde{\beta}$. We see that $\widetilde{\gamma}$ lies locally below $\widetilde{\alpha}^+$ and thus we need to show that $\widetilde{\gamma}_2$ also lies locally above $\widetilde{\beta}_2$ on the right half plane and locally below $\widetilde{\beta}_2$ on the left half plane. We calculate derivatives

$$D\tilde{q}_1\big|_{x=0} = 0$$

$$D^2\tilde{q}_1\big|_{x=0} = 0$$

$$D^3\tilde{q}_1\big|_{x=0}(v,v,v) = -6v_1^3 + 18v_1v_2^2$$

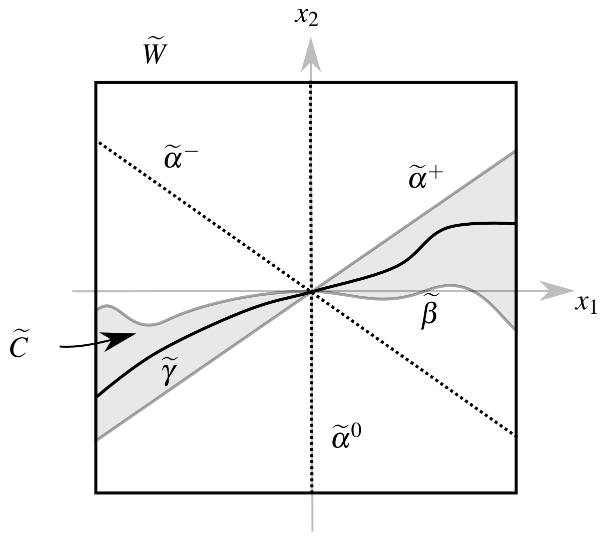


Figure 5.8: An overview of the situation in proposition 5.10.

and

$$
\begin{aligned}
D\tilde{q}_2\big|_{x=0} &= 2e_2^\top \\
D^2\tilde{q}_2\big|_{x=0} &= \begin{bmatrix} 2 & \\ & -2 \end{bmatrix} \\
D^3\tilde{q}_2\big|_{x=0}(v,v,v) &= -6\big(v_1^2 v_2 + v_2^3\big)
\end{aligned}
$$

and now by relation (5.29)

$$
\begin{aligned}
Dh\big|_{x=0} &= \varepsilon D\tilde{q}_2\big|_{x=0} \\
D^2h\big|_{x=0} &= \varepsilon D^2\tilde{q}_2\big|_{x=0} \\
D^3h\big|_{x=0}(v,v,v) &= -6v_1^3 + 18v_1v_2^2 - 6\varepsilon\big(v_1^2v_2 + v_2^3\big)
\end{aligned}
\tag{5.30}
$$

where we wrote $h(x) = \widetilde{P}_1(\varepsilon, x)$ for brevity. We have $0 = h \circ \widetilde{\gamma}$ and thus

$$
0 = \partial_t (h \circ \widetilde{\gamma})\big|_{t=0} = \Big(Dh\big|_{\widetilde{\gamma}}\widetilde{\gamma}'\Big)_{t=0} = 2\varepsilon\widetilde{\gamma}_2'(0)
\tag{5.31}
$$

so $\widetilde{\gamma}_2'(0) = 0$ and in particular $\widetilde{\gamma}'(0) = e_1$. Taking another derivative we get

$$
\begin{aligned}
0 &= \partial_t^2 (h \circ \widetilde{\gamma})\big|_{t=0} \\
&= \Big(D^2h\big|_{\widetilde{\gamma}}(\widetilde{\gamma}', \widetilde{\gamma}') + Dh\big|_{\widetilde{\gamma}}\widetilde{\gamma}''\Big)_{t=0} \\
&= \big(D^2h\big|_{x=0}(e_1, e_1) + Dh\big|_{x=0}\widetilde{\gamma}''\big)_{t=0} \\
&= \varepsilon\big(2 + 2\widetilde{\gamma}_2''\big)(0)
\end{aligned}
\tag{5.32}
$$

so $\widetilde{\gamma}'(0) = -e_2$. The third derivative then yields

$$\begin{aligned}
0 &= \partial_t^3 (h \circ \widetilde{\gamma})\big|_{t=0} \\
&= \left(D^3 h\big|_{\widetilde{\gamma}}(\widetilde{\gamma}', \widetilde{\gamma}', \widetilde{\gamma}') + 3 D^2 h\big|_{\widetilde{\gamma}}(\widetilde{\gamma}', \widetilde{\gamma}'') + D h\big|_{\widetilde{\gamma}} \widetilde{\gamma}''' \right)_{t=0} \\
&= \left(D^3 h\big|_{x=0}(e_1, e_1, e_1) - 3 D^2 h\big|_{x=0}(e_1, e_2) + D h\big|_{x=0} \widetilde{\gamma}''' \right)_{t=0} \\
&= \left(-6 + 2\varepsilon \widetilde{\gamma}_2''' \right)(0)
\end{aligned}$$

so $\widetilde{\gamma}_2'''(0) = 3/\varepsilon$. Analogously we observe that it follows from $0 = \tilde{q}_2 \circ \widetilde{\beta}$ that

$$0 = \varepsilon \partial_t \left(\tilde{q}_2 \circ \widetilde{\beta} \right)\big|_{x=0} = \varepsilon \left(D \tilde{q}_2\big|_{\widetilde{\beta}} \widetilde{\beta}' \right)_{t=0} \tag{5.33}$$

and

$$0 = \varepsilon \partial_t^2 \left(\tilde{q}_2 \circ \widetilde{\beta} \right)\big|_{x=0} = \varepsilon \left(D^2 \tilde{q}_2\big|_{\widetilde{\beta}} \left(\widetilde{\beta}', \widetilde{\beta}' \right) + D \tilde{q}_2\big|_{\widetilde{\beta}} \widetilde{\beta}'' \right)_{t=0}. \tag{5.34}$$

By the relations (5.30), equation (5.33) is identical to equation (5.31) and equation (5.34) is identical to equation (5.32) with $\widetilde{\gamma}$ replaced by $\widetilde{\beta}$. Now equations (5.31) and (5.32) determined $\widetilde{\gamma}(0)$ and $\widetilde{\gamma}'(0)$ uniquely and thus we have that $\widetilde{\gamma}(0) = \widetilde{\beta}(0)$, $\widetilde{\gamma}(0) = \widetilde{\beta}'(0)$ and $\widetilde{\gamma}'(0) = \widetilde{\beta}''(0)$. For the third derivative we observe that

$$\begin{aligned}
0 &= \partial_t^3 \left(\tilde{q}_2 \circ \widetilde{\beta} \right)\big|_{t=0} \\
&= \left(D^3 \tilde{q}_2\big|_{\widetilde{\beta}} \left(\widetilde{\beta}', \widetilde{\beta}', \widetilde{\beta}' \right) + 3 D^2 \tilde{q}_2\big|_{\widetilde{\beta}} \left(\widetilde{\beta}', \widetilde{\beta}'' \right) + D \tilde{q}_2\big|_{\widetilde{\beta}} \widetilde{\beta}''' \right)_{t=0} \\
&= \left(D^3 \tilde{q}_2\big|_{x=0}(e_1, e_1, e_1) - 3 D^2 \tilde{q}_2\big|_{x=0}(e_1, e_2) + D \tilde{q}_2\big|_{x=0} \widetilde{\beta}''' \right)_{t=0} \\
&= 2 \widetilde{\beta}_2'''(0)
\end{aligned}$$

so $\widetilde{\beta}'''(0) = 0$. Thus we obtain

$$\partial_t^k \left(\widetilde{\gamma}_2 - \widetilde{\beta}_2 \right)\big|_{t=0} = 0$$

for $k \leq 2$ and

$$\partial_t^3 \left(\widetilde{\gamma}_2 - \widetilde{\beta}_2 \right)\big|_{t=0} = 3/\varepsilon > 0\,.$$

Hence we indeed have that $\widetilde{\gamma}_2 > \widetilde{\beta}_2$ in a sufficiently small neighbourhood of the origin for $t > 0$ and also that $\widetilde{\gamma}_2 < \widetilde{\beta}_2$ in a sufficiently small neighbourhood for $t < 0$.

Now that we have established that $\widetilde{\gamma}$ lies locally in $\widetilde{C}$ it remains to be shown that $\widetilde{\gamma}$ indeed remains in $\widetilde{C}$ and reaches the boundary $\partial \widetilde{W}$. For this let $\widetilde{\gamma}\colon [0,b) \to \widetilde{C}$ be a maximally extended parametrisation of $\widetilde{\gamma}$. Since $\widetilde{C}$ is compact there exists a sequence $t_k \to b$ such that $x_k = \widetilde{\gamma}(t_k) \to x$ is convergent. Then we have by continuity that also $h(x) = 0$ so also $x \in \mathscr{V}_\varepsilon$. As $\mathscr{V}_\varepsilon$ is smooth we have that $Dh(x)$ is of full rank. Thus by the implicit function theorem we must in fact have that $b < \infty$ and the parametrisation of $\widetilde{\gamma}$ can be extended beyond the point b. But this means that $x \in \partial \widetilde{C}$. Again by smoothness x cannot lie in the origin. Now note that $\widetilde{P}_1(\varepsilon, \cdot) = \tilde{q}_1 < 0$ on the arc $\widetilde{\beta}$ and that $\widetilde{P}_1(\varepsilon, \cdot) = \varepsilon \tilde{q}_2 > 0$ on the arc $\widetilde{\alpha}^+$. So $\widetilde{\gamma}$ cannot intersect $\partial \widetilde{C}$ on $\widetilde{\beta}$ or on $\widetilde{\alpha}$. Thus we must have that $x \in \partial \widetilde{W}$ and hence $\widetilde{\gamma}$ splits $\widetilde{W}$ in the right half plane into two parts. On the left half plane the argumentation is analogous. Then $\widetilde{\gamma}$ divides the plane into two parts which yields the claim. Note that as a consequence $\mathscr{V}_\varepsilon$ looks similar to figure 5.7c. □

An example with positive distance between the entrant and emergent boundaries

In the previous example we showed that Σ^+ is separated from Σ^- by the variety $\Sigma^0 = V(p_1, P_2(r, \cdot))$. In the following we will show that it is possible to thicken the tangential boundary Σ^0 without loosing the fundamental properties of the solution. More concretely, we construct for $\rho > 0$ a harmonic function f_ρ for which the entrant and emergent boundaries are simply connected, have positive distance from one another and such that f_ρ has an interior stagnation point. For this we use the notation from the previous example. We need a preliminary result:

Proposition 5.11 (Existence of a cutoff function)**.** *Let Σ, $\Sigma^\pm$, Σ^0 and p_1 be as in example 5.3. For every $\rho > 0$ there exists a cutoff function $\theta_\rho : \Sigma \to [0,1]$ such that*

1. *$\theta_\rho = 1$ outside of the tubular neighbourhood $\mathrm{Tub}_\rho(\Sigma^0)$,*
2. *θ_ρ has support contained in $\Sigma \setminus \Sigma^0$ and*
3. *$\langle \theta_\rho p_1 \rangle_\Sigma = 0$, where $\langle h \rangle_X := \int_X h$ denotes the expectation value of a function $h \colon X \to Y$.*

Proof. Let $\theta_\lambda^+ : \Sigma \to [0,1]$ be a family of cutoff functions smoothly depending on $\lambda > 0$ such that $\theta_\lambda^+ = 1$ outside of the neighbourhood $\mathrm{Tub}_\lambda(\Sigma^{\leq 0})$ and θ_λ^+ has support contained in Σ^+. Analogously we let $\theta_\lambda^- : \Sigma \to [0,1]$ be a family of cutoff functions smoothly depending on $\lambda > 0$ such that $\theta_\lambda^- = 1$ outside of the neighbourhood $\mathrm{Tub}_\lambda(\Sigma^{\geq 0})$ and θ_λ^- has support contained in Σ^-. We can also assume that both θ_λ^+ and θ_λ^- are monotonously increasing as λ decreases. Set

$$\tilde{\theta}_\rho = \theta_\rho^+ + \theta_{\lambda(\rho)}^- \tag{5.35}$$

where $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a suitably chosen map which we will specify later. By construction the support of $\tilde{\theta}_\rho$ lies inside the set $\Sigma \setminus \Sigma^0$ so $\tilde{\theta}_\rho$ fulfils condition 2. For condition 3 we calculate

$$\langle \tilde{\theta}_\rho p_1 \rangle_\Sigma = \left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} + \left\langle \theta_{\lambda(\rho)}^- p_1 \right\rangle_{\Sigma^-}. \tag{5.36}$$

Now we have that $p_1 < 0$ on Σ^- and that $p_1 > 0$ on Σ^+. Thus $\left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} > 0$ is monotonically increasing as ρ decreases and

$$\left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} \longrightarrow \langle p_1 \rangle_{\Sigma^+} =: c^+$$

as $\rho \to 0$. Analogously $\left\langle \theta_\lambda^- p_1 \right\rangle_{\Sigma^-} < 0$ is monotonically and continuously decreasing as λ decreases and

$$\left\langle \theta_\lambda^- p_1 \right\rangle_{\Sigma^-} \longrightarrow \langle p_1 \rangle_{\Sigma^-} =: c^-$$

as $\lambda \to 0$. It follows from the divergence theorem that

$$0 = r \int_X \Delta f = \int_\Sigma r \nabla f \cdot n = \langle p_1 \rangle_\Sigma = c^+ + c^-$$

where $n \colon \Sigma \to S^2$ is the outer unit normal. It then follows from continuity that for each $\rho > 0$ there exists a $\lambda(\rho) > 0$ such that

$$\left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} + \left\langle \theta_{\lambda(\rho)}^- p_1 \right\rangle_{\Sigma^-} = 0$$

so by equation (5.36) $\tilde{\theta}_\rho$ also fulfils condition 3. In fact we can also assume that $\lambda(\rho) \to 0$ as $\rho \to 0$. Now choose $\tilde{\lambda}(\rho)$ such that $\max\big(\tilde{\lambda}, \lambda(\tilde{\lambda})\big) < \rho$. This defines a function $\tilde{\lambda} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and we set $\theta_\rho := \tilde{\theta}_{\tilde{\lambda}(\rho)}$. Property 1 then follows from the fact that $\tilde{\theta}_\rho = 1$ outside of the neighbourhood $\mathrm{Tub}_{\max(\rho, \lambda(\rho))}\big(\Sigma^0\big)$ by equation 5.35. □

With the help of this cutoff function θ_ρ we can now formulate our main result in this section:

Example 5.12 (Thickening Σ^0)**.** Let f and X be as in example 5.3 and the cutoff function θ_ρ be as in the previous proposition. Consider now for $\rho > 0$ the solution f_ρ to the system

$$\begin{aligned} \Delta f_\rho &= 0 && \text{on } X, \\ \nabla f_\rho \cdot n &= \theta_\rho (\nabla f \cdot n) && \text{on } \Sigma, \\ \langle f_\rho \rangle_X &= \langle f \rangle_X \end{aligned}$$

where $n \colon \Sigma \to S^2$ is the outer unit normal. By definition of f_ρ the entrant and emergent boundaries are simply connected and have positive distance from one another. We will show in proposition 5.13 that there exists $\rho > 0$ such that f_ρ has an interior critical point.

Proposition 5.13 (Structural stability)**.** *Let x be a non-degenerate interior critical point of f. Then there exist $\mathrm{P} > 0$ and $\varepsilon > 0$ such that for all $\rho \leq \mathrm{P}$ the function f_ρ from example 5.12 has an interior critical point in $\overline{B}_\varepsilon(x)$.*

Proof. The main idea is to apply a fixed-point theorem in a suitable manner. To simplify notation we only show this for the interior critical point of f at the origin $x = 0$. Define $g_\rho := f_\rho - f$. By linearity g_ρ is a solution to the system

$$\begin{aligned} \Delta g_\rho &= 0 && \text{on } X, \\ \nabla g_\rho \cdot n &= \big(\theta_\rho - 1\big)(\nabla f \cdot n) && \text{on } \Sigma, \\ \langle g_\rho \rangle_X &= 0 && . \end{aligned}$$

To simplify notation we write $x \lesssim y$ instead of $x \leq cy$ for some constant $c \geq 1$ which is independent of ρ. We have that

$$
\begin{aligned}
\left\|Dg_\rho\right\|^2_{L^2\left(X;\mathbb{R}^d\right)} &\overset{\text{divergence Theorem}}{=} \int_\Sigma g_\rho \left(\nabla g_\rho \cdot n\right) \\
&\overset{\text{Cauchy-Schwarz}}{\leq} \left\|g_\rho\right\|_{L^2(\Sigma)} \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)} \\
&\overset{\text{trace theorem}}{\lesssim} \left\|g_\rho\right\|_{W^{1,2}(X)} \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)} \\
&\overset{\text{Poincaré-Wirtinger}}{\lesssim} \left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^d\right)} \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)}
\end{aligned}
$$

and thus together with dominated convergence

$$
\left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^d\right)} \lesssim \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)} = \left\|\left(\theta_\rho - 1\right)\left(\nabla f \cdot n\right)\right\|_{L^2(\Sigma)} \longrightarrow 0
$$

as $\rho \to 0$. As a consequence of this we obtain on the one hand that for a compactly contained open set $\{0\} \Subset U \Subset X$ with smooth boundary we have

$$
\begin{aligned}
\left\|Dg_\rho\right\|_{L^\infty\left(U;\mathbb{R}^{d\times d}\right)} &\overset{\text{mean value property}}{\lesssim} \left\|Dg_\rho\right\|_{L^1\left(X;\mathbb{R}^{d\times d}\right)} \\
&\overset{\text{Cauchy-Schwarz}}{\lesssim} \left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^{d\times d}\right)} \longrightarrow 0
\end{aligned}
\tag{5.37}
$$

as $\rho \to 0$. Similarly it also follows for a compactly contained open set $U \Subset V \Subset X$ with smooth boundary that

$$
\begin{aligned}
\left\|D^2 g_\rho\right\|_{L^\infty\left(U;\mathbb{R}^{d\times d}\right)} &\overset{\text{mean value property}}{\lesssim} \left\|D^2 g_\rho\right\|_{L^1\left(V;\mathbb{R}^{d\times d}\right)} \\
&\overset{\text{Cauchy-Schwarz}}{\lesssim} \left\|D^2 g_\rho\right\|_{L^2\left(V;\mathbb{R}^{d\times d}\right)} \\
&\overset{\text{interior regularity}}{\lesssim} \left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^d\right)} \longrightarrow 0
\end{aligned}
\tag{5.38}
$$

as $\rho \to 0$. We would like to apply a fixed point theorem to the mapping

$$
\begin{aligned}
F_\rho \colon \mathbb{R}^d &\to \mathbb{R}^d \\
y &\mapsto y - D^2 f\big|_0^{-1} Df_\rho(y)
\end{aligned}
$$

where we used the fact that $D^2f\big|_0$ is bijective by non-degeneracy. We denote by $\|\cdot\|_{\mathrm{Op}}$ the operator norm induced by $L^\infty\big(U;\mathbb{R}^d\big)$, that is

$$\|h\|_{\mathrm{Op}} = \sup_{y\in L^\infty(U;\mathbb{R}^d)\setminus\{0\}} \frac{\|hy\|_{L^\infty(U;\mathbb{R}^d)}}{\|y\|_{L^\infty(U;\mathbb{R}^d)}}$$

for some function $h \in L^\infty\big(U;\mathbb{R}^{d\times d}\big)$. Set $U = B_\varepsilon \subset \mathbb{R}^3$ to be an open ball around the origin with radius $\varepsilon > 0$. We calculate the derivative

$$DF_\rho = \mathrm{Id} - D^2f\big|_0^{-1} D^2 f_\rho$$

and take the operator norm

$$\frac{1}{c}\|DF_\rho\|_{\mathrm{Op}} \leq \big\|D^2f\big|_0 - D^2f_\rho\big\|_{\mathrm{Op}} \leq \big\|D^2f\big|_0 - D^2f\big\|_{\mathrm{Op}} + \big\|D^2g_\rho\big\|_{\mathrm{Op}} \tag{5.39}$$

for some constant $c \geq 1$ independent of ρ. Now we have by continuity that

$$\big\|D^2f\big|_0 - D^2f\big\|_{\mathrm{Op}} \lesssim \big\|D^2f\big|_0 - D^2f\big\|_{L^\infty(U;\mathbb{R}^{d\times d})} \longrightarrow 0$$

as $\varepsilon \to 0$. Pick $\varepsilon > 0$ such that $\big\|D^2f\big|_0 - D^2f\big\|_{\mathrm{Op}} \leq 1/(4c)$. Now we have for $\rho \to 0$ that

$$\big\|D^2g_\rho\big\|_{\mathrm{Op}} \lesssim \big\|D^2g_\rho\big\|_{L^\infty(U;\mathbb{R}^{d\times d})} \longrightarrow 0$$

by equation (5.38). Thus there exists a $\mathrm{P} > 0$ such that $\big\|D^2g_\rho\big\|_{\mathrm{Op}} \leq 1/(4c)$ for all $\rho \leq \mathrm{P}$. By equation (5.39) it then follows that $\big\|DF_\rho\big\|_{\mathrm{Op}} \leq 1/2$ for all $\rho \leq \mathrm{P}$. Since we have by equation (5.37) that

$$\big|F_\rho(0)\big| = \Big|D^2f\big|_0^{-1} Df_\rho(0)\Big| \lesssim |Df_\rho(0)| = |Dg_\rho(0)| \leq \big\|Dg_\rho\big\|_{L^\infty(U;\mathbb{R}^d)} \longrightarrow 0$$

as $\rho \to 0$ we can also additionally choose $\mathrm{P} > 0$ such that for all $\rho \leq \mathrm{P}$ we have that $\big|F_\rho(0)\big| \leq \varepsilon/2$. We then have for all $\rho \leq \mathrm{P}$ that $F_\rho\big(\overline{B}_\varepsilon\big) \subseteq \overline{B}_\varepsilon$ and F_ρ is a contraction. It then follows from fixed point theorems that for every $\rho \leq \mathrm{P}$ the function F_ρ has a fixed point, say $y_\rho \in \overline{B}_\varepsilon$. This then fulfils by the construction of F_ρ the equation

$$y_\rho = F_\rho\big(y_\rho\big) = y_\rho - D^2f\big|_0^{-1} Df_\rho\big(y_\rho\big)$$

which implies that $Df_\rho\big(y_\rho\big) = 0$ and y_ρ is an interior critical point. $\square$

6 No in- or outflow

In the final chapter of this thesis we will discuss harmonic vector fields without inflow or outflow through the boundary. First we will state the Morse index theorem which generalises the Poincaré-Hopf index theorem in a certain sense. As a direct consequence we obtain that the number of interior stagnation points for such a harmonic vector field in $d = 2$ dimensions must equal the negative Euler characteristic $-\chi(X)$ of the domain. This relation between the domain topology in the plane and the number of critical points is very similar to a result on the number of critical points of Green functions in for example [31, p.133]. After that we will give examples which illustrate this result. In the final part we turn our attention to $d = 3$ dimension. In this case the Morse index theorem implies that the Euler characteristic of the domain has to vanish and the number of stagnation points of such a harmonic vector field has to be even.

The Morse index theorem

In this chapter we will discuss harmonic vector fields u which are not necessarily a gradient field of the form $u = \nabla f$. Hence we would like to state the Morse inequalities from theorem 3.3 for general vector fields u. For this we define another notion of index:

Definition 6.1 (Poincaré-Hopf index, [8, Definition 1.1.1])**.** Let X be an orientable manifold with corners and $u\colon X \to TX$ a continuous vector field with isolated stagnation point x of u_j on the stratum X_j. Let $\phi\colon U \to V$ be a chart on an open neighbourhood $U \subseteq X_j$ of x. We can define the *Poincaré-Hopf index* $\mathrm{Ind}^{\mathrm{PH}}_{j,x}(u)$ to be the Brouwer degree of the Gauss map

$$\frac{v}{|v|}\colon S^{d-1}_{\varepsilon}(\phi(x)) \to S^{d-1}$$

where $v := D\phi \circ u_j \circ \phi^{-1}$ and $\varepsilon > 0$ is chosen sufficiently small. Assume that X_j has at most finitely many isolated stagnation points. Then we can define the *total index* $\mathrm{Ind}^{\mathrm{PH}}_{j}(u)$ on the stratum X_j to be the sum of indexes $\mathrm{Ind}^{\mathrm{PH}}_{j,x}(u)$ for every isolated essential interior stagnation point x of u on X_j.

Note that unlike in the definition of the Morse index we do not require u to be irrotational. The reason for requiring X to be orientable is for the Brouwer degree to make sense. It is shown in [26, §6, Lemma 1] that this index is well-defined. The next lemma relates this notion of index to the Morse index. It is stated and proved in [26, §6, Lemma 4].

Lemma 6.2 (Relation to the Morse index, [26, §6, Lemma 4])**.** *A non-degenerate stagnation point of an irrotational vector field with Morse index k has Poincaré-Hopf index $(-1)^k$.*

Hence it follows for a Morse irrotational vector field $u\colon X \to TX$ that

$$\mathrm{Ind}_j^{\mathrm{PH}}(u) = \sum_k (-1)^k \,\mathrm{Ind}_{j,k}^{\mathrm{M}}(u)$$

for all strata X_j and so

$$\sum_j \mathrm{Ind}_j^{\mathrm{PH}}(u) = \sum_k (-1)^k \,\mathrm{Ind}_k^{\mathrm{M}}(u)\,. \tag{6.1}$$

We can now formulate the Morse index theorem which is proved for instance in [28]:

Theorem 6.3 (Morse index theorem, [28, Theorem A_0])**.** *Let X be a compact orientable manifold with smooth boundary and $u\colon X \to TX$ a continuous vector field. Assume that u is regular and that all essential stagnation points of u are isolated. Then we have that*

$$\sum_j \mathrm{Ind}_j^{PH}(u) = \chi(X)\,.$$

This theorem generalises the Poincaré-Hopf index theorem which can be found for instance in [26, §6]. It has been generalised in [33] to manifolds with boundary for Morse vector fields using terminology of convex optimisation. One can also find a proof of this theorem in [22] for more general stratified spaces. The theorem relates to the Morse inequalities in the following way: Assume that $u = \nabla f$ is a Morse gradient field. Since all stagnation points are non-degenerate they are also isolated and formula (6.1) holds. It then follows that the Morse index theorem reads as

$$\sum_k (-1)^k \,\mathrm{Ind}_k^{\mathrm{M}}(u) = \sum_j \mathrm{Ind}_j^{\mathrm{PH}}(u) = \chi(X)$$

which is precisely the equality (3.4) from the Morse inequalities. We now illustrate the Poincaré-Hopf index and the Morse index formula with a concrete example.

Example 6.4 (Vector field with rotation)**.** Consider the stream function $\psi(x) = x_1^2 + x_2^2$ and define the vector field $u = \nabla^\perp \psi$ on the domain $X = B_1 \subset \mathbb{R}^2$. Then u has one stagnation point at the origin and no boundary stagnation points, so $\mathrm{Ind}_\Sigma^{\mathrm{PH}} = 0$ vanishes. Since the domain has Euler characteristic $\chi(X) = 1$ it follows from the Morse index theorem that the origin in fact has Poincaré-Hopf index $\mathrm{Ind}_{\mathrm{int}(X),x}^{\mathrm{PH}}(u) = 1$. Since this vector field is not a gradient field and the eigenvalues of $Du(0)$ are imaginary the Morse index makes no sense in this case. A plot of the streamlines of u can be seen in figure 6.1.

A condition on the number of stagnation points in $\mathbb{R}^2$

Our first result relates the number of critical points with the domain topology. We shall give two proofs of this result, one involving the Morse index theorem and the other involving the argument principle to highlight the connection to complex analysis.

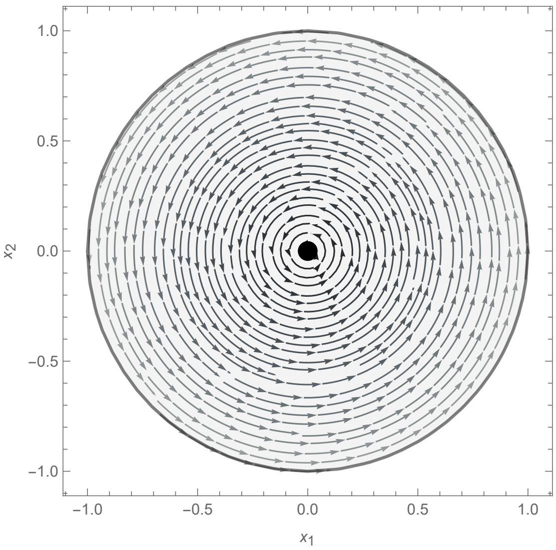


Figure 6.1: A plot of $u = \nabla^{\perp}\psi$ on the domain $X = B_1 \subset \mathbb{R}^2$ from example 6.4.

Proposition 6.5 (Condition on the number of stagnation points). *Let $X \subset \mathbb{R}^2$ be a compact planar manifold with smooth boundary and let $u\colon X \to TX$ be a harmonic vector field without boundary stagnation points and such that the interior stagnation points are isolated. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points counting multiplicities and $\chi(X)$ is the Euler characteristic of X.*

Proof. By assumption we have that $\mathrm{Ind}_{\Sigma}^{\mathrm{PH}}(u) = 0$. In the case that u is non-degenerate on the interior $\mathrm{int}(X)$ it follows from the harmonicity of u and equation (6.1) that

$$-M = M_2 - M_1 + M_0 = \sum_j \mathrm{Ind}_j^{\mathrm{PH}}(u)\,.$$

Thus for harmonic functions $-\sum_j \mathrm{Ind}_j^{\mathrm{PH}}(u)$ is the number of stagnation points counting multiplicities and the claim then follows from the Morse index theorem. $\square$

We now give an alternative and proof of proposition 6.5 using the argument principle.

Alternative proof. We slit the domain such that it is homeomorphic to a disk. By proposition 2.1 u is the gradient of a harmonic function f on this new domain $\tilde{X}$. Let $h \in \mathrm{Hol}(\tilde{X})$ be the holomorphic function given by $h = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ. We now determine the change of argument $\arg h$ along γ. For this consider first the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other, the contribution to the change of $\arg h$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h$ is 2π for every hole in the domain since $h \cdot \gamma' = u \cdot \gamma'$ does not change sign as γ traverses a hole in clockwise direction. Similarly the contribution to the change in argument of

$\arg h$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have b_1 holes in the domain the total change of $\arg h$ as γ traverses Σ is $2\pi(b_1-1)$. Since h has no poles it follows from the argument principle (see for example [13, Chapter VIII]) that

$$-2\pi\chi(X) = 2\pi(b_1-1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M$$

from which the claim follows. $\square$

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ if $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* if $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can give the following corollary.

Corollary 6.6 (No in- or outflow)**.** *Let $X \subset \mathbb{R}^2$ be a compact planar manifold with smooth boundary and $u\colon X \to \mathbb{R}^2$ a harmonic vector field without inflow or outflow or irregular stagnation points on Σ. Then we have the relation $M = -\chi(X)$ where M is the number of stagnation points counting multiplicities and $\chi(X)$ is the Euler characteristic of X.*

Examples of planar harmonic vector fields

We would like to illustrate the previous results with examples. We first give an example of a planar harmonic vector field without inflow or outflow and with one stagnation point.

Example 6.7 (No in- or outflow)**.** Consider the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x-e_1) + \Phi_2(x+e_1) \end{aligned} \tag{6.2}$$

where Φ_2 the fundamental solution of Δ in $\mathbb{R}^2$ as in example 4.5. A plot of the streamlines in figure 6.2 indicates that $u = \nabla^\perp\psi$ in the domain $X = \psi^{-1}([-1,1])$ has the desired properties. Indeed, since ψ is constant on each component of Σ the function u has neither inflow nor outflow. It follows from $\psi(-x) = \psi(x)$ that $u(-x) = -u(x)$ and thus the origin $x = 0$ is a stagnation point. By proposition 6.5 it is in fact the sole stagnation point of u on X.

In a second example given by [35] we start with the domain rather than the function.

Example 6.8 (No in- or outflow, [35])**.** Set $X = \overline{B}_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We let the stream function ψ be determined by the system

$$\begin{aligned} \Delta\psi &= 0 \quad \text{on } \operatorname{int}(X), \\ \psi &= 0 \quad \text{on the outer ring } 4S^1, \\ \psi &= 1 \quad \text{on the inner rings } S^1(-2e_1) \cup S^1(2e_1), \end{aligned} \tag{6.3}$$

and set $u = \nabla^\perp\psi$. A numerical solution to this system is plotted in figure 6.3. Again, it follows from symmetry that the origin is a stagnation point and from proposition 6.5 that it is in fact the sole stagnation point of u.

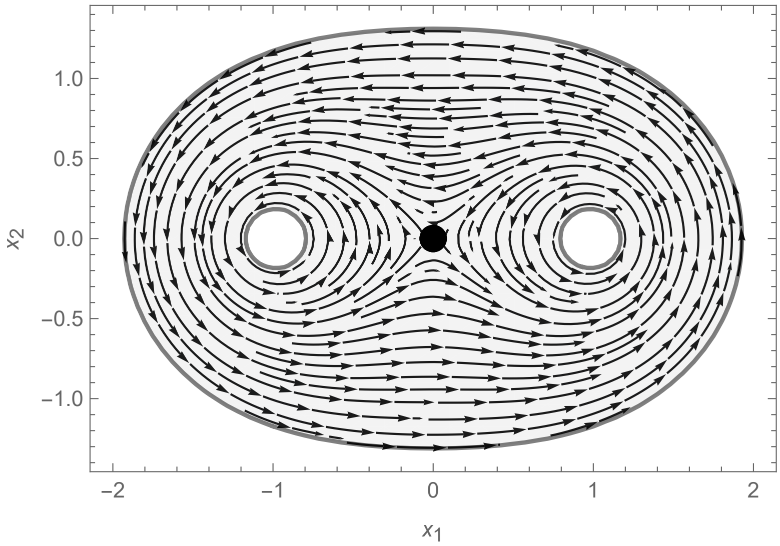


Figure 6.2: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here ψ is given by equation (6.2).

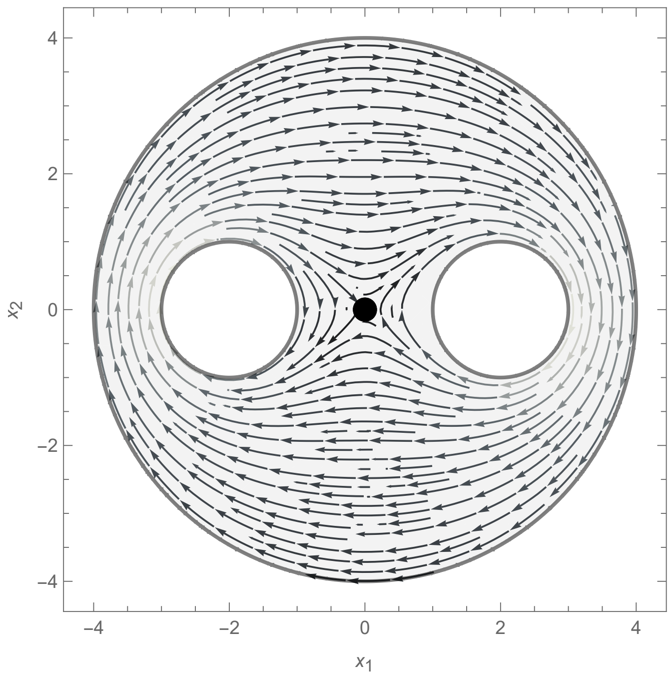


Figure 6.3: A plot of $u = \nabla^{\perp}\psi$ where ψ is the numerical solution to (6.3).

The next example highlights the importance of the assumption that u be strongly Morse in corollary 6.6.

Example 6.9 (Stagnation points on the boundary, [35])**.** In this example given by [35] we again start by fixing the domain. Let $X = \overline{B}_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We let the stream function ψ be determined by the system

$$\begin{aligned} \Delta\psi &= 0 && \text{on } \operatorname{int}(X), \\ \psi &= 0 && \text{on the outer ring } 4S^1, \\ \psi &= -1 && \text{on the left inner ring } S^1(-2e_1), \\ \psi &= 1 && \text{on the right inner ring } S^1(2e_1), \end{aligned} \tag{6.4}$$

and then set $u = \nabla^\perp \psi$. The numerical solution to this system is plotted in figure 6.4. Here we

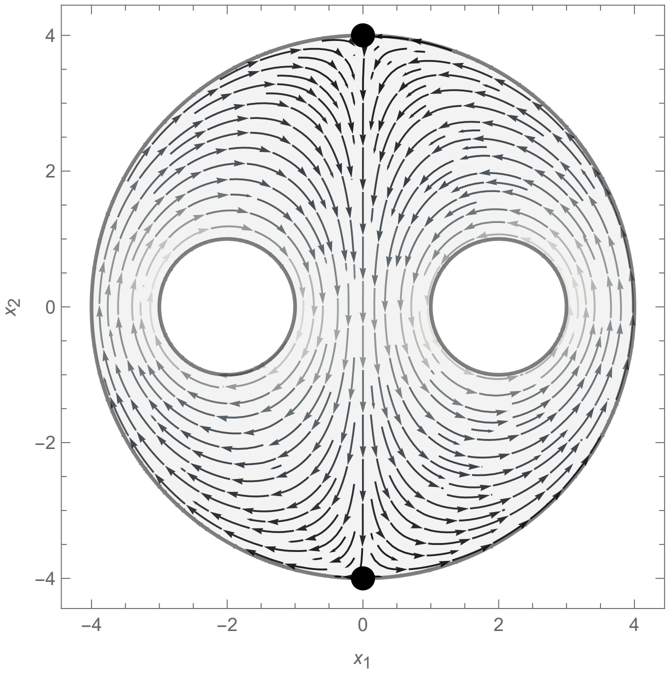


Figure 6.4: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (6.4).

obtain from the symmetry $\psi(-x_1, x_2) = \psi(x)$ that $\psi = 0$ on the x_2-axis. Since also $\psi = 0$ on $4S^1$ we have two stagnation points at $\pm 4e_2$. This function again has no in- or outflow through the boundary. The domain contains two holes so $\chi(X) = -1$. Now u has no interior stagnation point, seemingly contradicting corollary 6.6. But since the two stagnation points at $\pm 4e_2$ lie on the boundary u is in fact not strongly Morse and thus we cannot apply corollary 6.6. This shows the importance of the assumption that u is strongly Morse. Proposition 6.10 essentially states that in this case the stagnation points of u on the boundary count half as much as stagnation points in the interior. This explains why there are two critical points in this example.

Proposition 6.10 (Critical points on the boundary, [3, Theorem 1.1]). *Let $X \subset \mathbb{R}^2$ be bounded with piecewise $C^{1,\alpha}$ boundary. Let further $f: X \to \mathbb{R}$ be harmonic, not constant on each connectivity component and constant on each boundary component. Then there exist finitely many stagnation points points $x_1, \dots, x_k \in X$ of u_d with multiplicities $m_1, \dots, m_k$ and we have that*

$$\sum_{x_j \in \operatorname{int}(X)} m_j + \frac{1}{2} \sum_{x_j \in \Sigma} m_j = -\chi(X)\,.$$

The proof uses complex analysis techniques and can be found in [3].

No in-or outflow in $\mathbb{R}^3$

In $\mathbb{R}^3$ our results to question 1.3 remain unsatisfactory. Nonetheless we will state them in this final section. We leave unanswered the question from [23, p.198] whether it is possible to have a harmonic vector field in $\mathbb{R}^3$ with interior stagnation point and no in- or outflow through the boundary. This can be generalised to the question if it is possible to have a harmonic vector field $u: \mathbb{R}^3 \supset X \to \mathbb{R}^3$ with interior critical points and without critical points on the boundary.

Proposition 6.11 (Condition on the domain topology). *Let X be a compact orientable odd dimensional manifold with smooth boundary. Let further $u: X \to TX$ be a smooth vector field with isolated stagnation points on the interior and without boundary stagnation points. Then the Euler characteristic of the domain $\chi(X) = 0$ has to vanish.*

Proof. The Morse index formula for u reads as

$$\chi(X) = \operatorname{Ind}^{\mathrm{PH}}_{\operatorname{int}(X)}(u)\,. \tag{6.5}$$

Now note that we have for an interior stagnation point x

$$\operatorname{Ind}^{\mathrm{PH}}_x(-u) = (-1)^d \operatorname{Ind}^{\mathrm{PH}}_x(u) = -\operatorname{Ind}^{\mathrm{PH}}_x(u)$$

and thus the Morse index theorem applied to the vector field $-u$ yields

$$\chi(X) = \operatorname{Ind}^{\mathrm{PH}}_{\operatorname{int}(X)}(-u) = -\operatorname{Ind}^{\mathrm{PH}}_{\operatorname{int}(X)}(u)\,. \tag{6.6}$$

Adding equations (6.5) and (6.6) then yields the claim. □

This yields as a consequence

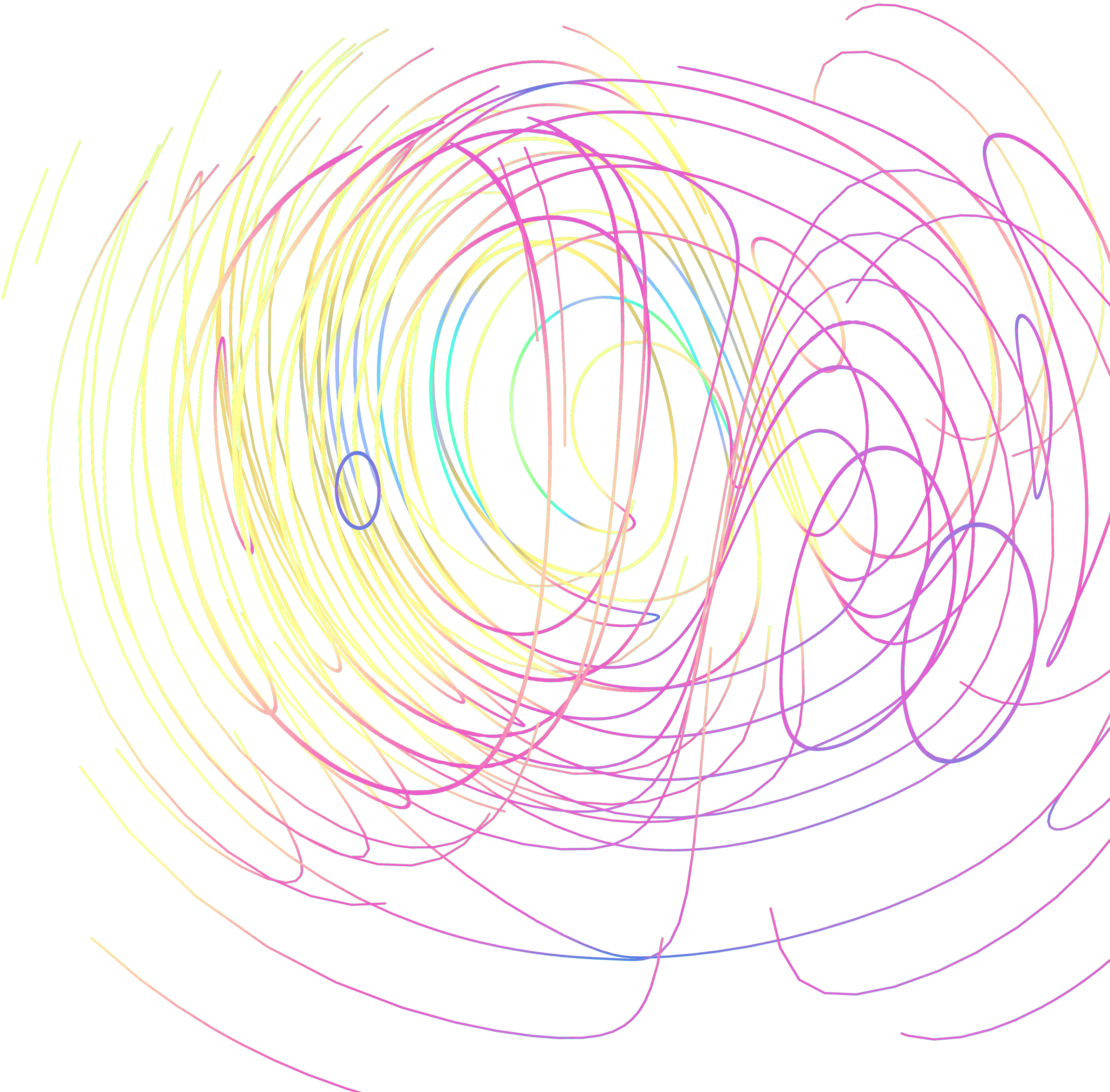
Corollary 6.12 (Condition on the type numbers and domain). *Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with smooth boundary and let further $u: X \to TX$ be a Morse harmonic vector field with no inflow or outflow through the boundary. Then we have the condition $M_1 = M_2$ between the interior type numbers and the Euler characteristic of the domain $\chi(X) = 0$ vanishes.*

Proof. Note that u fulfils the conditions of proposition 6.11 and thus the Euler characteristic of the domain $\chi(X) = 0$ has to vanish. We thus have by the Morse index theorem that $0 = \operatorname{Ind}^{\mathrm{PH}}_{\operatorname{int}(X)}(u)$ and the claim then follows from equation (6.1). □

Thus for any such vector field the number of stagnation points has to be even. We can also give a second condition on the domain topology which is a version of the hairy ball theorem.

Proposition 6.13 (Condition on the boundary topology)**.** *Let X be a orientable manifold with smooth boundary and $u\colon X \to TX$ be a smooth vector field without boundary stagnation points. Then the Euler characteristic $\chi(\Gamma) = 0$ vanishes for each boundary component $\Gamma \subseteq \Sigma$.*

Proof. Since the projection u_Γ onto Γ does not vanish on Γ by assumption it follows that the Poincaré-Hopf index $\mathrm{Ind}^{\mathrm{PH}}(u_\Gamma) = 0$ vanishes. The claim then follows from the Morse index theorem. □



Symbols

d	Dimensions $d = 2$ or $d = 3$
X	Compact domain in $\mathbb{R}^d$, often assumed to be a manifold with corners.
$f\colon X \to \mathbb{R}$	A harmonic function.
$u\colon X \to \mathbb{R}^d$ or TX	A harmonic vector field.
D	Derivative or differential.
Δ	Big capital delta. Here used for the Laplace operator. In Cartesian coordinates $\Delta = \sum_j \partial_j^2$.
∇	Gradient.
$\nabla^\perp$	Orthogonal gradient. Given by equation (4.11).
curl	Curl operator. Given in $\mathbb{R}^2$ by equation (4.12).
div	Divergence operator. In Cartesian coordinates $\operatorname{div} = \sum_j \partial_j$.
X_j	A stratification of X as given in definition 2.3. Often but not always assumed to be given by equation (2.1)
$\prec$	We write $X_j \prec X_k$ if $X_j \subseteq \overline{X}_k$. See definition 2.3 for details.
$\precsim$	We write $X_j \precsim X_k$ if $X_j \prec X_k$ and the strata differ in dimension by one. See definition 2.3 for details.
$\lesssim$	$x \lesssim y$ means there exists a constant $c \geq 1$ such that $x \leq cy$.
$\simeq$	$x \simeq y$ means that x is homotopic to y.
$\operatorname{rel\,int}$	Relative interior as in definition 2.7.
T_xX	Tangent space of X at x. See definition 2.4.
C_xX	Contingent cone of X at x. See definition 2.5.
A^*	Dual cone of a set A. Defined analogous to equation (2.4).
Σ	Boundary $\Sigma = \partial X$
Σ^-, $\Sigma^{\leq 0}$	(Strictly) entrant boundary. See definition 2.8.
Σ^+, $\Sigma^{\geq 0}$	(Strictly) emergent boundary. See definition 2.8.
Σ^0	Tangential boundary. See definition 2.8.
$B_r(x)$, B_r	Ball of radius r around the point x / the origin.
$S^{d-1}(x)$, S^{d-1}	$(d-1)$-dimensional sphere around x / the origin.
e_j	j-th unit vector in $\mathbb{R}^d$.
u_j	Projection of u to the tangent bundle TX_j. See equation (2.9).
π_j	Orthogonal projection onto the tangent bundle TX_j. See equation (2.8).
Cr_j	Set of essential stagnation points. See definition 2.10.
$\mathrm{Ind}^{\mathrm{M}}_{j,k}$	k-th type number on the stratum X_j. See equation (2.10).

$\mathrm{Ind}_k^{\mathrm{M}}$	k-th type number. See equation (2.15).
$\mathrm{Ind}_{j,x}^{\mathrm{PH}}$	Poincaré-Hopf index of x on the stratum X_j. See definition 6.1.
$\mathrm{Ind}_j^{\mathrm{PH}}$	Total Poincaré-Hopf index on the stratum X_j. See definition 6.1.
M_k	k-th interior type numbers. See equation (2.12).
M	Total number of stagnation points. See equation (2.13).
μ_k	k-th boundary type numbers of f. See equation (2.14).
ν_k	k-th boundary type numbers of $-f$. See definition 2.11.
u^ε	modification to u as in equation (2.17)
$b_k(X)$	k-th Betti number of X as defined in equation (3.1)
$H_k(X)$	k-th homology group of the space X.
$H_k(X,A)$	k-th relative homology group for spaces $A \subseteq X$.
$\chi(X)$	Euler characteristic of X. See equation (3.2).
$\chi(X,A)$	Euler characteristic for the relative homology group $H_k(X,A)$. See equation (3.6).
Φ_2	Multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$. Given by equation (4.14).
$V_U(\ldots)$, $V(\ldots)$	Algebraic variety.
p_1	Polynomial given by equation (5.9).
p_2	Polynomial given by equation (5.20).
P_1	Homogenisation of p_1 given by equation (5.21).
P_2	Homogenisation of p_2 given by equation (5.10).
$d_H(A,B)$	Hausdorff metric. See equation (5.23).
$\mathrm{dist}(x,A)$	Distance between x and A. See equation (5.24)
$\mathrm{Tub}_\delta(A)$	Tubular neighbourhood of A. See equation (2.20).
$\langle h\rangle_X$	Expectation value $\int_X h$ of a function $h\colon X \to Y$.
$L^p(X;Y)$	Lebesgue space of functions $u\colon X \to Y$.
$W^{k,p}(X;Y)$	Sobolev space of order k for functions $u\colon X \to Y$.

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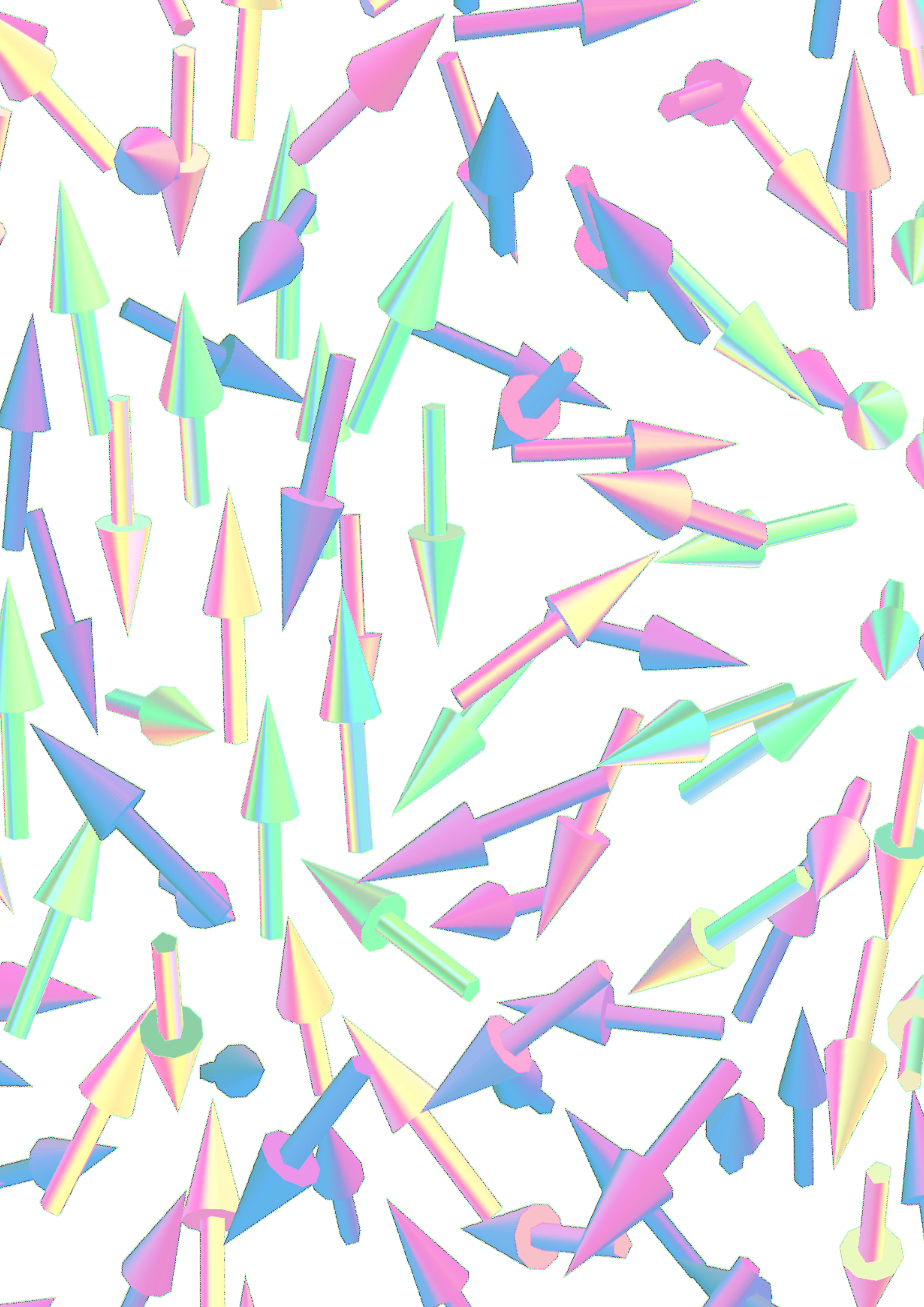

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