

Stagnation points of harmonic vector fields and the domain topology

Some applications of Morse theory

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Definition (Harmonic vector field)

Let $X \subset \mathbb{R}^d$ be a suitable domain. A function $f: X \rightarrow \mathbb{R}$ is *harmonic* if it satisfies the equation

$$0 = \Delta f = \sum_j \partial_j^2 f.$$

A vector field $u: X \rightarrow \mathbb{R}^d$ is *harmonic* if it is locally the gradient of a harmonic function, that is locally $u = \nabla f$.

Why harmonic vector fields?

- ▶ gravitational field in classical mechanics
- ▶ steady state heat flow
- ▶ irrotational flow of an inviscid incompressible medium
- ▶ electrostatic field in vacuum
- ▶ magnetostatic field in vacuum

Definition (Stagnation point)

We call the zeros of a vector field $u: X \rightarrow \mathbb{R}^d$ *stagnation points*. The stagnation points of $u = \nabla f$ are called *critical points* of $f: X \rightarrow \mathbb{R}$.

Why stagnation points?

vector field topology $\xleftarrow{\text{Morse theory}}$ stagnation points

The following question is inspired by [1]:

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u: X \rightarrow \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

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Answer

- ▶ For $d = 2$ dimensions: Not possible, see [4]. The proof uses Morse theory but one can also see this with the argument principle from complex analysis.

But if one allows for holes in $d = 2$ dimensions it becomes possible.

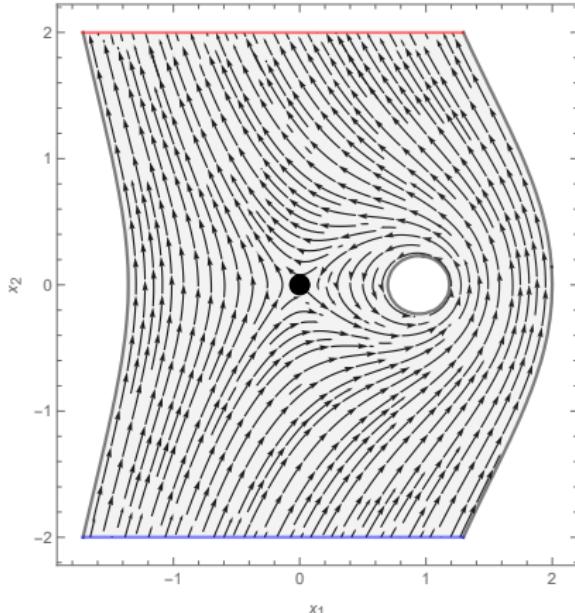


Figure: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.5, 2]) \cap (\mathbb{R} \times [-2, 2])$. Here $\psi := \Phi_2(x - e_1) + x_1$.

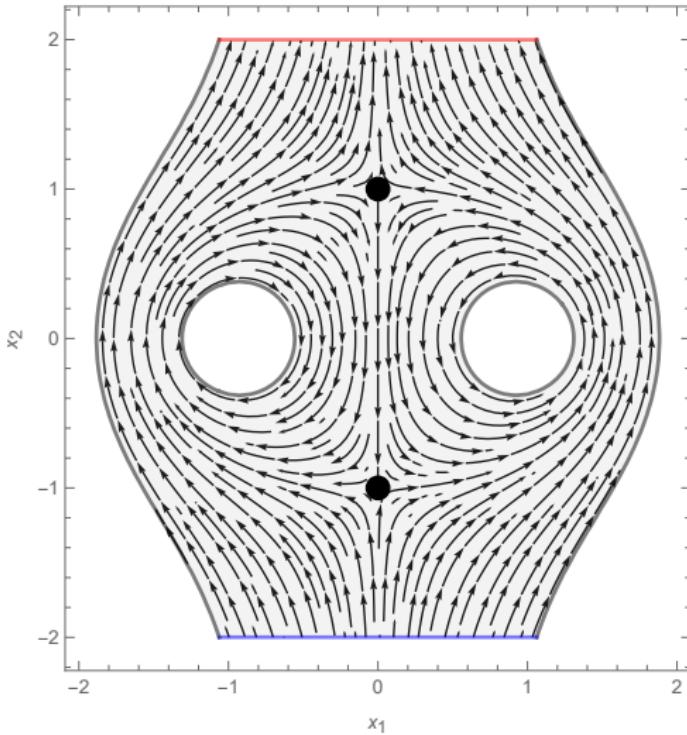


Figure: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2, 2])$.
Here $\psi := \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1$.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u: X \rightarrow \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- ▶ For $d = 2$ dimensions: No.
- ▶ For $d = 2$ dimensions for a domain with holes: Possible.

Proposition (Negative answer for cylinders, [5] or [4, Prop. 5.1])

Let $X = [0, 1] \times \overline{U} \subset \mathbb{R}^d$ be a cylinder where $U \subset \mathbb{R}^{d-1}$ is a bounded open set with C^1 boundary. Let further $f: X \rightarrow \mathbb{R}$ be non-constant and harmonic such that the sides $[0, 1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^{\leq 0}$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^{\geq 0}$ is the emergent boundary. Then f cannot have an interior critical point.

Proof.

See blackboard or [4].

□

\square Σ^0
 \blacksquare $\Sigma^{\leq 0}$
 \blacksquare $\Sigma^{\geq 0}$

\star interior stagnation point
 \rightarrow u

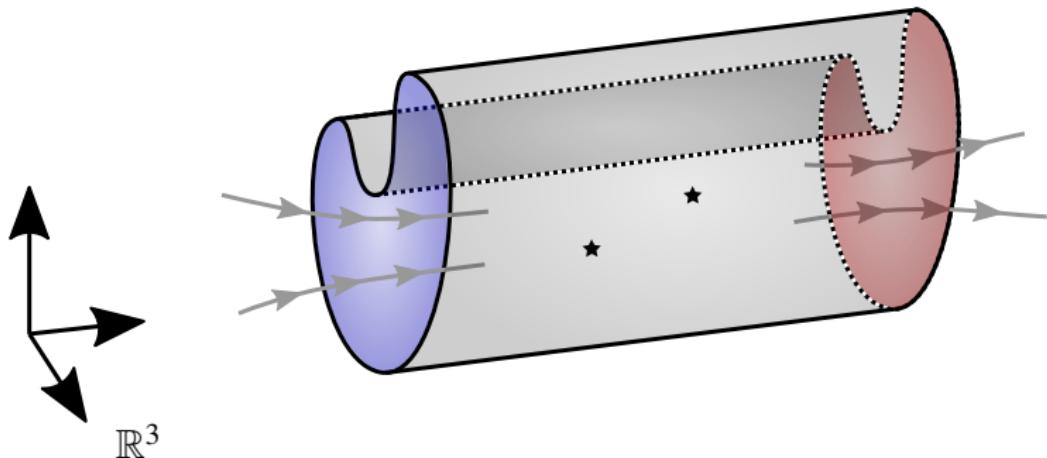


Figure: This kind of situation is not possible.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u: X \rightarrow \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- ▶ For $d = 2$ dimensions: No.
- ▶ For $d = 2$ dimensions for a domain with holes: Possible.
- ▶ For the cylinder: No.

Example (Connected entrant boundary in $d = 4$ dimensions, [4, Ex. 4.7])

Consider as domain $X = B_1 \subset \mathbb{R}^4$ the unit ball and the harmonic function

$$f: X \rightarrow \mathbb{R}$$

$$x \mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

This has a critical point at the origin. It is shown in [4, Prop. 4.8] that the entrant and emergent boundaries are in fact connected.

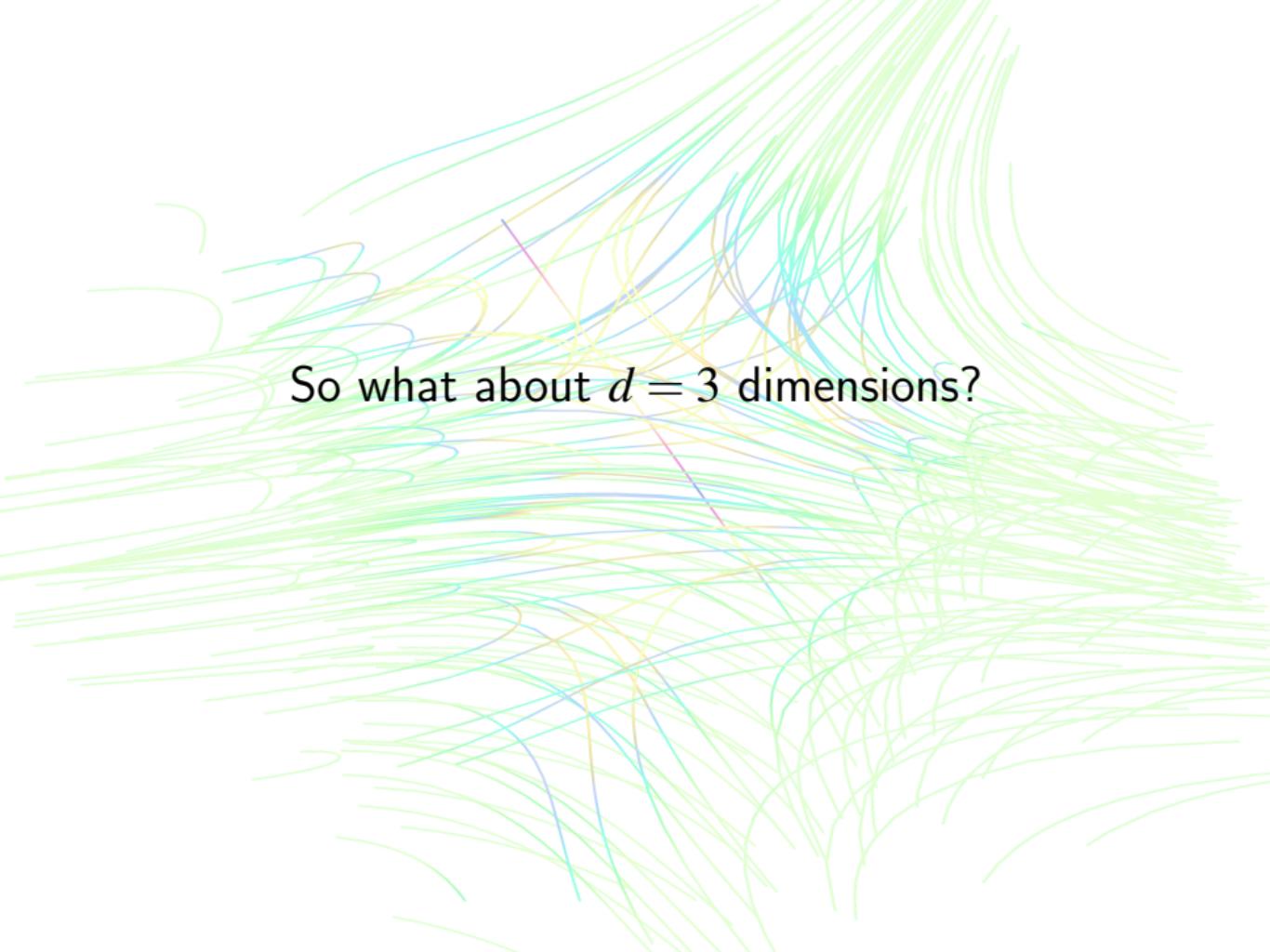
Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u: X \rightarrow \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- ▶ For $d = 2$ dimensions: No.
- ▶ For $d = 2$ dimensions for a domain with holes: Possible.
- ▶ For the cylinder: No.
- ▶ In $d = 4$ dimensions: Yes.



So what about $d = 3$ dimensions?

Definition (Interior type numbers)

Let $u: X \rightarrow \mathbb{R}^d$ be a vector field and x an interior stagnation point of u . We say that x is *non-degenerate* if the derivative Du is bijective. If in addition u is irrotational we say that x has *Morse index* k if Du has exactly k negative eigenvalues. The *interior type number* M_k denotes the number of interior stagnation points of index k .

Example (Interior type numbers)

Consider the harmonic function $f = x_1^2 + x_2^2 - x_3^2 - x_4^2$. One calculates

$$u = \nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

which has the sole stagnation point $x = 0$. Furthermore

$$Du = 2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

which implies that $x = 0$ is non-degenerate and has Morse index 2. This implies $M_2 = 1$ and $M_k = 0$ for $k \neq 2$.

Proposition (Condition on the interior type numbers, [4, Prop. 5.2])

Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with corners and let $f: X \rightarrow \mathbb{R}$ be a harmonic function without irregular stagnation points and such that f is non-degenerate on the interior $\text{int}(X)$. Let $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ be a disjoint decomposition of the boundary into simply connected nonempty sets such that we have for the strictly entrant boundary $\Sigma^- \subseteq \Sigma_{\leq 0}$ and for the strictly emergent boundary $\Sigma^+ \subseteq \Sigma_{\geq 0}$. Additionally we require that $\gamma := \partial \Sigma_{\leq 0}$ is a one-dimensional manifold diffeomorphic to the circle S^1 . Then we have the relation $M_1 = M_2$ between the interior type numbers.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u: X \rightarrow \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Answer

- ▶ For $d = 2$ dimensions: No.
- ▶ For $d = 2$ dimensions for a domain with holes: Possible.
- ▶ For the cylinder: No.
- ▶ In $d = 4$ dimensions: Yes.
- ▶ In $d = 3$ dimensions: Has to have an even number of stagnation points.

Example (A harmonic function with interior critical points and connected entrant and emergent boundaries, [4, Ex. 5.3])

For $d = 3$ dimensions we have for r sufficiently large the example
 $X = B_r$, $u = \nabla f$ with

$$f = \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_1x_2^2}{2} + x_1x_2^2 + x_2x_3$$

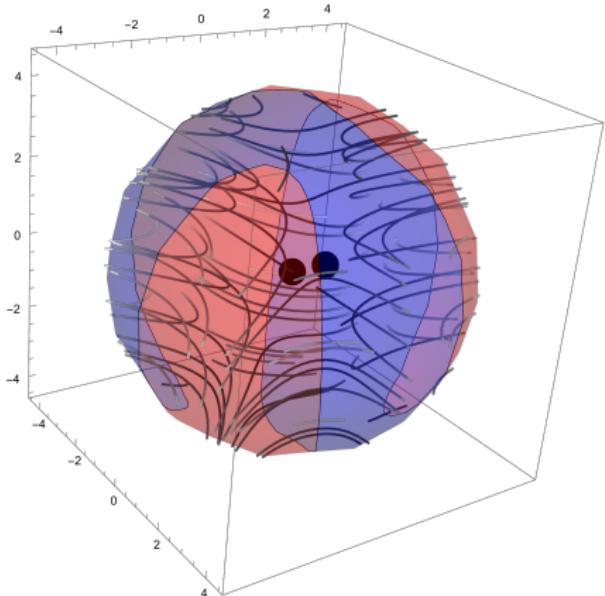


Figure: A stream plot of the function u . The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

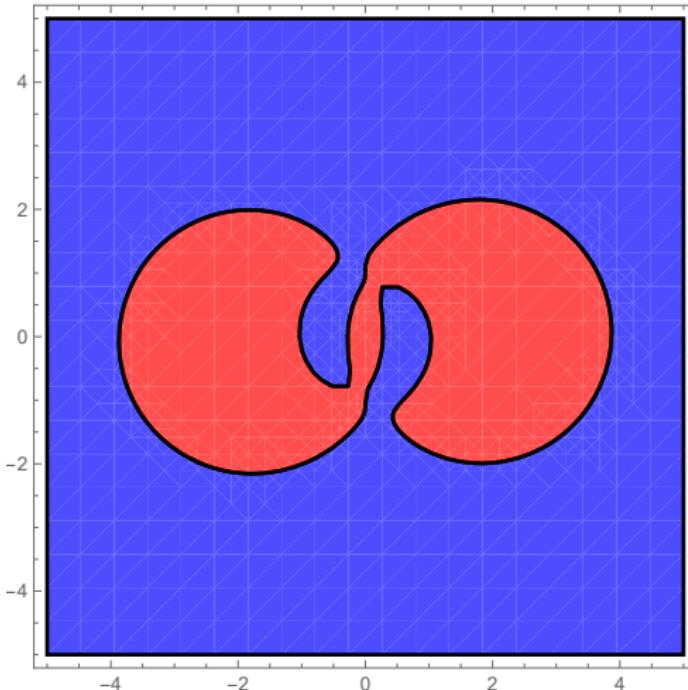


Figure: Stereographic projection of the surface Σ . Σ^+ is shaded red, Σ^- blue.

One can perturb this solution to show that there exists a harmonic vector field on B_r with interior stagnation point such that Σ^+ and Σ^- have positive distance from one another and are simply connected. This is done in [4, Ex. 5.12].

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Answer

- ▶ For $d = 2$ dimensions: No.
- ▶ For $d = 2$ dimensions for a domain with holes: Possible.
- ▶ For the cylinder: No.
- ▶ In $d = 3$ dimensions: Yes.
- ▶ In $d = 3$ dimensions: Has to have an even number of stagnation points.

The following question is inspired by [2]:

Question (Harmonic vector fields without inflow or outflow)

Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

The following answers the question in $d = 2$ dimensions:

Proposition (Condition on the number of stagnation points, [4, Prop. 6.5])

Let $X \subset \mathbb{R}^2$ be a compact connected planar manifold with corners and let $u: X \rightarrow \mathbb{R}^2$ be a Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of X .

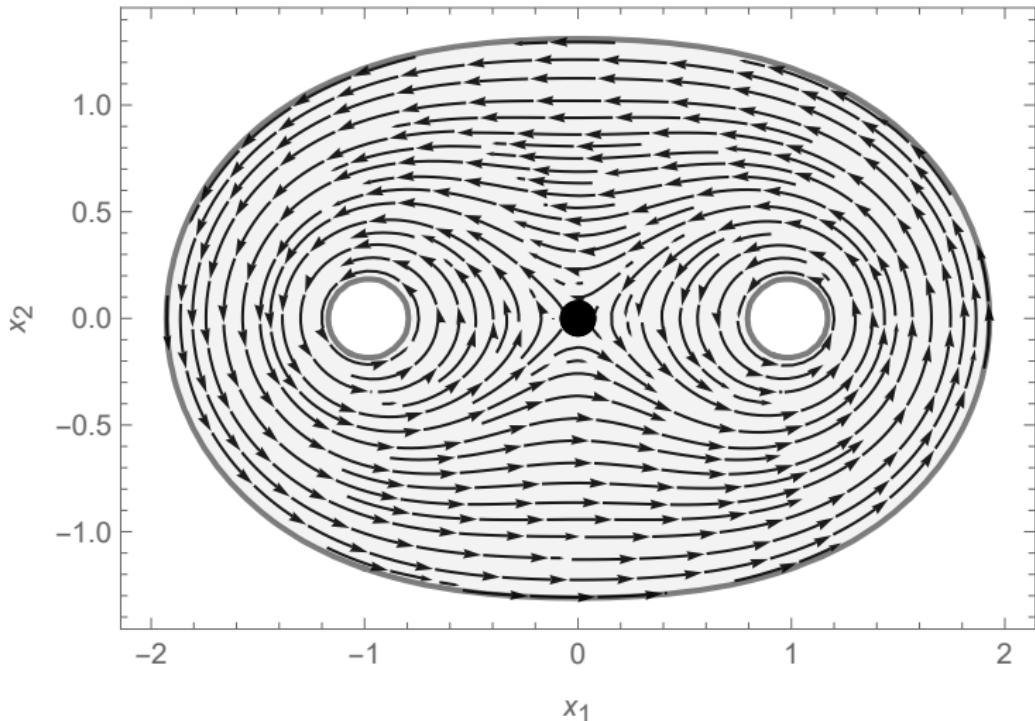


Figure: A plot of $u = \nabla^\perp \psi$ in the domain $\psi^{-1}([-1, 1])$. Here $\psi := \Phi_2(x - e_1) + \Phi_2(x + e_1)$.

Example (Stagnation points on the boundary, [4, Ex. 6.9])

Let $X = \overline{B}_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain and let the stream function ψ be given by

$$\Delta\psi = 0 \quad \text{on } \text{int}(X),$$

$$\psi = 0 \quad \text{on the outer ring } 4S^1,$$

$$\psi = -1 \quad \text{on the left inner ring } S^1(-2e_1),$$

$$\psi = 1 \quad \text{on the right inner ring } S^1(2e_1),$$

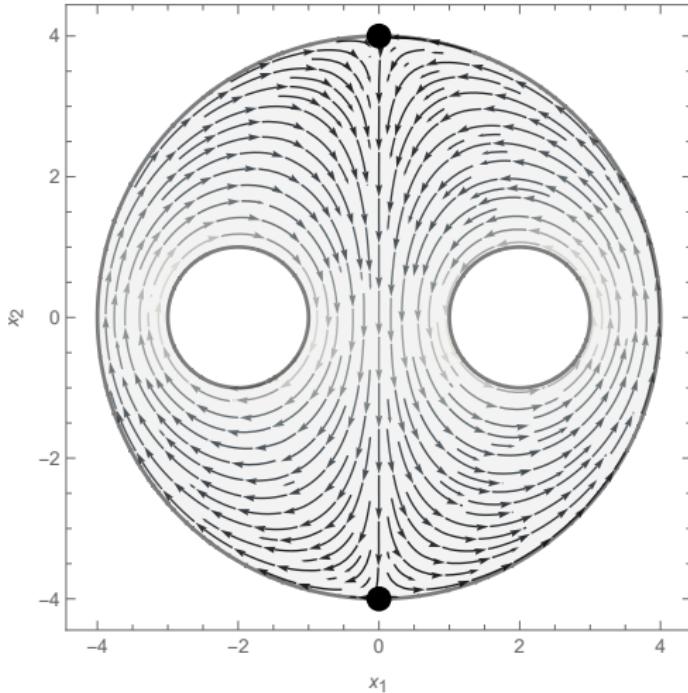


Figure: A plot of $u = \nabla^\perp \psi$ in the domain X given as in the previous slide.

Proposition (Condition on the domain topology, [4, Prop. 6.11])

Let X be a compact orientable odd dimensional manifold with smooth boundary. Let further $u: X \rightarrow TX$ a smooth vector field with isolated stagnation points on the interior and without boundary stagnation points. Then the Euler characteristic of the domain $\chi(X) = 0$ has to vanish.

Corollary (Condition on the type numbers and domain, [4, Cor. 6.12])

Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with smooth boundary and let further $u: X \rightarrow TX$ be a Morse harmonic vector field with no inflow or outflow through the boundary. Then we have the condition $M_1 = M_2$ between the type numbers and the Euler characteristic of the domain $\chi(X) = 0$ vanishes.

Summary

- ▶ Is it possible to have an interior stagnation point and simply connected entrant and emergent boundaries?
 - ▶ $d = 2$ dimensions: No, unless one allows for holes in the domain
 - ▶ $d = 3$ dimensions: Yes and $M_1 = M_2$, but not possible for cylinders
 - ▶ $d = 4$ dimensions: Yes by a simple example
 - ▶ techniques: Morse theory for manifolds with corners
- ▶ What is the relation between interior stagnation points and the domain topology for harmonic vector fields without inflow or outflow through the boundary?
 - ▶ $d = 2$ dimensions: $M = -\chi(X)$
 - ▶ $d = 3$ dimensions: $\chi(X) = 0$ and $M_1 = M_2$.
 - ▶ techniques: Morse index theorem

Sources I

- [1] H.-D. Alber, “Existence of three-dimensional, steady, inviscid, incompressible flows with nonvanishing vorticity,” *Math. Ann.*, vol. 292, no. 3, pp. 493–528, 1992, ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01444632. [Online]. Available: <https://doi.org/10.1007/BF01444632>.
- [2] D. Lortz, “Ueber die existenz toroidaler magnetohydrostatischer gleichgewichte ohne rotationstransformation,” *Z. Angew. Math. Phys.*, vol. 21, pp. 196–211, 1970, ISSN: 0044-2275,1420-9039. DOI: 10.1007/BF01590644. [Online]. Available: <https://doi.org/10.1007/BF01590644>.
- [3] master-thesis, *Github repository to the thesis*. Online, 2023. [Online]. Available: <https://github.com/TheoKoppenhoefer/master-thesis>.
- [4] Theo Koppenhöfer, *Stagnation points of harmonic vector fields and the domain topology: Some applications of Morse theory (to appear)*, Student Paper, Apr. 2024.

Sources II

- [5] Wahlén, Erik, *In private communication.* 2023.



Thank you for your attention.