Stagnation points of harmonic vector fields and the domain topology

Some applications of Morse theory

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Definition (Harmonic vector field)

Let $X \subset \mathbb{R}^d$ be a suitable domain. A function $f\colon X \to \mathbb{R}$ is *harmonic* if it satisfies the equation

$$0 = \Delta f = \sum_j \partial_j^2 f\,.$$

A vector field $u\colon X \to \mathbb{R}^d$ is *harmonic* if it is locally the gradient of a harmonic function, that is locally $u = \nabla f$.

Why harmonic vector fields?

- gravitational field in classical mechanics
- steady state heat flow
- irrotational flow of an inviscid incompressible medium
- electrostatic field in vacuum
- magnetostatic field in vacuum

Definition (Stagnation point)

We call the zeros of a vector field $u\colon X \to \mathbb{R}^d$ *stagnation points*.

Why stagnation points?

$$\text{vector field topology} \xleftrightarrow{\text{Morse theory}} \text{stagnation points}$$

The following question is inspired by [1]:

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are connected?

Proposition (Negative answer to this question in $d = 2$ dimensions)

Let $X \subset \mathbb{R}^2$ *be a simply connected planar compact manifold with corners and let* $f : X \to \mathbb{R}$ *be harmonic, non-degenerate on the interior* $\mathrm{int}(X)$ *and without irregular critical points. Let* $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ *be a disjoint decomposition of the boundary into simply connected nonempty sets such that we have for the strictly entrant boundary* $\Sigma^- \subseteq \Sigma_{\leq 0}$ *and for the strictly emergent boundary* $\Sigma^+ \subseteq \Sigma_{\geq 0}$*. Then* f *has no interior critical point.*

Proposition (Negative answer for cylinders, **Wahlen2023**)

Let $X = [0,1] \times \overline{U} \subset \mathbb{R}^d$ be a cylinder where $U \subset \mathbb{R}^{d-1}$ is a bounded open set with C^1 boundary. Let further $f : X \to \mathbb{R}$ be non-constant and harmonic such that the sides $[0,1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^{\leq 0}$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^{\geq 0}$ is the emergent boundary. Then f cannot have an interior critical point.

Proof.

See blackboard or [4]. $\square$

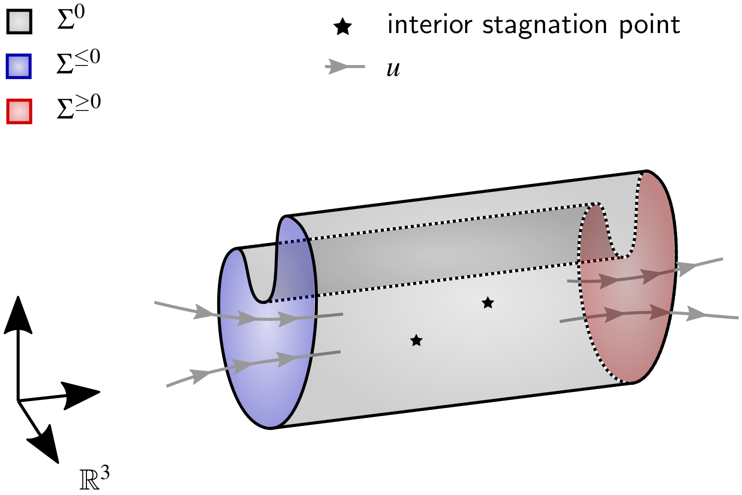

Figure: This kind of situation is not possible.

Example (Connected entrant boundary in $d = 4$ dimensions)

Consider as domain $X = B_1 \subset \mathbb{R}^4$ the unit ball and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned}$$

This has a critical point at the origin. It is shown in [4] that the entrant and emergent boundaries are in fact connected.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ on X such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are simply connected?

Answer

- $d = 2$ dimensions: Not possible (known).
- cylinders in $d = 3$ dimensions: Not possible (known).
- $d = 3$ dimensions: Number of stagnation points has to be even.
- $d = 4$ dimensions: Possible for $X = B_1$, $u = \nabla f$ with

$$f = x_1^2 + x_2^2 - x_3^2 - x_4^2 .$$

But if one allows for holes in $d = 2$ dimensions it becomes possible.

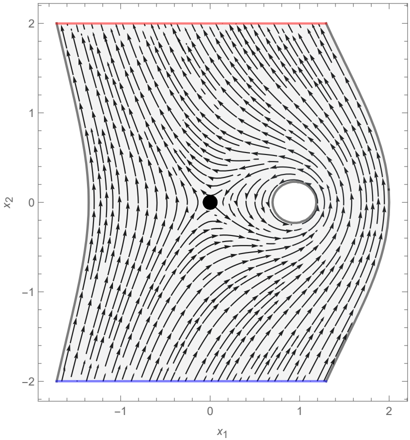

Figure: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap (\mathbb{R} \times [-2,2])$. Here $\psi := \Phi_2(x - e_1) + x_1$.

Example (A harmonic function with interior critical points and connected entrant and emergent boundaries)

For $d = 3$ dimensions we have for r sufficiently large the example $X = B_r$, $u = \nabla f$ with

$$f = \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_1 x_2^2}{2} + x_1 x_2^2 + x_2 x_3$$

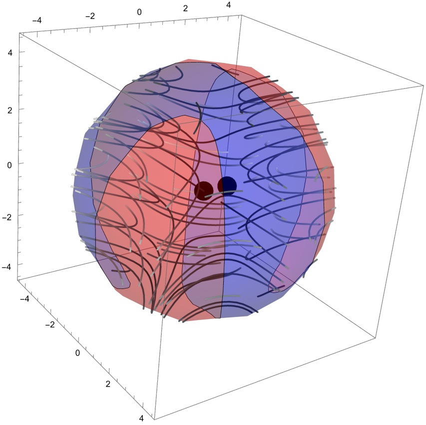

Figure: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

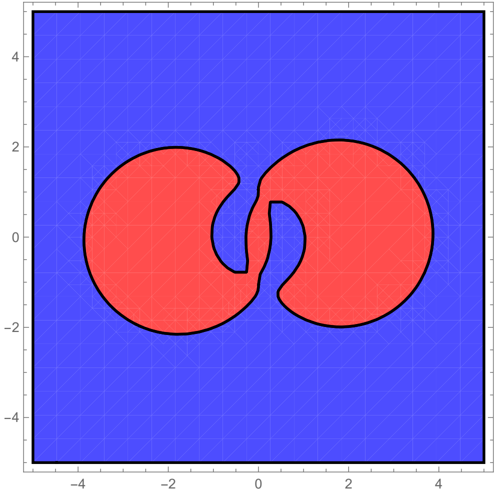

Figure: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

One can perturb this solution to show that there exists a harmonic vector field on B_r with interior stagnation point such that Σ^+ and Σ^- have positive distance from one another and are simply connected.

The following question is inspired by [2]:

Question (Harmonic vector fields without inflow or outflow)

Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

Definition (Interior stagnation points)

Let $X \subset \mathbb{R}^d$ be a d-dimensional compact manifold with corners and $u\colon X \to \mathbb{R}^d$ be a vector field without boundary stagnation points. We call an interior stagnation point x *non-degenerate* if the derivative $Du(x)$ is bijective. If all interior stagnation points are non-degenerate u is called *Morse*.

The following answers the question in $d = 2$ dimensions:

Proposition (Condition on the number of stagnation points, [4])

Let $X \subset \mathbb{R}^2$ be a compact connected planar manifold with corners and let $u \colon X \to \mathbb{R}^2$ be a Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of X.

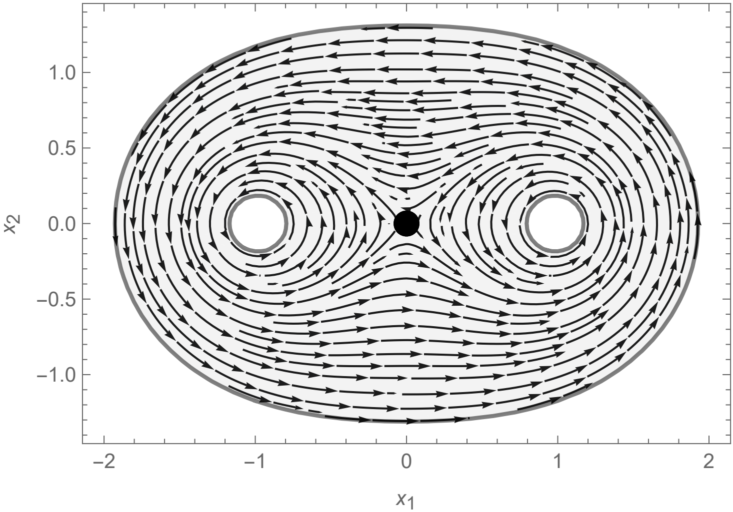

Figure: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here $\psi := \Phi_2(x - e_1) + \Phi_2(x + e_1)$.

Example (Stagnation points on the boundary)

Let $X = \overline{B}_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain and let the stream function ψ be given by

$$\begin{aligned} \Delta\psi &= 0 && \text{on } \operatorname{int}(X), \\ \psi &= 0 && \text{on the outer ring } 4S^1, \\ \psi &= -1 && \text{on the left inner ring } S^1(-2e_1), \\ \psi &= 1 && \text{on the right inner ring } S^1(2e_1), \end{aligned}$$

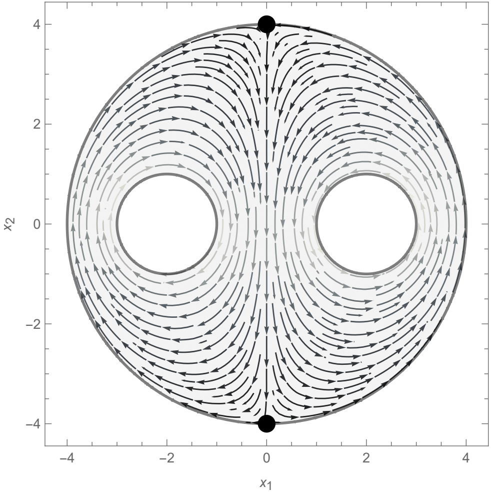

Figure: A plot of $u = \nabla^{\perp}\psi$ in the domain X given as in the previous slide.

Definition (Interior type number)

Let $u\colon X \to \mathbb{R}^d$ be a Morse vector field without boundary stagnation points. We say that a stagnation point x of u has *index* k if Du has exactly k negative eigenvalues. The *interior type number* M_k denotes the number of interior stagnation points of index k.

Proposition (Condition on the domain topology, [4])

Let X *be a compact orientable odd dimensional manifold with smooth boundary. Let further* $u\colon X \to TX$ *a smooth vector field with isolated stagnation points on the interior and without boundary stagnation points. Then the Euler characteristic of the domain* $\chi(X) = 0$ *has to vanish.*

Corollary (Condition on the type numbers and domain, [4])

Let $X \subset \mathbb{R}^3$ *be a compact three-dimensional manifold with smooth boundary and let further* $u\colon X \to TX$ *be a Morse harmonic vector field with no inflow or outflow through the boundary. Then we have the condition* $M_1 = M_2$ *between the type numbers and the Euler characteristic of the domain* $\chi(X) = 0$ *vanishes.*

Sources I

[1] H.-D. Alber, "Existence of three-dimensional, steady, inviscid, incompressible flows with nonvanishing vorticity," *Math. Ann.*, vol. 292, no. 3, pp. 493–528, 1992, ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01444632. [Online]. Available: https://doi.org/10.1007/BF01444632.

[2] D. Lortz, "Ueber die existenz toroidaler magnetohydrostatischer gleichgewichte ohne rotationstransformation," *Z. Angew. Math. Phys.*, vol. 21, pp. 196–211, 1970, ISSN: 0044-2275,1420-9039. DOI: 10.1007/BF01590644. [Online]. Available: https://doi.org/10.1007/BF01590644.

[3] master-thesis, *Github repository to the thesis.* Online, 2023. [Online]. Available: https://github.com/TheoKoppenhoefer/master-thesis.

[4] Theo Koppenhöfer, *Stagnation points of harmonic vector fields and the domain topology: Some applications of Morse theory (to appear)*, Student Paper, Apr. 2024.

Thank you for your attention.