Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

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January 25, 2024

Contents

Todo list

General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [23] and [20]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.

Some questions

- Should I state Hopf's Lemma?

1 Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\Omega = \operatorname{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. Unless otherwise stated we denote by

$$f: X \to \mathbb{R}$$

a C^2 function on X. Often f will be assumed to be harmonic. We also denote by

$$u: X \to \mathbb{R}^d$$

a vector field of class C^1. In the following we often assume that u is in fact *harmonic*, that is u fulfils $\operatorname{Div} u = 0$ and $\operatorname{curl} u = 0$. Often but not always we assume that in fact $u = \nabla f$ is a gradient field. One question we seek to answer in this thesis is the following:

Question 1.1 (Flowthrough with stagnation point)**.** Does there exist a region $X \subseteq \mathbb{R}^3$ homeomorphic to a ball with flow u through the region such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. the boundary on which u enters the region is simply connected?

The answer for this will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Another question we will consider is of the type:

Question 1.2 (stagnation points of harmonic vector fields without inflow or outflow)**.** Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary behaviour.

General definitions

We start by requiring some regularity for the boundary of X. More precisely, we require X to be a compact Riemannian manifold with corners:

Definition 1.3 (Manifolds with corners, [9]). We introduce the notation

$$H_j^d = \mathbb{R}^j_{\geq 0} \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d .$$

where $j \in \{0,\dots,d\}$. A *manifold with (convex) corners* is a topological space X together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exists an open neighbourhood U_x of x, a number $j = j(x)$ and a diffeomorphism $\phi \colon U_x \to H_j^d$ in $\mathscr{A}$ with $\phi(x) = 0$. We further define for $k \in \{0,\dots,d\}$ a collection of sets

$$X_k = \{x \in X \colon\, j(x) = d-k\}, \tag{1.1}$$

which form a stratification of X.

More generally we give the definition of a stratification as

Definition 1.4 (Stratified space, [9]). Let X be a topological space. A *stratum* is a subspace $X_j \subseteq X$, $j \in \mathscr{J}$, indexed by a partially ordered set $\mathscr{J}$ such that

1. each X_j is a manifold (without boundary) of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$ iff $j \prec k$.

The pair of X and the collection of strata is called a *stratified space*. In the case that $X_j \subseteq \overline{X}_k$ and additionally $n(k) = 0$ or $n(k) = n(j)+1$ we will write $X_j \precsim X_k$ or, abusing notation, we will write $X_k = X_{j+1}$.

In the case that the stratification arises through relation (1.1) we have precisely $X_j \precsim X_{j+1}$ for $j \in \{1,\dots,d\}$ and $X_0 \precsim X_0$. Note that in general for a given stratum X_j the stratum X_{j+1} such that $X_j \precsim X_{j+1}$ need not be unique. In the following we assume, unless otherwise stated, that the stratification is finite, that is $\#\mathscr{J} < \infty$ and that the interior Ω corresponds to a single stratum.

For completeness we also give the definition of the contingent cone for a stratification X_j of X:

Definition 1.5 (contingent cone, [13, Def. 4.6]). We denote the *(Bouligand) contingent cone* for a set $Y \subseteq X$ at $x \in \overline{Y}$ by $C_x Y$. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exists sequences $\lambda_n \to 0$ and $x_n \to x$ in Y such that

$$\lim_n \lambda_n (x_n - x) = v .$$

For clarification we give an example

Example 1.6 (Cubical domain). Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$. Then we have a stratification given by

$$\begin{aligned}
X_0 &= \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\
X_1 &= I \times \{-1\} \cup I \times \{1\} \cup \{-1\} \times I \cup \{1\} \times I \\
X_2 &= I \times I
\end{aligned}$$

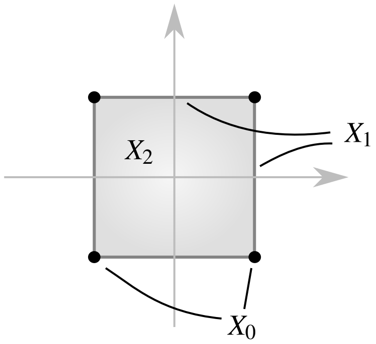


Figure 1.1: A stratification of X.

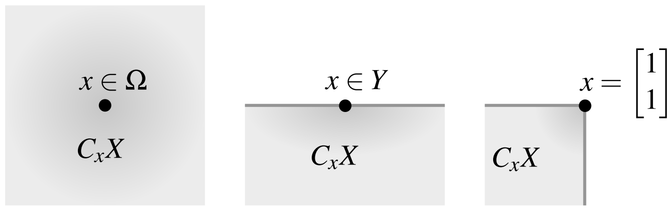


Figure 1.2: The contingent cones for various $x \in X$.

where $I = (-1,1) \subseteq \mathbb{R}$. The stratification is depicted in figure 1.1. For an interior point $x \in X_2$ we have the contingent cone $C_xX = T_x\mathbb{R}^d$. For a boundary point $x \in Y = I \times \{1\} \subset X_1$ we have the contingent cone

$$C_xX = \left\{ v \in T_x\mathbb{R}^2 \colon \, v \cdot n \leq 0 \right\}$$

where the basis vector $n = e_2$ is the outer unit normal. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top \in X_0$ we have

$$C_xX = \left\{ v \in T_x\mathbb{R}^2 \colon \, v_1 \leq 0 \text{ and } v_2 \leq 0 \right\}.$$

The situation is depicted in figure 1.2. The contingent cone on the other parts of the square $\Sigma = \partial X$ is given by similar formulas.

In the following we define the emergent and the entrant boundary in a way that generalises [20, p.282] for stratified manifolds.

Definition 1.7 (Emergent and entrant boundary)**.** We call a vector $v \in T_x\mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ if

1. v points into Ω or
2. v lies in the dual cone of the contingent cone C_xX, that is

$$v \in (C_xX)^* = \left\{ w \in T_x^*X \colon \, \langle w, w' \rangle \geq 0 \text{ for all } w' \in C_xX \right\}.$$

We call v *strictly entrant* if in addition v is not tangential to Σ and if $v \in (C_xX)^*$ then v lies in the relative interior $\operatorname{rel\,int}(C_xX)^*$. Analogously v is *(strictly) emergent* if $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which u is entrant. We define the *strictly entrant boundary* Σ^- to be the set of strictly entrant boundary points of u. In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* Σ^+. Further define the *tangential boundary* Σ^0 to be

$$\Sigma^0 = \Sigma^{\leq 0} \cup \Sigma^{\geq 0} \setminus \left(\Sigma^+ \cup \Sigma^- \right) \subseteq \Sigma. \tag{1.2}$$

We would now like to illustrate the preceding definitions.

Example 1.8. Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$ and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 - x_2^2 \,. \end{aligned} \tag{1.3}$$

This induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u\colon \Omega &\to \mathbb{R}^3 \\ x &\mapsto 2\begin{bmatrix} x_1 & -x_2 \end{bmatrix}^\top \,. \end{aligned} \tag{1.4}$$

For a boundary point $x \in I \times \{1\}$ the dual cone of the contingent cone C_xX is given by

$$(C_xX)^* = \{-re_2 \colon r \geq 0\} \,.$$

Now we have that $x \in \Sigma^{\leq 0}$ if

$$0 \geq n \cdot u = -2x_2$$

which is always fulfilled and thus $I \times \{1\} \subseteq \Sigma^{\leq 0}$. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ the dual of the contingent cone is given by

$$(C_xX)^* = C_xX \,.$$

Since $v = u(x) = 2\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ we have that $v \notin (C_xX)^*$ and $-v \notin (C_xX)^*$ and thus $x \notin \Sigma^{\geq 0} \cup \Sigma^{\leq 0}$. By analogous argumentation on the other sides of the square $\Sigma = \partial X$ one obtains that

$$\begin{aligned} \Sigma^{\leq 0} &= I \times \{1\} \cup I \times \{-1\} \\ \Sigma^{\geq 0} &= \{1\} \times I \cup \{-1\} \times I \,. \end{aligned}$$

A plot of the sets can be seen in figure 1.3.

Given a vector field $u\colon X \to \mathbb{R}^d$ and a stratification X_j of X we can construct for every $j \in \mathscr{J}$ a vector field

$$u_j \colon X_j \to T^*X_j \,.$$

Here T^*X_j denotes the cotangent space of the manifold X_j which is defined for instance in [11, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j\big|_x \colon \mathbb{R}^d \cong T_x^*\mathbb{R}^d \to T_x^*X_j \tag{1.5}$$

denote the orthogonal projection of a vector at x onto the cotangent space of the stratum X_j at x. Now let

$$u_j = \pi_j \circ u\big|_{X_j} \in C^1(T^*X_j) \tag{1.6}$$

be the projection of u onto the cotangent bundle T^*X_j.

The following are slight generalisation of definitions given in [23, p.138f], [21, §5] and [20, p.282f] to include harmonic vector fields.

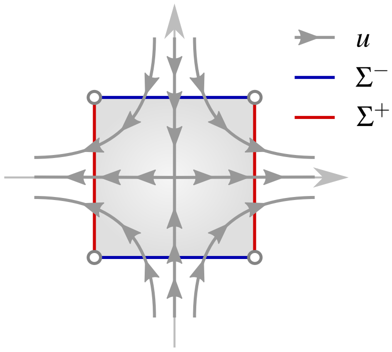


Figure 1.3: Depiction of the entrant and emergent boundaries for the function f given by equation (1.3)

Definition 1.9 (Stagnation points)**.** Let $u_j \colon X_j \to T^*X_j$ be a C^1 vector field on a stratum X_j of X. We call the zeroes $x \in X_j$ of u_j *stagnation points of* u_j *on* X_j. If $x \in \Omega$ then we call x an *interior stagnation point*. If $u(x) \in (C_xX)^*$ we call x an *essential stagnation point*. The set of all essential stagnation points of u_j is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$ and the essential stagnation points of u and $-u$ on X_j are called *stagnation points of* u *on* X_j. A stagnation point x of u_j is called *non-degenerate* if it is contained in the relative interior $\operatorname{rel\,int} X_j$ and the derivative

$$Du_j(x) = Du_j\big|_x \in T_xT^*X_j \cong \mathbb{R}^{n(j)\times n(j)}$$

is bijective. In addition we say that x has *index* k if $Du_j(x)$ has exactly k negative eigenvalues. u_j is called *(essentially) non-degenerate* if all its (essential) stagnation points are non-degenerate. Finally, we call a non-degenerate essential stagnation point of u_j such that additionally $u(x) \in \operatorname{rel\,int}(C_xX)^*$ *regular*. Boundary points which are non-regular essential stagnation points are called *irregular boundary points*. We call the set of all irregular boundary points the irregular boundary Σ^{irr}. u_j is called *regular* if it has no irregular boundary points. We can define the *k-th type number* $\mathrm{Ind}_{j,k}(u)$ of the stratum X_j to be the number of regular stagnation points of u_j of index k, that is

$$\mathrm{Ind}_{j,k}(u) = \#\big\{x \in \mathrm{Cr}_j(u) \colon x \text{ has index } k\big\}. \tag{1.7}$$

To illustrate the preceding definitions we return to our previous example.

Example 1.10. Let X, f and u be as in example 1.8. We have that $u_2 = u$ and thus one sees from equation (1.4) that the origin 0 is the sole stagnation point of u on the stratum X_2. Since we have that

$$Du(x) = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}$$

for all $x \in \Omega$ we see that $Du(0)$ is bijective and thus the origin is a non-degenerate interior stagnation point. Since $Du(0)$ has exactly one negative eigenvalue we see that the origin has index

1. Since an interior stagnation point is also an essential stagnation point we have $\mathrm{Ind}_{2,k} = \delta_{k1}$ where δ denotes the Kronecker delta. For $x \in I \times \{1\} = Y$ we calculate

$$u_1(x) = \pi_1 \circ u(x) = (u - n \cdot un)(x) = 2x_1 e_1$$

and thus we have that $x = e_2$ is the unique stagnation point of u on $I \times \{1\}$. Consider the curve

$$\begin{aligned} \gamma \colon I &\to Y \\ t &\mapsto te_1 + e_2 \end{aligned}$$

then $\gamma(0) = e_2$ and we have

$$Du_1(e_1)\big(\gamma'(0)\big) = (u_1 \circ \gamma)'(0) = (2te_1)'(0) = 2e_1 = 2\gamma'(0)$$

and thus e_1 is an eigenvector of $Du_1(e_2)$ to eigenvalue 2. Since e_1 spans the eigenspace $T_{e_2}Y$ it follows that e_2 is a non-degenerate stagnation point of u_1 with index 0. Now since $u(e_2) \in \operatorname{rel\,int}(C_xX)^*$ we have that e_2 is in fact a regular point. Proceeding in this manner for the other segments of the square Σ we obtain that $\mathrm{Ind}\, 1, k = 2\delta_{0k}$. If we now consider the point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ then we have that $u_0(x) = 0$ and thus x is a stagnation point. Now the derivative $Du_0 = 0 \in T_xT^*X_0 = 0$ is bijective and thus we have that x has index 0. Since however $u(x) \notin (C_xX)^*$ we have that x is not an essential stagnation point. Analogous argumentation on the other three corners yields that $\mathrm{Ind}_{0,0} = 0$.

The following characterisation of the irregular boundary will come in handy for showing the density of Morse functions later on:

Proposition 1.11 (Characterisation of the irregular boundary)**.** *The condition that the stagnation point $x \in X_j$ lies in Σ^{irr} is equivalent to that x is stagnation point of u_{j+1} for a stratum $X_j \precsim X_{j+1}$.*

Proof. We calculate for some boundary point $x \in \Sigma$:

$$(C_xX)^* \setminus \operatorname{rel\,int}(C_xX)^* = \left\{ w \in T_xX \;\middle|\; \begin{array}{l} \langle w, w' \rangle \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ \langle w, w' \rangle = 0 \text{ for some } w' \in \partial C_xX \setminus \{0\} \end{array} \right\}.$$

Now $w' \in \partial C_xX \setminus \{0\}$ iff $w' \in T_xX_{j+1}$ is orthogonal to T_xX_j at x. Thus we have that

$$u(x) \in (C_xX)^* \setminus \operatorname{rel\,int}(C_xX)^*$$

iff $u(x) \in (C_xX)^*$ and there exists a normal $n \in T_xX_{j+1}$ such that

$$0 = (n \cdot u\, n)(x) = (n \cdot u_{j+1}\, n)(x) = u_{j+1}(x) - u_j(x)$$

from which the claim follows. ☐

Definition 1.12 (Morse functions)**.** We call u *Morse* if for all $j \in \mathscr{J}$ we have that u_j is regular. If both u and $-u$ are Morse we call u *strongly Morse*. For a Morse function u we define the *interior type numbers M_k* to be the number of essential interior stagnation points of u of index k, that is

$$M_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \mathrm{Ind}_{j,k}(u) = \#\left\{ x \in \bigcup_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \mathrm{Cr}_j(u) \colon x \text{ has index } k \right\}. \tag{1.8}$$

The total number M of interior stagnation points of u is then given by

$$M = \sum_k M_k \,. \tag{1.9}$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of u of index k, that is

$$\mu_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) < d}} \mathrm{Ind}_{j,k}(u) \tag{1.10}$$

We further write ν_k for the k-th boundary type number of $-u$. We define the *type number* to be the number of essential stagnation points of u of index k, that is

$$\mathrm{Ind}_k(u) = \sum_{j \in \mathscr{J}} \mathrm{Ind}_{j,k}(u) = M_k + \mu_k \,. \tag{1.11}$$

We return to our example:

Example 1.13. Let X, f and u be as in example 1.8. By the calculations of the previous example 1.10 we have that u is Morse and we can calculate the interior type numbers

$$M_k = \mathrm{Ind}_{2,k} = \delta_{2k}$$

and the boundary type numbers

$$\mu_k = \mathrm{Ind}_{0,k}(u) + \mathrm{Ind}_{1,k}(u) = 2\delta_{0k}$$

This then yields the type numbers

$$\mathrm{Ind}_k(u) = M_k + \mu_k = \delta_{2k} + 2\delta_{0k} \,.$$

The previous definitions translate naturally to f. That is, we call f Morse, non-degenerate, et cetera if $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call x a *critical point* of f if it is a stagnation point of u. Note that most authors refer to regular and essential stagnation points as simply non-degenerate stagnation points and that this naming was introduced simply to distinguish between these different concepts.

Density of Morse functions

In the following section we argue that u and f being Morse is not a great restriction. Given u we define the modification

$$u^\varepsilon = u + \varepsilon \tag{1.12}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that the set E of ε for which u^ε is Morse is residual in $\mathbb{R}$. Recall that a *residual* set is a set whose complement is *meagre*, that is whose complement is the countable union of nowhere dense subsets. Since residual sets are dense in a Baire space by the Baire category theorem we can use u^ε to approximate a degenerate u. Our approach is to use Thom's theorem which is inspired by the approach in [11, Chapter 6].

Definition 1.14 (Transversality, [11, §3.2]). We call a function $g\colon Y_1 \to Y_2$ between two manifolds Y_1 and Y_2 (without boundary) *transverse* to a submanifold $A \subseteq Y_2$ if for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\mathrm{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y_2 \,.$$

As an application of this definition we make the following observation:

Proposition 1.15 (Transversal characterisation of non-degeneracy). *Let $u_j\colon X_j \to T^*X_j$ be a differentiable vector field. Then u_j is non-degenerate iff u_j is transverse to the zero section A_j of the cotangent space T^*X_j.*

Proof. First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A) = C$ is the set of stagnation points. Unravelling the definition of transversality we get that u_j is transverse to the zero section iff for all $x \in C = u_j^{-1}(A)$ we have that

$$\mathrm{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}TX \,. \tag{1.13}$$

As A is the zero section we have $T_{u_j(x)}A = 0$ and equation (1.13) is equivalent to stating that Du_j is of full rank at x. But Du_j being of full rank at all stagnation points is equivalent to u_j being non-degenerate. □

The alternative characterisation of non-degeneracy given in proposition 1.15 is sometimes used as a definition of non-degeneracy. We can now state a weakened version of the Thom's transversality theorem from [11, Theorem 2.7]:

Theorem 1.16 (Parametric transversality theorem, [11, §3 Theorem 2.7]). *Let $\mathscr{E}, Y_1, Y_2$ be C^r-manifolds (without boundary) and $A \subseteq Y_2$ a C^r submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A \,.$$

Let further $F\colon \mathscr{E} \to C^r(Y_1, Y_2)$ be such that the evaluation map

$$\begin{aligned} F^{ev}\colon \mathscr{E} \times Y_1 &\to Y_2 \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

is C^r and transverse to A. Then the set

$$E = \{\varepsilon \in \mathscr{E} \colon F_\varepsilon \textit{ is transverse to } A\}$$

is residual in $\mathscr{E}$.

Proof. See [11, Theorem 2.7] for details. □

From this we obtain a generalisation of the results in [20, §2] which will prove useful later:

Corollary 1.17 (Density of boundary generic functions)**.** *Let $u\colon X \to T^*X$ be a harmonic vector field on X and let X_j be a stratification of X. Assume that u has no irregular stagnation points. Then there exists a $\delta > 0$ and a residual (and thus dense) set $E \subseteq B_\delta \subseteq \mathbb{R}^d$ such that for every $\varepsilon \in E$ the following statements hold:*

1. *$u_j^\varepsilon \to u_j$ converge uniformly on all strata X_j as $\varepsilon \to 0$.*
2. *If $x_\varepsilon \to x$ is a convergent sequence of stagnation points of u_j^ε as $\varepsilon \to 0$ then x is a stagnation point of u_j.*
3. *u^ε is strongly Morse.*
4. *Additionally we can find for every $\eta > 0$ a $\delta > 0$ such that all stagnation points of u^ε are contained in an η-neighbourhood of the set of stagnation points of u.*
5. *the property of being entrant or emergent of stagnation points of u^ε is preserved, that is a stagnation point x^ε of u^ε lies in $\Sigma^\pm(u^\varepsilon)$ iff it lies in $\Sigma^\pm(u)$.*
6. *If u_j is non-degenerate on the stratum X_j we have for all k that*

$$\mathrm{Ind}_{X_j,k}(u^\varepsilon) = \mathrm{Ind}_{X_j,k}(u) \qquad \textit{and} \qquad \mathrm{Ind}_{X_j,k}(-u^\varepsilon) = \mathrm{Ind}_{X_j,k}(-u)\,.$$

Proof. Part 1. Follows from compactness of $\overline{X}_j \subseteq X$ and the continuity of π_j.

Part 2. Let $x_\varepsilon \to x$ be a convergent sequence of stagnation points on the stratum X_j. By part 1 $u_j^\varepsilon \to u_j$ as $\varepsilon \to 0$ uniformly which implies

$$0 = \lim_{\varepsilon} u_j^\varepsilon(x_\varepsilon) = u_j(x) \tag{1.14}$$

and thus x is a stagnation point of u_j.

Part 3. The following is essentially an adaptation of a proof given in [20, §2]. We first show that we can choose a $\delta > 0$ such that for all $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ the function u^ε has no irregular stagnation points. Assume not. Then there exists a sequence $\varepsilon_k \to 0$ and irregular stagnation points $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ of u^{ε_k}. By compactness of X we can assume that $x_k \to x$ for some $x \in X$ after taking a sub-sequence. After taking a further sub-sequence we can also assume that all x_k lie in a stratum X_j. The condition that the $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ are stagnation points means that after taking a further sub-sequence there exists a stratum X_{j-1} such that x_k is also stagnation point of this stratum by proposition 1.11. But then $x \in \overline{X}_j$ is also a stagnation point of X_{j-1} by part 2. Analogously x is also stagnation point of X_j. Thus $x \in \Sigma^{\mathrm{irr}}$ is an irregular stagnation point. A contradiction.

The next part of the proof is inspired by the use of transversality in [11, §6 Theorem 1.2] to show a similar statement. Set $r = 2$, $\mathscr{E} = B_\delta$ and $Y_2 = T^*X_j$ in the previous theorem. We initially set

$Y_1 = X_j = \Omega$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F\colon \mathscr{E} &\to C^\infty(X_j, T^*X_j) \\ \varepsilon &\mapsto u^\varepsilon \end{aligned}$$

and note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq T^*X_j$. Then the parametric transversality theorem yields a residual $E_j \subseteq \mathscr{E}$ on which $F_\varepsilon = u^\varepsilon$ is transverse to A. For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$\operatorname{Image}\left(DF^{\text{ev}}_{(\varepsilon,x)}\right) = T_xT^*X_j \tag{1.15}$$

since

$$DF^{\text{ev}}_{(\varepsilon,x)} = \left[\mathrm{Id}_{d\times d} \mid Du_x\right]$$

is surjective. Proposition 1.15 now implies that u^ε is non-degenerate on X_j for $\varepsilon \in E_j$.

Analogously we set $Y_1 = X_j$ to be an arbitrary strata in the previous proof and replace u^ε with the projection u^ε_j. To show that equation (1.15) holds we resort to the fact that

$$DF^{\text{ev}}_{(\varepsilon,x)} = D\big(u^\varepsilon_j(x)\big)_{(\varepsilon,x)} = D\pi_j \circ (Du^\varepsilon(x))_{(\varepsilon,x)}$$

is surjective as a concatenation of surjective functions. Thus there also exists a residual set $E_j \subseteq \mathscr{E}$ on which u^ε_j is non-degenerate on X_j.

Now the intersection

$$E = \bigcap_j E_j \subseteq \mathscr{E} = B_\delta$$

is residual and for every $\varepsilon \in E$ the function u^ε fulfils condition 3.

Part 4. Let C_η denote the open η-neighbourhood of the set of stagnation points of u. Since u has no irregular stagnation points we have for any stratum X_j that $u_j \neq 0$ on the compact set $\overline{X}_j \setminus C_\eta$ which implies that we can choose $\delta > 0$ so small that $|u_j| > 2\delta$ on $\overline{X}_j \setminus C_\eta$ for all strata X_j. For any $\varepsilon \in B_\delta$ it then follows that u^ε has no stagnation points on the set $\overline{X}_j \setminus C_\eta$ which yields the claim.

Part 5. Now consider a stratum X_j and the continuous mapping

$$\begin{aligned} \Phi\colon X_j &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \operatorname{dist}(u(x), \partial C_xX) \end{aligned}$$

on X_j. Φ is positive on the set of stagnation points C of u_j on X_j and thus we can choose $\eta > 0$ such that Φ is also positive in the neighbourhood $C_{2\eta}$. Choose $\delta > 0$ smaller than in part 4. Now the mapping Φ attains a positive minimum on the compact set $\overline{C_\eta}$. We can assume that $\delta > 0$ is less than this minimum. The choice of δ in this way ensures that emergent stagnation points of u_j are also emergent stagnation points of u^ε_j on X_j. Analogous argumentation with $-u$ then ensures that entrant stagnation points of u_j are also entrant stagnation points of u_j^ε on X_j. Since there are finitely many strata X_j we can choose $\delta > 0$ such that part 5 follows.

Part 6. Pick $\delta > 0$ as in part 5 and such that If x is a non-degenerate stagnation point of u on the stratum X_j it follows from the inverse function theorem that there exists for sufficiently small δ a neighbourhood around x on which there is a one-to-one correspondence between the stagnation points of u and u^ε. Since there are by proposition 2.1 at most finitely many non-degenerate stagnation points of u we can choose δ to be minimal over all these stagnation points. The equality of the indexes then follows from $Du^\varepsilon = Du$. □

2 Some general remarks

Bring order into this section.

The finiteness of the number of critical points is a known fact which is mentioned for example in [20]. For completeness we give the following proposition:

Proposition 2.1. *The number of non-degenerate stagnation points of u_j on X_j is finite.*

Proof. Let $x \in X_j$ be a non-degenerate stagnation point of u_j. Since $Du_j(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq X_j$ of x on which u_j is bijective. Hence x is the only stagnation point in U_x. Let C denote the set of all non-degenerate stagnation points of u_j. Then the sets U_x for $x \in C$ together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \tag{2.1}$$

form an open cover of $\overline{X}_j$. But $\overline{X}_j$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is contained the finite subcover. Thus it follows that $\#C < \infty$ is finite. $\square$

As a consequence we obtain the following observation:

Corollary 2.2. *For a Morse u the type numbers $M_0, \ldots, M_d$ and the boundary type numbers $\mu_0, \ldots, \mu_{d-1}$ are finite.*

State the theorem of Sard

We state Morse's lemma according to [11, §6, Lemma 1.1]

Lemma 2.3. *Let $f\colon X \to \mathbb{R}$ be C^{2+r} and x be a non-degenerate critical point of index k. Then there exists a C^r chart (φ, U) at x such that we have*

$$f\colon \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{d} y_j^2 .$$

Proof. See for example [11, §6]. $\square$

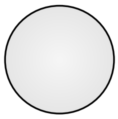
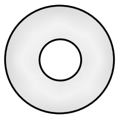
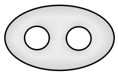
Domain	Picture	b_0	b_1	$b_k,\, k \geq 2$
Disk D		1	0	0
Annulus $2D \setminus D$		1	1	0
Two holed button		1	2	0

Table 2.1: Betti numbers for selected domains in $\mathbb{R}^2$.

Betti numbers

Let $H_k(X;\mathbb{R})$ denote the k-th homology space of X. For an introduction and definition of these we refer the reader to [10, Chapter 2]. We define the k-th Betti number as the dimension

$$b_k = \dim_{\mathbb{R}} H_k(X;\mathbb{R})\,. \tag{2.2}$$

We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

Example 2.4 (In flatland)**.** In $d = 2$ dimensions the 0-th Betti number counts the number of connected components of Ω and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely we give the Betti numbers for selected domains in table 2.1.

Example 2.5 (In spaceland)**.** In $d = 3$ dimensions the 0-th Betti number counts the number of connected components of Ω, the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 2.2.

Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.

The Morse inequalities

We state the Morse inequalities.

Theorem 2.6 (Strong Morse inequalities, [1, Theorem 2.4])**.** *Let X be a manifold with corners*

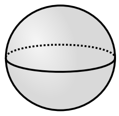
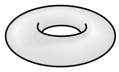
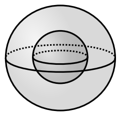
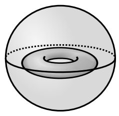
Domain	Picture	b_0	b_1	b_2	b_k, $k \geq 3$
Ball B		1	0	0	0
Solid torus $S^1 \times D$		1	1	0	0
Ball with bubble $2B \setminus B$		1	0	1	0
Ball with bubble in shape of torus		1	1	1	0

Table 2.2: Betti numbers for selected domains in $\mathbb{R}^3$.

and $f\colon X \to \mathbb{R}$ be Morse. Then we have for $l \in \{0,\dots,d\}$ the inequalities

$$\sum_{k=0}^{l} (-1)^{k+l} \operatorname{Ind}_j(f) \geq \sum_{k=0}^{l} (-1)^{k+l} b_j(X)\,.$$

For $l = d$ we in fact have equality

$$\sum_{k=0}^{d} (-1)^k \operatorname{Ind}_k(f) = \chi(X)$$

where the Euler characteristic

$$\chi(X) = \sum_{k=0}^{d} (-1)^k b_k(X)$$

is the alternating sum of the Betti numbers.

Proof. A for manifolds with C^1 boundary is given for instance in [21, Theorem 10.2']. The definition of regular critical points of f and their index given in definition 1.9 coincides with the definition of a critical point and its co-index of $-f$ given in [1]. The result then follows from [1, Theorem 2.4].

Give outline of proof idea.

□

Corollary 2.7 (Weak Morse inequalities)**.** *Let X be a manifold with corners and $f\colon X \to \mathbb{R}$ Morse. Then we have for $k \in \{0,\dots,d\}$ the inequalities*

$$\mathrm{Ind}_k(f) \geq b_k(X)\,.$$

Proof.

Write some proof.

□

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [21, Corollary 10.1].

Corollary 2.8 (Morse inequalities for f harmonic, $d = 2$)**.** *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned}\mu_0 &\geq b_0\\ M + \mu_1 - \mu_0 &= b_1 - b_0\,.\end{aligned}$$

In dimensions $d = 3$ we obtain [21, Corollary 10.2]

Corollary 2.9 (Morse inequalities for f harmonic, $d = 3$)**.** *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned}\mu_0 &\geq b_0\\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0\\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= b_2 - b_1 + b_0\,.\end{aligned}$$

Give a classical example of a Morse function to determine the Betti numbers.

On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2,3\}$.

Proposition 2.10 (Harmonic vector fields on simply connected domains)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f\colon \Omega \to \mathbb{R}$.*
2. *f is harmonic.*
3. *u is in fact C^∞.*
4. *The components $u_i = \partial_i f$ are harmonic.*

Proof.

1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.
2. This follows from $\Delta f = \operatorname{Div} u = 0$.
3. This follows from the fact that f is harmonic
4. This follows from $u_i = \partial_i f$.

□

If one considers not necessarily simply connected domains Ω then we obtain the properties of proposition 2.10 at least locally.

3 Harmonic functions, $d = 2$

The following result is essentially a negative to question 1.1 in $d = 2$ dimensions.

Proposition 3.1. *Let Ω be homeomorphic to $B_1 \subseteq \mathbb{R}^2$. Let further $f\colon \overline{\Omega} \to \mathbb{R}$ be regular harmonic with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \Sigma$ is not connected.*

We shall give two different proofs of this result. One involving level-sets and the other involving invariant manifolds

A proof involving level-sets

write omega-limit.

Sketch of Proof. Let $y_c = f(x_1)$ and $x_1, \dots, x_M$ be all the critical points such that $f(x_\cdot) = y_c$. We claim that the level set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

can be represented by a multigraph G which divides the boundary Σ into 4 components. To show this let $\gamma_i\colon (a_i, b_i) \to C$ for $i \in \{1, \dots, 4\}$ parametrise the curves in C intersecting at x_1. These can be constructed with the initial value problem

$$\gamma' = (\nabla f)^{\perp}\big|_{\gamma}$$
$$\gamma(0) = \gamma_0$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1. We assume that the intervals on which the γ_i are defined are maximal. We thus have for

$$\gamma_i^- = \lim_{t \to a_i} \gamma(t)$$
$$\gamma_i^+ = \lim_{t \to b_i} \gamma(t)$$

that $\gamma_i^{\pm} \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $\nabla f^{\perp} = 0$. This argument can be applied to all of the $x_1, \dots, x_M$. We therefore have a situation similar to the one depicted in figure 3.1.

Thus C can be represented by a multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq C$. In the following we identify the graph G with its planar embedding in $\overline{\Omega}$. Assume G contains a cycle with vertex sequence $v_{i_1}, \dots, v_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

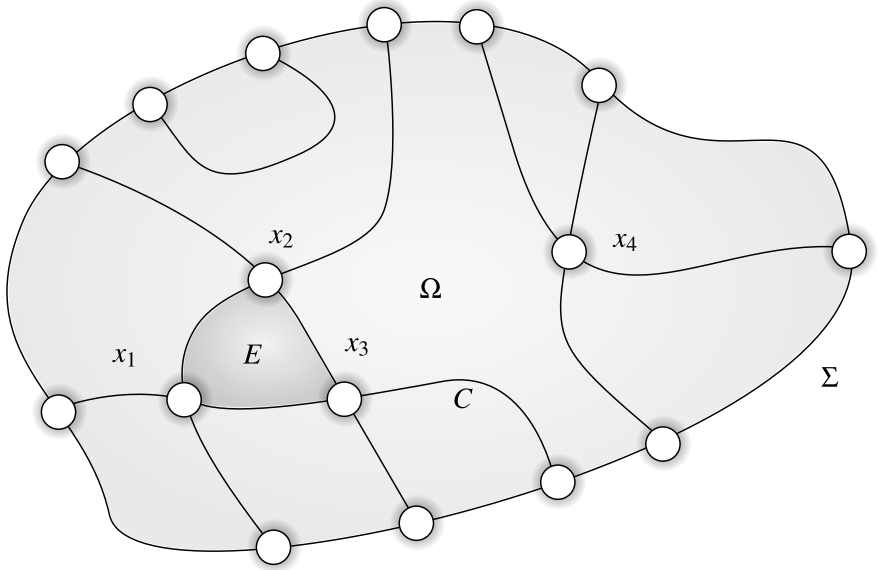


Figure 3.1: The situation at hand: The edges represent level curves and the interior vertices critical points.

is the boundary of a domain E for which $f = y_c$ on ∂E. By the maximum principle $f = y_c$ on E and thus $f = y_c$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with the boundary Σ is at least four and thus the boundary Σ is divided into at least four components.

Now choose four neighbouring components $\omega_1, \ldots, \omega_4$ as depicted in figure 3.2 Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle yields that ω_1 contains a local maximum or minimum of f since $f = y_c$ is constant on the other boundaries $\partial A \setminus \omega_1$. By the same argument $\omega_2, \ldots, \omega_4$ also contain local extrema. Since the $\partial \omega_i$ cannot be extremal points on Σ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3

use argument with ∇f here to show that extrema can be assumed to be alternating.

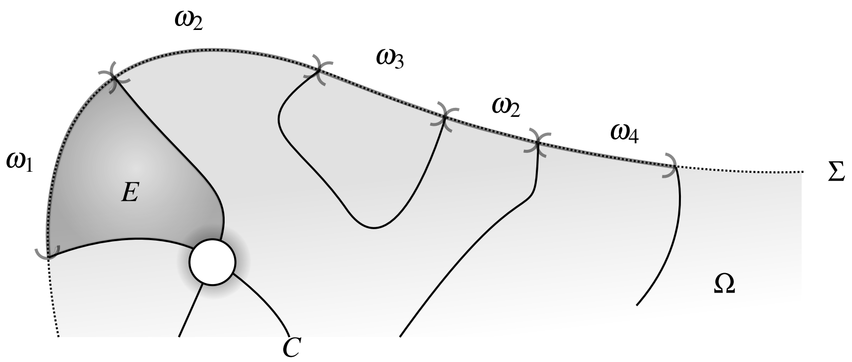


Figure 3.2: The choice of $\omega_1, \ldots, \omega_4$.

contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. □

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let $x_1, \dots, x_M$ denote the critical points of f. Let $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1,2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{3,4\}$ be chosen to parametrise the stable manifolds of x_1. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^{\pm} \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Once again we identify the graph with its planar embedding. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle A with vertices $x_{i_1}, \dots, x_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. The set of cycles forms a partial ordering with respect to the property 'contains another cycle'. We can assume that our chosen cycle A contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to one another. We also note that the edges cannot cross. We can thus describe the trail $x_{i_1}, \dots, x_{i_J}$ by a set of directives of the type

More precise.

$$(d_1, \dots, d_K) \in \{\mathrm{l}, \mathrm{r}, \mathrm{s}\}^J.$$

Here l, r and s stand for 'left', 'right' and 'straight' respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 3.3.

An example of the trail 'srsr' is given in figure 3.4. We now note that cycles of the type r,…,r or l,…,l cannot occur as we otherwise would have a directed cycle. Thus there exists a vertex where the chosen direction is s. Without loss of generality this vertex is x_{i_1}. Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1}. Now consider new cycle B starting at x_{i_1} with directives r,…,r. Since all vertices of B lie within the cycle A

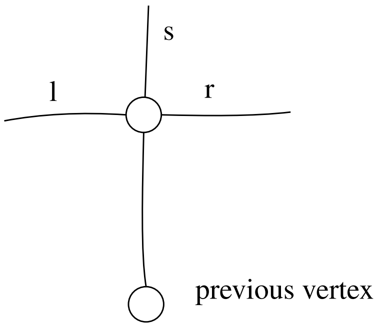


Figure 3.3: Explanation of the directives ‘l’,‘r’ and ‘r’.

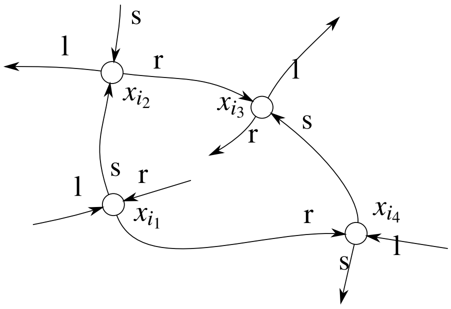


Figure 3.4: An example for a cycle.

we must at some step reach a vertex on the cycle A. But then cycle B is a new distinct cycle contained in cycle A, a contradiction to the minimality of A. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

Now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur. We now pick a tree $\widetilde{G}$ out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each ‘neighbouring’ leaf of this tree has opposite signage and the claim follows. $\square$

elaborate

elaborate

A proof involving Morse theory

We now give a proof involving Morse theory since the techniques of the proof generalise to the three dimensional case.

Proposition 3.2. *Let $d = 2$ and X be a simply connected manifold with corners and let $f\colon X \to \mathbb{R}$ have no irregular critical points. Assume further that Σ^- contains at least two points and is simply connected. Then f has no non-degenerate interior critical point.*

Proof. Let $\gamma = \{x_1, x_2\} = \partial\Sigma^-$. Then we can cut the domain along a curve Γ such that the endpoints $\gamma = \partial\Gamma$ of the cut coincide with x_1 and x_2, that is $\partial\Gamma = \{x_1, x_2\}$. Now we obtain two new domains X^+ and X^- such that $\partial X^+ \cap \Sigma^- \subseteq \gamma$ and $\partial X^- \cap \Sigma^+ \subseteq \gamma$. We can assume that Γ is a smooth manifold and corresponds to the stratum X_{Γ^+} for X^+ and X_{Γ^-} for X^-. Analogously γ corresponds to strata $X_{\gamma^\pm}$ on $X^\pm$. Locally around the corner point x_1 we have a situation depicted as in figure 3.5. We assume that we chose Γ in such a way that it forms an acute angle with $u = \nabla f$ at the boundary points γ. For the following argumentation we require that u is strongly Morse on both X^+ and X^-, so assume for a moment that this is the case. Since each point of γ is an essential critical point for either f or $-f$ on precisely one of the domains X^+ or X^- we have the relation

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^+,0}(-f) + \mathrm{Ind}_{\gamma^-,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) = 2. \tag{3.1}$$

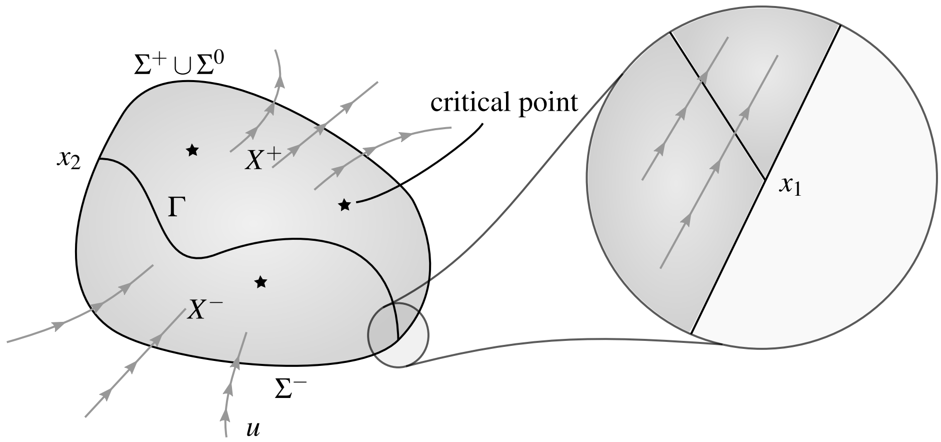


Figure 3.5: The situation at hand.

We now focus our attention on X^+. Since no essential critical points of f lie on $\Sigma^+ \setminus \gamma$ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f) \tag{3.2}$$

where δ_{ij} denotes the Kronecker delta. Analogously we have on X^- that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{3.3}$$

In addition we have on Γ that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^-, that is

$$\mathrm{Ind}_{\Gamma^+,0}(f) = \mathrm{Ind}_{\Gamma^-,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma^+,1}(f) = \mathrm{Ind}_{\Gamma^-,0}(-f)\,. \tag{3.4}$$

Using equations (3.2), (3.3) and (3.4) we obtain

$$\mu_0^+ - \mathrm{Ind}_{\gamma^+,0}(f) = \nu_1^- \qquad \text{and} \qquad \mu_1^+ = \nu_0^- - \mathrm{Ind}_{\gamma^-,0}(-f)\,. \tag{3.5}$$

Consider the Morse inequality for f on X^+

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) \tag{3.6}$$

and the Morse inequality for $-f$ on X^-

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X)\,. \tag{3.7}$$

We now add equations (3.6) and (3.7) and insert relations (3.5) to obtain

$$M^- + M^+ - \mathrm{Ind}_{\gamma^+,0}(f) - \mathrm{Ind}_{\gamma^-,0}(-f) = -2\chi(X) = -2\,.$$

Since $\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \leq 2$ by equation (3.1) and $M^\pm \geq 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 1.17 applied separately to the domains X^+ and X^-. Since $E^\pm$ are residual in B_δ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. It follows from the slanted angles at which Γ approaches γ that if the points x_1, x_2 are essential critical points of f that they then are in fact regular. Hence we obtain that

$$\mathrm{Ind}_{\gamma^+,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma^+,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma^-,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{3.8}$$

By the same corollary u^ε has no essential stagnation points on $\Sigma^+(u)$ and $-u$ has no essential stagnation points on $\Sigma^-(u)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M^\varepsilon = M$. □

Allowing for Inflow and outflow

The strategy in the above proofs can be generalised to show the following

Conjecture 3.3. *Let $X \subseteq \mathbb{R}^2$ be a manifold with corners with Betti numbers $b_0 = 1$ and b_1. Let further $f\colon X \to \mathbb{R}$ be Morse harmonic with M critical points. Assume that $\overline{\Sigma^-} \subseteq \Sigma$ on a given connected component of the boundary Σ consists of at most one connected component. Then we have*

$$\frac{4}{3}M \leq b_1 + 1\,.$$

This inequality can probably be improved considerably.

Let $J^\pm$ denote the number of connected components of $\Sigma^\pm$. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\geq 0} \sqcup \Sigma_{\leq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma^{\geq 0}$ of all such decompositions. We state a consequence of a result from [2, Theorem 2.1]

Proposition 3.4. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u\colon \overline{\Omega} \to \mathbb{R}$ be harmonic (with certain conditions on the boundary). Then we have*

$$M \leq b_1 - b_0 + \frac{J^+ + J^-}{2}\,.$$

If in addition we assume that there are no stagnation points on the boundary then we have

$$M \leq b_1 - b_0 + J^{\geq 0}\,.$$

Proof. See [2, Theorem 2.1]. □

4 Harmonic vector fields, $d = 2$

No inflow or outflow

Proposition 4.1. *Let $X \subset \mathbb{R}^2$ be a compact manifold with corners and Betti numbers $b_0 = 1$, and b_1 and let $u\colon X \to \mathbb{R}^2$ be a strongly Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of X.*

Proof. We slit the domain X such that it is homeomorphic to the disk as is depicted in figure 4.1. Denote the slit by Γ. Since the number of interior stagnation points is finite by proposition 2.1, we can choose Γ in such a way that it does not contain any interior stagnation points. We write denote the boundary of Γ by $\gamma = \partial\Gamma = \Gamma \cap \Sigma$ and define points $\{x_1, \ldots, x_{2b_1}\} = \gamma$. Note that there are $2b_1$ many such points. Without loss of generality we can assume that Γ and u form an acute angle at each point of γ. The situation is depicted in figure 4.1. For the following argumentation we require that u is strongly Morse on the new domain $\tilde{X}$ so assume for a moment that this is the case. Since each x_j is either an essential critical point of u or of $-u$ on the slit domain $\tilde{X}$ we have that

$$\mathrm{Ind}_{\gamma,0}(u) + \mathrm{Ind}_{\gamma,0}(-u) = 2b_1\,. \tag{4.1}$$

Since there are no stagnation points on Σ we have the relations

$$\mu_k = \mathrm{Ind}_{\Gamma,k}(u) + \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \nu_k = \mathrm{Ind}_{\Gamma,k}(-u) + \mathrm{Ind}_{\gamma,k}(-u) \tag{4.2}$$

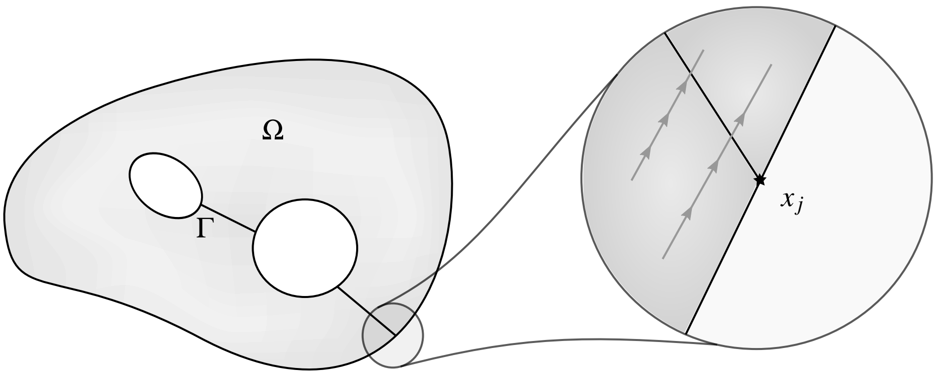


Figure 4.1: How we slit the domain.

for all $k \in \{0,1\}$. All entrant stagnation points of u on Γ are also emergent stagnation points of $-u$ on Γ (and vice versa) and hence we have the relations

$$\mathrm{Ind}_{\Gamma,0}(u) = \mathrm{Ind}_{\Gamma,1}(-u) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(u) = \mathrm{Ind}_{\Gamma,0}(-u)\,. \tag{4.3}$$

Equations (4.2) and (4.3) yield

$$\mu_0 - \mathrm{Ind}_{\gamma,0}(u) = \nu_1 \qquad \text{and} \qquad \mu_1 = \nu_0 - \mathrm{Ind}_{\gamma,0}(-u)\,. \tag{4.4}$$

Since $\tilde{X}$ is now simply connected u is by proposition 2.10 the gradient of a harmonic function f on this new domain. For this f we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \tag{4.5}$$

and for $-f$ the Morse inequalities

$$M + \nu_1 - \nu_0 = -\chi(\tilde{X}) = -1\,. \tag{4.6}$$

Adding equations (4.5) and (4.6) and using the relation (4.4) and then (4.1) we obtain

$$-2 = 2M - \mathrm{Ind}_{\gamma,0}(u) - \mathrm{Ind}_{\gamma,0}(u) = 2M - 2b_1$$

from which the claim follows.

The claim remains to be shown in the case that u is not strongly Morse on $\tilde{X}$. In this case let u^{ε} for $\varepsilon \in E$ be a strongly Morse function as in corollary 1.17. Since the $x_1,\ldots,x_{2b_1} \in \gamma$ are non-degenerate stagnation points of u due to the slanted angle at which Γ approaches γ we obtain that

$$\mathrm{Ind}_{\gamma,k}(u^{\varepsilon}) = \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-u^{\varepsilon}) = \mathrm{Ind}_{\gamma,k}(-u) \tag{4.7}$$

By the same corollary u^{ε} has no stagnation points on Σ. The claim then follows by the calculations above where we replace u with u^{ε} and then note that $M^{\varepsilon} = M$. □

We now give an alternative proof using the argument principle.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition 2.10 u is the gradient of a harmonic function f on this new domain $\tilde{X}$. Let $h \in \mathrm{Hol}(\tilde{X})$ be the holomorphic function given by $h = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ. We now determine the change of argument $\arg h$ along γ. For this consider first the parts of γ traversing the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other the contribution in the change of $\arg h$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h$ is 2π for every hole in the domain since $h \cdot \gamma' = u \cdot \gamma'$ does not change sign as γ traverses a hole clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have b_1 holes in the domain the total change of $\arg h$ as γ traverses Σ is $2\pi(b_1 - 1)$.

Since h has no poles it follows from the argument principle (see for example [7, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M$$

From this the claim follows. $\square$

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ if $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* if $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can give the following corollary.

Corollary 4.2. *Let X be a compact manifold with corners and $u\colon X \to \mathbb{R}^d$ a strongly Morse harmonic vector field without inflow or outflow on Σ. Then we have the relation $M = -\chi(X)$ where M is the number of stagnation points and $\chi(X)$ is the Euler characteristic of X.*

This result could also have been proved using the Poincaré-Hopf index theorem. For this define:

Definition 4.3 ([5, Definition 1.1.1])**.** The *Poincaré-Hopf index* $\mathrm{Ind}_{\mathrm{PH},x}(f)$ of an isolated interior stagnation point x of u is the degree of the Gauss map $u/|u|\colon S_\varepsilon^{d-1}(x) \to S^{d-1}$ for sufficiently small $\varepsilon > 0$. The *total index* $\mathrm{Ind}_{\mathrm{PH}}(u)$ is the sum of indexes $\mathrm{Ind}_{\mathrm{PH},x}(f)$ for every interior stagnation point x of u.

Note that in $d = 2$ dimensions a point x with Morse index k has Poincaré-Hopf index $(-1)^k$. Thus we have for a harmonic vector field that $M = -\mathrm{Ind}_{\mathrm{PH}}(u)$ and thus corollary 4.2 also follows from the Poincaré-Hopf index theorem:

Theorem 4.4 (Poincaré-Hopf index theorem, [18, §6])**.** *Let $u\colon X \to \mathbb{R}^d$ be a vector field on a manifold with corners without inflow. Then we have that the total index $\mathrm{Ind}_{PH}(u)$ equals the Euler characteristic $\chi(X)$.*

Proof. See for example [18, §6]. $\square$

In the following we would like to give examples for harmonic vector fields. For this define two differential operators for $d = 2$ dimensions by

$$\nabla^\perp f = \operatorname{Curl} f = \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix}$$

and

$$\operatorname{curl} u = -\partial_1 u_2 + \partial_2 u_1 \,.$$

Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

The following proposition gives us a recipe to generate harmonic vector fields in $d = 2$ dimensions.

Proposition 4.5. *Let $\psi\colon \Omega \to \mathbb{R}$ be harmonic then $\nabla^\perp f$ is a harmonic vector field.*

Proof. Since $\operatorname{Div}\nabla^{\perp} = 0$ we have

$$\operatorname{Div} u = \operatorname{Div}\nabla^{\perp}\psi = 0$$

and one calculates

$$\operatorname{curl} u = \operatorname{curl}\nabla^{\perp}\psi = -\Delta\psi = 0\,.$$

□

The function ψ is also called a *stream function*.

We now give an example of a harmonic vector field without inflow or outflow and with one stagnation point. For this consider the stream function

$$\begin{aligned}\psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x-e_1) + \Phi_2(x+e_1)\end{aligned} \tag{4.8}$$

where

$$\Phi_2 = \log(|\cdot|)$$

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and e_i is the i-th unit vector. Figure 4.2 indicates that $u = \nabla^{\perp}\psi$ in the region $X = \psi^{-1}([-1,1])$ has the desired properties. Since ψ is constant on each component of ∂X the function u has no inflow or outflow. It follows from $\psi(-x) = \psi(x)$ that $u(-x) = -u(x)$ and thus the origin $x = 0$ is a stagnation point. By proposition 4.1 it is in fact the sole stagnation point of u on X.

In a second example from [25] we fix the domain rather than the function. For this set $X = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We then have the system

$$\begin{aligned}\Delta\psi &= 0 \quad &&\text{, on } \Omega \\ \psi &= 0 \quad &&\text{, on the outer ring } 4S^1 \\ \psi &= 1 \quad &&\text{, on the inner rings } S^1(-2e_1) \cup S^1(2e_1)\end{aligned} \tag{4.9}$$

We solve this system numerically and set $u = \nabla^{\perp}\psi$. The result is plotted in figure 4.3. Again it follows from symmetry that the origin is a stagnation point and from proposition 4.1 that it is in fact the sole stagnation point of u.

Example of inflow on one side and outflow on the other

In the following we aim to give examples of domains in $d = 2$ dimensions for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component with stagnation points. For this consider first the stream function

$$\begin{aligned}\psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x-e_1) + x_1\end{aligned} \tag{4.10}$$

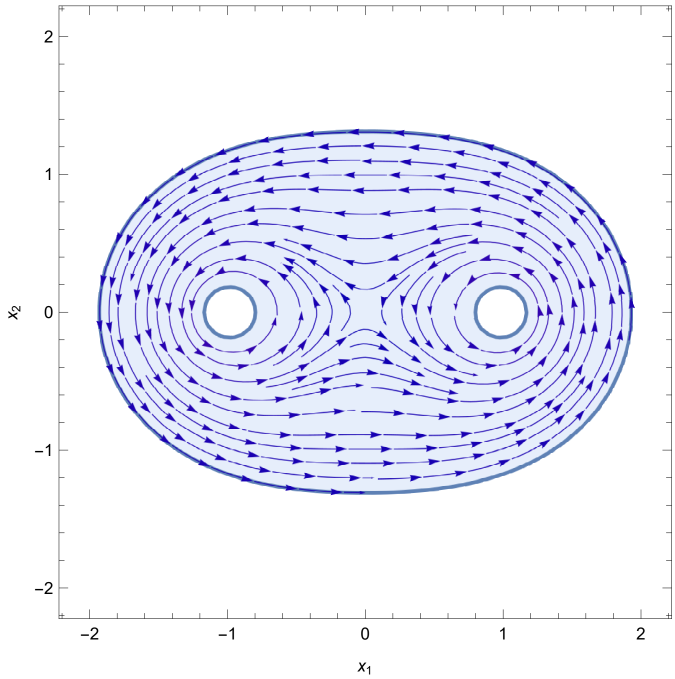


Figure 4.2: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-1,1])$. Here ψ is given by equation (4.8).

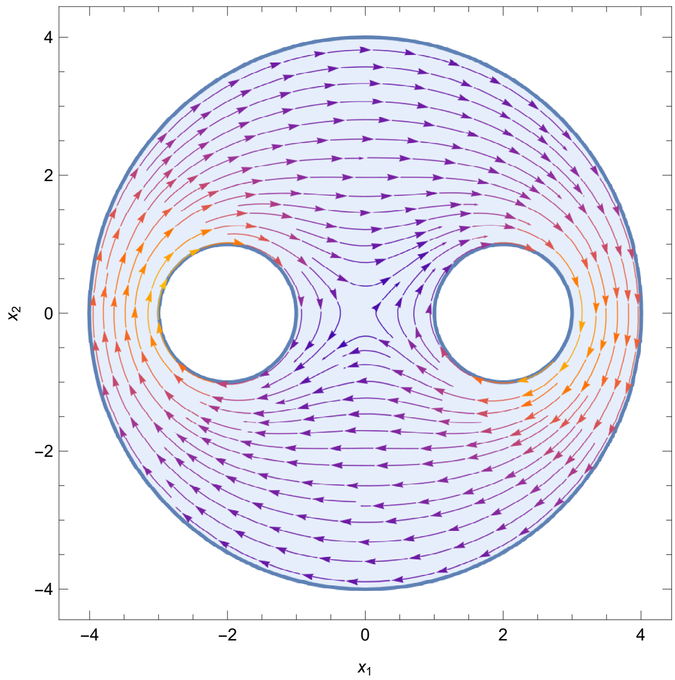


Figure 4.3: A plot of $u = \nabla^{\perp}\psi$ where ψ is the numerical solution to (4.9).

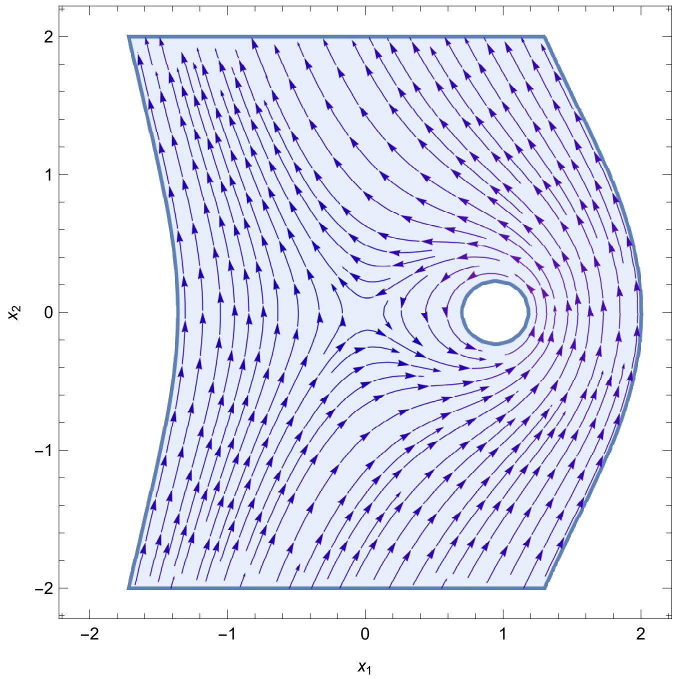


Figure 4.4: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap \mathbb{R} \times [-2,2]$. Here ψ is given by equation (4.10).

Figure 4.4 indicates that $u = \nabla^{\perp}\psi$ fulfils the requirements on the domain $X = \psi^{-1}([-0.5,2]) \cap \mathbb{R} \times [-2,2]$. An elementary calculation reveals that the origin is a stagnation point of u.

Now we would like to have a harmonic vector field similar to the example with two holes with inflow on the one side and outflow on the other. For this consider the streamline

$$\begin{aligned} u\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x-e_1) - \Phi_2(x+e_1) + x_1 \end{aligned} \tag{4.11}$$

Figure 4.5 indicates that $u = \nabla^{\perp}\psi$ on the domain $X = \psi^{-1}([-0.7,0.7]) \cap \mathbb{R} \times [-2,2]$ is the function we are looking for.

In another example given by [25] we fix the domain rather than the function. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We now have the system

$$\begin{aligned} \Delta\psi &= 0 && \text{, on } \Omega \\ \psi &= 0 && \text{, on the outer ring } 4S^1 \\ \psi &= -1 && \text{, on the left inner ring } S^1(-2e_1) \\ \psi &= 1 && \text{, on the right inner ring } S^1(2e_1) \end{aligned} \tag{4.12}$$

We solve this system numerically and set $u = \nabla^{\perp}\psi$. The result is plotted in figure 4.6. Here we obtain from the symmetry $\psi(-x_1, x_2) = \psi(x)$ that $\psi = 0$ on the x_2-axis. Since also $\psi = 0$ on $4S^1$ we have two stagnation points at $\pm 4e_2$.

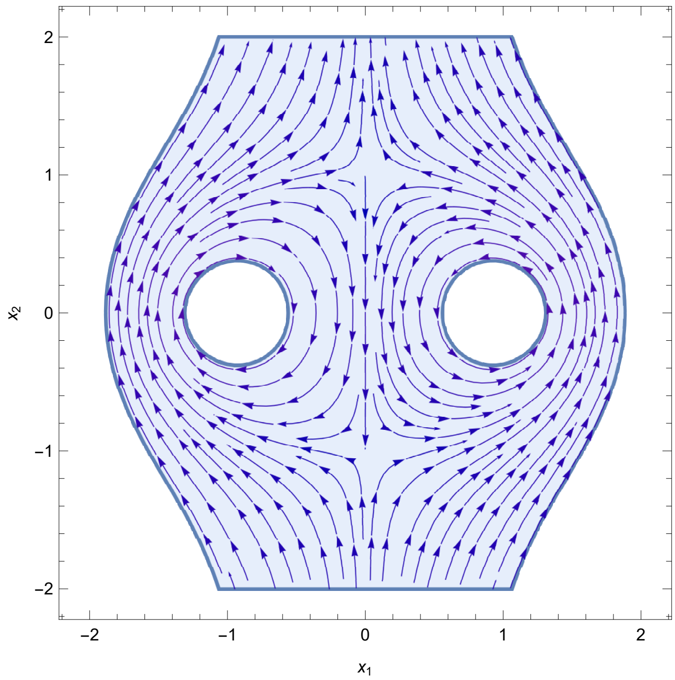


Figure 4.5: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap \mathbb{R} \times [-2, 2]$. Here ψ is given by equation (4.11).

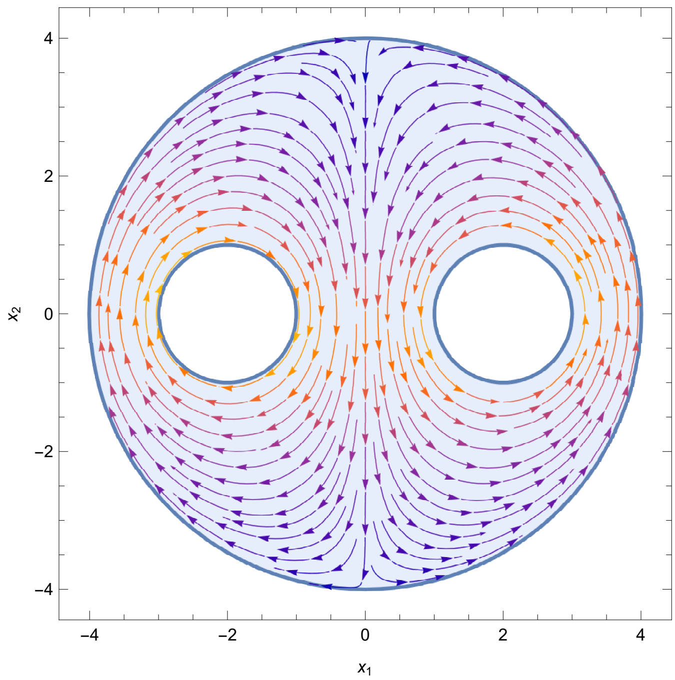


Figure 4.6: A plot of $u = \nabla^{\perp}\psi$ where ψ is the numerical solution to (4.12).

5 Harmonic functions, $d = 3$

The cylinder

The following proof comes from [25]

Proposition 5.1. *Let $\Omega = (0,1) \times U \subseteq \mathbb{R}^3$ be an open cylinder where $U \subseteq \mathbb{R}^2$ is an open set. Let further $f : X = \overline{\Omega} \to \mathbb{R}$ be harmonic such that the sides $[0,1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^+$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^-$ is the emergent boundary. Then f cannot have an interior critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ. Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times \partial U$ such that $\partial_1 f(x)$ is minimal on X. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. □

Simply connected entrant boundary

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 5.2. *Let $X \subset \mathbb{R}^3$ be a compact manifold with corners. Let $f : X \to \mathbb{R}$ be a Morse harmonic function. Assume that the strictly entrant boundary Σ^- is non-empty and simply connected and that $\gamma = \partial\Sigma^-$ is a one-dimensional manifold with corners homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Then we have that*

$$M_1 - M_2 = 0\,.$$

Proof. As in the two dimensional case we split the domain X with a surface Γ such that $\partial\Gamma = \gamma = \partial\Sigma^-$. Denote the two arising domains by X^+ and X^- where $\partial X^- \cap \Sigma^+ \subseteq \gamma$ and $\partial X^+ \cap \Sigma^- \subseteq \gamma$. Since by proposition 2.1 there are finitely many interior critical points in X we can also assume that no interior critical points lie on Γ. Furthermore we assume that Γ approaches

γ at a slanted angle. For the following argumentation we require that f is strongly Morse on both X^+ and X^- so assume for a moment that this is the case. By assumption we have that γ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Since f is non-degenerate the the number of maxima and minima of f on γ must be equal and thus

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) = \mathrm{Ind}_{\gamma^+,1}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \tag{5.1}$$

We now turn our attention to X^+. Since no essential critical points lie on Σ^+ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f)\,. \tag{5.2}$$

Analogously we have on X^- that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{5.3}$$

In addition we have that the emergent critical points on $\Gamma = \Gamma^+$ of f on X^+ are the entrant critical points on $\Gamma = \Gamma^-$ of $-f$ on X^-, that is

$$\begin{aligned} \mathrm{Ind}_{\Gamma^+,0}(f) &= \mathrm{Ind}_{\Gamma^-,2}(-f) \\ \mathrm{Ind}_{\Gamma^+,1}(f) &= \mathrm{Ind}_{\Gamma^-,1}(-f)\,. \\ \mathrm{Ind}_{\Gamma^+,2}(f) &= \mathrm{Ind}_{\Gamma^-,0}(-f) \end{aligned} \tag{5.4}$$

Using equations (5.2), (5.3) and (5.4) we obtain

$$\begin{aligned} \mu_0^+ - \nu_2^- &= \mathrm{Ind}_{\gamma^+,0}(f) \\ \mu_1^+ - \nu_1^- &= \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,1}(-f)\,. \\ \mu_2^+ - \nu_0^- &= -\mathrm{Ind}_{\gamma^-,0}(-f) \end{aligned} \tag{5.5}$$

We observe the Morse inequalities for f

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X)\,. \tag{5.6}$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \tag{5.7}$$

where the M_k continue to denote the interior type numbers of f. We now subtract equation (5.7) from (5.6) and insert relations (5.5) to obtain then with equation (5.1)

$$\begin{aligned} 0 &= M_1^- - M_2^- + M_1^+ - M_2^+ + \mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) - \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,0}(-f) \\ &= M_1 - M_2 \end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 1.17 applied separately to the domains X^+ and X^-. Since

$E^{\pm}$ are residual in B_δ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. Since x_1, x_2 are non-degenerate critical points of f due to the slanted angle at which Γ approaches γ we obtain that

$$\mathrm{Ind}_{\gamma,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-f) \tag{5.8}$$

By the same corollary we can assume that f^ε has no essential critical points on $\Sigma^+(f)$ and $-f^\varepsilon$ has no essential critical points on $\Sigma^-(f)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. ☐

A harmonic function with interior critical point and simply connected entrant boundary

In fact we can give an example for such a function with simply connected entrant boundary.

Example 5.3 (A harmonic function with interior critical point and simply connected entrant boundary)**.** Consider the domain $X = \overline{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large, and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1 x_2^2 + x_2 x_3 \end{aligned} \tag{5.9}$$

This induces the harmonic vector field

$$\begin{aligned} u\colon X &\to \mathbb{R}^3 \\ x &\mapsto \nabla f(x) = \begin{bmatrix} x_1(1-x_1) + x_2^2 \\ x_2(2x_1 - 1) + x_3 \\ x_2 \end{bmatrix} \end{aligned}$$

It follows from setting $u(x) = 0$ that $x_2 = 0$ and then $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of u. Conversely one sees that these are zeroes of u. Hence f has two interior critical points for $r > 1$. Figure 5.1 shows a stream plot of u with the two interior stagnation points highlighted as black dots. The boundary of the domain is shaded in blue for Σ^- and in red for Σ^+. The stereographic projection of the boundary is plotted in figure 5.2. This plot indicates that the entrant boundary and the emergent boundary are simply connected. We will give a formal proof of this in proposition 5.4.

We start by stating some required definitions. For polynomials $p_1, \dots, p_k \in \mathbb{R}[x_1, \dots, x_d]$ and a set $U \subseteq \mathbb{R}^d$ we denote the variety generated by these polynomials on U by $V_U(p_1, \dots, p_k)$. In the case that $U = \mathbb{R}^d$ we write $V(\dots) = V_U(\dots)$. We will also need a notion of convergence of subsets on a metric space. We define the *Hausdorff metric* for two sets $A, B \subseteq X$ to be given by

$$d_H(A,B) = \max\left\{ \sup_{x\in A} \mathrm{dist}(x,B), \sup_{y\in B} \mathrm{dist}(y,A) \right\} \tag{5.10}$$

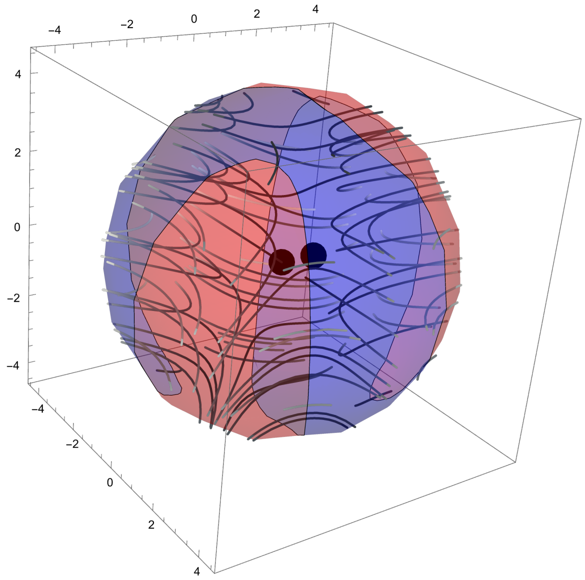


Figure 5.1: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

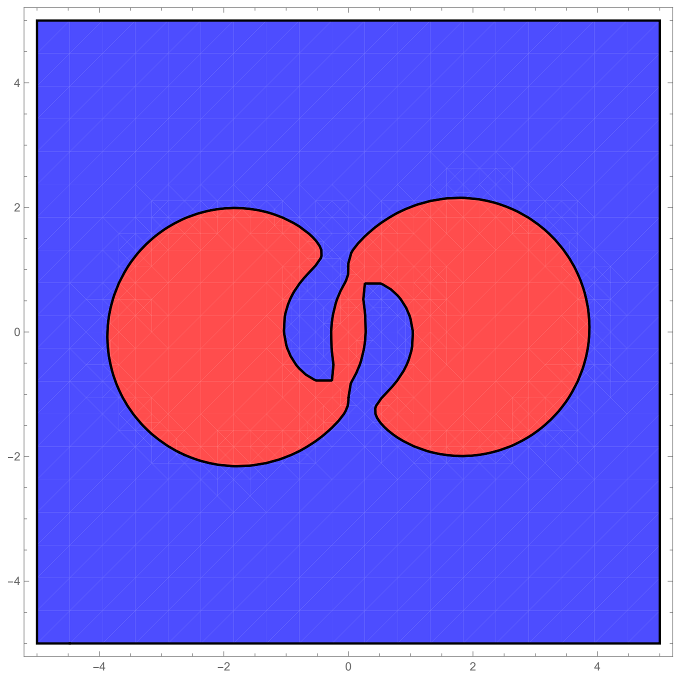


Figure 5.2: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

where

$$\operatorname{dist}(x,B) = \inf_{y \in B} d(x,y) \tag{5.11}$$

is the smallest distance from x to B. For a given $\delta > 0$ and a subset $A \subseteq \mathbb{R}^d$ we call the union of δ balls

$$\operatorname{Tub}_\delta(A) = \bigcup_{x \in A} B_\delta(x) \tag{5.12}$$

a *tubular neighbourhood* of A. We have the following proposition, the proof of which will span several propositions.

Proposition 5.4. *The harmonic function given by equation* (5.9) *has simply connected emergent and entrant boundaries.*

Proof. We calculate

$$rn \cdot u(x) = x_1^2(1-x_1) + x_2^2(3x_1-1) + 2x_2x_3 = p_1(x)$$

and define further

$$P_2(r,x) = x_1^2 + x_2^2 + x_3^2 - r^2\,.$$

Thus we have that the tangential boundary $\Sigma^0 = V(p_1, P_2(r,\cdot))$ is precisely the variety generated by the polynomials p_1 and $P_2(r,\cdot)$ for a fixed radius $r > 0$. It will be shown in proposition 5.5 that there exists $r > 0$ such that Σ^0 is smooth and connected. The stereographic projection of Σ^0 then defines a simple closed planar curve. This is indicated by the black curve in figure 5.2. By the Jordan curve theorem this curve then splits the plane into two connected regions, one of which is simply connected. The preimage of these connected regions under the stereographic projections then corresponds precisely to the entrant and emergent boundaries from which the claim follows. $\square$

The following definitions will play a central role in the upcoming propositions. We now define p_2 to be the dehomogenisation of P_2, that is

$$p_2 = P_2(1,\cdot)\,.$$

Analogously let P_1 denote the homogenisation of p_1, that is

$$P_1(\varepsilon,x) = \varepsilon^3 p_1(x/\varepsilon)\,.$$

Proposition 5.5. *There exists an $r > 0$ such that the planar variety $V(p_1, P_2(r,\cdot))$ is smooth and has one connectivity component.*

Proof. By proposition 5.6 there exists a $R > 0$ such that for all $r > R$ we have that the variety $V(p_1, P_2(r,\cdot))$ is smooth. By rescaling we obtain

$$V(p_1, P_2(r,\cdot)) = rV(p_1(r\cdot), p_2) = rV(P_1(\varepsilon,\cdot), p_2) = r\mathscr{V}_\varepsilon \tag{5.13}$$

where we set $\varepsilon = 1/r$ and $\mathscr{V}_\varepsilon = V(P_1(\varepsilon,\cdot), p_2)$. Motivated by taking the limit $r \to \infty$ we inspect the variety $\mathscr{V}_0$ closer. We observe that

$$P_1(0,x) = -x_1^3 + 3x_1x_2^2$$

which is the monkey saddle embedded into $\mathbb{R}^3$. We thus define curves

$$\widetilde{\alpha}^{\pm} = \left\{ t \begin{bmatrix} \pm\sqrt{3} & 1 \end{bmatrix}^\top : t \in \mathbb{R} \right\}$$

and $\widetilde{\alpha}^0 = \{0\} \times \mathbb{R}$. We then define $\alpha^\bullet = (\widetilde{\alpha}^\bullet \times \mathbb{R}) \cap S^2$. Setting $A = \alpha^- \cup \alpha^+ \cup \alpha^0$ we have the relation

$$\mathscr{V}_0 = V(P_1(\varepsilon,\cdot), p_2) = A\,.$$

Thus $\mathscr{V}_0$ consists of six smooth arcs originating at the singularity e_3 and ending at the singularity $-e_3$. Similar to the classical beach ball. Now consider for $\rho > 0$ the open sets $W_\rho = B_\rho \times \mathbb{R} \subseteq \mathbb{R}^3$ and $U_\rho = \mathbb{R}^3 \setminus W_\rho$. Since $\mathscr{V}_0$ is smooth in a neighbourhood of $\overline{U}_\rho$ we obtain from proposition 5.6 that in a certain sense $\mathscr{V}_\varepsilon$ is obtained from $\mathscr{V}_0$ by a small deformation on U_ρ. Thus in order to show connectedness of $\mathscr{V}_\varepsilon$ for sufficiently small $\varepsilon > 0$ we have to inspect what happens around the points $\pm e_3$. Now observe that we have the symmetry

$$p_1(x_1, -x_2, -x_3) = p_1(x) \tag{5.14}$$

and thus it suffices to inspect what happens around the point e_3. For this parametrise the neighbourhood $S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)$ of e_3 by the diffeomorphism

$$\begin{aligned} \psi\colon B_{1/2} &\to S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right) \\ \tilde{x} &\mapsto x = \begin{bmatrix} \tilde{x} & \sqrt{1 - |\tilde{x}|^2} \end{bmatrix}^\top . \end{aligned}$$

We set

$$\widetilde{\mathscr{V}}_\varepsilon = \psi^{-1}\left(\mathscr{V}_\varepsilon \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)\right) = V_{B_{1/2}}\left(P_1\left(\varepsilon, \cdot_1, \cdot_2, \sqrt{1 - \cdot_1^2 - \cdot_2^2}\right)\right) = V_{B_{1/2}}\left(\widetilde{P}_1(\varepsilon,\cdot)\right)$$

where we defined

$$\widetilde{P}_1(\varepsilon,x) = P_1\left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right) = -x_1^3 + 3x_1x_2^2 + \varepsilon\left(x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2}\right).$$

In a similar manner we define $\widetilde{\alpha}^\bullet = \psi^{-1}(\alpha^\bullet)$, $\widetilde{U}_\rho = \psi^{-1}\left(U_\rho\right)$, $\widetilde{W}_\rho = \psi^{-1}\left(W_\rho\right)$ and $\widetilde{A} = \psi^{-1}(A)$. Now let the sets W and $C \subseteq W$ be as in proposition 5.8. Pick $\rho > 0$ so small that $\widetilde{W}_{2\rho} \subseteq W$. Now we can pick η smaller than the minimum distance between two arcs of $\widetilde{\mathscr{V}_0}$ on $\widetilde{U}_\rho$. We also assume

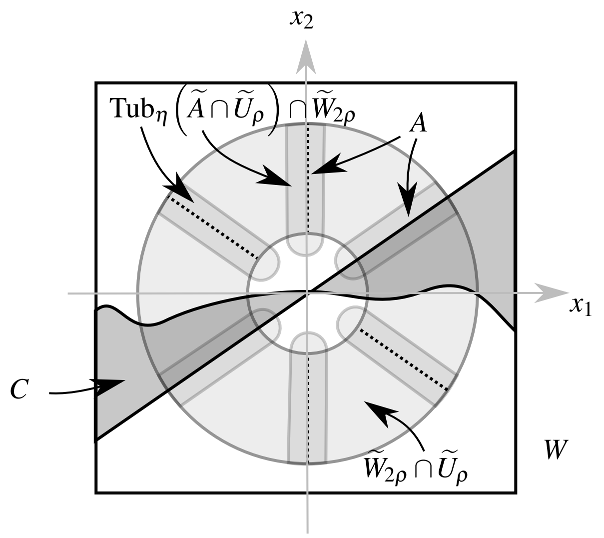


Figure 5.3: An overview of the situation around e_3.

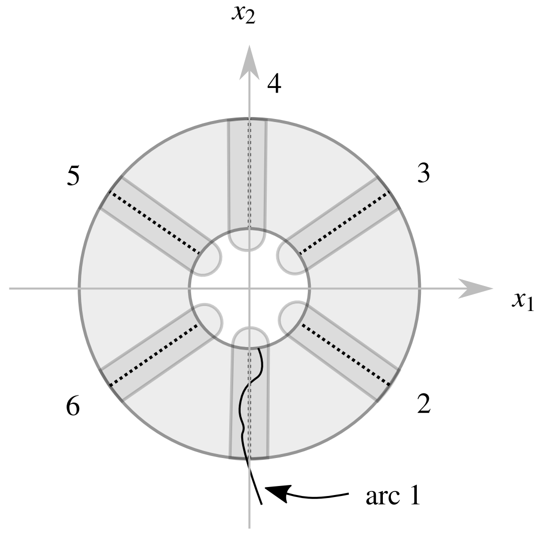


Figure 5.4: The numbering of the arcs around e_3.

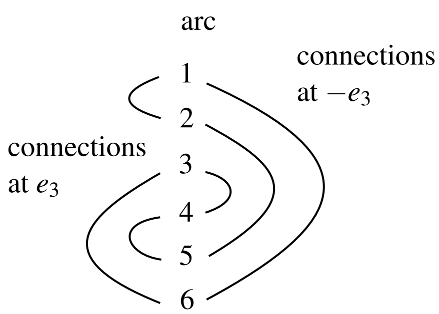


Figure 5.5: The connection of the arcs at $\pm e_3$.

that η is smaller than the Hausdorff distance between $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ and C. Now choose δ as in proposition 5.7. We can assume that $R > 1/\delta$. We thus have a situation as in figure 5.3. Number the arcs lying close to $A \cap U_\rho$ as in figure 5.4. Since $\widetilde{\gamma}$ by proposition 5.8 splits $\widetilde{W}_{2\rho}$ and lies in C we must have that $\widetilde{\gamma}$ in fact connects arcs 3 and 6. As the variety is smooth we must also have that arcs 1 and 2 are connected and analogously that arcs 4 and 5 are connected. By equation (5.14) the situation around $-e_3$ is analogous but mirrored. We thus obtain analogous connections between the six arcs at $-e_3$ which is summarised in figure 5.5. Thus we in fact have that $\mathscr{V}_\varepsilon$ is connected. The claim then follows from relation (5.13). $\square$

Proposition 5.6 (Smoothness)**.** *There exists $R > 0$ such that for every $r > R$ the variety $V(p_1, P_2(r,\cdot))$ is smooth.*

Proof. One calculates

$$T = \begin{bmatrix} \nabla p_1(x) & \frac{1}{2}\nabla_x P_2(r,x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 + 3x_2^2 + 2x_1 & x_1 \\ 6x_1x_2 - 2x_2 + 2x_3 & x_2 \\ 2x_2 & x_3 \end{bmatrix}$$

By the Jacobi criterion it is sufficient to show that this matrix is of full rank on $V(p_1, P_2(r,\cdot))$. This is equivalent to showing that

$$\begin{aligned} 0 \neq \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} &= -9x_1^2x_2 + 3x_2^3 + 4x_1x_2 - 2x_1x_3 = h_1(x)\,, \\ 0 \neq \det \begin{bmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} &= 6x_1x_2x_3 - 2x_2x_3 + 2x_3^2 - 2x_2^2 = h_2(x) \text{ or} \\ 0 \neq \det \begin{bmatrix} T_{31} & T_{32} \\ T_{11} & T_{12} \end{bmatrix} &= 2x_1x_2 + 3x_1^2x_3 - 3x_2^2x_3 - 2x_1x_3 = h_3(x) \end{aligned}$$

for any $x \in V(p_1, P_2(r,\cdot))$. This in turn is equivalent to showing that

$$V(p_1, P_2(r,\cdot), h_1, h_2, h_3) = \emptyset\,. \tag{5.15}$$

Indeed, consider the variety

$$V(p_1, h_1, h_2, h_3) \tag{5.16}$$

Maple calculates the Gröbner basis with lexicographic order $x < y < z$

$$\begin{aligned} 72x_1^8 - 198x_1^7 + 228x_1^6 - 153x_1^5 + 56x_1^4 - 5x_1^3 \\ 72x_1^5x_2 - 126x_1^4x_2 + 102x_1^3x_2 - 51x_1^2x_2 + 5x_1x_2 \\ -24x_1^7 + 42x_1^6 - 2x_1^5 - 23x_1^4 + 7x_1^3 + 10x_1x_2^2 \\ 48x_1^4x_2 - 60x_1^3x_2 + 13x_1^2x_2 + 15x_2^3 \\ 24x_1^4x_2 - 30x_1^3x_2 + 29x_1^2x_2 - 10x_1x_2 + 5x_1x_3 \\ 72x_1^7 - 126x_1^6 + 6x_1^5 + 69x_1^4 - 31x_1^3 + 10x_1^2 - 10x_2^2 + 20x_2x_3 \\ -72x_1^7 + 414x_1^6 - 654x_1^5 + 399x_1^4 - 97x_1^3 + 10x_1^2 - 30x_2^2 + 20x_3^2 \end{aligned}$$

from which we see that the variety (5.16) contains finitely many points. Thus if we choose R so large that all of these points are contained in the ball B_R then we have that (5.15) holds for all $r > R$. ☐

Proposition 5.7 (Convergence at smooth points)**.** *For $\varepsilon \in \mathbb{R}$ and an open set $W \subseteq \mathbb{R}^3$ define the variety $\mathscr{V}_\varepsilon = V_W(P_1(\varepsilon,\cdot), p_2)$. Let $U \subseteq \mathbb{R}^3$ be an open set such that $\mathscr{V}_0$ is smooth in an open neighbourhood of $\overline{U}$. Let further $\eta > 0$. Then there exists a $\delta > 0$ such that for all $\varepsilon < \delta$ we have that the Hausdorff distance satisfies*

$$d_H(\mathscr{V}_\varepsilon, \mathscr{V}_0) < \eta$$

and additionally $\mathscr{V}_\varepsilon$ is isotopic to $\mathscr{V}_0$.

Proof. Consider the mapping

$$F = \begin{bmatrix} P_1 \\ p_2 \end{bmatrix} : \mathbb{R}^4 \to \mathbb{R}^2 .$$

Since $V(P_1(0,\cdot), p_2)$ is smooth on an open neighbourhood of $\overline{U}$ we have by the Jacobi criterion that

$$DF = \left[\begin{array}{c|c} * & \begin{matrix} D_x P_1(0,\cdot) \\ Dp_2 \end{matrix} \end{array} \right] \tag{5.17}$$

is of full rank on $\overline{U}$. By the implicit function theorem there exists at every point $x \in \overline{U}$ open neighbourhoods $\Omega_x \subseteq \mathbb{R}^4$ of $(x,0)$ and $\omega_x \subseteq \mathbb{R}^3$, a coordinate permutation $I \in O(4)$ and a mapping $g_x \in C^1(\omega_x;\mathbb{R})$ such that

$$V(P_1, p_2) \cap \Omega_x = \{\varphi(x) = I(y, g_x(y)) \colon y \in \omega_x\} .$$

By equation (5.17) we can assume (possibly after shrinking the open sets) that I does not permute the ε-coordinate. We can also assume that $\Omega_x = B_{\delta_x} \times W_x \subseteq \mathbb{R}^4$ for some open $W_x \subseteq \mathbb{R}^3$ and some $\delta_x > 0$. Thus we also obtain that $\omega_x = B_{\delta_x} \times w_x$ for some open $w_x \subseteq \mathbb{R}^2$ and we can define our isotopy on Ω_x as

$$\begin{aligned} \varphi \colon B_{\delta_x} \times w_x &\to W_x \\ y &\mapsto \operatorname{proj}_{\mathbb{R}^4 \to W} I(y, g_x(y)) \end{aligned}$$

where $y = (\varepsilon, y_1, y_2)$. Note that $\varphi(\{\varepsilon\} \times w_x) = \mathscr{V}_\varepsilon \cap W_x$. From this it also follows that we can choose δ_x such that

$$d_H(\mathscr{V}_\varepsilon \cap W_x, \mathscr{V}_0 \cap W_x) < \eta .$$

Now for $x \in \overline{U}$ the Ω_x form an open cover of $\overline{U}$. By compactness there exists a finite subcover. Set $\delta > 0$ to be the minimum of all δ_x for the Ω_x in this finite subcover and the claim follows. ☐

Proposition 5.8. *Let $\widetilde{U}_\rho$, $\widetilde{\alpha}$, ε, et cetera be as in the proof of proposition 5.5. There exists an open cube $W \subseteq \mathbb{R}^2$ containing the origin and a set $C \subseteq W$ such that C has positive Hausdorff distance to the set $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that for any sufficiently small $\varepsilon > 0$ there is an arc $\widetilde{\gamma}$ entirely contained in $C \cap \widetilde{\mathscr{V}}_\varepsilon$ splitting W into two parts.*

Proof. We borrow notation from the proof of proposition 5.5. Let $W \subseteq \mathbb{R}^2$ be an open cube around the origin which we fix later. Define

$$\tilde{p}_1(x) = \widetilde{P}_1(1,x) = p_1\left(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right)$$

and observe that

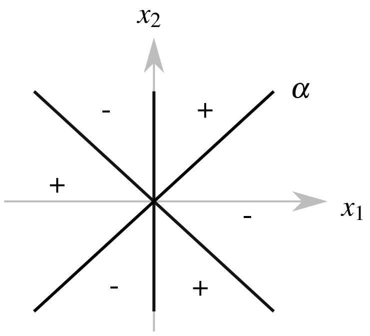
$$\widetilde{P}_1(\varepsilon, x) = (\tilde{p}_1)_3 + \varepsilon(\tilde{p}_1)_{\leq 2}. \tag{5.18}$$

Here we use $(\cdot)_k$ to refer to the parts homogeneous of degree k and $(\cdot)_{\leq k}$ to refer to the parts homogeneous of degree $\leq k$. As noted earlier $(\tilde{p}_1)_3 = -x_1^3 + 3x_1x_2^2$ is a monkey saddle and $V_W((\tilde{p}_1)_3) = \widetilde{A}$ is similar to figure 5.6a. The signs in the figure indicate the sign of $(\tilde{p}_1)_3$ in a given region. Since we have

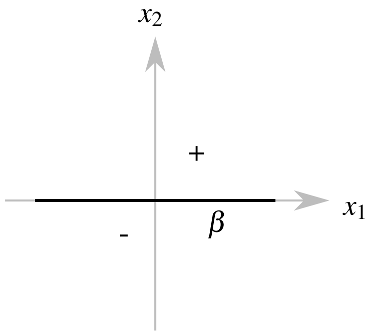
$$(\tilde{p}_1)_{\leq 2} = x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2}$$

we obtain that $\nabla(\tilde{p}_1)_{\leq 2}(0) = e_2$ and $(\tilde{p}_1)_{\leq 2}(0) = 0$. Hence we observe that the $V_W\left((\tilde{p}_1)_{\leq 2}\right)$ looks similar to figure 5.6b for a sufficiently small neighbourhood W. More concretely we choose W so small that the arc $\widetilde{\beta} = V_W\left((\tilde{p}_1)_{\leq 2}\right)$ has positive distance to $\widetilde{A} \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that a given vertical line in W intersects $\widetilde{\beta}$ in precisely one point. Now we claim that the set C consisting of the vertical lines between $\widetilde{\alpha}^+$ and $\widetilde{\beta}$ fulfills the claim.

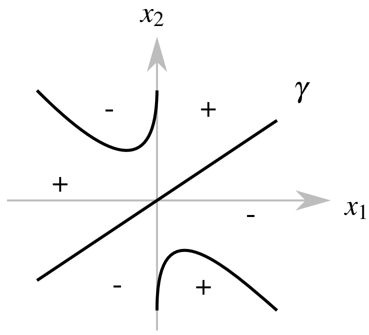
Indeed, consider the first quadrant. Here we have that $\widetilde{P}_1(\varepsilon, \cdot) = (\tilde{p}_1)_3 \leq 0$ on $\widetilde{\beta}$ and that $\widetilde{P}_1(\varepsilon, \cdot) = (\tilde{p}_1)_{\leq 2} \geq 0$ on $\widetilde{\alpha}^+$. By the intermediate value theorem thus there must be for each $(x_1, 0) \in W$ a corresponding point $(x_1, \widetilde{\gamma}(x_1)) \in V_W\left(\widetilde{P}_1(\varepsilon, \cdot)\right) = \widetilde{\mathscr{V}}_\varepsilon$ sandwiched between $\widetilde{\beta}$ and $\widetilde{\alpha}^+$. Since $\widetilde{\mathscr{V}}_\varepsilon$ is smooth $\widetilde{\gamma}$ in fact defines an arc which we also denote by $\widetilde{\gamma}$. An analogous argumentation using signage extends $\widetilde{\gamma}$ to the third quadrant. Then $\widetilde{\gamma}$ divides the plane into two parts and it follows that $\widetilde{\mathscr{V}}_\varepsilon$ which yields the claim. For clarity the idea of the proof is also depicted in figure 5.7. Also note that as a consequence $\widetilde{\mathscr{V}}_\varepsilon$ looks similar to figure 5.6c. □



(a) Sketch of the set $V((\tilde{p}_1)_3)$.



(b) Sketch of the set $V\big((\tilde{p}_1)_{\leq 2}\big)$.



(c) Sketch of the set $\widetilde{\mathscr{V}}_\varepsilon$.

Figure 5.6: Sketches of varieties.

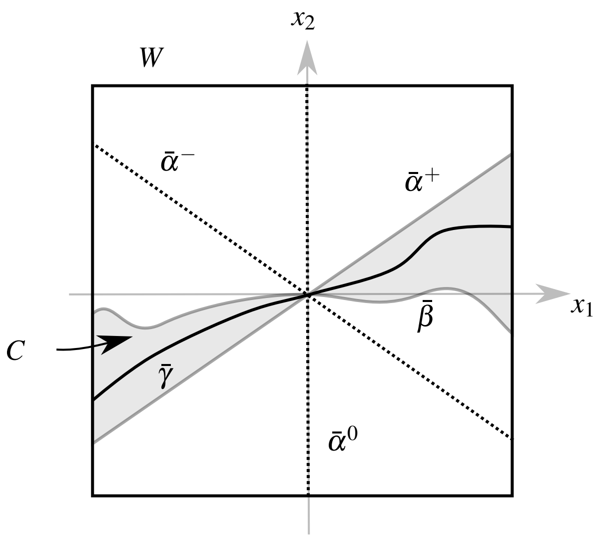


Figure 5.7: An overview of the situation in proposition 5.8.

6 Harmonic vector fields, $d = 3$

No inflow or outflow

We obtain as a quick consequence of the hairy ball theorem

Proposition 6.1. *Let X have Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without boundary stagnation points. Then we have*

$$b_2 \leq b_1 .$$

Proof. Assume not. Since Ω has b_2 bubbles and b_1 holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no boundary stagnation points on Γ we the restriction $u|_\Gamma$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. $\square$

We also obtain the following result:

Proposition 6.2. *Let $X \subseteq \mathbb{R}^3$ be a compact manifold with corners and Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a strongly Morse harmonic vector field without boundary critical points. Then the following relation for the interior type numbers of u holds:*

$$M_1 = M_2 .$$

Proof. As in the two dimensional case we cut the domain X with a surface Γ such that the slit domain is homeomorphic to a ball with bubbles. Since the number of interior stagnation points of u is finite by proposition 2.1, we can choose Γ in such a way that it does not contain any stagnation points. We also denote the arcs at which Γ meets Σ by $\gamma_1, \dots, \gamma_{b_1} \subseteq \partial\Gamma$. Note that there are b_1 many such curves. We can assume that Γ and the γ_j are manifolds with corners and that Γ approaches each γ_j at a slanted angle. The cut now yields a new domain $\tilde{X}$ which is a covering space of X. On this covering space we denote the cover of the cut Γ and the sets γ_j by Γ^i and γ_j^i with $i \in (1,2)$. Since this new domain $\tilde{X}$ is homeomorphic to a ball with bubbles the vector field $u = \nabla f$ is the gradient of a harmonic function f by proposition 2.10. For the following argumentation we require that u is strongly Morse on $\tilde{X}$, so assume for a moment that this is the case. Now we have that each γ_j is homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Since f is non-degenerate the number of maxima and minima of f on $\gamma_j^1 \cup \gamma_j^2$ must be equal and thus

$$\sum_i \Big(\mathrm{Ind}_{\gamma_j^i,0}(f) + \mathrm{Ind}_{\gamma_j^i,1}(-f)\Big) = \sum_i \Big(\mathrm{Ind}_{\gamma_j^i,1}(f) + \mathrm{Ind}_{\gamma_j^i,0}(-f)\Big) . \tag{6.1}$$

Since on Γ all entrant stagnation points of u are also emergent stagnation points of $-u$ (and vice versa) we have the relations

$$\begin{aligned}\mathrm{Ind}_{\Gamma^1,0}(\pm u) &= \mathrm{Ind}_{\Gamma^2,2}(\mp u)\\ \mathrm{Ind}_{\Gamma^1,1}(\pm u) &= \mathrm{Ind}_{\Gamma^2,1}(\mp u)\\ \mathrm{Ind}_{\Gamma^1,2}(\pm u) &= \mathrm{Ind}_{\Gamma^2,0}(\mp u)\,.\end{aligned} \tag{6.2}$$

As there are no boundary critical points on Σ it follows for the boundary type numbers that

$$\begin{aligned}\mu_k &= \sum_i \left(\mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma^i_j,k}\right)(f)\\ \nu_k &= \sum_i \left(\mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma^i_j,k}\right)(-f)\,.\end{aligned} \tag{6.3}$$

Equations (6.3) and (6.2) yield

$$\begin{aligned}\mu_0 - \nu_2 &= \sum_{i,j} \mathrm{Ind}_{\gamma^i_j,0}(f)\\ \mu_1 - \nu_1 &= \sum_{i,j} \Big(\mathrm{Ind}_{\gamma^i_j,1}(f) - \mathrm{Ind}_{\gamma^i_j,1}(-f)\Big)\\ \mu_2 - \nu_0 &= -\sum_{i,j} \mathrm{Ind}_{\gamma^i_j,0}(-f)\end{aligned} \tag{6.4}$$

For f we have the Morse inequalities

$$M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 = -\chi(\tilde{X}) \tag{6.5}$$

and for $-f$ the Morse inequalities

$$M_1 + \nu_2 - M_2 - \nu_1 + \nu_0 = -\chi(\tilde{X})\,. \tag{6.6}$$

Subtracting equation (6.6) from (6.5) and using the relation (6.4) we obtain together with equation (6.1) that

$$\begin{aligned}0 &= 2(M_2 - M_1) + \sum_{i,j} \Big(\mathrm{Ind}_{\gamma^i_j,0}(f) - \mathrm{Ind}_{\gamma^i_j,1}(f) + \mathrm{Ind}_{\gamma^i_j,1}(-f) - \mathrm{Ind}_{\gamma^i_j,0}(-f)\Big)\\ &= 2(M_2 - M_1)\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that f is not strongly Morse on X^+ and X^-. In this case let f^ε for $\varepsilon \in E$ be a strongly Morse function as in corollary 1.17. Since x_1, x_2 are non-degenerate critical points of f due to the slanted angle at which Γ approaches each γ_j we obtain that

$$\mathrm{Ind}_{\gamma_j,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma_j,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(-f) \tag{6.7}$$

By the same corollary we can assume that f^ε has no critical points on Σ. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. $\square$

As a consequence we obtain the following:

Corollary 6.3. *Let X be a compact manifold with corners and $u\colon X \to \mathbb{R}^d$ a strongly Morse harmonic vector field without inflow or outflow on the boundary Σ. The we have that*

$$M_1 = M_2\,.$$

7 Harmonic functions, $d = 4$

Define the harmonic function

$$\begin{aligned} f\colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 . \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{7.1}$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_\pm \colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{7.2}$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2\big(x_1^2 + x_2^2 - x_3^2 - x_4^2\big)$$

Using condition (7.1) we obtain the equivalent condition

$$0 < \pm 1 - 2\big(x_3^2 + x_4^2\big)$$

Define the cylinder

$$C = \big\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \big\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (7.2) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm(\Sigma^+)\,.$$

Analogously we have

$$B_1 \setminus C = \psi_\pm(\Sigma^-)\,.$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 7.1.

Check that the transition at the boundary is legal.

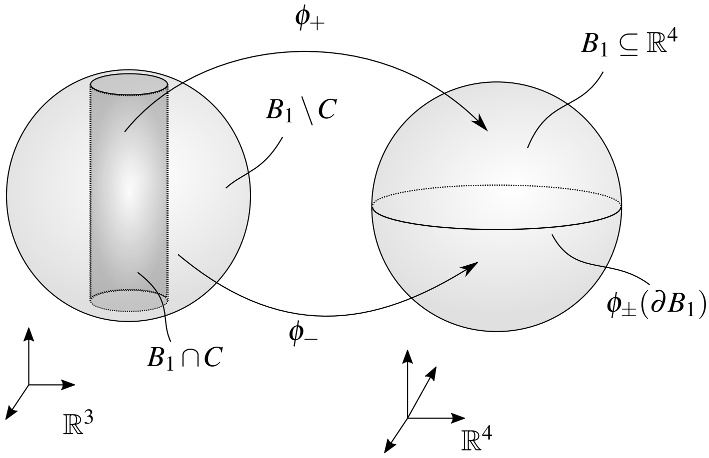


Figure 7.1: Visualisation of the situation.

$\square$

Symbols

d	Dimensions $d = 2$ or $d = 3$
X	Compact domain in $\mathbb{R}^d$, often assumed to be a manifold with corners.
Ω	Interior $\Omega = \mathrm{int}(X)$
$f\colon X \to \mathbb{R}$	A harmonic function.
$u\colon X \to \mathbb{R}^d$ or T^*X	A harmonic vector field.
X_j	A stratification of X as given in definition 1.4. Often but not always assumed to be given by equation (1.1)
Σ	Boundary $\Sigma = \partial X$
Σ^-, $\Sigma^{\leq 0}$	(Strictly) entrant boundary. See definition 1.7.
Σ^+, $\Sigma^{\geq 0}$	(Strictly) emergent boundary. See definition 1.7.
Σ^0	Tangential boundary. See definition 1.7.
Σ^{irr}	Irregular boundary. See definition 1.9.
$B_r(x)$, B_r	Ball of radius r around the point x / the origin.
$S^{d-1}(x)$, S^{d-1}	$(d-1)$-dimensional sphere around x / the origin.
u_j	Projection of u to the cotangent bundle T^*X_j. See equation (1.6).
π_j	Orthogonal projection onto the cotangent bundle T^*X_j. See equation (1.5).
Cr_j	Set of essential stagnation points. See definition 1.9.
$\mathrm{Ind}_{j,k}$	k-th type number on the stratum X_j. See equation (1.7).
Ind_k	k-th type number. See equation (1.11).
M_k	k-th interior type numbers. See equation (1.8).
M	Total number of stagnation points. See equation (1.9).
μ_k	k-th boundary type numbers of f. See equation (1.10).
ν_k	k-th boundary type numbers of $-f$. See definition 1.9.
u^ε	modification to u as in equation (1.12)
A	submanifold, can be thought of as the zero section of T^*X
b_k	Betti number as defined in equation (2.2)
$\chi(X)$	Euler characteristic.
$d_H(A,B)$	Hausdorff metric. See equation (5.10).
$\mathrm{dist}(x,A)$	Distance between x and A. See equation (5.11)
$\mathrm{Tub}_\delta(A)$	Tubular neighbourhood of A. See equation (5.12).

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