Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

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Contents

Todo list

General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.
- Define traversing vector field

Some questions

- Should I state Hopf's Lemma?

1 Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\Omega = \mathrm{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. Unless otherwise stated we denote by

$$f : X \to \mathbb{R}$$

a C^2 function on X. Often f will be assumed to be harmonic. We also denote by

$$u : X \to \mathbb{R}^d$$

a vector field of class C^1. In the following we often assume that u is in fact *harmonic*, that is u fulfils $\mathrm{Div}\, u = 0$ and $\mathrm{curl}\, u = 0$. Often but not always we assume that in fact $u = \nabla f$ is a gradient field. One question we seek to answer in this thesis is the following:

Question 1.1 (Flowthrough with stagnation point)**.** Does there exist a region $X \subseteq \mathbb{R}^3$ homeomorphic to a ball with flow u through the region such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. the boundary on which u enters the region is simply connected?

The answer for this will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Another question we will consider is of the type:

Question 1.2 (stagnation points of harmonic vector fields without inflow or outflow)**.** Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary behaviour.

General definitions

We start by requiring some regularity for the boundary of X. More precisely, we require X to be a compact Riemannian manifold with corners: