Some title

Master Thesis

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November 5, 2023

Contents

Todo list

General TODOs

- Look what happens if you add the examples
- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Fix issue with examples
- Find modern version of complex analysis counterpart
- Vector identities for curl and div in R2
- Harmonic vector fields, find up to date reference →prove used identities
- Look up Leau-Fatou flower
- Look up Bocher theorem
- Give a complex analysis proof of the number of critical points without inflow or outflow →does this generalise to one inflow, outflow?
- Look at application of Sperner's lemma

Some questions

- Should I state Hopf's Lemma?
- Weak formulation - a distraction? →Hartman, Wintner

Introduction

Some amazing introduction

Unless otherwise stated we denote by $\Omega \subseteq \mathbb{R}^d$ an open bounded subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial\Omega$. In the following we will work in dimensions $d \in \{2,3\}$. We denote with

$$f \colon \overline{\Omega} \to \mathbb{R}$$

a scalar function of class C^2. We also denote by

$$u \colon \overline{\Omega} \to \mathbb{R}^d$$

a vector field of class C^1. Often but not always u can be thought of as a *harmonic vector field*, that is u is of type C^1 and fulfils

$$\operatorname{Div} u = 0 \qquad \text{and} \qquad \operatorname{curl} u = 0\,.$$

Also often but not always we assume that globally $u = \nabla f$ is a gradient field, implying that f is harmonic. One question we seek to answer during this thesis is the following.

Question 1 (Flowthrough with stagnation point)**.** Does there exist a tube $\Omega \subseteq \mathbb{R}^3$ with flow u through the tube such that

1. u is a harmonic vector field
2. u has an interiour stagnation point
3. u enters the tube on the one end and exits the tube on the other?

To make the formulation more precise we begin with some general definitions regarding stagnation points and the boundary conditions.

General definitions

In the following we define the emergent and the entrant boundary as in [2, p.282]

Definition 2 (Emergent and entrant boundary)**.** We call a vector $v \in T_x\mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ iff v is not tangent to Σ and directed into the interiour of Ω. Analogously if v is not tangent and directed to the exteriour we call v *emergent*. We define the *entrant boundary* Σ^- to be the set of boundary points at which u is entrant. Analogously define the *emergent boundary* Σ^+ to be the set of boundary points at which u is emergent. Further define Σ^0 to contain all other boundary points such that we have a decomposition of the boundary

$$\Sigma = \Sigma^- \sqcup \Sigma^0 \sqcup \Sigma^+\,.$$

The following are slight generalisation of the definitions given in [1, p.138f], [3, §5] and [2, p.282f] to include harmonic vector fields. We call the zeroes of u *stagnation points*. A stagnation point $x \in \Omega$ is called *non-degenerate* if the derivative $Du(x)$ is invertible. We say that x has *index* k if $Du(x)$ has exactly k negative eigenvalues. u is called *non-degenerate* if all its stagnation points are non-degenerate.

We now define the *interiour type numbers* M_k to be the number of stagnation points of u of index k. The total number of stagnation points is thus given by

$$M = \sum_k M_k \,.$$

We call Ω a *regular domain* if Σ is a manifold of class C^2. In the following definition we require Ω to be regular.

Definition 3 (Boundary type numbers)**.** For a boundary point x let

$$\pi_x \colon \mathbb{R}^d \cong T_x\mathbb{R}^d \to T_x\Sigma$$

denote the projection of a vector at x onto the tangent space of Σ at x. Let

$$u_{\Sigma^-} = \pi \circ u|_{\Sigma^-} \in C^1\left(T\Sigma^-\right)$$

be the restriction and projection of u onto the tangent bundle of Σ^-. We define the *boundary type numbers* μ_k to be the number of stagnation points of u_{Σ^-} on the entrant boundary Σ^- of index k. We further write ν_k for the k-th boundary type number of $-u$.

This construction surely has a name in diffGeo.

We now call u *regular* iff u and u_Σ are non-degenerate and all stagnation points lie in Ω.

The previous definitions translate naturally to f. We call f regular, non-degenerate, et cetera iff $u = \nabla f$ is regular, non-degenerate, et cetera. Similarly we call x a critical point of index k if it is a stagnation point of u of index k.

On assuming non-degeneracy

We now comment on the assumption of non-degeneracy of u and f. Given u we define the modification

$$u_\varepsilon = u + \varepsilon$$

for some $\varepsilon \in \mathbb{R}^d$. The following is a slight adaptation of results in [2, §2].

We call a boundary point x *ordinary* if it is not tangent, that is $u(x) \notin T_x\Sigma$.

Proposition 4. *Assume all stagnation points of u lie in Ω and that the stagnation points on Ω are ordinary. Then there exists a $\delta > 0$ and a Lebesgue nullset N such that for all $\varepsilon \in B_\delta \setminus N$ such that*

- *u_ε is regular*
- *there is a one to one relation of the stagnation points in Ω and Σ preserving index and the property of being entrant or emergent.*

Proof.

insert proof here

□

Some general remarks

We make the following remarks

Proposition 5. *Let u be non-degenerate. Then the number of stagnation points is finite.*

Proof. Let x be a non-degenerate stagnation point. Since $Du(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq \Omega$ of x on which u is bijective. Hence x is the only stagnation point in U_x. Let C denote the set of all stagnation points of u. Then the sets U_x together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \tag{1}$$

form an open cover of $\overline{\Omega}$. But $\overline{\Omega}$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus ([)c]x$ and $x \notin U_C$ we must have that U_x is in the finite subcover. Thus it follows that $\#C < \infty$ is finite. □

As a consequence we obtain the following.

This was stated somewhere in Morse1969. Also, what is with the boundary stagnation points

Corollary 6. *For a non-degenerate u the type numbers $M_0, \ldots, M_d$ and the boundary type numbers $\mu_0, \ldots, \mu_{d-1}$ are finite.*

Betti numbers

For a formal definition of Betti numbers we refer to ??. We do however give some examples of Betti numbers for some domains.

complete this section

- betti numbers are finite

The Morse inequalities

We state the Morse inequalities.

How do we know that the betti numbers are finite?

More citations. Check if C^2 is really sufficient for Morse.

Theorem 7 (Morse inequalities)**.** *Let Ω and f be regular. Then we have the inequalities*

$$\begin{aligned}
&0 \leq M_0 + \mu_0 - R_0 \\
&0 \leq M_1 + \mu_1 - R_1 - (M_0 + \mu_0 - R_0) \\
&\quad\vdots \\
&0 \leq M_{d-1} + \mu_{d-1} - R_{d-1} - \cdots + (-1)^{d-1}(M_0 + \mu_0 - R_0) \\
&0 = M_d - R_d - (M_{d-1} + \mu_{d-1} - R_{d-1}) + \cdots + (-1)^d (M_0 + \mu_0 - R_0)\,.
\end{aligned}$$

Proof. See [3]. □

Give a more precise reference.

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$ and thus we obtain for the special case of dimensions $d = 2$.

Corollary 8 (Morse inequalities for f harmonic, $d = 2$)**.** *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} 0 &\leq \mu_0 - R_0 \\ 0 &= M + \mu_1 - R_1 - \mu_0 + R_0 \,. \end{aligned}$$

In dimensions $d = 3$ we obtain

Corollary 9 (Morse inequalities for f harmonic, $d = 3$)**.** *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} 0 &\leq \mu_0 - R_0 \\ 0 &\leq M_1 + \mu_1 - R_1 - \mu_0 + R_0 \\ 0 &= M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 - \mu_0 + R_0 \,. \end{aligned}$$

Give a classical example of a Morse function to determine the betti numbers.

On harmonic vector fields

We state the following result for harmonic vector fields in $d = 2$ dimensions. In the following we deduce some basic relations for harmonic vector fields.

Proposition 10 (Harmonic vector fields on simply connected domains)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f\colon \Omega \to \mathbb{R}$.*
2. *f is harmonic.*
3. *u is in fact C^∞.*
4. *The components $u_i = \partial_i f$ are harmonic.*

Proof.
1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.
2. This follows from $\Delta f = \operatorname{Div} u = 0$.
3. This follows from the fact that f is harmonic
4. This follows from $u_i = \partial_i f$.

□

If one considers not necessarily simply connected domains Ω then we obtain the previous properties at least locally.

Harmonic functions, $d = 2$

The following result is essentially a negative to question 1 in $d = 2$ dimensions.

Proposition 11. *Let Ω be homeomorphic to $B_1 \subseteq \mathbb{R}^2$. Let further $f : \overline{\Omega} \to \mathbb{R}$ be regular harmonic with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \Sigma$ is not connected.*

We shall give two different proofs of this result. One involving level-sets and the other involving invariant manifolds

A proof involving level-sets

write omega-limit.

Sketch of Proof. Let $y_c = f(x_1)$ and $x_1, \ldots, x_M$ be all the critical points such that $f(x_\cdot) = y_c$. We claim that the level set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

can be represented by a multigraph G which divides the boundary Σ into 4 components. To show this let $\gamma_i : (a_i, b_i) \to C$ for $i \in \{1, \ldots, 4\}$ parametrise the curves in C intersecting at x_1. These can be constructed with the initial value problem

$$\gamma' = (\nabla f)^{\perp}\big|_{\gamma}$$
$$\gamma(0) = \gamma_0$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1. We assume that the intervals on which the γ_i are defined are maximal. We thus have for

$$\gamma_i^- = \lim_{t \to a_i} \gamma(t)$$
$$\gamma_i^+ = \lim_{t \to b_i} \gamma(t)$$

that $\gamma_i^{\pm} \in \{x_1, \ldots, x_M, \Sigma\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $\nabla f^{\perp} = 0$. This argument can be applied to all of the $x_1, \ldots, x_M$. We therefore have a situation similar to the one depicted in figure 1.

This figure is not quite accurate as it does not contain loops on the boundary. Also, add E to the figure.

Thus C can be represented by a multigraph G with vertices $v_1, \ldots, v_K$ and edges $e_1, \ldots, e_L \subseteq C$. In the following we identify the graph G with its planar embedding in $\overline{\Omega}$. Assume G contains a cycle with vertex sequence $v_{i_1}, \ldots, v_{i_J}$ and edges $e_{i_1}, \ldots, e_{i_J}$. Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E. By the maximum principle $f = y_c$ on E and thus $f = y_c$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with the boundary Σ is at least four and thus the boundary Σ is divided into at least four components.

Now choose four neighbouring components $\omega_1, \ldots, \omega_4$ as depicted in figure 2 Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle

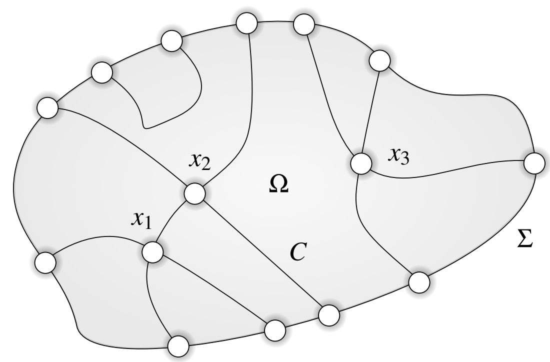

Figure 1: The situation at hand: The edges represent level curves and the interiour vertices critical points.

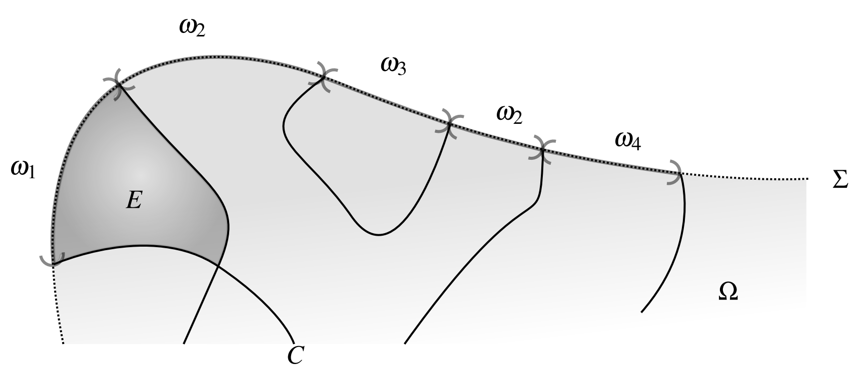

Figure 2: The choice of $\omega_1, \ldots, \omega_4$.

yields that ω_1 contains a local maximum or minimum of f since $f = y_c$ is constant on the other boundaries $\partial A \setminus \omega_1$. By the same argument $\omega_2, \dots, \omega_4$ also contain local extrema. Since the $\partial \omega_i$ cannot be extremal points on Σ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

use argument with ∇f here to show that extrema can be assumed to be alternating.

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. □

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let $x_1, \dots, x_M$ denote the critical points of f. Let $\lambda_i\colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1,2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i\colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{3,4\}$ be chosen to parametrise the stable manifolds of x_1. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Once again we identify the graph with its planar embedding. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle A with vertices $x_{i_1}, \dots, x_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. The set of cycles forms a partial ordering with respect to the property 'contains another cycle'. We can assume that our chosen cycle A contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to one another. We also note that the edges cannot cross. We can thus describe the trail $x_{i_1}, \dots, x_{i_J}$ by a set of directives of the type

More precise.

$$(d_1, \dots, d_K) \in \{\mathrm{l}, \mathrm{r}, \mathrm{s}\}^J.$$

Here l, r and s stand for 'left', 'right' and 'straight' respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 3.

An example of the trail 'srsr' is given in figure 4. We now note that cycles of the type r,…,r or l,…,l cannot occur as we otherwise would have a directed cycle. Thus there

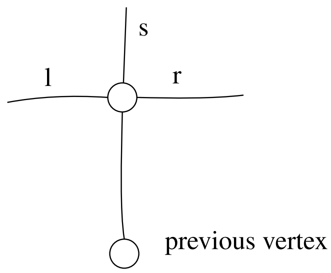

Figure 3: Explanation of the directives ‘l’,‘r’ and ‘r’.

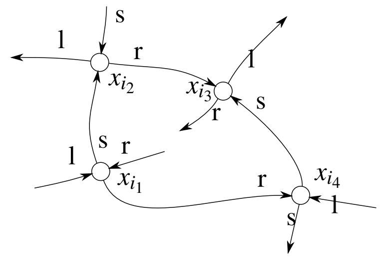

Figure 4: An example for a cycle.

exists a vertex where the chosen direction is s. Without loss of generality this vertex is x_{i_1}. Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1}. Now consider new cycle B starting at x_{i_1} with directives r,…,r. Since all vertices of B lie within the cycle A we must at some step reach a vertex on the cycle A. But then cycle B is a new distinct cycle contained in cycle A, a contradiction to the minimality of A. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

Now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur. elaborate We now pick a tree $\widetilde{G}$ out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each ‘neighbouring’ leaf of this tree has opposite elaborate signage and the claim follows. □

The strategy in the above proofs can be generalised to show the following

Conjecture 12. *Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with Betti numbers $R_0 = 1$ and R_1. Let further $f : \overline{\Omega} \to \mathbb{R}$ be regular harmonic with M critical points. Assume that $\overline{\Sigma^-} \subseteq \Sigma$ on a given connected component of the boundary Σ consists of at most one connected component. Then we have*

$$\frac{4}{3}M \leq R_1 + 1\,.$$

This inequality can probably be improved considerably. Check with A

Harmonic vector fields, $d = 2$

No inflow or outflow

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ iff $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* iff $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can state the following result.

Proposition 13 (Upper bound on M)**.** *Let $d = 2$ and Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1. Let further $u\colon \overline{\Omega} \to \mathbb{R}^2$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$M + 1 \leq R_1 \,.$$

Sketch of proof. As in the second proof of proposition 11 the critical manifolds form a directed multigraph. Since no critical manifold can intersect with the boundary each vertex of the graph has degree 4 and we thus have $2M$ edges. Now we obtain with Euler's polyhedron formula for a planar graph with multiple components

$$\begin{aligned} \#\text{ minimal cycles} &= \#\text{faces} - 1 \\ &= 1 + \#\text{components} - \#\text{vertices} + \#\text{edges} - 1 \\ &\geq 1 + 1 - M + 2M - 1 = M + 1 \end{aligned}$$

Here we use the term 'minimal' as in the second proof of proposition 11. Note that each minimal cycle must contain a hole of the domain since else we could restrict u to a simply connected region containing this cycle. Then by proposition 10 u would correspond to the gradient of a harmonic function in this region and we would obtain a contradiction as in the proof of proposition 11. Hence the number of minimal cycles is a lower bound on the number of holes R_1 of the domain. $\square$

In fact using the Morse inequalities we can obtain the stronger result.

Proposition 14. *Let Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1 and let $u\colon \overline{\Omega} \to \mathbb{R}^2$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$M + 1 = R_1$$

.

Sketch of proof. We slit the domain such that it is homeomorphic to the disk. By proposition 10 u is the gradient of a harmonic function f on this new domain. f has no critical points on the boundary Σ and fulfils on the cuts the conditions

$$\mu_0 = \nu_1 \qquad \mu_1 = \nu_0 \tag{2}$$

since every entrant critical point is also an emergent critical point on the other side of the cut of shifted index. We have for this new cut domain the Morse inequalities for f and $-f$

$$M + \mu_1 - R_1 - \mu_0 + R_0 = 0 \tag{3}$$
$$M + \nu_1 - R_1 - \nu_0 + R_0 = 0 \,. \tag{4}$$

Adding equations (3) and (4) and using the relation (2) we obtain

$$2(M - R_1 + R_0) = 0$$

from which the claim follows. □

We now give an alternative proof using the argument principle.

One could use the argument principle for Riemann surfaces.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition ?? u is the gradient of a harmonic function f on this new domain. Let $h \in \mathrm{Hol}(\mathbb{C})$ be a holomorphic function such that $h' = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ. We now determine the change of argument $\arg h'$ along γ. For this consider first the parts of γ traversing the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other the contribution in the change of $\arg h'$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h'$ is 2π for every hole in the domain since $h' = u$ is tangent to Σ and traverses the holes clockwise direction. Similarly the contribution to the change in argument of $\arg h'$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have R_1 holes in the domain the total change of $\arg h'$ as γ traverses Σ is $2\pi(R_1 - 1)$. Since h has no poles it follows from the argument principle (see for example **Gamelin2001**) that

$$2\pi(R_1 - 1) = \int_\gamma \mathrm{d}\arg(h'(z)) = 2\pi M \tag{5}$$

From this the claim follows. □

We now give an example of a harmonic vector field for which $M = R_1 - 1$. In order to do this we define two differential operators for $d = 2$ by

$$\nabla^\perp f = \operatorname{Curl} f = \begin{bmatrix} -\partial_2 f \\ \partial_2 f \end{bmatrix}$$

and

$$\operatorname{curl} u = -\partial_1 u_2 + \partial_2 u_1$$

Now consider the field defined by

Find source of this in Gedicke script. Relation to differential forms.

$$\begin{aligned} u \colon \mathbb{R}^2 \setminus (\{-e_1, e_1\}) &\to \mathbb{R}^2 \\ x &\mapsto \nabla^\perp \Phi_2(x - e_1) + \nabla^\perp \Phi_2(x + e_1) \end{aligned}$$

where

$$\Phi_2 = -\frac{1}{2\pi} \log(|\cdot|)$$

is the fundamental solution of Δ on $\mathbb{R}^2$ and $e_i = \delta_{i\cdot}$ is the first unit vector. This is a harmonic vector field since

$$\operatorname{curl} \nabla^\perp \Phi(\cdot - y) = -\Delta \Phi(\cdot - y) = 0$$

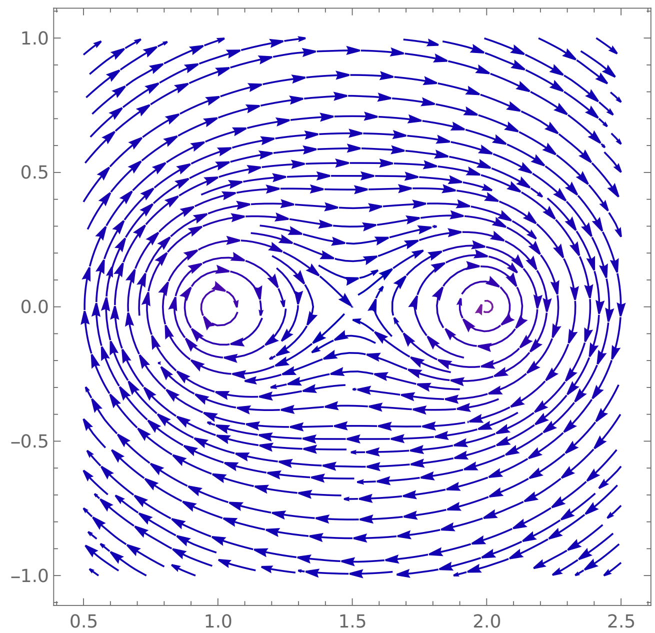

Figure 5: A plot of u for $M = 1$

and

$$\operatorname{Div} \nabla^{\perp} = 0 .$$

Figure 5 indicates that u has the desired properties.

Update this graphic.

An example of inflow on one side and outflow on the other

Consider

$$\begin{aligned} u\colon \mathbb{R}^2 \setminus (\{-e_1, e_1\}) &\to \mathbb{R}^2 \\ x &\mapsto \nabla^{\perp}\Phi_2(x - e_1) - \nabla^{\perp}\Phi_2(x + e_1) + x_1 \end{aligned}$$

As before u is a harmonic vector field. One can show that u has critical points at $-e_2$ and e_2. One can further see from figure 6 that on a fitting domain with two holes u has inflow from one side and outflow on the other.

Show this.

Restrict the plots and functions above to suitable domains.

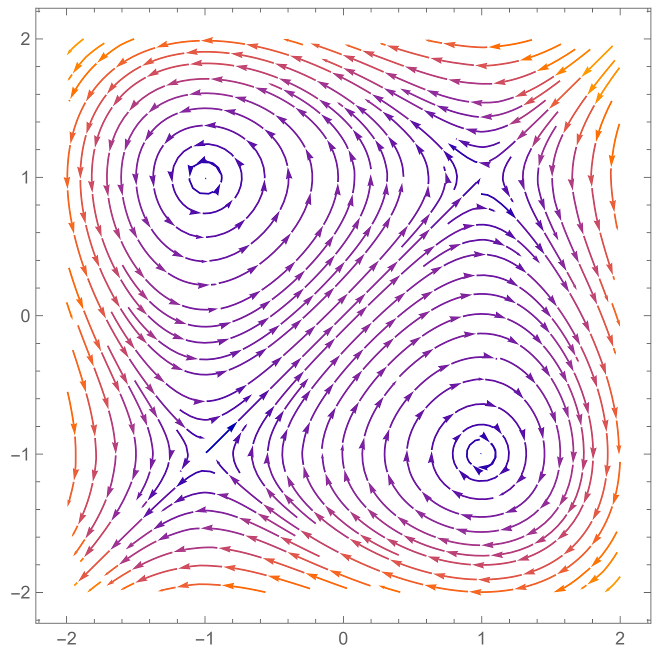

Figure 6: A plot of u as given by equation (**??**).

Harmonic functions, $d = 3$

The cylinder

The following proof comes from [5]

Proposition 15. *Let $\Omega = (0,1) \times B_1 \subseteq \mathbb{R}^3$ be the cylinder. Let further $f : \overline{\Omega} \to \mathbb{R}$ be regular harmonic with no inflow or outflow on the sides $\partial(0,1) \times B_1$, no outflow on $\{0\} \times B_1$ and no inflow on $\{1\} \times B_1$. Then f cannot have a critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ. Since $\partial_1 f(x) = 0$ for some interiour point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times S^1$ such that $\partial_1 f(x)$ is minimal on $\overline{\Omega}$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. □

Harmonic vector fields, $d = 3$

We obtain as a quick consequence of the hairy ball theorem

Proposition 16. *Let Ω have Betti numbers R_0, R_1 and R_2. Let $u\colon \overline{\Omega} \to \mathbb{R}$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$R_2 \leq R_1 \,.$$

Proof. Assume not. Since Ω has R_2 bubbles and R_1 holes there exists by the pidgeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no inflow or outflow on Γ we have that the restriction $u|_\Gamma \in T\Gamma$ is a vector field on Γ. Since u is regular $u|_\Gamma$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. □

A little more rigour would not harm.

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 17. *Let Ω have Betti numbers R_0, R_1 and R_2. Let $u\colon \overline{\Omega} \to \mathbb{R}$ be a regular harmonic vector field without inflow or outflow. Then we have the following relation for critical points of u*

$$M_2 = M_1$$

Attempt at proof. As in the two-dimensional case we begin by cutting up the domain such that the slit domain is homeomorphic to the ball with bubbles. Once again by proposition 10 u is the gradient of a harmonic function u on this new domain. f has no critical points on the boundary Σ and on the cut boundary it fulfils the conditions

$$\mu_0 = \nu_2 \qquad \mu_1 = \nu_1 \qquad \mu_2 = \nu_0 \tag{6}$$

by the same reasoning. We now have the Morse inequalities for f and $-f$

$$M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 + \mu_0 - R_0 = 0 \tag{7}$$
$$M_1 + \nu_2 - R_2 - M_2 - \nu_1 + R_1 + \nu_0 - R_0 = 0 \tag{8}$$

It then follows by subtracting equation (8) from (7) and using relations (6) that

$$2(M_2 - M_1) = 0\,.$$

□

Introduce Morse inequalities for $-f$.

Harmonic functions, $d = 4$

Define the harmonic function

$$
\begin{aligned}
f\colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\
x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,.
\end{aligned}
$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$
\sum_i x_i^2 = 1 \tag{9}
$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$
\begin{aligned}
\phi_\pm \colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\
\bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0
\end{aligned} \tag{10}
$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$
\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top
$$

and the normal to ∂B_1

$$
n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .
$$

Thus we have $x \in \Sigma^\pm$ iff

$$
0 < \pm \nabla f \cdot n = \pm 2\big(x_1^2 + x_2^2 - x_3^2 - x_4^2\big)
$$

Using condition (9) we obtain the equivalent condition

$$
0 < \pm 1 - 2\big(x_3^2 + x_4^2\big)
$$

Define the cylinder

$$
C = \left\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \right\} = \mathbb{R} \times B_{1/\sqrt{2}}
$$

If we return to our parametrisation (10) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(x) \in \Sigma^+$ and hence

$$
B_1 \cap C = \psi_\pm\big(\Sigma^+\big) \,.
$$

Analogously we have

$$
B_1 \setminus C = \psi_\pm\big(\Sigma^-\big) \,.
$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 7.

□

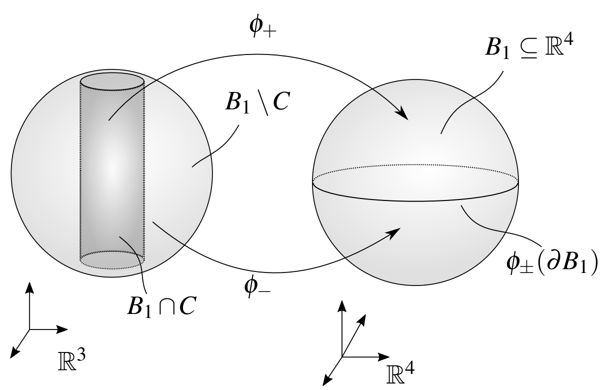

Figure 7: Visualisation of the situation.

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