

Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

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General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.
- Define traversing vector field

Some questions

- Should I state Hopf's Lemma?

Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of \mathbb{R}^d with boundary $\Sigma = \partial X$ and interior $\Omega = \text{int}(X)$. In the following we will work in dimensions $d \in \{2, 3\}$. We denote by

$$f: X \rightarrow \mathbb{R}$$

a scalar function of class C^2 . We also denote by

$$u: X \rightarrow \mathbb{R}^d$$

a vector field of class C^1 . Often but not always u can be thought of as a *harmonic vector field*, that is u is smooth and fulfils

$$\text{Div } u = 0 \quad \text{and} \quad \text{curl } u = 0.$$

Also often but not always we assume that globally $u = \nabla f$ is a gradient field, implying that f is harmonic. One question we seek to answer in this thesis is the following:

Question 1 (Flowthrough with stagnation point). Does there exist a region $X \subseteq \mathbb{R}^3$ with flow u through the region such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. the boundary on which u enters or exits the region are both simply connected?

The answer for this will turn out to be yes for dimensions $d \geq 3$ and no for the dimension $d = 2$. In the case of $d = 2$ dimensions we will look at what happens if we allow for holes in the domain. other questions are of the type:

Question 2 (stagnation points of harmonic vector fields without inflow or outflow). Let u be a flow in a domain X such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of critical points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary conditions.

General definitions

We start by requiring some regularity for the boundary of X . More precisely we require X to be a compact manifold with corners as in [3].

Definition 3 (Manifolds with corners). We introduce the notation

$$H_j^d = \mathbb{R}_{\geq 0}^j \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d.$$

A *manifold with (convex) corners* is a topological space X together with an atlas \mathcal{A} such that for every point $x \in X$ there exists an open neighbourhood U_x of x , a number $j = j(x)$ and a diffeomorphism $\phi: U_x \rightarrow H_j^d$ in \mathcal{A} with $\phi(x) = 0$. We further define sets

$$X_k = \{x \in X: j(x) = k\}, \quad (0.1)$$

which form a stratification of X .

More generally we give the definition of a stratification as

Definition 4 (Stratified space). A *stratified space* is a collection of a topological space X and a collection of subspaces $X_j \subseteq X$, $j \in \mathcal{J}$, called *strata*, indexed by a partially ordered set \mathcal{J} such that

1. each X_j is a manifold (without boundary) of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$.

In the case that $X_j \subseteq \overline{X}_k$ and additionally $n(j) = 0$ or $n(j) = n(k) + 1$ we will write $X_j \preceq X_k$ or, abusing notation, write $X_k = X_{j-1}$.

In the case that the stratification arises through relation (0.1) we have precisely $X_j \preceq X_{j-1}$ for $j \in \{1, \dots, d\}$ and $X_0 \preceq X_0$.

For completeness we also give the definition of the contingent cone for a stratification X_j of X

Definition 5 (contingent cone). We denote the (*Bouligand*) *contingent cone* for a set $Y \subseteq X$ at $x \in \overline{Y}$ by $C_x Y$. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exists sequences $\lambda_n \rightarrow 0$ and $x_n \rightarrow x$ in Y such that

$$\lim_n \lambda_n (x - x_n) = v.$$

Given a vector field $u: X \rightarrow \mathbb{R}^d$ and the above stratification X_k of X we can construct for every $j \in \mathcal{J}$ a vector field

$$u_j: X_j \rightarrow T^*X_j.$$

Here T^*X_j denotes the cotangent space of the manifold X_j as defined for example in [4, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j|_x: \mathbb{R}^d \cong T_x^* \mathbb{R}^d \rightarrow T_x^* X_j \quad (0.2)$$

denote the orthogonal projection of a vector at x onto the cotangent space of the stratum X_j at x . Now let

$$u_j = u|_{T^*X_j} = \pi_j \circ u|_{X_j} \in C^1(T^*X_j) \quad (0.3)$$

be the restriction of u onto the cotangent bundle T^*X_j .

In the following we define the emergent and the entrant boundary in a way that generalises [2, p.282] for stratified manifolds.

Definition 6 (Emergent and entrant boundary). We call a vector $v \in T_x \mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ iff

1. v points into Ω or
2. v lies in the dual cone of the contingent cone $C_x X$, that is

$$v \in (C_x X)^* = \{w \in T_x^* X : \langle w, w' \rangle \leq 0 \text{ for all } w' \in C_x X\}.$$

We call v *strictly entrant* iff in addition v is not tangential to Σ or v lies in the relative interior $\text{relint}(C_x X)^*$. Analogously v is *(strictly) emergent* iff $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which u is entrant. We define the *strictly entrant boundary* Σ^- to be the set of strictly entrant boundary points of u . In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* Σ^+ . Further define the *tangential boundary* Σ^0 to be

$$\Sigma^0 = \Sigma^{\leq 0} \cup \Sigma^{\geq 0} \setminus (\Sigma^+ \cup \Sigma^-) \subseteq \Sigma. \quad (0.4)$$

illustrate on boundary with corners

We would now like to illustrate the preceding definitions.

Example 7. We now consider our domain to be the ball $B_1 \subseteq \mathbb{R}^3$ around the origin in $d = 3$ dimensions. Now consider the harmonic function

$$\begin{aligned} f: \Omega &\rightarrow \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - 2x_3^2 \end{aligned} \quad (0.5)$$

Which induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u: \Omega &\rightarrow \mathbb{R} \\ x &\mapsto [2x_1 \quad 2x_2 \quad -4x_3]^\top. \end{aligned} \quad (0.6)$$

We have that the normal to the boundary $\Sigma = S^2$ is given by

$$\begin{aligned} n: S^2 &\rightarrow S^2 \\ x &\mapsto x \end{aligned}$$

and thus we have that $x \in \Sigma^-$ iff

$$0 > n \cdot u = 2(x_1^2 + x_2^2 - 2x_3^2) = 2f(x)$$

A plot of the sets can be seen in figure 0.1.

The following are slight generalisation of definitions given in [1, p.138f], [5, §5] and [2, p.282f] to include harmonic vector fields.



Figure 0.1: Plots of the entrant, emergent and tangential boundary for the function f given by equation (0.5)

Definition 8 (Stagnation points). Let $u_j: X_j \rightarrow T^*X_j$ be a C^1 vector field on a stratification of X . We call the zeroes $x \in X_j$ of u_j *stagnation points*. If $x \in \Omega$ then we call x an *interior stagnation point*. If x lies in the entrant boundary $\Sigma^{\leq 0}$ or is an interior stagnation point we call x an *essential stagnation point*. The set of all essential stagnation points of u_j is denoted by $\text{Cr}_j = \text{Cr}_j(u)$. A stagnation point x is called *non-degenerate* iff x does not lie in the tangential boundary Σ^0 and additionally the derivative

$$Du_j(x) = Du_j|_x \in T_x T^*X \cong \mathbb{R}^{n \times n}$$

is bijective. In addition we say that x has *index k* if $Du_j(x)$ has exactly k negative eigenvalues. u_j is called (*essentially*) *non-degenerate* if all its (essential) stagnation points are non-degenerate. Assume u_j is non-degenerate then we can define the *k -th type number* of the stratum X_j to be the number of essential critical points of u_j of index k , that is

$$\text{Ind}_{j,k}(u) = \#\{x \in \text{Cr}_j(u) : x \text{ has index } k\}.$$

We define the *interior type numbers* by

$$M_k = \sum_{j: n(j)=d} \text{Ind}_{j,k}(u).$$

The total number of interior stagnation points of u is then given by

$$M = \sum_k M_k.$$

Analogously we define the *k -th boundary type numbers* to be the number of essential boundary stagnation points of u of index k , that is

$$\mu_k = \sum_{j: n(j)<d} \text{Ind}_{j,k}(u) \tag{0.7}$$

We further write ν_k for the k -th boundary type number of $-u$.

The following definition is inspired by [2]. We call a boundary point $x \in X_j$ on a strata X_j *ordinary* iff $u(x)$ is not critical point of a stratum x_{j-1} .

Proposition 9. *The condition that the critical point $x \in X_j$ does not lie in Σ^0 is equivalent to that x is ordinary.*

Proof.

Some proof

□

Definition 10 (Morse functions). We call u (essentially) *Morse* iff for all j we have that u_j is (essentially) non-degenerate. For an essentially Morse function u we will denote the number of essential stagnation points of u of index k by

$$\text{Ind}_k(u) = \sum_{j=0}^d \text{Ind}_{j,k}(u) = \# \left\{ x \in \bigcup_j \text{Cr}_j(u) : x \text{ has index } k \right\}.$$

To better describe the boundary of the emergent boundary $\partial\Sigma^+$ we introduce the concept of tangency regularity, which is inspired by similar definitions made in [6]. In order to avoid the technical intricacies involved with manifolds with corners we assume that Σ is a differentiable manifold for the following definitions to make sense.

Definition 11 (Normal bundle). We define the *normal bundle* for a stratification of X to be the quotient space

$$NX_j = TX_{j-1}/TX_j, \quad (0.8)$$

where TX_j is the tangent space of X_j . This is well-defined if X_{j-1} is uniquely determined by X_j . This is the case if Σ is a C^1 manifold. The *conormal bundle* N^*X_j is defined as its pointwise dual, that is

$$N_x^*X_j = (N_xX_j)'.$$

Analogously to the definition of u_j we can define the vector field

$$u_j^N : X_j \rightarrow N^*X_j$$

as the restriction of u to the conormal bundle N^*X_j , that is

$$u_j^N = u|_{N^*X_j}. \quad (0.9)$$

The following definition is inspired by [6].

Definition 12 (Tangency points). Let u_j^N be as in equation (0.9). We call the zeroes $x \in X_j$ of u_j^N *tangency points*. A tangency point is called *regular* iff the derivative $Du_j^N(x)$ is bijective. We call a function $u : X \rightarrow \mathbb{R}^d$ *tangency regular* iff every tangency point is regular.

As in [6] we call u *boundary generic* iff u is Morse and tangency regular.

The previous definitions translate naturally to f . That is we call f Morse, non-degenerate, et cetera iff $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call x an critical point of f of index k if it is a stagnation point of u of index k .

Rewrite: discuss index on manifold with corners.

To illustrate the preceding definitions we return to our previous example.

Example 13. Let f and u be as in example 7. One sees from equation (0.6) that the origin 0 is the sole interior critical point of f . Since we have that

$$Du(x) = \begin{bmatrix} 2 & & \\ & 2 & \\ & & -4 \end{bmatrix}$$

for all $x \in \Omega$ we see that $Du(0)$ is bijective and thus a non-degenerate critical point. Since $Du(0)$ has exactly one negative eigenvalue we see that the origin has index 1. Since there are no other critical points we have $M = 1$ and

$$M_k = \delta_{k1}$$

where δ denotes the Kronecker delta. We now calculate for $x \in S^2$

$$\tilde{u}(x) = (u - (n \cdot u)n)(x) = (u - 2fn)(x) = 2 \begin{bmatrix} (1 - f(x))x_1 \\ (1 - f(x))x_2 \\ (-2 - f(x))x_2 \end{bmatrix}$$

Hence we see that $x \in \Sigma$ is a critical point iff

$$f(x) = 1 \text{ and } x_3 = 0 \text{ or} \tag{0.10}$$

$$f(x) = -2 \text{ and } x_1 = 0 = x_2. \tag{0.11}$$

The former equation (0.10) gives that every point belonging to $S^1 \times \{0\} \subseteq \mathbb{R}^3$ is in fact a critical point of f . But since $f = 1$ on this set these points are degenerate. We will discuss a fix to this issue in the upcoming section. We now consider equation (0.11) and take $f(x) = -2$ then we must have that $x = \pm e_3$ where $e_k = \delta_k$ is the k -th basis vector in \mathbb{R}^d . We now determine their index. For this consider the curves

$$\begin{aligned} \gamma_k: \mathbb{R} &\rightarrow S^2 \\ t &\mapsto \sin(t)e_k \pm \cos(t)e_3 \end{aligned}$$

for $k \in \{1, 2\}$. Note that $\gamma'_k(0) = e_k$ and $\gamma_k(0) = \pm e_3$. We see that

$$Du(e_1)(\gamma'_k(0)) = (u \circ \gamma_k)'(0) = (\sin(t)e_k \mp 2\cos(t)e_3)'(0) = e_k = \gamma'_k(0)$$

and thus $e_k \in T_{\pm e_3}S^2$ are eigenvectors of $Du(e_k)$ to eigenvalues 1. Since the e_k span the tangent space $T_{\pm e_3}S^2$ it follows that the $\pm e_3$ are non-degenerate critical points of f with index 0.

On assuming non-degeneracy

In the following section we argue that assuming non-degeneracy of u and f is not a great restriction. Given u we define the modification

$$u^\varepsilon = u + \varepsilon \tag{0.12}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that u_ε is for almost all choices of ε non-degenerate and can thus be used to approximate a degenerate u . Our approach is to use Thom's theorem which is inspired by the approach in [4, Chapter 6].

Definition 14 (Transversality). We call a function $g: Y_1 \rightarrow Y_2$ between to manifolds Y_1 and Y_2 (without boundary) *transverse* to a submanifold $A \subseteq Y_2$ iff for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\text{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y.$$

As an application we make the following observation.

Proposition 15 (Transversal characterisation of non-degeneracy). *Let $u_j: X_j \rightarrow T^*X_j$ be a differentiable vector field. Then u_j is non-degenerate iff u_j is transverse to the zero section A_j of T^*X_j and contains no critical points in Σ^0 .*

Proof. First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A) = C$. Unravelling the definition of transversality we get that u_j is transverse to the zero section iff for all $x \in C = u_j^{-1}(A)$ we have that

$$\text{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}TX. \quad (0.13)$$

As A is the zero section we have $T_{u_j(x)}A = 0$ and equation (0.13) is equivalent to stating that Du_j is of full rank. But Du_j being of full rank at all points in C and u_j having no critical points in Σ^0 is equivalent to u_j being non-degenerate. \square

Analogously we make the following observation.

Proposition 16 (Transversal characterisation of tangency regularity). *Let u_j^N be given as in equation (0.9). Then we have that u is tangency regular on a stratum X_j iff u_j^N is transverse to the zero section of N^*X_j .*

The following version of Thom's transversality theorem is an adaption (i.e. weakening) of [4, Theorem 2.7] to our needs.

Theorem 17 (Parametric transversality theorem.). *Let E, Y_1, Y_2 be C^r -manifolds (without boundary) and $A \subseteq Y_2$ a C^r submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A.$$

Let further $F: E \rightarrow C^r(Y_1, Y_2)$ be such that the evaluation map

$$\begin{aligned} F^{ev}: E \times Y_1 &\rightarrow Y_2 \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

is C^r and transverse to A . Then the set

$$\pitchfork(F; A) = \{\varepsilon \in E : F_\varepsilon \text{ is transverse to } A\}$$

is dense.

Proof. See [4, Theorem 2.7] for details. \square

Using proposition 15 we get a generalisation of the results in [2, §2].

Corollary 18 (Density of boundary generic functions). *Let $u: X \rightarrow T^*X$ be a harmonic vector field on X and let X_j be a stratification of X . Assume that u has no critical points on Σ^0 . Then there exists a $\delta > 0$ such that for almost every $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ we have that*

1. u^ε given is non-degenerate on X_j
2. if in addition Σ is a differentiable manifold then u^ε is tangency regular on X_j

Here u^ε is given by equation (??). In addition we have as $\varepsilon \rightarrow 0$ that

1. $u^\varepsilon \rightarrow u$ uniformly
2. The critical points of u^ε converge uniformly to critical points of u
3. The critical points of u^ε and u can be put into a one-to-one correspondence preserving index and the property of being emergent or entrant

Proof.

Fill in the details for the following. Note the parallels to Morse's proof in [2].

We first show that we can choose a $\delta > 0$ such that for all $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ we have no critical points on $\Sigma^0(u^\varepsilon)$. Assume not. Then there exist sequences $\varepsilon_k \rightarrow 0$ and x_k of critical points of u^{ε_k} on $\Sigma^0(u^{\varepsilon_k})$. By compactness of X we can assume that $x_k \rightarrow x$ for some $x \in X$ after taking a subsequence. After taking a further subsequence we can also assume that all x_k lie in a stratum X_j . The condition that x_j are critical points and lie in $\Sigma^0(u^{\varepsilon_j})$ means that there exists a stratum X_{j-1} such that x_j is also critical point of this stratum. But then x is also a critical point of X_{j-1} for u since $u^\varepsilon \rightarrow u$. This implies that x is a critical point of X_{j-1} , but $x \in \bar{X}_j$ and hence $x \in \Sigma^0$ is a critical point. A contradiction.

Set $r = 2$, $E = B_\delta$ and $Y_2 = T^*X_j$ in the previous theorem. We initially set $Y_1 = X_j = \Omega$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F: E &\rightarrow C^\infty(X_j, T^*X_j) \\ \varepsilon &\mapsto u^\varepsilon \end{aligned}$$

We note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq T^*X_j$. Then the parametric transversality theorem yields a dense $E_j \subseteq E$ on which F is transverse to A . For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$\text{Image}\left(DF_{(\varepsilon, x)}^{\text{ev}}\right) = T_x T^*X_j \tag{0.14}$$

since

$$DF_{(\varepsilon, x)}^{\text{ev}} = [\text{Id}_{d \times d} \mid Du_x]$$

is surjective. Proposition 15 now yields that u^ε is non-degenerate on E_j .

Analogously we set $Y_1 = X_j$ to be an arbitrary strata in the previous proof and replace u^ε with the restriction u_j^ε . To show that equation (0.14) holds we resort to the fact that

$$DF_{(\varepsilon,x)}^{\text{ev}} = D(u_j^\varepsilon(x))_{(\varepsilon,x)} = D\pi_j \circ (Du^\varepsilon(x))_{(\varepsilon,x)}$$

is surjective as a concatenation of surjective functions. Thus there also exists a dense set $E_j \subseteq \mathbb{R}$ on which u_j^ε is non-degenerate on X_j .

Now to the tangency regularity. For this we set $Y_2 = N^*X_j$ and $Y_1 = X_j$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F: E &\rightarrow C^\infty(X_j, N^*X_j) \\ \varepsilon &\mapsto u_j^{\varepsilon,N} \end{aligned}$$

Again F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq N^*X_j$. Then the parametric transversality theorem yields a dense $E_j^N \subseteq E$ on which F is transverse to A . It now follows for all $(\varepsilon, x) \in F^{-1}(A)$ that

$$\text{Image}\left(DF_{(\varepsilon,x)}^{\text{ev}}\right) = T_x N^*X_j \quad (0.15)$$

since we have that

$$DF_{(\varepsilon,x)}^{\text{ev}} = D(u_j^{\varepsilon,N}(x))_{(\varepsilon,x)} = D\pi_j^N \circ (Du^\varepsilon(x))_{(\varepsilon,x)}$$

is surjective as a concatenation of surjective mappings. Proposition 16 yields that $u^{\varepsilon,N}$ is tangency regular on E_j^N .

Now the set

$$\bar{E} = \left(\bigcap_j E_j\right) \cap \left(\bigcap_j E_j^N\right) \subseteq \mathbb{R} \quad (0.16)$$

is dense in \mathbb{R} and for every $\varepsilon \in \bar{E}$ the function u^ε fulfils conditions 1 and 2.

Show a version of the second part of the corollary

□

One of the reasons for introducing boundary tangency is the following proposition

Proposition 19. *Assume that Σ is a C^1 manifold and $u: X \rightarrow T^*X$ a tangency regular vector field. Then we have that $\partial\Sigma^+$ is a submanifold.*

Proof. In this case we have that $\partial\Sigma^+ = \Sigma^0$. Since

$$\Sigma^0 = (u_j^N)^{-1}(0) \quad (0.17)$$

And since u_j^N is transverse to the zero section of $N^*\Sigma$ the claim follows.

Elaborate

□

In the following we need a notion of convergence of subsets on a metric space. We define the *Hausdorff metric* for two sets $A, B \subseteq X$ to be given by

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \quad (0.18)$$

where

$$\text{dist}(x, B) = \inf_{y \in B} d(x, y) \quad (0.19)$$

is the smallest distance from x to B . We are now able to state the following proposition:

What happens if one of the sets involved is empty?

Proposition 20. *Assume that Σ is a C^1 manifold and that u is tangency regular. Then we have*

$$\lim_{\varepsilon} \Sigma^0(u^\varepsilon) = \Sigma^0(u) \quad (0.20)$$

in the topology induced by the Hausdorff metric.

Proof. Set $E = \mathbb{R}^d$. For every $x \in \Sigma^0(u)$ there exists a neighbourhood $U \subseteq \Sigma$ of x and charts $\phi: U \rightarrow V \subseteq \mathbb{R}^d - 1$ and $\psi: N^*U \rightarrow V \times W \subseteq \mathbb{R}^{d-1} \times \mathbb{R}$ such that ϕ is a diffeomorphism and ψ a vector space isomorphism. Define f via the following diagram:

$$\begin{array}{ccccc} U \times E \subseteq \Sigma \times E & \xrightarrow{u^2(\cdot)} & T^*U & \xrightarrow{\pi^N} & N^*U \\ \downarrow \phi & & & & \downarrow \pi_W \circ \psi \\ V \times E \subseteq \mathbb{R}^{d-1} \times E & \xrightarrow{f} & & & W \subseteq \mathbb{R} \end{array}$$

Since u is transverse to the zero section of N^*U the function f is of full rank around the point $(x, 0)$ in the x variable. By the implicit function theorem or constant rank theorem there exists a coordinate permutation and a differentiable function $g: \pi_{\mathbb{R}^{d-2}}V \times E \rightarrow \mathbb{R}$ on V such that

$$\{(y, g(y, \varepsilon), \varepsilon) : (y, \varepsilon) \in \pi_{\mathbb{R}^{d-2}}V \times E\} \quad (0.21)$$

is precisely the set around $(x, 0)$ on which f vanishes. Here the sets U , E and V potentially shrink and we assume that the coordinate permutation was incorporated into the mapping ϕ . Since g is continuous it follows at least locally that the set $\Sigma^0(u^\varepsilon)$ depends continuously on the parameter ε . This shows that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \Sigma^0(u)} \text{dist}(x, \Sigma^0(u^\varepsilon)) = 0. \quad (0.22)$$

It remains to be shown that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \Sigma^0(u^\varepsilon)} \text{dist}(x, \Sigma^0(u)) = 0. \quad (0.23)$$

Assume not. Then there exists sequences ε_k and $x_k \in \Sigma^0(u^{\varepsilon_k})$ and a $\delta > 0$ such that

$$\text{dist}(x_k, \Sigma^0(u)) \geq \delta. \quad (0.24)$$

By compactness of Σ we can assume that $x_k \rightarrow x$ in Σ after taking a subsequence. But now by continuity of π^N we obtain

$$0 = \lim_k u^{N, \varepsilon_k}(x_k) = \pi^N \circ \lim_k (u(x_k) + \varepsilon_k) = \pi^N \circ u \left(\lim_k x_k \right) = u^{N, 0}(x) \quad (0.25)$$

and thus $x \in \Sigma^0(u)$. But $x_k \rightarrow x$ is a contradiction to $\text{dist}(x_k, x) \geq \delta$.

Now equations (0.22) and (0.23) imply that

$$\lim_{\varepsilon} d_H(\Sigma^0(u^{\varepsilon}), \Sigma^0(u)) = 0 \quad (0.26)$$

□

From the previous proof we obtain the corollary

Corollary 21. *Assume that Σ is a C^1 manifold and that u is tangency regular. Then there exists a $\delta > 0$, such that for every $\varepsilon \in B_{\delta}$ the set $\Sigma^+(u^{\varepsilon})$ is homotopic to $\Sigma^+(u)$.*

Proof. Since Σ^0 is compact we obtain as in the proof above finitely points $x \in \Sigma(u)$, open sets $U = U_x$, $g = g_x$, $\phi = \phi_x$ etc. as in the proof above such that the U_x cover $\Sigma^0(u)$. Choose η sufficiently small such that the set

$$(\Sigma^0(u))_{\eta} = \{y \in \Sigma: \text{dist}(y, \Sigma^0(u)) < \delta\} \subseteq \bigcup_x U_x \quad (0.27)$$

is contained in the union of the U_x . By the previous proposition we can choose δ such that

$$d_H(\Sigma^0(u^{\varepsilon}), \Sigma^0(u)) < \eta \quad (0.28)$$

for all $\varepsilon \in B_{\delta}$. Now fix $\varepsilon \in E$. We now construct the homotopy

$$G: [0, 1] \times \Sigma^+(u) \rightarrow \Sigma^+(u^{\varepsilon}) \quad (0.29)$$

as $G(t, \cdot) = \text{Id}$ is the identity outside of $(\Sigma^0(u))_{\eta}$

complete construction

□

Some general remarks

We make the following remarks

Proposition 22. *Let u be non-degenerate. Then the number of stagnation points is finite.*

Proof. Let x be a non-degenerate stagnation point. Since $Du(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq \Omega$ of x on which u is bijective. Hence x is the only stagnation point in U_x . Let C denote the set of all stagnation points of u . Then the sets U_x together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \quad (0.30)$$

form an open cover of $\overline{\Omega}$. But $\overline{\Omega}$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is in the finite subcover. Thus it follows that $\#C < \infty$ is finite. \square

As a consequence we obtain the following.

Corollary 23. *For a non-degenerate u the type numbers M_0, \dots, M_d and the boundary type numbers μ_0, \dots, μ_{d-1} are finite.*

State the theorem of Sard

We state Morse's lemma according to [4, p.145]

Lemma 24. *Let $f: X \rightarrow \mathbb{R}$ be C^{2+r} and x be a non-degenerate critical point of index k . Then there exists a C^r chart (ϕ, U) at x such that we have*

$$f \circ \phi^{-1}(y) = f(x) - \sum_{j=1}^k y_j^2 + \sum_{j=k+1}^d y_j^2.$$

State proof.

Bring order into this section.

This was stated somewhere in Morse1969. Also, what is with the boundary stagnation points

Betti numbers

Let $H_k(X; \mathbb{R})$ denote the k -th homology space of X . For an introduction and definition of these we refer the reader to [7, Chapter 2]. We define the k -th Betti number as the dimension

$$b_k = \dim_{\mathbb{R}} H_k(X; \mathbb{R}). \quad (0.31)$$

We proceed to give examples for Betti numbers of selected connected domains in \mathbb{R}^d .

Example 25 (In flatland). In $d = 2$ dimensions the 0-th Betti number counts the number of connected components of Ω and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in \mathbb{R}^2 . More concretely we give the Betti numbers for selected domains in table 0.1.

Example 26 (In spaceland). In $d = 3$ dimensions the 0-th Betti number counts the number of connected components of Ω , the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 0.2.

Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.

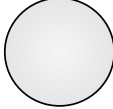
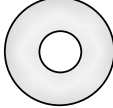

Domain	Picture	b_0	b_1	$b_k, k \geq 2$
Disk D		1	0	0
Annulus $2D \setminus D$		1	1	0
Two holed button		1	2	0

Table 0.1: Betti numbers for selected domains in \mathbb{R}^2 .

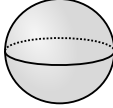
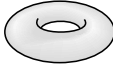
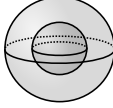
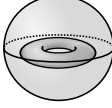
Domain	Picture	b_0	b_1	b_2	$b_k, k \geq 3$
Ball B		1	0	0	0
Solid torus $S^1 \times D$		1	1	0	0
Ball with bubble $2B \setminus B$		1	0	1	0
Ball with bubble in shape of torus		1	1	1	0

Table 0.2: Betti numbers for selected domains in \mathbb{R}^3 .

The Morse inequalities

We state the Morse inequalities.

More citations.

Theorem 27 (Strong Morse inequalities). *Let X be a manifold with corners and $f: X \rightarrow \mathbb{R}$ be essentially Morse. Then we have for $k \in \{0, \dots, d\}$ the inequalities*

$$\sum_{j=0}^k (-1)^{j+k} \text{Ind}_j(f) \geq \sum_{j=0}^k (-1)^{k+j} b_j(X).$$

For $k = d$ we in fact have equality

$$\sum_{j=0}^d (-1)^j \text{Ind}_j(f) = \chi(X)$$

where the Euler characteristic

$$\chi(X) = \sum_{j=0}^d (-1)^j b_j(X)$$

is the alternating sum of the Betti numbers.

Proof. See [5, Theorem 10.2].

□

Corollary 28 (Weak Morse inequalities). *Let X be a manifold with corners and $f: X \rightarrow \mathbb{R}$ be essentially Morse. Then we have for $k \in \{0, \dots, d\}$ the inequalities*

$$\text{Ind}_k(f) \geq b_k(X).$$

Proof.

Write some proof.

□

Give outline of proof idea. The citation for this version is no longer up to date.

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [5, Corollary 10.1].

Corollary 29 (Morse inequalities for f harmonic, $d = 2$). *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M + \mu_1 - \mu_0 &= b_1 - b_0. \end{aligned}$$

In dimensions $d = 3$ we obtain [5, Corollary 10.2]

Corollary 30 (Morse inequalities for f harmonic, $d = 3$). *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0 \\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= b_2 - b_1 + b_0. \end{aligned}$$

Give a classical example of a Morse function to determine the Betti numbers.

Give an outline of the proof.

On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2, 3\}$.

Proposition 31 (Harmonic vector fields on simply connected domains). *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. $u = \nabla f$ is the gradient field of some function $f: \Omega \rightarrow \mathbb{R}$.
2. f is harmonic.
3. u is in fact C^∞ .
4. The components $u_i = \partial_i f$ are harmonic.

Proof. 1. Since $\text{curl } u = 0$ this is a direct consequence of Stokes theorem.

2. This follows from $\Delta f = \text{Div } u = 0$.
3. This follows from the fact that f is harmonic
4. This follows from $u_i = \partial_i f$.

□

If one considers not necessarily simply connected domains Ω then we obtain the previous properties at least locally.

Harmonic functions, $d = 2$

The following result is essentially a negative to question 1 in $d = 2$ dimensions.

Proposition 32. *Let Ω be homeomorphic to $B_1 \subseteq \mathbb{R}^2$. Let further $f: \overline{\Omega} \rightarrow \mathbb{R}$ be regular harmonic with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \Sigma$ is not connected.*

We shall give two different proofs of this result. One involving level-sets and the other involving invariant manifolds

A proof involving level-sets

Sketch of Proof. Let $y_c = f(x_1)$ and x_1, \dots, x_M be all the critical points such that $f(x_i) = y_c$. We claim that the level set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

can be represented by a multigraph G which divides the boundary Σ into 4 components. To show this let $\gamma_i: (a_i, b_i) \rightarrow C$ for $i \in \{1, \dots, 4\}$ parametrise the curves in C intersecting at x_1 . These can be constructed with the initial value problem

$$\begin{aligned} \gamma' &= (\nabla f)^\perp|_\gamma \\ \gamma(0) &= \gamma_0 \end{aligned}$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1 . We assume that the intervals on which the γ_i are defined are maximal. We thus have for

$$\begin{aligned} \gamma_i^- &= \lim_{t \rightarrow a_i} \gamma(t) \\ \gamma_i^+ &= \lim_{t \rightarrow b_i} \gamma(t) \end{aligned}$$

that $\gamma_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $\nabla f^\perp = 0$. This argument can be applied to all of the x_1, \dots, x_M . We therefore have a situation similar to the one depicted in figure 0.2.

Thus C can be represented by a multigraph G with vertices v_1, \dots, v_K and edges $e_1, \dots, e_L \subseteq C$. In the following we identify the graph G with its planar embedding in $\overline{\Omega}$. Assume G contains a cycle with vertex sequence v_{i_1}, \dots, v_{i_j} and edges e_{i_1}, \dots, e_{i_j} . Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E . By the maximum principle $f = y_c$ on E and thus $f = y_c$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with the boundary Σ is at least four and thus the boundary Σ is divided into at least four components.

Now choose four neighbouring components $\omega_1, \dots, \omega_4$ as depicted in figure 0.3 Let $A \subseteq \Omega$ be

write
omega-
limit.

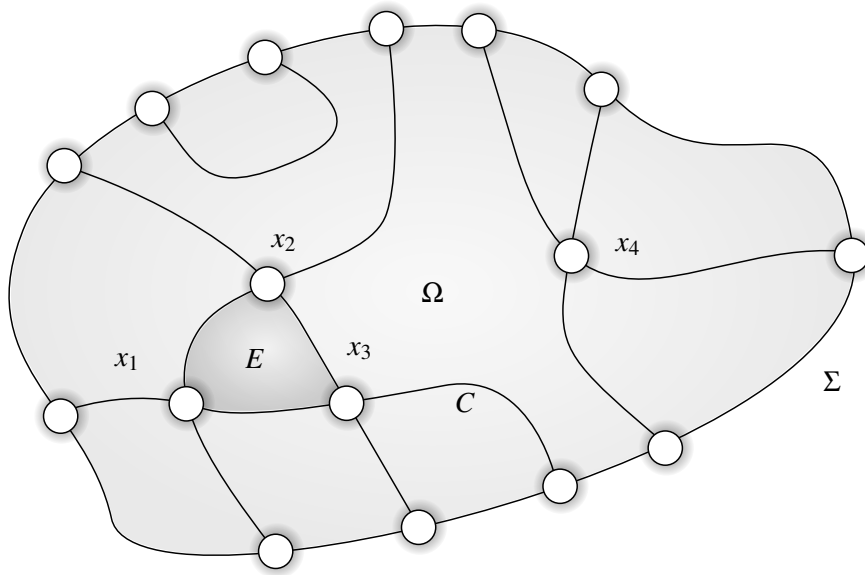


Figure 0.2: The situation at hand: The edges represent level curves and the interior vertices critical points.

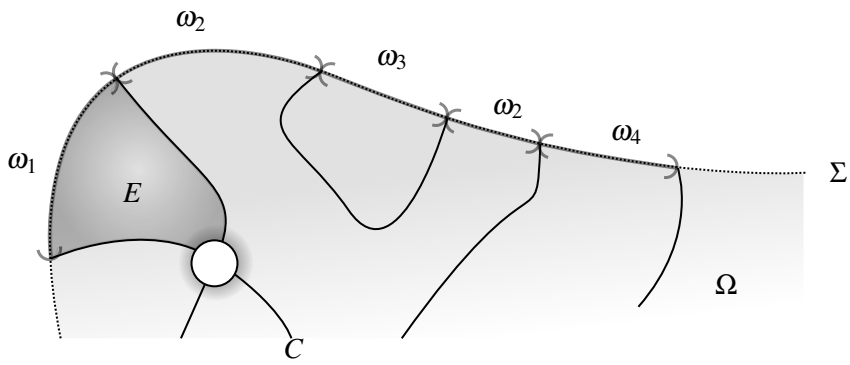


Figure 0.3: The choice of $\omega_1, \dots, \omega_4$.

the domain bounded by ω_1 and C as in the figure. The maximum principle yields that ω_1 contains a local maximum or minimum of f since $f = y_c$ is constant on the other boundaries $\partial A \setminus \omega_1$. By the same argument $\omega_2, \dots, \omega_4$ also contain local extrema. Since the $\partial\omega_i$ cannot be extremal points on Σ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. \square

use argument with ∇f here to show that extrema can be assumed to be alternating.

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let x_1, \dots, x_M denote the critical points of f . Let $\lambda_i: (a_i, b_i) \rightarrow \overline{\Omega}$ for $i \in \{1, 2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i: (a_i, b_i) \rightarrow \overline{\Omega}$ for $i \in \{3, 4\}$ be chosen to parametrise the stable manifolds of x_1 . As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\begin{aligned}\lambda_i^- &= \lim_{t \rightarrow a_i} \lambda(t) \\ \lambda_i^+ &= \lim_{t \rightarrow b_i} \lambda(t)\end{aligned}$$

that $\lambda_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices v_1, \dots, v_K and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Once again we identify the graph with its planar embedding. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle A with vertices x_{i_1}, \dots, x_{i_j} and edges e_{i_1}, \dots, e_{i_j} . The set of cycles forms a partial ordering with respect to the property ‘contains another cycle’. We can assume that our chosen cycle A contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to one another. We also note that the edges cannot cross. We can thus describe the trail x_{i_1}, \dots, x_{i_j} by a set of directives of the type

$$(d_1, \dots, d_K) \in \{l, r, s\}^J.$$

Here l , r and s stand for ‘left’, ‘right’ and ‘straight’ respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 0.4.

An example of the trail ‘srsr’ is given in figure 0.5. We now note that cycles of the type r, \dots, r or l, \dots, l cannot occur as we otherwise would have a directed cycle. Thus there exists a vertex where the chosen direction is s . Without loss of generality this vertex is x_{i_1} . Since we can swap f

More precise.

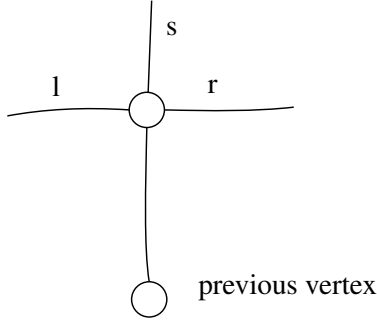


Figure 0.4: Explanation of the directives 'l', 'r' and 'r'.

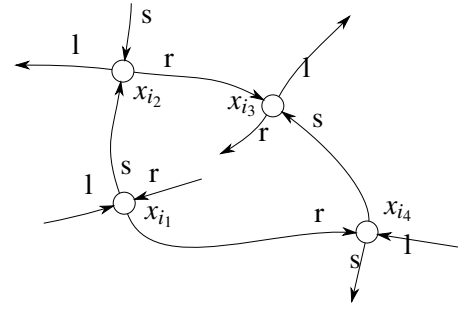


Figure 0.5: An example for a cycle.

with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1} . Now consider new cycle B starting at x_{i_1} with directives r, \dots, r . Since all vertices of B lie within the cycle A we must at some step reach a vertex on the cycle A. But then cycle B is a new distinct cycle contained in cycle A, a contradiction to the minimality of A. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

Now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur. We now pick a tree \tilde{G} out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each 'neighbouring' leaf of this tree has opposite signage and the claim follows. \square

elaborate

elaborate

A proof involving Morse theory

We now give a proof involving Morse theory since this generalises to the three dimensional case.

Proposition 33. *Let $d = 2$ and X be simply connected such that Σ is a differentiable manifold and let $f: X \rightarrow \mathbb{R}$ be tangency regular. Assume further that Σ^- and Σ^+ are nonempty and simply connected. Then f has no interior critical point.*

fix the following proof

Proof. Let $x_1, x_2 \in \Sigma^0(f)$ be two points on different connectivity components which we will fix later. Then we can cut the domain along a curve Γ such that the endpoints of the cut coincide with x_1 and x_2 , that is $\partial\Gamma = \{x_1, x_2\}$. Now we obtain two new domains X^+ and X^- such that $\partial X^+ \subseteq \Sigma^+ \cup \Sigma^0 \cup \Gamma$ and $\partial X^- \subseteq \Sigma^- \cup \Sigma^0 \cup \Gamma$. We can assume that Γ is a smooth manifold and corresponds to the stratum X_Γ for X^+ and X^- . We also assume that the corner points $x_1, x_2 \in \partial\Gamma$ correspond to the strata X_1 and X_2 . Locally around the corner point x_1 we have a situation depicted as in figure 0.6. In words, the vector field $u = \nabla f$ is essentially parallel to the boundary Σ in a neighbourhood of x_1 . We assume that we chose Γ such that the acute angle between u and Γ at x_1 is on the side where u flows into the new domain. Thus we have that x_1 is not an essential critical point of $-f$ on X^- . x_1 may or may not however be a critical point of f on X^+ . Analogously we can choose Γ in such a way around x_2 . For the following argumentation we require that u is strictly Morse on both X^+ and X^- , so assume for a moment that this is the case. We now focus

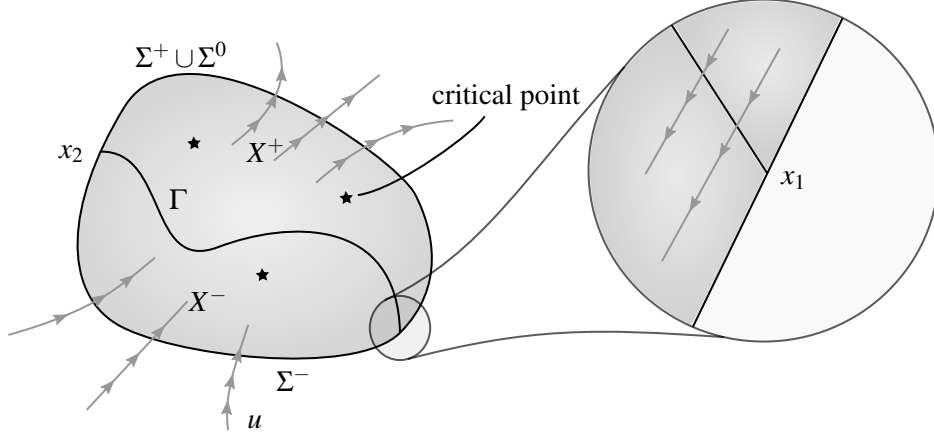


Figure 0.6: The situation at hand.

our attention on X^+ . Since no essential critical points lie on Σ^+ it follows for the boundary type numbers that

$$\mu_j^+ = \text{Ind}_{\Gamma,j}(f) + \delta_{j0} \text{Ind}_{\Sigma^0,j}(f) \quad (0.32)$$

where δ_{ij} denotes the Kronecker delta. Analogously we have on X^- that

$$v_j^- = \text{Ind}_{\Gamma,j}(-f) \quad (0.33)$$

where we took into account that $\text{Ind}_{\Sigma^0,j}(-f) = 0$. In addition we have on Γ that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^- , that is

$$\text{Ind}_{\Gamma,0}(f) = \text{Ind}_{\Gamma,1}(-f) \quad \text{and} \quad \text{Ind}_{\Gamma,1}(f) = \text{Ind}_{\Gamma,0}(-f) \quad (0.34)$$

Using equations (0.32), (0.33) and (0.34) we obtain

$$\mu_0^+ - \text{Ind}_{\Sigma^0,0}(f) = v_1^- \quad \text{and} \quad \mu_1^+ = v_0^- . \quad (0.35)$$

Consider the Morse inequality for f

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) . \quad (0.36)$$

and the Morse inequality for $-f$

$$M^- + v_1^- - v_0^- = -\chi(X^-) = -\chi(X) . \quad (0.37)$$

We now add equations (0.36) and (0.37) and insert relations (0.35) to obtain

$$M^- + M^+ - \text{Ind}_{\Sigma^0,0}(f) = -2\chi(X) = -2 .$$

Since $\text{Ind}_{\Sigma^0,0}(f) \leq 2$ and $M^\pm \geq 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

The statement now remains to be shown for u which are not Morse on Γ . Let $u^\varepsilon \rightarrow u$ for $\varepsilon \in E$ be a family of boundary generic functions with respect to the stratification $X^+ \cup X^-$ as in corollary 18. By proposition 20 the set $\Sigma^0(u^\varepsilon)$ converges to $\Sigma^0(u)$ as $\varepsilon \rightarrow 0$. Choose δ sufficiently small according to corollary 21 and now assume that $\varepsilon \in E_\delta = B_\delta \cap E$. Now take a sufficiently small $\eta > 0$ such that Γ has no critical points in $\overline{B_\eta(x_1)} \cup \overline{B_\eta(x_2)}$. We modify for every $\varepsilon \in E_\delta$ the cut Γ to yield a cut Γ_ε such that $\Gamma = \Gamma_\varepsilon$ outside of $B_\eta(x_1) \cup B_\eta(x_2)$ and such that Γ_ε intersects the boundary Σ precisely at $\Sigma^0(u^\varepsilon)$. Furthermore by choosing δ sufficiently small we can also ensure that Γ has no critical points of u^ε inside of $\overline{B_\eta(x_1)} \cup \overline{B_\eta(x_2)}$ and that Γ_ε is slanted in such a way that u_ε has no critical points on $\Gamma_\varepsilon \cap B_\eta(x_1) \cap B_\eta(x_2)$.

Now u^ε is by construction strictly Morse on Γ^ε . By the above argument the statement holds for u^ε with the modified cut Γ^ε . \square

Allowing for Inflow and outflow

The strategy in the above proofs can be generalised to show the following

Conjecture 34. *Let $X \subseteq \mathbb{R}^2$ be a manifold with corners with Betti numbers $b_0 = 1$ and b_1 . Let further $f: X \rightarrow \mathbb{R}$ be Morse harmonic with M critical points. Assume that $\overline{\Sigma^-} \subseteq \Sigma$ on a given connected component of the boundary Σ consists of at most one connected component. Then we have*

$$\frac{4}{3}M \leq b_1 + 1.$$

This inequality can probably be improved considerably.

Let J^\pm denote the number of connected components of Σ^\pm . Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\geq 0} \sqcup \Sigma_{\leq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma^{\geq 0}$ of all such decompositions. We state a consequence of a result from [8, Theorem 2.1]

Proposition 35. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u: \overline{\Omega} \rightarrow \mathbb{R}$ be harmonic (with certain conditions on the boundary). Then we have*

$$M \leq b_1 - b_0 + \frac{J^+ + J^-}{2}.$$

If in addition we assume that there are no critical points on the boundary then we have

$$M \leq b_1 - b_0 + J^{\geq 0}.$$

Proof. See [8, Theorem 2.1]. \square

Harmonic vector fields, $d = 2$

No inflow or outflow

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ iff $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* iff $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can state the following result.

Proposition 36 (Upper bound on M). *Let $d = 2$ and Ω be a compact manifold with corners with Betti numbers $b_0 = 1$, and b_1 . Let further $u: X \rightarrow \mathbb{R}^2$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$M + 1 \leq b_1.$$

Sketch of proof. As in the second proof of proposition 32 the critical manifolds form a directed multigraph. Since no critical manifold can intersect with the boundary each vertex of the graph has degree 4 and we thus have $2M$ edges. Now we obtain with Euler's polyhedron formula for a planar graph with multiple components

$$\begin{aligned} \# \text{ minimal cycles} &= \# \text{ faces} - 1 \\ &= 1 + \# \text{ components} - \# \text{ vertices} + \# \text{ edges} - 1 \\ &\geq 1 + 1 - M + 2M - 1 = M + 1 \end{aligned}$$

Here we use the term 'minimal' as in the second proof of proposition 32. Note that each minimal cycle must contain a hole of the domain since else we could restrict u to a simply connected region containing this cycle. Then by proposition 31 u would correspond to the gradient of a harmonic function in this region and we would obtain a contradiction as in the proof of proposition 32. Hence the number of minimal cycles is a lower bound on the number of holes b_1 of the domain. \square

In fact using the Morse inequalities we can obtain the stronger result.

Proposition 37. *Let $X \subset \mathbb{R}^2$ be a compact manifold with corners and Betti numbers $b_0 = 1$, and b_1 and let $u: X \rightarrow \mathbb{R}^2$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$M + 1 = b_1$$

.

Sketch of proof. We slit Ω such that it is homeomorphic to the disk as is depicted in figure 0.7. Denote the slit by Γ . Since the number of critical points is finite by proposition ??, we can choose Γ in such a way that it does not contain any critical points. We also denote the points at which Γ meets Σ by $x_1, \dots, x_{2b_1} \in \partial\Gamma$. Note that there are $2b_1$ many such points. We can assume that Γ is a smooth manifold. Now at the point x_1 we have that u is almost parallel to the boundary Σ . Thus we can slant the cut in such a way such that x_1 is an essential critical point of index 0 of u on the stratification of \tilde{X} . Here \tilde{X} denotes the covering space of X generated by the cut Γ . We denote the induced strata by Γ also with Γ . Note that x_1 then is no essential critical point for $-u$. We modify the cut for the other points x_2, \dots, x_{2b_1} as with x_1 . The situation is depicted in figure 0.7. Since

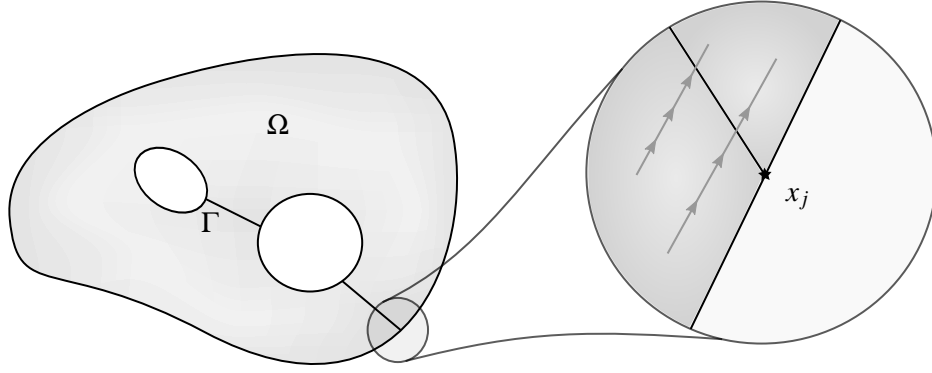


Figure 0.7: How we slit the domain.

there are no critical points on Σ all boundary critical points of u are on the strata induced by Γ and x_1, \dots, x_{2b_1} . Hence we have relations

$$\mu_k = \text{Ind}_{\Gamma,k}(u) + 2b_1 \delta_{k0} \quad \text{and} \quad \nu_k = \text{Ind}_{\Gamma,k}(-u) \quad (0.38)$$

for all $k \in \{0, 1\}$. Since on Γ all entrant critical points of u are also emergent critical points of $-u$ (and vice versa) we have the relations

$$\text{Ind}_{\Gamma,0}(u) = \text{Ind}_{\Gamma,1}(-u) \quad \text{and} \quad \text{Ind}_{\Gamma,1}(u) = \text{Ind}_{\Gamma,0}(-u). \quad (0.39)$$

Equations (0.38) and (0.39) yield

$$\mu_0 = \nu_1 + 2b_1 \quad \text{and} \quad \mu_1 = \nu_0. \quad (0.40)$$

Since Ω is now simply connected u is by proposition 31 the gradient of a harmonic function f on this new domain. For this f we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \quad (0.41)$$

and for $-f$ the Morse inequalities

$$M + \nu_1 - \nu_0 = -\chi(\tilde{X}) = -1. \quad (0.42)$$

Adding equations (0.41) and (0.42) and using the relation (0.40) we obtain

$$2M - 2b_1 = -2$$

from which the claim follows. □

We now give an alternative proof using the argument principle.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition ?? u is the gradient of a harmonic function f on this new domain. Let $h \in \text{Hol}(\mathbb{C})$ be the holomorphic function given by $h = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain

One could use the argument principle for Riemann surfaces.

lies to the left of γ . We now determine the change of argument $\arg h$ along γ . For this consider first the parts of γ traversing the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other the contribution in the change of $\arg h$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h$ is 2π for every hole in the domain since $h = u$ is tangent to Σ and traverses the holes clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have b_1 holes in the domain the total change of $\arg h$ as γ traverses Σ is $2\pi(b_1 - 1)$. Since h has no poles it follows from the argument principle (see for example [9, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_{\gamma} d\arg(h(z)) = 2\pi M \quad (0.43)$$

From this the claim follows. \square

In the following we would like to give examples for harmonic vector fields. In order to do this we define two differential operators for $d = 2$ by

$$\nabla^{\perp} f = \text{Curl } f = \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix}$$

and

$$\text{curl } u = -\partial_1 u_2 + \partial_2 u_1$$

Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

The following proposition gives us a recipe to generate harmonic vector fields in $d = 2$ dimensions.

Proposition 38. *Let $\psi: \Omega \rightarrow \mathbb{R}$ be harmonic then $\nabla^{\perp} \psi$ is a harmonic vector field.*

Proof. Since $\text{Div } \nabla^{\perp} \psi = 0$ we have

$$\text{Div } u = \text{Div } \nabla^{\perp} \psi = 0$$

and one calculates

$$\text{curl } u = \text{curl } \nabla^{\perp} \psi = -\Delta \psi = 0.$$

\square

The function ψ is also called a stream function.

We now give an example of a harmonic vector field without inflow or outflow and with one critical point. For this consider the stream function

$$\begin{aligned} \psi: \mathbb{R}^2 \setminus \{-e_1, e_1\} &\rightarrow \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + \Phi_2(x + e_1) \end{aligned} \quad (0.44)$$

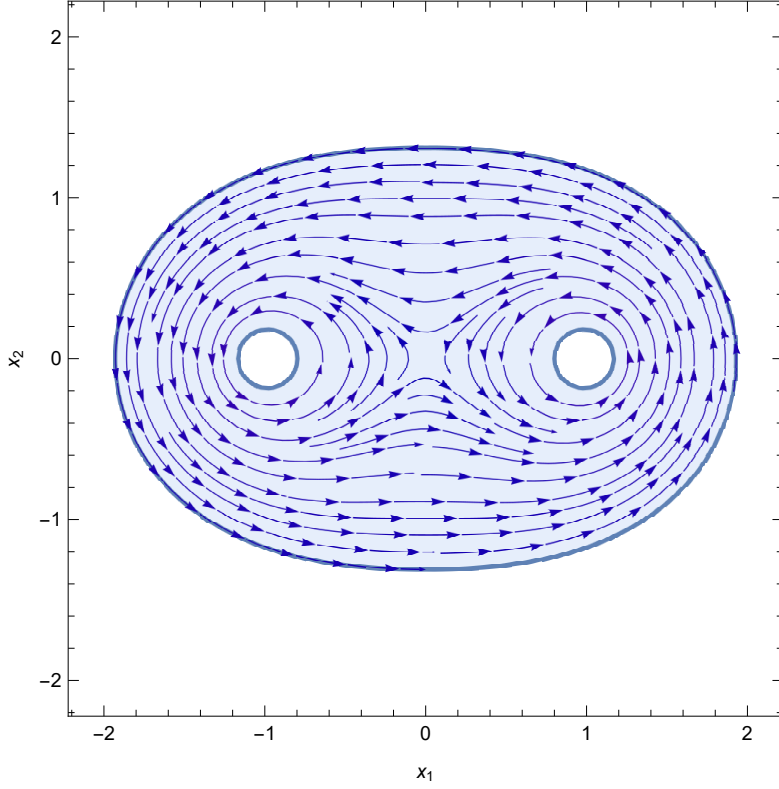


Figure 0.8: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-1, 1])$. Here ψ is given by equation (0.44).

where

$$\Phi_2 = \log(|\cdot|)$$

is a multiple of the fundamental solution of the Laplace equation on \mathbb{R}^2 and $e_i = \delta_i$ are the unit vectors. Figure 0.8 indicates that $u = \nabla^\perp \psi$ has the desired properties.

In a second example given by [10] we fix the domain rather than the function. For this set $\overline{\Omega} = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We then have the system

$$\begin{aligned} \Delta \psi &= 0 \quad , \text{ on } \Omega \\ \psi &= 0 \quad , \text{ on the outer ring } 4S^1 \\ \psi &= 1 \quad , \text{ on the inner rings } S^1(-2e_1) \cup S^1(2e_1) \end{aligned} \tag{0.45}$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 0.9.

An example of inflow on one side and outflow on the other

In the following we aim to give examples of domains in $d = 2$ dimensions for which we have inflow on one simply connected boundary component and outflow on another simply connected

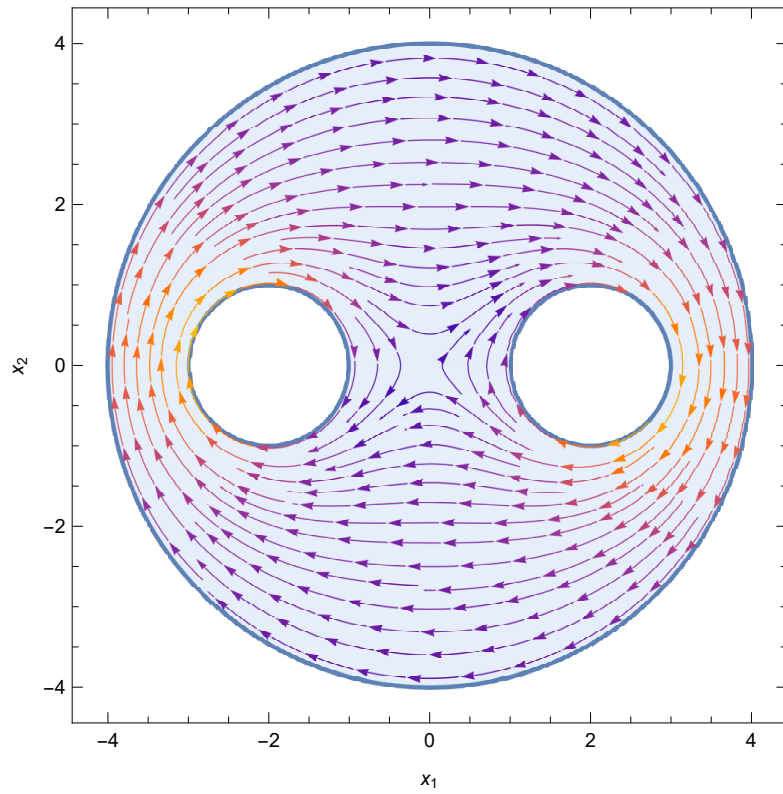


Figure 0.9: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (0.45).

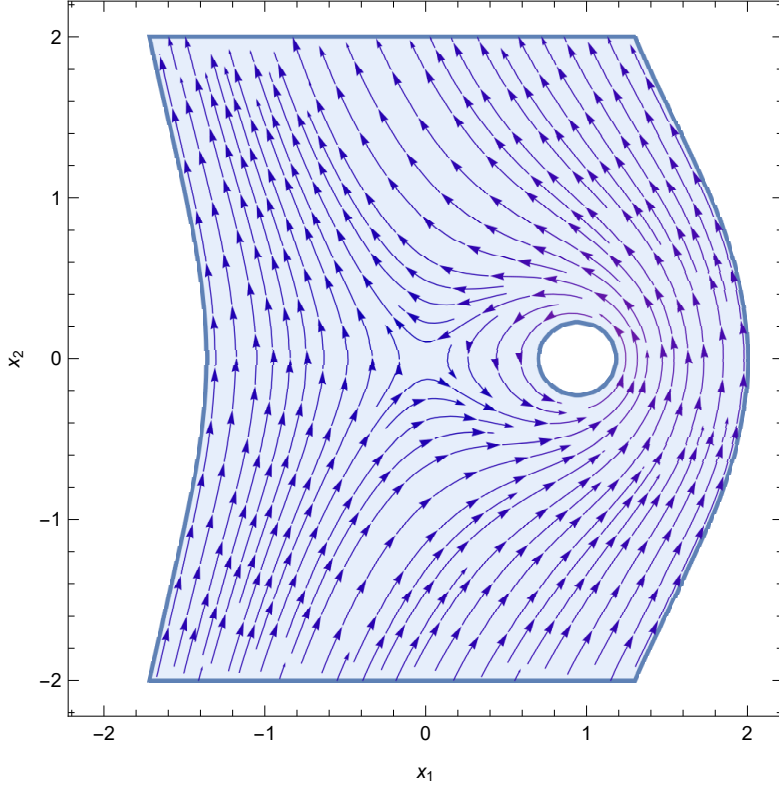


Figure 0.10: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.5, 2]) \cap \mathbb{R} \times [-2, 2]$. Here ψ is given by equation (0.46).

boundary component. For this consider first the stream function

$$\begin{aligned} \psi: \mathbb{R}^2 \setminus \{-e_1, e_1\} &\rightarrow \mathbb{R}^2 \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \quad (0.46)$$

Figure 0.10 indicates that $u = \nabla^\perp \psi$ fulfils the requirements.

Now we would like to have a harmonic vector field similar to the example with two holes with inflow on the one side and outflow on the other. For this consider the streamline

$$\begin{aligned} u: \mathbb{R}^2 \setminus \{-e_1, e_1\} &\rightarrow \mathbb{R}^2 \\ x &\mapsto \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1 \end{aligned} \quad (0.47)$$

Figure 0.11 indicates that $u = \nabla^\perp \psi$ is the function we are looking for.

In another example given by [10] we once again fix the domain rather than the function. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We now have the system

$$\begin{aligned} \Delta \psi &= 0 && , \text{ on } \Omega \\ \psi &= 0 && , \text{ on the outer ring } 4S^1 \\ \psi &= -1 && , \text{ on the left inner ring } S^1(-2e_1) \\ \psi &= 1 && , \text{ on the right inner ring } S^1(2e_1) \end{aligned} \quad (0.48)$$

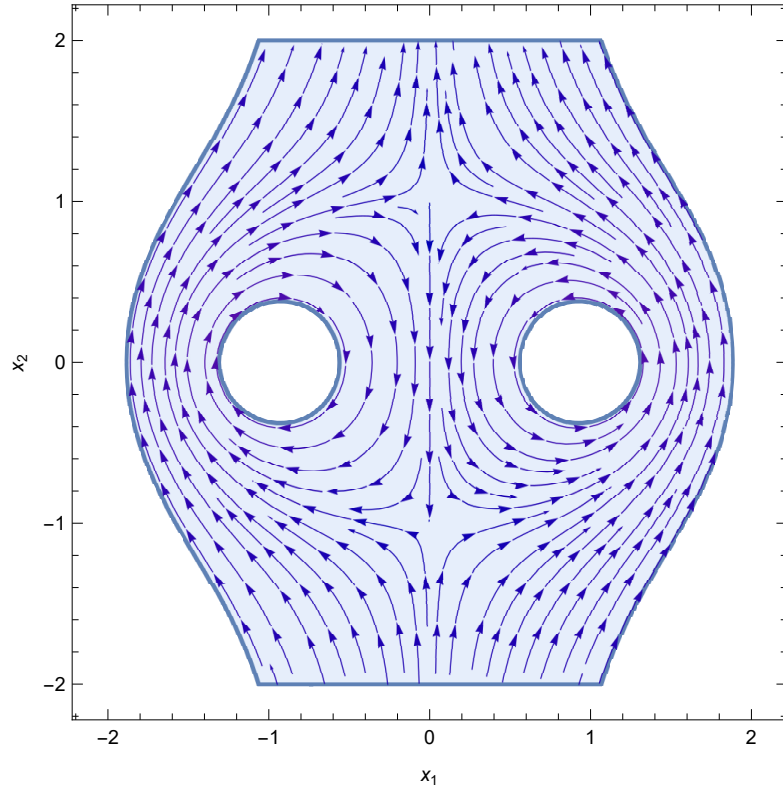


Figure 0.11: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap \mathbb{R} \times [-2, 2]$. Here ψ is given by equation (0.47).

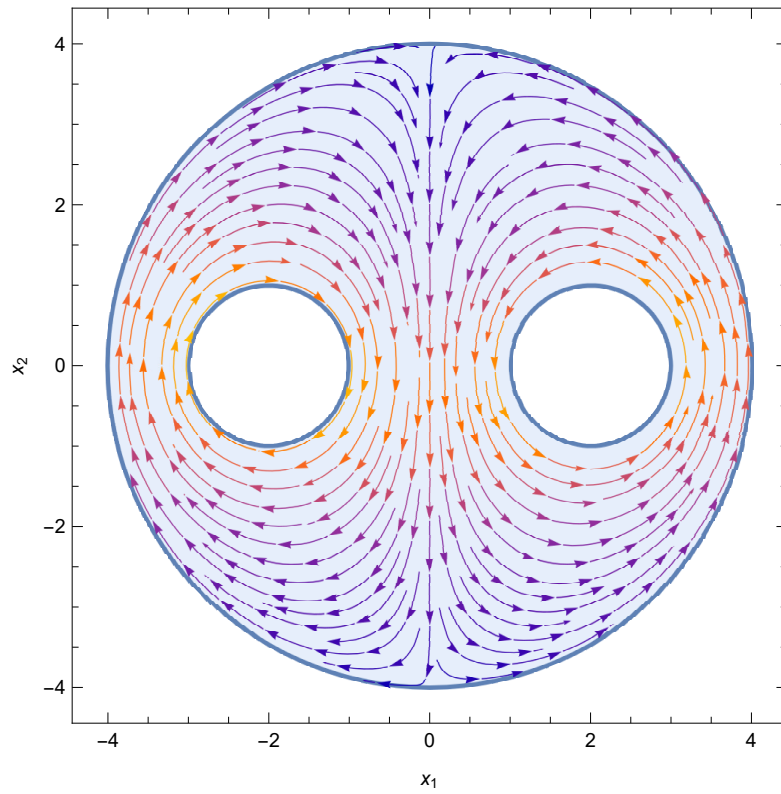


Figure 0.12: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (0.48).

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 0.12.

Check the signs of this example. Give explanation for why it works.

Harmonic functions, $d = 3$

The cylinder

The following proof comes from [10]

Proposition 39. *Let $\Omega = (0, 1) \times B_1 \subseteq \mathbb{R}^3$ be the cylinder. Let further $f: \overline{\Omega} \rightarrow \mathbb{R}$ be regular harmonic with no inflow or outflow on the sides $\partial(0, 1) \times B_1$, no outflow on $\{0\} \times B_1$ and no inflow on $\{1\} \times B_1$. Then f cannot have a critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ . Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0, 1) \times S^1$ such that $\partial_1 f(x)$ is minimal on $\overline{\Omega}$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0,$$

a contradiction. □

Harmonic vector fields, $d = 3$

We obtain as a quick consequence of the hairy ball theorem

Proposition 40. *Let Ω have Betti numbers b_0 , b_1 and b_2 . Let $u: X \rightarrow \mathbb{R}$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$b_2 \leq b_1.$$

Proof. Assume not. Since Ω has b_2 bubbles and b_1 holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no inflow or outflow on Γ we have that the restriction $u|_{\Gamma} \in T\Gamma$ is a vector field on Γ . Since u is regular $u|_{\Gamma}$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. \square

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 41. *Let $X \subset \mathbb{R}^3$ be a compact manifold with corners homeomorphic to the ball B . Let $f: X \rightarrow \mathbb{R}$ be a Morse harmonic function. Assume that Σ^- is simply connected. Then we have that*

$$M_1 = M_2$$

Proof. As in the two dimensional case we split the domain Ω with a plane Γ such that $\partial\Gamma = \gamma = \partial\Sigma^-$. Denote the two arising domains X^+ and X^- where $\partial X^+ = \Sigma^+ \cup \Sigma^0 \cup \bar{\Gamma}$ and $\partial X^- = \Sigma^- \cup \bar{\Gamma}$. We can assume that Γ is a smooth manifold. Since by proposition ?? there are finitely many critical points in Ω we can also assume that no interior critical points lie on Ω . We now look at a critical point x on Γ . We choose the slant at which Γ approaches γ in such a way that x is neither an essential critical point of f nor of $-f$. We now turn our attention to X^+ . Since no essential critical points lie on Σ^+ or γ it follows for the boundary type numbers that

$$\mu_j^+ = \text{Ind}_{j,\Gamma}(f). \quad (0.49)$$

Analogously we have on X^- that

$$\nu_j^- = \text{Ind}_{j,\Gamma}(-f). \quad (0.50)$$

In addition we have that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^- , that is

$$\begin{aligned} \text{Ind}_{0,\Gamma}(u) &= \text{Ind}_{2,\Gamma}(-f) \\ \text{Ind}_{1,\Gamma}(u) &= \text{Ind}_{1,\Gamma}(-f) \\ \text{Ind}_{2,\Gamma}(u) &= \text{Ind}_{0,\Gamma}(-f) \end{aligned} \quad (0.51)$$

Using equations (0.49), (0.50) and (0.51) we obtain

$$\begin{aligned} \mu_0^+ &= \nu_2^- \\ \mu_1^+ &= \nu_1^- \\ \mu_2^+ &= \nu_0^- \end{aligned} \quad (0.52)$$

A little more rigour would not harm.

Why are there finitely many?

More details here.

We observe the Morse inequalities for f

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X). \quad (0.53)$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \quad (0.54)$$

where the M_j continue to denote the interior type numbers of f . We now subtract equation (0.53) from (0.54) and insert relations (0.52) to obtain

$$M_1^- - M_2^- + M_1^+ - M_2^+ = 0$$

from which the claim follows. \square

In fact we can give an example for such a function with simply connected entrant boundary.

Example 42 (A harmonic vector field with simply connected entrant boundary). Consider the domain $X = \bar{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large and the harmonic function

$$\begin{aligned} f: X &\rightarrow \mathbb{R} \\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1x_2^2 + x_2x_3 \end{aligned}$$

This induces the harmonic vector field

$$\begin{aligned} u: X &\rightarrow \mathbb{R}^3 \\ x &\mapsto \begin{bmatrix} x_1(1-x_1) + x_2^2 \\ x_2(2x_1-1) + x_3 \\ x_2 \end{bmatrix} \end{aligned}$$

It follows from setting $u(x) = 0$ implies that $x_2 = 0$ and then that $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of u . Conversely these are zeroes of u .

Figure 0.13 indicates that f has the desired properties. Consider now the inverse stereographic projection given by

$$\psi: \mathbb{R}^2 \rightarrow B_1 \setminus \{e_3\} \quad (0.55)$$

$$x \mapsto \frac{1}{s^2+1} \begin{bmatrix} x \\ s^2-1 \end{bmatrix} \quad (0.56)$$

where $s = |x|$. We calculate

$$\frac{1}{r} n \cdot u(x) = x_1^2(1-x_1) + x_2^2(3x_1-1) + 2x_2x_3 \quad (0.57)$$

If we now precompose with the stereographic projection we obtain

$$\left(\frac{1}{r} n \cdot u \right) \circ \psi(x) \quad (0.58)$$

$$= \frac{1}{(s^2+1)^3} (x_1^2((s^2+1)-x_1) + x_2^2(3x_1-(s^2+1)) + 2x_2(s^2+1)(s^2-1)) \quad (0.59)$$

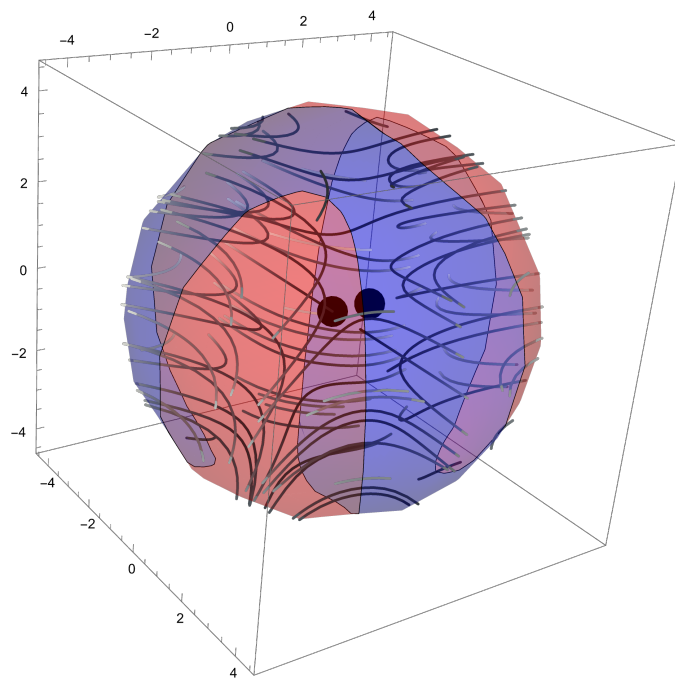


Figure 0.13: An plot of the function u

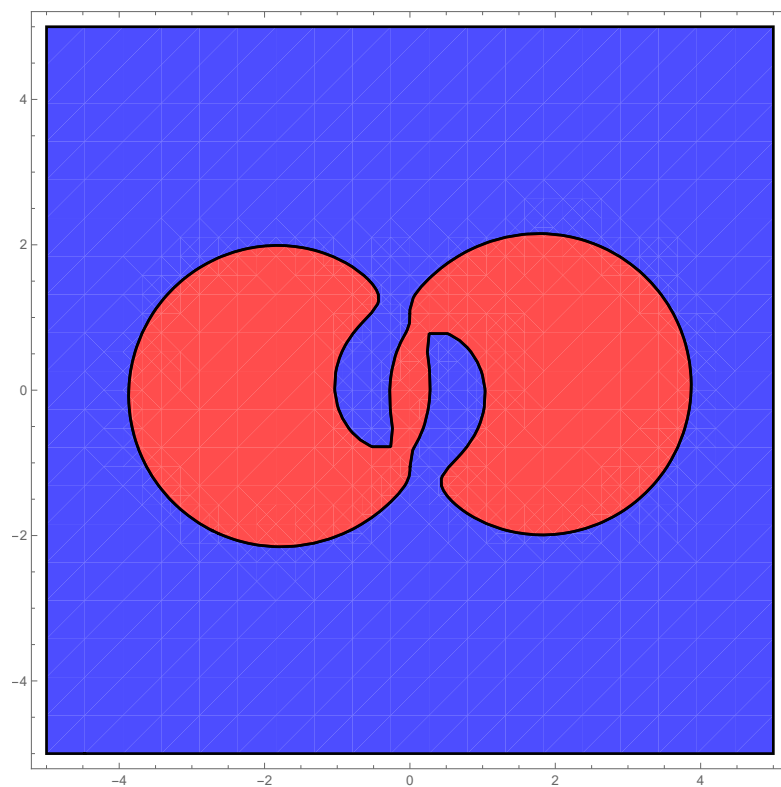


Figure 0.14: Stereographic projection of the surface Σ .

which vanishes precisely iff

$$0 = x_1^2((s^2 + 1) - x_1) + x_2^2(3x_1 - (s^2 + 1)) + 2x_2(s^2 + 1)(s^2 - 1) \quad (0.60)$$

$$= (s^2 + 1)(x_1^2 - x_2^2) - x_1^3 + 3x_1x_2^2 + 2x_2(s^4 - 1) \quad (0.61)$$

$$(0.62)$$

complete this section.

Try to say something about the case without inflow or outflow.

Harmonic functions, $d = 4$

Define the harmonic function

$$\begin{aligned} f: B_1 \subseteq \mathbb{R}^4 &\rightarrow \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2. \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in \mathbb{R}^4 with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{0.63}$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_{\pm}: B_1 \subseteq \mathbb{R}^3 &\rightarrow \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{0.64}$$

with inverses $\psi_{\pm} = (\phi_{\pm})^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^{\top}$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^{\top}.$$

Thus we have $x \in \Sigma^{\pm}$ iff

$$0 < \pm \nabla f \cdot n = \pm 2(x_1^2 + x_2^2 - x_3^2 - x_4^2)$$

Using condition (0.63) we obtain the equivalent condition

$$0 < \pm 1 - 2(x_3^2 + x_4^2)$$

Define the cylinder

$$C = \{\bar{x} \in \mathbb{R}^3: x_3^2 + x_4^2 < 1/2\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (0.64) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_{\pm}(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_{\pm}(\Sigma^+).$$

Analogously we have

$$B_1 \setminus C = \psi_{\pm}(\Sigma^-).$$

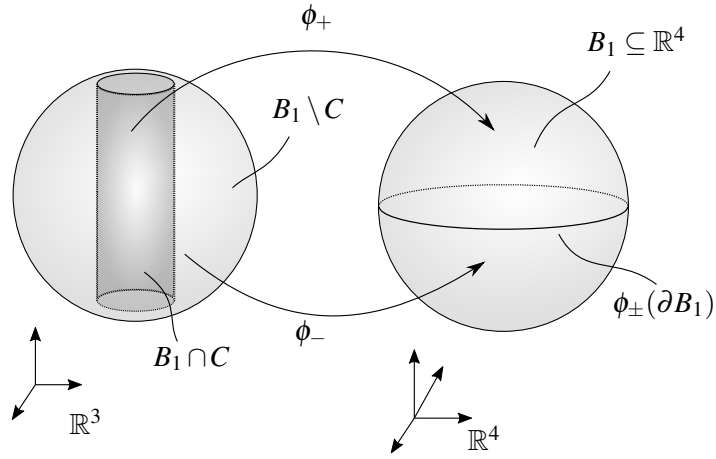


Figure 0.15: Visualisation of the situation.

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 0.15.

Check that the transition at the boundary is legal.

□

Symbols

d	Dimensions $d = 2$ or $d = 3$
Ω	Domain in \mathbb{R}^d , assumed to be $\text{int}(X)$
Σ	Boundary of Ω or X
$f: X \rightarrow \mathbb{R}$	A C^2 mapping, often assumed harmonic
$u: X \rightarrow \mathbb{R}^d$ or $T^*\overline{\Omega}$	A C^1 vector field, often assumed harmonic
X	A compact manifold with corners, assumed to be $X = \overline{\Omega}$
Y	A manifold
X_j	A stratification of X as given in definition 4. Often but not always assumed to be given by equation 0.1
u_j	Restriction of u to the cotangent bundle T^*X_j , see equation 0.3
Σ^-	entrant boundary, see definition 6
Σ^+	emergent boundary, see definition 6
Σ^0	tangential boundary, see definition 6
M_k	interior type numbers
M	Total number of stagnation points
μ_k	boundary type numbers of f , see equation (0.7)
ν_k	boundary type numbers of $-f$, see definition 8
u_ε	modification to u as in equation (0.12)
A	submanifold, can be thought of as the zero section of T^*X
b_k	Betti number as defined in equation (0.31)

Change Gamelin to Lang, complex analysis

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