Stagnation points of harmonic vector fields and the domain topology

Some applications of Morse theory

Theo Koppenhöfer

March 20, 2024

Why harmonic vector fields?

- gravitational field in classical mechanics
- steady state heat flow
- irrotational flow of an inviscid incompressible medium
- electrostatic field in vacuum
- magnetostatic field in vacuum

Why stagnation points?

$$\text{vector field topology} \xleftrightarrow{\text{Morse theory}} \text{stagnation points}$$

The following question is inspired by [1]:

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are simply connected?

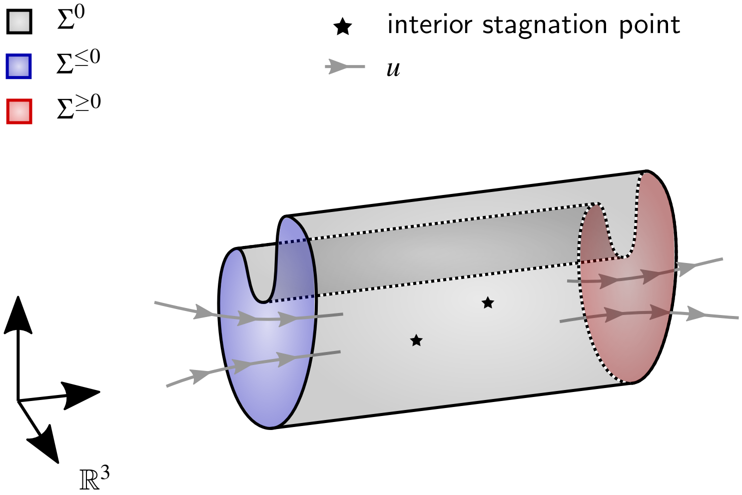

Figure: This kind of situation is not possible.

Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ on X such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are simply connected?

Answer

- $d = 2$ dimensions: Not possible (known).
- cylinders in $d = 3$ dimensions: Not possible (known).
- $d = 3$ dimensions: Number of stagnation points has to be even.
- $d = 4$ dimensions: Possible for $X = B_1$, $u = \nabla f$ with

$$f = x_1^2 + x_2^2 - x_3^2 - x_4^2 .$$

But if one allows for holes in $d = 2$ dimensions it becomes possible.

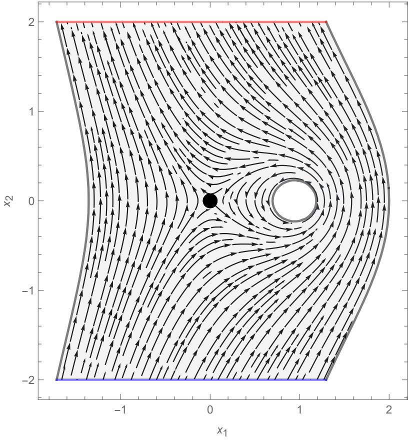

Figure: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5, 2]) \cap (\mathbb{R} \times [-2, 2])$. Here $\psi := \Phi_2(x - e_1) + x_1$.

For $d = 3$ dimensions we have for r sufficiently large the example $X = B_r$, $u = \nabla f$ with

$$f = \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_1 x_2^2}{2} + x_1 x_2^2 + x_2 x_3$$

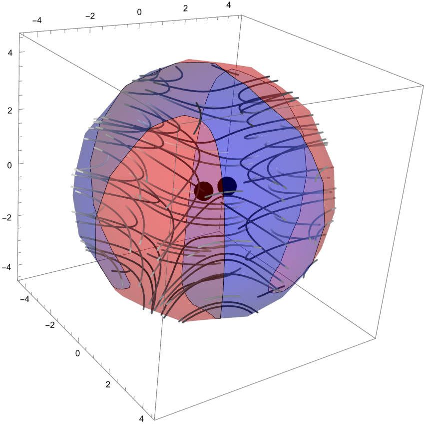

Figure: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

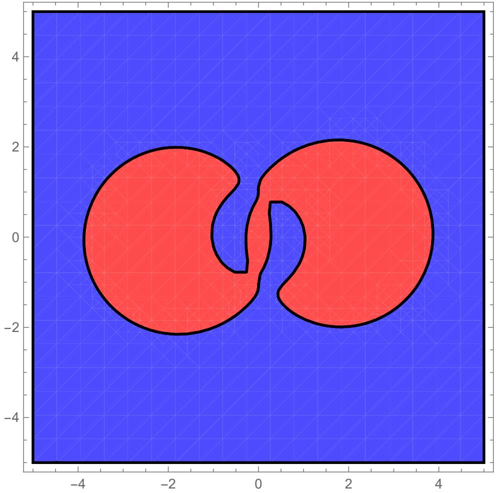

Figure: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

One can perturb this solution to show that there exists a harmonic vector field on B_r with interior stagnation point such that Σ^+ and Σ^- have positive distance from one another and are simply connected.

The following question is inspired by [2]:

Question (Harmonic vector fields without inflow or outflow)

Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

Definition (Interior stagnation points)

Let $X \subset \mathbb{R}^d$ be a d-dimensional compact manifold with corners and $u\colon X \to \mathbb{R}^d$ be a vector field without boundary stagnation points. We call an interior stagnation point x *non-degenerate* if the derivative $Du(x)$ is bijective. If all interior stagnation points are non-degenerate u is called *Morse*.

The following answers the question in $d = 2$ dimensions:

Proposition (Condition on the number of stagnation points, [4])

Let $X \subset \mathbb{R}^2$ be a compact connected planar manifold with corners and let $u\colon X \to \mathbb{R}^2$ be a Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where M denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of X.

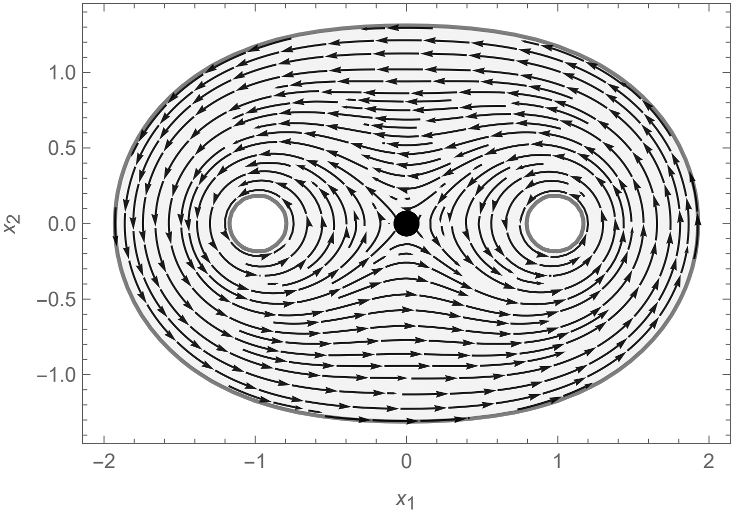

Figure: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here $\psi := \Phi_2(x - e_1) + \Phi_2(x + e_1)$.

Definition (Interior type number)

Let $u\colon X \to \mathbb{R}^d$ be a Morse vector field without boundary stagnation points. We say that a stagnation point x of u has *index* k if Du has exactly k negative eigenvalues. The *interior type number* M_k denotes the number of interior stagnation points of index k.

In $d = 3$ dimensions one has essentially an even number of stagnation points:

Proposition (Condition on the type numbers, [4])

Let $X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with corners. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without boundary stagnation points. Then the following relation for the interior type numbers of u holds:

$$M_1 = M_2 \,.$$

It then follows with the Poincaré-Hopf index theorem that

Corollary (Condition on the domain, [4])

$X \subset \mathbb{R}^3$ be a compact three-dimensional manifold with corners. Let $u\colon X \to \mathbb{R}$ be a strongly Morse harmonic vector field without inflow or outflow through the boundary. Then the Euler characteristic $\chi(X) = 0$ vanishes.

Sources I

[1] H.-D. Alber, "Existence of three-dimensional, steady, inviscid, incompressible flows with nonvanishing vorticity," *Math. Ann.*, vol. 292, no. 3, pp. 493–528, 1992, ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01444632. [Online]. Available: https://doi.org/10.1007/BF01444632.

[2] D. Lortz, "Ueber die existenz toroidaler magnetohydrostatischer gleichgewichte ohne rotationstransformation," *Z. Angew. Math. Phys.*, vol. 21, pp. 196–211, 1970, ISSN: 0044-2275,1420-9039. DOI: 10.1007/BF01590644. [Online]. Available: https://doi.org/10.1007/BF01590644.

[3] master-thesis, *Github repository to the thesis.* Online, 2023. [Online]. Available: https://github.com/TheoKoppenhoefer/master-thesis.

[4] Theo Koppenhöfer, *Stagnation points of harmonic vector fields and the domain topology: Some applications of Morse theory (to appear)*, Student Paper, Apr. 2024.

Thank you for your attention.