Some title

Master Thesis

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Some amazing introduction

Some general remarks

remarks:

- only finitely many critical points possible
- state Hopf's lemma

General definitions

Define:

- emergent, entrant boundary
- admissable function, non-degeneracy

On assuming non-degeneracy

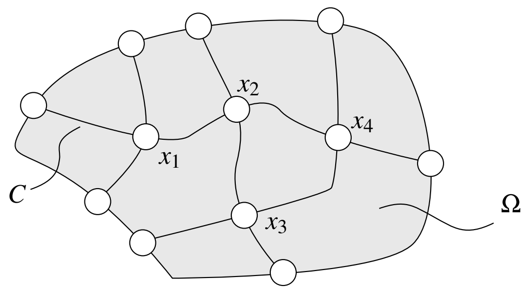

Figure 1: The situation at hand: The edges represent level curves and the vertices critical points.

Harmonic functions, $n = 2$

Claim. *Let Ω be homoemorph to $B_1 \subseteq \mathbb{R}^2$. Let further $f : \overline{\Omega} \to \mathbb{R}$ be harmonic and admissable as in Morse with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \partial\Omega$ consists of at least 2 components.*

A proof involving level-sets

Sketch of Proof. Let $y_c = f(x_1)$ and $x_1, \ldots, x_M$ be all the critical points such that $f(x_.) = y_c$. We claim that the set of level curves

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

divides the boundary $\partial\Omega$ into 4 components. To show this let $\gamma_i : (a_i, b_i) \to C$ for $i \in \{1, \ldots, 4\}$ parametrise the curves in C intersecting at x_1. These can be constructed with the initial value problem

$$\gamma' = (Df)^{\perp}\big|_{\gamma}$$
$$\gamma(0) = \gamma_0$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1. Without loss of generality the intervals on which the γ_i are defined are maximal. We thus have for

$$\gamma_i^- = \lim_{t \to a_i} \gamma(t)$$
$$\gamma_i^+ = \lim_{t \to b_i} \gamma(t)$$

that $\gamma_i^{\pm} \in \{x_1, \ldots, x_M, \partial\Omega\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $Df^{\perp} = 0$. This argument can be applied to all of the $x_1, \ldots, x_M$. We therefore have a situation similar to the one depicted in figure 1.

One sees that C can thus be represented by a multigraph G with vertices $v_1, \ldots, v_K$ and edges $e_1, \ldots, e_L \subseteq C$. Assume G contains a cycle with vertex sequence $v_{i_1}, \ldots, v_{i_J}$

and edges $e_{i_1}, \dots, e_{i_J}$. Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E. By the maximum principle $f = 0$ on E and thus $f = 0$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with $\partial\Omega$ is at least 4 and thus $\partial\Omega$ is divided into 4 components.

Now choose 4 neighbouring components as depicted in figure [TODO: insert figure]. Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle yields that ω_1 contains a local maximum or minimum of f since f is constant on the other boundaries of A. By the same argument $\omega_2, \dots, \omega_4$ also contain local extrema. TODO: use argument with ∇f here to show that extrema can be assumed to be alternating. Since the $\partial\omega_i$ cannot be extremal points on $\partial\Omega$ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. □

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let $x_1, \dots, x_M$ denote the critical points of f. Let $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1,2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{3,4\}$ be chosen to parametrise the stable manifolds of x_1. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^{\pm} \in \{x_1, \dots, x_M, \partial\Omega\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Here we exclude edges along the boundary $\partial\Omega$. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle with vertices $x_{i_1}, \dots, x_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. Since the set of cycles forms a partial ordering with

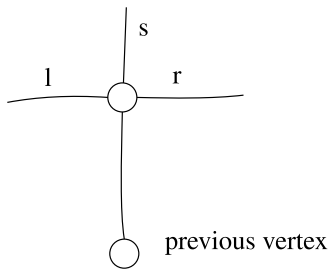

Figure 2: Explanation of the directives ‘l’,‘r’ and ‘r’.

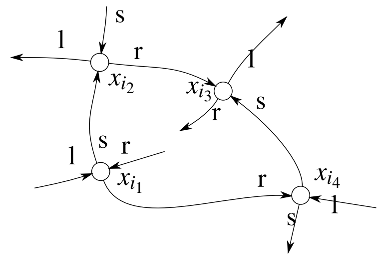

Figure 3: An example for a cycle.

respect to the property ‘contains another cycle’ we can choose this cycle such that it contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to each other. We also note that the edges cannot cross. We can thus describe the trail $x_{i_1}, \dots, x_{i_J}$ by a set of directives of the type

$$(d_1, \dots, d_K) \in \{\mathrm{l}, \mathrm{r}, \mathrm{s}\}^J .$$

Here l, r and s stand for ‘left’, ‘right’ and ‘straight’ respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 2.

An example of the trail ‘srsr’ is given in figure 3. We now note that cycles of the type r,. . . ,r or l,. . . ,l cannot occur as we otherwise would have a directed cycle. Thus there exists a vertex where the chosen direction is s. Without loss of generality this vertex is x_{i_1}. Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1}. Now we look at the directives r,. . . ,r Since all vertices lie within the cycle we must at some step reach a vertex on the cycle. But then this cycle is a new distinct cycle contained in the outer cycle, a contradiction. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

We now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur (elaborate). We now pick a tree $\widetilde{G}$ out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each ‘neighbouring’ (elaborate) leaf of this tree has opposite signage and the claim follows. $\square$

The strategy in the above proofs can be generalised to show the following

Conjecture 1. *Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with Betti numbers $R_0 = 1$ and R_1. Let further $f: \overline{\Omega} \to \mathbb{R}$ be harmonic and admissable as in Morse with M critical points. Assume further that $\overline{\Sigma^-} \subseteq \partial\Omega$ on a given connected component of the boundary $\partial\Omega$ consists of at most 1 connected component. Then we have*

$$\frac{4}{3} M \leq R_1 + 1 .$$

This inequality can probably be improved considerably.

Harmonic vector fields, $n = 2$

No inflow or outflow

TODO

- define inflow, outflow
- define harmonic vector field
- define minimal cycle
- show that u is the gradient of a harmonic function if the domain is simply connected.
- discuss what it means for a critical point to be non-degenerate.
- Introduce Morse inequalities for f and for $-f$.

Proposition 2. *Let Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1 and let $u\colon \Omega \to \mathbb{R}^2$ be a harmonic vector field without inflow or outflow on $\partial\Omega$. Let M denote the number of critical points of u. Then we have*

$$M + 1 \leq R_1$$

.

Sketch of proof. As in previous proofs the critical manifolds form a directed multigraph. Since no critical manifold can intersect with the boundary each vertex of the graph has degree 4 and we thus have $2M$ edges. Now we obtain with Euler's polyhedron formula for a planar graph with multiple components

$$\begin{aligned}
\#\text{ interiour minimal cycles} &= \#\text{faces} - 1 \\
&= 1 + \#\text{components} - \#\text{vertices} + \#\text{edges} - 1 \\
&\geq 1 + 1 - M + 2M - 1 = M + 1
\end{aligned}$$

Now note that each interiour minimal cycle must contain a hole of the domain since else we could restrict u to a simply connected region containing this cycle. Then u would correspond to the gradient of a harmonic function and we would obtain a contradiction as in the previous proof. Hence the number of minimal cycles is a lower bound on the number of holes R_1 of the domain. □

In fact using the Morse inequalities we can obtain the stronger result

Proposition 3. *Let Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1 and let $u\colon \Omega \to \mathbb{R}^2$ be a harmonic vector field without inflow or outflow on $\partial\Omega$. Let M denote the number of critical points of u. Then we have*

$$M + 1 = R_1$$

.

Sketch of proof. We cut down the domain such that it is homeomorphic to the disk. By some previous lemma (to appear) u is the gradient of a harmonic function f on this new domain. f has no critical points on the boundary of the uncut domain Ω and fulfills on the cuts the conditions

$$\mu_0 = \nu_1 \qquad \mu_1 = \nu_0 \tag{1}$$

since every entrant critical point is also an emergent critical point on the other side of the cut of shifted index. We have for this new cut domain the Morse inequalities

$$M + \mu_0 - R_1 - \mu_1 + R_0 = 0 \tag{2}$$
$$M + \nu_0 - R_1 - \nu_0 + R_0 = 0\,. \tag{3}$$

Adding eqations (2) and (3) and using the relation (1) we obtain

$$2(M - R_1 + R_0) = 0$$

from which the claim follows. $\square$

We now give an example of a harmonic vector field for which $M = R_1 - 1$. For this consider the field defined by

$$\begin{aligned} u\colon \mathbb{R}^2 \setminus \left(\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} m \\ 0 \end{bmatrix} \right\} \right) &\to \mathbb{R}^2 \\ x &\mapsto \sum_{m=1}^{M} \nabla^{\perp} \Phi_2 \left(x - \begin{bmatrix} m \\ 0 \end{bmatrix} \right) \end{aligned}$$

where

$$\Phi_2 = -\frac{1}{2\pi} \log(|\cdot|)$$

is the fundamental solution of Δ on $\mathbb{R}^2$. This is a harmonic vector field since

$$\operatorname{curl} \nabla^{\perp} \Phi_2(\cdot - y) = -\Delta \Phi_2(\cdot - y) = 0$$

and by the spherical symmetry of Φ_2

$$\operatorname{Div} \nabla^{\perp} \Phi_2(\cdot - y) = \left(\partial_1^2 - \partial_2^2 \right) \Phi_2(\cdot - y) = 0\,.$$

Figure 4 with $M = 1$ indicates that u has the desired properties. One can see that the plots for larger M also have the desired properties (but I am too lazy to show them here).

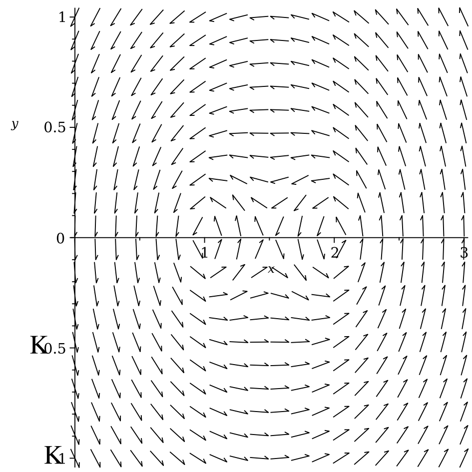

Figure 4: A plot of u for $M = 1$ (TODO: This looks awful).

Harmonic functions, $n = 3$

The cylinder

Proposition 4. *Let $\Omega = (0,1) \times B_1 \subseteq \mathbb{R}^3$ be the cylinder. Let further $f \colon \overline{\Omega} \to \mathbb{R}$ be nontrivial and harmonic with no inflow or outflow on the sides $\partial(0,1) \times B_1$, inflow on $\{0\} \times B_1$ and outflow on $\{1\} \times B_1$. Then f cannot have a critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have that ∂f attains its minimum on $\partial\Omega$. Since $\partial_1 f(x) = 0$ for some interiour point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times S^1$ such that $\partial_1 f(x)$ is minimal on $\overline{\Omega}$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. $\square$

Harmonic vector fields, $n = 3$

Mimiking the proof in 2 dimensions we obtain the following proposition.

Proposition 5. *Let Ω have Betti numbers R_0,R_1 and R_2 Let $u\colon \overline{\Omega} \to \mathbb{R}$ be a harmonic vector field without inflow or outflow. Then we have the following relation for critical points of u*

$$M_2 = M_1$$

Proof. As in the two-dimensional case we begin by cutting up the boundary such that Ω is homeomorphic to the ball with bubbles. Once again by a lemma (TODO: now I'm not quite sure about this statement in 3D) u is the gradient of a harmonic function u on this new domain. f has no critical points on the boundary $\partial\Omega$ and on the cut boundary it fulfills the conditions

$$\mu_0 = \nu_2 \qquad \mu_1 = \nu_1 \qquad \mu_2 = \nu_0 \tag{4}$$

by the same reasoning. We now have the Morse inequalities

$$M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 + \mu_0 - R_0 = 0 \tag{5}$$
$$M_1 + \nu_2 - R_2 - M_2 - \nu_1 + R_1 + \nu_0 - R_0 = 0 \tag{6}$$

It then follows by subtracting equation (6) from (5) and using relations (4) that

$$2(M_2 - M_1) = 0\,.$$

$\square$

Harmonic functions, $n = 4$

Define the harmonic function

$$\begin{aligned} f\colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{7}$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_\pm \colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{8}$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2 \big(x_1^2 + x_2^2 - x_3^2 - x_4^2 \big)$$

Using the condition (7) we obtain the equivalent condition

$$0 < \pm 1 - 2 \big(x_3^2 + x_4^2 \big)$$

Define the cylinder

$$C = \big\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \big\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (8) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm \big(\Sigma^+ \big) .$$

Analogously we have

$$B_1 \setminus C = \psi_\pm \big(\Sigma^- \big) .$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 5.

□

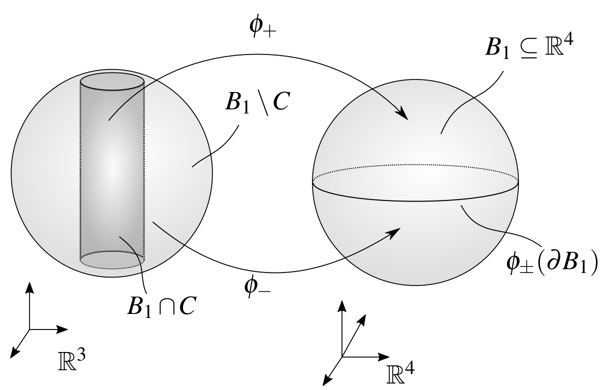

Figure 5: Visualisation of the situation.

Bibliography

[1] master-thesis, *Github repository to the thesis.* Online, 2023. [Online]. Available: https://github.com/TheoKoppenhoefer/master-thesis.

[2] A. Banyaga and D. Hurtubise, *Lectures on Morse homology*, ser. Kluwer Texts in the Mathematical Sciences. Kluwer Academic Publishers Group, Dordrecht, 2004, vol. 29, pp. x+324, ISBN: 1-4020-2695-1. DOI: 10.1007/978-1-4020-2696-6. [Online]. Available: https://doi.org/10.1007/978-1-4020-2696-6.

[3] M. C. Irwin, *Smooth dynamical systems*, ser. Advanced Series in Nonlinear Dynamics. World Scientific Publishing Co., Inc., River Edge, NJ, 2001, vol. 17, pp. xii+259, Reprint of the 1980 original, With a foreword by R. S. MacKay, ISBN: 981-02-4599-8. DOI: 10.1142/9789812810120. [Online]. Available: https://doi.org/10.1142/9789812810120.

[4] M. Morse, "Relations between the critical points of a real function of n independent variables," *Trans. Amer. Math. Soc.*, vol. 27, no. 3, pp. 345–396, 1925, ISSN: 0002-9947,1088-6850. DOI: 10.2307/1989110. [Online]. Available: https://doi.org/10.2307/1989110.

[5] M. Morse and S. S. Cairns, *Critical point theory in global analysis and differential topology: An introduction*, ser. Pure and Applied Mathematics. Academic Press, New York-London, 1969, vol. Vol. 33, pp. xii+389.

[6] M. Morse, "Equilibrium points of harmonic potentials," *J. Analyse Math.*, vol. 23, pp. 281–296, 1970, ISSN: 0021-7670,1565-8538. DOI: 10.1007/BF02795505. [Online]. Available: https://doi.org/10.1007/BF02795505.

[7] R. Shelton, "Critical points of harmonic functions on domains in $\mathbf{R}^3$," *Trans. Amer. Math. Soc.*, vol. 261, no. 1, pp. 137–158, 1980, ISSN: 0002-9947,1088-6850. DOI: 10.2307/1998322. [Online]. Available: https://doi.org/10.2307/1998322.

[8] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544, ISBN: 0-521-79160-X; 0-521-79540-0.