Some title

Master Thesis

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Contents

Todo list

General TODOs

- Disproof using hairy ball theorem.
- Write other example
- Look what happens if you add the examples
- Check that the signs of the examples are correct
- Make the plots of the examples look nice.
- Change n to d, consistent notation.
- Check for typos.

Some amazing introduction

General definitions

Unless otherwise stated we denote by $\Omega \subseteq \mathbb{R}^d$ an open bounded subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial\Omega$. We denote with

$$f\colon \overline{\Omega} \to \mathbb{R}$$

a scalar function of class C^2. We also denote by

$$u\colon \overline{\Omega} \to \mathbb{R}^d$$

a vector field of class C^1. Often but not always u can be thought of as a *harmonic vector field*, that is u fulfills

$$\operatorname{Div} u = 0 \qquad \text{and} \qquad \operatorname{curl} u = 0\,.$$

Also often but not always we assume that locally $u = \nabla f$, implying that f is harmonic.

In the following we define the emergent and the entrant boundary as in [1, p.282]

Definition 1 (Emergent and entrant boundary)**.** We call a vector $v \in T_x\mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ iff v is not tangent to Σ and directed into the interiour of Ω. Analogously if v is not tangent and directed to the exteriour we call v *emergent*. We define the *entrant boundary* Σ^- to be the set of boundary points at which u is entrant. Analogously define the *emergent boundary* Σ^+ to be the set of boundary points at which u is emergent. Further define Σ^0 to contain all other boundary points such that we have a decomposition of the boundary

$$\Sigma = \Sigma^- \sqcup \Sigma^0 \sqcup \Sigma^+\,.$$

The following are slight generalisation of the definitions given in [2, p.138f], [3, §5] and [1, p.282f] to include harmonic vector fields. We call the zeroes of u *critical points*. A critical point $x \in \Omega$ is called *non-degenerate* if the derivative $Du(x)$ is invertible. We say that x has *index* k if $Du(x)$ has exactly k negative eigenvalues. u is called *non-degenerate* if all its critical points are non-degenerate.

We now define the *interiour type numbers* M_k to be the number of critical points of u of index k. The total number of critical points is thus given by

$$M = \sum_k M_k\,.$$

We call Ω a *regular domain* if Σ is a manifold of class C^2. In the following definition we require Ω to be regular.

Definition 2 (Boundary type numbers)**.** For a boundary point x let

$$\pi_x\colon \mathbb{R}^d \cong T_x\mathbb{R}^d \to T_x\Sigma$$

denote the projection of a vector at x onto the tangent space of Σ at x. Let

$$u_{\Sigma^-} = \pi \circ u|_{\Sigma^-} \in C^1\left(T\Sigma^-\right)$$

to be the restriction and projection of u onto the tangent bundle of Σ^-. We define the *boundary type numbers* μ_k to be the number of critical points of u_{Σ^-} on the entrant boundary Σ^- of index k. We further write ν_k for the k-th boundary type number of $-u$.

We now call u *regular* iff u and u_Σ are non-degenerate and all critical points lie in Ω.

The previous definitions translate naturally to f. We call f regular, non-degenerate, et cetera iff $u = \nabla f$ is regular, non-degenreate, et cetera.

On assuming non-degeneracy

Some general remarks

- only finitely many critical points possible
- state Hopf's lemma

Betti numbers

For a formal definition of Betti numbers we refer to ??. We do however give some examples of Betti numbers for some domains.

The Morse inequalities

We state the Morse inequalities.

More citations.

Theorem 3 (Morse inequalities)**.** *Let Ω and f be regular. Then we have the inequalities*

$$
\begin{aligned}
&0 \leq M_0 + \mu_0 - R_0 \\
&0 \leq M_1 + \mu_1 - R_1 - (M_0 + \mu_0 - R_0) \\
&\vdots \\
&0 \leq M_{d-1} + \mu_{d-1} - R_{d-1} - \cdots + (-1)^{d-1}(M_0 + \mu_0 - R_0) \\
&0 = M_d - R_d - (M_{d-1} + \mu_{d-1} - R_{d-1}) + \cdots + (-1)^d (M_0 + \mu_0 - R_0)\,.
\end{aligned}
$$

Proof. See [3]. □

Give a more precise reference.

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$ and thus we obtain for the special case of dimensions $d = 2$.

Corollary 4 (Morse inequalities for f harmonic, $d = 2$)**.** *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$
\begin{aligned}
&0 \leq \mu_0 - R_0 \\
&0 = M + \mu_1 - R_1 - \mu_0 + R_0\,.
\end{aligned}
$$

In dimensions $d = 3$ we obtain

Corollary 5 (Morse inequalities for f harmonic, $d = 3$)**.** *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned}
0 &\leq \mu_0 - R_0 \\
0 &\leq M_1 + \mu_1 - R_1 - \mu_0 + R_0 \\
0 &= M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 - \mu_0 + R_0 \,.
\end{aligned}$$

- Introduce Morse inequalities for f and for $-f$.

Harmonic functions, $n = 2$

Proposition 6. *Let Ω be homoemorph to $B_1 \subseteq \mathbb{R}^2$. Let further $f\colon \overline{\Omega} \to \mathbb{R}$ be harmonic and admissable as in Morse with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \partial\Omega$ consists of at least 2 components.*

On minimal cycles

The following proofs make use of graph theory. Here note that we identify the graph G with its planar embedding which we assume to be unique. We call a cycle in G minimal if it contains no distinct cycle. More precise.

A proof involving level-sets

Sketch of Proof. Let $y_c = f(x_1)$ and $x_1, \dots, x_M$ be all the critical points such that $f(x_.) = y_c$. We claim that the set of level curves

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

divides the boundary $\partial\Omega$ into 4 components. To show this let $\gamma_i\colon (a_i, b_i) \to C$ for $i \in \{1, \dots, 4\}$ parametrise the curves in C intersecting at x_1. These can be constructed with the initial value problem

$$\begin{aligned} \gamma' &= (Df)^\perp\big|_\gamma \\ \gamma(0) &= \gamma_0 \end{aligned}$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1. Without loss of generality the intervals on which the γ_i are defined are maximal. We thus have for

$$\begin{aligned} \gamma_i^- &= \lim_{t \to a_i} \gamma(t) \\ \gamma_i^+ &= \lim_{t \to b_i} \gamma(t) \end{aligned}$$

that $\gamma_i^\pm \in \{x_1, \dots, x_M, \partial\Omega\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $Df^\perp = 0$. This argument can be applied to all of the $x_1, \dots, x_M$. We therefore have a situation similar to the one depicted in figure 1.

One sees that C can thus be represented by a multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq C$. Assume G contains a cycle with vertex sequence $v_{i_1}, \dots, v_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E. By the maximum principle $f = 0$ on E and thus $f = 0$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with $\partial\Omega$ is at least 4 and thus $\partial\Omega$ is divided into 4 components.

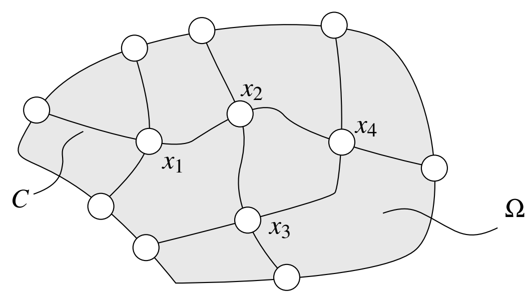

Figure 1: The situation at hand: The edges represent level curves and the vertices critical points.

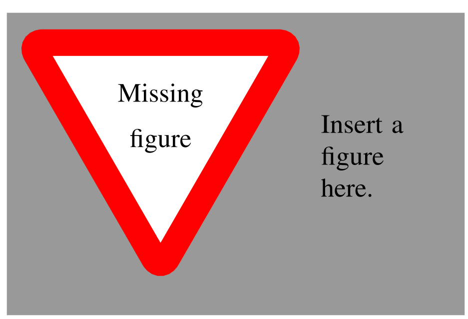

Figure 2: The choice of $\omega_1, \ldots, \omega_4$.

Now choose 4 neighbouring components as depicted in figure 2 Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle yields that ω_1 contains a local maximum or minimum of f since f is constant on the other boundaries of A. By the same argument $\omega_2, \dots, \omega_4$ also contain local extrema. Since the $\partial\omega_i$ cannot be extremal points on $\partial\Omega$ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

use argument with ∇f here to show that extrema can be assumed to be alternating.

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. □

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let $x_1, \dots, x_M$ denote the critical points of f. Let $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1, 2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{3, 4\}$ be chosen to parametrise the stable manifolds of x_1. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^\pm \in \{x_1, \dots, x_M, \partial\Omega\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Here we exclude edges along the boundary $\partial\Omega$. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle with vertices $x_{i_1}, \dots, x_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. Since the set of cycles forms a partial ordering with respect to the property 'contains another cycle' we can choose this cycle such that it contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to each other. We also note that the edges cannot cross. We can thus describe the trail $x_{i_1}, \dots, x_{i_J}$ by a set of directives of the type

$$(d_1, \dots, d_K) \in \{\mathrm{l}, \mathrm{r}, \mathrm{s}\}^J .$$

Here l, r and s stand for 'left', 'right' and 'straight' respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 3.

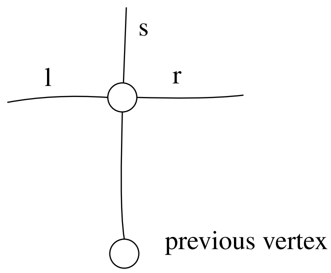

Figure 3: Explanation of the directives ‘l’,‘r’ and ‘r’.

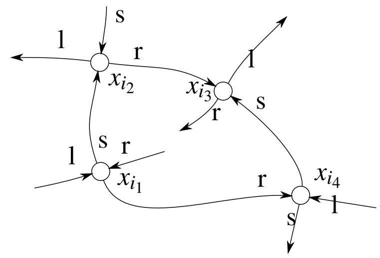

Figure 4: An example for a cycle.

An example of the trail ‘srsr’ is given in figure 4. We now note that cycles of the type r,. . . ,r or l,. . . ,l cannot occur as we otherwise would have a directed cycle. Thus there exists a vertex where the chosen direction is s. Without loss of generality this vertex is x_{i_1}. Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1}. Now we look at the directives r,. . . ,r Since all vertices lie within the cycle we must at some step reach a vertex on the cycle. But then this cycle is a new distinct cycle contained in the outer cycle, a contradiction. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

We now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur (elaborate). We now pick a tree $\widetilde{G}$ out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each ‘neighbouring’ (elaborate) leaf of this tree has opposite signage and the claim follows. □

The strategy in the above proofs can be generalised to show the following

Conjecture 7. *Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with Betti numbers $R_0 = 1$ and R_1. Let further $f : \overline{\Omega} \to \mathbb{R}$ be harmonic and admissable as in Morse with M critical points. Assume further that $\overline{\Sigma^-} \subseteq \partial\Omega$ on a given connected component of the boundary $\partial\Omega$ consists of at most 1 connected component. Then we have*

$$\frac{4}{3}M \leq R_1 + 1\,.$$

This inequality can probably be improved considerably.

Harmonic vector fields, $n = 2$

We state the following result for harmonic vector fields in $d = 2$ dimensions.

Proposition 8 (Harmonic vector fields on simply connected domains)**.** *Let Ω be simply connected and u be a harmonic vector field. Then there exists a harmonic function $f\colon \Omega \to \mathbb{R}$ such that $u = \nabla f$.*

Proof. □

State some proof

This implies

Corollary 9. *Let u be a harmonic vector field and $x \in \Omega$. Then there exists a harmonic function f such that locally around x we have $u = \nabla f$.*

No inflow or outflow

We say that u has no inflow on a subset $S \subseteq \Sigma$ iff $\Sigma^- \cap S = \emptyset$ and that it has no outflow iff $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can state the following result.

Proposition 10 (Upper bound on M)**.** *Let $d = 2$ and Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1. Let further $u\colon \Omega \to \mathbb{R}^2$ be a regular harmonic vector field without inflow or outflow. Then we have*

$$M + 1 \leq R_1$$

.

Sketch of proof. As in previous proofs the critical manifolds form a directed multigraph. Since no critical manifold can intersect with the boundary each vertex of the graph has degree 4 and we thus have $2M$ edges. Now we obtain with Euler's polyhedron formula for a planar graph with multiple components

$$\begin{aligned}
\text{\# interiour minimal cycles} &= \text{\#faces} - 1 \\
&= 1 + \text{\#components} - \text{\#vertices} + \text{\#edges} - 1 \\
&\geq 1 + 1 - M + 2M - 1 = M + 1
\end{aligned}$$

Now note that each interiour minimal cycle must contain a hole of the domain since else we could restrict u to a simply connected region containing this cycle. Then u would correspond to the gradient of a harmonic function and we would obtain a contradiction as in the previous proof. Hence the number of minimal cycles is a lower bound on the number of holes R_1 of the domain. □

In fact using the Morse inequalities we can obtain the stronger result

Proposition 11. *Let Ω be a regular domain with Betti numbers $R_0 = 1$, and R_1 and let $u\colon \Omega \to \mathbb{R}^2$ be a harmonic vector field without inflow or outflow. Then we have*

$$M + 1 = R_1$$

.

Sketch of proof. We cut down the domain such that it is homeomorphic to the disk. By some previous lemma (to appear) u is the gradient of a harmonic function f on this new domain. f has no critical points on the boundary of the uncut domain Ω and fulfills on the cuts the conditions

$$\mu_0 = \nu_1 \qquad \mu_1 = \nu_0 \tag{1}$$

since every entrant critical point is also an emergent critical point on the other side of the cut of shifted index. We have for this new cut domain the Morse inequalities

$$M + \mu_0 - R_1 - \mu_1 + R_0 = 0 \tag{2}$$
$$M + \nu_0 - R_1 - \nu_0 + R_0 = 0\,. \tag{3}$$

Adding eqations (2) and (3) and using the relation (1) we obtain

$$2(M - R_1 + R_0) = 0$$

from which the claim follows. □

We now give an example of a harmonic vector field for which $M = R_1 - 1$. In order to do this we define two differential operators for $d = 2$ by

$$\nabla^{\perp} f = \operatorname{Curl} f = \begin{bmatrix} -\partial_2 f \\ \partial_2 f \end{bmatrix}$$

and

$$\operatorname{curl} u = -\partial_1 u_2 + \partial_2 u_1$$

Now consider the field defined by

Find source of this in Gedicke script. Relation to differential forms.

$$u\colon \mathbb{R}^2 \setminus \left(\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} m \\ 0 \end{bmatrix} \right\} \right) \to \mathbb{R}^2$$
$$x \mapsto \sum_{m=1}^{M} \nabla^{\perp} \Phi_2 \left(x - \begin{bmatrix} m \\ 0 \end{bmatrix} \right)$$

where

$$\Phi_2 = -\frac{1}{2\pi} \log(|\cdot|)$$

is the fundamental solution of Δ on $\mathbb{R}^2$. This is a harmonic vector field since

$$\operatorname{curl} \nabla^{\perp} \Phi_2(\cdot - y) = -\Delta \Phi_2(\cdot - y) = 0$$

and by the spherical symmetry of Φ_2

$$\operatorname{Div} \nabla^{\perp} \Phi_2(\cdot - y) = \left(\partial_1^2 - \partial_2^2\right) \Phi_2(\cdot - y) = 0\,.$$

Figure 5 with $M = 1$ indicates that u has the desired properties. One can see that the plots for larger M also have the desired properties (but I am too lazy to show them here).

This figure looks awful

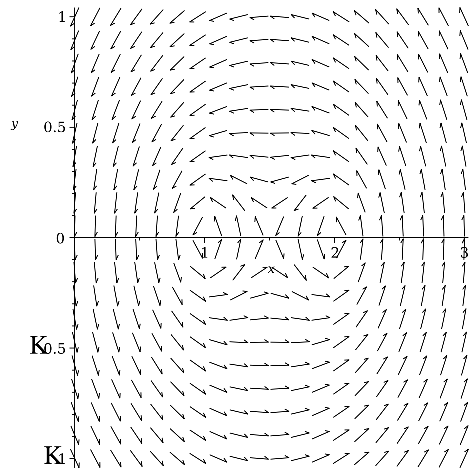

Figure 5: A plot of u for $M = 1$

An example of inflow on one side and outflow on the other

Consider the degerate harmonic function

$$\begin{aligned} f_1 \colon \mathbb{C} \cong \mathbb{R}^2 &\to \mathbb{R} \\ z &\mapsto \Re(z^3)\,. \end{aligned}$$

Since we are only interested in non-degenerate functions we modify this to

$$f_2(z) = \Re(z^3 + (1 - \mathrm{i})z)$$

Now we set

$$\begin{aligned} u \colon \mathbb{R}^2 \setminus (\{\dots\}) &\to \mathbb{R}^2 \\ x &\mapsto \nabla^\perp f(x_1 + \mathrm{i}x_2) \end{aligned}$$

As before u is a harmonic vector field and a plot shows that it has the desired properties

Harmonic functions, $n = 3$

The cylinder

Proposition 12. *Let $\Omega = (0,1) \times B_1 \subseteq \mathbb{R}^3$ be the cylinder. Let further $f\colon \overline{\Omega} \to \mathbb{R}$ be regular harmonic with no inflow or outflow on the sides $\partial(0,1) \times B_1$, no outflow on $\{0\} \times B_1$ and no inflow on $\{1\} \times B_1$. Then f cannot have a critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have that ∂f attains its minimum on $\partial\Omega$. Since $\partial_1 f(x) = 0$ for some interiour point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times S^1$ such that $\partial_1 f(x)$ is minimal on $\overline{\Omega}$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. $\square$

Harmonic vector fields, $n = 3$

Mimiking the proof in 2 dimensions we obtain the following proposition.

Proposition 13. *Let Ω have Betti numbers R_0,R_1 and R_2 Let $u\colon \overline{\Omega} \to \mathbb{R}$ be a harmonic vector field without inflow or outflow. Then we have the following relation for critical points of u*

$$M_2 = M_1$$

Proof. As in the two-dimensional case we begin by cutting up the boundary such that Ω is homeomorphic to the ball with bubbles. Once again by a lemma u is the gradient of a harmonic function u on this new domain. f has no critical points on the boundary $\partial\Omega$ and on the cut boundary it fulfills the conditions

now I'm not quite sure about this statement in 3D

$$\mu_0 = \nu_2 \qquad \mu_1 = \nu_1 \qquad \mu_2 = \nu_0 \tag{4}$$

by the same reasoning. We now have the Morse inequalities

$$M_2 + \mu_2 - R_2 - M_1 - \mu_1 + R_1 + \mu_0 - R_0 = 0 \tag{5}$$
$$M_1 + \nu_2 - R_2 - M_2 - \nu_1 + R_1 + \nu_0 - R_0 = 0 \tag{6}$$

It then follows by subtracting equation (6) from (5) and using relations (4) that

$$2(M_2 - M_1) = 0\,.$$

□

Harmonic functions, $n = 4$

Define the harmonic function

$$\begin{aligned} f\colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{7}$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_\pm \colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{8}$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right)$$

Using the condition (7) we obtain the equivalent condition

$$0 < \pm 1 - 2\left(x_3^2 + x_4^2\right)$$

Define the cylinder

$$C = \left\{\bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2\right\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (8) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm\left(\Sigma^+\right).$$

Analogously we have

$$B_1 \setminus C = \psi_\pm\left(\Sigma^-\right).$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 6.

□

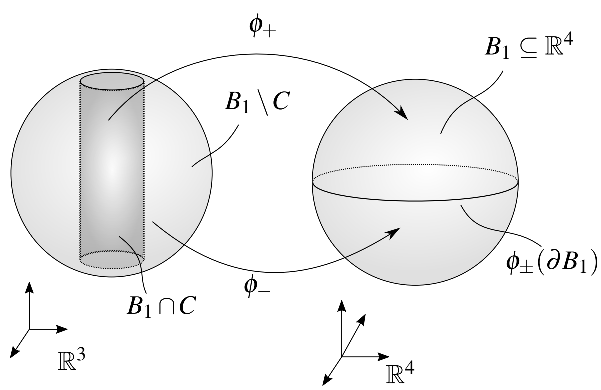

Figure 6: Visualisation of the situation.

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