Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

Theo Koppenhöfer

Lund

November 30, 2023

Contents

Todo list

General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.
- Define traversing vector field
- Look at regularity requirements. In particular: Riemannian mfs

Some questions

- Should I state Hopf's Lemma?
- Weak formulation - a distraction? →Hartman, Wintner

Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and interior $\Omega = \mathrm{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. We denote by

$$f : X \to \mathbb{R}$$

a scalar function of class C^2. We also denote by

$$u : \overline{\Omega} \to \mathbb{R}^d$$

a vector field of class C^1. Often but not always u can be thought of as a *harmonic vector field*, that is u is of type C^1 and fulfils

$$\mathrm{Div}\, u = 0 \qquad \text{and} \qquad \mathrm{curl}\, u = 0\,.$$

Also often but not always we assume that globally $u = \nabla f$ is a gradient field, implying that f is harmonic. One question we seek to answer in this thesis is the following.

Question 1 (Flowthrough with stagnation point)**.** Does there exist a tube $\Omega \subseteq \mathbb{R}^3$ with flow u through the tube such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. u enters the tube on the one side and exits the tube on the other?

The answer for this will turn out to be yes for dimensions $d \geq 3$ and no for the dimension $d = 2$. In the case of $d = 2$ dimensions we will look at what happens if we allow for holes in the domain. other questions are of the type

Question 2 (stagnation points of harmonic vector fields without inflow or outflow)**.** Let u be a flow in a domain X and such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of critical points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary conditions.

General definitions

We start by requiring some regularity for the boundary of X. More precisely we require X to be a compact manifold with corners as in [3].

Definition 3 (Manifolds with corners)**.** We introduce the notation

$$H_j^d = \mathbb{R}_{\geq 0}^j \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d .$$

A *manifold with (convex) corners* is a topological space X together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exists an open neighbourhood U_x of x, a number $j = j(x)$ and a diffeomorphism $\phi : U_x \to H_j^d$ in $\mathscr{A}$ with $\phi(x) = 0$. We further define sets

$$X_k = \{x \in X : \ j(x) = k\}, \tag{1}$$

which form a stratification of X.

More generally we give the definition of a stratification as

Definition 4 (Stratified space)**.** A *stratified space* is a collection of a topological space X and a collection of subspaces $X_j \subseteq X$, $j \in \mathscr{J}$, called *strata*, indexed by a partially ordered set $\mathscr{J}$ such that

1. each X_j is a manifold (without boundary) of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$.

In the case that $X_j \subseteq \overline{X}_k$ and additionally $n(j) = 0$ or $n(j) = n(k) + 1$ we will write $X_j \precsim X_k$ or, abusing notation, write $X_k = X_{j-1}$.

In the case that the stratification arises through relation (1) we have precisely $X_j \precsim X_{j-1}$ for $j \in \{1, \dots, d\}$ and $X_0 \precsim X_0$.

For completeness we also give the definition of the conormal bundle and the contingent cone for a stratification X_j of X

Definition 5 (Normal bundle, contingent cone)**.** We define the *normal bundle* for a stratification of X to be the quotient space

$$NX_j = TX_{j-1}/TX_j, \tag{2}$$

where TX_j is the tangent space of X_j. The *conormal bundle* N^*X_j is defined as its pointwise dual, that is

$$N_x^* X_j = (N_x X_j)' .$$

We define the *contingent cone* (according to Bouligand) for a set $Y \subseteq X$ at $x \in \overline{Y}$ as

$$C_x Y = \left\{ v \in \mathbb{R}^d : \text{ there exists } \lambda_n \to 0 \text{ and } Y \ni x_n \to x \text{ s.t. } \lim_n \lambda_n (v - v_n) = 0 \right\}. \tag{3}$$

Given a vector field $u : X \to \mathbb{R}^d$ and the above stratification X_k of X we can construct for every $j \in \mathscr{J}$ a vector field

$$u_j : X_j \to T^* X_j .$$

Here T^*X_j denotes the cotangent space of the manifold X_j as defined for example in [4, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j\big|_x \colon \mathbb{R}^d \cong T_x^*\mathbb{R}^d \to T_x^*X_j \tag{4}$$

denote the orthogonal projection of a vector at x onto the cotangent space of the stratum X_j at x. Now set

$$u_j = u\big|_{T^*X_j} = \pi_j \circ u\big|_{X_j} \in C^1(T^*X_j) \tag{5}$$

be the restriction of u onto the cotangent bundle T^*X_j.

In the following we define the emergent and the entrant boundary in a way that generalises [2, p.282] for stratified manifolds.

Definition 6 (Emergent and entrant boundary)**.**

rewrite the following

We call a vector $v \in T_x\mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ iff v lies in the relative interiour of the polar cone of the contingent cone C_xX, that is

$$v \in \operatorname{rel\,int}(C_xX)^o = \left\{w \in T_xX \colon w \cdot w' < 0 \text{ for all } w' \in C_xX\right\}$$

This can be thought of as that v is not tangent to the boundary Σ and points into the interiour of Ω. Analogously we call v *emergent* iff v belongs to the relative interiour of the dual cone of the contingent cone $C_x\Sigma$, that is

$$v \in \operatorname{rel\,int}(C_xX)^* = \left\{w \in T_xX \colon w \cdot w' > 0 \text{ for all } w' \in C_xX\right\}$$

We define the *entrant boundary* Σ^- to be the set of boundary points at which u is entrant. Analogously define the *emergent boundary* Σ^+ to be the set of boundary points at which u is emergent. Further define the *tangential boundary* Σ^0 to be

$$\Sigma^0 = \Sigma \setminus \left(\Sigma^+ \cup \Sigma^-\right) \cup \partial\Sigma^+ \cup \partial\Sigma^- \tag{6}$$

For convenience we also introduce the *non-entrant boundary* $\Sigma^{\geq 0} = \Sigma^+ \sqcup \Sigma^0$ and the *non-emergent* boundary $\Sigma^{\leq} = \Sigma^- \sqcup \Sigma^0$.

illustrate on boundary with corners

We would now like to illustrate the preceding definitions.

Example 7. We now consider our domain to be the ball $B_1 \subseteq \mathbb{R}^3$ around the origin in $d = 3$ dimensions. Now consider the harmonic function

$$\begin{aligned} f \colon \Omega &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - 2x_3^2 \end{aligned} \tag{7}$$

Which induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u \colon \Omega &\to \mathbb{R} \\ x &\mapsto \begin{bmatrix} 2x_1 & 2x_2 & -4x_3 \end{bmatrix}^\top . \end{aligned} \tag{8}$$

Figure 1: Plots of the entrant, emergent and tangential boundary for the function f given by equation (7)

We have that the normal to the boundary $\Sigma = S^2$ is given by

$$\begin{aligned} n\colon S^2 &\to S^2 \\ x &\mapsto x \end{aligned}$$

and thus we have that $x \in \Sigma^-$ iff

$$0 > n \cdot u = 2\left(x_1^2 + x_2^2 - 2x_3^2\right) = 2f(x)$$

A plot of the sets can be seen in figure 1.

The following are slight generalisation of definitions given in [1, p.138f], [5, §5] and [2, p.282f] to include harmonic vector fields.

Have a closer look at the regularity in the following.

Definition 8 (Stagnation points)**.** Let $u_j\colon X_j \to T^*X_j$ be a C^1 vector field on a stratification of X. We call $x \in X_j$ a *stagnation point* iff $u_{j-1}(x)$ belongs to the polar cone of the contingent cone C_xX_j or to the dual cone of the contingent cone C_xX_j, that is

$$u_{j-1}(x) \in (C_xX_j)^o = \left\{v \in T_x^*X \colon v \cdot w \leq 0 \text{ for all } w \in C_xX_j\right\}$$

or

$$u_{j-1}(x) \in (C_xX_j)^* = \left\{v \in T_x^*X \colon v \cdot w \geq 0 \text{ for all } w \in C_xX_j\right\}.$$

A necessary condition for this is that x is a zero of u_j. If X_j has dimension $n(j) = d$ then we call x an *interior stagnation point*. If x does not lie in the emergent boundary Σ^+ we call x an *essential stagnation point*. The set of all essential stagnation points of u_j is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$. A stagnation point x is called *non-degenerate* iff

$$u_{j-1}(x) \in \operatorname{rel\,int}(C_xX_j)^* \cup \operatorname{rel\,int}(C_xX_j)^o \tag{9}$$

and additionally the derivative

$$Du_j(x) = Du_j\big|_x \in T_xT^*X \cong \mathbb{R}^{n\times n}$$

is bijective. In addition we say that x has *index* k if $Du_j(x)$ has exactly k negative eigenvalues. u_j is called *(essentially) non-degenerate* if all its stagnation points are (essentially) non-degenerate. Assume u_j is non-degenerate then we can define the *k-th type number* of the stratum X_j to be the number of essential critical points of u_j of index k, that is

$$\mathrm{Ind}_{j,k}(u) = \#\{x \in \mathrm{Cr}_j(u) \colon x \text{ has index } k\}\,.$$

We define the *interior type numbers* by

$$M_k = \sum_{j\colon n(j)=d} \mathrm{Ind}_{j,k}(u)\,.$$

The total number of interior stagnation points of u is then given by

$$M = \sum_k M_k\,.$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of u of index k, that is

$$\mu_k = \sum_{j\colon n(j)<d} \mathrm{Ind}_{j,k}(u)$$

We further write ν_k for the k-th boundary type number of $-u$.

Definition 9 (Morse functions)**.** We call u *(essentially) Morse* iff for all j we have that u_j is (essentially) non-degenerate. For an essentially Morse function u we will denote the number of essential stagnation points of u of index k by

$$\mathrm{Ind}_k(u) = \sum_{j=0}^{d} \mathrm{Ind}_{j,k}(u) = \#\left\{x \in \bigcup_j \mathrm{Cr}_j(u) \colon x \text{ has index } k\right\}.$$

The following definition is inspired by [6].

this definition is floating freely, also incorporate strata.

Definition 10 (Tangency regular)**.** We call a point $x \in \Sigma^0$ a *tangency point*. We call a function $u\colon X \to \mathbb{R}^d$ tangency regular iff at every tangency point $x \in \Sigma^0$ we have that $Du_x(u) \neq 0$.

The previous definitions translate naturally to f. That is we call f Morse, non-degenerate, et cetera iff $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call x an critical point of f of index k if it is a stagnation point of u of index k.

Rewrite: discuss index on manifold with corners.

To illustrate the preceding definitions we return to our previous example.

Example 11. Let f and u be as in example 7. One sees from equation (8) that the origin 0 is the sole interior critical point of f. Since we have that

$$Du(x) = \begin{bmatrix} 2 & & \\ & 2 & \\ & & -4 \end{bmatrix}$$

for all $x \in \Omega$ we see that $Du(0)$ is bijective and thus a non-degenerate critical point. Since $Du(0)$ has exactly one negative eigenvalue we see that the origin has index 1. Since there are no other critical points we have $M = 1$ and

$$M_k = \delta_{k1} .$$

We now calculate for $x \in S^2$

$$\tilde{u}(x) = (u - (n \cdot u)n)(x) = (u - 2fn)(x) = 2 \begin{bmatrix} (1-f(x))x_1 \\ (1-f(x))x_2 \\ (-2-f(x))x_2 \end{bmatrix}$$

Hence we see that $x \in \Sigma$ is a critical point iff

$$f(x) = 1 \text{ and } x_3 = 0 \text{ or} \tag{10}$$

$$f(x) = -2 = \text{ and } x_1 = 0 = x_2 . \tag{11}$$

The former equation (10) gives that every point belonging to $S^1 \times \{0\} \subseteq \mathbb{R}^3$ is in fact a critical point of f. But since $f = 1$ on this set these points are degenerate. We will discuss a fix to this issue in the upcoming section. We now consider equation (11) and take $f(x) = -2$ then we must have that $x = \pm e_3$ where $e_k = \delta_{k \cdot}$ is the k-th basis vector in $\mathbb{R}^d$. We now determine their index. For this consider the curves

$$\begin{aligned} \gamma_k \colon \mathbb{R} &\to S^2 \\ t &\mapsto \sin(t) e_k \pm \cos(t) e_3 \end{aligned}$$

for $k \in \{1,2\}$. Note that $\gamma_k'(0) = e_k$ and $\gamma_k(0) = \pm e_3$. We see that

$$Du(e_1)(\gamma_k'(0)) = (u \circ \gamma_k)'(0) = (\sin(t)e_k \mp 2\cos(t)e_3)'(0) = e_k = \gamma_k'(0)$$

and thus $e_k \in T_{\pm e_3} S^2$ are eigenvectors of $Du(e_k)$ to eigenvalues 1. Since the e_k span the tangent space $T_{\pm e_3} S^2$ it follows that the $\pm e_3$ are non-degenerate critical points of f with index 0.

On assuming non-degeneracy

Rewrite: Use updated notation from previous section. Change for manifolds with corners.

In the following section we argue that assuming non-degeneracy of u and f is not a great restriction. Given u we define the modification

$$u^\varepsilon = u + \varepsilon \tag{12}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that u_ε is for almost all choices of ε non-degenerate and can thus be used to approximate a degenerate u. Our approach is to use Thom's theorem which is inspired by the approach in [4, Chapter 6]. In this section we refer to X and Y as generic manifolds without boundary .

of which class?

Definition 12 (Transversality)**.** We call a function $g\colon X \to Y$ transverse to a submanifold $A \subseteq Y$ iff for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\mathrm{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y\,.$$

As an application we make the following observation.

Proposition 13 (Transversal characterisation of non-degeneracy)**.** *Let $u\colon X \to T^*X$ be a differentiable vector field. Then u is non-degenerate iff u is transverse to the zero section A of T^*X.*

Proof. First note that we have that $x \in u^{-1}(A)$ iff $u(x) = 0$ and thus $u^{-1}(A) = C$. Unravelling the definition of transversality we get that u is transverse to the zero section iff for all $x \in C = u^{-1}(A)$ we have that

$$\mathrm{Image}(Du_x) + T_{u(x)}A = T_{u(x)}TX\,. \tag{13}$$

As A is the zero section we have $T_{u(x)}A = 0$ and equation (13) is equivalent to stating that Du is of full rank. But Du being of full rank at all points in C is equivalent to u being non-degenerate. □

A further application is given by the following proposition.

Proposition 14 (Transversal characterisation of tangency regularity)**.** *Let $u\colon X \to T^*X$ be a differentiable vector field on a manifold with corners X. Let $X_j = \partial_j X$ be a stratification of X. Then u is tangency regular iff u viewed as a section of the normal bundle N^*X_j is transverse to the zero section of N^*X_j for all $j > 0$.*

Why do we require $j > 0$

Proof.

Some proof

□

The following version of Thom's transversality theorem is an adaption (i.e. weakening) of [4, Theorem 2.7] to our needs.

Theorem 15 (Parametric transversality theorem.)**.** *Let E, X, Y be C^r-manifolds (without boundary) and $A \subseteq Y$ a C^r submanifold such that*

$$r > \dim X - \dim Y + \dim A\,.$$

Let further $F\colon E \to C^r(X,Y)$ be such that the evaluation map

$$\begin{aligned} F^{ev}\colon E \times X &\to Y \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

is C^r and transverse to A. Then the set

$$\pitchfork(F;A) = \{\varepsilon \in E\colon F_\varepsilon \text{ is transverse to } A\}$$

is dense.

Proof. See [4, Theorem 2.7] for details. □

Using proposition 13 we obtain the corollary

Corollary 16. *Let $u\colon X \to T^*X$ be a harmonic vector field on X and let X_j be a stratification of X. Then for almost every $\varepsilon \in \mathbb{R}^d$ we have that*

1. *u_j^ε given by equations* (6) *and* (6) *is non-degenerate*
2. *u_j^ε is tangency regular*

Proof. Set $r = 2$, $E = \mathbb{R}^d$ and $Y = T^*X_j$. We initially set $X = \Omega$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F\colon E &\to C^\infty(X, T^*X) \\ \varepsilon &\mapsto u^\varepsilon \end{aligned}$$

We note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq T^*X$. Then the parametric transversality theorem yields a dense $E_j \subseteq E$ on which F is transverse to A. For this note that $E \times C = F^{-1}(A)$. It then follows for all $(\varepsilon, x) \in F^{-1}(A)$ that

$$\mathrm{Image}\Big(DF^{\mathrm{ev}}_{(\varepsilon,x)}\Big) = T_x T^*X \tag{14}$$

since we have that

$$DF^{\mathrm{ev}}_{(\varepsilon,x)} = \big[Du_x \mid \mathrm{Id}_{d\times d}\big]$$

is surjective. Proposition 13 now yields that u^ε non-degenerate on E_j.

Analogously we set $X = X_j$ in the previous proof and replace u^ε with the restriction u_j^ε. To show that equation (14) holds we resort to the fact that

$$DF^{\mathrm{ev}}_{(\varepsilon,x)} = D\big(u_j^\varepsilon(x)\big)_{(\varepsilon,x)} = D\pi_j \circ (Du^\varepsilon(x))_{(\varepsilon,x)}$$

is surjective as a concatenation of surjective functions. Thus there also exists a dense set $E_\Sigma \subseteq \mathbb{R}$ on which u_j^ε is non-degenerate on X_j. Now the set

$$\overline{E} = \cap_j E_j \subseteq \mathbb{R} \tag{15}$$

is dense and for every $\varepsilon \in \overline{E}$ the function u^ε fulfils condition 1.

Now to the tangency regularity. □

As a consequence we get a version of the results in [2, §2].

We call a boundary point $x \in X_j$ on a strata X_j *ordinary* iff x lies in the interior of X_j and if there exists a stratum X_{j-1} that x is not critical point of X_{j-1}.

Proposition 17. *Assume all stagnation points of u are ordinary. Then there exists a positive $\delta > 0$ such that for (Lebesgue) almost every $\varepsilon \in B_\delta$ we have that*

- *u^ε is non-degenerate*
- *u^ε is tangency regular*
- *there is a one to one relation of the stagnation points in Ω and Σ preserving index and the property of being entrant or emergent.*

Proof. We follow [2, text]. First choose δ so small that for all stagnation points of u^ε lie in Ω. If there did not exist such a δ we could choose a sequence $\varepsilon_n \to 0$ in $\mathbb{R}^d$ such that u^{ε_n} had a stagnation point x_n on Σ. Since Σ is compact we can assume that x_n converges to a stagnation point x. But then x is also a stagnation point of u. A contradiction.

We note that since $\overline{\Omega}$ is compact we have that $\nabla u_\varepsilon \to \nabla u$ uniformly as $|\varepsilon| \to 0$.

complete proof

□

One of the reasons for introducing boundary tangency is the following proposition

Proposition 18. *Let u be tangency regular.*

Formulate that Σ^0 is a substratification with desired properties.

Proof. Let $u^\varepsilon \to u$. Then we have that □

Some general remarks

We make the following remarks

Proposition 19. *Let u be non-degenerate. Then the number of stagnation points is finite.*

Proof. Let x be a non-degenerate stagnation point. Since $Du(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq \Omega$ of x on which u is bijective. Hence x is the only stagnation point in U_x. Let C denote the set of all stagnation points of u. Then the sets U_x together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \tag{16}$$

form an open cover of $\overline{\Omega}$. But $\overline{\Omega}$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is in the finite subcover. Thus it follows that $\#C < \infty$ is finite. □

This was stated somewhere in Morse1969. Also, what is with the boundary stagnation points

As a consequence we obtain the following.

Corollary 20. *For a non-degenerate u the type numbers $M_0,\dots,M_d$ and the boundary type numbers $\mu_0,\dots,\mu_{d-1}$ are finite.*

State the theorem of Sard

We state Morse's lemma according to [4, p.145]

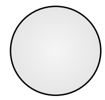
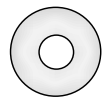
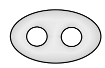
Domain	Picture	b_0	b_1	b_k, $k \geq 2$
Disk D		1	0	0
Annulus $2D \setminus D$		1	1	0
Two holed button		1	2	0

Table 1: Betti numbers for selected domains in $\mathbb{R}^2$.

Lemma 21. *Let $f\colon X \to \mathbb{R}$ be C^{2+r} and x be a non-degenerate critical point of index k. Then there exists a C^r chart (φ, U) at x such that we have*

$$f\colon \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{d} y_j^2 .$$

State proof.

Bring order into this section.

Betti numbers

Let $H_k(X;\mathbb{R})$ denote the k-th homology space of X. For an introduction and definition of these we refer the reader to [7, Chapter 2]. We define the k-th Betti number as the dimension

$$b_k = \dim_{\mathbb{R}} H_k(X;\mathbb{R}) . \tag{17}$$

We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

Example 22 (In flatland)**.** In $d = 2$ dimensions the 0-th Betti number counts the number of connected components of Ω and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely we give the Betti numbers for selected domains in table 1.

Example 23 (In spaceland)**.** In $d = 3$ dimensions the 0-th Betti number counts the number of connected components of Ω, the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 2.

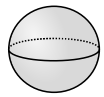
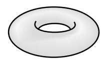
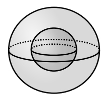
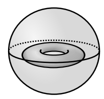
Domain	Picture	b_0	b_1	b_2	b_k, $k \geq 3$
Ball B		1	0	0	0
Solid torus $S^1 \times D$		1	1	0	0
Ball with bubble $2B \setminus B$		1	0	1	0
Ball with bubble in shape of torus		1	1	1	0

Table 2: Betti numbers for selected domains in $\mathbb{R}^3$.

Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.

The Morse inequalities

We state the Morse inequalities.

More citations.

Theorem 24 (Strong Morse inequalities)**.** *Let X be a manifold with corners and $f\colon X \to \mathbb{R}$ be essentially Morse. Then we have for $k \in \{0,\dots,d\}$ the inequalities*

$$\sum_{j=0}^{k} (-1)^{j+k} \operatorname{Ind}_j(f) \geq \sum_{j=0}^{k} (-1)^{k+j} b_j(X) \,.$$

For $k = d$ we in fact have equality

$$\sum_{j=0}^{d} (-1)^{j} \operatorname{Ind}_j(f) = \chi(X)$$

where the Euler characteristic

$$\chi(X) = \sum_{j=0}^{d} (-1)^{j} b_j(X)$$

is the alternating sum of the Betti numbers.

Proof. See [5, Theorem 10.2']. $\square$

Give outline of proof idea. The citation for this version is no longer up to date.

Corollary 25 (Weak Morse inequalities)**.** *Let X be a manifold with corners and $f\colon X \to \mathbb{R}$ essentially Morse. Then we have for $k \in \{0,\dots,d\}$ the inequalities*

$$\operatorname{Ind}_k(f) \geq b_k(X)\,.$$

Proof.

Write some proof.

□

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [5, Corollary 10.1].

Corollary 26 (Morse inequalities for f harmonic, $d = 2$)**.** *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M + \mu_1 - \mu_0 &= b_1 - b_0\,. \end{aligned}$$

In dimensions $d = 3$ we obtain [5, Corollary 10.2]

Corollary 27 (Morse inequalities for f harmonic, $d = 3$)**.** *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0 \\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= b_2 - b_1 + b_0\,. \end{aligned}$$

Give a classical example of a Morse function to determine the Betti numbers.

Give an outline of the proof.

On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2,3\}$.

Proposition 28 (Harmonic vector fields on simply connected domains)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f\colon \Omega \to \mathbb{R}$.*
2. *f is harmonic.*
3. *u is in fact C^∞.*
4. *The components $u_i = \partial_i f$ are harmonic.*

Proof. 1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.

2. This follows from $\Delta f = \operatorname{Div} u = 0$.

3. This follows from the fact that f is harmonic

4. This follows from $u_i = \partial_i f$.

$\square$

If one considers not necessarily simply connected domains Ω then we obtain the previous properties at least locally.

Harmonic functions, $d = 2$

The following result is essentially a negative to question 1 in $d = 2$ dimensions.

Proposition 29. *Let Ω be homeomorphic to $B_1 \subseteq \mathbb{R}^2$. Let further $f\colon \overline{\Omega} \to \mathbb{R}$ be regular harmonic with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \Sigma$ is not connected.*

We shall give two different proofs of this result. One involving level-sets and the other involving invariant manifolds

A proof involving level-sets

write omega-limit.

Sketch of Proof. Let $y_c = f(x_1)$ and $x_1, \dots, x_M$ be all the critical points such that $f(x_\cdot) = y_c$. We claim that the level set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

can be represented by a multigraph G which divides the boundary Σ into 4 components. To show this let $\gamma_i\colon (a_i, b_i) \to C$ for $i \in \{1, \dots, 4\}$ parametrise the curves in C intersecting at x_1. These can be constructed with the initial value problem

$$\begin{aligned} \gamma' &= (\nabla f)^{\perp}\big|_{\gamma} \\ \gamma(0) &= \gamma_0 \end{aligned}$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1. We assume that the intervals on which the γ_i are defined are maximal. We thus have for

$$\begin{aligned} \gamma_i^- &= \lim_{t \to a_i} \gamma(t) \\ \gamma_i^+ &= \lim_{t \to b_i} \gamma(t) \end{aligned}$$

that $\gamma_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $\nabla f^\perp = 0$. This argument can be applied to all of the $x_1, \dots, x_M$. We therefore have a situation similar to the one depicted in figure 2.

Thus C can be represented by a multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq C$. In the following we identify the graph G with its planar embedding in $\overline{\Omega}$. Assume G contains a cycle with vertex sequence $v_{i_1}, \dots, v_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

is the boundary of a domain E for which $f = y_c$ on ∂E. By the maximum principle $f = y_c$ on E and thus $f = y_c$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with the boundary Σ is at least four and thus the boundary Σ is divided into at least four components.

Now choose four neighbouring components $\omega_1, \dots, \omega_4$ as depicted in figure 3 Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle

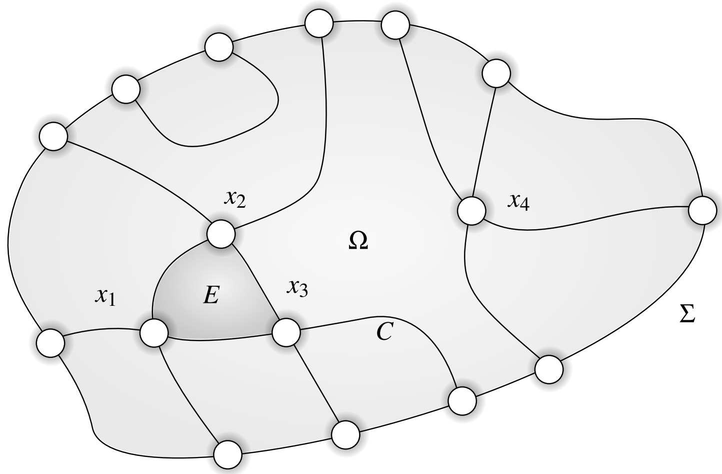


Figure 2: The situation at hand: The edges represent level curves and the interior vertices critical points.

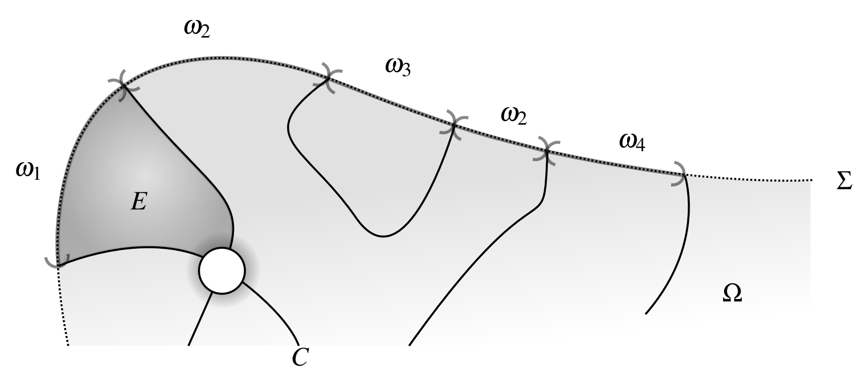


Figure 3: The choice of $\omega_1, \ldots, \omega_4$.

yields that ω_1 contains a local maximum or minimum of f since $f = y_c$ is constant on the other boundaries $\partial A \setminus \omega_1$. By the same argument $\omega_2, \dots, \omega_4$ also contain local extrema. Since the $\partial\omega_i$ cannot be extremal points on Σ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3 contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

use argument with ∇f here to show that extrema can be assumed to be alternating.

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. $\square$

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let $x_1, \dots, x_M$ denote the critical points of f. Let $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{1,2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i \colon (a_i, b_i) \to \overline{\Omega}$ for $i \in \{3,4\}$ be chosen to parametrise the stable manifolds of x_1. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^\pm \in \{x_1, \dots, x_M, \Sigma\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices $v_1, \dots, v_K$ and edges $e_1, \dots, e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Once again we identify the graph with its planar embedding. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle A with vertices $x_{i_1}, \dots, x_{i_J}$ and edges $e_{i_1}, \dots, e_{i_J}$. The set of cycles forms a partial ordering with respect to the property 'contains another cycle'. We can assume that our chosen cycle A contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to one another. We also note that the edges cannot cross. We can thus describe the trail $x_{i_1}, \dots, x_{i_J}$ by a set of directives of the type

More precise.

$$(d_1, \dots, d_K) \in \{\mathrm{l}, \mathrm{r}, \mathrm{s}\}^J .$$

Here l, r and s stand for 'left', 'right' and 'straight' respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 4.

An example of the trail 'srsr' is given in figure 5. We now note that cycles of the type r,...,r or l,...,l cannot occur as we otherwise would have a directed cycle. Thus there

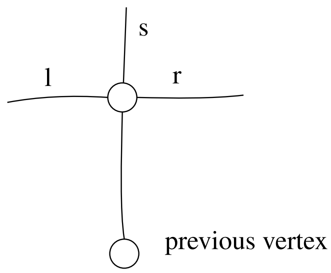


Figure 4: Explanation of the directives ‘l’,‘r’ and ‘r’.

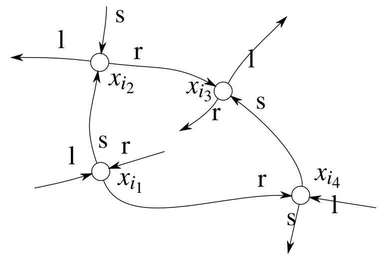


Figure 5: An example for a cycle.

exists a vertex where the chosen direction is s. Without loss of generality this vertex is x_{i_1}. Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1}. Now consider new cycle B starting at x_{i_1} with directives r,...,r. Since all vertices of B lie within the cycle A we must at some step reach a vertex on the cycle A. But then cycle B is a new distinct cycle contained in cycle A, a contradiction to the minimality of A. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

Now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur. We now pick a tree $\widetilde{G}$ out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each ‘neighbouring’ leaf of this tree has opposite signage and the claim follows. □

elaborate

elaborate

A proof involving Morse theory

fix the following proof

Proof. Assume that Σ^- is connected. Then we can cut the domain along Γ such that the endpoints of the cut coincide with the endpoints of Σ^-, that is $\partial\Gamma = \partial\Sigma^-$. Now we obtain two new domains X^+ and X^- such that $\partial X^+ = \Sigma^+ \cup \Sigma^0 \cup \overline{\Gamma}$ and $\partial X^- = \Sigma^- \cup \overline{\Gamma}$. We can assume that Γ is a smooth manifold and corresponds to the stratum X_Γ for X^+ and X^-. We also assume that the corner points $x_1, x_2 \in \partial\Gamma$ correspond to the strata X_1 and X_2. For the following argumentation we require that u is Morse on both X^+ and X^- so assume this is the case. Locally around the corner point x_1 we have a situation depicted as in figure 6 That is $u = \nabla f$ is essentially parallel to the boundary Σ. We assume that we chose Γ such that the acute angle is on the side where u flows into the new domain. Thus we have that x_1 is not a critical point of either f nor $-f$. Analogously we can choose Γ in such a way around x_2. We now focus our attention on X^+. Since no essential critical points lie on Σ^+ or $\partial\Gamma$ it follows for the boundary type numbers that

$$\mu_j^+ = \mathrm{Ind}_{\Gamma,j}(f)\,. \tag{18}$$

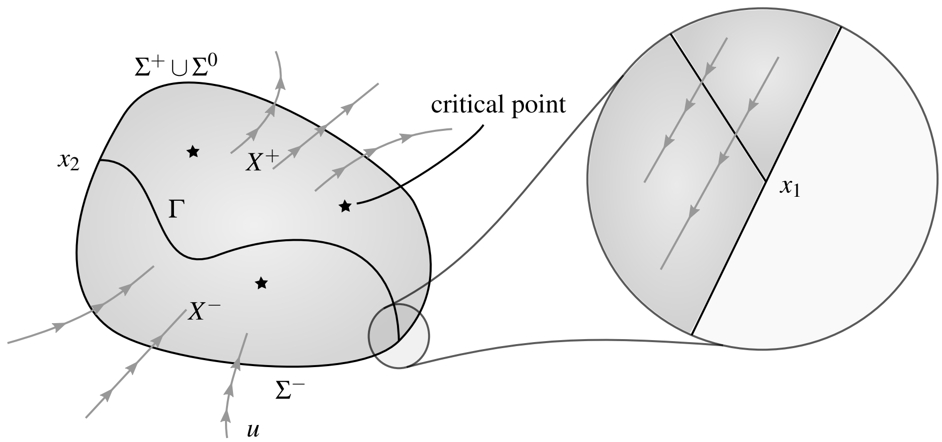


Figure 6: The situation at hand.

Analogously we have on X^- that

$$\nu_j^- = \mathrm{Ind}_{\Gamma,j}(-f)\,. \tag{19}$$

In addition we have that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^-, that is

$$\mathrm{Ind}_{\Gamma,0}(f) = \mathrm{Ind}_{\Gamma,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(f) = \mathrm{Ind}_{\Gamma,0}(-f) \tag{20}$$

Using equations (35), (36) and (37) we obtain

$$\mu_0^+ = \nu_1^- \qquad \text{and} \qquad \mu_1^+ = \nu_0^-\,. \tag{21}$$

Consider the Morse inequality for f

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X)\,. \tag{22}$$

and the Morse inequality for $-f$

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X)\,. \tag{23}$$

We now add equations (39) and (40) and insert relations (38) to obtain

$$M^- + M^+ = -2\chi(X) < 0$$

in contradiction to $M^\pm \geq 0$.

□

Allowing for Inflow and outflow

The strategy in the above proofs can be generalised to show the following

Conjecture 30. *Let $X \subseteq \mathbb{R}^2$ be a manifold with corners with Betti numbers $b_0 = 1$ and b_1. Let further $f\colon X \to \mathbb{R}$ be Morse harmonic with M critical points. Assume that $\overline{\Sigma^-} \subseteq \Sigma$ on a given connected component of the boundary Σ consists of at most one connected component. Then we have*

$$\frac{4}{3}M \leq b_1 + 1\,.$$

This inequality can probably be improved considerably.

Let $J^{\pm}$ denote the number of connected components of $\Sigma^{\pm}$. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\geq 0} \sqcup \Sigma_{\leq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma^{\geq 0}$ of all such decompositions. We state a consequence of a result from [8, Theorem 2.1]

Proposition 31. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u\colon \overline{\Omega} \to \mathbb{R}$ be harmonic (with certain conditions on the boundary). Then we have*

$$M \leq b_1 - b_0 + \frac{J^+ + J^-}{2}\,.$$

If in addition we assume that there are no critical points on the boundary then we have

$$M \leq b_1 - b_0 + J^{\geq 0}\,.$$

Proof. See [8, Theorem 2.1]. ☐

Harmonic vector fields, $d = 2$

No inflow or outflow

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ iff $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* iff $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can state the following result.

Proposition 32 (Upper bound on M)**.** *Let $d = 2$ and Ω be a compact manifold with corners with Betti numbers $b_0 = 1$, and b_1. Let further $u\colon X \to \mathbb{R}^2$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$M + 1 \leq b_1 \,.$$

Sketch of proof. As in the second proof of proposition 29 the critical manifolds form a directed multigraph. Since no critical manifold can intersect with the boundary each vertex of the graph has degree 4 and we thus have $2M$ edges. Now we obtain with Euler's polyhedron formula for a planar graph with multiple components

$$\begin{aligned}
\#\text{ minimal cycles} &= \#\text{faces} - 1 \\
&= 1 + \#\text{components} - \#\text{vertices} + \#\text{edges} - 1 \\
&\geq 1 + 1 - M + 2M - 1 = M + 1
\end{aligned}$$

Here we use the term 'minimal' as in the second proof of proposition 29. Note that each minimal cycle must contain a hole of the domain since else we could restrict u to a simply connected region containing this cycle. Then by proposition 28 u would correspond to the gradient of a harmonic function in this region and we would obtain a contradiction as in the proof of proposition 29. Hence the number of minimal cycles is a lower bound on the number of holes b_1 of the domain. $\square$

In fact using the Morse inequalities we can obtain the stronger result.

Proposition 33. *Let $X \subset \mathbb{R}^2$ be a compact manifold with corners and Betti numbers $b_0 = 1$, and b_1 and let $u\colon X \to \mathbb{R}^2$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$M + 1 = b_1$$

.

Sketch of proof. We slit Ω such that it is homeomorphic to the disk as is depicted in figure 7. Denote the slit by Γ. Since the number of critical points is finite by proposition ??, we can choose Γ in such a way that it does not contain any critical points. We also denote the points at which Γ meets Σ by $x_1, \ldots, x_{2b_1} \in \partial\Gamma$. Note that there are $2b_1$ many such points. We can assume that Γ is a smooth manifold. Now at the point x_1 we have that u is almost parallel to the boundary Σ. Thus we can slant the cut in such a way such that x_1 is an essential critical point of index 0 of u on the stratification of $\tilde{X}$. Here $\tilde{X}$ denotes the covering space of X generated by the cut Γ. We denote the induced strata by Γ also with Γ. Note that x_1 then is no essential critical point for $-u$. We modify the cut for the other points $x_2, \ldots, x_{2b_1}$ as with x_1. The situation is depicted in figure 7. Since

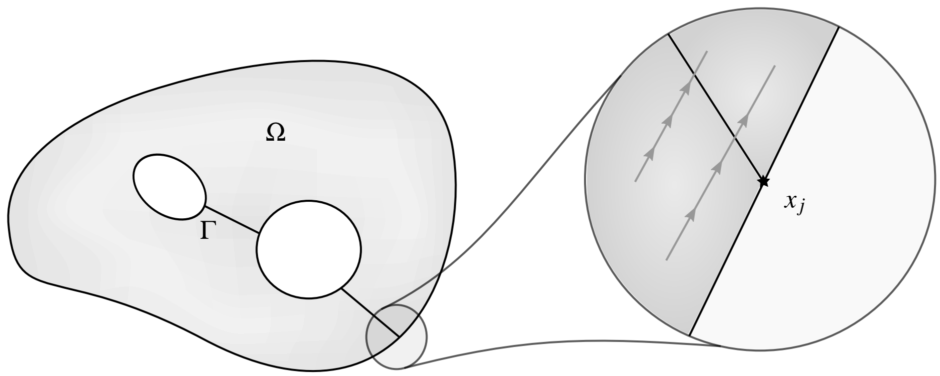


Figure 7: How we slit the domain.

there are no critical points on Σ all boundary critical points of u are on the strata induced by Γ and $x_1,\ldots,x_{2b_1}$. Hence we have relations

$$\mu_k = \mathrm{Ind}_{\Gamma,k}(u) + 2b_1\delta_{k0} \qquad \text{and} \qquad \nu_k = \mathrm{Ind}_{\Gamma,k}(-u) \tag{24}$$

for all $k \in \{0,1\}$. Since on Γ all entrant critical points of u are also emergent critical points of $-u$ (and vice versa) we have the relations

$$\mathrm{Ind}_{\Gamma,0}(u) = \mathrm{Ind}_{\Gamma,1}(-u) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(u) = \mathrm{Ind}_{\Gamma,0}(-u)\,. \tag{25}$$

Equations (24) and (25) yield

$$\mu_0 = \nu_1 + 2b_1 \qquad \text{and} \qquad \mu_1 = \nu_0\,. \tag{26}$$

Since Ω is now simply connected u is by proposition 28 the gradient of a harmonic function f on this new domain. For this f we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \tag{27}$$

and for $-f$ the Morse inequalities

$$M + \nu_1 - \nu_0 = -\chi(\tilde{X}) = -1\,. \tag{28}$$

Adding equations (27) and (28) and using the relation (26) we obtain

$$2M - 2b_1 = -2$$

from which the claim follows. $\square$

We now give an alternative proof using the argument principle.

One could use the argument principle for Riemann surfaces.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition ?? u is the gradient of a harmonic function f on this new domain. Let $h \in \mathrm{Hol}(\mathbb{C})$ be the holomorphic function given by $h = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ. We now determine the change of argument $\arg h$ along γ. For this consider first the parts of γ traversing the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other the contribution in the change of $\arg h$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h$ is 2π for every hole in the domain since $h = u$ is tangent to Σ and traverses the holes clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have b_1 holes in the domain the total change of $\arg h$ as γ traverses Σ is $2\pi(b_1 - 1)$. Since h has no poles it follows from the argument principle (see for example [9, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M \tag{29}$$

From this the claim follows. □

In the following we would like to give examples for harmonic vector fields. In order to do this we define two differential operators for $d = 2$ by

$$\nabla^\perp f = \operatorname{Curl} f = \begin{bmatrix} -\partial_2 f \\ \partial_2 f \end{bmatrix}$$

and

$$\operatorname{curl} u = -\partial_1 u_2 + \partial_2 u_1$$

Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

The following proposition gives us a recipe to generate harmonic vector fields in $d = 2$ dimensions.

Proposition 34. *Let $\psi\colon \Omega \to \mathbb{R}$ be harmonic then $\nabla^\perp f$ is a harmonic vector field.*

Proof. Since $\operatorname{Div} \nabla^\perp = 0$ we have

$$\operatorname{Div} u = \operatorname{Div} \nabla^\perp \psi = 0$$

and one calculates

$$\operatorname{curl} u = \operatorname{curl} \nabla^\perp \psi = -\Delta \psi = 0\,.$$

□

The function ψ is also called a stream function.

We now give an example of a harmonic vector field without inflow or outflow and with one critical point. For this consider the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + \Phi_2(x + e_1) \end{aligned} \tag{30}$$

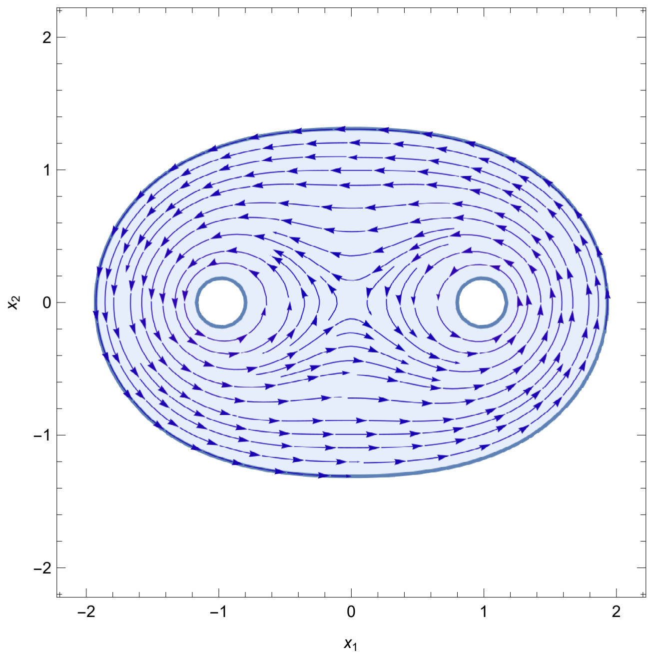


Figure 8: A plot of $u = \nabla^\perp \psi$ in the region $\psi^{-1}([-1,1])$. Here ψ is given by equation (30).

where

$$\Phi_2 = \log(|\cdot|)$$

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and $e_i = \delta_{i\cdot}$ are the unit vectors. Figure 8 indicates that $u = \nabla^\perp \psi$ has the desired properties.

In a second example given by [10] we fix the domain rather than the function. For this set $\overline{\Omega} = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We then have the system

$$\begin{aligned} \Delta\psi &= 0 \quad \text{, on } \Omega \\ \psi &= 0 \quad \text{, on the outer ring } 4S^1 \\ \psi &= 1 \quad \text{, on the inner rings } S^1(-2e_1) \cup S^1(2e_1) \end{aligned} \tag{31}$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 9.

An example of inflow on one side and outflow on the other

In the following we aim to give examples of domains in $d = 2$ dimensions for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component. For this consider first the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R}^2 \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{32}$$

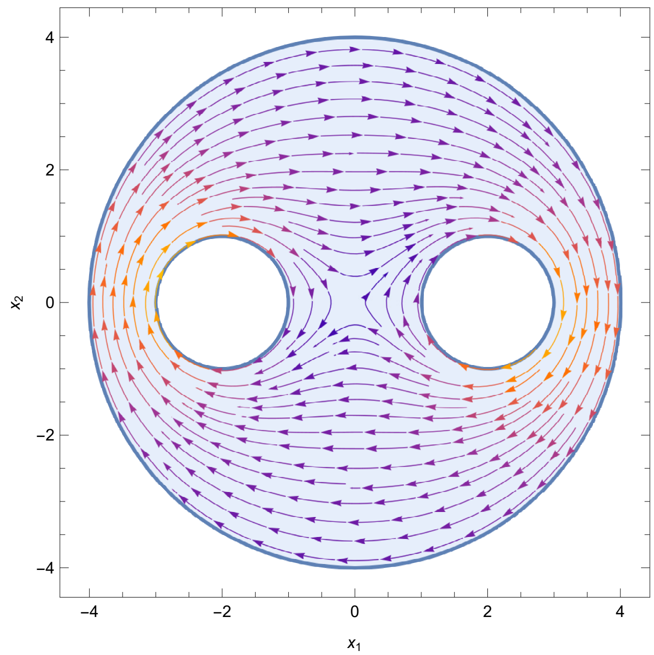


Figure 9: A plot of $u = \nabla^\perp \psi$ where ψ is the numerical solution to (31).

Figure 10 indicates that $u = \nabla^\perp \psi$ fulfils the requirements.

Now we would like to have a harmonic vector field similar to the example with two holes with inflow on the one side and outflow on the other. For this consider the streamline

$$\begin{aligned} u\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R}^2 \\ x &\mapsto \Phi_2(x-e_1) - \Phi_2(x+e_1) + x_1 \end{aligned} \tag{33}$$

Figure 11 indicates that $u = \nabla^\perp \psi$ is the function we are looking for.

In another example given by [10] we once again fix the domain rather than the function. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We now have the system

$$\begin{aligned} \Delta\psi &= 0 && \text{, on } \Omega \\ \psi &= 0 && \text{, on the outer ring } 4S^1 \\ \psi &= -1 && \text{, on the left inner ring } S^1(-2e_1) \\ \psi &= 1 && \text{, on the right inner ring } S^1(2e_1) \end{aligned} \tag{34}$$

We solve this system numerically and set $u = \nabla^\perp \psi$. The result is plotted in figure 12.

Check the signs of this example. Give explanation for why it works.

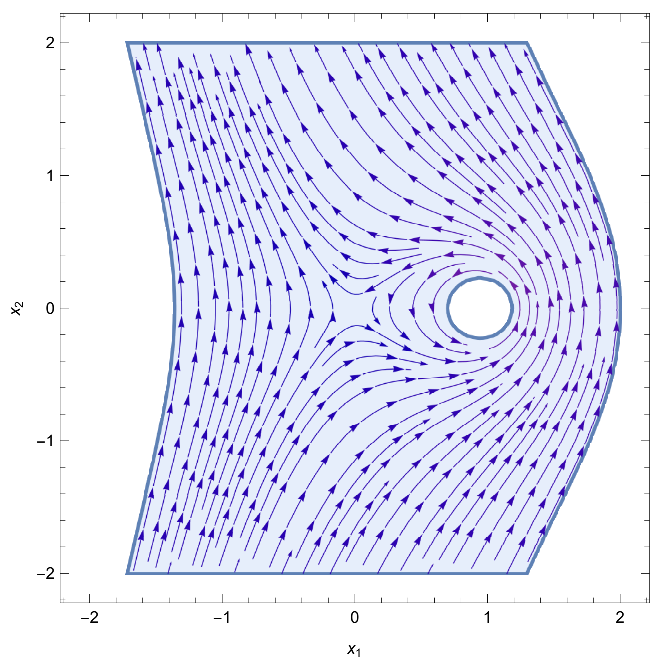


Figure 10: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap \mathbb{R} \times [-2,2]$. Here ψ is given by equation (32).

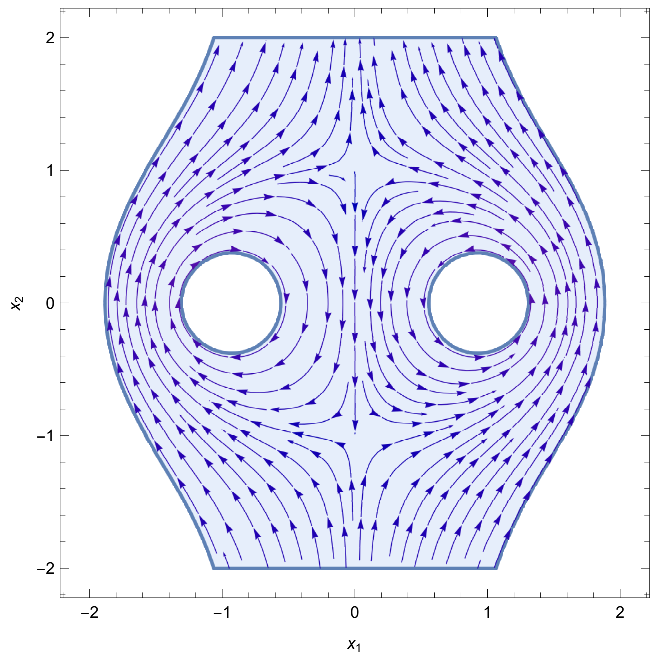


Figure 11: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap \mathbb{R} \times [-2, 2]$. Here ψ is given by equation (33).

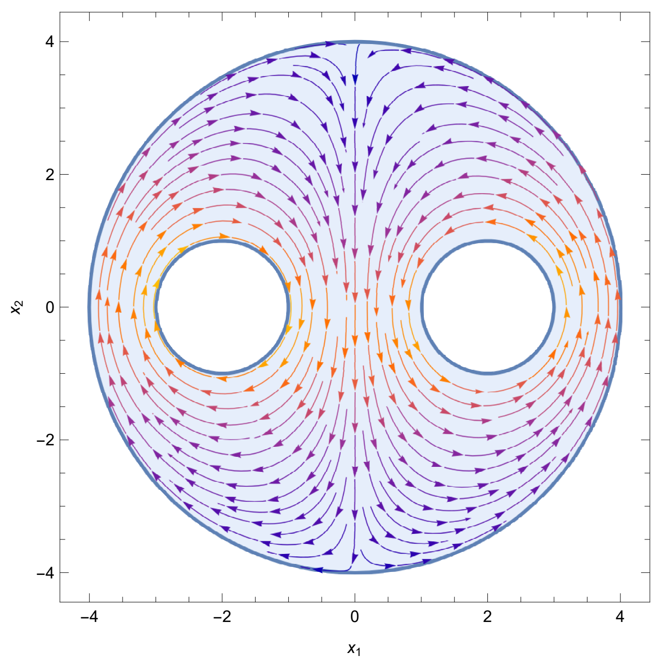


Figure 12: A plot of $u = \nabla^{\perp}\psi$ where ψ is the numerical solution to (34).

Harmonic functions, $d = 3$

The cylinder

The following proof comes from [10]

Proposition 35. *Let $\Omega = (0,1) \times B_1 \subseteq \mathbb{R}^3$ be the cylinder. Let further $f \colon \overline{\Omega} \to \mathbb{R}$ be regular harmonic with no inflow or outflow on the sides $\partial(0,1) \times B_1$, no outflow on $\{0\} \times B_1$ and no inflow on $\{1\} \times B_1$. Then f cannot have a critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ. Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times S^1$ such that $\partial_1 f(x)$ is minimal on $\overline{\Omega}$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. $\square$

Harmonic vector fields, $d = 3$

We obtain as a quick consequence of the hairy ball theorem

Proposition 36. *Let Ω have Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$b_2 \leq b_1\,.$$

Proof. Assume not. Since Ω has b_2 bubbles and b_1 holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no inflow or outflow on Γ we have that the restriction $u|_\Gamma \in T\Gamma$ is a vector field on Γ. Since u is regular $u|_\Gamma$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. $\square$

A little more rigour would not harm.

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 37. *Let $X \subset \mathbb{R}^3$ be a compact manifold with corners homeomorphic to the ball B. Let $f\colon X \to \mathbb{R}$ be a Morse harmonic function. Assume that Σ^- is simply connected. Then we have that*

$$M_1 = M_2$$

Proof. As in the two dimensional case we split the domain Ω with a plane Γ such that $\partial\Gamma = \gamma = \partial\Sigma^-$. Denote the two arising domains X^+ and X^- where $\partial X^+ = \Sigma^+ \cup \Sigma^0 \cup \overline{\Gamma}$ and $\partial X^- = \Sigma^- \cup \overline{\Gamma}$. We can assume that Γ is a smooth manifold. Since by proposition ?? there are finitely many critical points in Ω we can also assume that no interior critical points lie on Ω. We now look at a critical point x on Γ. We choose the slant at which Γ approaches γ in such a way that x is neither an essential critical point of f nor of $-f$.

Why are there finitely many?

We now turn our attention to X^+. Since no essential critical points lie on Σ^+ or γ it follows for the boundary type numbers that

More details here.

$$\mu_j^+ = \mathrm{Ind}_{j,\Gamma}(f)\,. \tag{35}$$

Analogously we have on X^- that

$$\nu_j^- = \mathrm{Ind}_{j,\Gamma}(-f)\,. \tag{36}$$

In addition we have that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^-, that is

$$\begin{aligned} \mathrm{Ind}_{0,\Gamma}(u) &= \mathrm{Ind}_{2,\Gamma}(-f) \\ \mathrm{Ind}_{1,\Gamma}(u) &= \mathrm{Ind}_{1,\Gamma}(-f) \\ \mathrm{Ind}_{2,\Gamma}(u) &= \mathrm{Ind}_{0,\Gamma}(-f) \end{aligned} \tag{37}$$

Using equations (35), (36) and (37) we obtain

$$\begin{aligned} \mu_0^+ &= \nu_2^- \\ \mu_1^+ &= \nu_1^- \\ \mu_2^+ &= \nu_0^- \end{aligned} \tag{38}$$

We observe the Morse inequalities for f

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X)\,. \tag{39}$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \tag{40}$$

where the M_j continue to denote the interior type numbers of f. We now subtract equation (39) from (40) and insert relations (38) to obtain

$$M_1^- - M_2^- + M_1^+ - M_2^+ = 0$$

from which the claim follows. $\square$

In fact we can give an example for such a function with simply connected entrant boundary.

Example 38 (Example of a harmonic vector field with simply connected entrant boundary)**.** Consider the domain $X = \overline{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1 x_2^2 + x_2 x_3 \end{aligned}$$

This induces the harmonic vector field

$$\begin{aligned} u\colon X &\to \mathbb{R}^3 \\ x &\mapsto \begin{bmatrix} x_1(1-x_1) + x_2^2 \\ x_2(2x_1 - 1) + x_3 \\ x_2 \end{bmatrix} \end{aligned}$$

It follows from setting $u(x) = 0$ implies that $x_2 = 0$ and then that $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of u. Conversely these are zeroes of u.

Figure 13 indicates that f has the desired properties.

complete this section.

Try to say something about the case without inflow or outflow.

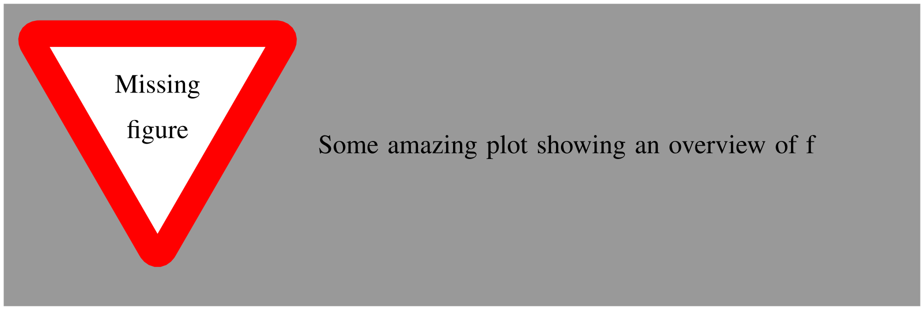


Figure 13: An plot of the function u

Harmonic functions, $d = 4$

Define the harmonic function

$$\begin{aligned} f\colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{41}$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_\pm \colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{42}$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right)$$

Using condition (41) we obtain the equivalent condition

$$0 < \pm 1 - 2\left(x_3^2 + x_4^2\right)$$

Define the cylinder

$$C = \left\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \right\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (42) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm\left(\Sigma^+\right).$$

Analogously we have

$$B_1 \setminus C = \psi_\pm\left(\Sigma^-\right).$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 14.

Check that the transition at the boundary is legal.

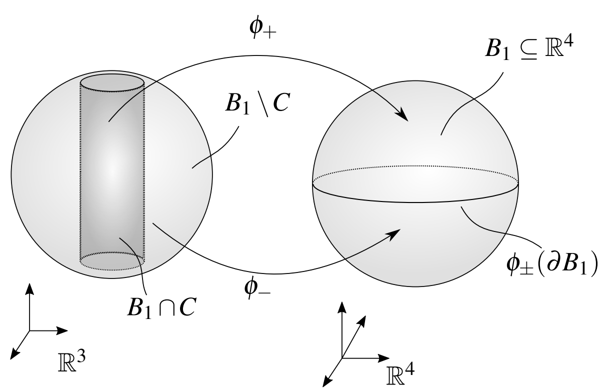


Figure 14: Visualisation of the situation.

□

Symbols

d	Dimensions $d = 2$ or $d = 3$
Ω	Domain in $\mathbb{R}^d$, assumed to be $\mathrm{int}(X)$
Σ	Boundary of Ω or X
$f\colon X \to \mathbb{R}$	A C^2 mapping, often assumed harmonic
$u\colon X \to \mathbb{R}^d$ or $T^*\overline{\Omega}$	A C^1 vector field, often assumed harmonic
X	A compact manifold with corners, assumed to be $X = \overline{\Omega}$
Y	A manifold
X_j	A stratification of X as given in definition 4. Often but not always assumed to be given by equation 1
u_j	Restriction of u to the cotangent bundle T^*X_j, see equation 5
Σ^-	entrant boundary, see definition 6
Σ^+	emergent boundary, see definition 6
Σ^0	tangential boundary, see definition 6
M_k	interior type numbers
M	Total number of stagnation points
μ_k	boundary type numbers of f, see definition **??**
ν_k	boundary type numbers of $-f$, see definition **??**
u_ε	modification to u as in equation (12)
A	submanifold, can be thought of as the zero section of T^*X
b_k	Betti number as defined in equation (17)

Change Gamelin to Lang, complex analysis

Bibliography

[1] R. Shelton, “Critical points of harmonic functions on domains in $\mathbf{R}^3$,” *Trans. Amer. Math. Soc.*, vol. 261, no. 1, pp. 137–158, 1980, ISSN: 0002-9947,1088-6850. DOI: 10.2307/1998322. [Online]. Available: https://doi.org/10.2307/1998322.

[2] M. Morse, “Equilibrium points of harmonic potentials,” *J. Analyse Math.*, vol. 23, pp. 281–296, 1970, ISSN: 0021-7670,1565-8538. DOI: 10.1007/BF02795505. [Online]. Available: https://doi.org/10.1007/BF02795505.

[3] D. G. C. Handron, “Generalized billiard paths and Morse theory for manifolds with corners,” *Topology Appl.*, vol. 126, no. 1-2, pp. 83–118, 2002, ISSN: 0166-8641,1879-3207. DOI: 10.1016/S0166-8641(02)00036-6. [Online]. Available: https://doi.org/10.1016/S0166-8641(02)00036-6.

[4] M. W. Hirsch, *Differential topology*, ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994, vol. 33, pp. x+222, Corrected reprint of the 1976 original, ISBN: 0-387-90148-5.

[5] M. Morse and S. S. Cairns, *Critical point theory in global analysis and differential topology: An introduction*, ser. Pure and Applied Mathematics. Academic Press, New York-London, 1969, vol. Vol. 33, pp. xii+389.

[6] G. Katz, “Traversally generic & versal vector flows: Semi-algebraic models of tangency to the boundary,” *Asian J. Math.*, vol. 21, no. 1, pp. 127–168, 2017, ISSN: 1093-6106,1945-0036. DOI: 10.4310/AJM.2017.v21.n1.a3. [Online]. Available: https://doi.org/10.4310/AJM.2017.v21.n1.a3.

[7] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544, ISBN: 0-521-79160-X; 0-521-79540-0.

[8] G. Alessandrini and R. Magnanini, “The index of isolated critical points and solutions of elliptic equations in the plane,” *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, vol. 19, no. 4, pp. 567–589, 1992, ISSN: 0391-173X,2036-2145. [Online]. Available: http://www.numdam.org/item?id=ASNSP_1992_4_19_4_567_0.

[9] T. W. Gamelin, *Complex analysis*, ser. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001, pp. xviii+478, ISBN: 0-387-95093-1; 0-387-95069-9. DOI: 10.1007/978-0-387-21607-2. [Online]. Available: https://doi.org/10.1007/978-0-387-21607-2.

[10] Wahlén, Erik, *In private communication.* 2023.

[11] master-thesis, *Github repository to the thesis.* Online, 2023. [Online]. Available: https://github.com/TheoKoppenhoefer/master-thesis.

[12] A. Banyaga and D. Hurtubise, *Lectures on Morse homology*, ser. Kluwer Texts in the Mathematical Sciences. Kluwer Academic Publishers Group, Dordrecht, 2004, vol. 29, pp. x+324, ISBN: 1-4020-2695-1. DOI: 10.1007/978-1-4020-2696-6. [Online]. Available: https://doi.org/10.1007/978-1-4020-2696-6.

[13] L. C. Evans, *Partial differential equations*, Second, ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, vol. 19, pp. xxii+749, ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019. [Online]. Available: https://doi.org/10.1090/gsm/019.

[14] M. C. Irwin, *Smooth dynamical systems*, ser. Advanced Series in Nonlinear Dynamics. World Scientific Publishing Co., Inc., River Edge, NJ, 2001, vol. 17, pp. xii+259, Reprint of the 1980 original, With a foreword by R. S. MacKay, ISBN: 981-02-4599-8. DOI: 10.1142/9789812810120. [Online]. Available: https://doi.org/10.1142/9789812810120.

[15] M. Morse, "Relations between the critical points of a real function of n independent variables," *Trans. Amer. Math. Soc.*, vol. 27, no. 3, pp. 345–396, 1925, ISSN: 0002-9947,1088-6850. DOI: 10.2307/1989110. [Online]. Available: https://doi.org/10.2307/1989110.

[16] J. L. Walsh, *The Location of Critical Points of Analytic and Harmonic Functions*, ser. American Mathematical Society Colloquium Publications. American Mathematical Society, New York, 1950, vol. Vol. 34, pp. viii+384.

[17] Z. Nehari, *Conformal mapping*. Dover Publications, Inc., New York, 1975, pp. vii+396, Reprinting of the 1952 edition.

[18] J. Morgan, *Morse theory lectures*, Columbia University, 2020. [Online]. Available: https://www.math.columbia.edu/~jmorgan/syllabus.pdf.

[19] H. Deng, H. Liu, and X. Yang, "Critical points of solutions to a kind of linear elliptic equations in multiply connected domains," *Israel J. Math.*, vol. 249, no. 2, pp. 935–971, 2022, ISSN: 0021-2172,1565-8511. DOI: 10.1007/s11856-022-2330-6. [Online]. Available: https://doi.org/10.1007/s11856-022-2330-6.

[20] A. A. Agrachëv and S. A. Vakhrameev, "Morse theory and optimal control problems," in *Nonlinear synthesis (Sopron, 1989)*, ser. Progr. Systems Control Theory, vol. 9, Birkhäuser Boston, Boston, MA, 1991, pp. 1–11, ISBN: 0-8176-3484-3.

[21] G. Katz, "Flows in Flatland: A romance of few dimensions," *Arnold Math. J.*, vol. 3, no. 2, pp. 281–317, 2017, ISSN: 2199-6792,2199-6806. DOI: 10.1007/s40598-016-0059-1. [Online]. Available: https://doi.org/10.1007/s40598-016-0059-1.