

Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

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Todo list

2do	3
Some amazing introduction	5
Some proof	10
Fill in the details for the following.	13
The following implication is false, look into [5] for fix.	14
Bring order into this section.	15
This was stated somewhere in [6].	15
State the theorem of Sard	15
State proof.	15
Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.	16
Give outline of proof idea.	17
Write some proof.	18
Give a classical example of a Morse function to determine the Betti numbers.	18
write omega-limit.	20
use argument with ∇f here to show that extrema can be assumed to be alternating.	21
More precise.	22
elaborate	23
elaborate	23
One could use the argument principle for Riemann surfaces.	27
Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.	28
continue this argument	37
continue proof	38
Rewrite: It should be much easier arguing in \mathbb{R}^3	38
insert picture here, remove quadrant notation.	39
fill in the details.	41
complete this section.	42
A little more rigour would not harm.	44
Check that the transition at the boundary is legal.	47
Change Gamelin to Lang, complex analysis	48

General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.
- Define traversing vector field

Some questions

- Should I state Hopf's Lemma?

1 Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of \mathbb{R}^d with boundary $\Sigma = \partial X$ and nonempty interior $\Omega = \text{int}(X)$. In the following we will work in dimensions $d \in \{2, 3\}$. Unless otherwise stated we denote by

$$f: X \rightarrow \mathbb{R}$$

a C^2 function on X . Often f will be assumed to be harmonic. We also denote by

$$u: X \rightarrow \mathbb{R}^d$$

a vector field of class C^1 . In the following we often assume that u is in fact *harmonic*, that is u fulfils $\text{Div } u = 0$ and $\text{curl } u = 0$. Often but not always we assume that in fact $u = \nabla f$ is a gradient field. One question we seek to answer in this thesis is the following:

Question 1.1 (Flowthrough with stagnation point). Does there exist a region $X \subseteq \mathbb{R}^3$ homeomorphic to a ball with flow u through the region such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. the boundary on which u enters the region is simply connected?

The answer for this will turn out to be ‘yes’ for dimensions $d \geq 3$ and ‘no’ for $d = 2$ dimensions. Another question we will consider is of the type:

Question 1.2 (stagnation points of harmonic vector fields without inflow or outflow). Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary behaviour.

General definitions

We start by requiring some regularity for the boundary of X . More precisely, we require X to be a compact Riemannian manifold with corners: