# Stagnation points of harmonic vector fields and the domain topology

Some applications of Morse theory

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Master's thesis
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## Abstract

Given a harmonic vector field $u$ on a bounded domain we call the set on which $u$ enters the domain entrant boundary and the set on which $u$ exits the domain emergent boundary. In the first part of this thesis we ask the question of when it is possible to have interior stagnation points and connected entrant and connected emergent boundaries. The answer in two dimensions is essentially 'no'. If one allows for holes in the domain the answer becomes a 'yes' for which we give explicit examples. In four dimensions a simple example shows that this is possible with the ball as the domain. In three dimensions we use Morse theory to argue that the number of stagnation points is even. With the help of this result and numerical methods we found a harmonic polynomial on the ball which has interior stagnation points and simply connected entrant and emergent sets.

The second part revolves around whether there exist harmonic vector fields with interior stagnation points and without boundary stagnation points. This question in two dimensions yields a very elegant relation between the domain topology and the number of interior critical points. As a special case we obtain statements about harmonic vector fields which are tangential at the boundary. We also give examples of vector fields illustrating this point. In three dimensions we do not find such an example but we use Morse theory to infer that for such an example the number of critical points has to be even.

As part of the thesis we formulate a version of Morse theory by [1] and [11] for manifolds with boundaries. This relates the domain topology with the number of critical points. Additionally we give a proof for the density of Morse functions for this type of Morse theory.



## Populärvetenskaplig sammanfattning

Givet ett harmoniskt vektorfält $u$ på en begränsad domän kallar vi den mängd där $u$ träder in i domänen för entrantrand och den mängd där $u$ lämnar domänen emergentrand. I den första delen av detta examensarbete ställer vi frågan om när det är möjligt att ha inre stagnationspunkter och sammanhängande entrant- och sammanhängande emergentrand. Svaret i två dimensioner är i huvudsak 'nej'. Om man tillåter hål i domänen blir svaret ett 'ja', vilket vi ger explicita exempel på. I fyra dimensioner visar ett enkelt exempel att detta är möjligt med bollen som domän. I tre dimensioner använder vi Morse-teorin för att visa att antalet stagnationspunkter är jämnt. Med hjälp av detta resultat och numeriska metoder fann vi ett harmoniskt polynom på bollen som har inre stagnationspunkter och enkelt sammanhängande entrant- och emergentrand.

Den andra delen handlar om huruvida det finns harmoniska vektorfält med inre stagnationspunkter och utan stagnationspunkter på gränsen. Denna fråga i två dimensioner ger ett mycket elegant samband mellan domänens topologi och antalet inre kritiska punkter. Som ett specialfall får vi uttalanden om harmoniska vektorfält som är tangentiella vid randen. Vi ger också exempel på vektorfält som illustrerar denna punkt. I tre dimensioner hittar vi inget sådant exempel, men vi använder Morse-teorin för att dra slutsatsen att för ett sådant exempel antalet kritiska punkter måste vara jämnt.

Som en del av examensarbetet formulerar vi en version av Morse-teorin enligt [1] och [11] för mångfalder med kantiga ränder. Denna relaterar domäntopologin till antalet kritiska punkter. Dessutom ger vi ett bevis för tätheten av Morsefunktioner för denna typ av Morse-teori.


## Acknowledgements

Heap praise on people here. In particular: Erik Wahlén, Martin Löfström, Ludvig Sundell, Anne-Lisa Rathsmann, Thomas Renström, others

# Contents

General TODOs

- Check for typos.
- add Alber to sources
- Should I call thesis ‘Some applications of Morse theory to harmonic vector fields’?
- Make sure tangential to boundary is excluded.

# 1 Introduction

Harmonic vector fields arise naturally within certain areas of mathematics and physics. Classical electrostatic, magnetic and gravitational fields in vacuum are examples of harmonic vector fields. Other examples are the heat flow in a system that has reached steady state or an irrotational flow through an inviscid incompressible medium. The null points of for instance the electrostatic or magnetic fields are particularly interesting. These are examples of stagnation points of harmonic vector fields. If one considers the underlying potentials which exist at least locally then these stagnation points are critical points of a harmonic function. The study of these points has a long tradition. Walsh gives in [30] a comprehensive overview of the state of knowledge regarding critical points of harmonic functions in 1950. In particular complex analysis by that time had clarified the situation in two dimensions significantly though far less was (and is) known about the situation in three dimensions. 20 years later Morse applies his theory in [25] and [24] to the critical points of harmonic functions. Morse gives in both works a set of inequalities for two and three dimensions relating the number and type of critical points on the boundary and the interior with the domain topology. He then poses in [24] the question: To a given set of numbers of critical points and a domain topology fulfilling these inequalities does there exists a harmonic function in $\mathbb{R}^3$ with precisely these numbers of critical points? This question is answered affirmatively in 1980 in [27] with a construction using line charges. Inspired by this we will apply Morse theory to harmonic functions in an attempt to answer the following question:

**Question 1.1** (Flowthrough with stagnation point)**.** Does there exist a domain $X \subseteq \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ on $X$ such that

1. $u$ has an interior stagnation point
2. the boundary on which $u$ enters the region is simply connected?

The answer to this will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Somewhat related, [19, p.198] poses the question whether it is possible to have a harmonic vector field without inflow or outflow through the boundary with interior stagnation points in $\mathbb{R}^3$. Inspired by this we will also consider the following question:

**Question 1.2** (Harmonic vector fields without inflow or outflow)**.** Let $u$ be a harmonic vector field in a domain $X$ such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

In fact we will use Morse theory to deal with the following more general question:

**Question 1.3** (Harmonic vector fields without boundary stagnation points)**.** Let $u\colon X \to \mathbb{R}^d$ be a harmonic vector field on a domain $X$ without boundary stagnation points. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. More concretely we will show that the number of stagnation points equals the negative Euler characteristic of the domain. In $d = 3$ dimensions we will show that the number of stagnation points has to be even.

The thesis starts out with some mathematical preliminaries. Here we discuss some important definitions to make the formulation of these questions more precise and show the density of Morse functions. In the next chapter we state the Morse inequalities for manifolds with corners and give a motivation as to why they hold. The next chapter gives an essentially negative answer to question 1.1 in the case of two dimensions. Here we also give examples of planar harmonic vector fields on domains with holes and an example in four dimensions. Chapter 5 then deals with the three-dimensional case where we give an explicit example which answers question 1.1 affirmatively in three dimensions. Chapter 6 then deals with question 1.3 in two dimensions where we give an elegant relation between the number of critical points and the domain topology. In the final chapter we discuss the case of question 1.3 in three dimensions where we give a condition on the number of interior stagnation points.

This thesis is directed to an audience with basic knowledge of harmonic functions such as the maximum principle, the mean value property and the existence and uniqueness of solutions. For a comprehensive introduction to the theory of partial differential equations more generally we refer the reader to [8]. We also assume the reader has had exposure to basic topology and some form of differential topology or differential geometry. In some parts knowledge of complex analysis as in for example [9] is required though it is not essential to understand the thesis. Elementary knowledge of convex optimisation is helpful but not a prerequisite. Despite Morse theory playing a central role in this thesis we do not assume that the reader has had prior exposure to this topic. To simplify things we avoid the language of homology theory where possible and instead give the theory in the classical sense as a set of inequalities. Thus we do not require prior knowledge of algebraic topology.

# 2 Mathematical preliminaries

In this chapter we give some important definitions. We start of by defining harmonic vector fields. After this we define Manifolds with corners, stratified spaces and the entrant and emergent boundaries. We then give a definition of stagnation points, type numbers and Morse functions for vector fields. Throughout we give examples and elementary results to illustrate the definitions. In the final part of this chapter we prove that harmonic functions can essentially be approximated by harmonic Morse functions whilst preserving certain properties.

## Harmonic vector fields

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\operatorname{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. Throughout the thesis we denote by

$$f \colon X \to \mathbb{R}$$

a $C^2$ function on $X$. Often $f$ will be assumed to be *harmonic*, that is $\Delta f = 0$ on $\operatorname{int}(X)$ where $\Delta$ denotes the Laplace operator. We also denote by

$$u \colon X \to \mathbb{R}^d$$

a vector field of class $C^1$. In the following we often assume that $u$ is in fact a *harmonic vector field*, that is $u$ fulfils $\operatorname{div} u = 0$ and $\operatorname{curl} u = 0$ on $\operatorname{int}(X)$. Note that a harmonic function $f$ gives rise to a harmonic vector field via its gradient $u = \nabla f$. In general a harmonic vector field $u$ is not the gradient of a harmonic function which one sees for instance in example 4.5. On simply connected domains this implication is however true:

**Proposition 2.1** (Harmonic vector fields on simply connected domains)**.** *Let $d \in \{2,3\}$ and $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and $u \colon \Omega \to \mathbb{R}^d$ be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f \colon \Omega \to \mathbb{R}$.*
2. *$f$ is harmonic.*
3. *$u$ is in fact smooth.*
4. *The components $u_i$ are harmonic.*

*Proof.* 1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.

2. This follows from $\Delta f = \operatorname{div} u = 0$.
3. This follows from the fact that $f$ is harmonic.
4. This follows from $u_i = \partial_i f$. □

If one considers not necessarily simply connected domains $\Omega$ then we obtain the properties of proposition 2.1 at least locally.

## Stratified spaces and the entrant and emergent boundary

We start by requiring some regularity for the boundary of $X$. More precisely, we require $X$ to be a compact Riemannian manifold with corners:

**Definition 2.2** (Manifolds with corners, [11])**.** We introduce the notation

$$H^d_j := \mathbb{R}^j_{\geq 0} \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d .$$

where $j \in \{0, \ldots, d\}$. A *manifold with (convex) corners* is a topological space $X$ together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exists an open neighbourhood $U_x$ of $x$, a number $j = j(x)$ and a diffeomorphism $\phi_x \colon U_x \to H^d_j$ in $\mathscr{A}$ with $\phi_x(x) = 0$. We further define for $k \in \{0, \ldots, d\}$ a collection of sets

$$X_k := \{x \in X \colon\ j(x) = d - k\}, \tag{2.1}$$

which form a stratification of $X$.

More generally we give the definition of a stratification as follows:

**Definition 2.3** (Stratified space, [11])**.** Let $X$ be a topological space. A *stratum* is a subspace $X_j \subseteq X$, $j \in \mathscr{J}$, indexed by a partially ordered set $\mathscr{J}$ such that

1. each $X_j$ is a manifold (without boundary) of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$ iff $j \prec k$.

The pair of $X$ and the collection of strata is called a *stratified space* and we call $n$ the *dimension of the stratum* $X_j$. In the case that $X_j \subseteq \overline{X}_k$ we will write $X_j \prec X_k$ using the notation of [11]. If additionally $n(k) = n(j) + 1$ we will write $X_j \precapprox X_k$ or, indicating that the strata differ by one in their dimension, we may write $X_k = X_{j+1}$.

In the case that the stratification arises through relation (2.1) we have precisely $X_j \precapprox X_{j+1}$ for $j \in \{1, \ldots, d-1\}$. Note that in general for a given stratum $X_j$ the stratum $X_{j+1}$ such that $X_j \precapprox X_{j+1}$ need not be unique. In the following we assume, unless otherwise stated, that the stratification is finite, that is $\#\mathscr{J} < \infty$ and that the interior $\operatorname{int}(X)$ corresponds to a single stratum.

We now give the definition of the contingent cone:

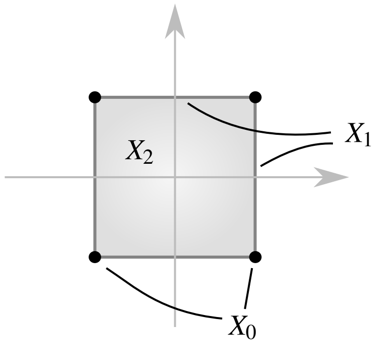



Figure 2.1: A stratification of $X$.

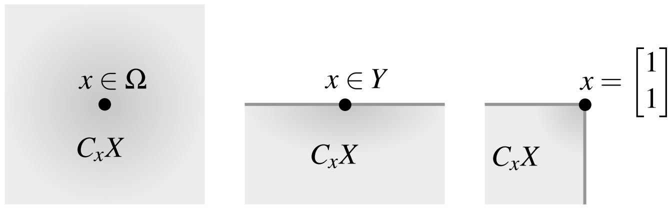



Figure 2.2: The contingent cones for various $x \in X$.

**Definition 2.4** (Contingent cone, [16, Def. 4.6])**.** Denote the *(Bouligand) contingent cone* for a set $Y \subseteq X$ at the point $x \in \overline{Y}$ by $C_xY$. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exist sequences $\lambda\colon \mathbb{N} \to \mathbb{R}_{>0}$ and $x_n \to x$ in $Y$ such that

$$\lim_n \lambda_n (x_n - x) = v\,.$$

To clarify we give an example:

**Example 2.5** (Cubical domain)**.** Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$. Then we have a stratification given by

$$\begin{aligned} X_0 &= \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\ X_1 &= (I \times \{-1\}) \cup (I \times \{1\}) \cup (\{-1\} \times I) \cup (\{1\} \times I) \\ X_2 &= I \times I \end{aligned}$$

where $I = (-1,1) \subseteq \mathbb{R}$. The stratification is depicted in figure 2.1. For an interior point $x \in X_2$ we have the contingent cone $C_xX = \mathbb{R}^2$. For a boundary point $x \in Y := I \times \{1\} \subset X_1$ we have the contingent cone

$$C_xX = \left\{ v \in T_x\mathbb{R}^2 \colon v \cdot n \leq 0 \right\} \tag{2.2}$$

where the basis vector $n = e_2$ is the outer unit normal. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top \in X_0$ we have

$$C_xX = \left\{ v \in T_x\mathbb{R}^2 \colon v_1 \leq 0 \text{ and } v_2 \leq 0 \right\}. \tag{2.3}$$

The situation is depicted in figure 2.2. The contingent cone on the other parts of the square $\Sigma = \partial X$ is given by similar formulas.

In the following we define the emergent and the entrant boundary in a way that generalises [24, p.282] for stratified manifolds.

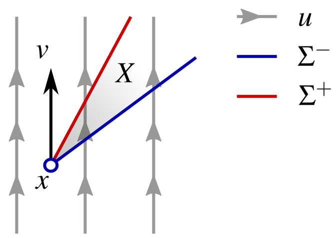



Figure 2.3: The vector $v$ is entrant.

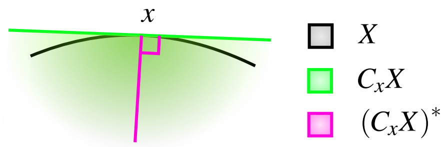



Figure 2.4: Depiction of $(C_xX)^*$ for a manifold with $C^1$ boundary.

**Definition 2.6** (Emergent and entrant boundary)**.** We call a vector $v \in T_xX$ *entrant* at a boundary point $x \in \Sigma$ if

1. $v$ lies in the contingent cone $C_xX$ or
2. $v$ lies in the dual cone of the contingent cone $C_xX$, that is

$$v \in (C_xX)^* = \left\{ w \in T_xX : w \cdot w' \geq 0 \text{ for all } w' \in C_xX \right\}.$$

We call $v$ *strictly entrant* if in addition $v$ lies in the relative interior $\operatorname{rel\,int} C_xX$ or if $v \in (C_xX)^*$ then $v$ lies in the relative interior $\operatorname{rel\,int}(C_xX)^*$. Analogously $v$ is *(strictly) emergent* if $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which $u$ is entrant. We define the *strictly entrant boundary* $\Sigma^-$ to be the set of strictly entrant boundary points of $u$. In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* $\Sigma^+$. Further define the *tangential boundary* $\Sigma^0$ to be

$$\Sigma^0 := \left(\Sigma^{\leq 0} \cup \Sigma^{\geq 0}\right) \setminus \left(\Sigma^+ \cup \Sigma^-\right) \subseteq \Sigma.$$

Although the condition that we call a vector entrant if it lies in the contingent cone $C_xX$ is more or less intuitive the second criteria requires a little motivation. For this consider a vector field $u$ and a tangent vector $v$ at the corner point $x$ as in figure 2.3. Since $v$ does not point into $X$ it does not satisfy the first condition. Now assume we place a test particle at $x$ which is confined to $X$ and $u$ describes a force acting on this particle. This test particle will then move away from the point $x$ and in this manner we would like to call $u$ at $x$ entrant. This type of reasoning is made precise in the proof of the Morse inequalities in [1] and [11] where a flow on $X$ is constructed. This motivates why we added a second criteria for calling $v$ entrant.

We would also like to motivate the naming of the tangential boundary. Assume that $X$ is a manifold with $C^1$ boundary and without corners. Let $x \in \Sigma$ be a boundary point and let $n$ denote the outer unit normal at $x$. Then we have that

$$(C_xX)^* = \left\{ w \in T_xX : w \cdot w' \geq 0 \text{ for all } w' \in C_xX \right\} = \{-rn : r \geq 0\}.$$

This situation is depicted in figure 2.4. If the vector $v$ lies in $(C_xX)^*$ but not in the relative interior $\operatorname{rel\,int}(C_xX)^*$ we in fact have that $v = 0$. Thus the tangential boundary $\Sigma^0$ consists in this case of precisely those boundary points at which $u$ is tangential to the boundary or vanishes.

We would now like to illustrate definition 2.6 with a more concrete example.

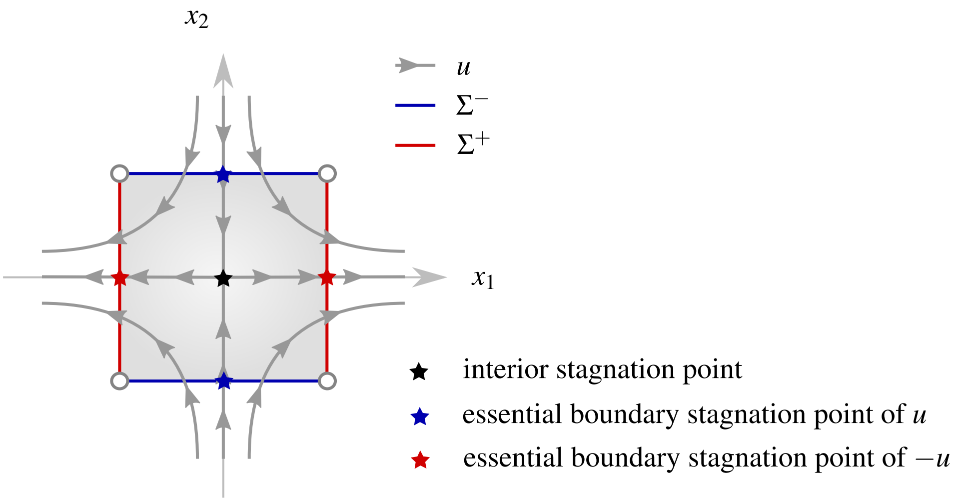



Figure 2.5: The entrant and emergent boundaries in example (2.7)

**Example 2.7** (A vector field on the cube)**.** Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$ and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 - x_2^2 . \end{aligned} \tag{2.4}$$

This induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u\colon X &\to \mathbb{R}^3 \\ x &\mapsto 2\begin{bmatrix} x_1 & -x_2 \end{bmatrix}^\top . \end{aligned} \tag{2.5}$$

For a boundary point $x \in I \times \{1\}$ we have that

$$0 > -2x_2 = n \cdot u$$

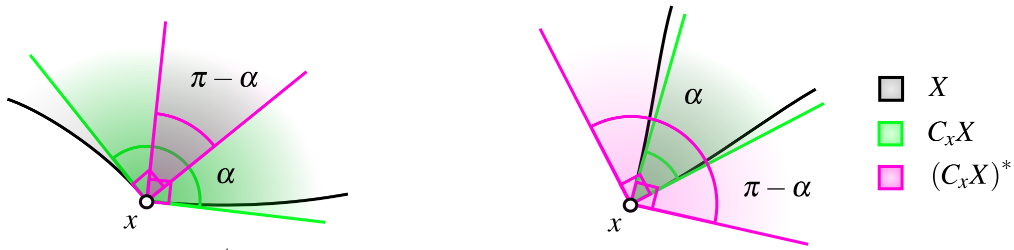
and thus by equation (2.2) it follows that $x \in \Sigma^-$ so $I \times \{1\} \subseteq \Sigma^-$. At the corner point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ the dual of the contingent cone is given by

$$(C_x X)^* = C_x X \tag{2.6}$$

where we used the characterisation of equation (2.3). Since $v = u(x) = 2\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ we have that $v \notin (C_x X)^*$ and $-v \notin (C_x X)^*$ and thus $x \notin \Sigma^{\geq 0} \cup \Sigma^{\leq 0}$. By analogous argumentation on the other sides of the square $\Sigma = \partial X$ one obtains that

$$\begin{aligned} \Sigma^- &= \Sigma^{\leq 0} = I \times \{-1, 1\} \\ \Sigma^+ &= \Sigma^{\geq 0} = \{-1, 1\} \times I . \end{aligned}$$

A sketch of the sets can be seen in figure 2.5.




(a) $\alpha > \pi/2$ (b) $\alpha < \pi/2$

Figure 2.6: Illustration of a planar contingent cone and its dual cone at a corner.

Since the dual of the contingent cone $(C_xX)^*$ will also play a role later on we make a further remark to illustrate the concept.

*Remark* (Contingent cone). If the angle of the cone $C_xX$ at a corner point $x$ in the plane is $\alpha$ the angle formed by the cone $(C_xX)^*$ is $\pi - \alpha$. Thus equation (2.6) in the previous example is a special case for $\alpha = \pi/2$. The more general case is illustrated in figure 2.6.

## Stagnation points

Given a vector field $u\colon X \to \mathbb{R}^d$ and a stratification $X_j$ of $X$ we can construct for every $j \in \mathscr{J}$ a vector field

$$u_j\colon X_j \to T^*X_j\,.$$

Here $T^*X_j$ denotes the cotangent space of the manifold $X_j$ which is defined for instance in [15, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j\big|_x\colon T^*_xX \to T^*_xX_j \tag{2.7}$$

denote the orthogonal projection of a vector at $x$ onto the cotangent space of the stratum $X_j$ at $x$. Let

$$u_j := \pi_j \circ u\big|_{X_j} \in C^1(T^*X_j) \tag{2.8}$$

be the projection of $u$ onto the cotangent bundle $T^*X_j$. We can now give a definition of stagnation points and their index. This generalises definitions given in [27, p.138f], [25, §5] and [24, p.282f] to include vector fields which are not necessarily gradient fields.

**Definition 2.8** (Stagnation points)**.** Let $u_j\colon X_j \to T^*X_j$ be a $C^1$ vector field on a stratum $X_j$ of $X$. We call the zeroes $x \in X_j$ of $u_j$ *stagnation points of* $u_j$ *on* $X_j$. If $x \in \operatorname{int}(X)$ is a stagnation point of a stratum with dimension $d$ then we call $x$ an *interior stagnation point* else $x$ is a *boundary stagnation point*. If $u(x) \in (C_xX)^*$, that is $x$ lies in the entrant boundary $\Sigma^{\leq 0}$, we call $x$ an *essential stagnation point*. The set of all essential stagnation points of $u_j$ is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$ and the essential stagnation points of $u$ and $-u$ on $X_j$ are called *stagnation points of $u$ on $X_j$*. A stagnation point $x$ of $u_j$ is called *non-degenerate* if the derivative

$$Du_j(x) = Du_j\big|_x \in T_x T^* X_j \cong \mathbb{R}^{n(j)\times n(j)}$$

is bijective. In addition we say that $x$ has *index $k$* if $Du_j(x)$ has exactly $k$ negative eigenvalues. $u_j$ is called *(essentially) non-degenerate* if all its (essential) stagnation points are non-degenerate. Finally, we call a non-degenerate essential stagnation point $x$ of $u_j$ *regular* if additionally $u(x) \in \operatorname{rel\,int}(C_x X)^*$, that is $x$ lies in the strictly entrant boundary $\Sigma^-$. Boundary points which are non-regular essential stagnation points are called *irregular boundary points*. We call the set of all irregular boundary points the irregular boundary $\Sigma^{\mathrm{irr}}$. $u_j$ is called *regular* if it has no irregular boundary points. We can define the *k-th type number* $\operatorname{Ind}_{j,k}(u)$ of the stratum $X_j$ to be the number of regular stagnation points of $u_j$ of index $k$, that is

$$\operatorname{Ind}_{j,k}(u) := \#\big\{x \in \operatorname{Cr}_j(u) \colon x \text{ has index } k\big\}\,. \tag{2.9}$$

The reason for only considering essential stagnation points has to do with the Morse inequalities. In the proof given by [11] a flow along the vector field $-u$ on $X$ is constructed. Since this flow would leave the domain at the entrant boundary this flow is modified on $\Sigma^{\leq 0}$ so as to not leave $X$ and becomes discontinuous. For more details we refer the reader to chapter 3. To illustrate the preceding definitions we return to our previous example.

**Example 2.9.** Let $X$, $f$ and $u$ be as in example 2.7. We have that $u_2 = u$ and thus one sees from equation (2.5) that the origin 0 is the sole stagnation point of $u$ on the stratum $X_2$. Since we have that

$$Du(x) = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}$$

for all $x \in \operatorname{int}(X)$ we see that $Du(0)$ is bijective and thus the origin is a non-degenerate interior stagnation point. As $Du(0)$ has exactly one negative eigenvalue we see that the origin has index 1. Since an interior stagnation point is also an essential stagnation point we have $\operatorname{Ind}_{2,k} = \delta_{k1}$ where $\delta$ denotes the Kronecker delta. For $x \in I \times \{1\} =: Y$ we calculate

$$u_1(x) = \pi_1 \circ u(x) = (u - (n\cdot u)n)(x) = 2x_1 e_1$$

with $n$ the outer unit normal. Thus we have that $x = e_2$ is the unique stagnation point of $u$ on $Y$. Consider the curve

$$\begin{aligned} \gamma\colon I &\to Y \\ t &\mapsto te_1 + e_2 \end{aligned}$$

then $\gamma(0) = e_2$ and we have

$$Du_1(e_1)\big(\gamma'(0)\big) = (u_1 \circ \gamma)'(0) = (2te_1)'(0) = 2e_1 = 2\gamma'(0)$$

and thus $e_1$ is an eigenvector of $Du_1(e_2)$ to the eigenvalue 2. Since $e_1$ spans the eigenspace $T_{e_2}Y$ it follows that $e_2$ is a non-degenerate stagnation point of $u_1$ with index 0. Now since $u(e_2) \in \operatorname{rel\,int}(C_xX)^*$ we have that $e_2$ is in fact a regular point. Proceeding in this manner for the other segments of the square $\Sigma$ we obtain that $\mathrm{Ind}_{1,k} = 2\delta_{0k}$. If we now consider the point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ then we have that $u_0(x) = 0$ and thus $x$ is a stagnation point. Now the derivative $Du_0 = 0 \in T_xT^*X_0 = 0$ is bijective and thus we have that $x$ has index 0. Since however $u(x) \notin (C_xX)^*$ we have that $x$ is not an essential stagnation point. Analogous argumentation on the other three corners yields that $\mathrm{Ind}_{0,0} = 0$.

The following characterisation of the irregular boundary will come in handy for showing the density of Morse functions later on:

**Proposition 2.10** (Characterisation of the irregular boundary)**.** *The condition that the stagnation point $x \in X_j$ lies in $\Sigma^{irr}$ is equivalent to that $x$ is stagnation point of $u_{j+1}$ for a stratum $X_j \precsim X_{j+1}$.*

We first remark that $x$ being a stagnation point of $u_{j+1}$ makes at first sight no sense since $x \notin X_{j+1}$. We have however that $x \in \overline{X}_{j+1}$. The concept of tangent space $T_xX_{j+1}$ and the projection $\pi_{j+1}$ can be extended to $\overline{X}_{j+1}$ by continuity and so we can reasonably speak of evaluating $u_{j+1}$ at the point $x$.

*Proof.* We first claim that for some boundary point $x \in \Sigma \cap X_j$ the relations

$$\begin{aligned}
\partial(C_xX)^* &:= (C_xX)^* \setminus \operatorname{rel\,int}(C_xX)^* \\
&\overset{(+)}{=} \left\{ w \in T_xX \;\middle|\; \begin{array}{l} w \cdot w' \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ w \cdot w' = 0 \text{ for some } w' \in C_xX \setminus T_xX_j \end{array} \right\} \\
&\overset{(*)}{=} \left\{ w \in T_xX \;\middle|\; \begin{array}{l} w \cdot w' \geq 0 \text{ for all } w' \in C_xX \text{ and} \\ w \cdot w' = 0 \text{ for some } w' \in T_xX_{j+1} \setminus T_xX_j \\ \text{where } X_j \precsim X_{j+1} \end{array} \right\}
\end{aligned} \tag{2.10}$$

hold. To see the equality $(+)$ let $w \in \partial(C_xX)^*$ and $w_k \in T_xX \setminus (C_xX)^*$ be a sequence converging to $w$. Then there exists for each $w_k$ a $w'_k \in C_xX$ such that $w_k \cdot w'_k < 0$. We can assume that $|w'_k| = 1$ and after taking a subsequence that the $w'_k$ converge to a $w' \in C_xX$. But then we have by continuity that $w \cdot w' \leq 0$ so the inclusion '$\subseteq$' follows. On the other hand if $w \in \operatorname{rel\,int}(C_xX)^*$ then there exists a relatively open neighbourhood $U \subseteq (C_xX)^*$ of $w$. Assume that $w \cdot w' = 0$ for some $C_xX \setminus T_xX_j$. Since we have that $T_xX_j \perp (C_xX)^*$ are perpendicular we can assume that the component of $w'$ projected onto the space $T_xX_j$ vanishes. Then we have for $\lambda < 0$ small enough that $w + \lambda w' \in U$ and it follows that $(w + \lambda w') \cdot w' = \lambda |w'|^2 < 0$, a contradiction. Thus '$\supseteq$' follows.

Argue why $\supseteq$ follows precisely. Argue why in the first part $w' \notin T_xX$.

For $(*)$ we show the inclusion '$\subseteq$'. Let $w \in \partial(C_xX)^*$ and $w' \in (T_xX_k \cap C_xX) \setminus T_xX_j$ for some stratum $X_j \prec X_k$ with $w \cdot w' = 0$. We assume that $X_k$ is chosen to be of minimal dimension. Now if $X_j \precsim X_k$ we are finished. Else there exist strata $X^1_{k-1}, X^2_{k-1} \precsim X_k$ adjacent to $X_j$. Then there further exist $w'_1 \in \left(T_xX^1_{k-1} \cap C_xX\right) \setminus T_xX_j$ and $w'_2 \in \left(T_xX^2_{k-1} \cap C_xX\right) \setminus T_xX_j$ such that $w' = w'_1 + w'_2$. But

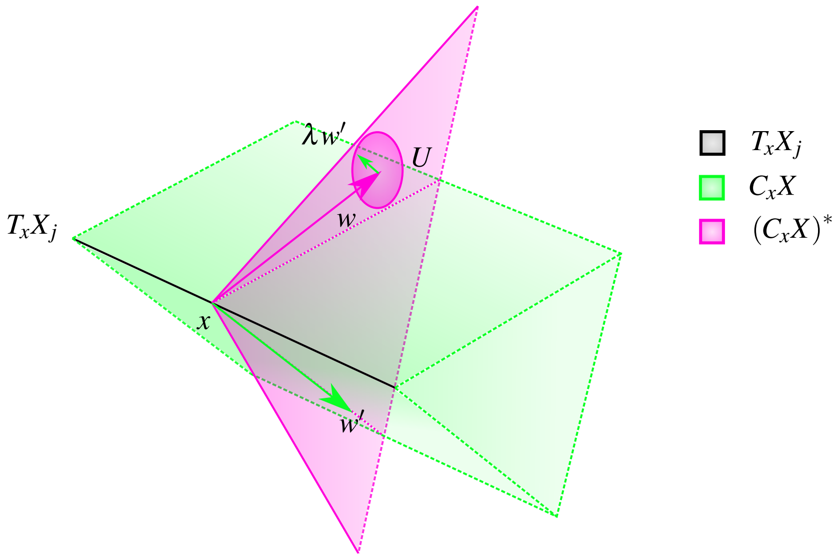



Figure 2.7: An overview of the proof of $(+)$ in equation (2.10).

then it follows from

$$0 = w \cdot w' = \underbrace{w \cdot w_1'}_{\geq 0} + \underbrace{w \cdot w_2'}_{\geq 0}$$

that also $0 = w \cdot w_1'$ which contradicts the choice of $X_k$ having minimal dimension. Hence the inclusion '$\subseteq$' follows.

Update figure to the notation of the proof. Argue why $\supseteq$ follows. Also update the figure.

Now let $x$ be an irregular stagnation point of $u_j$ then $u(x) \in \partial(C_xX)^*$. Hence there exists a $w' \in T_xX_{j+1} \setminus T_xX_j$ such that $u(x) \cdot w' = 0$. Additionally we have that $u(x) \cdot w = 0$ for all $w \in T_xX_j$ since $x$ is stagnation point of $u_j$. Since we have that $\mathrm{Span}(w') + T_xX_j = T_xX_{j+1}$ by dimension reasons it then follows that $u(x) \cdot w = 0$ for all $w \in T_xX_{j+1}$ which means that $u_{j+1}(x) = 0$.

Conversely assume that $x \in X_j$ is stagnation point for some stratum $X_{j+1}$. Then we have that $u(x) \cdot w = 0$ for all $w \in T_xX_{j+1}$. Since $\mathrm{Span}(w') + T_xX_j = T_xX_{j+1}$ for some $w' \in T_xX_{j+1} \setminus T_xX_j$ it then follows from equation (2.10) that $u(x) \in \partial(C_xX)^*$ and $x$ is an irregular stagnation point. $\square$

## Morse functions

The following definition of type numbers is inspired by [25, Definition 10.3]:

**Definition 2.11** (Morse functions, [25, Definition 10.3])**.** We call $u$ *Morse* if for all $j \in \mathscr{J}$ we have that $u_j$ is regular. If both $u$ and $-u$ are Morse we call $u$ *strongly Morse*. For a Morse function $u$ we define the *interior type numbers* $M_k$ to be the number of essential interior stagnation points

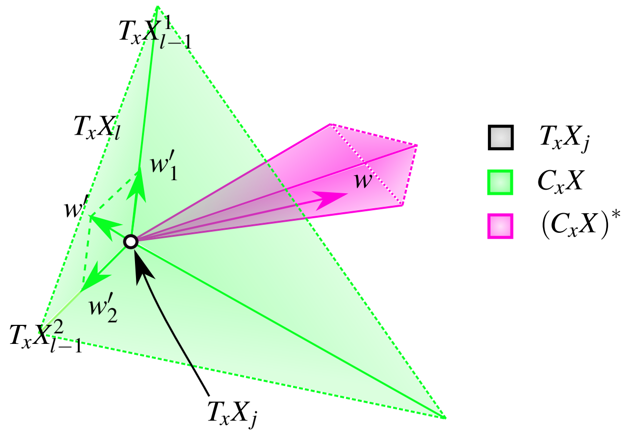



Figure 2.8: An overview of the proof of $(*)$ in equation (2.10).

of $u$ of index $k$, that is

$$M_k := \sum_{\substack{j\in\mathscr{J} \text{ s.t.}\\ n(j)=d}} \mathrm{Ind}_{j,k}(u) = \#\left\{x \in \bigcup_{\substack{j\in\mathscr{J} \text{ s.t.}\\ n(j)=d}} \mathrm{Cr}_j(u) \colon x \text{ has index } k\right\}. \tag{2.11}$$

The total number $M$ of interior stagnation points of $u$ is then given by

$$M := \sum_k M_k\,. \tag{2.12}$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of $u$ of index $k$, that is

$$\mu_k := \sum_{\substack{j\in\mathscr{J} \text{ s.t.}\\ n(j)<d}} \mathrm{Ind}_{j,k}(u) \tag{2.13}$$

We further write $\nu_k$ for the $k$-th boundary type number of $-u$. We define the *type number* to be the number of essential stagnation points of $u$ of index $k$, that is

$$\mathrm{Ind}_k(u) := \sum_{j\in\mathscr{J}} \mathrm{Ind}_{j,k}(u) = M_k + \mu_k\,. \tag{2.14}$$

We illustrate this with our example:

**Example 2.12.** Let $X$, $f$ and $u$ be as in example 2.7. By the calculations of the previous example 2.9 we have that $u$ is Morse and we can calculate the interior type numbers

$$M_k = \mathrm{Ind}_{2,k} = \delta_{1k}$$

and the boundary type numbers

$$\mu_k = \mathrm{Ind}_{0,k}(u) + \mathrm{Ind}_{1,k}(u) = 2\delta_{0k}$$

This then yields the type numbers

$$\mathrm{Ind}_k(u) = M_k + \mu_k = \delta_{1k} + 2\delta_{0k}\,.$$

The finiteness of the number of critical points for Morse functions is a known fact which is mentioned for example in [24]. For completeness we give the following proposition:

**Proposition 2.13.** *The number of non-degenerate stagnation points of $u_j$ on $X_j$ is finite.*

*Proof.* Let $x \in X_j$ be a non-degenerate stagnation point of $u_j$. Since $Du_j(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq X_j$ of $x$ on which $u_j$ is bijective. Hence $x$ is the only stagnation point in $U_x$. Let $C$ denote the set of all non-degenerate stagnation points of $u_j$. Then the sets $U_x$ for $x \in C$ together with

$$U_C = \overline{X}_j \setminus \overline{C} \tag{2.15}$$

form an open cover of $\overline{X}_j$. But $\overline{X}_j$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that $U_x$ is contained in the finite subcover. Thus it follows that $\#C < \infty$ is finite. □

As a consequence we obtain the following observation:

**Corollary 2.14.** *For a Morse vector field $u\colon X \to \mathbb{R}^d$ the type numbers $M_0,\dots,M_d$ and the boundary type numbers $\mu_0,\dots,\mu_{d-1}$ are finite.*

The previous definitions translate naturally to a skalar function $f\colon X \to \mathbb{R}$. That is, we call $f$ Morse, non-degenerate, et cetera if $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call a point $x$ a *critical point* of $f$ if it is a stagnation point of $u$. Note that most authors refer to regular and essential stagnation points as non-degenerate stagnation points and that this naming was introduced simply to distinguish between these different concepts.

## Density of Morse functions

In the following section we argue that $u$ and $f$ being Morse is not a great restriction. Given $u$ we define the modification

$$u^\varepsilon := u + \varepsilon \tag{2.16}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that the set $E$ of $\varepsilon$ for which $u^\varepsilon$ is Morse is residual in $\mathbb{R}^d$. Recall that a *residual* set is a set whose complement is *meagre*, that is whose complement is the countable union of nowhere dense subsets. Since residual sets are dense in a Baire space by the Baire category theorem we can use $u^\varepsilon$ to approximate a degenerate $u$. Our approach is to use Thom's transversality theorem which is inspired by the approach in [15, Chapter 6].

**Definition 2.15** (Transversality, [15, §3.2] and [1]). Let $X$ be a manifold with corners and $Y$ a manifold without boundary. We call a function $g\colon X \to Y$ *transverse* to a submanifold $A \subseteq Y$ with corners if for all points in the preimage $x \in g^{-1}(A)$ we have that

$$Dg\big|_x(C_x X) + C_{g(x)}A = T_{g(x)}Y\,. \tag{2.17}$$

In the remainder of this section we will deal with manifolds without boundaries and then condition (2.17) reduces to

$$\mathrm{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y\,.$$

As an application of this definition we make the following observation:

**Proposition 2.16** (Transversal characterisation of non-degeneracy). *Let $u_j\colon X_j \to T^*X_j$ be a differentiable vector field. Then $u_j$ is non-degenerate iff $u_j$ is transverse to the zero section $A$ of the cotangent space $T^*X_j$.*

*Proof.* First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A) = C$ is the set of stagnation points. Unravelling the definition of transversality we get that $u_j$ is transverse to the zero section iff for all $x \in C = u_j^{-1}(A)$ we have that

$$\mathrm{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}T^*X_j\,. \tag{2.18}$$

As $A$ is the zero section we have $T_{u_j(x)}A = 0$ and equation (2.18) is equivalent to stating that $Du_j$ is of full rank at $x$. But $Du_j$ being of full rank at all stagnation points is equivalent to $u_j$ being non-degenerate. □

The alternative characterisation of non-degeneracy given in proposition 2.16 is sometimes used as a definition of non-degeneracy. We can now state a weakened version of Thom's transversality theorem which is proven in [15, §3 Theorem 2.7]:

**Theorem 2.17** (Parametric transversality theorem, [15, §3 Theorem 2.7]). *Let $\mathscr{E}, Y_1, Y_2$ be $C^r$-manifolds (without boundary) and $A \subseteq Y_2$ a $C^r$ submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A\,.$$

*Let further $F: \mathscr{E} \to C^r(Y_1, Y_2)$ be such that the evaluation map*

$$\begin{aligned} F^{ev}\colon \mathscr{E} \times Y_1 &\to Y_2 \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

*is $C^r$ and transverse to $A$. Then the set*

$$E = \{\varepsilon \in \mathscr{E}\colon F_\varepsilon \text{ is transverse to } A\}$$

*is residual in $\mathscr{E}$.*

From this we obtain a generalisation of the results in [24, §2] which will be useful later:

**Corollary 2.18** (Density of harmonic Morse functions)**.** *Let $u\colon X \to T^*X$ be a vector field on $X$ and let $X_j$ be a stratification of $X$. Assume that $u$ has no irregular stagnation points. Then there exists a $\delta > 0$ and a residual (and thus dense) set $E \subseteq B_\delta \subseteq \mathbb{R}^d$ such that for every $\varepsilon \in E$ the following statements hold:*

1. *$u_j^\varepsilon \to u_j$ converge uniformly on all strata $X_j$ as $\varepsilon \to 0$.*
2. *If $x_\varepsilon \to x$ is a convergent sequence of stagnation points of $u_j^\varepsilon$ as $\varepsilon \to 0$ then $x$ is a stagnation point of $u_j$.*
3. *$u^\varepsilon$ is strongly Morse.*
4. *Additionally we can find for every $\eta > 0$ a $\delta > 0$ such that all stagnation points of $u^\varepsilon$ are contained in an $\eta$-neighbourhood of the set of stagnation points of $u$.*
5. *the property of being strictly entrant or strictly emergent of stagnation points of $u^\varepsilon$ is preserved, that is a stagnation point $x^\varepsilon$ of $u^\varepsilon$ lies in $\Sigma^\pm(u^\varepsilon)$ iff it lies in $\Sigma^\pm(u)$.*
6. *If $u_j$ is non-degenerate on the stratum $X_j$ we have for all $k$ that*

$$\mathrm{Ind}_{X_j,k}(u^\varepsilon) = \mathrm{Ind}_{X_j,k}(u) \qquad \textit{and} \qquad \mathrm{Ind}_{X_j,k}(-u^\varepsilon) = \mathrm{Ind}_{X_j,k}(-u)\,.$$

*In addition note that if $u$ is harmonic then by construction $u^\varepsilon$ is also harmonic.*

We remark that the fact that $u$ has no irregular stagnation points is essential to the point 6 of this corollary. Assume for instance $u_d$ has a non-degenerate stagnation point $x$ on the boundary $\Sigma$, which means by proposition 2.10 that $x$ is an irregular critical point. It may then follow that for $\varepsilon$ contained in a cone at the origin the corresponding critical point of $u^\varepsilon$ moves outside of the domain $X$ which in turn changes the index. Example 6.7 gives a concrete illustration of this point.

*Proof. Part 1*. Follows from compactness of $\overline{X}_j \subseteq X$ and the continuity of $\pi_j$.

*Part 2*. Let $x_\varepsilon \to x$ be a convergent sequence of stagnation points on the stratum $X_j$. By part 1 $u_j^\varepsilon \to u_j$ as $\varepsilon \to 0$ uniformly which implies

$$0 = \lim_{\varepsilon} u_j^\varepsilon(x_\varepsilon) = u_j(x) \tag{2.19}$$

and thus $x$ is a stagnation point of $u_j$.

*Part 3*. The following is essentially an adaptation of a proof given in [24, §2]. We first show that we can choose a $\delta > 0$ such that for all $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ the function $u^\varepsilon$ has no irregular stagnation points. Assume not. Then there exists a sequence $\varepsilon_k \to 0$ and irregular stagnation points $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ of $u^{\varepsilon_k}$. By compactness of $X$ we can assume that $x_k \to x$ for some $x \in X$ after taking a sub-sequence. After taking a further sub-sequence we can also assume that all $x_k$ lie in a stratum $X_j$. The condition that the $x_k \in \Sigma^{\mathrm{irr}}(u^{\varepsilon_k})$ are stagnation points means that after taking a further sub-sequence there exists a stratum $X_{j-1}$ such that $x_k$ is also stagnation point of this stratum by proposition 2.10. But then $x \in \overline{X}_j$ is also a stagnation point of $X_{j-1}$ by part

2. Analogously $x$ is also stagnation point of $X_j$. Thus $x \in \Sigma^{\text{irr}}$ is an irregular stagnation point, a contradiction.

The next part of the proof is inspired by the use of transversality in [15, §6 Theorem 1.2] to show a similar statement. Set $r = 2$, $\mathscr{E} = B_\delta$ and $Y_2 = T^*X_j$ in the previous theorem. We initially set $Y_1 = X_j = \text{int}(X)$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F\colon \mathscr{E} &\to C^\infty(X_j, T^*X_j) \\ \varepsilon &\mapsto u^\varepsilon \end{aligned}$$

and note that $F^{\text{ev}}$ is sufficiently smooth. We need to show that $F^{\text{ev}}$ is transverse to the zero section $A \subseteq T^*X_j$. Then the parametric transversality theorem yields a residual $E_j \subseteq \mathscr{E}$ on which $F_\varepsilon = u^\varepsilon$ is transverse to $A$. For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$\text{Image}\Big(DF^{\text{ev}}_{(\varepsilon,x)}\Big) = T_xT^*X_j \tag{2.20}$$

since

$$DF^{\text{ev}}_{(\varepsilon,x)} = \big[\text{Id}_{d\times d} \mid Du_x\big]$$

is surjective. Proposition 2.16 now implies that $u^\varepsilon$ is non-degenerate on $X_j$ for $\varepsilon \in E_j$. Analogously we set $Y_1 = X_j$ to be an arbitrary stratum in the previous proof and replace $u^\varepsilon$ with the projection $u_j^\varepsilon$. To show that equation (2.20) holds we resort to the fact that

$$DF^{\text{ev}}_{(\varepsilon,x)} = D\big(u_j^\varepsilon(x)\big)_{(\varepsilon,x)} = D\pi_j \circ (Du^\varepsilon(x))_{(\varepsilon,x)}$$

is surjective as a composition of surjective functions. Thus there also exists a residual set $E_j \subseteq \mathscr{E}$ on which $u_j^\varepsilon$ is non-degenerate on $X_j$. Now the intersection

$$E = \bigcap_j E_j \subseteq \mathscr{E} = B_\delta$$

is residual and for every $\varepsilon \in E$ the function $u^\varepsilon$ fulfils condition 3.

*Part 4.* Let $C_\eta$ denote the open $\eta$-neighbourhood of the set of stagnation points of $u$. We have for any stratum $X_j$ that $u_j \neq 0$ on the compact set $\overline{X}_j \setminus C_\eta$ which implies that we can choose $\delta > 0$ so small that $|u_j| > 2\delta$ on $\overline{X}_j \setminus C_\eta$ for all strata $X_j$. For any $\varepsilon \in B_\delta$ it then follows that $u^\varepsilon$ has no stagnation points on the set $\overline{X}_j \setminus C_\eta$ which yields the claim.

*Part 5.* Now consider a stratum $X_j$ and the continuous mapping

$$\begin{aligned} \Phi\colon X_j &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \text{dist}\big(u(x), \partial(C_xX)^*\big) \end{aligned}$$

on $X_j$. Since $u$ has no irregular stagnation points $\Phi$ is positive on the set of stagnation points $C$ of $u_j$ on $X_j$. Thus we can choose $\eta > 0$ such that $\Phi$ is also positive in the neighbourhood $C_{2\eta}$. Choose $\delta > 0$ smaller than in part 4. Now the mapping $\Phi$ attains a positive minimum on the compact set $\overline{C_\eta}$. We can assume that $\delta > 0$ is less than this minimum. The choice of $\delta$ in this way ensures that emergent stagnation points of $u_j$ are also emergent stagnation points of $u_j^\varepsilon$ on $X_j$. Analogous argumentation with $-u$ then ensures that entrant stagnation points of $u_j$ are also entrant stagnation points of $u_j^\varepsilon$ on $X_j$. Since there are finitely many strata $X_j$ we can choose $\delta > 0$ such that part 5 follows.

*Part 6.* Pick $\delta > 0$ as in part 5. If $x$ is a non-degenerate stagnation point of $u$ on the stratum $X_j$ it follows from the inverse function theorem that there exists for sufficiently small $\delta$ a neighbourhood around $x$ on which there is a one-to-one correspondence between the stagnation points of $u$ and $u^\varepsilon$. Since there are by proposition 2.13 at most finitely many non-degenerate stagnation points of $u$ we can choose $\delta$ to be minimal over all these stagnation points such that this one-to-one correspondence holds. The equality of the indexes then follows from $Du^\varepsilon = Du$. $\square$

Some parts of the proof above are not so clean. Also, do I use the notation $X_j \precsim X_{j+1}$ anywhere?

# 3 The Morse inequalities

The aim of this chapter is to state and motivate the Morse inequalities for manifolds with corners. For this we first introduce the Betti numbers of the domain $X$, then we state the Morse inequalities and finally we give a motivation as to why these inequalities hold. After that we state the Morse inequalities in the special case of $d \in \{2,3\}$ dimensions for a harmonic Morse function $f$.

## Betti numbers

Motivate homology a little more.

Let $H_k(X)$ denote the $k$-th homology group of $X$ with coefficients in the field $\mathbb{K}$. For an introduction and definition of these we refer the reader to [14, Chapter 2]. We define the $k$-th Betti number as the dimension

$$b_k = b_k(X) := \dim_{\mathbb{K}} H_k(X)\,. \tag{3.1}$$

Now we can define the *Euler characteristic* to be

$$\chi = \chi(X) := \sum_{k=0}^{d} (-1)^k b_k(X) \tag{3.2}$$

the alternating sum of the Betti numbers. We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

**Example 3.1** (In flatland)**.** In $d = 2$ dimensions the zeroth Betti number counts the number of connected components of $X$ and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely, we give the Betti numbers for selected domains in table 3.1.

**Example 3.2** (In spaceland)**.** In $d = 3$ dimensions the zeroth Betti number counts the number of connected components of $X$, the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 3.2.

clarify final example.

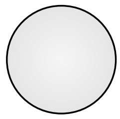
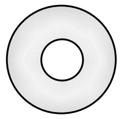
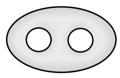
| Domain | Picture | $b_0$ | $b_1$ | $b_k,\, k \geq 2$ | $\chi(X)$ |
|---|---|---|---|---|---|
| Disk $D$ |  | 1 | 0 | 0 | 1 |
| Annulus $2D \setminus D$ |  | 1 | 1 | 0 | 0 |
| Two holed button |  | 1 | 2 | 0 | -1 |

Table 3.1: Betti numbers for selected domains in $\mathbb{R}^2$.

## The Morse inequalities

We state the Morse inequalities for manifolds with corners:

**Theorem 3.3** (Morse inequalities, [1, Theorem 2.4])**.** *Let $X$ be a manifold with corners and $f\colon X \to \mathbb{R}$ be smooth and Morse. Then we have for $l \in \{0,\ldots,d\}$ the inequalities*

$$\sum_{k=0}^{l} (-1)^{k+l} \operatorname{Ind}_k(f) \geq \sum_{k=0}^{l} (-1)^{k+l} b_k(X)\,. \tag{3.3}$$

*For $l = d$ we in fact have equality*

$$\sum_{k=0}^{d} (-1)^{k} \operatorname{Ind}_k(f) = \chi(X) \tag{3.4}$$

*where $\chi(X)$ denotes the Euler characteristic.*

The definition of regular critical points of $f$ and their index given in definition 2.8 coincides with the definition of a critical point and its co-index of $-f$ given in [1]. Theorem 3.3 then follows from [1, Theorem 2.4]. Nonetheless we will give an idea of the proof.

Define for $a \in \mathbb{R}$ the manifold $X^a := \{x \in X : f(x) \leq a\}$. The main idea in the proof of the Morse inequalities for manifolds with corners is to inspect how the topology of the manifold $X^a$ changes as we vary $a$. This was the idea in Morse's original proof of these inequalities for manifolds without boundary in [23]. Later proofs of the Morse inequalities for manifolds without boundary using this approach added more terminology from algebraic topology and differential geometry. Proofs of the Morse inequalities can be found in [21, §5] for manifolds without boundary and in [25, Theorem 10.2'] for manifolds with $C^1$ boundary. There also exists proofs using dynamical system techniques which can be found for instance in [28]. We require however the Morse inequalities on manifolds with corners. This is a special case of stratified Morse theory by

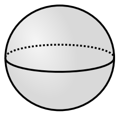
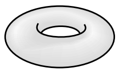
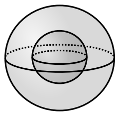
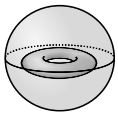
| Domain | Picture | $b_0$ | $b_1$ | $b_2$ | $b_k$, $k \geq 3$ | $\chi(X)$ |
|---|---|---|---|---|---|---|
| Ball $B$ |  | 1 | 0 | 0 | 0 | 1 |
| Solid torus $S^1 \times D$ |  | 1 | 1 | 0 | 0 | 0 |
| Ball with bubble $2B \setminus B$ |  | 1 | 0 | 1 | 0 | 2 |
| Ball with bubble in shape of torus |  | 1 | 1 | 1 | 0 | 1 |

Table 3.2: Betti numbers for selected domains in $\mathbb{R}^3$.

Goresky and McPherson which is outlined for instance in [6, §5.5]. We will try to avoid the fancy tools that are used in stratified Morse theory and instead resort to [1] and [11] where the Morse inequalities are shown for manifolds with corners. In the following we will outline the main ideas of the proof to convince the reader as to why equation (3.4) is reasonable as this will be the only part of the Morse inequalities we will use.

Reference Banyaga here.

We first cite a result from [1].

**Proposition 3.4** ([1, Theorem 2.1])**.** *Let $t_0, t_1 \in \mathbb{R}$ and $F\colon \mathbb{R} \times X \to Y$ be a smooth map such that for each $t_0 \leq t \leq t_1$ the function $f_t = F(t, \cdot)$ is transverse to a submanifold with corners $W \subseteq Y$. Then the sets $f_{t_0}^{-1}(W) \simeq f_{t_1}^{-1}(W)$ are homotopic.*

*Idea of proof.* See [1, Theorem 2.1]. The proof involves the construction of a flow $P$ defined on a neighbourhood of $X$ for which $P_t\big(f_{t_0}^{-1}(W)\big) \subseteq f_t^{-1}(W)$ for all $t_0 \leq t \leq t_1$. $\square$

With this result one can show the following:

**Lemma 3.5** (Interval without critical value, [11, Theorem 7] and [1, p.7])**.** *Assume that the interval $[a,b]$ contains no critical value of $f$. Then $X^a \simeq X^b$ are homotopic.*

*Proof.* One sets $f_t = f - t$ in the previous lemma and $W = \mathbb{R}_{\leq 0}$. Note that $X^c = f_c^{-1}(W)$ for all $c \in \mathbb{R}$. We have for every $a \leq t \leq b$ and and $x \in f_t^{-1}(W)$ that

$$Df_t\big|_x(C_x X) + C_{f(x)} W = C_{f(x)}\mathbb{R} = \mathbb{R}$$

since $C_{f(x)}W = \mathbb{R}$ for $f(x) < t$ and

$$Df_t\big|_x(C_xX) = Df\big|_x(C_xX) = \mathbb{R}$$

for $f(x) = t$ since $f$ has no critical values on the interval $[a,b]$. Thus $f_t$ is transverse to $W$ for each $a \le t \le b$ and the result follows. □

A similar result is given in [11, Theorem 7]. The proof in [11, Theorem 7] also involves the construction of a flow on the manifold $X$. It requires however an additional technical property, named property (3), which is why we gave the proof by [1] for this lemma.

We state Morse's lemma according to [11, Lemma 5].

**Proposition 3.6** (Morse's lemma, [11, Lemma 5]). *Let $f\colon X \to \mathbb{R}$ be $C^2$ and $x$ be an essential regular non-degenerate critical point of index $k$ on the stratum $X_j$ of dimension $n$. Then there exists a local chart $\varphi\colon U \to V \subseteq H^d_{d-n}$ such that*

$$f \circ \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{n} y_j^2 + \sum_{j=n+1}^{d} y_j\,. \tag{3.5}$$

*Idea of proof.* The case that $n = d$ is the classical version of the Morse lemma and can be found for instance in [15, §6, Lemma 1.1]. Note that the definition of an essential critical point in [11] fulfilling property (3) also encompasses our definition of a non-degenerate regular essential critical point. Here the regularity condition ensures that property (3) in [11] is fulfilled. Hence this statement follows from [11, Lemma 5]. The idea of the proof involves first applying the Morse lemma on the stratum of the critical point. Then this coordinate chart is extended with Taylor in a suitable manner to a full coordinate chart on $X$ such that equation (3.5) holds. □

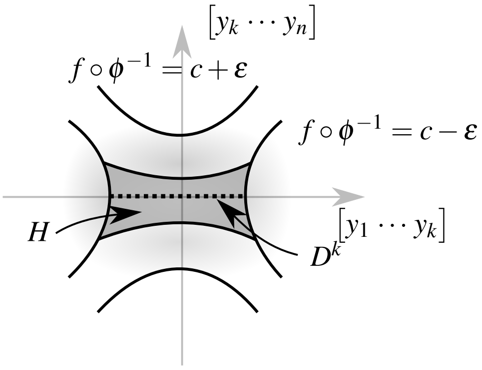
Using the Morse lemma one can then show the following result:

**Lemma 3.7** (Interval with critical value, [11, Theorem 8] and [1, Theorem 2.2]). *If the interval $[a,b]$ contains one critical value $a < c < b$ with corresponding essential regular non-degenerate critical point $x$ of index $k$ then $X^b$ is homotopic to $X^a$ by attaching the $k$-cell $D^k$.*

*Idea of proof.* This is proven in [11, Theorem 8]. Here we note that we do not require what [11] calls property (3) since this was only used to show the analogue of lemma 3.5 and in the formulation of the Morse lemma. The proof is in structure very similar to [21]. One first uses the Morse lemma to choose a suitable coordinate system in a neighbourhood of the critical point. Then one chooses $\varepsilon > 0$ so small such that $B_{2\varepsilon} \subseteq V$. The proof then proceeds in three steps. First one defines a region $H$ as in figure 3.1. Then one shows that $X^{c-\varepsilon}$ with $H$ attached is homotopic to $X^{c+\varepsilon}$. Finally one shows that $X^{c-\varepsilon}$ with the $k$-cell $D^k$ attached is homotopic to $X^{c-\varepsilon}$ with $H$ attached. The claim then follows from

$$X^a \cup D^k \simeq X^{c-\varepsilon} \cup D^k \simeq X^{c-\varepsilon} \cup H \simeq X^{c+\varepsilon} \simeq X^b$$

where we also used the result of lemma 3.5. □



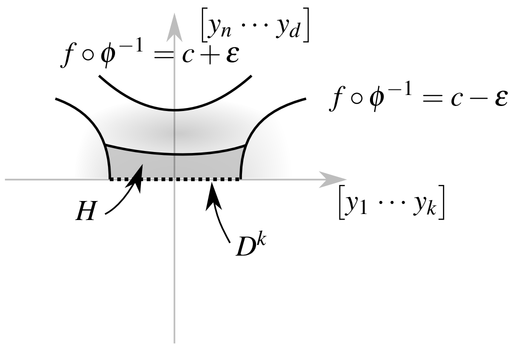




Figure 3.1

In the original proof [23] of the Morse inequalities Morse inspected in the analogues to lemmas 3.5 and 3.7 directly how the Betti numbers of the manifold changed from $X^a$ to $X^b$. The modern proofs however lean on the language of algebraic topology which is why we now need to argue how lemmas 3.5 and 3.7 imply the Morse inequalities. This final part of the proof is given in all details in [21, §I.5]. Since we only use the equality in equation (3.4) we shall outline its proof.

Denote the $k$-th relative homology group by $H_k(X,A)$ for spaces $A \subseteq X$. Once again we refer the reader to [14, Chapter 2] for details regarding their definition. We require a result from algebraic topology proven in [14, Corollary 2.24]:

**Proposition 3.8** (Corollary of the excision theorem, [14, Corollary 2.24])**.** *Let $X = A \cup B$ be a CW-complex. Then we have for every $k$ that $H_k(X,A) \cong H_k(B, A\cap B)$ are isomorphic.*

Now define the Euler characteristic for relative homology group by

$$\chi(X,Y) := \sum_k (-1)^k \dim_{\mathbb{K}} H_k(X,Y)\,. \tag{3.6}$$

It is shown in [21, §5] that this is additive, in the sense that given suitable $X \supseteq Y \supseteq Z$ we have that

$$\chi(X,Z) = \chi(X,Y) + \chi(Y,Z) \tag{3.7}$$

Let $a_0 < \cdots < a_L$ be such that $[a_l, a_{l+1}]$ contains precisely one critical value in its interior with index $k_l$ and such that $X = X^{a_L}$ and $X^{a_0} = \emptyset$. It now follows from lemma 3.7 and proposition 3.8 that

$$\begin{aligned} H_k(X^{a_{l+1}}, X^{a_l}) &= H_k\left(X^{a_l} \cup D^{k_l}, X^{a_l}\right) \\ &= H_k\left(D^{k_l}, S^{k_l}\right) \\ &= \begin{cases} \mathbb{K} & \text{if } k = k_l, \\ 0 & \text{else} \end{cases} \end{aligned}$$

so

$$\chi(X^{a_{l+1}}, X^{a_l}) = (-1)^{k_l}\,. \tag{3.8}$$

Hence we obtain by inductively applying equation (3.7) and then (3.8) that

$$\chi(X) = \chi(X^{a_L}, X^{a_0}) = \sum_l \chi(X^{a_{l+1}}, X^{a_l}) = \sum_k (-1)^k \operatorname{Ind}_k(f)$$

so equation (3.4) of the Morse inequalities follows for the case that no two critical points lie in the same level set. In case there are multiple critical points on the same level set the notation becomes a little more involved but the idea of the proof is the same. The proof of the inequalities (3.3) involves defining the Betti numbers for relative homology groups and then making a similar argumentation as above with these Betti numbers instead of the Euler characteristic. For details we refer the reader to [21, §5].

In the final part of this chapter we inspect the Morse inequalities in the special case that $f$ is harmonic. If we assume that $f$ is harmonic the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain from the Morse inequalities (theorem 3.3) a result from [25, Corollary 10.1].

**Corollary 3.9** (Morse inequalities for harmonic $f$ in $\mathbb{R}^2$, [25, Corollary 10.1])**.** *Let $X \subseteq \mathbb{R}^2$ be a manifold with corners and $f\colon X \to \mathbb{R}$ be harmonic and Morse. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M + \mu_1 - \mu_0 &= -\chi(X)\,. \end{aligned}$$

In dimensions $d = 3$ we obtain from theorem 3.3 a result from [25, Corollary 10.2].

**Corollary 3.10** (Morse inequalities for harmonic $f$ in $\mathbb{R}^3$, [25, Corollary 10.2])**.** *Let $X \subseteq \mathbb{R}^3$ be a manifold with corners and $f\colon X \to \mathbb{R}$ be harmonic and Morse. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0 \\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= \chi(X)\,. \end{aligned}$$

Perhaps give a classical example of a Morse function to determine the Betti numbers.

# 4 Connected entrant boundaries in $\mathbb{R}^2$

We will start this section by giving an essentially negative answer to question 1.1 in $d = 2$ dimensions. Thus it is not possible to have a harmonic function with interior critical point on a simply connected planar domain with connected entrant and connected emergent boundaries. The proof of this will involve Morse theory on manifolds with corners which was introduced in the previous chapter. We shall then state a result from [2] which more generally relates the number of entrant components with the number of critical points in the plane using tools from complex analysis. After that we give examples showing that dropping the condition of simply connectedness on the domain yields a harmonic function with interior stagnation point. We will also consider the analogous problem in $d = 4$ dimensions and show that there exists a function and domain with the desired properties.

## A negative result for simply connected domains

We now give a result which is essentially a negative answer to question 1.1. It was possible to prove this statement using level sets of critical points or using invariant manifolds. The following proof involves Morse theory since the techniques of the proof generalise to the three dimensional case.

**Proposition 4.1** (Negative answer to question 1.1 in $d = 2$ dimensions)**.** *Let $X \subseteq \mathbb{R}^2$ be a simply connected planar manifold with corners and let $f\colon X \to \mathbb{R}$ have no irregular critical points. Let $\Sigma = \Sigma_{\leq 0} \sqcup \Sigma_{\geq 0}$ be a disjoint decomposition of the boundary into simply connected nonempty sets such that we have for the entrant boundary $\Sigma^- \subseteq \Sigma_{\leq 0}$ and for the emergent boundary $\Sigma^+ \subseteq \Sigma_{\geq 0}$. Then $f$ has no interior critical point.*

*Proof.* Let $\gamma = \{x_1, x_2\} = \partial\Sigma_{\leq 0}$. Then we can cut the domain along a curve $\Gamma$ such that the endpoints $\gamma = \partial\Gamma$ of the cut coincide with $x_1$ and $x_2$, that is $\partial\Gamma = \{x_1, x_2\}$. Now we obtain two new domains $X^+$ and $X^-$ such that $\partial X^+ \cap \Sigma_{\leq 0} \subseteq \gamma$ and $\partial X^- \cap \Sigma_{\geq 0} \subseteq \gamma$. We can assume that $\Gamma$ is a smooth manifold and corresponds to the stratum $X_{\Gamma^+}$ for $X^+$ and $X_{\Gamma^-}$ for $X^-$. Analogously $\gamma$ corresponds to strata $X_{\gamma^\pm}$ on $X^\pm$. Locally around the corner point $x_1$ we have a situation depicted as in figure 4.1. We assume that we chose $\Gamma$ in such a way that it forms an acute angle with $u = \nabla f$ at the boundary points $\gamma$. For the following argumentation we require that $u$ is strongly Morse on both $X^+$ and $X^-$, so assume for a moment that this is the case. Since each point of $\gamma$ is an essential critical point for either $f$ or $-f$ on precisely one of the domains $X^+$ or $X^-$ we have the relation

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^+,0}(-f) + \mathrm{Ind}_{\gamma^-,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) = 2\,. \tag{4.1}$$

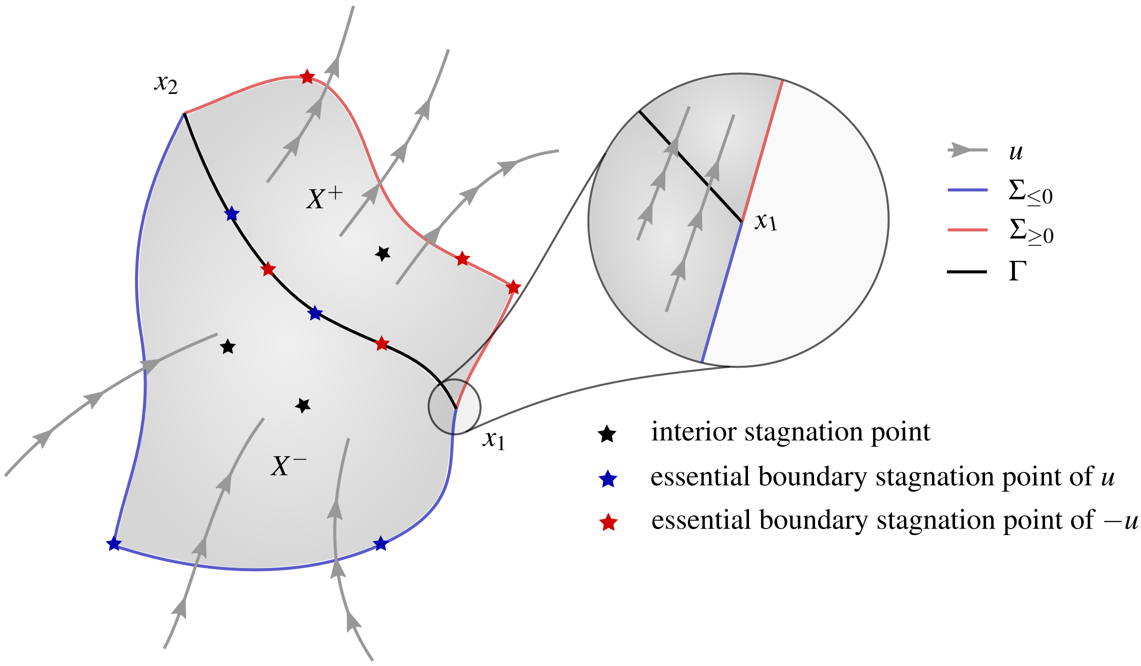



Figure 4.1: The situation at hand.

We now focus our attention on $X^+$. Since no essential critical points of $f$ lie on $\Sigma^+ \setminus \gamma$ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f)\,. \tag{4.2}$$

Analogously we have on $X^-$ that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{4.3}$$

In addition we have on $\Gamma$ that the emergent critical points of $f$ on $X^+$ are the entrant critical points of $-f$ on $X^-$, that is

$$\mathrm{Ind}_{\Gamma^+,0}(f) = \mathrm{Ind}_{\Gamma^-,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma^+,1}(f) = \mathrm{Ind}_{\Gamma^-,0}(-f)\,. \tag{4.4}$$

Using equations (4.2), (4.3) and (4.4) we obtain

$$\mu_0^+ - \mathrm{Ind}_{\gamma^+,0}(f) = \nu_1^- \qquad \text{and} \qquad \mu_1^+ = \nu_0^- - \mathrm{Ind}_{\gamma^-,0}(-f)\,. \tag{4.5}$$

Consider the Morse inequality for $f$ on $X^+$

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) \tag{4.6}$$

and the Morse inequality for $-f$ on $X^-$

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X)\,. \tag{4.7}$$

We now add equations (4.6) and (4.7) and insert relations (4.5) to obtain

$$M^- + M^+ - \mathrm{Ind}_{\gamma^+,0}(f) - \mathrm{Ind}_{\gamma^-,0}(-f) = -2\chi(X) = -2\,.$$

Since $\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \leq 2$ by equation (4.1) and $M^\pm \geq 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

The claim remains to be shown in the case that $f$ is not strongly Morse on $X^+$ and $X^-$. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 2.18 applied separately to the domains $X^+$ and $X^-$. Since $E^+$ and $E^-$ are residual in $B_\delta$ we can in particular pick an $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. It follows from the slanted angles at which $\Gamma$ approaches $\gamma$ that if the points $x_1$, $x_2$ are essential critical points of $f$ that they then are in fact regular. Hence we obtain that

$$\mathrm{Ind}_{\gamma^+,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma^+,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma^-,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma^-,k}(-f)\,.$$

By the same corollary $u^\varepsilon$ has no essential stagnation points on $\Sigma^+(u)$ and $-u$ has no essential stagnation points on $\Sigma^-(u)$. The claim then follows by the calculations above where we replace $f$ with $f^\varepsilon$ and then note that $M^\varepsilon = M$. □

## A generalisation of this result

We could also have used tools from complex analysis to show proposition 4.1. In fact, using complex analysis [2] gives a more refined result for which we require the next definition.

**Definition 4.2** (Number of entrant / emergent boundary components, [2])**.** Let $J^\pm$ denote the number of connected components of $\Sigma^\pm$ which are proper subsets of a component of $\Sigma$. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\geq 0} \sqcup \Sigma_{\leq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma_{\geq 0}$ which are proper subsets of a component of $\Sigma$.

We state a consequence of a result from [2, Theorem 2.1 and 2.2]:

**Proposition 4.3** (Special case of [2, Theorem 2.1 and 2.2])**.** *Let $X \subseteq \mathbb{R}^d$ be a bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u\colon X \to \mathbb{R}$ be a harmonic vector field, nonzero on each component. Then we have the relation*

$$M \leq -\chi(X) + \frac{J^+ + J^-}{2} \tag{4.8}$$

*where $M$ denotes the number of stagnation points of $u$, counting multiplicities. If in addition we assume that there are no irregular stagnation points then we have*

$$M \leq -\chi(X) + J^{\geq 0}\,. \tag{4.9}$$

For a proof we refer the reader to [2, Theorem 2.1] and [2, Theorem 2.2]. Here one sets $\underline{\alpha} = n$ to be the outer unit normal and $D = \chi(X)$ to be the Euler characteristic.

If we set $X$ to be homeomorphic to the disk such that $\chi(X) = 1$ and have a decomposition of the boundary $\Sigma$ as in proposition 4.1, that is $J^\pm = J^{\geq 0} = 1$, we then obtain from proposition 4.3 that $M \leq 0$ and $f$ has no interior critical point.

## The case of holes in the domain

If we set $J^{\pm} = J^{\geq 0} = 1$ in relations (4.8) or (4.9) we obtain the condition on the number of interior stagnation points

$$M \leq -\chi(X) + 1\,. \tag{4.10}$$

This indicates that if we allow for holes in the domain $X$ it is possible to have a vector field with simply connected entrant boundary and interior stagnation points. In fact we will give two examples where we have equality in equation (4.10) for the cases $\chi(X) = 0$ and $\chi(X) = -1$. For this define two differential operators in $d = 2$ dimensions by

$$\nabla^{\perp} f = \operatorname{Curl} f := \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix} \tag{4.11}$$

and

$$\operatorname{curl} u := -\partial_1 u_2 + \partial_2 u_1\,.$$

The next proposition gives us a recipe to generate harmonic vector fields.

**Proposition 4.4.** *Let $\psi \colon \mathbb{R}^2 \supseteq X \to \mathbb{R}$ be harmonic then $u = \nabla^{\perp}\psi$ is a harmonic vector field.*

*Proof.* We have

$$\operatorname{div} u = \operatorname{div} \nabla^{\perp}\psi = 0$$

and one calculates

$$\operatorname{curl} u = \operatorname{curl} \nabla^{\perp}\psi = -\Delta\psi = 0$$

from which the claim follows. □

The function $\psi$ is also called a *stream function*. The next example is our first example of a harmonic vector fields in $d = 2$ dimensions with an interior stagnation points for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component.

**Example 4.5** (Flow through tube with hole and stagnation point)**.** Consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{4.12}$$

where

$$\Phi_2 = \log(|\cdot|) \tag{4.13}$$

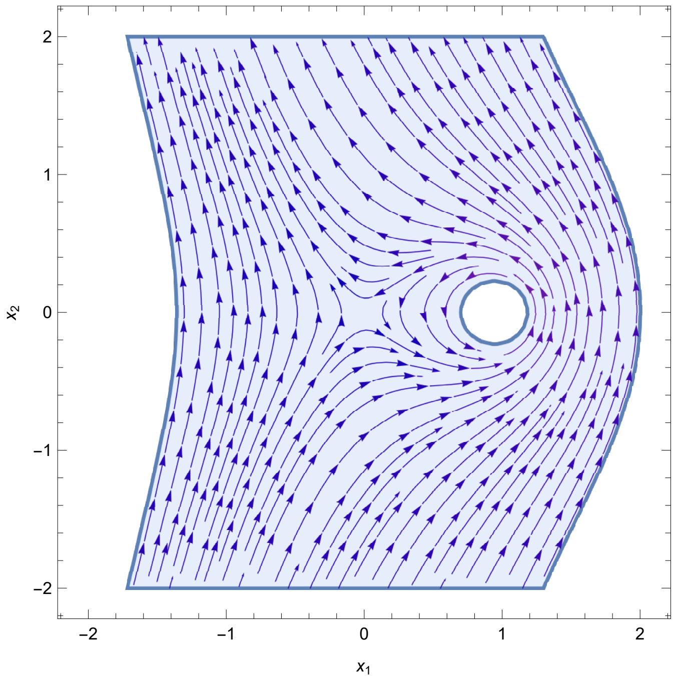



Figure 4.2: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap (\mathbb{R} \times [-2,2])$. Here $\psi$ is given by equation (4.12).

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and $e_i$ is the $i$-th unit vector. A plot of the streamlines in figure 4.2 indicates that $u = \nabla^{\perp}\psi$ fulfils the requirements on the domain

$$X = \psi^{-1}([-0.5,2]) \cap (\mathbb{R} \times [-2,2]) .$$

Indeed, an elementary calculation reveals that the origin is a stagnation point of $u$.

Example 4.5 highlights the importance of the requirement in proposition 4.1 that the domain be simply connected. We now give a similar example, this time with two holes in the domain and two interior stagnation points.

**Example 4.6** (Flow through tube with two holes and stagnation points)**.** We consider the stream function

$$\begin{aligned} \psi \colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1 \end{aligned} \tag{4.14}$$

A plot of the streamlines in figure 4.3 indicates that $u = \nabla^{\perp}\psi$ on the domain

$$X = \psi^{-1}([-0.7,0.7]) \cap (\mathbb{R} \times [-2,2])$$

has the desired properties.

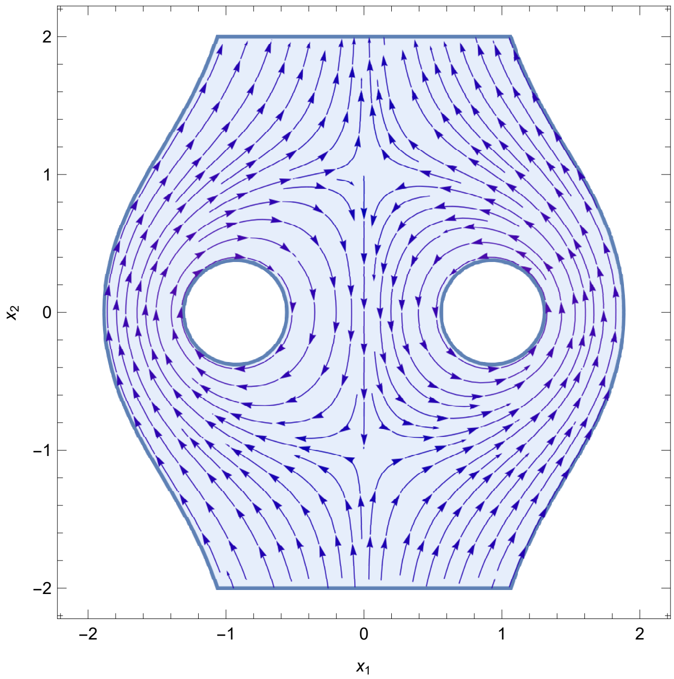



Figure 4.3: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.7, 0.7]) \cap (\mathbb{R} \times [-2,2])$. Here $\psi$ is given by equation (4.14).

## The case of $d = 4$ dimensions

We ask whether there exists a harmonic vector field on a simply connected domain with connected entrant boundary and with interior stagnation point in a dimension higher than $d = 2$. Indeed, in $d = 4$ dimensions we can readily give an example of such a harmonic vector field.

**Example 4.7** (Connected entrant boundary in $d = 4$ dimensions)**.** Consider as domain $X = B_1 \subseteq \mathbb{R}^4$ the unit ball and the harmonic function

$$\begin{aligned} f \colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned} \tag{4.15}$$

This has a critical point at the origin. We will show in proposition 4.8 that the entrant boundary $\Sigma^-$ is in fact connected.

**Proposition 4.8** (Simply connected entry boundary in example 4.7)**.** *The harmonic function given by equation* (4.15) *has simply connected entrant boundary.*

*Proof.* First observe that the boundary $\Sigma = S^3$ can be away from the equator locally parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{4.16}$$

on the boundary $\Sigma = S^3$ we have that $x_1$ is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_\pm \colon \mathbb{R}^3 \supseteq B_1 &\to \Sigma \subseteq \mathbb{R}^4 \\ \bar{x} &\mapsto x = (x_1, \bar{x}) \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{4.17}$$

with inverses $\psi_\pm = (\phi_\pm)^{-1}$. We now calculate the gradient of $f$

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the outer unit normal to the boundary $\Sigma$ is given by

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^\pm$ iff

$$0 < \pm \nabla f \cdot n = \pm 2\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) .$$

Using condition (4.16) we obtain the equivalent condition

$$0 < \pm 1 - 2\left(x_3^2 + x_4^2\right)$$

Define the cylinder

$$C = \left\{\bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2\right\} = \mathbb{R} \times B_{1/\sqrt{2}} \subseteq \mathbb{R}^3$$

If we return to our parametrisation (4.17) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(\bar{x}) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm\left(\Sigma^+\right)$$

and

$$B_1 \setminus C = \psi_\pm\left(\Sigma^-\right) .$$

Since $\phi_+$ is a homeomorphism onto the northern hemisphere and $\phi_-$ a homeomorphism onto the southern hemisphere and the surface $S^2$ corresponds to the equatorial sphere the claim then follows. The situation is depicted in figure 4.4. $\square$

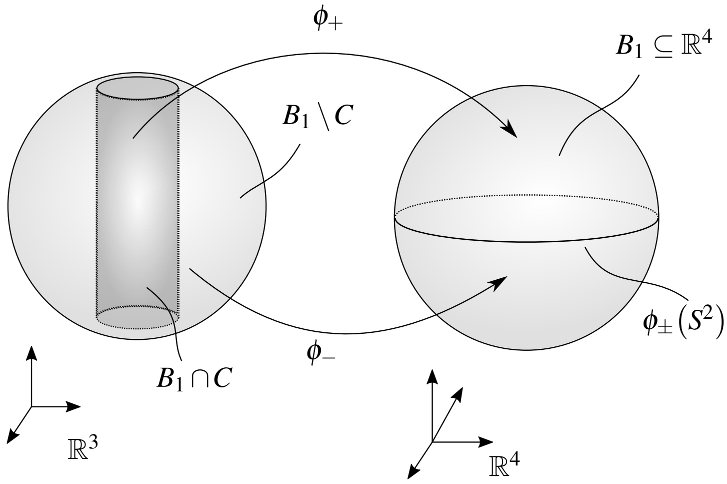



Figure 4.4: Visualisation of the situation in the proof of proposition 4.8.

# 5 Connected entrant boundaries in $\mathbb{R}^3$

In the following chapter we will discuss question 1.1 in the case of $d = 3$ dimensions. That is, we are looking for a harmonic function with interior stagnation point on a simply connected domain and connected entrant boundary. We will first give a negative answer in the case of a cylinder. After that we will use Morse theory to essentially argue that the number of interior stagnation points of such an example must be an even number. Finally we will give an example of a function and a domain with the desired properties.

## A negative result for the cylinder

The following proof comes from [29] and is a negative answer to question 1.1 for the cylinder.

**Proposition 5.1.** *Let $X = [0,1] \times \overline{U} \subseteq \mathbb{R}^3$ be a cylinder where $U \subseteq \mathbb{R}^2$ is an open set. Let further $f\colon X \to \mathbb{R}$ be harmonic such that the sides $[0,1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^+$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^-$ is the emergent boundary. Then $f$ cannot have an interior critical point.*

*Proof.* Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary $\Sigma$. Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times \partial U$ such that $\partial_1 f(x)$ is minimal on $X$. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. $\square$

## A condition on the interior type numbers

Mimicking the proof of proposition 4.1 in two dimensions we obtain a condition on the type numbers for a harmonic function with interior stagnation point and simply connected entrant boundary.

**Proposition 5.2.** *Let $X \subset \mathbb{R}^3$ be a compact manifold with corners. Let $f\colon X \to \mathbb{R}$ be a Morse harmonic function. Assume that the strictly entrant boundary $\Sigma^-$ is non-empty and simply connected and that $\gamma = \partial\Sigma^-$ is a one-dimensional manifold with corners homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Then we have that*

$$M_1 - M_2 = 0\,.$$

*Proof.* As in the two dimensional case we split the domain $X$ with a surface $\Gamma$ such that $\partial\Gamma = \gamma = \partial\Sigma^-$. Denote the two arising domains by $X^+$ and $X^-$ where $\partial X^- \cap \Sigma^+ \subseteq \gamma$ and $\partial X^+ \cap \Sigma^- \subseteq \gamma$. Since by proposition 2.13 there are finitely many interior critical points in $X$ we can also assume that no interior critical points lie on $\Gamma$. Furthermore we assume that $\Gamma$ approaches $\gamma$ at a slanted angle. For the following argumentation we require that $f$ is strongly Morse on both $X^+$ and $X^-$ so assume for a moment that this is the case. By assumption we have that $\gamma$ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Since $f$ is non-degenerate the the number of maxima and minima of $f$ on $\gamma$ must be equal and thus

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) = \mathrm{Ind}_{\gamma^+,1}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \tag{5.1}$$

We now turn our attention to $X^+$. Since no essential critical points lie on $\Sigma^+$ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f)\,. \tag{5.2}$$

Analogously we have on $X^-$ that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{5.3}$$

In addition we have that the emergent critical points on $\Gamma = \Gamma^+$ of $f$ on $X^+$ are the entrant critical points on $\Gamma = \Gamma^-$ of $-f$ on $X^-$, that is

$$\begin{aligned} \mathrm{Ind}_{\Gamma^+,0}(f) &= \mathrm{Ind}_{\Gamma^-,2}(-f) \\ \mathrm{Ind}_{\Gamma^+,1}(f) &= \mathrm{Ind}_{\Gamma^-,1}(-f)\,. \\ \mathrm{Ind}_{\Gamma^+,2}(f) &= \mathrm{Ind}_{\Gamma^-,0}(-f) \end{aligned} \tag{5.4}$$

Using equations (5.2), (5.3) and (5.4) we obtain

$$\begin{aligned} \mu_0^+ - \nu_2^- &= \mathrm{Ind}_{\gamma^+,0}(f) \\ \mu_1^+ - \nu_1^- &= \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,1}(-f)\,. \\ \mu_2^+ - \nu_0^- &= -\mathrm{Ind}_{\gamma^-,0}(-f) \end{aligned} \tag{5.5}$$

We observe the Morse inequalities for $f$

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X)\,. \tag{5.6}$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \tag{5.7}$$

where the $M_k$ continue to denote the interior type numbers of $f$. We now subtract equation (5.7) from (5.6) and insert relations (5.5) to obtain then with equation (5.1)

$$\begin{aligned}0 &= M_1^- - M_2^- + M_1^+ - M_2^+ + \operatorname{Ind}_{\gamma^+,0}(f) + \operatorname{Ind}_{\gamma^-,1}(-f) - \operatorname{Ind}_{\gamma^+,1}(f) - \operatorname{Ind}_{\gamma^-,0}(-f)\\ &= M_1 - M_2\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that $f$ is not strongly Morse on $X^+$ and $X^-$. In this case let $E^+, E^- \subseteq B_\delta$ be as in corollary 2.18 applied separately to the domains $X^+$ and $X^-$. Since $E^\pm$ are residual in $B_\delta$ we can in particular pick a $\varepsilon \in E^+ \cap E^-$ by the Baire category theorem. Since $x_1, x_2$ are non-degenerate critical points of $f$ due to the slanted angle at which $\Gamma$ approaches $\gamma$ we obtain that

$$\operatorname{Ind}_{\gamma,k}(f^\varepsilon) = \operatorname{Ind}_{\gamma,k}(f) \qquad \text{and} \qquad \operatorname{Ind}_{\gamma,k}(-f^\varepsilon) = \operatorname{Ind}_{\gamma,k}(-f)$$

By the same corollary we can assume that $f^\varepsilon$ has no essential critical points on $\Sigma^+(f)$ and $-f^\varepsilon$ has no essential critical points on $\Sigma^-(f)$. The claim then follows by the calculations above where we replace $f$ with $f^\varepsilon$ and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. □

## THE example

Based on example 4.7 in $d = 4$ dimensions of a harmonic vector field with interior stagnation point the author argued that it would be simplest to construct such a vector field in $d = 3$ dimensions in a similar manner. More specifically we choose as domain the ball and polynomials as our harmonic function. In choosing the ball as domain we also take into account the result from proposition 5.1 which is a negative result to question 1.1 for the cylinder. Based on the result from proposition 5.2 we see that the number of stagnation points must be at least two. The author then implemented a Mathematica routine to generate harmonic polynomials with two stagnation points and a plotting function to inspect what occurs. Indeed, this approach yielded a function with the desired properties as we shall discuss in this section.

**Example 5.3** (A harmonic function with interior critical point and simply connected entrant boundary)**.** Consider the domain $X = \overline{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large, and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R}\\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1x_2^2 + x_2x_3\end{aligned} \tag{5.8}$$

This induces the harmonic vector field

$$\begin{aligned} u\colon X &\to \mathbb{R}^3\\ x &\mapsto \nabla f(x) = \begin{bmatrix} x_1(1-x_1) + x_2^2\\ x_2(2x_1 - 1) + x_3\\ x_2\end{bmatrix}\end{aligned}$$

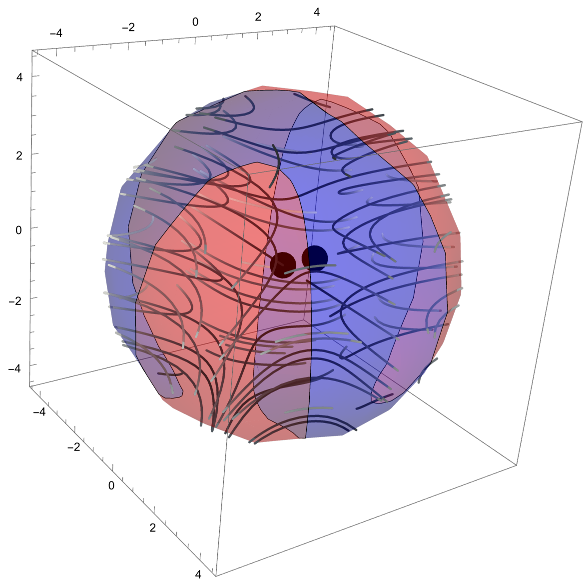



Figure 5.1: A stream plot of the function $u$. The interior stagnation points are highlighted in black. $\Sigma^+$ is shaded red, $\Sigma^-$ blue.

It follows from setting $u(x) = 0$ that $x_2 = 0$ and then $x_3 = 0$ and $x_1 \in \{0,1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of $u$. Conversely one sees that these are zeroes of $u$. Hence $f$ has two interior critical points for $r > 1$. Figure 5.1 shows a stream plot of $u$ with the two interior stagnation points highlighted as black dots. The boundary of the domain is shaded in blue for $\Sigma^-$ and in red for $\Sigma^+$. The stereographic projection of the boundary is plotted in figure 5.2. This plot indicates that the entrant boundary and the emergent boundary are simply connected. Indeed, we will show this in theorem 5.4.

The remainder of this section will be devoted to the proof of the following theorem:

**Theorem 5.4.** *The harmonic function given by equation* (5.8) *on the sphere has interior stagnation points and connected emergent and entrant boundaries for sufficiently large $r > 0$.*

Before we proceed to the proof we require some definitions from algebraic geometry. For an introduction this subject we refer the reader to [10], [12] or [13]. For polynomials $p_1, \dots, p_k \in \mathbb{R}[x_1, \dots, x_d]$ and a set $U \subseteq \mathbb{R}^d$ we denote the variety generated by these polynomials on $U$ by $V_U(p_1, \dots, p_k)$. In the case that $U = \mathbb{R}^d$ we write $V(\dots) = V_U(\dots)$.

**Definition 5.5** (Smoothness, [13, §5])**.** We call a $k$ dimensional algebraic variety $V_U(p_1, \dots, p_n)$ *smooth* or *non-singular* if the the Jacobian

$$\begin{bmatrix} \nabla p_1 & \cdots & \nabla p_n \end{bmatrix}^\top$$

is of rank $d-k$ on $V_U(p_1, \dots, p_n)$. This criterion of smoothness is also called the *Jacobi criterion*.

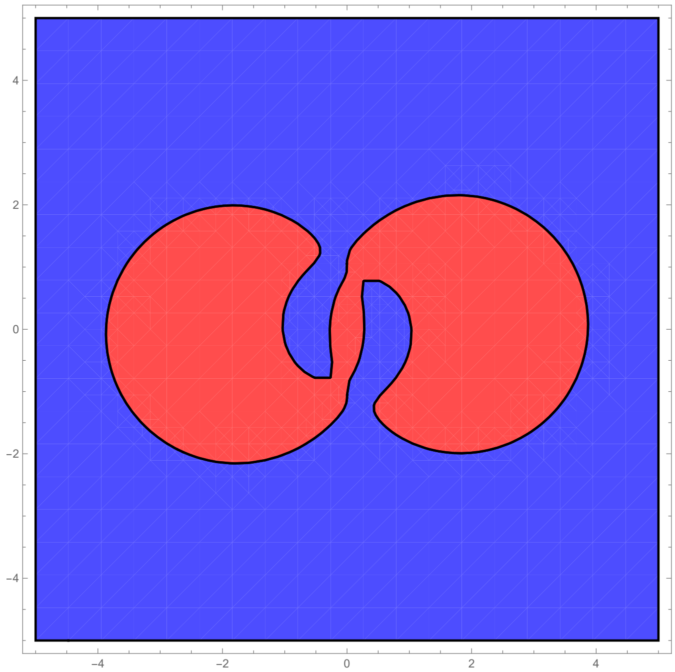



Figure 5.2: Stereographic projection of the surface $\Sigma$. $\Sigma^+$ is shaded red, $\Sigma^-$ blue.

By the implicit function theorem this means that the variety $V_U(p_1, \dots, p_n)$ in fact defines a $d-k$ dimensional submanifold of $\mathbb{R}^d$.

The proof of theorem 5.4 requires lemmas 5.6 and 5.9 which we will show later on.

*Proof of theorem 5.4.* It was already discussed in example 5.3 that $f$ has an interior critical point at the origin and is harmonic. For the connectedness of the entrant and emergent boundaries we calculate

$$rn \cdot u(x) = x_1^2(1-x_1) + x_2^2(3x_1 - 1) + 2x_2x_3 =: p_1(x) \tag{5.9}$$

and define further

$$P_2(r,x) := x_1^2 + x_2^2 + x_3^2 - r^2 \,. \tag{5.10}$$

Thus we have that the tangential boundary $\Sigma^0 = V(p_1, P_2(r,\cdot))$ is precisely the variety generated by the polynomials $p_1$ and $P_2(r,\cdot)$ for a fixed radius $r > 0$. In lemma 5.6 we will show that the variety $V(p_1, P_2(r,\cdot))$ is in fact smooth and in lemma 5.9 we will then show that it is in fact connected. Thus $\Sigma^0$ then defines a simple closed curve on $\Sigma$ and the stereographic projection of $\Sigma^0$ defines a simple closed planar curve. This is indicated by the black curve in figure 5.2. By the Jordan curve theorem this curve then splits the plane into two connected regions, one of which is simply connected. The preimage of these connected regions under the stereographic projections then corresponds precisely to the entrant and emergent boundaries. From this it follows that the entrant and emergent boundaries are simply connected which proves the theorem. $\square$

From now onwards we assume that $p_1$ and $P_2$ are given by equations 5.9 and 5.10. We first show the smoothness which was required in the proof of theorem 5.4.

**Lemma 5.6** (Smoothness)**.** *There exists $R > 0$ such that for every $r > R$ the variety $V(p_1, P_2(r,\cdot))$ is smooth.*

*Proof.* One calculates

$$T := \begin{bmatrix} \nabla p_1(x) & \frac{1}{2}\nabla_x P_2(r,x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 + 3x_2^2 + 2x_1 & x_1 \\ 6x_1x_2 - 2x_2 + 2x_3 & x_2 \\ 2x_2 & x_3 \end{bmatrix}$$

By the Jacobi criterion it is sufficient to show that this matrix is of full rank on $V(p_1, P_2(r,\cdot))$. This is equivalent to showing that

$$0 \neq \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = -9x_1^2x_2 + 3x_2^3 + 4x_1x_2 - 2x_1x_3 =: h_1(x)\,,$$

$$0 \neq \det \begin{bmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} = 6x_1x_2x_3 - 2x_2x_3 + 2x_3^2 - 2x_2^2 =: h_2(x) \text{ or}$$

$$0 \neq \det \begin{bmatrix} T_{31} & T_{32} \\ T_{11} & T_{12} \end{bmatrix} = 2x_1x_2 + 3x_1^2x_3 - 3x_2^2x_3 - 2x_1x_3 =: h_3(x)$$

for any $x \in V(p_1, P_2(r,\cdot))$. This in turn is equivalent to showing that

$$V(p_1, P_2(r,\cdot), h_1, h_2, h_3) = \emptyset\,. \tag{5.11}$$

Indeed, consider the variety

$$V(p_1, h_1, h_2, h_3)\,. \tag{5.12}$$

Maple calculates the Gröbner basis with lexicographic order $x_1 < x_2 < x_3$

$$72x_1^8 - 198x_1^7 + 228x_1^6 - 153x_1^5 + 56x_1^4 - 5x_1^3\,, \tag{5.13}$$

$$72x_1^5x_2 - 126x_1^4x_2 + 102x_1^3x_2 - 51x_1^2x_2 + 5x_1x_2\,, \tag{5.14}$$

$$-24x_1^7 + 42x_1^6 - 2x_1^5 - 23x_1^4 + 7x_1^3 + 10x_1x_2^2\,, \tag{5.15}$$

$$48x_1^4x_2 - 60x_1^3x_2 + 13x_1^2x_2 + 15x_2^3\,, \tag{5.16}$$

$$24x_1^4x_2 - 30x_1^3x_2 + 29x_1^2x_2 - 10x_1x_2 + 5x_1x_3\,, \tag{5.17}$$

$$72x_1^7 - 126x_1^6 + 6x_1^5 + 69x_1^4 - 31x_1^3 + 10x_1^2 - 10x_2^2 + 20x_2x_3\,, \tag{5.18}$$

$$-72x_1^7 + 414x_1^6 - 654x_1^5 + 399x_1^4 - 97x_1^3 + 10x_1^2 - 30x_2^2 + 20x_3^2\,. \tag{5.19}$$

For an introduction to Gröbner bases we refer the reader to for example [7]. We will however only use the fact that the polynomials (5.13)-(5.19) generate the variety (5.12). We see from the basis vector (5.13) that for $x \in V(p_1, h_1, h_2, h_3)$ the coordinate $x_1$ can take only finitely many values. It then follows with (5.14) that also $x_2$ can take only finitely many values and finally with (5.15) that $x_3$ can also take only finitely many values. So the variety (5.12) contains finitely many points. Thus if we choose $R$ so large that all of these points are contained in the ball $B_R$ then we have that (5.11) holds for all $r > R$. □

We now define $p_2$ to be the dehomogenisation of $P_2$, that is

$$p_2 := P_2(1,\cdot)\,.$$

Analogously let $P_1$ denote the homogenisation of $p_1$, that is

$$P_1(\varepsilon,x) := \varepsilon^3 p_1(x/\varepsilon)\,.$$

By rescaling the variety $V(p_1,P_2(r,\cdot))$ we obtain

$$V(p_1,P_2(r,\cdot)) = rV(x\mapsto p_1(rx),p_2) = rV(P_1(\varepsilon,\cdot),p_2) = r\mathscr{V}_\varepsilon \tag{5.20}$$

where we set $\varepsilon = 1/r$ and $\mathscr{V}_\varepsilon := V(P_1(\varepsilon,\cdot),p_2)$. Motivated by taking the limit $r\to\infty$ we inspect the variety $\mathscr{V}_0$ closer. The next proposition is required in the proof of lemma 5.9. It essentially states that the varieties $\mathscr{V}_\varepsilon \to \mathscr{V}_0$ converge as $\varepsilon \to 0$ outside of singular points. We will thus also need a notion of convergence of subsets on a metric space.

**Definition 5.7** (Hausdorff metric, tubular neighbourhood)**.** The *Hausdorff metric* for two sets $A,B\subseteq X$ is given by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\operatorname{dist}(x,B),\, \sup_{y\in B}\operatorname{dist}(y,A)\right\} \tag{5.21}$$

where

$$\operatorname{dist}(x,B) = \inf_{y\in B} d(x,y) \tag{5.22}$$

is the smallest distance from $x$ to $B$. For a given $\delta>0$ and a subset $A\subseteq\mathbb{R}^d$ we call the union of $\delta$ balls

$$\operatorname{Tub}_\delta(A) = \bigcup_{x\in A} B_\delta(x) \tag{5.23}$$

a *tubular neighbourhood* of $A$.

**Proposition 5.8** (Convergence of $\mathscr{V}_\varepsilon$ at smooth points)**.** *Let $U\subseteq\mathbb{R}^3$ be an open set such that $\mathscr{V}_0$ is smooth in an open neighbourhood of $\overline{U}$. Let further $\eta>0$. Then there exists a $\delta>0$ such that for all $\varepsilon<\delta$ we have that the Hausdorff distance satisfies*

$$d_H(\mathscr{V}_\varepsilon\cap U,\mathscr{V}_0\cap U) < \eta$$

*and additionally $\mathscr{V}_\varepsilon\cap U$ is isotopic to $\mathscr{V}_0\cap U$.*

*Proof.* Consider the mapping

$$F = \begin{bmatrix} P_1\\ p_2\end{bmatrix} : \mathbb{R}^4\to\mathbb{R}^2\,.$$

Since $V(P_1(0,\cdot),p_2)$ is smooth on an open neighbourhood of $\overline{U}$ we have by the Jacobi criterion that

$$DF(0,\cdot) = \left[\begin{array}{c|c} D_\varepsilon P_1(0,\cdot) & D_x P_1(0,\cdot) \\ 0 & Dp_2 \end{array}\right] \tag{5.24}$$

is of full rank on $\overline{U}$. By the implicit function theorem there exists at every point $x \in \overline{U}$ open neighbourhoods $\Omega_x \subseteq \mathbb{R}^4$ of $(0,x)$ and $\omega_x \subseteq \mathbb{R}^3$, a coordinate permutation $I \in O(4)$ and a continuously differentiable mapping $g_x \colon \omega_x \to \mathbb{R}$ such that

$$V(P_1,p_2) \cap \Omega_x = \{I(y,g_x(y)) \colon y \in \omega_x\}\,.$$

Since $D_xF(0,x)$ is of full rank we can assume (possibly after shrinking the open sets) that $I$ does not permute the $\varepsilon$-coordinate. Thus we can write $y = (\varepsilon, y_1, y_2) \in \omega_x$. We can also assume that $\Omega_x = B_{\delta_x} \times W_x \subseteq \mathbb{R}^4$ for some open $W_x \subseteq \mathbb{R}^3$ and some $\delta_x > 0$. Hence we also obtain that $\omega_x = B_{\delta_x} \times w_x$ for some open $w_x \subseteq \mathbb{R}^2$ and we can define our isotopy on $\Omega_x$ as

$$\begin{aligned} \varphi_x \colon B_{\delta_x} \times w_x &\to W_x \\ y &\mapsto \operatorname{proj}_{\mathbb{R}^4 \to W_x} I(y,g_x(y))\,. \end{aligned}$$

Note that $\varphi_x(\{\varepsilon\} \times w_x) = \mathscr{V}_\varepsilon \cap W_x$. From this it also follows that we can choose $\delta_x$ such that

$$d_H(\mathscr{V}_\varepsilon \cap W_x, \mathscr{V}_0 \cap W_x) < \eta\,.$$

Now for $x \in \overline{U}$ the $\Omega_x$ form an open cover of $\overline{U}$. By compactness there exists a finite subcover. Set $\delta > 0$ to be the minimum of all $\delta_x$ for the $\Omega_x$ in this finite subcover and the claim follows. □

The next lemma shows the connectedness of the variety $V(p_1,P_2(r,\cdot))$. Because the proof is quite lengthy, a part of the proof has been split off into proposition 5.10 which will be proved later on.

**Lemma 5.9** (Connectedness)**.** *There exists an $r > 0$ such that the planar variety $V(p_1,P_2(r,\cdot))$ has one connectivity component.*

*Proof.* By lemma 5.6 there exists a $R > 0$ such that for all $r > R$ we have that the variety $V(p_1,P_2(r,\cdot))$ is smooth and by equation (5.20) $\mathscr{V}_\varepsilon$ is also smooth for $\varepsilon < 1/R$. We inspect $\mathscr{V}_0$ closer. Observe that

$$P_1(0,x) = -x_1^3 + 3x_1x_2^2$$

which is the monkey saddle embedded into $\mathbb{R}^3$. We thus define curves

$$\widetilde{\alpha}^\pm := \left\{ t \begin{bmatrix} \pm\sqrt{3} & 1 \end{bmatrix}^\top : t \in \mathbb{R} \right\}$$

and $\widetilde{\alpha}^0 := \{0\} \times \mathbb{R}$. We then define $\alpha^\bullet := (\widetilde{\alpha}^\bullet \times \mathbb{R}) \cap S^2$. Setting $A := \alpha^- \cup \alpha^+ \cup \alpha^0$ we have the relation

$$\mathscr{V}_0 = V(P_1(\varepsilon,\cdot),p_2) = A\,.$$

Thus $\mathscr{V}_0$ consists of six smooth arcs originating at the singularity $e_3$ and ending at the singularity $-e_3$. Similar to the classical beach ball. Now consider for $\rho > 0$ the open sets $W_\rho := B_\rho \times \mathbb{R} \subseteq \mathbb{R}^3$ and $U_\rho := \mathbb{R}^3 \setminus W_\rho$. Since $\mathscr{V}_0$ is smooth in a neighbourhood of $\overline{U}_\rho$ we obtain from proposition 5.8 that in a certain sense $\mathscr{V}_\varepsilon$ is obtained from $\mathscr{V}_0$ by a small deformation on $U_\rho$. Thus in order to show connectedness of $\mathscr{V}_\varepsilon$ for sufficiently small $\varepsilon > 0$ we have to inspect what happens around the points $\pm e_3$. Now observe that we have the symmetry

$$p_1(x_1, -x_2, -x_3) = p_1(x) \tag{5.25}$$

and thus it suffices to inspect what happens around the point $e_3$. For this parametrise the neighbourhood $S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)$ of $e_3$ by the diffeomorphism

$$\begin{aligned} \psi\colon B_{1/2} &\to S^2 \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right) \\ \tilde{x} &\mapsto x = \begin{bmatrix} \tilde{x} & \sqrt{1 - |\tilde{x}|^2} \end{bmatrix}^\top . \end{aligned}$$

We set

$$\widetilde{\mathscr{V}}_\varepsilon := \psi^{-1}\left(\mathscr{V}_\varepsilon \cap \left(B_{1/2} \times \mathbb{R}_{\geq 0}\right)\right) = V_{B_{1/2}}\left(x \mapsto P_1\left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right)\right) = V_{B_{1/2}}\left(\widetilde{P}_1(\varepsilon, \cdot)\right)$$

where we defined

$$\widetilde{P}_1(\varepsilon, x) := P_1\left(\varepsilon, x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right) = -x_1^3 + 3x_1x_2^2 + \varepsilon\left(x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2}\right).$$

In a similar manner we define $\widetilde{\alpha}^\bullet := \psi^{-1}(\alpha^\bullet)$, $\widetilde{U}_\rho := \psi^{-1}\left(U_\rho\right)$, $\widetilde{W}_\rho := \psi^{-1}\left(W_\rho\right)$ and $\widetilde{A} := \psi^{-1}(A)$. Now let the sets $W$ and $C \subseteq W$ be as in proposition 5.10. Pick $\rho > 0$ so small that $\widetilde{W}_{2\rho} \subseteq W$. Now we can pick $\eta$ smaller than the minimum distance between two arcs of $\widetilde{\mathscr{V}_0}$ on $\widetilde{U}_\rho$. We also assume that $\eta$ is smaller than the Hausdorff distance between $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ and $C$. Now choose $\delta$ as in proposition 5.8. We can assume that $0 < \varepsilon < \delta$. We thus have a situation as in figure 5.3. Number the arcs lying close to $A \cap U_\rho$ as in figure 5.4. Since $\widetilde{\gamma}$ by proposition 5.10 splits $\widetilde{W}_{2\rho}$ and lies in $C \cap \widetilde{\mathscr{V}}_\varepsilon$ we must have that $\widetilde{\gamma}$ in fact connects arcs 3 and 6. As the variety is smooth we must also have that arcs 1 and 2 are connected and analogously that arcs 4 and 5 are connected. By equation (5.25) the situation around $-e_3$ is analogous but mirrored. We thus obtain analogous connections between the six arcs at $-e_3$ which is summarised in figure 5.5. Thus we in fact have that $\mathscr{V}_\varepsilon$ is connected. The claim then follows from relation (5.20). □

The following proposition is part of the proof of lemma 5.9:

**Proposition 5.10.** *Let $\widetilde{U}_\rho$, $\widetilde{\alpha}^\bullet$, $\widetilde{A}$, $\widetilde{\mathscr{V}}_\varepsilon$ and $\widetilde{P}_1$ be as in the proof of lemma 5.9. There exists an open cube $W \subseteq \mathbb{R}^2$ containing the origin and a set $C \subseteq W$ such that $C$ has positive Hausdorff distance to the set $\left(\widetilde{\alpha}^0 \cup \widetilde{\alpha}^-\right) \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that for any sufficiently small $\varepsilon > 0$ there is an arc $\widetilde{\gamma}$ entirely contained in $C \cap \widetilde{\mathscr{V}}_\varepsilon$ splitting $W$ into two parts.*

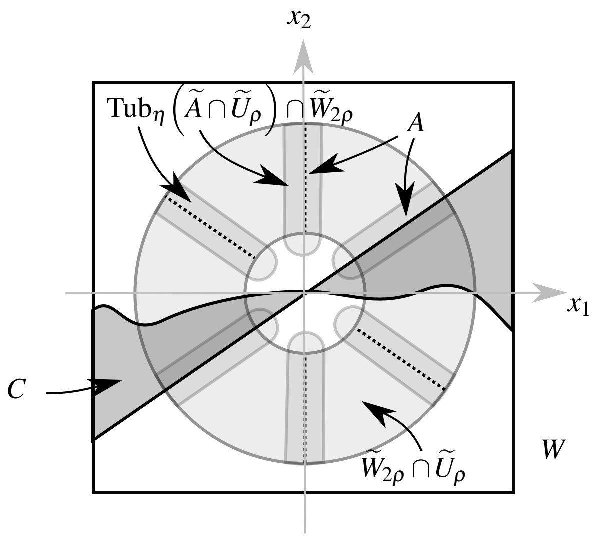



Figure 5.3: An overview of the situation around $e_3$.

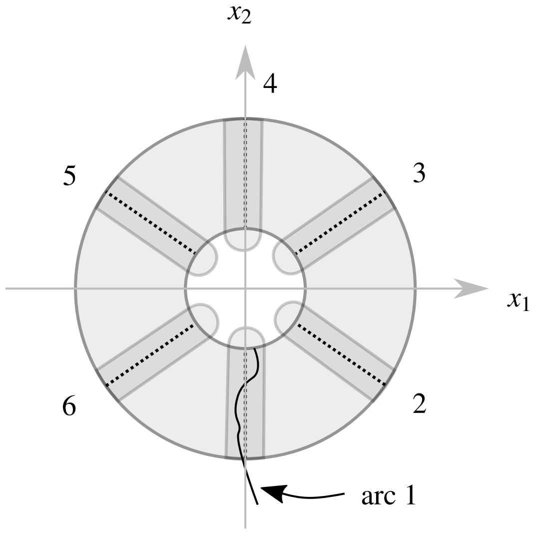



Figure 5.4: The numbering of the arcs around $e_3$.

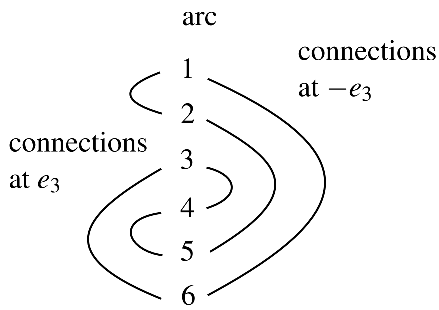



Figure 5.5: The connection of the arcs at $\pm e_3$.

*Proof.* We borrow notation from the proof of lemma 5.9. Let $W \subseteq \mathbb{R}^2$ be an open cube around the origin which we fix later. Define polynomials

$$\tilde{q}_1(x) := -x_1^3 + 3x_1x_2^2 \tag{5.26}$$

$$\tilde{q}_2(x) := x_1^2 - x_2^2 + x_2\sqrt{1 - x_1^2 - x_2^2} \tag{5.27}$$

and observe that

$$\widetilde{P}_1(\varepsilon,\cdot) = \tilde{q}_1 + \varepsilon\tilde{q}_2\,. \tag{5.28}$$

By equation (5.26) $\tilde{q}_1$ is a monkey saddle and $V_W(\tilde{q}_1) = \widetilde{A}$ is similar to figure 5.6a. The signs in the figure indicate the sign of $\tilde{q}_1$ in a given region. From equation (5.27) we obtain that $\nabla\tilde{q}_2(0) = e_2$ and $\tilde{q}_2(0) = 0$. Hence we observe that the $V_W(\tilde{q}_2)$ looks similar to figure 5.6b for a sufficiently small neighbourhood $W$. More concretely we choose $W$ so small that the arc $\widetilde{\beta} = V_W(\tilde{q}_2)$ has positive distance to $\widetilde{A} \cap \widetilde{U}_\rho$ for any $\rho > 0$ and such that a given vertical line in $W$ intersects $\widetilde{\beta}$ in precisely one point. Now we claim that the set $C$ consisting of the vertical lines between $\widetilde{\alpha}^+$ and $\widetilde{\beta}$ fulfils the claim.

We first show that there exists a curve $\widetilde{\gamma}$ through the origin which is entirely contained in $C \cap \widetilde{\mathscr{V}}_\varepsilon$. For this note that we have $\widetilde{P}_1(\varepsilon,0) = 0$ and $\nabla_x\widetilde{P}_1(\varepsilon,x)\big|_{x=0} = \varepsilon e_2$ and thus there exists locally around the origin a parametrisation $\widetilde{\gamma}(t) = (t,\widetilde{\gamma}_2(t))$. By a similar argument there also exists locally a parametrisation $\widetilde{\beta}(t) = \big(t,\widetilde{\beta}_2(t)\big)$. We see that $\widetilde{\gamma}$ lies locally below $\widetilde{\alpha}^+$ and thus we need to show that $\widetilde{\gamma}_2$ also lies locally above $\widetilde{\beta}_2$ on the right half plane and locally below $\widetilde{\beta}_2$ on the left half plane. We calculate derivatives

$$\begin{aligned} D\tilde{q}_1\big|_{x=0} &= 0 \\ D^2\tilde{q}_1\big|_{x=0} &= 0 \\ D^3\tilde{q}_1\big|_{x=0}(v,v,v) &= -6v_1^3 + 18v_1v_2^2 \end{aligned}$$

and

$$\begin{aligned} D\tilde{q}_2\big|_{x=0} &= e_2^\top \\ D^2\tilde{q}_2\big|_{x=0} &= \begin{bmatrix} 2 & \\ & -2 \end{bmatrix} \\ D^3\tilde{q}_2\big|_{x=0}(v,v,v) &= 3v_1^2v_2 + v_2^3 \end{aligned}$$

and now by relation (5.28)

$$\begin{aligned} Dh\big|_{x=0} &= \varepsilon D_x\tilde{q}_2 \\ D^2h\big|_{x=0} &= \varepsilon D_x^2\tilde{q}_2 \\ D^3h\big|_{x=0}(v,v,v) &= -6v_1^3 + 18v_1v_2^2 + \varepsilon\big(3v_1^2v_2 + v_2^3\big)\,. \end{aligned} \tag{5.29}$$

where we wrote $h(x) = \widetilde{P}_1(\varepsilon,x)$ for brevity. We have $0 = h\circ\widetilde{\gamma}$ and thus

$$0 = \partial_t(h\circ\widetilde{\gamma})\big|_{t=0} = \Big(Dh\big|_{\widetilde{\gamma}}\widetilde{\gamma}'\Big)_{t=0} = \varepsilon\widetilde{\gamma}_2'(0) \tag{5.30}$$

so $\widetilde{\gamma}_2(0) = 0$ and in particular $\widetilde{\gamma}'(0) = e_1$. Taking another derivative we get

$$\begin{aligned}
0 &= \partial_t^2 (h \circ \widetilde{\gamma})\big|_{t=0} \\
&= \Big( D^2 h\big|_{\widetilde{\gamma}} (\widetilde{\gamma}', \widetilde{\gamma}') + Dh\big|_{\widetilde{\gamma}} \widetilde{\gamma}'' \Big)_{t=0} \\
&= \big( D^2 h\big|_{x=0} (e_1, e_1) + Dh\big|_{x=0} \widetilde{\gamma}'' \big)_{t=0} \\
&= \varepsilon \big( 2 + \widetilde{\gamma}_2'' \big)(0)
\end{aligned} \tag{5.31}$$

so $\widetilde{\gamma}''(0) = -2e_2$. The third derivative then yields

$$\begin{aligned}
0 &= \partial_t^3 (h \circ \widetilde{\gamma})\big|_{t=0} \\
&= \Big( D^3 h\big|_{\widetilde{\gamma}} (\widetilde{\gamma}', \widetilde{\gamma}', \widetilde{\gamma}') + 3D^2 h\big|_{\widetilde{\gamma}} (\widetilde{\gamma}', \widetilde{\gamma}'') + Dh\big|_{\widetilde{\gamma}} \widetilde{\gamma}''' \Big)_{t=0} \\
&= \big( D^3 h\big|_{x=0} (e_1, e_1, e_1) - 6D^2 h\big|_{x=0} (e_1, e_2) + Dh\big|_{x=0} \widetilde{\gamma}''' \big)_{t=0} \\
&= \big( -6 + \varepsilon \big( \widetilde{\gamma}_2''' \big) \big)(0)
\end{aligned}$$

so $\widetilde{\gamma}_2'''(0) = 6/\varepsilon$. Analogously we observe that it follows from $0 = \tilde{q}_2 \circ \widetilde{\beta}$ that

$$0 = \varepsilon \partial_t \Big( \tilde{q}_2 \circ \widetilde{\beta} \Big)\big|_{x=0} = \varepsilon \Big( D\tilde{q}_2\big|_{\widetilde{\gamma}} \widetilde{\beta}' \Big)_{t=0} \tag{5.32}$$

and

$$0 = \varepsilon \partial_t^2 \Big( \tilde{q}_2 \circ \widetilde{\beta} \Big)\big|_{x=0} = \varepsilon \Big( D^2 \tilde{q}_2\big|_{\widetilde{\gamma}} \Big( \widetilde{\beta}', \widetilde{\beta}' \Big) + D\tilde{q}_2\big|_{\widetilde{\gamma}} \widetilde{\beta}'' \Big)_{t=0}. \tag{5.33}$$

By the relations (5.29), equation (5.32) is identical to equation (5.30) and equation (5.33) is identical to equation (5.31) with $\widetilde{\gamma}$ replaced by $\widetilde{\beta}$. Now equations (5.30) and (5.31) determined $\widetilde{\gamma}'(0)$ and $\widetilde{\gamma}''(0)$ uniquely and thus we have that $\widetilde{\gamma}(0) = \widetilde{\beta}(0)$, $\widetilde{\gamma}'(0) = \widetilde{\beta}'(0)$ and $\widetilde{\gamma}''(0) = \widetilde{\beta}''(0)$. For the third derivative we observe that

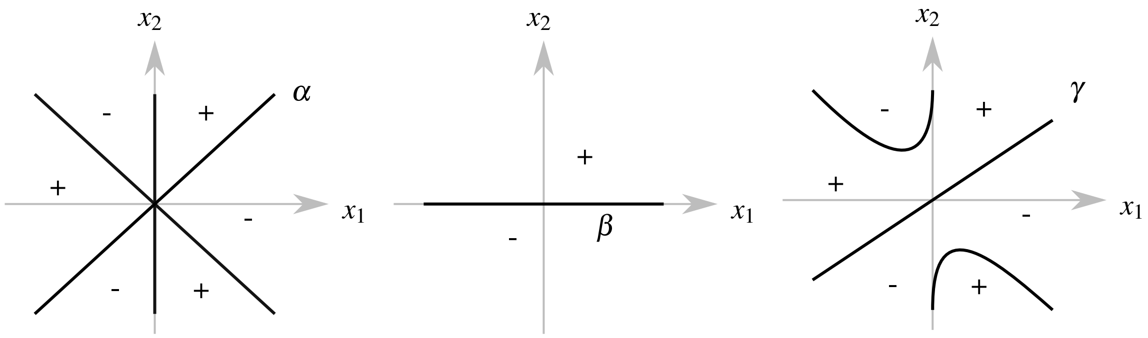
$$\begin{aligned}
0 &= \partial_t^3 \Big( \tilde{q}_2 \circ \widetilde{\beta} \Big)\big|_{t=0} \\
&= \Big( D^3 \tilde{q}_2\big|_{\widetilde{\gamma}} \Big( \widetilde{\beta}', \widetilde{\beta}', \widetilde{\beta}' \Big) + 3D^2 \tilde{q}_2\big|_{\widetilde{\gamma}} \Big( \widetilde{\beta}', \widetilde{\beta}'' \Big) + D\tilde{q}_2\big|_{\widetilde{\gamma}} \widetilde{\beta}''' \Big)_{t=0} \\
&= \Big( D^3 \tilde{q}_2\big|_{x=0} (e_1, e_1, e_1) - 6D^2 \tilde{q}_2\big|_{x=0} (e_1, e_2) + D\tilde{q}_2\big|_{x=0} \widetilde{\beta}''' \Big)_{t=0} \\
&= \widetilde{\beta}_2'''(0)
\end{aligned}$$

so $\widetilde{\beta}'''(0) = 0$. Thus we obtain

$$\partial_t^k \Big( \widetilde{\gamma}_2 - \widetilde{\beta}_2 \Big)\big|_{t=0} = 0$$

for $k \leq 2$ and

$$\partial_t^3 \Big( \widetilde{\gamma}_2 - \widetilde{\beta}_2 \Big)\big|_{t=0} = 6/\varepsilon > 0\,.$$




(a) Sketch of the set $V(\tilde{q}_1)$. (b) Sketch of the set $V(\tilde{q}_2)$. (c) Sketch of the set $\widetilde{\mathscr{V}}_\varepsilon$.

Figure 5.6: Sketches of varieties.

Hence we indeed have that $\widetilde{\gamma}_2 > \widetilde{\beta}_2$ in a sufficiently small neighbourhood of the origin for $t > 0$ and also that $\widetilde{\gamma}_2 < \widetilde{\beta}_2$ in a sufficiently small neighbourhood for $t < 0$.

Now that we have established that $\widetilde{\gamma}$ lies locally in $C$ it remains to be shown that $\widetilde{\gamma}$ indeed reaches the boundary $\partial W$. For this let $\widetilde{\gamma}\colon [0,b) \to C$ be a maximally extended parametrisation of $\widetilde{\gamma}$. Since $C$ is compact there exists a sequence $t_k \to b$ such that $x_k = \widetilde{\gamma}(t_k) \to x$ is convergent. Then we have by continuity that also $h(x) = 0$ so also $x \in \mathscr{V}_\varepsilon$. As $\mathscr{V}_\varepsilon$ is smooth we have that $Dh(x)$ is of full rank. Thus by the implicit function theorem we must in fact have that $b < \infty$ and the parametrisation of $\widetilde{\gamma}$ can be extended beyond the point $b$. But this means that $x \in \partial C$. Again by smoothness $x$ cannot lie in the origin. Now note that $\widetilde{P}_1(\varepsilon,\cdot) = \tilde{q}_1 < 0$ on the arc $\widetilde{\beta}$ and that $\widetilde{P}_1(\varepsilon,\cdot) = \tilde{q}_2 > 0$ on the arc $\widetilde{\alpha}^+$. So $\widetilde{\gamma}$ cannot intersect $\partial C$ on $\widetilde{\beta}$ or on $\widetilde{\alpha}$. Thus we must have that $x \in \partial W$ and hence $\widetilde{\gamma}$ splits $W$ in the right half plane into two parts. On the left half plane the argumentation is analogous. Then $\widetilde{\gamma}$ divides the plane into two parts which yields the claim. For clarity the idea of the proof is also depicted in figure 5.7. Also note that as a consequence $\widetilde{\mathscr{V}}_\varepsilon$ looks similar to figure 5.6c. □

## An example with positive distance between the entrant and emergent boundaries

In the previous example we showed that $\Sigma^+$ is separated from $\Sigma^-$ by the curve $\Sigma^0 = V(p_1, P_2(r,\cdot))$. In the following we will show that it is possible to thicken this curve $\Sigma^0$ without loosing the fundamental properties of the solution. More concretely, we construct a harmonic function $f_\rho$ for which the entrant and emergent boundaries are simply connected, have positive distance from one another and such that $f_\rho$ has an interior stagnation point. For this we use the notation from the previous example. We need a preliminary result:

**Proposition 5.11** (Existence of a cutoff function)**.** *Let* $\Sigma$ *and* $\Sigma^0$ *be as in example 5.3. For every* $\rho > 0$ *there exists a cutoff function* $\theta_\rho : \Sigma \to [0,1]$ *such that*

1. $\theta_\rho = 1$ *outside of the tubular neighbourhood* $\mathrm{Tub}_\rho\left(\Sigma^0\right)$,

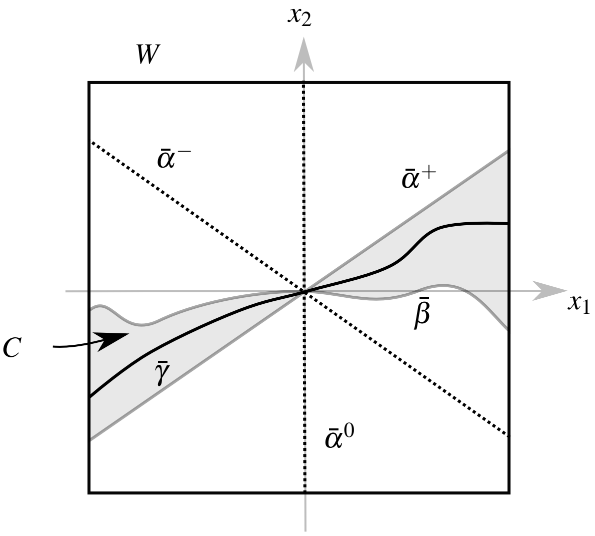



Figure 5.7: An overview of the situation in proposition 5.10.

2. *$\theta_\rho$ has support contained in $\Sigma \setminus \Sigma^0$ and*
3. *$\langle \theta_\rho p_1 \rangle_\Sigma = 0$, where $\langle h \rangle_X := \fint_X h$ denotes the mean of a function $h\colon X \to Y$.*

*Proof.* Let $\theta_\lambda^+ : \Sigma \to [0,1]$ be a family of cutoff functions smoothly depending on $\lambda$ such that $\theta_\lambda^+ = 1$ outside of $\mathrm{Tub}_\lambda\left(\Sigma^0\right) \cup \Sigma^-$ and $\theta_\lambda^+$ has support contained in $\Sigma^+$. Analogously we let $\theta_\lambda^- : \Sigma \to [0,1]$ be a family of cutoff functions smoothly depending on $\lambda$ such that $\theta_\lambda^- = 1$ outside of $\mathrm{Tub}_\lambda\left(\Sigma^0\right) \cup \Sigma^+$ and $\theta_\lambda^-$ has support contained in $\Sigma^-$. We can also assume that both $\theta_\lambda^+$ and $\theta_\lambda^-$ are monotonously increasing in $\lambda$. Set

$$\tilde{\theta}_\rho = \theta_\rho^+ + \theta_{\lambda(\rho)}^- \tag{5.34}$$

where $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a suitably chosen map. By construction the support of $\tilde{\theta}_\rho$ lies outside of the set $\Sigma \setminus \Sigma^0$ so $\tilde{\theta}_\rho$ fulfils condition 2. For condition 3 we calculate

$$\langle \tilde{\theta}_\rho p_1 \rangle_\Sigma = \left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} + \left\langle \theta_{\lambda(\rho)}^- p_1 \right\rangle_{\Sigma^-} \tag{5.35}$$

Now we have that $p_1 < 0$ on $\Sigma^-$ and that $p_1 > 0$ on $\Sigma^+$. Thus $\left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} > 0$ is monotonically increasing with $\rho$ and

$$\left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} \longrightarrow \langle p_1 \rangle_{\Sigma^+} =: c^+$$

as $\rho \to 0$. Analogously $\langle \theta_\lambda^- p_1 \rangle_{\Sigma^-} < 0$ is monotonically and continuously decreasing with $\lambda$ and

$$\langle \theta_\lambda^- p_1 \rangle_{\Sigma^-} \longrightarrow \langle p_1 \rangle_{\Sigma^-} =: c^-$$

as $\lambda \to 0$. It follows from the divergence theorem that

$$0 = r\int_X \Delta f = \int_\Sigma r\nabla f \cdot n = \langle p_1 \rangle_\Sigma = c^+ + c^-$$

where $n\Sigma \to S^2$ is the outer unit normal. It then follows that for each $\rho > 0$ there exists a $\lambda(\rho) > 0$ such that

$$\left\langle \theta_\rho^+ p_1 \right\rangle_{\Sigma^+} = \left\langle \theta_{\lambda(\rho)}^- p_1 \right\rangle_{\Sigma^-}$$

so by equation (5.35) $\tilde{\theta}_\rho$ also fulfils condition 3. In fact we can also assume that $\lambda(\rho) \to 0$ as $\rho \to 0$. Now choose $\tilde{\lambda}(\rho)$ such that $\max\left(\tilde{\lambda}, \lambda(\tilde{\lambda})\right) < \rho$. This defines a function $\tilde{\lambda} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and we set $\theta_\rho := \tilde{\theta}_{\tilde{\lambda}(\rho)}$. Property 1 then follows from the fact that $\tilde{\theta}_\rho = 1$ outside of the neighbourhood $\mathrm{Tub}_{\max(\rho,\lambda(\rho))}\left(\Sigma^0\right)$ by equation 5.34. □

With the help of this cutoff function $\theta_\rho$ we can now formulate our main result in this section:

**Example 5.12** (Thickening $\Sigma^0$)**.** Let $f$ and $X$ be as in example 5.3 and the cutoff function $\theta_\rho$ be as in the previous proposition. Consider now the solution $f_\rho$ to the system

$$\begin{aligned} \Delta f_\rho &= 0 && \text{on } X, \\ \nabla f_\rho \cdot n &= \theta_\rho(\nabla f \cdot n) && \text{on } \Sigma, \\ \left\langle f_\rho \right\rangle_X &= \langle f \rangle_X \end{aligned}$$

where $n \colon \Sigma \to S^2$ is the outer unit normal. By definition of $f_\rho$ the entrant and emergent boundaries are simply connected and have positive distance from one another. We will show in proposition 5.13 that there exists $\rho > 0$ such that $f_\rho$ has an interior critical point.

**Proposition 5.13** (Structural stability)**.** *Let $x$ be a non-degenerate interior critical point of $f$. Then there exist $\mathrm{P} > 0$ and $\varepsilon > 0$ such that for all $\rho \leq \mathrm{P}$ the function $f_\rho$ from example 5.12 has an interior critical point in $\overline{B}_\varepsilon(x)$.*

*Proof.* To simplify notation we only show this for the interior critical point of $f$ at the origin $x = 0$. Define $g_\rho := f_\rho - f$. By linearity $g_\rho$ is a solution the system

$$\begin{aligned} \Delta g_\rho &= 0 && \text{on } X, \\ \nabla g_\rho \cdot n &= \left(\theta_\rho - 1\right)(\nabla f \cdot n) && \text{on } \Sigma, \\ \left\langle g_\rho \right\rangle_X &= 0 \end{aligned}$$

To simplify notation we write $x \lesssim y$ instead of $x \leq cy$ for some constant $c \geq 1$ which is independent of $\rho$. We have that

$$
\begin{aligned}
\left\|Dg_\rho\right\|^2_{L^2\left(X;\mathbb{R}^d\right)} &\overset{\text{divergence Theorem}}{=} \int_\Sigma g_\rho \left(\nabla g_\rho \cdot n\right) \\
&\overset{\text{Cauchy-Schwarz}}{\leq} \left\|g_\rho\right\|_{L^2(\Sigma)} \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)} \\
&\overset{\text{trace theorem}}{\lesssim} \left\|g_\rho\right\|_{W^{1,2}(X)} \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)} \\
&\overset{\text{Poincaré-Wirtinger}}{\lesssim} \left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^d\right)} \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)}
\end{aligned}
$$

and thus together with dominated convergence

$$
\left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^d\right)} \lesssim \left\|\nabla g_\rho \cdot n\right\|_{L^2(\Sigma)} = \left\|\left(\theta_\rho - 1\right)\left(\nabla f \cdot n\right)\right\|_{L^2(\Sigma)} \longrightarrow 0
$$

as $\rho \to 0$. As a consequence of this we obtain on the one hand that for a compactly contained open set $\{0\} \Subset U \Subset X$ with smooth boundary we have

$$
\begin{aligned}
\left\|Dg_\rho\right\|_{L^\infty\left(U;\mathbb{R}^{d\times d}\right)} &\overset{\text{mean value property}}{\lesssim} \left\|Dg_\rho\right\|_{L^1\left(X;\mathbb{R}^{d\times d}\right)} \\
&\overset{\text{Cauchy-Schwarz}}{\lesssim} \left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^{d\times d}\right)} \longrightarrow 0
\end{aligned}
\tag{5.36}
$$

as $\rho \to 0$. It also follows for a compactly contained open set $U \Subset V \Subset X$ with smooth boundary that

$$
\begin{aligned}
\left\|D^2 g_\rho\right\|_{L^\infty\left(U;\mathbb{R}^{d\times d}\right)} &\overset{\text{mean value property}}{\lesssim} \left\|D^2 g_\rho\right\|_{L^1\left(V;\mathbb{R}^{d\times d}\right)} \\
&\overset{\text{Cauchy-Schwarz}}{\lesssim} \left\|D^2 g_\rho\right\|_{L^2\left(V;\mathbb{R}^{d\times d}\right)} \\
&\overset{\text{interior regularity}}{\lesssim} \left\|Dg_\rho\right\|_{L^2\left(X;\mathbb{R}^d\right)} \longrightarrow 0
\end{aligned}
\tag{5.37}
$$

as $\rho \to 0$.

We would like to apply a fixed point theorem. For this we define a mapping

$$
F_\rho(y) = y - D^2 f\big|_0^{-1} Df_\rho(y)
$$

where we used the fact that $D^2 f\big|_0$ is bijective by non-degeneracy. We denote by $\|\cdot\|_{\mathrm{Op}}$ the operator norm induced by $L^\infty\big(U;\mathbb{R}^d\big)$, that is

$$\|h\|_{\mathrm{Op}} = \sup_{y\in L^\infty\left(U;\mathbb{R}^d\right)\setminus\{0\}} \frac{\|hy\|_{L^\infty\left(U;\mathbb{R}^d\right)}}{\|y\|_{L^\infty\left(U;\mathbb{R}^d\right)}}$$

for some function $h\in L^\infty\big(U;\mathbb{R}^{d\times d}\big)$. Set $U=B_\varepsilon\subseteq\mathbb{R}^3$ to be an open ball around the origin with radius $\varepsilon>0$. We calculate the derivative

$$DF_\rho = \mathrm{Id} - D^2 f\big|_0^{-1} D^2 f_\rho$$

and take the operator norm

$$\frac{1}{c}\left\|DF_\rho\right\|_{\mathrm{Op}} \le \left\|D^2 f|_0 - D^2 f_\rho\right\|_{\mathrm{Op}} \le \left\|D^2 f|_0 - D^2 f\right\|_{\mathrm{Op}} + \left\|D^2 g_\rho\right\|_{\mathrm{Op}} \tag{5.38}$$

for some constant $c\ge 1$ independent of $\rho$. Now we have by continuity that

$$\left\|D^2 f|_0 - D^2 f\right\|_{\mathrm{Op}} \lesssim \left\|D^2 f|_0 - D^2 f\right\|_{L^\infty\left(U;\mathbb{R}^{d\times d}\right)} \longrightarrow 0$$

as $\varepsilon\to 0$. Pick $\varepsilon>0$ such that $\left\|D^2 f|_0 - D^2 f\right\|_{\mathrm{Op}} \le 1/(4c)$. Now we have for $\rho\to 0$ that

$$\left\|D^2 g_\rho\right\|_{\mathrm{Op}} \lesssim \left\|D^2 g_\rho\right\|_{L^\infty\left(U;\mathbb{R}^{d\times d}\right)} \longrightarrow 0$$

by equation (5.37). Thus there exists a $\mathrm{P}>0$ such that $\left\|D^2 g_\rho\right\|_{\mathrm{Op}} \le 1/(4c)$ for all $\rho\le\mathrm{P}$. By equation (5.38) it then follows that $\left\|DF_\rho\right\|_{\mathrm{Op}} \le 1/2$ for all $\rho\le\mathrm{P}$. Since we have by equation (5.36) that

$$\left|F_\rho(0)\right| = \left|D^2 f\big|_0^{-1} Df_\rho(0)\right| \lesssim |Df_\rho(0)| = |Dg_\rho(0)| \le \left\|Dg_\rho\right\|_{L^\infty\left(U;\mathbb{R}^d\right)} \longrightarrow 0$$

as $\rho\to 0$ we can also additionally choose $\mathrm{P}>0$ such that for all $\rho\le\mathrm{P}$ we have that $\left|F_\rho(0)\right|\le\varepsilon/2$. We then have for all $\rho\le\mathrm{P}$ that $F_\rho\big(\overline{B}_\varepsilon\big)\subseteq\overline{B}_\varepsilon$ and $F_\rho$ is a contraction. It then follows from fixed point theorems that for every $\rho\le\mathrm{P}$ the function $F_\rho$ has a fixed point, say $y_\rho\in\overline{B}_\varepsilon$. This then fulfils by the construction of $F_\rho$ the equation

$$y_\rho = F_\rho\big(y_\rho\big) = y_\rho - D^2 f\big|_0^{-1} Df_\rho\big(y_\rho\big)$$

which implies that $Df_\rho\big(y_\rho\big)=0$ and $y_\rho$ is an interior critical point. □

# 6 No in- or outflow in $\mathbb{R}^2$

In the second part of this thesis we will discuss harmonic vector fields without inflow or outflow through the boundary. More generally we will first consider harmonic vector fields without boundary critical points. Here we will show in proposition 6.1 that in $d = 2$ dimensions there is a strong relation between the number of critical points and the domain topology. More concretely we will show that the number of critical points $M$ equals under certain conditions the negative Euler characteristic $-\chi(X)$. This relation between the domain topology in the plane and the number of critical points also shows up for instance for Green functions for instance in [26, p.133]. We will then discuss briefly how this relates to the Poincaré-Hopf index theorem. After that we will give examples which illustrate these results.

## A condition on the number of stagnation points

Our first result relates the number of critical points with the domain topology. We shall give two proofs of this result, one involving Morse theory and the other involving the argument principle. The reason for this is that the result follows quickly from the argument principle. However, the techniques of the proof involving Morse theory generalise more readily to $d = 3$ dimensions.

**Proposition 6.1.** *Let $X \subset \mathbb{R}^2$ be a compact manifold with corners and Betti numbers $b_0 = 1$, and $b_1$ and let $u\colon X \to \mathbb{R}^2$ be a strongly Morse harmonic vector field without boundary stagnation points. Then we have the relation $M = -\chi(X)$ where $M$ denotes the number of stagnation points and $\chi(X)$ is the Euler characteristic of $X$.*

*Proof.* We slit the domain $X$ such that it is homeomorphic to the disk as is depicted in figure 6.1. Denote the slit by $\Gamma$. Since the number of interior stagnation points is finite by proposition 2.13, we can choose $\Gamma$ in such a way that it does not contain any interior stagnation points. We write denote the boundary of $\Gamma$ by $\gamma = \partial\Gamma = \Gamma \cap \Sigma$ and define points $\{x_1, \ldots, x_{2b_1}\} = \gamma$. Note that there are $2b_1$ many such points. Without loss of generality we can assume that $\Gamma$ and $u$ form an acute angle at each point of $\gamma$. The situation is depicted in figure 6.1. For the following argumentation we require that $u$ is strongly Morse on the new domain $\tilde{X}$ so assume for a moment that this is the case. Since each $x_j$ is either an essential critical point of $u$ or of $-u$ on the slit domain $\tilde{X}$ we have that

$$\mathrm{Ind}_{\gamma,0}(u) + \mathrm{Ind}_{\gamma,0}(-u) = 2b_1\,. \tag{6.1}$$

Since there are no stagnation points on $\Sigma$ we have the relations

$$\mu_k = \mathrm{Ind}_{\Gamma,k}(u) + \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \nu_k = \mathrm{Ind}_{\Gamma,k}(-u) + \mathrm{Ind}_{\gamma,k}(-u) \tag{6.2}$$

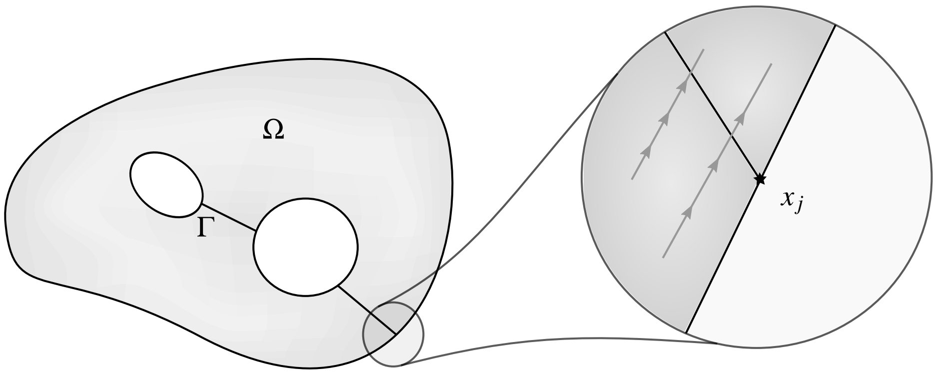



Figure 6.1: How we slit the domain.

for all $k \in \{0,1\}$. All entrant stagnation points of $u$ on $\Gamma$ are also emergent stagnation points of $-u$ on $\Gamma$ (and vice versa) and hence we have the relations

$$\mathrm{Ind}_{\Gamma,0}(u) = \mathrm{Ind}_{\Gamma,1}(-u) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(u) = \mathrm{Ind}_{\Gamma,0}(-u)\,. \tag{6.3}$$

Equations (6.2) and (6.3) yield

$$\mu_0 - \mathrm{Ind}_{\gamma,0}(u) = \nu_1 \qquad \text{and} \qquad \mu_1 = \nu_0 - \mathrm{Ind}_{\gamma,0}(-u)\,. \tag{6.4}$$

Since $\tilde{X}$ is now simply connected $u$ is by proposition 2.1 the gradient of a harmonic function $f$ on this new domain. For this $f$ we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \tag{6.5}$$

and for $-f$ the Morse inequalities

$$M + \nu_1 - \nu_0 = -\chi(\tilde{X}) = -1\,. \tag{6.6}$$

Adding equations (6.5) and (6.6) and using the relation (6.4) and then (6.1) we obtain

$$-2 = 2M - \mathrm{Ind}_{\gamma,0}(u) - \mathrm{Ind}_{\gamma,0}(u) = 2M - 2b_1$$

from which the claim follows.

The claim remains to be shown in the case that $u$ is not strongly Morse on $\tilde{X}$. In this case let $u^\varepsilon$ for $\varepsilon \in E$ be a strongly Morse function as in corollary 2.18. Since the $x_1,\dots,x_{2b_1} \in \gamma$ are non-degenerate stagnation points of $u$ due to the slanted angle at which $\Gamma$ approaches $\gamma$ we obtain that

$$\mathrm{Ind}_{\gamma,k}(u^\varepsilon) = \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-u^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-u) \tag{6.7}$$

By the same corollary $u^\varepsilon$ has no stagnation points on $\Sigma$. The claim then follows by the calculations above where we replace $u$ with $u^\varepsilon$ and then note that $M^\varepsilon = M$. □

We now give an alternative and simpler proof of proposition 6.1 using the argument principle.

*Alternative proof.* As before we slit the domain such that it is homeomorphic to a disk. By proposition 2.1 $u$ is the gradient of a harmonic function $f$ on this new domain $\tilde{X}$. Let $h \in \mathrm{Hol}(\tilde{X})$ be the holomorphic function given by $h = \nabla f$. Let $\gamma$ traverse the boundary of the slit domain such that the domain lies to the left of $\gamma$. We now determine the change of argument $\arg h$ along $\gamma$. For this consider first the the slits. Since $\nabla f$ is continuously differentiable along the slit and $\gamma$ traverses the slit once in one direction and once in the other, the contribution to the change of $\arg h$ from the slits vanishes. On the other hand as $\gamma$ traverses the boundary $\Sigma$ the contribution to the change in argument of $\arg h$ is $2\pi$ for every hole in the domain since $h \cdot \gamma' = u \cdot \gamma'$ does not change sign as $\gamma$ traverses a hole in clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is $-2\pi$ for the outer boundary component which is traversed counterclockwise. Since we have $b_1$ holes in the domain the total change of $\arg h$ as $\gamma$ traverses $\Sigma$ is $2\pi(b_1 - 1)$. Since $h$ has no poles it follows from the argument principle (see for example [9, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M$$

from which the claim follows. □

We say that $u$ has no *inflow* on a boundary subset $S \subseteq \Sigma$ if $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* if $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can give the following corollary.

**Corollary 6.2.** *Let $X$ be a compact manifold with corners and $u\colon X \to \mathbb{R}^2$ a strongly Morse harmonic vector field without inflow or outflow on $\Sigma$. Then we have the relation $M = -\chi(X)$ where $M$ is the number of stagnation points and $\chi(X)$ is the Euler characteristic of $X$.*

This corollary could also have been proved using the Poincaré-Hopf index theorem. For this define:

**Definition 6.3** ([5, Definition 1.1.1])**.** The *Poincaré-Hopf index* $\mathrm{Ind}_{\mathrm{PH},x}(f)$ of an isolated interior stagnation point $x$ of $u$ is the degree of the Gauss map $u/|u|\colon S_\varepsilon^{d-1}(x) \to S^{d-1}$ for sufficiently small $\varepsilon > 0$. The *total index* $\mathrm{Ind}_{\mathrm{PH}}(u)$ is the sum of indexes $\mathrm{Ind}_{\mathrm{PH},x}(f)$ for every interior stagnation point $x$ of $u$.

Note that in $d = 2$ dimensions a point $x$ with Morse index $k$ has Poincaré-Hopf index $(-1)^k$. Thus we have for a harmonic vector field in $d = 2$ dimensions that $M = -\mathrm{Ind}_{\mathrm{PH}}(u)$ and thus corollary 6.2 also follows from the Poincaré-Hopf index theorem:

**Theorem 6.4** (Poincaré-Hopf index theorem, [22, §6])**.** *Let $u\colon X \to \mathbb{R}^d$ be a vector field on a manifold with corners without inflow. Then we have that the total index $\mathrm{Ind}_{PH}(u)$ equals the Euler characteristic $\chi(X)$.*

For a proof we refer the reader for instance to [22, §6].

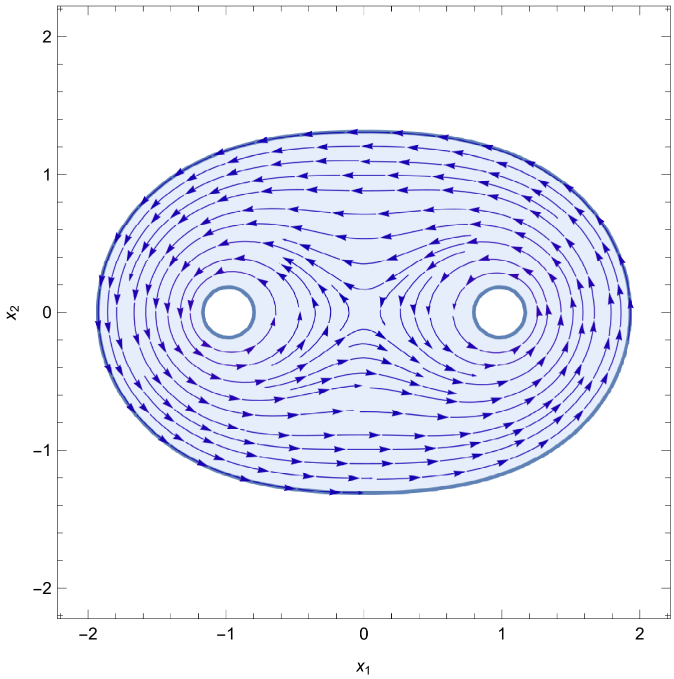



Figure 6.2: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here $\psi$ is given by equation (6.8).

## Examples of harmonic vector fields

We would like to illustrate the previous results with examples. We first give an example of a harmonic vector field in $d = 2$ dimensions without inflow or outflow and with one stagnation point.

**Example 6.5** (No in- or outflow)**.** Consider the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x-e_1) + \Phi_2(x+e_1) \end{aligned} \tag{6.8}$$

where $\Phi_2$ is as in example 4.5. A plot of the streamlines in figure 6.2 indicates that $u = \nabla^{\perp}\psi$ in the domain $X = \psi^{-1}([-1,1])$ has the desired properties. Indeed, since $\psi$ is constant on each component of $\Sigma$ the function $u$ has neither inflow nor outflow. It follows from $\psi(-x) = \psi(x)$ that $u(-x) = -u(x)$ and thus the origin $x = 0$ is a stagnation point. By proposition 6.1 it is in fact the sole stagnation point of $u$ on $X$.

In a second example given by [29] we start with the domain rather than the function.

**Example 6.6** (No in- or outflow)**.** Set $X = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We let the stream function $\psi$ be determined by the system

$$\begin{aligned} \Delta\psi &= 0 \quad \text{on } \operatorname{int}(X), \\ \psi &= 0 \quad \text{on the outer ring } 4S^1, \\ \psi &= 1 \quad \text{on the inner rings } S^1(-2e_1) \cup S^1(2e_1), \end{aligned} \tag{6.9}$$

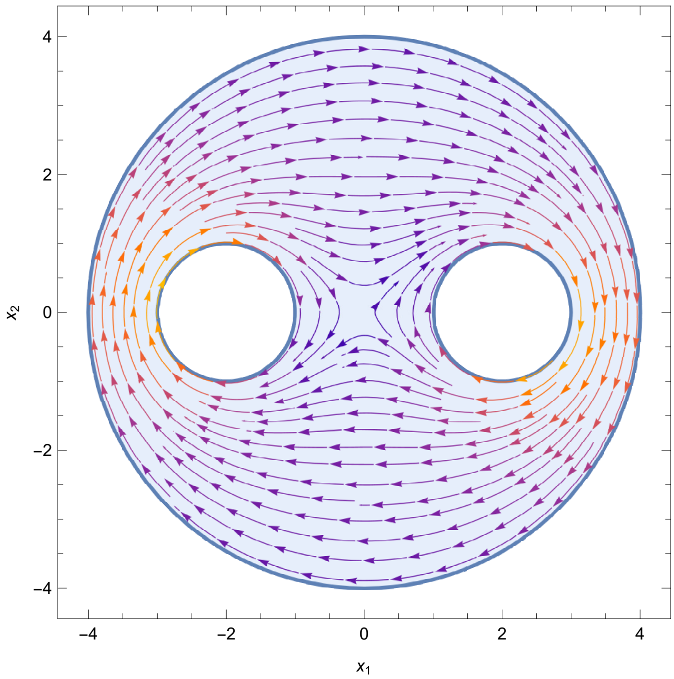



Figure 6.3: A plot of $u = \nabla^{\perp}\psi$ where $\psi$ is the numerical solution to (6.9).

and set $u = \nabla^{\perp}\psi$. A numerical solution to this system is plotted in figure 6.3. Again, it follows from symmetry that the origin is a stagnation point and from proposition 6.1 that it is in fact the sole stagnation point of $u$.

Our third example highlights the importance of assuming that $u$ be Morse in corollary 6.2.

**Example 6.7** (Stagnation points on the boundary)**.** In this example given by [29] we again start by fixing the domain. Let $X = \overline{B}_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$ be the domain as before. We let the stream function $\psi$ be determined by the system

$$\begin{aligned} \Delta\psi &= 0 && \text{on } \operatorname{int}(X), \\ \psi &= 0 && \text{on the outer ring } 4S^1, \\ \psi &= -1 && \text{on the left inner ring } S^1(-2e_1), \\ \psi &= 1 && \text{on the right inner ring } S^1(2e_1), \end{aligned} \tag{6.10}$$

and then set $u = \nabla^{\perp}\psi$. The numerical solution to this system is plotted in figure 6.4. Here we obtain from the symmetry $\psi(-x_1, x_2) = \psi(x)$ that $\psi = 0$ on the $x_2$-axis. Since also $\psi = 0$ on $4S^1$ we have two stagnation points at $\pm 4e_2$. This function again has no in- or outflow through the boundary. The domain contains two holes so $\chi(X) = -1$. Now $u$ has no interior stagnation point, seemingly contradicting corollary 6.2. But since the two stagnation points at $\pm 4e_2$ lie on the boundary $u$ is in fact not Morse and thus we cannot apply corollary 6.2. This shows the importance of the assumption that $u$ is Morse. Proposition 6.8 essentially states that in this case

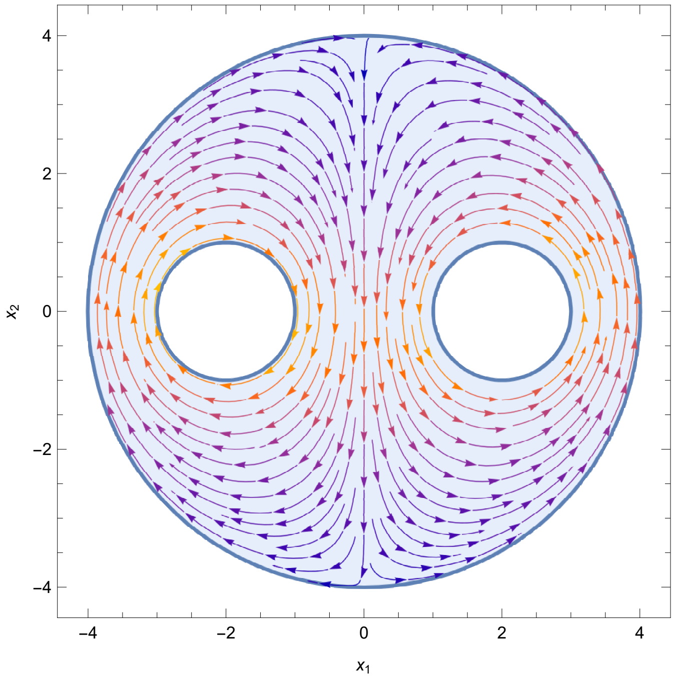



Figure 6.4: A plot of $u = \nabla^\perp \psi$ where $\psi$ is the numerical solution to (6.10).

the stagnation points of $u$ on the boundary count half as much as stagnation points in the interior. This explains why there are two critical points in this example.

**Proposition 6.8** (Critical points on the boundary, [2, Theorem 1.1])**.** *Let $X \subseteq \mathbb{R}^2$ be bounded with piecewise $C^{1,\alpha}$ boundary. Let further $f \colon X \to \mathbb{R}$ be harmonic, not constant on each connectivity component and constant on each boundary component. Then there exist finitely many critical points $x_1, \dots, x_k \in X$ of $u$ with multiplicities $m_1, \dots, m_k$ and we have that*

$$\sum_{x_j \in \mathrm{int}(X)} m_j + \frac{1}{2} \sum_{x_j \in \Sigma} m_j = -\chi(X) \,.$$

*Proof.* The proof uses complex analysis techniques and can be found in [2]. □

# 7 No in- or outflow in $\mathbb{R}^3$

In $\mathbb{R}^3$ our results to question 1.3 remain unsatisfactory. Nonetheless we will state them in this final chapter. The question from [19, p.198] remains if it is possible to have a harmonic vector field in $\mathbb{R}^3$ with interior stagnation point and no in- or outflow through the boundary. This can be generalised to the question if it is possible to have a harmonic vector field $u\colon \mathbb{R}^3 \supseteq X \to \mathbb{R}^3$ with interior critical points and without critical points on the boundary. We remark that the boundary $\Sigma$ cannot have a component homeomorphic to the sphere $S^2$ as a direct consequence of the hairy ball theorem. We can also give a nontrivial condition on the interior critical points of such a $u$ using the techniques of previous proofs:

**Proposition 7.1.** *Let $X \subseteq \mathbb{R}^3$ be a compact manifold with corners and Betti numbers $b_0$, $b_1$ and $b_2$. Let $u\colon X \to \mathbb{R}$ be a strongly Morse harmonic vector field without boundary critical points. Then the following relation for the interior type numbers of $u$ holds:*

$$M_1 = M_2\,.$$

*Proof.* As in the two dimensional case we cut the domain $X$ with a surface $\Gamma$ such that the slit domain is homeomorphic to a ball with bubbles. Since the number of interior stagnation points of $u$ is finite by proposition 2.13, we can choose $\Gamma$ in such a way that it does not contain any stagnation points. We also denote the arcs at which $\Gamma$ meets $\Sigma$ by $\gamma_1,\dots,\gamma_{b_1} \subseteq \partial\Gamma$. Note that there are $b_1$ many such curves. We can assume that $\Gamma$ and the $\gamma_j$ are manifolds with corners and that $\Gamma$ approaches each $\gamma_j$ at a slanted angle. The cut now yields a new domain $\tilde{X}$ which is a covering space of $X$. On this covering space we denote the cover of the cut $\Gamma$ and the sets $\gamma_j$ by $\Gamma^i$ and $\gamma_j^i$ with $i \in (1,2)$. Since this new domain $\tilde{X}$ is homeomorphic to a ball with bubbles the vector field $u = \nabla f$ is the gradient of a harmonic function $f$ by proposition 2.1. For the following argumentation we require that $u$ is strongly Morse on $\tilde{X}$, so assume for a moment that this is the case. Now we have that each $\gamma_j$ is homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Since $f$ is non-degenerate the number of maxima and minima of $f$ on $\gamma_j^1 \cup \gamma_j^2$ must be equal and thus

$$\sum_i \Big(\mathrm{Ind}_{\gamma_j^i,0}(f) + \mathrm{Ind}_{\gamma_j^i,1}(-f)\Big) = \sum_i \Big(\mathrm{Ind}_{\gamma_j^i,1}(f) + \mathrm{Ind}_{\gamma_j^i,0}(-f)\Big)\,. \tag{7.1}$$

Since on $\Gamma$ all entrant stagnation points of $u$ are also emergent stagnation points of $-u$ (and vice versa) we have the relations

$$\begin{aligned}
\mathrm{Ind}_{\Gamma^1,0}(\pm u) &= \mathrm{Ind}_{\Gamma^2,2}(\mp u)\\
\mathrm{Ind}_{\Gamma^1,1}(\pm u) &= \mathrm{Ind}_{\Gamma^2,1}(\mp u)\\
\mathrm{Ind}_{\Gamma^1,2}(\pm u) &= \mathrm{Ind}_{\Gamma^2,0}(\mp u)\,.
\end{aligned} \tag{7.2}$$

As there are no boundary critical points on $\Sigma$ it follows for the boundary type numbers that

$$\begin{aligned}\mu_k &= \sum_i \left( \mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma^i_j,k} \right)(f) \\ \nu_k &= \sum_i \left( \mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma^i_j,k} \right)(-f)\,.\end{aligned} \tag{7.3}$$

Equations (7.3) and (7.2) yield

$$\begin{aligned}\mu_0 - \nu_2 &= \sum_{i,j} \mathrm{Ind}_{\gamma^i_j,0}(f) \\ \mu_1 - \nu_1 &= \sum_{i,j} \Big( \mathrm{Ind}_{\gamma^i_j,1}(f) - \mathrm{Ind}_{\gamma^i_j,1}(-f) \Big) \\ \mu_2 - \nu_0 &= -\sum_{i,j} \mathrm{Ind}_{\gamma^i_j,0}(-f)\end{aligned} \tag{7.4}$$

For $f$ we have the Morse inequalities

$$M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 = -\chi\big(\tilde{X}\big) \tag{7.5}$$

and for $-f$ the Morse inequalities

$$M_1 + \nu_2 - M_2 - \nu_1 + \nu_0 = -\chi\big(\tilde{X}\big)\,. \tag{7.6}$$

Subtracting equation (7.6) from (7.5) and using the relation (7.4) we obtain together with equation (7.1) that

$$\begin{aligned}0 &= 2(M_2 - M_1) + \sum_{i,j} \Big( \mathrm{Ind}_{\gamma^i_j,0}(f) - \mathrm{Ind}_{\gamma^i_j,1}(f) + \mathrm{Ind}_{\gamma^i_j,1}(-f) - \mathrm{Ind}_{\gamma^i_j,0}(-f) \Big) \\ &= 2(M_2 - M_1)\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that $f$ is not strongly Morse on $X^+$ and $X^-$. In this case let $f^\varepsilon$ for $\varepsilon \in E$ be a strongly Morse function as in corollary 2.18. Since $x_1, x_2$ are non-degenerate critical points of $f$ due to the slanted angle at which $\Gamma$ approaches each $\gamma_j$ we obtain that

$$\mathrm{Ind}_{\gamma_j,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma_j,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(-f) \tag{7.7}$$

By the same corollary we can assume that $f^\varepsilon$ has no critical points on $\Sigma$. The claim then follows by the calculations above where we replace $f$ with $f^\varepsilon$ and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. □

As a consequence we obtain the following:

**Corollary 7.2.** *Let $X$ be a compact manifold with corners and $u\colon X \to \mathbb{R}^d$ a strongly Morse harmonic vector field without inflow or outflow on the boundary $\Sigma$. The we have that*

$$M_1 = M_2\,.$$

# Symbols

| | |
|---|---|
| $d$ | Dimensions $d = 2$ or $d = 3$ |
| $X$ | Compact domain in $\mathbb{R}^d$, often assumed to be a manifold with corners. |
| $f : X \to \mathbb{R}$ | A harmonic function. |
| $u : X \to \mathbb{R}^d$ or $T^*X$ | A harmonic vector field. |
| $X_j$ | A stratification of $X$ as given in definition 2.3. Often but not always assumed to be given by equation (2.1) |
| $\prec$ | We write $X_j \prec X_k$ if $X_j \subseteq \overline{X}_k$. See definition 2.3 for details. |
| $\precsim$ | We write $X_j \precsim X_k$ if $X_j \prec X_k$ and the strata differ in dimension by one. See definition 2.3 for details. |
| $\Sigma$ | Boundary $\Sigma = \partial X$ |
| $\Sigma^-$, $\Sigma^{\leq 0}$ | (Strictly) entrant boundary. See definition 2.6. |
| $\Sigma^+$, $\Sigma^{\geq 0}$ | (Strictly) emergent boundary. See definition 2.6. |
| $\Sigma^0$ | Tangential boundary. See definition 2.6. |
| $\Sigma^{\mathrm{irr}}$ | Irregular boundary. See definition 2.8. |
| $B_r(x)$, $B_r$ | Ball of radius $r$ around the point $x$ / the origin. |
| $S^{d-1}(x)$, $S^{d-1}$ | $(d-1)$-dimensional sphere around $x$ / the origin. |
| $e_j$ | $j$-th unit vector in $\mathbb{R}^d$. |
| $u_j$ | Projection of $u$ to the cotangent bundle $T^*X_j$. See equation (2.8). |
| $\pi_j$ | Orthogonal projection onto the cotangent bundle $T^*X_j$. See equation (2.7). |
| $\mathrm{Cr}_j$ | Set of essential stagnation points. See definition 2.8. |
| $\mathrm{Ind}_{j,k}$ | $k$-th type number on the stratum $X_j$. See equation (2.9). |
| $\mathrm{Ind}_k$ | $k$-th type number. See equation (2.14). |
| $M_k$ | $k$-th interior type numbers. See equation (2.11). |
| $M$ | Total number of stagnation points. See equation (2.12). |
| $\mu_k$ | $k$-th boundary type numbers of $f$. See equation (2.13). |
| $\nu_k$ | $k$-th boundary type numbers of $-f$. See definition 2.8. |
| $u^\varepsilon$ | modification to $u$ as in equation (2.16) |
| $A$ | submanifold, can be thought of as the zero section of $T^*X$ |
| $b_k(X)$ | $k$-th Betti number of $X$ as defined in equation (3.1) |
| $H_k(X)$ | $k$-th homology group of the space $X$. |
| $H_k(X,A)$ | $k$-th relative homology group for spaces $A \subseteq X$. |
| $\chi(X)$ | Euler characteristic of $X$. See equation (3.2). |

| | |
|---|---|
| $\chi(X,A)$ | Euler characteristic for the relative homology group $H_k(X,A)$. See equation (3.6). |
| $\nabla^{\perp}$ | Orthogonal gradient. Given by equation (4.11). |
| $\Phi_2$ | Multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$. Given by equation (4.13). |
| $V_U(\ldots), V(\ldots)$ | Algebraic variety. |
| $d_H(A,B)$ | Hausdorff metric. See equation (5.21). |
| $\operatorname{dist}(x,A)$ | Distance between $x$ and $A$. See equation (5.22) |
| $\operatorname{Tub}_\delta(A)$ | Tubular neighbourhood of $A$. See equation (5.23). |
| $\langle h\rangle_X$ | Mean $\fint_X h$ of a function $h\colon X\to Y$. |
| $\lesssim$ | $x\lesssim y$ means there exists a constant $c\geq 1$ such that $x\leq cy$. |

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