Stagnation points of harmonic vector fields and the domain topology

Some applications of Morse theory

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Question (Flowthrough with stagnation point)

Does there exist a domain $X \subset \mathbb{R}^d$ homeomorphic to a ball and a harmonic vector field $u\colon X \to \mathbb{R}^d$ on X such that

1. u has an interior stagnation point
2. the boundaries on which u enters and leaves the region are simply connected?

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Answer

- $d = 2$ dimensions: Not possible (known).
- cylinders in $d = 3$ dimensions: Not possible (known).
- $d = 3$ dimensions: Number of stagnation points has to be even.
- $d = 4$ dimensions: Possible for $X = B_1$, $u = \nabla f$ with

$$f = x_1^2 + x_2^2 - x_3^2 - x_4^2 .$$

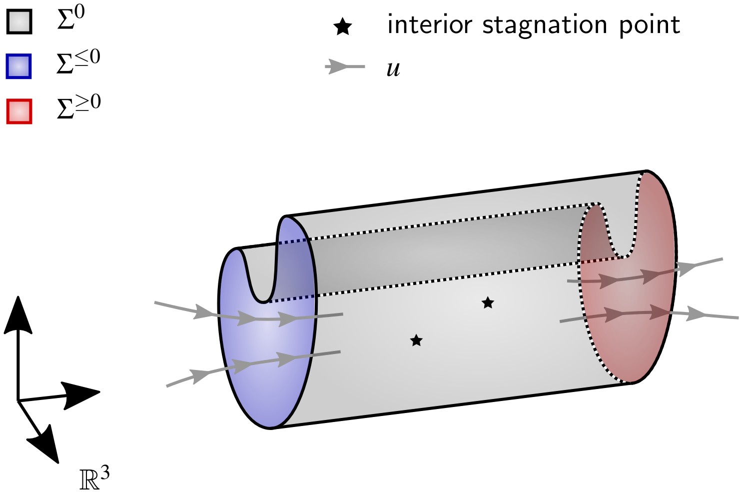

Figure: This kind of situation is not possible.

But if one allows for holes in $d=2$ dimensions it becomes possible.

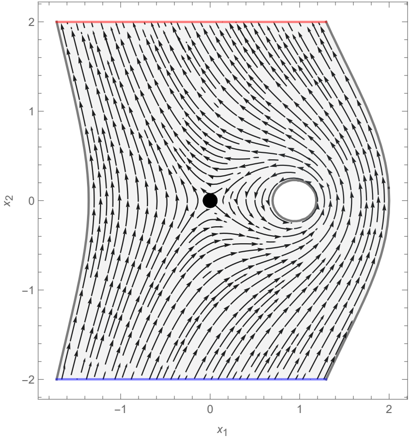

Figure: A plot of $u=\nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2])\cap(\mathbb{R}\times[-2,2])$. Here $\psi:=\Phi_2(x-e_1)+x_1$.

For $d = 3$ dimensions we have for r sufficiently large the example $X = B_r$, $u = \nabla f$ with

$$f = \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_1 x_2^2}{2} + x_1 x_2^2 + x_2 x_3$$

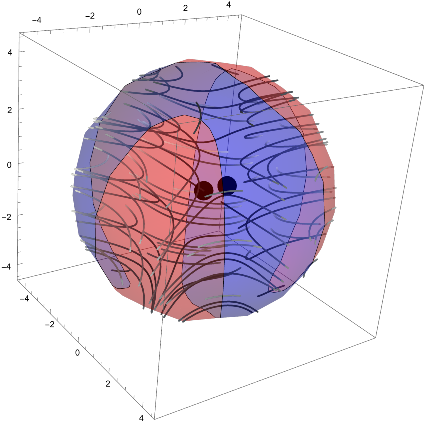

Figure: A stream plot of the function u. The interior stagnation points are highlighted in black. Σ^+ is shaded red, Σ^- blue.

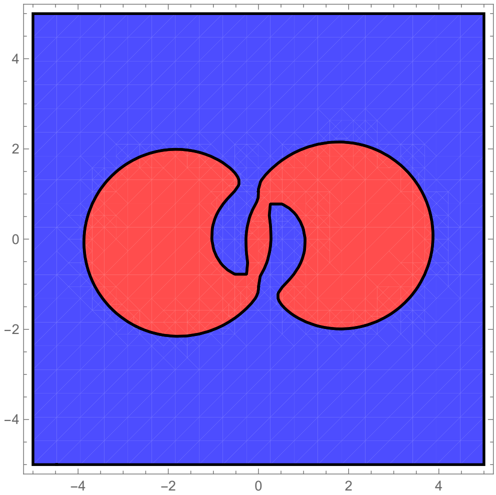

Figure: Stereographic projection of the surface Σ. Σ^+ is shaded red, Σ^- blue.

One can perturb this solution to show that there exists a harmonic vector field on B_r with interior stagnation point such that Σ^+ and Σ^- have positive distance from one another and are simply connected.

Question (Harmonic vector fields without inflow or outflow)

Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary and non-vanishing. What can be said about the relation between the number of stagnation points and the domain topology?

In $d = 2$ dimensions one essentially has the relation

$$M = \chi(X)$$

where M is the number of stagnation points and $\chi(X)$ the Euler characteristic of X.

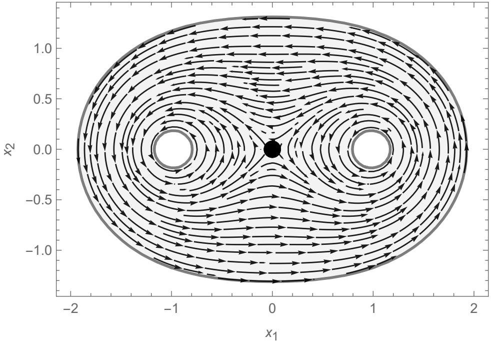

Figure: A plot of $u = \nabla^{\perp}\psi$ in the domain $\psi^{-1}([-1,1])$. Here $\psi := \Phi_2(x - e_1) + \Phi_2(x + e_1)$.

In $d = 3$ dimensions one has essentially an even number of stagnation points.

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Thank you for your attention.