Some relations between equilibria of harmonic vector fields and the domain topology.

Master Thesis

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Contents

Todo list

General TODOs

- Check for typos.
- Does Girault-Raviart theorem with Helmholtz decomp. help?
- bring in results from [1] and [2]
- Harmonic vector fields, find up to date reference
- Mention Sard's theorem
- Does Bocher's theorem help?
- Look at application of Sperner's lemma
- C is used once for critical points, once for level sets.
- Define traversing vector field

Some questions

- Should I state Hopf's Lemma?

1 Introduction

Some amazing introduction

Unless otherwise stated we denote by $X \subseteq \mathbb{R}^d$ a compact subset of $\mathbb{R}^d$ with boundary $\Sigma = \partial X$ and nonempty interior $\Omega = \mathrm{int}(X)$. In the following we will work in dimensions $d \in \{2,3\}$. Unless otherwise stated we denote by

$$f\colon X \to \mathbb{R}$$

a C^2 function on X. Often f will be assumed to be harmonic. We also denote by

$$u\colon X \to \mathbb{R}^d$$

a vector field of class C^1. In the following we often assume that u is in fact *harmonic*, that is u fulfils $\mathrm{Div}\, u = 0$ and $\mathrm{curl}\, u = 0$. Often but not always we assume that in fact $u = \nabla f$ is a gradient field. One question we seek to answer in this thesis is the following:

Question 1.1 (Flowthrough with stagnation point)**.** Does there exist a region $X \subseteq \mathbb{R}^3$ homeomorphic to a ball with flow u through the region such that

1. u is a harmonic vector field
2. u has an interior stagnation point
3. the boundary on which u enters the region is simply connected?

The answer for this will turn out to be 'yes' for dimensions $d \geq 3$ and 'no' for $d = 2$ dimensions. Another question we will consider is of the type:

Question 1.2 (stagnation points of harmonic vector fields without inflow or outflow)**.** Let u be a harmonic vector field in a domain X such that at every boundary point it is tangential to the boundary. What can be said about the relation between the number of stagnation points and the domain topology?

This question yields a very nice result in the case of $d = 2$ dimensions. To make the formulation of these questions more precise we begin with some general definitions regarding stagnation points and the boundary behaviour.

General definitions

We start by requiring some regularity for the boundary of X. More precisely, we require X to be a compact Riemannian manifold with corners:

Definition 1.3 (Manifolds with corners, [3])**.** We introduce the notation

$$H_j^d = \mathbb{R}_{\geq 0}^j \times \mathbb{R}^{d-j} \subseteq \mathbb{R}^d .$$

where $j \in \{0, \dots, d\}$. A *manifold with (convex) corners* is a topological space X together with an atlas $\mathscr{A}$ such that for every point $x \in X$ there exists an open neighbourhood U_x of x, a number $j = j(x)$ and a diffeomorphism $\phi \colon U_x \to H_j^d$ in $\mathscr{A}$ with $\phi(x) = 0$. We further define for $k \in \{0, \dots, d\}$ a collection of sets

$$X_k = \{x \in X \colon j(x) = d - k\}, \tag{1.1}$$

which form a stratification of X.

More generally we give the definition of a stratification as

Definition 1.4 (Stratified space, [3])**.** Let X be a topological space. A *stratum* is a subspace $X_j \subseteq X$, $j \in \mathscr{J}$, indexed by a partially ordered set $\mathscr{J}$ such that

1. each X_j is a manifold (without boundary) of dimension $n = n(j)$
2. $X = \bigcup_j X_j$
3. $X_j \cap \overline{X}_k \neq \emptyset$ iff $X_j \subseteq \overline{X}_k$ iff $j \prec k$.

The pair of X and the collection of strata is called a *stratified space*. In the case that $X_j \subseteq \overline{X}_k$ and additionally $n(k) = 0$ or $n(k) = n(j) + 1$ we will write $X_j \precsim X_k$ or, abusing notation, we will write $X_k = X_{j+1}$.

In the case that the stratification arises through relation (1.1) we have precisely $X_j \precsim X_{j+1}$ for $j \in \{1, \dots, d\}$ and $X_0 \precsim X_0$. Note that in general for a given stratum X_j the stratum X_{j+1} such that $X_j \precsim X_{j+1}$ need not be unique.

For completeness we also give the definition of the contingent cone for a stratification X_j of X:

Definition 1.5 (contingent cone, [4, Def. 4.6])**.** We denote the *(Bouligand) contingent cone* for a set $Y \subseteq X$ at $x \in \overline{Y}$ by $C_x Y$. It is defined as the set of all $v \in \mathbb{R}^d$ such that there exists sequences $\lambda_n \to 0$ and $x_n \to x$ in Y such that

$$\lim_n \lambda_n (x - x_n) = v .$$

For clarification we give an example

Example 1.6 (Cubical domain)**.** Consider the domain to be the cube $X = [-1, 1]^2 \subseteq \mathbb{R}^2$. Then we have a stratification given by

$$\begin{aligned}
X_0 &= \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\
X_1 &= I \times \{-1\} \cup I \times \{1\} \cup \{-1\} \times I \cup \{1\} \times I \\
X_2 &= I \times I
\end{aligned}$$

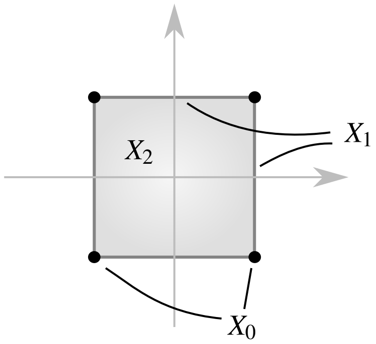


Figure 1.1: A stratification of X.

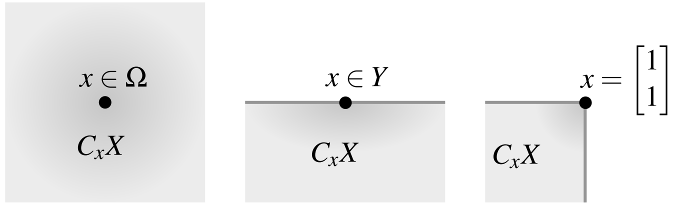


Figure 1.2: The contingent cones for various $x \in X$.

where $I = (-1,1) \subseteq \mathbb{R}$. The stratification is depicted in figure 1.1. For an interior point $x \in X_2$ we have the contingent cone $C_xX = T_x\mathbb{R}^d$. For a boundary point $x \in Y = I \times \{1\} \subset X_1$ we have the contingent cone

$$C_xX = \left\{v \in T_x\mathbb{R}^2 \colon v \cdot n \leq 0\right\}$$

where the basis vector $n = e_2$ is the outer unit normal. At the boundary point $x = \begin{bmatrix}1 & 1\end{bmatrix}^\top \in X_0$ we have

$$C_xX = \left\{v \in T_x\mathbb{R}^2 \colon v_1 \leq 0 \text{ and } v_2 \leq 0\right\}.$$

The situation is depicted in figure 1.2. The contingent cone on the other parts of the square $\Sigma = \partial X$ is given by similar formulas.

In the following we define the emergent and the entrant boundary in a way that generalises [2, p.282] for stratified manifolds.

Definition 1.7 (Emergent and entrant boundary)**.** We call a vector $v \in T_x\mathbb{R}^d$ *entrant* at a boundary point $x \in \Sigma$ if

1. v points into Ω or
2. v lies in the dual cone of the contingent cone C_xX, that is

$$v \in (C_xX)^* = \left\{w \in T_x^*X \colon \left\langle w, w' \right\rangle \geq 0 \text{ for all } w' \in C_xX\right\}.$$

We call v *strictly entrant* if in addition v is not tangential to Σ or v lies in the relative interior $\operatorname{rel\,int}(C_xX)^*$. Analogously v is *(strictly) emergent* if $-v$ is (strictly) entrant. Now define the *entrant boundary* $\Sigma^{\leq 0}$ to be the set of boundary points at which u is entrant. We define the *strictly entrant boundary* Σ^- to be the set of strictly entrant boundary points of u. In the same manner we define the *emergent boundary* $\Sigma^{\geq 0}$ and the *strictly emergent boundary* Σ^+. Further define the *tangential boundary* Σ^0 to be

$$\Sigma^0 = \Sigma^{\leq 0} \cup \Sigma^{\geq 0} \setminus \left(\Sigma^+ \cup \Sigma^-\right) \subseteq \Sigma. \tag{1.2}$$

We would now like to illustrate the preceding definitions.

Example 1.8. Consider the domain to be the cube $X = [-1,1]^2 \subseteq \mathbb{R}^2$ and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto x_1^2 - x_2^2 \,. \end{aligned} \tag{1.3}$$

This induces the harmonic vector field $u = \nabla f$, or more precisely

$$\begin{aligned} u\colon \Omega &\to \mathbb{R}^3 \\ x &\mapsto 2\begin{bmatrix} x_1 & -x_2 \end{bmatrix}^\top \,. \end{aligned} \tag{1.4}$$

For a boundary point $x \in I \times \{1\}$ the dual cone of the contingent cone C_xX is given by

$$(C_xX)^* = C_xX \,.$$

Hence we have that $x \in \Sigma^-$ iff

$$0 > n \cdot u = -2x_2$$

which is always fulfilled and thus $I \times \{1\} \subseteq \Sigma^-$. At the boundary point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ the dual of the contingent cone is also given by

$$(C_xX)^* = C_xX \,.$$

Since $v = u(x) = 2\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ we have that $v \notin (C_xX)^*$ and $-v \notin (C_xX)^*$ and thus $x \notin \Sigma^{\geq 0} \cup \Sigma^{\leq 0}$. By analogous argumentation on the other sides of the square $\Sigma = \partial X$ one obtains that

$$\begin{aligned} \Sigma^- &= \Sigma \leq 0 = I \times \{1\} \cup I \times \{-1\} \\ \Sigma^+ &= \Sigma \geq 0 = \{1\} \times I \cup \{-1\} \times I \,. \end{aligned}$$

In particular the tangential boundary $\Sigma^0 = \emptyset$ is empty. A plot of the sets can be seen in figure 1.3.

Given a vector field $u\colon X \to \mathbb{R}^d$ and a stratification X_j of X we can construct for every $j \in \mathscr{J}$ a vector field

$$u_j\colon X_j \to T^*X_j \,.$$

Here T^*X_j denotes the cotangent space of the manifold X_j which is defined for instance in [5, Chapter 6]. More precisely, for $x \in X_j$ let

$$\pi_j\big|_x\colon \mathbb{R}^d \cong T_x^*\mathbb{R}^d \to T_x^*X_j \tag{1.5}$$

denote the orthogonal projection of a vector at x onto the cotangent space of the stratum X_j at x. Now let

$$u_j = \pi_j \circ u\big|_{X_j} \in C^1(T^*X_j) \tag{1.6}$$

be the projection of u onto the cotangent bundle T^*X_j.

The following are slight generalisation of definitions given in [1, p.138f], [6, §5] and [2, p.282f] to include harmonic vector fields.

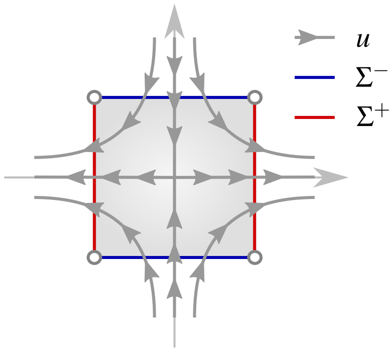


Figure 1.3: Depiction of the entrant and emergent boundaries for the function f given by equation (1.3)

Definition 1.9 (Stagnation points)**.** Let $u_j \colon X_j \to T^*X_j$ be a C^1 vector field on a stratum X_j of X. We call the zeroes $x \in X_j$ of u_j *stagnation points*. If $x \in \Omega$ then we call x an *interior stagnation point*. If x lies in the entrant boundary $\Sigma^{\leq 0}$ or is an interior stagnation point we call x an *essential stagnation point*. The set of all essential stagnation points of u_j is denoted by $\mathrm{Cr}_j = \mathrm{Cr}_j(u)$. A stagnation point x is called *non-degenerate* if x does not lie in the tangential boundary Σ^0 and additionally the derivative

$$Du_j(x) = Du_j\big|_x \in T_xT^*X_j \cong \mathbb{R}^{n(j)\times n(j)}$$

is bijective. In addition we say that x has *index k* if $Du_j(x)$ has exactly k negative eigenvalues. u_j is called *(essentially) non-degenerate* if all its (essential) stagnation points are non-degenerate. Assume u_j is non-degenerate then we can define the *k-th type number* $\mathrm{Ind}_{j,k}(u)$ of the stratum X_j to be the number of essential stagnation points of u_j of index k, that is

$$\mathrm{Ind}_{j,k}(u) = \#\{x \in \mathrm{Cr}_j(u) \colon x \text{ has index } k\}. \tag{1.7}$$

To illustrate the preceding definitions we return to our previous example.

Example 1.10. Let X, f and u be as in example 1.8. We have that $u_2 = u$ and thus one sees from equation (1.4) that the origin 0 is the sole stagnation point of u on the stratum X_2. Since we have that

$$Du(x) = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}$$

for all $x \in \Omega$ we see that $Du(0)$ is bijective and thus the origin is a non-degenerate interior stagnation point. Since $Du(0)$ has exactly one negative eigenvalue we see that the origin has index 1. Hence we have $\mathrm{Ind}_{2,k} = \delta_{k1}$ where δ denotes the Kronecker delta. For $x \in I \times \{1\} = Y$ we calculate

$$u_1(x) = \pi_1 \circ u(x) = (u - n \cdot un)(x) = 2x_1 e_1$$

and thus we have that $x = e_2$ is the unique stagnation point of u on $I \times \{1\}$. Consider the curve

$$\begin{aligned} \gamma \colon I &\to Y \\ t &\mapsto te_1 + e_2 \end{aligned}$$

then $\gamma(0) = e_2$ and we have

$$Du_1(e_1)\big(\gamma'(0)\big) = (u_1 \circ \gamma)'(0) = (2te_1)'(0) = 2e_1 = 2\gamma'(0)$$

and thus e_1 is an eigenvector of $Du_1(e_2)$ to eigenvalue 2. Since e_1 spans the eigenspace $T_{e_2}Y$ it follows that e_2 is a non-degenerate stagnation point of u_1 with index 0. Now since $e_2 \in \Sigma^-$ we have that e_2 is in fact an essential stagnation point. Proceeding in this manner for the other segments of the square Σ we obtain that $\operatorname{Ind} 1, k = 2\delta_{0k}$. If we now consider the point $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ then we have that $u_0(x) = 0$ and thus x is a stagnation point. Now since x does not lie in Σ^0 and the derivative $Du_0 = 0 \in T_x T^* X_0 = 0$ is bijective we have that x is a non-degenerate critical point of index 0. Since however $x \notin \Sigma^{\leq 0}$ we have that x is not an essential stagnation point. Analogous argumentation on the other three corners yields that $\operatorname{Ind}_{0,0} = 0$.

We call a boundary point $x \in X_j$ on a strata X_j *ordinary* if $u(x)$ is not stagnation point of a stratum X_{j+1}. This definition of ordinary points is inspired by [2].

Proposition 1.11. *The condition that the stagnation point $x \in X_j$ does not lie in Σ^0 is equivalent to that x is ordinary.*

Proof.

Some proof

□

Definition 1.12 (Morse functions)**.** We call u *(essentially) Morse* if for all $j \in \mathscr{J}$ we have that u_j is (essentially) non-degenerate. For an essentially Morse function u we define the *interior type numbers* M_k to be the number of essential interior stagnation points of u of index k, that is

$$M_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \operatorname{Ind}_{j,k}(u) = \#\left\{ x \in \bigcup_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) = d}} \operatorname{Cr}_j(u) \colon x \text{ has index } k \right\}. \tag{1.8}$$

The total number M of interior stagnation points of u is then given by

$$M = \sum_k M_k \,. \tag{1.9}$$

Analogously we define the *k-th boundary type numbers* to be the number of essential boundary stagnation points of u of index k, that is

$$\mu_k = \sum_{\substack{j \in \mathscr{J} \text{ s.t.} \\ n(j) < d}} \operatorname{Ind}_{j,k}(u) \tag{1.10}$$

We further write ν_k for the k-th boundary type number of $-u$. We define the *type number* to be the number of essential stagnation points of u of index k, that is

$$\mathrm{Ind}_k(u) = \sum_{j \in \mathscr{J}} \mathrm{Ind}_{j,k}(u) = M_k + \mu_k\,. \tag{1.11}$$

We return to our example:

Example 1.13. Let X, f and u be as in example 1.8. By the calculations of the previous example 1.10 we have that u is Morse and we can calculate the interior type numbers

$$M_k = \mathrm{Ind}_{2,k} = \delta_{2k}$$

and the boundary type numbers

$$\mu_k = \mathrm{Ind}_{0,k}(u) + \mathrm{Ind}_{1,k}(u) = 2\delta_{0k}$$

This then yields the type numbers

$$\mathrm{Ind}_k(u) = M_k + \mu_k = \delta_{2k} + 2\delta_{0k}\,.$$

The previous definitions translate naturally to f. That is we call f Morse, non-degenerate, et cetera if $u = \nabla f$ is Morse, non-degenerate, et cetera. Similarly we call x a *critical point* of f of index k if it is a stagnation point of u of index k.

On assuming non-degeneracy

In the following section we argue that assuming non-degeneracy of u and f is not a great restriction. Given u we define the modification

$$u^{\varepsilon} = u + \varepsilon \tag{1.12}$$

for some $\varepsilon \in \mathbb{R}^d$. We would like to show that u_ε is for almost all choices of ε non-degenerate and can thus be used to approximate a degenerate u. Our approach is to use Thom's theorem which is inspired by the approach in [5, Chapter 6].

Definition 1.14 (Transversality)**.** We call a function $g\colon Y_1 \to Y_2$ between to manifolds Y_1 and Y_2 (without boundary) *transverse* to a submanifold $A \subseteq Y_2$ if for all points in the preimage $x \in g^{-1}(A)$ we have that

$$\mathrm{Image}(Dg_x) + T_{g(x)}A = T_{g(x)}Y\,.$$

As an application we make the following observation.

Proposition 1.15 (Transversal characterisation of non-degeneracy). *Let $u_j\colon X_j \to T^*X_j$ be a differentiable vector field. Then u_j is non-degenerate iff u_j is transverse to the zero section A_j of T^*X_j and contains no stagnation points in Σ^0.*

Proof. First note that we have that $x \in u_j^{-1}(A)$ iff $u_j(x) = 0$ and thus $u_j^{-1}(A) = C$. Unravelling the definition of transversality we get that u_j is transverse to the zero section iff for all $x \in C = u_j^{-1}(A)$ we have that

$$\operatorname{Image}(Du_j(x)) + T_{u_j(x)}A = T_{u_j(x)}TX\,. \tag{1.13}$$

As A is the zero section we have $T_{u_j(x)}A = 0$ and equation (1.13) is equivalent to stating that Du_j is of full rank. But Du_j being of full rank at all points in C and u_j having no stagnation points in Σ^0 is equivalent to u_j being non-degenerate. $\square$

The following version of Thom's transversality theorem is an adaption (i.e. weakening) of [5, Theorem 2.7] to our needs.

Theorem 1.16 (Parametric transversality theorem.). *Let E, Y_1, Y_2 be C^r-manifolds (without boundary) and $A \subseteq Y_2$ a C^r submanifold such that*

$$r > \dim Y_1 - \dim Y_2 + \dim A\,.$$

Let further $F\colon E \to C^r(Y_1, Y_2)$ be such that the evaluation map

$$\begin{aligned} F^{ev}\colon E \times Y_1 &\to Y_2 \\ (\varepsilon, x) &\mapsto F_\varepsilon(x) \end{aligned}$$

is C^r and transverse to A. Then the set

$$\pitchfork(F; A) = \{\varepsilon \in E\colon F_\varepsilon \text{ is transverse to } A\}$$

is dense.

Proof. See [5, Theorem 2.7] for details. $\square$

Using proposition 1.15 we get a generalisation of the results in [2, §2].

Corollary 1.17 (Density of boundary generic functions). *Let $u\colon X \to T^*X$ be a harmonic vector field on X and let X_j be a stratification of X. Assume that u has no stagnation points on Σ^0. Then there exists a $\delta > 0$ such that for almost every $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ we have that*

1. *u^ε is non-degenerate on X_j*
2. *$u^\varepsilon \to u$ uniformly*
3. *if $x_\varepsilon \to x$ then x is a stagnation point of u*
4. *Additionally we can find for every $\eta > 0$ a $\delta > 0$ such that all stagnation points of u^ε are contained in a η-neighbourhood of the set of stagnation points of u.*

5. *the property of being entrant or emergent of stagnation points of u^ε is preserved, that is a stagnation point x^ε of u^ε lies in $\Sigma^+(u^\varepsilon)$ iff it lies in $\Sigma^-(u)$.*
6. *If x is a non-degenerate stagnation point on the stratum X_j of u we have that*

$$\mathrm{Ind}_{X_j,k}(u^\varepsilon) = \mathrm{Ind}_{X_j,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{X_j,k}(-u^\varepsilon) = \mathrm{Ind}_{X_j,k}(-u)$$

for all k.

Proof.

Fill in the details for the following. .

The following is essentially an adaptation of a proof given in [2, §2]. We first show that we can choose a $\delta > 0$ such that for all $\varepsilon \in B_\delta \subseteq \mathbb{R}^d$ we have no stagnation points on $\Sigma^0(u^\varepsilon)$. Assume not. Then there exist sequences $\varepsilon_k \to 0$ and x_k of stagnation points of u^{ε_k} on $\Sigma^0(u^{\varepsilon_k})$. By compactness of X we can assume that $x_k \to x$ for some $x \in X$ after taking a sub-sequence. After taking a further sub-sequence we can also assume that all x_k lie in a stratum X_j. The condition that x_j are stagnation points and lie in $\Sigma^0(u^{\varepsilon_j})$ means that there exists a stratum X_{j-1} such that x_j is also stagnation point of this stratum. But then x is also a stagnation point of X_{j-1} for u since $u^\varepsilon \to u$. This implies that x is a stagnation point of X_{j-1}, but $x \in \overline{X}_j$ and hence $x \in \Sigma^0$ is a stagnation point. A contradiction.

The next part of the proof is inspired by [5] use of transversality to show a similar statement. Set $r = 2$, $E = B_\delta$ and $Y_2 = T^*X_j$ in the previous theorem. We initially set $Y_1 = X_j = \Omega$. We would like to apply the parametric transversality theorem to the function

$$\begin{aligned} F\colon E &\to C^\infty(X_j, T^*X_j) \\ \varepsilon &\mapsto u^\varepsilon \end{aligned}$$

We note that F^{ev} is sufficiently smooth. We need to show that F^{ev} is transverse to the zero section $A \subseteq T^*X_j$. Then the parametric transversality theorem yields a dense $E_j \subseteq E$ on which F is transverse to A. For this note that for all $(\varepsilon, x) \in F^{-1}(A)$ we have

$$\mathrm{Image}\left(DF^{\mathrm{ev}}_{(\varepsilon,x)}\right) = T_x T^* X_j \tag{1.14}$$

since

$$DF^{\mathrm{ev}}_{(\varepsilon,x)} = \left[\mathrm{Id}_{d\times d} \mid Du_x\right]$$

is surjective. Proposition 1.15 now yields that u^ε is non-degenerate on E_j.

Analogously we set $Y_1 = X_j$ to be an arbitrary strata in the previous proof and replace u^ε with the restriction u^ε_j. To show that equation (1.14) holds we resort to the fact that

$$DF^{\mathrm{ev}}_{(\varepsilon,x)} = D\big(u^\varepsilon_j(x)\big)_{(\varepsilon,x)} = D\pi_j \circ (Du^\varepsilon(x))_{(\varepsilon,x)}$$

is surjective as a concatenation of surjective functions. Thus there also exists a dense set $E_j \subseteq \mathbb{R}$ on which u_j^ε is non-degenerate on X_j.

The following implication is false, look into [5] for fix.

Now the set

$$\overline{E} = \bigcap_j E_j \subseteq \mathbb{R} \tag{1.15}$$

is dense in $\mathbb{R}$ and for every $\varepsilon \in \overline{E}$ the function u^ε fulfils condition 1.

Let $x_\varepsilon \to x$ on the stratum X_j. The uniform convergence $u^\varepsilon \to u$ as $\varepsilon \to 0$ and the continuity of π_j imply that x is a stagnation point of u. More concretely let $x_\varepsilon \to x$ be convergent sequence of stagnation points for u^ε as $\varepsilon \to 0$ then we have that

$$0 = \lim_\varepsilon u_j^\varepsilon(x_\varepsilon) = u_j(x) \tag{1.16}$$

and thus x is a stagnation point.

Let U_η denote the open η-neighbourhood of the set of stagnation points of u. Since u has no stagnation points in Σ^0 we have for any stratum X_j that $u_j \neq 0$ on the compact set $\overline{X}_j \setminus U_\eta$ which implies that we can choose $\delta > 0$ so small that $|u_j| > \delta$ on $\overline{X}_j \setminus U_\eta$ for all strata X_j. For any $\varepsilon \in B_\delta$ it then follows that u^ε has no stagnation points on the set $\overline{X}_j \setminus U_\eta$ which yields the claim.

Since the boundary stagnation points of u have a positive distance to the tangential boundary Σ^0, say 2η we can choose δ as in the previous part of the proof. Now consider the continuous mapping

$$x \mapsto \operatorname{dist}(u(x), \partial C_x X_j)$$

which is positive on $X_j \setminus \left(\Sigma^0\right)_\eta$ and thus attains a positive minimum over all strata X_j. We can assume that $\delta > 0$ is less than this minimum. The choice of δ in this way ensures that the property of being entrant or emergent is preserved.

Since x is not contained in Σ^0, by continuity x will be entrant or emergent. On the other hand if x is a non-degenerate stagnation point of u on the stratum X_j it follows from the inverse function theorem that there exists for sufficiently small δ a neighbourhood around x on which there is a one-to-one correspondence between the stagnation points of u and u^ε. Since there are by proposition 2.1 at most finitely many non-degenerate stagnation points of u we can choose δ to be minimal over all these stagnation points. The equality of the indexes then follows from $Du^\varepsilon = Du$.

$\square$

2 Some general remarks

Bring order into this section.

We make the following remarks

This was stated somewhere in [6].

Proposition 2.1. *The number of non-degenerate stagnation points of u_j on X_j is finite.*

Proof. Let $x \in X_j$ be a non-degenerate stagnation point of u_j. Since $Du_j(x)$ is invertible there exists by the inverse function theorem an open neighbourhood $U_x \subseteq X_j$ of x on which u_j is bijective. Hence x is the only stagnation point in U_x. Let C denote the set of all non-degenerate stagnation points of u_j. Then the sets U_x for $x \in C$ together with

$$U_C = \mathbb{R}^d \setminus \overline{C} \tag{2.1}$$

form an open cover of $\overline{X}_j$. But $\overline{X}_j$ is compact and thus there exists a finite subcover. Since we have for every stagnation point $x \in C$ that $x \notin U_y$ for all other $y \in C \setminus \{x\}$ and $x \notin U_C$ we must have that U_x is contained the finite subcover. Thus it follows that $\#C < \infty$ is finite. □

As a consequence we obtain the following observation:

Corollary 2.2. *For a Morse u the type numbers $M_0, \dots, M_d$ and the boundary type numbers $\mu_0, \dots, \mu_{d-1}$ are finite.*

State the theorem of Sard

We state Morse's lemma according to [5, p.145]

Lemma 2.3. *Let $f\colon X \to \mathbb{R}$ be C^{2+r} and x be a non-degenerate critical point of index k. Then there exists a C^r chart (φ, U) at x such that we have*

$$f\colon \varphi^{-1}(y) = f(x) - \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{d} y_j^2 .$$

State proof.

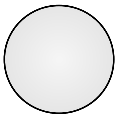
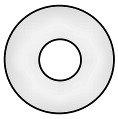
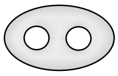
Domain	Picture	b_0	b_1	b_k, $k \geq 2$
Disk D		1	0	0
Annulus $2D \setminus D$		1	1	0
Two holed button		1	2	0

Table 2.1: Betti numbers for selected domains in $\mathbb{R}^2$.

Betti numbers

Let $H_k(X;\mathbb{R})$ denote the k-th homology space of X. For an introduction and definition of these we refer the reader to [7, Chapter 2]. We define the k-th Betti number as the dimension

$$b_k = \dim_{\mathbb{R}} H_k(X;\mathbb{R})\,. \tag{2.2}$$

We proceed to give examples for Betti numbers of selected connected domains in $\mathbb{R}^d$.

Example 2.4 (In flatland)**.** In $d = 2$ dimensions the 0-th Betti number counts the number of connected components of Ω and the first Betti number counts the number of holes of this domain. All other Betti numbers vanish in $\mathbb{R}^2$. More concretely we give the Betti numbers for selected domains in table 2.1.

Example 2.5 (In spaceland)**.** In $d = 3$ dimensions the 0-th Betti number counts the number of connected components of Ω, the first Betti number counts the number of holes and the second Betti number counts the number of bubbles of the domain. All other Betti numbers vanish. The Betti numbers for selected domains can be seen in table 2.2.

Comment on the finiteness of Betti numbers. Check numbers for ball with torus bubble.

The Morse inequalities

We state the Morse inequalities.

Theorem 2.6 (Strong Morse inequalities, [11, Theorem 2.4])**.** *Let X be a manifold with corners*

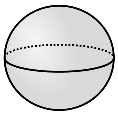
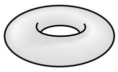
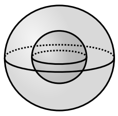
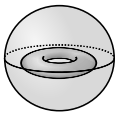
Domain	Picture	b_0	b_1	b_2	$b_k,\, k \geq 3$
Ball B		1	0	0	0
Solid torus $S^1 \times D$		1	1	0	0
Ball with bubble $2B \setminus B$		1	0	1	0
Ball with bubble in shape of torus		1	1	1	0

Table 2.2: Betti numbers for selected domains in $\mathbb{R}^3$.

and $f\colon X \to \mathbb{R}$ be essentially Morse. Then we have for $k \in \{0,\dots,d\}$ the inequalities

$$\sum_{j=0}^{k} (-1)^{j+k} \operatorname{Ind}_j(f) \geq \sum_{j=0}^{k} (-1)^{k+j} b_j(X)\,.$$

For $k = d$ we in fact have equality

$$\sum_{j=0}^{d} (-1)^{j} \operatorname{Ind}_j(f) = \chi(X)$$

where the Euler characteristic

$$\chi(X) = \sum_{j=0}^{d} (-1)^{j} b_j(X)$$

is the alternating sum of the Betti numbers.

Proof. A for manifolds with C^1 boundary is given for instance in [6, Theorem 10.2'] The definition of essential critical points of f and its index given in definition **??** coincides with the definition of a critical point and its co-index of $-f$ given in [11]. The result then follows from [11, Theorem 2.4].

Give outline of proof idea.

□

Corollary 2.7 (Weak Morse inequalities)**.** *Let X be a manifold with corners and $f\colon X \to \mathbb{R}$ essentially Morse. Then we have for $k \in \{0,\dots,d\}$ the inequalities*

$$\mathrm{Ind}_k(f) \geq b_k(X)\,.$$

Proof.

Write some proof.

□

If we now assume that f is harmonic then the maximum principle implies that $M_0 = 0 = M_d$. If we additionally assume that we have dimensions $d = 2$ we obtain [6, Corollary 10.1].

Corollary 2.8 (Morse inequalities for f harmonic, $d = 2$)**.** *Let $d = 2$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M + \mu_1 - \mu_0 &= b_1 - b_0\,. \end{aligned}$$

In dimensions $d = 3$ we obtain [6, Corollary 10.2]

Corollary 2.9 (Morse inequalities for f harmonic, $d = 3$)**.** *Let $d = 3$, Ω and f be regular and assume that f is harmonic. Then we have*

$$\begin{aligned} \mu_0 &\geq b_0 \\ M_1 + \mu_1 - \mu_0 &\geq b_1 - b_0 \\ M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 &= b_2 - b_1 + b_0\,. \end{aligned}$$

Give a classical example of a Morse function to determine the Betti numbers.

On harmonic vector fields

In the following we deduce some basic relations for harmonic vector fields in dimensions $d \in \{2,3\}$.

Proposition 2.10 (Harmonic vector fields on simply connected domains)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and simply connected and u be a harmonic vector field. Then*

1. *$u = \nabla f$ is the gradient field of some function $f\colon \Omega \to \mathbb{R}$.*
2. *f is harmonic.*
3. *u is in fact C^∞.*
4. *The components $u_i = \partial_i f$ are harmonic.*

Proof.

1. Since $\operatorname{curl} u = 0$ this is a direct consequence of Stokes theorem.
2. This follows from $\Delta f = \operatorname{Div} u = 0$.
3. This follows from the fact that f is harmonic
4. This follows from $u_i = \partial_i f$.

$\square$

If one considers not necessarily simply connected domains Ω then we obtain the previous properties at least locally.

3 Harmonic functions, $d = 2$

The following result is essentially a negative to question 1.1 in $d = 2$ dimensions.

Proposition 3.1. *Let Ω be homeomorphic to $B_1 \subseteq \mathbb{R}^2$. Let further $f : \overline{\Omega} \to \mathbb{R}$ be regular harmonic with critical point $x_1 \in \Omega$. Then $\Sigma^- \subseteq \Sigma$ is not connected.*

We shall give two different proofs of this result. One involving level-sets and the other involving invariant manifolds

A proof involving level-sets

write omega-limit.

Sketch of Proof. Let $y_c = f(x_1)$ and $x_1, \ldots, x_M$ be all the critical points such that $f(x_.) = y_c$. We claim that the level set

$$C = \{f = y_c\} \subseteq \overline{\Omega}$$

can be represented by a multigraph G which divides the boundary Σ into 4 components. To show this let $\gamma_i\colon (a_i, b_i) \to C$ for $i \in \{1, \ldots, 4\}$ parametrise the curves in C intersecting at x_1. These can be constructed with the initial value problem

$$\begin{aligned} \gamma' &= (\nabla f)^{\perp}\big|_{\gamma} \\ \gamma(0) &= \gamma_0 \end{aligned}$$

where $\gamma_0 \in C$ is chosen sufficiently near x_1. We assume that the intervals on which the γ_i are defined are maximal. We thus have for

$$\begin{aligned} \gamma_i^- &= \lim_{t \to a_i} \gamma(t) \\ \gamma_i^+ &= \lim_{t \to b_i} \gamma(t) \end{aligned}$$

that $\gamma_i^{\pm} \in \{x_1, \ldots, x_M, \Sigma\}$ since the x_j are the sole points on $\Omega \cap \overline{C}$ at which $\nabla f^{\perp} = 0$. This argument can be applied to all of the $x_1, \ldots, x_M$. We therefore have a situation similar to the one depicted in figure 3.1.

Thus C can be represented by a multigraph G with vertices $v_1, \ldots, v_K$ and edges $e_1, \ldots, e_L \subseteq C$. In the following we identify the graph G with its planar embedding in $\overline{\Omega}$. Assume G contains a cycle with vertex sequence $v_{i_1}, \ldots, v_{i_J}$ and edges $e_{i_1}, \ldots, e_{i_J}$. Then

$$\partial E = \bigcup_j e_{i_j} \subseteq C$$

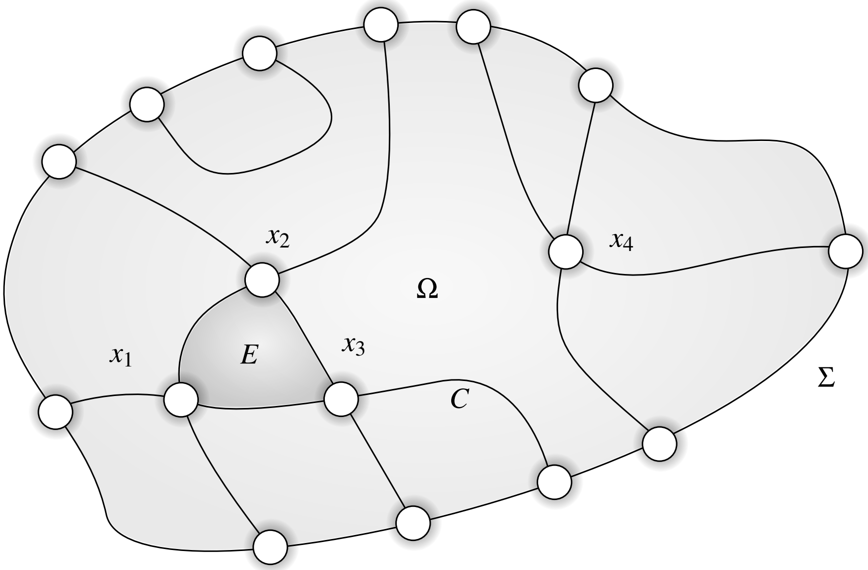


Figure 3.1: The situation at hand: The edges represent level curves and the interior vertices critical points.

is the boundary of a domain E for which $f = y_c$ on ∂E. By the maximum principle $f = y_c$ on E and thus $f = y_c$ on $\overline{\Omega}$, a contradiction to the non-degeneracy. Hence G is acyclic and the number of intersections of C with the boundary Σ is at least four and thus the boundary Σ is divided into at least four components.

Now choose four neighbouring components $\omega_1, \ldots, \omega_4$ as depicted in figure 3.2 Let $A \subseteq \Omega$ be the domain bounded by ω_1 and C as in the figure. The maximum principle yields that ω_1 contains a local maximum or minimum of f since $f = y_c$ is constant on the other boundaries $\partial A \setminus \omega_1$. By the same argument $\omega_2, \ldots, \omega_4$ also contain local extrema. Since the $\partial \omega_i$ cannot be extremal points on Σ we can assume without loss of generality (by switching f for $-f$) that ω_1 and ω_3

use argument with ∇f here to show that extrema can be assumed to be alternating.

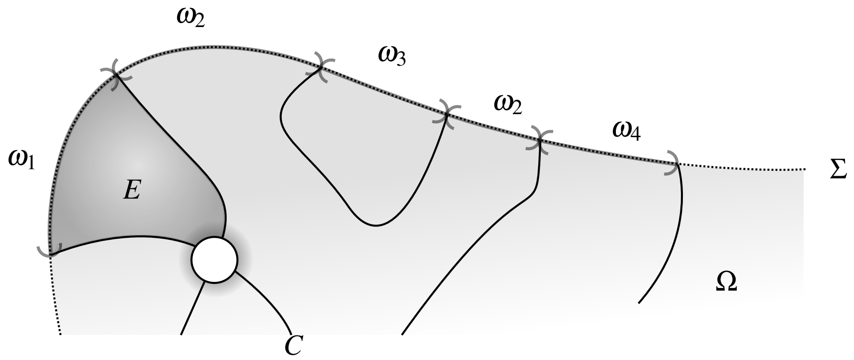


Figure 3.2: The choice of $\omega_1, \ldots, \omega_4$.

contain local maxima and ω_2 and ω_4 local minima. By Hopf's lemma we thus have

$$\Sigma^- \cap \omega_2 \neq \emptyset \neq \Sigma^- \cap \omega_4$$

and

$$\Sigma^+ \cap \omega_1 \neq \emptyset \neq \Sigma^+ \cap \omega_3$$

From this the claim follows. ☐

A proof involving invariant manifolds

Using invariant manifolds we obtain the following proof.

Sketch of Proof. Let $x_1,\dots,x_M$ denote the critical points of f. Let $\lambda_i\colon (a_i,b_i) \to \overline{\Omega}$ for $i \in \{1,2\}$ parametrise the unstable manifolds of the critical point x_1 and $\lambda_i\colon (a_i,b_i) \to \overline{\Omega}$ for $i \in \{3,4\}$ be chosen to parametrise the stable manifolds of x_1. As in the previous proof we can assume the interval on which the λ_i are defined to be maximal. We thus have for

$$\lambda_i^- = \lim_{t \to a_i} \lambda(t)$$
$$\lambda_i^+ = \lim_{t \to b_i} \lambda(t)$$

that $\lambda_i^\pm \in \{x_1,\dots,x_M,\Sigma\}$ since the x_j are the sole points on $\overline{\Omega}$ at which $Df = 0$. Thus all invariant manifolds of all critical points form a directed multigraph G with vertices $v_1,\dots,v_K$ and edges $e_1,\dots,e_L \subseteq \overline{\Omega}$. Here the direction of the edge is determined by whether f increases or decreases along the edge. Once again we identify the graph with its planar embedding. By construction graph is acyclic directed. We claim that the underlying undirected graph is in fact a forest. Thus it remains to be shown that the underlying undirected graph is acyclic. Assume not, i.e. we have a undirected cycle A with vertices $x_{i_1},\dots,x_{i_J}$ and edges $e_{i_1},\dots,e_{i_J}$. The set of cycles forms a partial ordering with respect to the property 'contains another cycle'. We can assume that our chosen cycle A contains no other distinct cycles, i.e. it is a minimal cycle. We note that each vertex has 2 incoming and 2 outgoing arcs which lie opposite to one another. We also note that the edges cannot cross. We can thus describe the trail $x_{i_1},\dots,x_{i_J}$ by a set of directives of the type

More precise.

$$(d_1,\dots,d_K) \in \{\mathrm{l},\mathrm{r},\mathrm{s}\}^J.$$

Here l, r and s stand for 'left', 'right' and 'straight' respectively. The underlying idea is that we follow a particular trail and orient all vertices as in figure 3.3.

An example of the trail 'srsr' is given in figure 3.4. We now note that cycles of the type r,…,r or l,…,l cannot occur as we otherwise would have a directed cycle. Thus there exists a vertex where the chosen direction is s. Without loss of generality this vertex is x_{i_1}. Since we can swap f with $-f$ we can assume without loss of generality that the cycle lies to right of x_{i_1}. Now consider new cycle B starting at x_{i_1} with directives r,…,r. Since all vertices of B lie within the cycle A

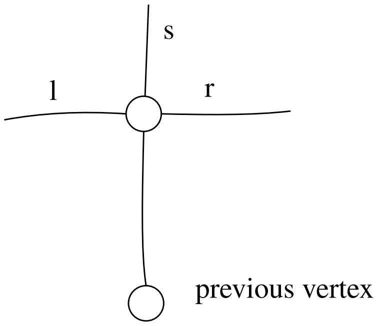


Figure 3.3: Explanation of the directives ‘l’,‘r’ and ‘r’.

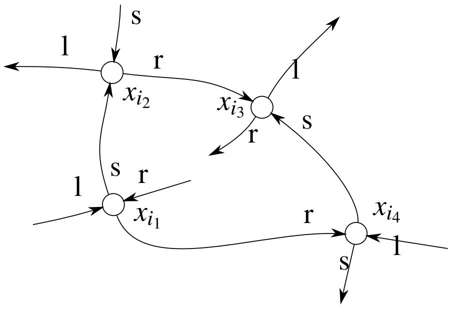


Figure 3.4: An example for a cycle.

we must at some step reach a vertex on the cycle A. But then cycle B is a new distinct cycle contained in cycle A, a contradiction to the minimality of A. Hence every case considered leads to a contradiction and it follows that the underlying undirected multigraph of G is acyclic.

Now call a leaf positive if it lies on the emergent boundary and negative if it lies on the entrant boundary. The case that a leaf is neither positive or negative cannot occur. We now pick a tree $\widetilde{G}$ out of G and note that there are at least 4 boundary vertices to this tree. By construction we see that each ‘neighbouring’ leaf of this tree has opposite signage and the claim follows. ☐

elaborate

elaborate

A proof involving Morse theory

We now give a proof involving Morse theory since the techniques of the proof generalise to the three dimensional case.

Proposition 3.2. *Let $d = 2$ and X be simply connected such that Σ is a differentiable manifold and let $f\colon X \to \mathbb{R}$ have no critical points on Σ^0. Assume further that Σ^- and Σ^+ are nonempty and simply connected. Then f has no non-degenerate interior critical point.*

Proof. Let $x_1, x_2 \in \Sigma^0(f)$ be two points on different connectivity components which we will fix later. Then we can cut the domain along a curve Γ such that the endpoints $\gamma = \partial\Gamma$ of the cut coincide with x_1 and x_2, that is $\partial\Gamma = \{x_1, x_2\}$. Now we obtain two new domains X^+ and X^- such that $\partial X^+ \subseteq \Sigma^+ \cup \Sigma^0 \cup \Gamma$ and $\partial X^- \subseteq \Sigma^- \cup \Sigma^0 \cup \Gamma$. We can assume that Γ is a smooth manifold and corresponds to the stratum X_Γ for X^+ and X^-. We also assume that the corner points $x_1, x_2 \in \partial\Gamma$ correspond to the strata X_1 and X_2. Locally around the corner point x_1 we have a situation depicted as in figure 3.5. We assume that we chose Γ in such a way that it forms an acute angle with Σ at the boundary points γ. Without loss of generality x_1 is not an essential critical point of $-f$ on X^-. The point x_1 may or may not however be a critical point of f on X^+. Analogously we can choose Γ in such a way around x_2. For the following argumentation we require that u is strictly Morse on both X^+ and X^-, so assume for a moment that this is the case. We now focus our attention on X^+. Since no essential critical points lie on Σ^+ it follows for the

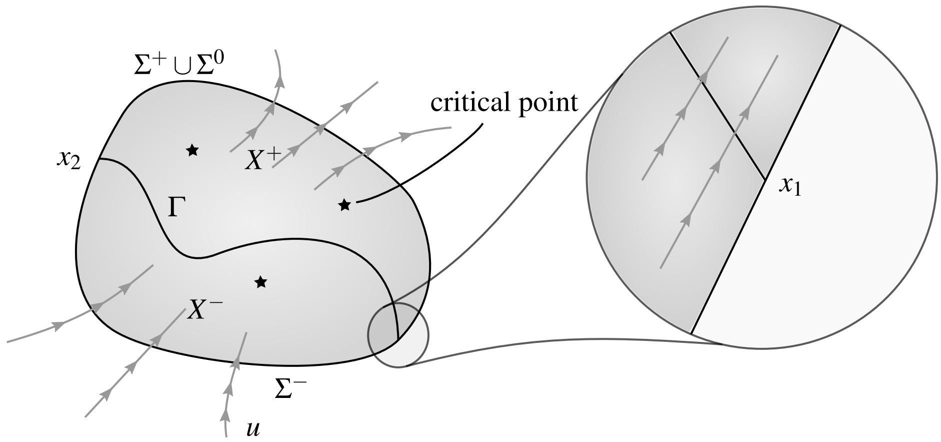


Figure 3.5: The situation at hand.

boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma,k}(f) + \delta_{k0}\,\mathrm{Ind}_{\gamma,k}(f) \tag{3.1}$$

where δ_{ij} denotes the Kronecker delta. Analogously we have on X^- that

$$\nu_k^- = \mathrm{Ind}_{\Gamma,k}(-f) \tag{3.2}$$

where we took into account that $\mathrm{Ind}_{\gamma,k}(-f) = 0$. In addition we have on Γ that the emergent critical points of f on X^+ are the entrant critical points of $-f$ on X^-, that is

$$\mathrm{Ind}_{\Gamma,0}(f) = \mathrm{Ind}_{\Gamma,1}(-f) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(f) = \mathrm{Ind}_{\Gamma,0}(-f)\,. \tag{3.3}$$

Using equations (3.1), (3.2) and (3.3) we obtain

$$\mu_0^+ - \mathrm{Ind}_{\gamma,0}(f) = \nu_1^- \qquad \text{and} \qquad \mu_1^+ = \nu_0^-\,. \tag{3.4}$$

Consider the Morse inequality for f on X^+

$$M^+ + \mu_1^+ - \mu_0^+ = -\chi(X^+) = -\chi(X) \tag{3.5}$$

and the Morse inequality for $-f$ on X^-

$$M^- + \nu_1^- - \nu_0^- = -\chi(X^-) = -\chi(X)\,. \tag{3.6}$$

We now add equations (3.5) and (3.6) and insert relations (3.4) to obtain

$$M^- + M^+ - \mathrm{Ind}_{\gamma,0}(f) = -2\chi(X) = -2\,.$$

Since $\mathrm{Ind}_{\gamma,0}(f) \leq 2$ and $M^\pm \geq 0$ we must in fact have $M^\pm = 0$ from which the claim follows.

The claim remains to be shown in the case that f is not strictly Morse on X^+ and X^-. In this case let u^ε be as in corollary 1.17. It follows from the slanted angles at which Γ approaches γ that if the points x_1, x_2 are critical points of f that they then are in fact non-degenerate. Hence we obtain that

$$\mathrm{Ind}_{\gamma,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-f)\,. \tag{3.7}$$

By the same corollary u^ε has no essential stagnation points on $\Sigma^+(u)$ and $-u$ has no essential stagnation points on $\Sigma^-(u)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M^\varepsilon = M$. □

Allowing for Inflow and outflow

The strategy in the above proofs can be generalised to show the following

Conjecture 3.3. *Let $X \subseteq \mathbb{R}^2$ be a manifold with corners with Betti numbers $b_0 = 1$ and b_1. Let further $f\colon X \to \mathbb{R}$ be Morse harmonic with M critical points. Assume that $\overline{\Sigma^-} \subseteq \Sigma$ on a given connected component of the boundary Σ consists of at most one connected component. Then we have*

$$\frac{4}{3}M \leq b_1 + 1\,.$$

This inequality can probably be improved considerably.

Let $J^\pm$ denote the number of connected components of $\Sigma^\pm$. Consider a disjoint decomposition of the boundary $\Sigma = \Sigma_{\geq 0} \sqcup \Sigma_{\leq 0}$ such that $\Sigma_{\geq 0} \subseteq \Sigma^{\geq 0}$ and $\Sigma_{\leq 0} \subseteq \Sigma^{\leq 0}$. Let now $J^{\geq 0}$ denote the minimal number of connected components of $\Sigma^{\geq 0}$ of all such decompositions. We state a consequence of a result from [8, Theorem 2.1]

Proposition 3.4. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain with a boundary consisting of simple closed $C^{1,\alpha}$ curves. Let $u\colon \overline{\Omega} \to \mathbb{R}$ be harmonic (with certain conditions on the boundary). Then we have*

$$M \leq b_1 - b_0 + \frac{J^+ + J^-}{2}\,.$$

If in addition we assume that there are no stagnation points on the boundary then we have

$$M \leq b_1 - b_0 + J^{\geq 0}\,.$$

Proof. See [8, Theorem 2.1]. □

4 Harmonic vector fields, $d = 2$

No inflow or outflow

We say that u has no *inflow* on a boundary subset $S \subseteq \Sigma$ if $\Sigma^- \cap S = \emptyset$ and that it has no *outflow* if $\Sigma^+ \cap S = \emptyset$. Armed with this definition we can state the following result.

Proposition 4.1 (Special case of the Poincaré-Hopf index theorem.)**.** *Let $X \subset \mathbb{R}^2$ be a compact manifold with C^1 boundary and Betti numbers $b_0 = 1$, and b_1 and let $u \colon X \to \mathbb{R}^2$ be a strictly Morse harmonic vector field without inflow or outflow. Then we have*

$$M + 1 = b_1$$

.

Sketch of proof. We slit Ω such that it is homeomorphic to the disk as is depicted in figure 4.1. Denote the slit by Γ. Since the number of stagnation points is finite by proposition ??, we can choose Γ in such a way that it does not contain any stagnation points. We also denote the points at which Γ meets Σ by $x_1, \ldots, x_{2b_1} \in \partial\Gamma = \gamma$. Note that there are $2b_1$ many such points. We can assume that Γ is a smooth manifold. Now at the point x_1 we have that u is almost parallel to the boundary Σ. Thus we can slant the cut in such a way such that x_1 is an essential stagnation point of index 0 of u on the stratification of $\tilde{X}$. Here $\tilde{X}$ denotes the covering space of X generated by the cut Γ. We denote the induced strata by Γ also with Γ. Note that then x_1 is no essential stagnation point for $-u$. We modify the cut for the other points $x_2, \ldots, x_{2b_1}$ as with x_1. The situation is depicted in figure 4.1. For the following argumentation we require that u is strictly Morse on the

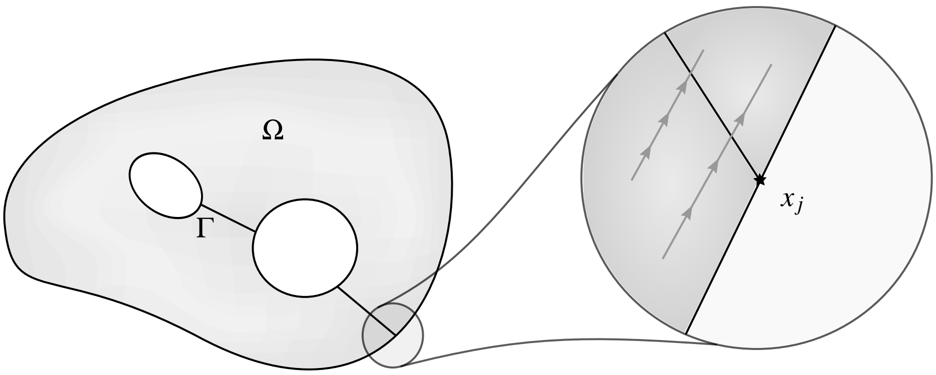


Figure 4.1: How we slit the domain.

new domain $\tilde{X}$ so assume for a moment that this is the case. Since there are no stagnation points on Σ all boundary stagnation points of u are on the strata induced by Γ and $x_1, \ldots, x_{2b_1}$. Hence we have relations

$$\mu_k = \mathrm{Ind}_{\Gamma,k}(u) + 2b_1\delta_{k0} \qquad \text{and} \qquad \nu_k = \mathrm{Ind}_{\Gamma,k}(-u) \tag{4.1}$$

for all $k \in \{0,1\}$. Since on Γ all entrant stagnation points of u are also emergent stagnation points of $-u$ (and vice versa) we have the relations

$$\mathrm{Ind}_{\Gamma,0}(u) = \mathrm{Ind}_{\Gamma,1}(-u) \qquad \text{and} \qquad \mathrm{Ind}_{\Gamma,1}(u) = \mathrm{Ind}_{\Gamma,0}(-u)\,. \tag{4.2}$$

Equations (4.1) and (4.2) yield

$$\mu_0 = \nu_1 + 2b_1 \qquad \text{and} \qquad \mu_1 = \nu_0\,. \tag{4.3}$$

Since Ω is now simply connected u is by proposition 2.10 the gradient of a harmonic function f on this new domain. For this f we have the Morse inequalities

$$M + \mu_1 - \mu_0 = -\chi(\tilde{X}) = -1 \tag{4.4}$$

and for $-f$ the Morse inequalities

$$M + \nu_1 - \nu_0 = -\chi(\tilde{X}) = -1\,. \tag{4.5}$$

Adding equations (4.4) and (4.5) and using the relation (4.3) we obtain

$$2M - 2b_1 = -2$$

from which the claim follows.

The claim remains to be shown in the case that u is not strictly Morse on $\tilde{X}$. In this case let u^ε for $\varepsilon \in E$ be a family of strictly Morse functions as in corollary 1.17. Since the $x_1, \ldots, x_{2b_1} \in \gamma$ are non-degenerate stagnation points of u due to the slanted angle at which Γ approaches γ we obtain that

$$\mathrm{Ind}_{\gamma,k}(u^\varepsilon) = \mathrm{Ind}_{\gamma,k}(u) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-u^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-u) \tag{4.6}$$

By the same corollary u^ε has no stagnation points on $\Sigma^0(u)$. The claim then follows by the calculations above where we replace u with u^ε and then note that $M^\varepsilon = M$. □

We now give an alternative proof using the argument principle.

One could use the argument principle for Riemann surfaces.

Proof. As before we slit the domain such that it is homeomorphic to a disk. By proposition ?? u is the gradient of a harmonic function f on this new domain. Let $h \in \mathrm{Hol}(\mathbb{C})$ be the holomorphic function given by $h = \nabla f$. Let γ traverse the boundary of the slit domain such that the domain lies to the left of γ. We now determine the change of argument $\arg h$ along γ. For this consider first the parts of γ traversing the slits. Since ∇f is continuously differentiable along the slit and γ traverses the slit once in one direction and once in the other the contribution in the change of

$\arg h$ from the slits vanishes. On the other hand as γ traverses the boundary Σ the contribution to the change in argument of $\arg h$ is 2π for every hole in the domain since $h = u$ is tangent to Σ and traverses the holes clockwise direction. Similarly the contribution to the change in argument of $\arg h$ is -2π for the outer boundary component which is traversed counterclockwise. Since we have b_1 holes in the domain the total change of $\arg h$ as γ traverses Σ is $2\pi(b_1 - 1)$. Since h has no poles it follows from the argument principle (see for example [9, Chapter VIII]) that

$$2\pi(b_1 - 1) = \int_\gamma \mathrm{d}\arg(h(z)) = 2\pi M \tag{4.7}$$

From this the claim follows. □

In the following we would like to give examples for harmonic vector fields. For this define two differential operators for $d = 2$ dimensions by

$$\nabla^\perp f = \operatorname{Curl} f = \begin{bmatrix} -\partial_2 f \\ \partial_1 f \end{bmatrix}$$

and

$$\operatorname{curl} u = -\partial_1 u_2 + \partial_2 u_1 \,.$$

Look into James Kelliher, stream functions for divergence free vector fields. Relation to differential forms.

The following proposition gives us a recipe to generate harmonic vector fields in $d = 2$ dimensions.

Proposition 4.2. *Let $\psi\colon \Omega \to \mathbb{R}$ be harmonic then $\nabla^\perp f$ is a harmonic vector field.*

Proof. Since $\operatorname{Div} \nabla^\perp = 0$ we have

$$\operatorname{Div} u = \operatorname{Div} \nabla^\perp \psi = 0$$

and one calculates

$$\operatorname{curl} u = \operatorname{curl} \nabla^\perp \psi = -\Delta \psi = 0 \,.$$

□

The function ψ is also called a *stream function*.

We now give an example of a harmonic vector field without inflow or outflow and with one stagnation point. For this consider the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + \Phi_2(x + e_1) \end{aligned} \tag{4.8}$$

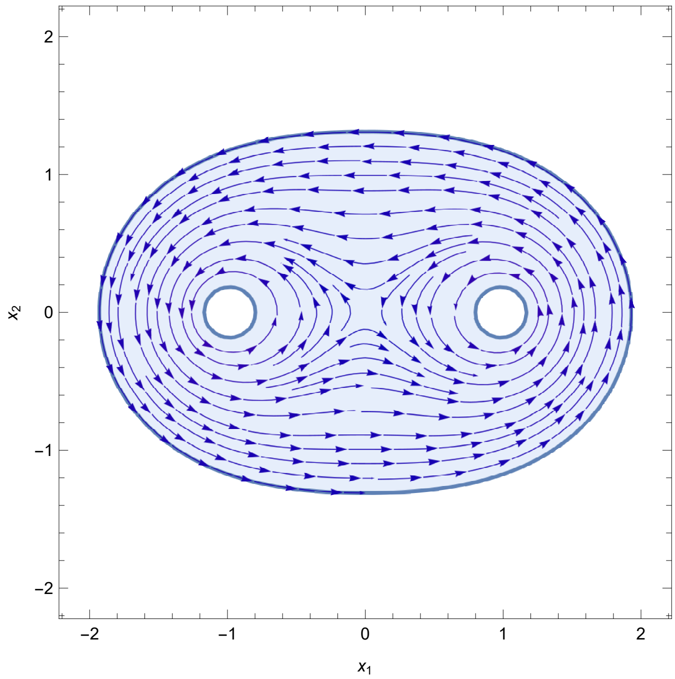


Figure 4.2: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-1,1])$. Here ψ is given by equation (4.8).

where

$$\Phi_2 = \log(|\cdot|)$$

is a multiple of the fundamental solution of the Laplace equation on $\mathbb{R}^2$ and e_i is the i-th unit vector. Figure 4.2 indicates that $u = \nabla^{\perp}\psi$ in the region $X = \psi^{-1}([-1,1])$ has the desired properties. Since ψ is constant on each component of ∂X the function u has no inflow or outflow. It follows from $\psi(-x) = \psi(x)$ that $u(-x) = -u(x)$ and thus the origin $x = 0$ is a stagnation point. By proposition 4.1 it is in fact the sole stagnation point of u on X.

In a second example from [10] we fix the domain rather than the function. For this set $X = \overline{B_4} \setminus (B_1(2e_1) \cup B_1(-2e_1))$ to be the domain. We then have the system

$$\begin{aligned} \Delta\psi &= 0 \quad &&\text{, on } \Omega \\ \psi &= 0 \quad &&\text{, on the outer ring } 4S^1 \\ \psi &= 1 \quad &&\text{, on the inner rings } S^1(-2e_1) \cup S^1(2e_1) \end{aligned} \tag{4.9}$$

We solve this system numerically and set $u = \nabla^{\perp}\psi$. The result is plotted in figure 4.3. Again it follows from symmetry that the origin is a stagnation point and from proposition 4.1 that it is in fact the sole stagnation point of u.

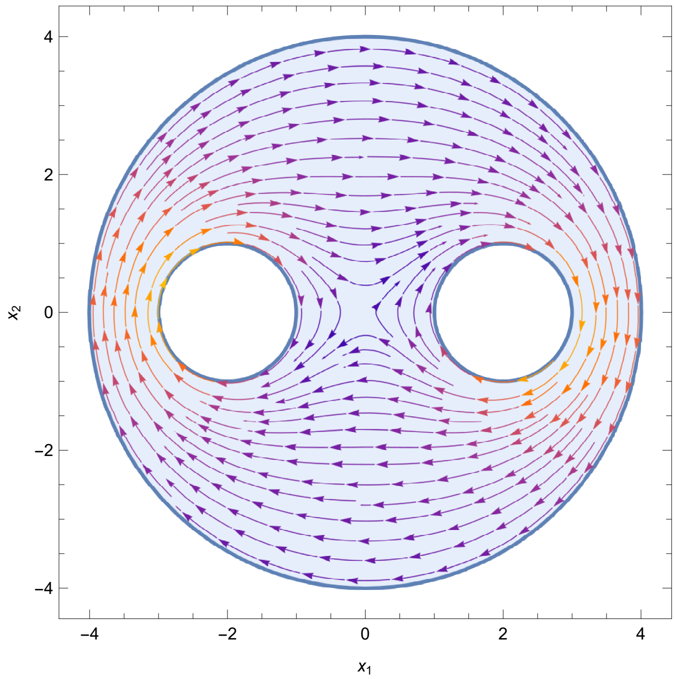


Figure 4.3: A plot of $u = \nabla^{\perp}\psi$ where ψ is the numerical solution to (4.9).

Example of inflow on one side and outflow on the other

In the following we aim to give examples of domains in $d = 2$ dimensions for which we have inflow on one simply connected boundary component and outflow on another simply connected boundary component with stagnation points. For this consider first the stream function

$$\begin{aligned} \psi\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) + x_1 \end{aligned} \tag{4.10}$$

Figure 4.4 indicates that $u = \nabla^{\perp}\psi$ fulfils the requirements on the domain $X = \psi^{-1}([-0.5, 2]) \cap \mathbb{R} \times [-2, 2]$. An elementary calculation reveals that the origin is a stagnation point of u.

Now we would like to have a harmonic vector field similar to the example with two holes with inflow on the one side and outflow on the other. For this consider the streamline

$$\begin{aligned} u\colon \mathbb{R}^2 \setminus \{-e_1, e_1\} &\to \mathbb{R} \\ x &\mapsto \Phi_2(x - e_1) - \Phi_2(x + e_1) + x_1 \end{aligned} \tag{4.11}$$

Figure 4.5 indicates that $u = \nabla^{\perp}\psi$ on the domain $X = \psi^{-1}([-0.7, 0.7]) \cap \mathbb{R} \times [-2, 2]$ is the function we are looking for.

In another example given by [10] we fix the domain rather than the function. Let $\Omega = B_4 \setminus (B_1(2e_1) \cup B_1(-2e_1))$

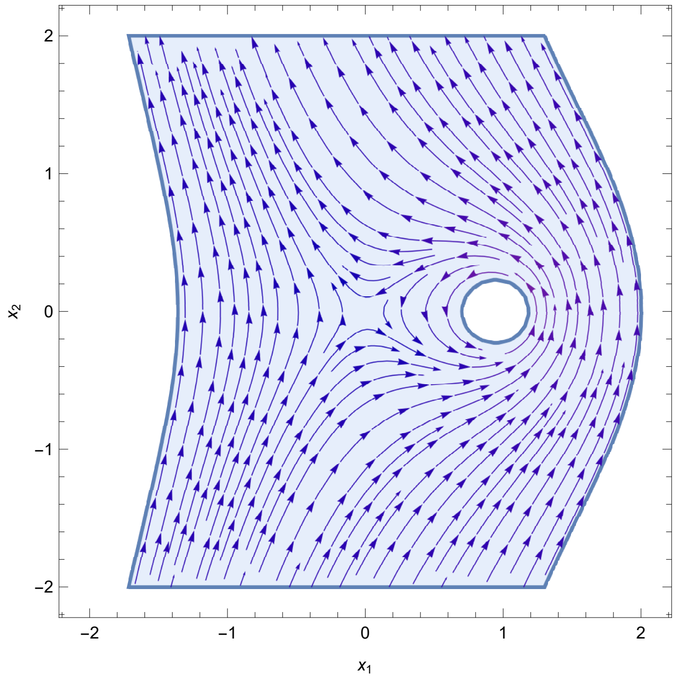


Figure 4.4: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.5,2]) \cap \mathbb{R} \times [-2,2]$. Here ψ is given by equation (4.10).

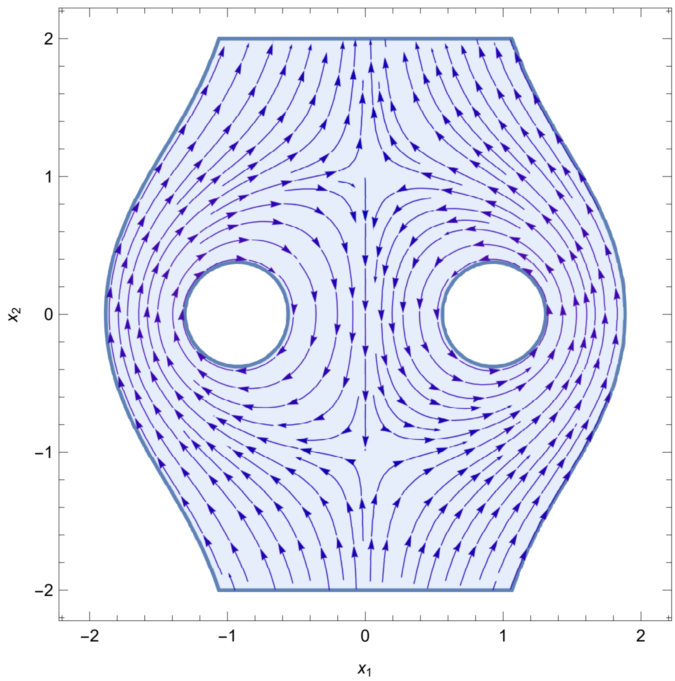


Figure 4.5: A plot of $u = \nabla^{\perp}\psi$ in the region $\psi^{-1}([-0.7,0.7]) \cap \mathbb{R} \times [-2,2]$. Here ψ is given by equation (4.11).

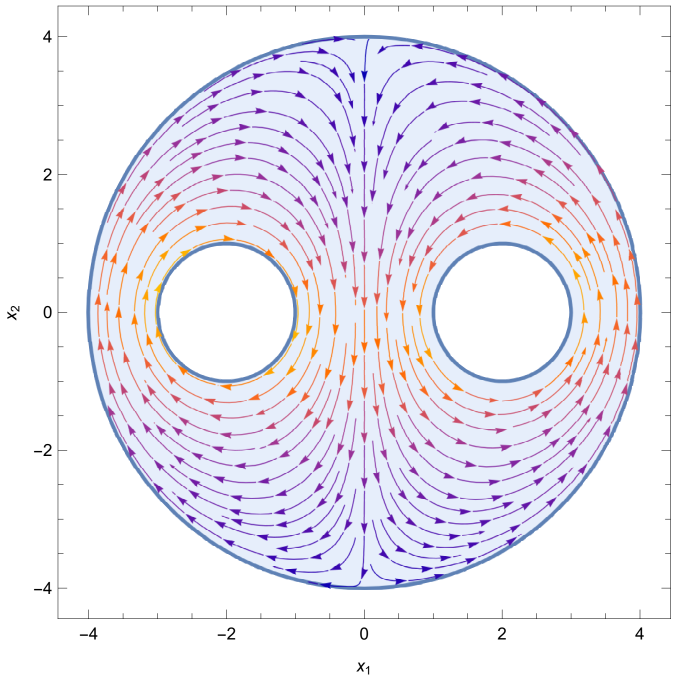


Figure 4.6: A plot of $u = \nabla^{\perp}\psi$ where ψ is the numerical solution to (4.12).

be the domain as before. We now have the system

$$\begin{aligned} \Delta\psi &= 0 && \text{, on } \Omega \\ \psi &= 0 && \text{, on the outer ring } 4S^1 \\ \psi &= -1 && \text{, on the left inner ring } S^1(-2e_1) \\ \psi &= 1 && \text{, on the right inner ring } S^1(2e_1) \end{aligned} \tag{4.12}$$

We solve this system numerically and set $u = \nabla^{\perp}\psi$. The result is plotted in figure 4.6. Here we obtain from the symmetry $\psi(-x_1, x_2) = \psi(x)$ that $\psi = 0$ on the x_2-axis. Since also $\psi = 0$ on $4S^1$ we have two stagnation points at $\pm 4e_2$.

5 Harmonic functions, $d = 3$

The cylinder

The following proof comes from [10]

Proposition 5.1. *Let $\Omega = (0,1) \times U \subseteq \mathbb{R}^3$ be an open cylinder where $U \subseteq \mathbb{R}^2$ is an open set. Let further $f\colon X = \overline{\Omega} \to \mathbb{R}$ be harmonic such that the sides $[0,1] \times \partial U = \Sigma^0$ are the tangential boundary, the lid $\{0\} \times U = \Sigma^+$ is the entrant boundary and the lid $\{1\} \times U = \Sigma^-$ is the emergent boundary. Then f cannot have an interior critical point.*

Proof. Assume not. Since

$$\Delta(\partial_1 f) = \partial_1(\Delta f) = 0$$

we have by the maximum principle that $\partial_1 f$ attains its minimum on the boundary Σ. Since $\partial_1 f(x) = 0$ for some interior point by assumption and $\partial_1 f > 0$ on the lids $\{x_1 = 0\} \cup \{x_1 = 1\}$ there exists a point $x \in (0,1) \times \partial U$ such that $\partial_1 f(x)$ is minimal on X. But then we have by Hopf's lemma that

$$0 < \nabla(\partial_1 f) \cdot n = \partial_1(\nabla f \cdot n) = 0\,,$$

a contradiction. □

Simply connected entrant boundary

Mimicking the proof in 2 dimensions we obtain the following proposition.

Proposition 5.2. *Let $X \subset \mathbb{R}^3$ be a compact manifold homeomorphic to the ball B such that Σ is a differentiable manifold. Let $f\colon X \to \mathbb{R}$ be a Morse harmonic function. Assume that Σ^- is simply connected. Then we have that*

$$M_1 - M_2 = 0\,.$$

Proof. As in the two dimensional case we split the domain Ω with a plane Γ such that $\partial\Gamma = \gamma \subseteq \Sigma^0$. Denote the two arising domains X^+ and X^- where $\partial X^+ \subseteq \Sigma^+ \cup \Sigma^0 \cup \overline{\Gamma}$ and $\partial X^- = \Sigma^- \cup \overline{\Gamma}$. We can assume that Γ as well as γ are smooth manifold. Since by proposition ?? there are finitely many critical points in Ω we can also assume that no interior critical points lie on Γ. Furthermore we assume that Γ is bent towards X^+ at γ. For the following argumentation we require that f is strictly Morse on both X^+ and X^- so assume for a moment that this is the case. Now we have that γ is diffeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Since f is non-degenerate the the number of maxima and minima of f on γ must be equal and thus

$$\mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) = \mathrm{Ind}_{\gamma^+,1}(f) + \mathrm{Ind}_{\gamma^-,0}(-f) \tag{5.1}$$

We now turn our attention to X^+. Since no essential critical points lie on Σ^+ it follows for the boundary type numbers that

$$\mu_k^+ = \mathrm{Ind}_{\Gamma^+,k}(f) + \mathrm{Ind}_{\gamma^+,k}(f)\,. \tag{5.2}$$

Analogously we have on X^- that

$$\nu_k^- = \mathrm{Ind}_{\Gamma^-,k}(-f) + \mathrm{Ind}_{\gamma^-,k}(-f)\,. \tag{5.3}$$

In addition we have that the emergent critical points on $\Gamma = \Gamma^+$ of f on X^+ are the entrant critical points on $\Gamma = \Gamma^-$ of $-f$ on X^-, that is

$$\begin{aligned} \mathrm{Ind}_{\Gamma^+,0}(f) &= \mathrm{Ind}_{\Gamma^-,2}(-f) \\ \mathrm{Ind}_{\Gamma^+,1}(f) &= \mathrm{Ind}_{\Gamma^-,1}(-f)\,. \\ \mathrm{Ind}_{\Gamma^+,2}(f) &= \mathrm{Ind}_{\Gamma^-,0}(-f) \end{aligned} \tag{5.4}$$

Using equations (5.2), (5.3) and (5.4) we obtain

$$\begin{aligned} \mu_0^+ - \nu_2^- &= \mathrm{Ind}_{\gamma^+,0}(f) \\ \mu_1^+ - \nu_1^- &= \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,1}(-f)\,. \\ \mu_2^+ - \nu_0^- &= -\mathrm{Ind}_{\gamma^-,0}(-f) \end{aligned} \tag{5.5}$$

We observe the Morse inequalities for f

$$M_2^+ + \mu_2^+ - M_1^+ - \mu_1^+ + \mu_0^+ = \chi(X^+) = \chi(X)\,. \tag{5.6}$$

and the Morse inequalities for $-f$

$$M_1^- + \nu_2^- - M_2^- - \nu_1^- + \nu_0^- = \chi(X^-) = \chi(X) \tag{5.7}$$

where the M_k continue to denote the interior type numbers of f. We now subtract equation (5.7) from (5.6) and insert relations (5.5) to obtain then with equation (5.1)

$$\begin{aligned} 0 &= M_1^- - M_2^- + M_1^+ - M_2^+ + \mathrm{Ind}_{\gamma^+,0}(f) + \mathrm{Ind}_{\gamma^-,1}(-f) - \mathrm{Ind}_{\gamma^+,1}(f) - \mathrm{Ind}_{\gamma^-,0}(-f) \\ &= M_1 - M_2 \end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that f is not strictly Morse on X^+ and X^-. In this case let f^ε for $\varepsilon \in E$ be a family of strictly Morse functions as in corollary 1.17. Since x_1, x_2 are non-degenerate critical points of f due to the slanted angle at which Γ approaches γ we obtain that

$$\mathrm{Ind}_{\gamma,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma,k}(-f) \tag{5.8}$$

By the same corollary we can assume that f^ε has no essential critical points on $\Sigma^+(f)$ and $-f^\varepsilon$ has no essential critical points on $\Sigma^-(f)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. $\square$

A harmonic function with interior critical point and simply connected entrant boundary

In fact we can give an example for such a function with simply connected entrant boundary.

Example 5.3 (A harmonic function with interior critical point and simply connected entrant boundary)**.** Consider the domain $X = \overline{B}_r \subseteq \mathbb{R}^3$ with $r > 0$ sufficiently large and the harmonic function

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto \frac{x_1^2}{2} - \frac{x_1^3}{3} - \frac{x_2^2}{2} + x_1 x_2^2 + x_2 x_3 \end{aligned}$$

This induces the harmonic vector field

$$\begin{aligned} u\colon X &\to \mathbb{R}^3 \\ x &\mapsto \begin{bmatrix} x_1(1-x_1)+x_2^2 \\ x_2(2x_1-1)+x_3 \\ x_2 \end{bmatrix} \end{aligned}$$

It follows from setting $u(x) = 0$ implies that $x_2 = 0$ and then that $x_3 = 0$ and $x_1 \in \{0, 1\}$. Thus we have that $x \in \{0, e_1\}$ are the sole possible zeroes of u. Conversely these are zeroes of u.

Figure 5.1 indicates that f has the desired properties.

Proposition 5.4. *The function given by equation* (**??**) *has simply connected emergent and entrant boundaries.*

Proof. Consider the inverse stereographic projection given by

$$\begin{aligned} \psi\colon \mathbb{R}^2 &\to B_r \setminus \{re_1\} \\ x &\mapsto \frac{r}{s^2+1} \begin{bmatrix} s^2-1 \\ x \end{bmatrix} \end{aligned}$$

where $s = |x|$. We calculate

$$rn \cdot u(x) = x_1^2(1-x_1) + x_2^2(3x_1-1) + 2x_2x_3$$

If we precompose with the inverse stereographic projection we obtain

$$\begin{aligned} &(rn \cdot u) \circ \psi(x) \\ &= \frac{r^2}{(s^2+1)^3}\Big(\big(s^2-1\big)^2\big(\big(s^2+1\big) - r\big(s^2-1\big)\big) + x_1^2\big(3r\big(s^2-1\big) - \big(s^2+1\big)\big) + 2x_1\big(s^2+1\big)x_2\Big) \end{aligned}$$

which vanishes precisely iff

$$\begin{aligned} 0 &= \frac{1}{r}\big(s^2-1\big)^2\big(\big(s^2+1\big) - \big(s^2-1\big)\big) + x_1^2\left(3\big(s^2-1\big) - \frac{1}{r}\big(s^2+1\big)\right) + \frac{1}{r}2x_1\big(s^2+1\big)x_2 \\ &= \frac{1}{r}\big(s^2+1\big)\Big(\big(s^2-1\big)^2 - x_1^2\Big) - \big(s^2-1\big)^3 + 3\big(s^2-1\big)x_1^2 + \frac{1}{r}2x_1x_2\big(s^2+1\big) \end{aligned}$$

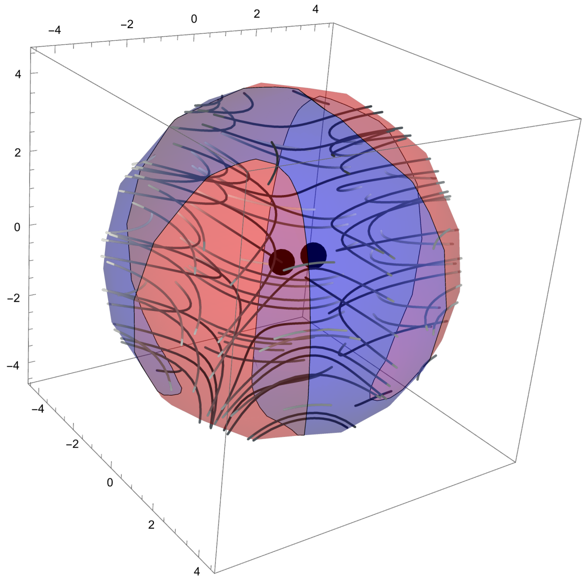


Figure 5.1: A plot of the function u

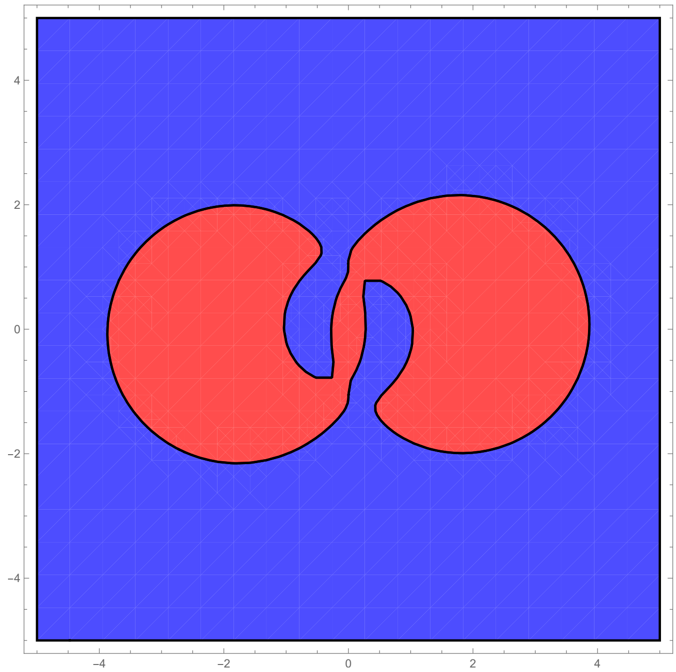


Figure 5.2: Stereographic projection of the surface Σ.

Writing $\varepsilon = 1/r$ and setting

$$p^{\varepsilon}(x) = \varepsilon\Big(\big(s^2+1\big)\Big(\big(s^2-1\big)^2 - x_1^2\Big) + 2x_1x_2\big(s^2+1\big)\Big) - \big(s^2-1\big)^3 + 3\big(s^2-1\big)x_1^2$$

we see that this is equivalent to the condition $p^{\varepsilon}(x) = 0$. Our aim is to show that the real planar variety defined in this way is compact and has precisely one connectivity component. It then follows from the Jordan curve theorem that this variety splits the plane into two components, namely one where p^{ε} is positive and one where it is negative. The image of these sets under the stereographic projection is then precisely Σ^+ and Σ^- respectively and it follows that these sets are simply connected.

$\square$

Proposition 5.5. *There exists an $\varepsilon > 0$ such that the planar variety $p^{\varepsilon}(x)$ given by equation* (**??**) *has one connectivity component.*

Proof. By proposition 5.6 we have that $V\left(p^0\right)$ consists of six arcs which all meet at the points e_2 and $-e_2$. The set $V\left(p^0\right)$ is sketched in figure **??**. We will argue that the corresponding figure for a small enough ε is given by **??**. Away from the singularities at $\pm e_2$ we obtain the result by proposition **??** and picking ε small enough. By symmetry of p^{ε}, that is we have

$$p^{\varepsilon}(-x) = p^{\varepsilon}(x) \tag{5.9}$$

it suffises to inspect what happens around the point e_2 as $\varepsilon \to 0$.

Now define the polynomial q by

$$p^{\varepsilon}(x) = \varepsilon q(x) + p^0(x)\,. \tag{5.10}$$

For simplicity we would like to move the singular point to the origin so we give

$$p^0(x-e_2) = (x_1^2+x_2^2-2x_2)(x_1^4+2x_1^2x_2^2+x_2^4-4x_1^2x_2-4x_2^3-3x_1^2+4x_2^2)$$

and

$$q(x-e_2) = (x_1^2+x_2^2-2x_2+2)(2x_1x_2+x_2^2-2x_1-2x_2)$$

By Tougeron's theorem there exists a neighbourhood U around the origin such that

$$p\colon\, (\varepsilon,x) \mapsto p^{\varepsilon}(x) \tag{5.11}$$

continue this argument

Taking the homogeneous parts of lowest order we obtain

$$h_3(x) = \big(p^0(x-e_2)\big)_3 = 2x_2(3x_1^2-4x_2^2)$$

and

$$h_1(x) = (q(x - e_2))_1 = -4(x_1 + x_2)\,.$$

In proposition 5.9 it will be shown that the polynomial

$$h^{\varepsilon}(x) = \varepsilon h_1(x) + h_3(x)\,. \tag{5.12}$$

has a variety of the shape given in figure 5.5c Moreover $V(h^{\varepsilon})$ converges to $V\left(h^0\right)$ with respect to the Hausdorff metric as $\varepsilon \to 0$. Now

continue proof

□

Proposition 5.6 (The set $V\left(p^0\right)$). *The set $V\left(p^0\right)$ where p^0 given by equation* (**??**) *consists of six arcs which all meet at the points $\pm e_2$ and nowhere else.*

Proof.

Rewrite: It should be much easier arguing in $\mathbb{R}^3$.

From the factorisation

$$p^0(x) = \underbrace{(s^2-1)}_{=q_1(x)} \underbrace{\left(\left(s^2-1\right)^2 - 3x_1^2\right)}_{=q_2(x)}$$

we see that

$$V\left(p^0\right) = V(q_1) \cup V(q_2) = S^1 \cup V(q_2)$$

We claim that the set of points $V(q_2)$ consists of four curves intersecting the circle S^1 precisely at the points $\pm e_2$ and that there are no other singular points. From this the claim will then follow. To show this take a closer look at

$$q_2(x) = \left(s^2-1\right)^2 - 3x_1^2\,.$$

Substituting $\tilde{x}_k = x_k^2$ we can write this as

$$\begin{aligned} q_2(x) &= (\tilde{x}_1 + \tilde{x}_2 - 1)^2 - 3\tilde{x}_1 \\ &= \tilde{x}_1^2 + 2\tilde{x}_1\tilde{x}_2 + \tilde{x}_2^2 - 5\tilde{x}_1 - 2\tilde{x}_2 + 1 \\ &= (\tilde{x}_1 + \tilde{x}_2)^2 - 5\tilde{x}_1 - 2\tilde{x}_2 + 1 \\ &= \tilde{q}_2(\tilde{x})\,. \end{aligned}$$

Now the equation

$$0 = \tilde{q}_2(\tilde{x})$$

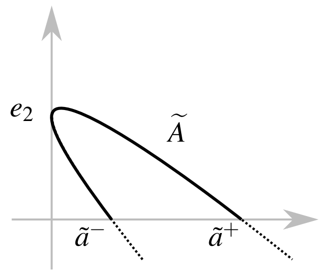


Figure 5.3: The arc $\widetilde{A}$.

is that of a conic section. Since we are assuming $\tilde{x} \in \mathbb{R}^2_{\geq 0}$ we obtain the connected arc $\widetilde{A}$ in the upper right quadrant intersecting the $\tilde{x}_1$-axis at the points

$$\tilde{a}^{\pm} = \begin{bmatrix} \frac{5 \pm 2\sqrt{6}}{2} & 0 \end{bmatrix}^\top$$

and touching the $\tilde{x}_2$-axis at the point e_2. The arc $\widetilde{A}$ is shown in figure 5.3. Since there are only two points where the curve intersects $\partial\mathbb{R}^2_{\geq 0}$ and the curve does not leave this quadrant at infinity it follows that $\tilde{C}$ is a connected compact arc. Define the mappings

$$ {}^{\circ_1\circ_2}\!\!\sqrt{\cdot}\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2 \tag{5.13}$$

$$\tilde{x} \mapsto x = \begin{bmatrix} \circ_1\sqrt{\tilde{x}_1} & \circ_2\sqrt{\tilde{x}_2} \end{bmatrix}^\top \tag{5.14}$$

for signs $\circ_1, \circ_2 \in \{+, -\}$. Now taking the image of the arc $\widetilde{A}$ under this square root we obtain four arcs

$$A^{\circ_1\circ_2} = \left\{ {}^{\circ_1\circ_2}\!\!\sqrt{\tilde{x}} \colon \tilde{x} \in \widetilde{A} \right\}$$

for signs $\circ_1, \circ_2 \in \{+, -\}$. Note that by construction the arc $A^{\mathrm{I}} = A^{++}$ lies in the first quadrant and analogously $A^{\mathrm{II}} = A^{-+}$, $A^{\mathrm{III}} = A^{--}$ and $A^{\mathrm{IV}} = A^{+-}$ lie in the second, third and fourth quadrants. Now define four points on the x_1-axis by

$$a^{\circ_1\circ_2} = {}^{\circ_1 +}\!\!\sqrt{a^{\circ_2}}\,. \tag{5.15}$$

We see that $A^{\mathrm{I}} \cap A^{\mathrm{IV}} = \{a^{++}, a^{+-}\}$ and $A^{\mathrm{II}} \cap A^{\mathrm{III}} = \{a^{--}, a^{-+}\}$. We also see that $A^{\mathrm{I}} \cap A^{\mathrm{II}} = \{e_2\}$ and $A^{\mathrm{III}} \cap A^{\mathrm{IV}} = \{-e_2\}$ Since the origin is not contained in any arc it thus follows that $A^{\mathrm{I}} \cap A^{\mathrm{III}} = \emptyset$ and $A^{\mathrm{II}} \cap A^{\mathrm{IV}} = \emptyset$. Thus the union

$$A = A^{++} \cup A^{+-} \cup A^{-+} \cup A^{--} \tag{5.16}$$

consists of four arcs which meet at the points $\pm e_2$ and hence also $V(p_2) = A$.

insert picture here, remove quadrant notation.

It now remains to be shown that $A \cap S^1 = \{\pm e_2\}$. For this let $x \in S^1$ such that

$$0 = q_2(x) \tag{5.17}$$

but then this claim follows immediately from $q_2(x) = -3x_1^2$.

□

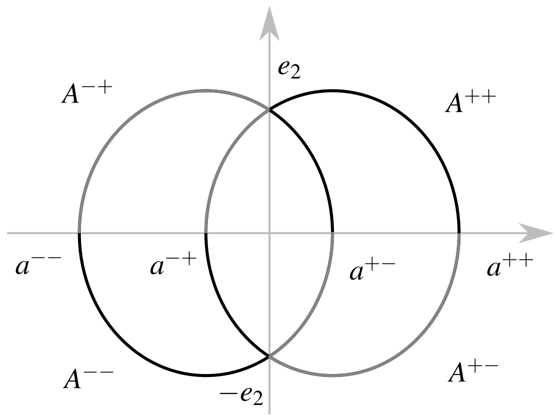


Figure 5.4: Sketch of the set $V(q_2)$.

Proposition 5.7. *For $\varepsilon > 0$ sufficiently small the variety $V(p^\varepsilon)$ is smooth.*

Proof. We inspect the equations

$$P(x) = -x_1^3 + x_1x_2^2 + x_1^2 - x_2^2 + 2x_2x_3 \tag{5.18}$$

and

$$Q(x) = x_1^2 + x_2^2 + x_3^2 - r^2 \tag{5.19}$$

closer. One calculates

$$T = \begin{bmatrix} \nabla P(x) & \frac{1}{2}\nabla Q(x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 + x_2^2 + 2x_1 & x_1 \\ 2x_1x_2 - 2x_2 + 2x_3 & x_2 \\ 2x_2 & x_3 \end{bmatrix} \tag{5.20}$$

By the Jacobi criterion it is sufficient to show that this matrix is of full rank on $V(P,Q)$. This is equivalent to showing that

$$0 \neq \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = -5x_1^2x_2 + x_2^3 + 4x_1x_2 - 2x_1x_3 = R_1(x)\,, \tag{5.21}$$

$$0 \neq \det \begin{bmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} = 2x_1x_2x_3 - 2x_2x_3 + 2x_3^2 - 2x_2^2 = R_2(x) \text{ or} \tag{5.22}$$

$$0 \neq \det \begin{bmatrix} T_{31} & T_{32} \\ T_{11} & T_{12} \end{bmatrix} = 2x_1x_2 + 3x_1^2x_3 - x_2^2x_3 - 2x_1x_3 = R_3(x) \tag{5.23}$$

for any $x \in V(P,Q)$. This in turn is equivalent to showing that

$$V(P,Q,R_1,R_2,R_3) = \emptyset\,. \tag{5.24}$$

□

In the following we need a notion of convergence of subsets on a metric space. We define the *Hausdorff metric* for two sets $A, B \subseteq X$ to be given by

$$d_H(A,B) = \max\left\{ \sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A) \right\} \tag{5.25}$$

where

$$\operatorname{dist}(x,B) = \inf_{y\in B} d(x,y) \tag{5.26}$$

is the smallest distance from x to B. We also require the definition of tubular neighbourhoods. For a given $\delta > 0$ and a subset $A \subseteq \mathbb{R}^d$ we call the union of δ balls

$$\operatorname{Tub}_\delta(A) = \bigcup_{x\in A} B_\delta(x) \tag{5.27}$$

a *tubular neighbourhood* of A. We have the following proposition.

Proposition 5.8 (Convergence at non-singular points)**.** *Let U be an open bounded set such that p is non-singular on $\overline{U}$ Then there exists a $\delta > 0$ such that for all $\varepsilon < \delta$ we have that the Hausdorff distance satisfies $d_H\left(V_U(p^\varepsilon), V_U\left(p^0\right)\right) < \delta$ and $V_U(p^\varepsilon)$ is homotopic to $V_U\left(p^0\right)$.*

Proof. Since p is non-singular on $\overline{U}$ it follows that $D^2 p^0$ is bijective on every point on $\overline{U}$. By the implicit function theorem there exists around every point $x \in \overline{U}$ a neighbourhood U_x, a $\delta_x > 0$ and a mapping $h\colon B_{\delta_x} \times V_{U_x}\left(p^0\right) \to U_x$ such that h_ε parametrises $V_{U_x}(p^\varepsilon)$. Since $\overline{U}$ is compact we can choose finitely many such x, pick δ to be the minimum of all δ_x. By the pasting lemma we can then construct our desired homotopy h.

fill in the details.

□

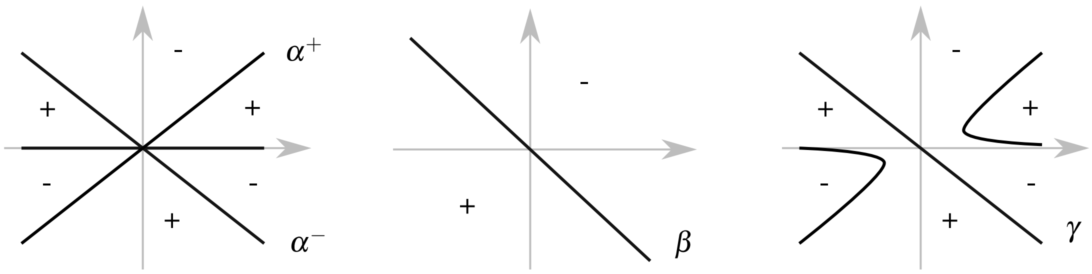
Proposition 5.9. *The variety given by h has three arcs arranged as in figure 5.5c.*

Proof. We start by showing that the set $V(h^\varepsilon)$ has no singular points. By Jacobi's formula we can calculate the singular points by determining the critical points of h^ε. Hence we calculate

$$0 = \nabla h^\varepsilon(x) = 2\begin{bmatrix} 6x_1x_2 - 2\varepsilon \\ 3x_1^2 - 12x_2^2 - 2\varepsilon \end{bmatrix} \tag{5.28}$$

Now x must also fulfil the condition

$$\begin{aligned} 0 &= h^\varepsilon(x) && (5.29)\\ &= 2x_2\left(3x_1^2 - 4x_2^2\right) - 4\varepsilon(x_1 + x_2) && (5.30)\\ &= 6x_2x_1^2 - 4\varepsilon x_1 - 8x_2^3 - 4\varepsilon x_2 && (5.31) \end{aligned}$$



(a) Sketch of the set $V(h_3)$. (b) Sketch of the set $V(h_1)$. (c) Sketch of the set $V(h^\varepsilon)$.

Figure 5.5: Sketches of varieties.

This set of equations has no solution in $\mathbb{R}^2$ and thus all points in $V(h^\varepsilon)$ are non-singular. Now consider the derivative at the origin It now Hence h^ε has no singular points and hence the six arcs have to join in a manner similar to figure 5.5a. The behavior of h^ε at infinity is determined by the higher order terms and thus by the function h_3. Now h_3 is a monkey saddle and thus there are six rays emanating to infinity. From the formula one sees that these are the x_1-axis and the lines given by

$$\alpha^\pm = \left\{ t \begin{bmatrix} \pm 2 & \sqrt{3} \end{bmatrix}^\top : t \in \mathbb{R} \right\} \tag{5.32}$$

Now the arc

$$\beta = \left\{ t \begin{bmatrix} 1 & -1 \end{bmatrix}^\top : t \in \mathbb{R} \right\} \tag{5.33}$$

is the zero set of h_1. We now look at the second quadrant. Here we have that $h^\varepsilon = h_3 \leq 0$ on β and that $h^\varepsilon = h_1 \geq 0$ on α^-. By the intermediate value theorem thus there must be an arc $\gamma \subseteq V(h^\varepsilon)$ starting at the origin and sandwiched in this quadrant between α^- and β and tending to α^- at $-\infty$. An analogous argumentation on the third quadrant shows that here too there must be an arc γ starting at the origin and tending to ∞. Denote the union of these two arcs by γ. Then γ divides the plane into two parts and since h^ε has no singular points it follows that the zeroes of h^ε must look similar to figure 5.5c. $\square$

For a polynomial $p \in \mathbb{R}[x_1, x_2]$ and a set $U \subseteq \mathbb{R}^2$ we denote the set of zeroes of p by

$$V_U(p) = (x \in U : p(x) = 0)$$

We call the points at which multiple arcs meet *singular*.

complete this section.

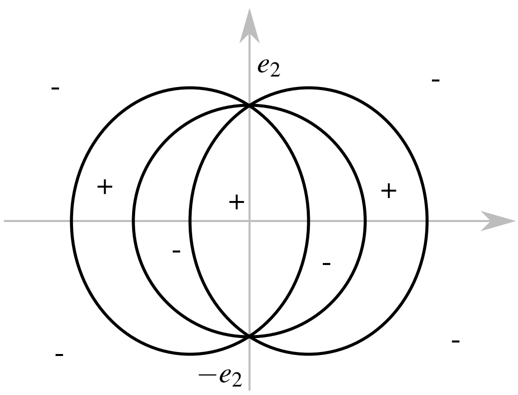


Figure 5.6: A sketch of the set $V\left(p^0\right)$.

6 Harmonic vector fields, $d = 3$

No inflow or outflow

We obtain as a quick consequence of the hairy ball theorem

Proposition 6.1. *Let Ω have Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without inflow or outflow. Then we have*

$$b_2 \leq b_1 .$$

Proof. Assume not. Since Ω has b_2 bubbles and b_1 holes there exists by the pigeon hole principle a bubble $\Gamma \subseteq \Sigma$ without a hole. Since u has no inflow or outflow on Γ we have that the restriction $u|_\Gamma \in T\Gamma$ is a vector field on Γ. Since u is regular $u|_\Gamma$ does not vanish. But Γ is homeomorphic to the Ball in contradiction to the hairy ball theorem. $\square$

A little more rigour would not harm.

We also obtain the following result:

Proposition 6.2. *Let $X \subseteq \mathbb{R}^3$ be a differentiable manifold with Betti numbers b_0, b_1 and b_2. Let $u\colon X \to \mathbb{R}$ be a Morse harmonic vector field without inflow or outflow. Then we have the following relation for the interior type numbers of u*

$$M_2 = M_1 .$$

Proof. As in the two dimensional case we cut the domain X with planes Γ such that the slit domain is homeomorphic to a ball with bubbles. Since the number of stagnation points is finite by proposition ??, we can choose Γ in such a way that it does not contain any stagnation points. We also denote the curves at which Γ meets Σ by $\gamma_1, \dots, \gamma_{b_1} \subseteq \partial\Gamma$. Note that there are b_1 many such curves. We can assume that Γ and the γ_j are smooth manifolds and that Γ approaches each γ_j at a slanted angle. The cut now yields a new domain $\tilde{X}$ which is a covering space of X. On this covering space we denote the cover of the cut Γ and the sets γ_j by Γ^i and γ_j^i with $i \in (1,2)$. Since this new domain $\tilde{X}$ is homeomorphic to a ball with bubbles by proposition 2.10 the vector field u is the gradient of a harmonic function f. For the following argumentation we require that u is strictly Morse on $\tilde{X}$. Now we have that each γ_j is diffeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$. Since f is non-degenerate the number of maxima and minima of f on $\gamma_j^1 \cup \gamma_j^2$ must be equal and thus

$$\sum_i \Big(\mathrm{Ind}_{\gamma_j^i,0}(f) + \mathrm{Ind}_{\gamma_j^i,1}(-f)\Big) = \sum_i \Big(\mathrm{Ind}_{\gamma_j^i,1}(f) + \mathrm{Ind}_{\gamma_j^i,0}(-f)\Big) . \tag{6.1}$$

Since on Γ all entrant stagnation points of u are also emergent stagnation points of $-u$ (and vice versa) we have the relations

$$\begin{aligned}
\mathrm{Ind}_{\Gamma^1,0}(\pm u) &= \mathrm{Ind}_{\Gamma^2,2}(\mp u)\\
\mathrm{Ind}_{\Gamma^1,1}(\pm u) &= \mathrm{Ind}_{\Gamma^2,1}(\mp u)\\
\mathrm{Ind}_{\Gamma^1,2}(\pm u) &= \mathrm{Ind}_{\Gamma^2,0}(\mp u)\,.
\end{aligned} \tag{6.2}$$

Since there are no boundary critical points on Σ^0 it follows for the boundary type numbers that

$$\begin{aligned}
\mu_k &= \sum_i \left(\mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma^i_j,k} \right)(f)\\
\nu_k &= \sum_i \left(\mathrm{Ind}_{\Gamma^i,k} + \sum_j \mathrm{Ind}_{\gamma^i_j,k} \right)(-f)\,.
\end{aligned} \tag{6.3}$$

Equations (6.3) and (6.2) yield

$$\begin{aligned}
\mu_0 - \nu_2 &= \sum_{i,j} \mathrm{Ind}_{\gamma^i_j,0}(f)\\
\mu_1 - \nu_1 &= \sum_{i,j} \Big(\mathrm{Ind}_{\gamma^i_j,1}(f) - \mathrm{Ind}_{\gamma^i_j,1}(-f) \Big)\\
\mu_2 - \nu_0 &= -\sum_{i,j} \mathrm{Ind}_{\gamma^i_j,0}(-f)
\end{aligned} \tag{6.4}$$

Since Ω is now simply connected u is by proposition 2.10 the gradient of a harmonic function f on this new domain. For this f we have the Morse inequalities

$$M_2 + \mu_2 - M_1 - \mu_1 + \mu_0 = -\chi(\tilde{X}) \tag{6.5}$$

and for $-f$ the Morse inequalities

$$M_1 + \nu_2 - M_2 - \nu_1 + \nu_0 = -\chi(\tilde{X})\,. \tag{6.6}$$

Subtracting equation (6.6) from (6.5) and using the relation (6.4) we obtain together with equation (6.1)

$$\begin{aligned}
0 &= 2(M_2 - M_1) + \sum_{i,j} \Big(\mathrm{Ind}_{\gamma^i_j,0}(f) - \mathrm{Ind}_{\gamma^i_j,1}(f) + \mathrm{Ind}_{\gamma^i_j,1}(-f) - \mathrm{Ind}_{\gamma^i_j,0}(-f) \Big)\\
&= 2(M_2 - M_1)
\end{aligned}$$

from which the claim follows.

The claim remains to be shown in the case that f is not strictly Morse on X^+ and X^-. In this case let f^ε for $\varepsilon \in E$ be a family of strictly Morse functions as in corollary 1.17. Since x_1, x_2 are non-degenerate critical points of f due to the slanted angle at which Γ approaches each γ_j we obtain that

$$\mathrm{Ind}_{\gamma_j,k}(f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(f) \qquad \text{and} \qquad \mathrm{Ind}_{\gamma_j,k}(-f^\varepsilon) = \mathrm{Ind}_{\gamma_j,k}(-f) \tag{6.7}$$

By the same corollary we can assume that f^ε has no critical points on $\Sigma^0(f)$. The claim then follows by the calculations above where we replace f with f^ε and then note that $M_1^\varepsilon = M_1$ and $M_2^\varepsilon = M_2$. □

7 Harmonic functions, $d = 4$

Define the harmonic function

$$\begin{aligned} f\colon B_1 \subseteq \mathbb{R}^4 &\to \mathbb{R} \\ x &\mapsto x_1^2 + x_2^2 - x_3^2 - x_4^2 \,. \end{aligned}$$

This has a stagnation point at the origin. We now claim that the sets Σ^+ and Σ^- are both simply connected, i.e. we have a tube in $\mathbb{R}^4$ with throughflow and a stagnation point.

Proof. To prove this claim we observe that the boundary ∂B_1 can be parametrised by the coordinates $\bar{x} = (x_2, x_3, x_4)$ for which we have $|\bar{x}| \leq 1$. By the condition

$$\sum_i x_i^2 = 1 \tag{7.1}$$

on the boundary ∂B_1 we have that x_1 is then uniquely determined up to sign. Thus we have have defined parametrisations

$$\begin{aligned} \phi_{\pm}\colon B_1 \subseteq \mathbb{R}^3 &\to \mathbb{R} \\ \bar{x} &\mapsto x \text{ such that } \pm x_1 \geq 0 \end{aligned} \tag{7.2}$$

with inverses $\psi_{\pm} = (\phi_{\pm})^{-1}$. We now calculate the gradient of f

$$\nabla f = 2 \begin{bmatrix} x_1 & x_2 & -x_3 & -x_4 \end{bmatrix}^\top$$

and the normal to ∂B_1

$$n = \begin{bmatrix} x_1 & \cdots & x_4 \end{bmatrix}^\top .$$

Thus we have $x \in \Sigma^{\pm}$ iff

$$0 < \pm \nabla f \cdot n = \pm 2\big(x_1^2 + x_2^2 - x_3^2 - x_4^2\big)$$

Using condition (7.1) we obtain the equivalent condition

$$0 < \pm 1 - 2\big(x_3^2 + x_4^2\big)$$

Define the cylinder

$$C = \left\{ \bar{x} \in \mathbb{R}^3 \colon x_3^2 + x_4^2 < 1/2 \right\} = \mathbb{R} \times B_{1/\sqrt{2}}$$

If we return to our parametrisation (7.2) we see that we have $\bar{x} \in B_1 \cap C$ iff $\phi_\pm(x) \in \Sigma^+$ and hence

$$B_1 \cap C = \psi_\pm\left(\Sigma^+\right).$$

Analogously we have

$$B_1 \setminus C = \psi_\pm\left(\Sigma^-\right).$$

The claim then follows from the fact that ϕ is a homeomorphism onto its image and $x_1 = 0$ is equivalent to $\bar{x} \in \partial B_1 \subseteq \mathbb{R}^2$. The situation is depicted in figure 7.1.

Check that the transition at the boundary is legal.

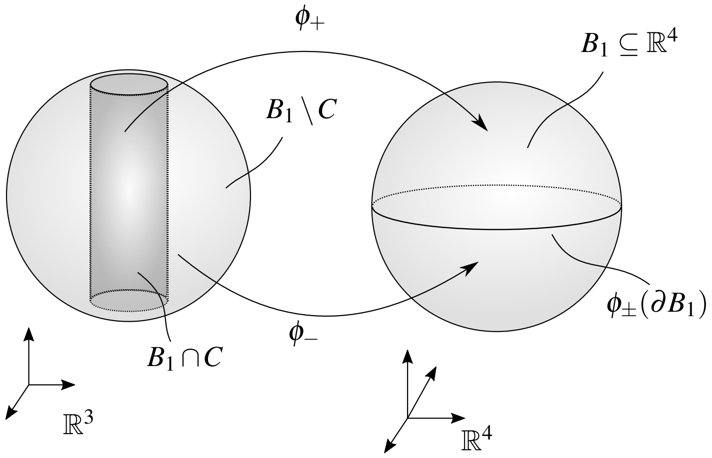


Figure 7.1: Visualisation of the situation.

$\square$

Symbols

d	Dimensions $d = 2$ or $d = 3$
X	Compact domain in $\mathbb{R}^d$, often assumed to be a manifold with corners.
Ω	Interior $\Omega = \text{int}(X)$
$f: X \to \mathbb{R}$	A harmonic function.
$u: X \to \mathbb{R}^d$ or T^*X	A harmonic vector field.
X_j	A stratification of X as given in definition 1.4. Often but not always assumed to be given by equation (1.1)
Σ	Boundary $\Sigma = \partial X$
Σ^-, $\Sigma^{\leq 0}$	(strictly) entrant boundary. See definition 1.7.
Σ^+, $\Sigma^{\geq 0}$	(strictly) emergent boundary. See definition 1.7.
Σ^0	tangential boundary. See definition 1.7.
$B_r(x)$, B_r	Ball of radius r around the point x / the origin.
$S^{d-1}(x)$, S^{d-1}	$(d-1)$-dimensional sphere around x / the origin.
u_j	Projection of u to the cotangent bundle T^*X_j. See equation (1.6).
π_j	Orthogonal projection onto the cotangent bundle T^*X_j. See equation (1.5).
Cr_j	Number of essential stagnation points. See definition 1.9.
$\text{Ind}_{j,k}$	k-th type number on the stratum X_j. See equation (1.7).
Ind_k	k-th type number. See equation (1.11).
M_k	k-th interior type numbers. See equation (1.8).
M	Total number of stagnation points. See equation (1.9).
μ_k	k-th boundary type numbers of f. See equation (1.10).
ν_k	k-th boundary type numbers of $-f$. See definition 1.9.
u_ε	modification to u as in equation (1.12)
A	submanifold, can be thought of as the zero section of T^*X
b_k	Betti number as defined in equation (2.2)
$d_H(A,B)$	Hausdorff metric. See equation (5.25).
$\text{dist}(x,A)$	Distance between x and A. See equation (5.26)
$\text{Tub}_\delta(A)$	Tubular neighbourhood of A. See equation (5.27).

Change Gamelin to Lang, complex analysis

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