Project Report for Seminar Course in Numerical Analysis, VT23

Junzi Zhang, Brendan O'Donoghue, Stephen Boyd: Globally Convergent Type-I Anderson Acceleration for Non-Smooth Fixed-Point Iterationsy

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Proof. The proof follows [1][Theorem 6]. We partition $\mathbb{N} = K_{AA} \sqcup K_{KM}$ where $K_{AA} = \{k_0, k_1, \dots\}$ denote the indices k where the algorithm chose an AA-step (a) and $K_{KM} = \{l_0, l_1, \dots\}$ where the algorithm chose a KM-step (b).

if $\|g_k\| \leq D\bar{U}(n_{AA}+1)^{-(1+\varepsilon)}$ **then**
    Set $x_{k+1} = \tilde{x}_{k+1}$ and $n_{AA} = n_{AA}+1$. (a)
else
    Set $x_{k+1} = (1-\alpha)x_k + \alpha f(x_k)$. (b)
end

Algorithm 1: The two cases for x_{k+1}.

Let y be a fixed point. We distinguish

case (1) $k \in K_{AA}$ then

$$\begin{aligned} \|x_{k+1} - y\| &\leq \|x_k - y\| + \|H_k g_k\| \\ &\leq \|x_k - y\| + c_1 \|g_k\| \\ &\leq \|x_k - y\| + c_2 (k+1)^{-(1+\varepsilon)} \end{aligned} \tag{1}$$

case (2) $k \in K_{KM}$ then (motivate this)

$$\|x_{k+1} - y\|^2 \leq \|x_k - y\|^2 - \alpha(1-\alpha)\|g_k\|^2 \leq \|x_k - y\|^2 \tag{2}$$

Hence in any case

$$\begin{aligned} \|x_k - y\| &\leq \|x_0 - y\| + \sum_{l=0}^{k-1} \|x_{l+1} - x_l\| \\ &\leq \|x_0 - y\| + c_2 \sum_k (k+1)^{-(1+\varepsilon)} = c_3 < \infty. \end{aligned}$$

It then follows that

$$\begin{aligned} a_{k+1} &= \|x_{k+1} - y\|^2 \\ &\overset{(1),(2)}{\leq} \underbrace{\|x_k - y\|^2}_{=a_k} + \underbrace{c_2^2 (k+1)^{-2(1+\varepsilon)} + 2c_2 \underbrace{\|x_k - y\|}_{\leq c_3} (k+1)^{-(1+\varepsilon)}}_{=b_k} \\ &= a_k + b_k \end{aligned} \tag{3}$$

and hence

$$\alpha(1-\alpha)\sum_i \|g_{l_i}\|^2 \overset{(2)}{\leq} \sum_i a_{l_i} - a_{l_i+1} \overset{(3)}{\leq} a_0 + \sum_k b_k < \infty$$

We therefore have $\lim_i \|g_{l_i}\| = 0$. It also follows from $\|g_{k_i}\| \leq D\bar{U}(i+1)^{-(1+\varepsilon)}$ that $\lim_i \|g_{k_i}\| = 0$. Thus indeed $\lim_k \|g_k\| = 0$.

(part 2) Let now n_j and $N_j \geq n_j$ be such that

$$a_{n_j} \xrightarrow{j\to\infty} \liminf_k a_k = \underline{a}$$

$$a_{N_j} \xrightarrow{j\to\infty} \limsup_k a_k = \overline{a}$$

Then it follows that

$$\overline{a} - \underline{a} \xleftarrow{n_j\to\infty} \overline{a} - a_{n_j} \xleftarrow{N_j\to\infty} a_{N_j} - a_{n_j} = \sum_{k=n_j}^{N_j-1} a_{k+1} - a_k \leq \sum_{k=n_j}^{\infty} b_k \xrightarrow{n_j\to\infty} 0$$

so

$$\limsup_k a_k = \overline{a} \leq \underline{a} = \liminf_k a_k$$

and thus $a_k = \|x_k - y\|$ converges to some b.

(part 3) Let k_j and l_j be convergent subsequences of x_k convergent against y_1 and y_2 respectively. Since by continuity of g

$$\|g(y_1)\| = \lim_j \|g(x_{k_j})\| = 0$$

we have that y_1 is a fixed point and y_2 too. Now

$$\|y_1\| \xleftarrow{j\to\infty} \|x_{k_j}\|^2 = \|x_k - y\|^2 + \|y\|^2 + 2y^\top x_{k_j} \xrightarrow{j\to\infty} b^2 + \|y\|^2 + 2y^\top y_1$$

and analogously for y_2. Thus

$$\|y_i\| = b^2 + \|y\|^2 + 2y^\top y_i$$

which implies

$$2y^\top (y_1 - y_2) = \|y_1\|^2 - \|y_2\|^2$$

It then follows from $y \in \{y_i\}_i$ that

$$y_1^\top (y_1 - y_2) = y_2^\top (y_1 - y_2)$$

and further

$$(y_1 - y_2)^\top (y_1 - y_2) = 0$$

and thus $y_1 = y_2$. We have shown that two convergent subsequences have the same limit and hence x_k is convergent and the solution must be a fixed point of f. $\square$

Sources

Bibliography

[1] J. Zhang, B. O'Donoghue, and S. Boyd, "Globally convergent type-I Anderson acceleration for nonsmooth fixed-point iterations," *SIAM J. Optim.*, vol. 30, no. 4, pp. 3170–3197, 2020, ISSN: 1052-6234. DOI: `10.1137/18M1232772`. [Online]. Available: `https://doi-org.ludwig.lub.lu.se/10.1137/18M1232772`.