# Linear (Autonomous) Equations of Order n

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June 6, 2020

# Outline

# (I) General remarks

General case:

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)\dot{x} + a_0(t) = b(t)$$

where $a_i(t), b(t)$ are continuous and $x(t)$ is the unknown.
Corresponding first order system:

$$\begin{aligned}
[x(t)]' &= \dot{x}(t) \\
[\dot{x}(t)]' &= \ddot{x}(t) \\
&\vdots \\
[x^{(n-1)}(t)]' &= -a_0(t)x(t) - a_1(t)\dot{x}(t) \cdots - a_{n-1}(t)x^{(n-1)}(t) + b(t)
\end{aligned}$$

Written as a matrix we get

$$\begin{pmatrix} x \\ \dot{x} \\ \vdots \\ \\ x^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ -a_0 & & \cdots & & -a_{n-1} & \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ \\ x^{(n-1)} \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$$

In short $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$.

If $b(t) = 0$: *homogeneous* equation, else *inhomogeneous* equation. If $a_j(t) = a_j =$ const. ($j = 0, \ldots, n-1$), $A(t) = A$: *autonomous* equation.

## Note

The solutions form a $n$ dimensional vector space. A solution is uniquely determined by the initial conditions

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= x_1 \\ &\vdots \\ x^{(n-1)}(0) &= x_{n-1} \end{aligned}$$

In short: $\mathbf{x}(0) = \mathbf{x}_0$

*From now on:* autonomous equations

If we write

$$P(z) \coloneqq z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

our equation becomes

$$\begin{aligned} P\left(\frac{d}{dt}\right)(x) &= \left[\frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1\frac{d}{dt} + a_0\right](x) \\ &= x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1\dot{x} + a_0 \\ &= b(t) \end{aligned}$$

We receive

$$\chi_A(z) = \det(z\,\mathrm{Id} - A) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

through Laplace Expansion on the last row of

$$\begin{vmatrix} z & -1 & & & \\ & z & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ a_0 & & \cdots & & z + a_{n-1} \end{vmatrix}$$

And hence we have in $\mathbb{C}$

$$\chi_A(z) = P(z) = \prod_{j=1}^{m} (z - \lambda_j)^{\mu_j}$$

where $\lambda_j$ is an Eigenvalue with multiplicity $\mu_j$.

# (II) Homogeneous Autonomous Equations of Order n

Now we look at equations of the form

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1\dot{x} + a_0 = 0$$

Example (1)

$x^{(4)} + 4\ddot{x} = 0.$

The corresponding system is given by

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ x^{(3)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ x^{(3)} \end{pmatrix}$$

$$\chi_A(z) = P(z) = z^4 + 4z^2 = z^2(z^2+4) = z^2(z-2i)(z+2i) \overset{!}{=} 0$$

has solutions $\lambda_1 = 0$, $\lambda_2 = 2i$, $\lambda_3 = -2i$ with multiplicities $\mu_1 = 2$ and $\mu_2 = \mu_3 = 1$.

## Theorem (Solutions in $\mathbb{C}$)

*The functions*

$$x_{j,k}(t) := t^k \exp(\lambda_j t) \quad (0 \le k < \mu_j,\ 1 \le j \le m)$$

*are $n$ linearly independent solutions. These solutions therefore form a basis of the space of all solutions. The general solution thus has the form*

$$x(t) = \sum_{j,k} C_{j,k} t^k \exp(\lambda_j t) \quad (C_{j,k} \in \mathbb{C})$$

## Example (2)

$x^{(4)} + 4\ddot{x} = 0$.

This has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2i$, $\lambda_3 = -2i$ with multiplicities $\mu_1 = 2$ and $\mu_2 = \mu_3 = 1$. Therefore

$$
\begin{aligned}
x_{1,0}(t) &= t^0 \exp(0t) = 1 \\
x_{1,1}(t) &= t^1 \exp(0t) = t \\
x_2(t) &= t^0 \exp(i2t) = e^{i2t} \\
x_3(t) &= t^0 \exp(-i2t) = e^{-i2t}
\end{aligned}
$$

are linearly independent solutions and the general solution is of the form

$$x(t) = C_{1,0} + C_{1,1}t + C_2 e^{i2t} + C_3 e^{-i2t}$$

with $C_{1,0}, C_{1,1}, C_2, C_3 \in \mathbb{C}$.

## Proof Theorem.

We have

$$\begin{aligned}
\mathbf{x}(t) &= \exp(tA)\mathbf{x}_0 = U\exp(tU^{-1}AU)U^{-1}\mathbf{x}_0 \\
&= U\exp\left(t\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}\right)\mathbf{y}_0 \\
&= U\begin{pmatrix} \exp(tJ_1) & & \\ & \ddots & \\ & & \exp(tJ_m) \end{pmatrix}\mathbf{y}_0
\end{aligned}$$

where $U$ is a fitting change of coordinates

## Proof Theorem (cont.)

We have

$$\exp(tJ_i) = \exp\left( t \begin{pmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{pmatrix} + t \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \right)$$

$$= \exp\left( \begin{pmatrix} \lambda_j t & & \\ & \ddots & \\ & & \lambda_j t \end{pmatrix} \right) \exp\left( \begin{pmatrix} 0 & t & & \\ & 0 & \ddots & \\ & & 0 & t \\ & & & 0 \end{pmatrix} \right)$$

## Proof Theorem (cont.)

We see with $x_{j,k}(t) = t^k e^{\lambda_j t}$ further that

$$\exp(tJ_i) = \exp\left(\begin{pmatrix} \lambda_j t & & \\ & \ddots & \\ & & \lambda_j t \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & t & & \\ & 0 & \ddots & \\ & & 0 & t \\ & & & 0 \end{pmatrix}\right)$$

$$= e^{\lambda_j t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{\mu_j - 1}}{(\mu_j - 1)!} \\ & 1 & t & & \vdots \\ & & 1 & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} x_{j,0} & x_{j,1} & \frac{x_{j,2}}{2!} & \cdots & \frac{x_{j,\mu_j - 1}}{(\mu_j - 1)!} \\ & x_{j,0} & x_{j,1} & & \vdots \\ & & x_{j,0} & \ddots & \frac{x_{j,2}}{2!} \\ & & & \ddots & x_{j,1} \\ & & & & x_{j,0} \end{pmatrix}$$

## Proof Theorem (cont.)

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ x^{(n-1)}(t) \end{pmatrix} = \mathbf{x}(t) = U \begin{pmatrix} \exp(tJ_1) & & \\ & \ddots & \\ & & \exp(tJ_m) \end{pmatrix} \mathbf{y}_0$$

implies that $x(t)$ is a linear combination of

$$x_{j,k}(t) = t^k \exp(\lambda_j t) \quad (0 \leq k < \mu_j,\ 1 \leq j \leq m)$$

Because $x(t)$ uniquely determines $\mathbf{x}(t)$ and vice versa we can identify $x(t)$ with $\mathbf{x}(t)$. The space of solutions $\mathbf{x}(t)$ is $n$-dimensional and hence the space of solutions $x(t)$ is also $n$-dimensional. As these $n$ functions span the Space of solutions $x(t)$ they must form a basis. This means they are linearly independent. $\square$

**Proposition.**

$$P\left(\frac{d}{dt}\right)\left(e^{\lambda t}\right)=\sum_{j=1}^{n} a_j\left(e^{\lambda t}\right)^{(j)}=\sum_{j=1}^{n} a_j(\lambda)^j e^{\lambda t}=P\left(\lambda\right)\left(e^{\lambda t}\right)$$

$\square$

**Alternative Proof of the Theorem.**
We give an alternative proof that $x_{j,k}(t)=t^k e^{\lambda_j t}$ solves the homogeneous equation. For $k=0$ we have

$$P\left(\frac{d}{dt}\right)\left(x_{j,0}(t)\right)=P\left(\frac{d}{dt}\right)\left(e^{\lambda_j t}\right)=P\left(\lambda_j\right)e^{\lambda_j t}=0$$

## Alternative Proof of the Theorem (cont.)

For $1 \le k < \mu_j$ we observe

$$t^k e^{\lambda_j t} = \frac{d^k}{d\lambda_j^k} e^{\lambda_j t}$$

As $\lambda_j$ is a zero of $P(z)$ with multiplicity $\mu_j$ we have for $0 \le i \le k < \mu_j$: $\frac{d^i}{d\lambda_j^i} P(\lambda_j) = 0$. It then follows from Schwarz's theorem and Leibniz's formula

$$\begin{aligned} P\left(\frac{d}{dt}\right)(x_{j,k}(t)) &= P\left(\frac{d}{dt}\right)(t^k e^{\lambda_j t}) = P\left(\frac{d}{dt}\right)\left(\frac{d^k}{d\lambda_j^k} e^{\lambda_j t}\right) \\ &= \frac{d^k}{d\lambda_j^k} P\left(\frac{d}{dt}\right)\left(e^{\lambda_j t}\right) = \frac{d^k}{d\lambda_j^k}\left[P\left(\lambda_j\right) e^{\lambda_j t}\right] \\ &= \sum_{i=0}^{k} \binom{k}{i} \underbrace{\frac{d^i}{d\lambda_j^i} P\left(\lambda_j\right)}_{=0} \frac{d^{k-i}}{d\lambda_j^{k-i}} e^{\lambda_j t} = 0 \end{aligned}$$

## Proposition (Solutions in $\mathbb{R}$, Complexification)

*For $a_j \in \mathbb{R}$ we know: If $\lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$ is a root of $P(z)$ then $\overline{\lambda} = \alpha - i\beta$ is also a root with the same multiplicity. We can replace*

$$\begin{cases} t^k e^{\lambda t} \\ t^k e^{\overline{\lambda} t} \end{cases}$$

*with*

$$\begin{cases} \Re(t^k e^{\lambda t}) = t^k e^{\alpha t} \sin(\beta t) \\ \Im(t^k e^{\lambda t}) = t^k e^{\alpha t} \cos(\beta t) \end{cases}$$

*This insures all solutions are real.*

## Proof Proposition.

This amounts to a change of basis:

$$
\begin{aligned}
e^{\alpha t}\cos(\beta t) &= \Re\left(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\right) \\
&= \Re\left(e^{\lambda t}\right) = \frac{1}{2}\left(e^{\lambda t} + e^{\overline{\lambda} t}\right) \\
e^{\alpha t}\sin(\beta t) &= \Im\left(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\right) \\
&= \Im\left(e^{\lambda t}\right) = \frac{1}{2i}\left(e^{\lambda t} - e^{\overline{\lambda} t}\right)
\end{aligned}
$$

$$
\begin{aligned}
e^{\lambda t} &= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) \\
e^{\overline{\lambda} t} &= e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)
\end{aligned}
$$

$\square$

## Example (3)

$x^{(4)} + 4\ddot{x} = 0.$

This has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2i$, $\lambda_3 = -2i$ with multiplicities $\mu_1 = 2$ and $\mu_2 = \mu_3 = 1$. The complex solutions

$$x_2(t) = e^{i2t}$$
$$x_3(t) = e^{-i2t}$$

can be transformed into

$$\tilde{x}_2(t) = \Re(e^{i2t}) = e^0 \cos(2t) = \cos(2t)$$
$$\tilde{x}_3(t) = \Im(e^{i2t}) = e^0 \sin(2t) = \sin(2t)$$

The real solution can therefore be written as

$$x(t) = C_{1,0} + C_{1,1}t + C_2 \cos(2t) + C_3 \sin(2t)$$

with $C_{1,0}, C_{1,1}, C_2, C_3 \in \mathbb{R}$

# (III) Inhomogeneous Autonomous Equations of Order n

We shall now consider equations of the form

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1\dot{x} + a_0 = b(t) \neq 0$$

## Idea

Find the general solution $x_h(t)$ to the corresponding homogeneous equation $P\left(\frac{d}{dt}\right)(x_h) = 0$ and a particular solution $x_p(t)$ to $P\left(\frac{d}{dt}\right)(x_p(t)) = b(t)$. Set

$$x(t) = x_h(t) + x_p(t)$$

Then it follows from linearity

$$P\left(\frac{d}{dt}\right)(x) = P\left(\frac{d}{dt}\right)(x_h + x_p) = \underbrace{P\left(\frac{d}{dt}\right)(x_h)}_{=0} + P\left(\frac{d}{dt}\right)(x_p) = b(t)$$

## Proposition (Particular solution)

$$x_p(t) = \int_0^t x_u(t-s)b(s)ds$$

*is a solution to the inhomogeneous equation.* $x_u(t)$ *is a solution to the corresponding homogeneous equation* $P\left(\frac{d}{dt}\right)(x_u) = 0$ *with initial values*

$$x_u(0) = \dot{x}_u(0) = \cdots = x_u^{(n-2)}(0) = 0$$

*and*

$$x_u^{(n-1)}(0) = 1.$$

Example (4)

$x^{(4)} + 4\ddot{x} = 16$.

We know

$$x_u(t) = C_{1,0} + C_{1,1}t + C_2\cos(2t) + C_3\sin(2t)$$

with $C_{1,0}, C_{1,1}, C_2, C_3 \in \mathbb{C}$ solves the equation. Our initial value problem states that $x_u(0) = \dot{x}_u(0) = \ddot{x}_u(0) = 0$ and $x_u^{(3)}(0) = 1$. We get $x_u$ by solving the system of linear equations.

## Example (4)

We have $x_u(t) = C_{1,0} + C_{1,1}t + C_2\cos(2t) + C_3\sin(2t)$

$$1 = x_u^{(3)}(0) = C_2 2^3 \sin(2\cdot 0) - C_3 2^3\cos(2\cdot 0) = -2^3 C_3$$
$$\implies C_3 = -\frac{1}{8}$$
$$0 = \ddot{x}_u(0) = -C_2 2^2\cos(2\cdot 0) - C_3 2^2\sin(2\cdot 0) = -2^2 C_2$$
$$\implies C_2 = 0$$
$$0 = \dot{x}_u(0) = C_{1,1} - \frac{1}{8}2\cos(2\cdot 0) = C_{1,1} - \frac{1}{4} \implies C_{1,1} = \frac{1}{4}$$
$$0 = x_u(0) = C_{1,0} + \frac{1}{4}\cdot 0 - \frac{1}{8}\sin(2\cdot 0) = C_{1,0} \implies C_{1,0} = 0$$

Thus $x_u(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$

## Example (4)

So our particular solution is

$$
\begin{aligned}
x_p(t) &= \int_0^t x_u(t-s)b(s)ds \\
&= \int_0^t \left[\frac{1}{4}(t-s) - \frac{1}{8}\sin(2(t-s))\right] \cdot 16ds \\
&= \int_0^t \left[4t - 4s - 2\sin(2(t-s))\right] ds \\
&= \left[4ts - 2s^2 - \cos(2(t-s))\right]_0^t \\
&= \left[4t^2 - 2t^2 - \cos(2(t-t))\right] + \cos(2(t-0)) \\
&= \cos(2t) + 2t^2 - 1
\end{aligned}
$$

### Proposition (Duhamel's Formula)

*For $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t)$ there is a particular solution of the form*

$$\mathbf{x}_p(t) = \int_0^t \exp\left((t-s)A\right)\mathbf{b}(s)ds$$

## Proof Duhamel's Formula.

As $tAsA = sAtA$ commutes it follows that

$$\begin{aligned}
\dot{\mathbf{x}}_p(t) &= \left[\int_0^t \exp\left((t-s)A\right)\mathbf{b}(s)ds\right]' \\
&= \left[\int_0^t \exp\left(tA\right)\exp\left(-sA\right)\mathbf{b}(s)ds\right]' \\
&= \left[\exp\left(tA\right)\int_0^t \exp\left(-sA\right)\mathbf{b}(s)ds\right]' \\
&= A\exp\left(tA\right)\int_0^t \exp\left(-sA\right)\mathbf{b}(s)ds + \exp\left(tA\right)\exp\left(-tA\right)\mathbf{b}(t) \\
&= A\int_0^t \exp\left(tA\right)\exp\left(-sA\right)\mathbf{b}(s)ds + \exp\left(tA - tA\right)\mathbf{b}(t) \\
&= A\mathbf{x}_p(t) + \mathbf{b}(t)
\end{aligned}$$

$\square$

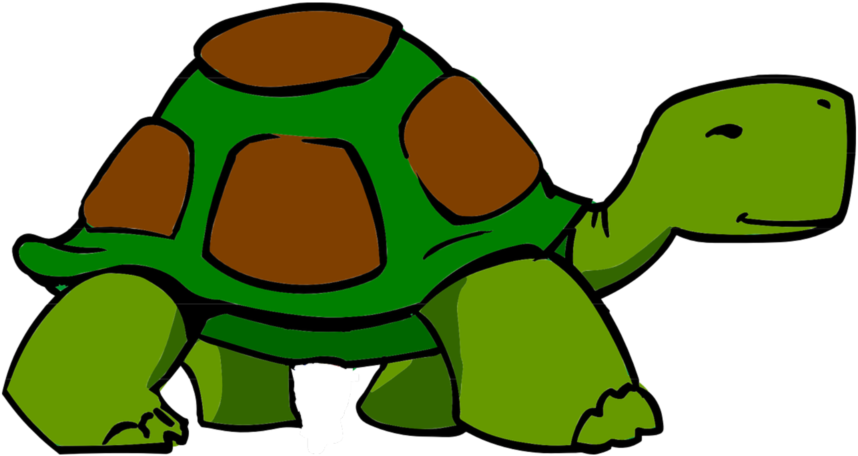
## Proof Particular Solution.

We know from Duhamel's formula that for $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t)$ there is a particular solution of the form

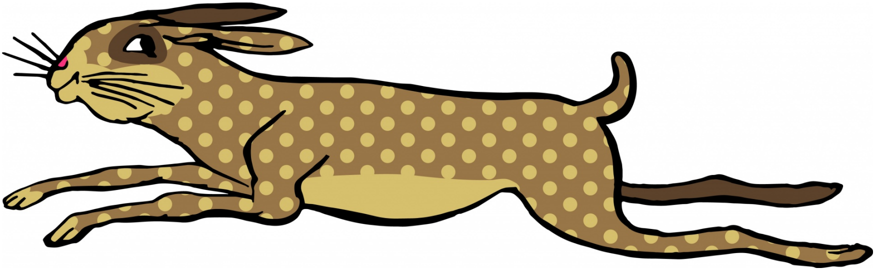
$$\begin{aligned}
\mathbf{x}_p(t) &= \int_0^t \exp\left((t-s)A\right)\mathbf{b}(s)ds \\
&= \int_0^t \exp\left((t-s)A\right)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} b(s)ds \\
&= \int_0^t \mathbf{x}_u(t-s)b(s)ds
\end{aligned}$$

And by taking the first component we get our claim. $\square$

This was more like...



We want to be like...



## Theorem (Ansatz for solutions)

*If we have*

$$b(t) = R(t)\exp(ct), \quad c \in \mathbb{C},\ R(t) \neq 0$$

*then* $P\left(\frac{d}{dt}\right)(x) = b$ *has a solution of the form*

$$x_p(t) = t^{\mu} Q(t)\exp(ct)$$

*where* $\deg Q = \deg R$ *and*

$$\mu := \begin{cases} \mu_j\ , & \textit{if } c = \lambda_j \\ 0 & \textit{else.} \end{cases}$$

## Example (5)

$x^{(4)} + 4\ddot{x} = t = te^{0t}$.

As $c = 0 = \lambda_1$ is a root with multiplicity $\mu_1 = 2 = \mu$, we are looking for a solution of the form $t^2 Q(t) e^{0t}$ with $\deg Q = \deg t = 1$. Substituting

$$x_p(t) = t^2(at+b)e^{0t} = at^3 + bt^2$$

into the equation we receive

$$0 + 4(3 \cdot 2at + 2b) \stackrel{!}{=} t$$
$$\rightsquigarrow b = 0,\ a = \frac{1}{24}$$

Thus

$$x_p(t) = \frac{1}{24} t^3.$$

## Lemma

*Let $f$, $g$ be $n$ times differentiable, $P(z)$ polynomial, $\deg P \leq n$. Then*

$$P\left(\frac{d}{dt}\right)(fg) = \sum_{j=0}^{n} \frac{1}{j!} f^{(j)} P^{(j)}\left(\frac{d}{dt}\right)(g)$$

## Proof Lemma.

It is enough to prove this for $P(z) = z^k$, $k \leq n$. The general case follows by taking the linear combination. We have

$$P^{(j)}\left(\frac{d}{dt}\right) = \begin{cases} k(k-1)\dots(k-j+1)\left(\frac{d}{dt}\right)^{k-j} & \text{, if } j \leq k \\ 0 & \text{, if } j > k \end{cases}$$

It follows from Leibniz's formula that

$$\begin{aligned} P\left(\frac{d}{dt}\right)(fg) = (fg)^{(k)} &= \sum_{j=0}^{k} \binom{k}{j} f^{(k)} g^{(k-j)} \\ &= \sum_{j=0}^{k} \frac{k(k-1)\dots(k-j+1)}{j!} f^{(k)} g^{(k-j)} \\ &= \sum_{j=0}^{n} \frac{1}{j!} f^{(k)} P^{(j)}\left(\frac{d}{dt}\right)(g) \end{aligned}$$

## Proof Theorem.

The lemma implies that

$$\begin{aligned} P\left(\frac{d}{dt}\right)\left(t^{\mu}Q(t)e^{ct}\right) &= \sum_{j=0}^{n}\frac{1}{j!}(t^{\mu}Q(t))^{(j)}P^{(j)}\left(\frac{d}{dt}\right)\left(e^{ct}\right) \\ &= \sum_{j\geq\mu}\frac{1}{j!}(t^{\mu}Q(t))^{(j)}P^{(j)}(c)e^{ct} \end{aligned}$$

As $P^{(j)}(c) = 0$ for $j \leq \mu - 1$ and $P^{(j)} = 0$ for $j \geq n$.

## Proof Theorem (cont.)

It follows further that

$$
\begin{aligned}
P\left(\frac{d}{dt}\right)\left(t^\mu Q(t)e^{ct}\right) &= \sum_{j\geq\mu}\frac{1}{j!}\underbrace{\left(t^\mu Q(t)\right)^{(j)}}_{(t^\mu Q(t))^{(\mu)}=:y(t)} P^{(j)}(c)e^{ct} \\
&= \sum_{j\geq\mu}\frac{1}{j!}y^{(j-\mu)}(t)P^{(j)}(c)e^{ct} \\
&= \sum_{j\geq 0}\underbrace{\frac{1}{(j+\mu)!}P^{(j+\mu)}(c)e^{ct}}_{=:b_j} y^{(j)}(t) \\
&= \sum_{j\geq 0} b_j y^{(j)}(t) = b_0 y(t) + b_1\dot{y}(t) + \dots \\
&\overset{!}{=} R(t)
\end{aligned}
$$

## Proof Theorem (cont.)

Note that if a polynomial $y(t)$ exists, so that $\deg y = \deg R$ and

$$b_0 y(t) + b_1 \dot{y}(t) + \cdots = R(t) = r_k t^k + \cdots + r_0$$

we are finished because $y(t) = (t^\mu Q(t))^{(\mu)}$. We get $Q(t)$ by integrating $y$ $\mu$ times and then dividing by $t^\mu$. Then it follows that

$$P\left(\frac{d}{dt}\right)\left(t^\mu Q(t) e^{ct}\right) = b_0 y(t) + b_1 \dot{y}(t) + \cdots = R(t)$$

We prove the existence of $y(t)$ by induction to $k = \deg R$

$k = 0$: Set for $R(t) = r_0$

$$y(t) \coloneqq \frac{r_0}{b_0} \quad \left(b_0 = \frac{1}{(\mu)!} P^{(\mu)}(c) e^{ct} \neq 0\right)$$

Then $\deg y = 0 = \deg R$ and

$$b_0 y(t) + b_1 \dot{y}(t) + \cdots = b_0 \frac{r_0}{b_0} = r_0 = R(t)$$

## Proof Theorem (cont.)

$k-1 \rightsquigarrow k$: Set $y(t) := \frac{r_k}{b_0}t^k + z(t)$ with $\deg z < k$. Then

$$\begin{aligned} & b_0\left(\frac{r_k}{b_0}t^k + z(t)\right) + b_1\left(\frac{r_k}{b_0}t^k + z(t)\right)' + \dots \\ & = r_k t^k + b_0 z(t) + b_1 k \frac{r_k}{b_0} t^{k-1} + b_1 \dot{z}(t) + \dots \\ & = R(t) = r_k t^k + r_{k-1}t^{k-1} + \dots + r_0 \end{aligned}$$

This is equivalent to

$$\begin{aligned} & b_0 z(t) + b_1 \dot{z}(t) + \dots \\ & \quad = r_{k-1}t^{k-1} + \dots + r_0 - b_1 k \frac{r_k}{b_0} t^{k-1} - \dots \end{aligned}$$

By induction hypothesis $z(t)$ exists and therefore $y(t) = \frac{r_k}{b_0}t^k + z(t)$ exists and $\deg y = k = \deg R$.

$\square$

## Proposition (linear combination of $b(t)$)

*If* $b(t) = c_1 b_1(t) + \cdots + c_k b_k(t)$ *and* $P\left(\frac{d}{dt}\right)(x_j) = b_j$ *then* $x = c_1 x_1 + \cdots + c_k x_k$ *solves* $P\left(\frac{d}{dt}\right)(x) = b$.

## Proof.

$$
\begin{aligned}
P\left(\frac{d}{dt}\right)(x) = P\left(\frac{d}{dt}\right)\left(\sum_j c_j x_j\right) &= \sum_j c_j P\left(\frac{d}{dt}\right)(x_j) \\
&= \sum_j c_j b_j = b
\end{aligned}
$$

$\square$

## Note

All right-hand-sides of the form

$$b(t) = \sum_j R_j(t) \exp(c_j t), \quad R_j(t) \neq 0, c_j \in \mathbb{C} \quad \text{(“quasi-polynomials”)}$$

can be solved. In particular $\sin(t)$ and $\cos(t)$ are quasi-polynomials because:

$$\cos(t) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \sin(t) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

## Example (6)

$x^{(4)} + 4\ddot{x} = \sin(t)$.

Because $\sin(t) = \Im(e^{it})$, we complexify the problem and then take the imaginary part of the solution: $x^{(4)} + 4\ddot{x} = e^{it}$. As $c = i$ is not a root of $P(z)$ and $\deg Q = \deg 1 = 0$ we are looking for a solution of the form $x_p(t) = t^0 a e^{it} = a e^{it}$. Substituting into the equation we receive:

$$(i)^4 a e^{it} + 4(i)^2 a e^{it} = a e^{it} - 4 a e^{it} = -3 a e^{it} \stackrel{!}{=} e^{it}$$
$$\rightsquigarrow a = -\frac{1}{3}$$

And taking the imaginary part of the solution

$$x_p(t) = \Im\left(-\frac{1}{3} e^{it}\right) = -\frac{1}{3}\sin(t)$$

## Example (7)

$x^{(4)} + 4\ddot{x} = 6 \cdot \sin(t) - 24t.$

We know:

$$P\left(\frac{d}{dt}\right)\left(-\frac{1}{3}\sin(t)\right) = \sin(t), \qquad P\left(\frac{d}{dt}\right)\left(\frac{1}{24}t^3\right) = t$$

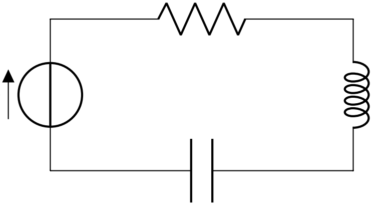
The proposition thus tells us that

$$x_p(t) = 6\left(-\frac{1}{3}\sin(t)\right) - 24\left(\frac{1}{24}t^3\right) = 2 \cdot \sin(t) - t^3$$

is a particular solution to our equation.

# (IV) Some Aspects of 2nd Order Autonomous Equations

Example (8)




| | | |
|---|---|---|
| Ohm's law: | $v_R = RI$ | |
| Faraday's law: | $v_L = L\dot{I}$ | |
| Capacitor: | $v_C = \frac{Q}{C}$ | $(I = \dot{Q},$ charge) |

We are looking at a periodic forcing term (Voltage) $V(t) = V_0 \cos(\omega t)$. $\omega$ is the *external angular frequency*.

We know that

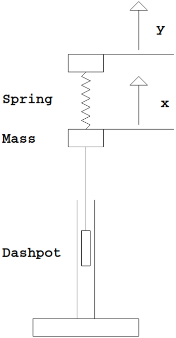
$$\frac{\dot{V}(t)}{L} = \frac{[V_0 \cos(\omega t)]'}{L} = \underbrace{-\frac{V_0 \omega}{L}}_{=:A} \sin(\omega t) = A \sin(\omega t)$$

Together with Kirchhoff's law this yields:

$$\begin{aligned} V(t) &= v_L + v_R + v_C = L\dot{I} + RI + \frac{Q}{C} \\ \implies \dot{V}(t) &= L\ddot{I} + R\dot{I} + \frac{1}{C} I \\ \implies \frac{\dot{V}(t)}{L} &= \ddot{I} + \underbrace{\frac{R}{L}}_{=:2\eta > 0} \dot{I} + \underbrace{\frac{1}{LC}}_{=:\omega_n^2 > 0} I \\ \implies A \sin(\omega t) &= \ddot{x} + 2\eta \dot{x} + \omega_n^2 x \end{aligned}$$

Where $x = I$, $2\eta$ is the *damping constant* and $\omega_n$ is the *natural angular frequency*.

# Example (9)





Swinging saloon doors and a spring/mass/dashpot system also lead to an equation of the form

$$\ddot{x} + 2\eta\dot{x} + \omega_n^2 x = A\sin(\omega t)$$

## Homogeneous equation

For $\eta > 0$, $\omega_n^2 > 0$ the homogeneous equation is

$$\ddot{x} + 2\eta\dot{x} + \omega_n^2 x = 0$$

This has the eigenvalues

$$\lambda_{1,2} = -\eta \pm \sqrt{\eta^2 - \omega_n^2}$$

We distinguish 3 cases

Over damping: $\eta > \omega_n$ $\lambda_1, \lambda_2 < 0$. This has the real solution

$$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad (C_1, C_2 \in \mathbb{R})$$

Critical damping: $\eta = \omega_n$ $\lambda_1 = \lambda_2 = -\eta < 0$. This has the real solution

$$x_h(t) = (C_{1,0} + C_{1,1} t) e^{\lambda_1 t} \quad (C_{1,0}, C_{1,1} \in \mathbb{R})$$

Under damping: $\eta < \omega_n$ $\lambda_{1,2} = -\eta \pm i\omega_d$, $\omega_d = \sqrt{\omega_n^2 - \eta^2}$. This has the real solution $(C_{1,0}, C_{1,1}, C, \varphi \in \mathbb{R})$

$$\begin{aligned} x_h(t) &= C_1 \Re(e^{\lambda_1 t}) + C_2 \Im(e^{\lambda_1 t}) \\ &= C_1 e^{-\eta t} \cos(\omega_d t) + C_2 e^{-\eta t} \sin(\omega_d t) \\ &= C e^{-\eta t} \sin(\omega_d t + \varphi) \end{aligned}$$

Here $\omega_d$ is called the *damped angular frequency (pseudo-frequency)*. It is the frequency of an unforced oscillation.

We notice that in all cases $x_h(t) \to 0$ as $t \to \infty$. Therefore

$$x(t) = x_h(t) + x_p(t) \approx x_p(t)$$

## Inhomogeneous Equation

Now we will look for a particular solution of the inhomogeneous equation. Because $A\sin(\omega t) = \Im(Ae^{i\omega t})$ we can complexify the problem to

$$\ddot{x} + 2\eta\dot{x} + \omega_n^2 x = Ae^{i\omega t}$$

We know that $\Re(\lambda_{1,2}) \neq 0$. Therefore $i\omega$ is not a root and we can try the ansatz $x_p(t) = t^0 Q(t)e^{i\omega t} = qe^{i\omega t}$ $(c \in \mathbb{C})$ which leads to

$$P\left(\frac{d}{dt}\right)\left(qe^{i\omega t}\right) = qP\left(\frac{d}{dt}\right)\left(e^{i\omega t}\right) = qP\left(i\omega\right)e^{i\omega t} \stackrel{!}{=} Ae^{i\omega t}$$

$$\implies q = \frac{A}{P(i\omega)} = \frac{A}{(i\omega)^2 + 2\eta i\omega + \omega_n^2} =: \alpha + i\beta$$

We get

$$(\alpha + i\beta)e^{i\omega t} = (\alpha\cos(\omega t) - \beta\sin(\omega t)) + i(\alpha\sin(\omega t) + \beta\cos(\omega t))$$

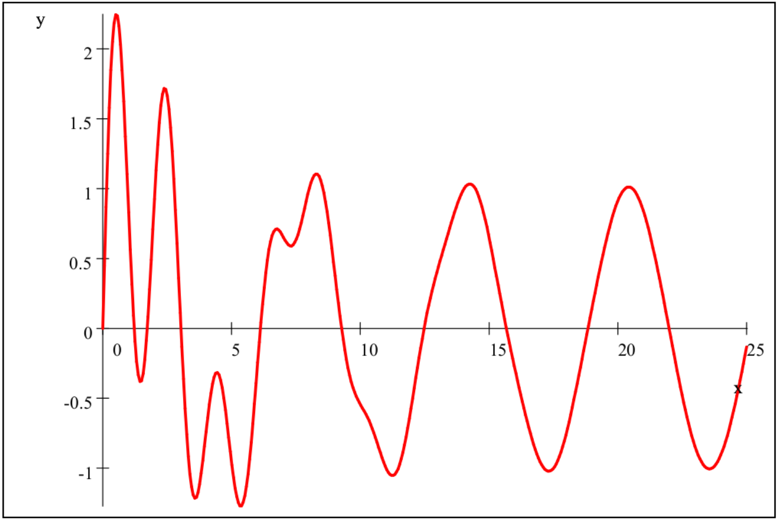
And by taking the imaginary part the solution

$$x_p(t) = \alpha\sin(\omega t) + \beta\cos(\omega t) = B\sin(\omega t + \psi)$$

with $B = |\alpha + i\beta| = |q|$ and $\cos(\psi) = \frac{\alpha}{B}$

## Example (10)

Plot of a possible solution: $x(t) = \underbrace{2e^{-t/4}\sin(t)}_{=x_h} + \underbrace{\sin(t)}_{=x_p}$



## For which $\omega$ is the amplitude $B$ maximal?

$$B^2 = |q|^2 = \frac{|A|^2}{|(i\omega)^2 + 2\eta i\omega + \omega_n^2|^2}$$

is maximal, if

$$\begin{aligned} |(i\omega)^2 + 2\eta i\omega + \omega_n^2|^2 &= (-\omega^2 + \omega_n^2)^2 + (2\eta\omega)^2 \\ &= \omega^4 - 2\omega^2\omega_n^2 + \omega_n^4 + (2\eta)^2\omega^2 \\ &= \omega^4 + ((2\eta)^2 - 2\omega_n^2)\omega^2 + \omega_n^4 \end{aligned}$$

is minimal.

This occurs if $\omega^4 + \underbrace{\left((2\eta)^2 - 2\omega_n^2\right)}_{=:-2\omega_r^2}\omega^2 = \omega^4 - 2\omega_r^2\omega^2$ is minimal:

case 1: $(2\eta)^2 \geq 2\omega_n^2$, when $\omega = 0$.

case 2: $(2\eta)^2 < 2\omega_n^2$, when

$$\omega^2 = \omega_r^2$$

$$\implies \omega = \omega_r = \sqrt{-\frac{1}{2}\left((2\eta)^2 - 2\omega_n^2\right)} = \sqrt{\omega_n^2 - 2\eta^2}$$

$\omega_r$ is called the *resonant angular frequency*. If $\omega = \omega_r$ resonance occurs (greatest response). Note that in general $\omega_d = \sqrt{\omega_n^2 - \eta^2} \neq \sqrt{\omega_n^2 - 2\eta^2} = \omega_r$ (damped angular frequency $\neq$ resonant angular frequency)

In the second case we get for $\omega = \omega_r$

$$\begin{aligned}
B_{\mathsf{max}}^2 &= \frac{|A|^2}{\omega_r^4 - 2\omega_r^2\omega_r^2 + \omega_n^4} = \frac{|A|^2}{\omega_n^4 - \omega_r^4} \\
&= \frac{|A|^2}{\omega_n^4 - \left(\omega_n^2 - 2\eta^2\right)^2} = \frac{|A|^2}{2\omega_n^2 \cdot 2\eta^2 - (2\eta)^2\eta^2} \\
&= \frac{|A|^2}{(2\eta)^2\left(\omega_n^2 - \eta^2\right)} = \frac{|A|^2}{(2\eta)^2\omega_d^2}
\end{aligned}$$

The maximal amplitude thus is

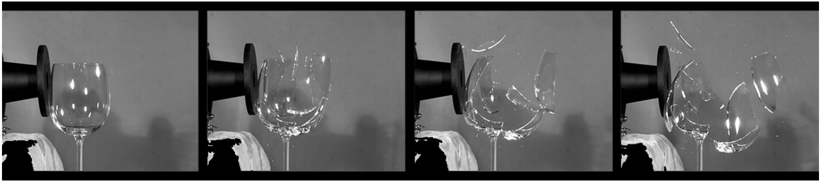
$$B_{\mathsf{max}} = \frac{|A|}{(2\eta)|\omega_d|}$$

### Resonance catastrophe

We see that for $\eta \to 0$ we have $B_{\text{max}} = \frac{|A|}{(2\eta)|\omega_d|} \to \infty$ in the case of resonance. In particular if $\eta = 0$, $x(t)$ grows beyond all bounds. We then have complete resonance and $\omega = \omega_r = \omega_d$. This leads to resonance catastrophe.

Example (Resonance catastrophe)

Glass breaking as a result of resonance:



# (V) Linear Equations of Order n

As at the beginning we have

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)\dot{x} + a_0(t) = b(t)$$

Now $a_j(t)$ is not necessarily constant.

**Note**

There is no general way of solving n'th order equations.

## Reduction of Order (d'Alambert)

We have a solution $\phi$ of the equation $P\left(\frac{d}{dt}\right)(\phi) = 0$. We try a variation of constants ansatz $x(t) = c(t)\phi(t)$. Setting $d \coloneqq \dot{c}$ our lemma leads to an equation of order $(n-1)$:

$$
\begin{aligned}
0 &\stackrel{!}{=} P\left(\frac{d}{dt}\right)(x) = P\left(\frac{d}{dt}\right)(c\phi) = \sum_{j=0}^{n} \frac{1}{j!} c^{(j)} P^{(j)}\left(\frac{d}{dt}\right)(\phi) \\
&= \frac{1}{0!} c \underbrace{P\left(\frac{d}{dt}\right)(\phi)}_{=0} + \sum_{j=1}^{n} \frac{1}{j!} \underbrace{c^{(j)}}_{=d^{(j-1)}} P^{(j)}\left(\frac{d}{dt}\right)(\phi) \\
&= \sum_{j=1}^{n} \underbrace{\left[\frac{1}{j!} P^{(j)}\left(\frac{d}{dt}\right)(\phi)\right]}_{=:b_j} d^{(j-1)} = \sum_{j=0}^{n-1} b_{j-1} d^{(j)}
\end{aligned}
$$

## Example (11)

$\ddot{x} - 2t\dot{x} - 2x = 0$ where $\phi(t) = e^{t^2}$ is a solution
Now we set $x(t) = c(t)e^{t^2}$ and obtain

$$
\begin{aligned}
0 &\overset{!}{=} \left[ce^{t^2}\right]'' - 2t\left[ce^{t^2}\right]' - 2ce^{t^2} \\
&= \left[\ddot{c}e^{t^2} + 2\dot{c}e^{t^2}(2t) + c\left[e^{t^2}(2t)\right]'\right] - 2t\left[\dot{c}e^{t^2} + ce^{t^2}(2t)\right] - 2ce^{t^2} \\
&= \left[\ddot{c}e^{t^2} + (2t)\dot{c}e^{t^2} + c\left[e^{t^2}(2t)^2 + 2e^{t^2}\right]\right] - (2t)^2\left[ce^{t^2}\right] - 2ce^{t^2} \\
&= \ddot{c}e^{t^2} + (2t)\dot{c}e^{t^2} = e^{t^2}\left(\ddot{c} + (2t)\dot{c}\right) = e^{t^2}\left(\dot{d} + (2t)d\right)
\end{aligned}
$$

$\dot{d}(t) = -(2t)d(t)$ has the solution $d(t) = e^{-t^2}$.

## Example (11)

We get $c$ by integrating

$$c = \int_0^t d(s)ds = \int_0^t e^{-s^2} ds = \frac{\sqrt{\pi}}{2} \operatorname{erf}(t)$$

where $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$ is the Gauss error function. We have the second solution

$$\psi(t) = e^{t^2} \operatorname{erf}(t)$$

# (VI) Summary

Algorithm to solve an autonomous equation:

1. Complexify, if in $\mathbb{R}$
2. Find $x_h$
   - 2.1 Find the eigenvalues of the corresponding first order system
   - 2.2 Find $x_{j,k}(t) = t^k e^{\lambda_j t}$
   - 2.3 Change basis to get real solution, if in $\mathbb{R}$
   - 2.4 If there are initial values find $C_{j,k}$ so that $x = \sum_{j,k} C_{j,k} x_{j,k}$ is a solution
3. Find $x_p$
   - 3.1 With an ansatz
   - 3.2 With Duhamel's Formula
   - 3.3 Take the real part, if in $\mathbb{R}$
4. Your solution is $x = x_h + x_p$

If the equation is not autonomous and a solution $\phi$ is given, try $c\phi$ as an ansatz.

Thank you for listening

Questions?

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