ON THE IMPORTANCE OF 26 DIMENSIONS FOR BOSONIC STRING THEORY

A semester project in the research group

QUANTUM FIELD THEORY AND STRINGS
of the INSTITUTE FOR THEORETICAL PHYSICS
at ETH ZÜRICH

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June 20, 2024

ABSTRACT

String theory has the potential to unify general relativity with quantum mechanics. One of its advantages is that it has relatively few tunable parameters, offering a form of uniqueness. For instance, the number of spacetime dimensions is derived from the theory rather than being an assumption. This thesis, primarily following the notation from "A First Course in String Theory" by Barton Zwiebach, provides a brief review of the preliminaries and explicitly calculates the commutator of Lorentz generators $[M^{-I}, M^{-J}]$ to determine the critical dimension and normal ordering constant of bosonic open string theory.

CONTENTS

NOTATION

FREQUENTLY USED SYMBOLS

D	spacetime dimension of the $(1, D-1)$ Minkowski space
μ, ν	$\in \{0, \ldots, D-1\}$, Lorentz indices
$\eta^{\mu\nu}$	Minkowski metric
$+, -$	indices for the first two spacetime dimensions in light-cone coordinates
I, J	$\in \{2, \ldots, D-1\}$, transverse Lorentz indices, repetition of the same index implies summation
x_0^μ	(center of mass) position
p^μ	(center of mass) momentum
α_n^I	creation (or annihilation) operator when $n > 0$ (or $n < 0$)
$\alpha_{-n}^{[I}\alpha_n^{J]}$	$= \alpha_{-n}^I\alpha_n^J - \alpha_{-n}^J\alpha_n^I$, antisymmetrization in the Lorentz indices
L_n	n-th transverse (normal-ordered) Virasoro mode
L_0'	$= L_0 + a$, transverse Virasoro zero mode with normal ordering constant a
$: \alpha_n^I \ldots \alpha_m^J :$	normal ordering, all creation operators to the left of all annihilation operators
$\theta(n)$	Heaviside step function, 1 for $n > 0$ and 0 for $n \leq 0$

PHYSICAL CONSTANTS

$c = 1$	speed of light in vacuum in natural units
$\hbar = 1$	Planck's constant in natural units
α'	slope parameter

1 INTRODUCTION

Many of the most significant advances in theoretical physics have come from unifying different phenomena into a common framework. For instance, in 1865, James Clerk Maxwell unified electric and magnetic forces into a single electromagnetic theory. In the late 1960s, Steven Weinberg and Abdus Salam developed the electroweak model, which unifies the electromagnetic interaction with the weak force within a relativistic quantum framework. Additionally, quantum chromodynamics provided a quantized theory for the strong force. Together, these theories form the Standard Model of particle physics, encompassing a wide array of elementary particles (fermions) and force carriers (bosons).

While the Standard Model has been remarkably successful in explaining phenomena at small scales, it has yet to be consistently unified with Einstein's theory of general relativity. Moreover, the Standard Model is highly non-unique, requiring numerous continuous parameters as inputs. In the quest for a Grand Unified Theory that unifies all four fundamental forces, string theory has emerged as a promising candidate.

In string theory, elementary particles are viewed as oscillation modes of a fundamental, microscopic string. The only essential input for this theory is a length scale ℓ_s. An additional advantage of string theory, which we aim to illustrate in this thesis, is that the number of spacetime dimensions D is not an assumed parameter but rather a calculable quantity from the theory. Specifically, in the context of bosonic string theory, we will demonstrate that the only consistent number of dimensions is 26, comprising one time dimension and 25 spatial dimensions.

This thesis is structured as follows: In chapter 2, we will review the necessary preliminaries, including light-cone gauge, quantization, oscillators, and the Virasoro algebra. The main computation is presented in chapter 3, where it is divided into multiple digestible parts. Finally, we compare our approach to previous literature and discuss the physical implications of our findings in the Summary and Outlook.

2

PREREQUISITES

2.1 LIGHT-CONE GAUGE

We describe our string in light cone coordinates by introducing the plus and minus coordinates

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^1).$$

In those coordinates, the Minkowski metric takes the form

$$X^2 \equiv X^\mu X^\nu \eta_{\mu\nu} = -2X^+X^- + X^I X^I \tag{2.1}$$

An introduction to light-cone coordinates can be found in section 2.2 of [1].
We use the gauge invariance for the parametrization in (τ, σ) of the world sheet, to fix a τ-parametrization given by

$$X^+ = 2\alpha' p^+ \tau,$$

and by fixing a σ-parametrization of constant light-cone energy density with range $\sigma \in [0, \pi]$. Those choices can be elegantly summarized as

$$(\dot{X} \pm X')^2 = 0, \tag{2.2}$$

with the derivatives denoted as $\dot{X} = \partial_\tau X$, $X' = \partial_\sigma X$. This is called the light cone gauge. With this choice of gauge the field equation derived from the Nambu-Goto action (section 6.4 of [1]) takes the form of a wave equation:

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0, \tag{2.3}$$

as shown in section 9.3 of [1]. For the free open string with a space-filling D-brane, the Neumann boundary conditions are

$$X^{I\prime}(\tau, \sigma) = 0, \qquad \text{for } \sigma = 0, \pi \tag{2.4}$$

2.2 MODE EXPANSION AND QUANTIZATION

The general solution to this wave equation eq. (2.3) with the boundary conditions eq. (2.4) can be written in terms of oscillator modes

$$X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'} p^I \tau + i\sqrt{2\alpha'} \sum_{n \neq 0}^{\infty} \frac{1}{n} \alpha_n^I \cos(n\sigma) e^{-in\tau}$$

since X^I is real, we demand $a_n^I = (a_{-n}^I)^*$. To have a more concise notation we introduce the zero-mode

$$\sqrt{2\alpha'} p^I \equiv \alpha_0^I \tag{2.5}$$

Upon quantization, the x_0^I, p^I, α_n^I are promoted to operators which satisfy the commutation relations

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ} \delta_{m+n,0} \tag{2.6}$$

$$[x_0^I, \alpha_n^J] = \sqrt{2\alpha'} i\eta^{IJ} \delta_{n,0} \tag{2.7}$$

We identify α_n^I for $n > 1$ as annihilation operators and $\alpha_{-n}^I = (\alpha_n^I)^\dagger$ as creation operators. Using eq. (2.5), we find the second commutator eq. (2.7) to agree with the canonical commutator $[x^\mu, p^\nu] = i\eta^{\mu\nu}$. A detailed derivation of those commutators is given in section 12.3 of [1].

2.3 VIRASORO ALGEBRA

Since the X^+ and X^- components only appear linearly in the eq. (2.1), this allows us to write the τ evolution of the X^- in terms of the transverse coordinates I using eq. (2.2). This completely fixed the X^- dynamics up to an integration constant x_0^-. The mode expansion is hence given by

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}p^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty}\frac{1}{n}\alpha_n^- \cos(n\sigma)e^{-in\tau}$$

$$\begin{aligned}\sqrt{2\alpha'}\alpha_n^- &= \frac{1}{p^+}L_n \\ 2\alpha' p^- &= \frac{1}{p^+}L_0'\end{aligned}$$

with the transverse Virasoro operators

$$\begin{aligned}L_n &= \frac{1}{2}\sum_{p\neq 0}\alpha_{n-p}^I\alpha_p^I \\ L_0 &= \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I,\end{aligned} \tag{2.8}$$

which are by definition, normal ordered. Normal ordering means that all creation operators are to the left of the annihilation operators. To compensate for this ambiguity of L_0, we include an ordering constant a into $L_0' = L_0 + a$.

The Virasoro operators and their commutators define a Virasoro algebra with central extension. The following useful properties follow from eq. (2.6) as shown in section 12.5 of [1]:

$$\begin{aligned}(L_n)^\dagger &= L_{-n} \\ [x_0^I, L_n] &= i\sqrt{2\alpha'}\alpha_n^I \\ [\alpha_m^J, L_n] &= m\alpha_{m+n}^J \qquad (2.9)\\ [L_m, L_n] &= (m-n)L_{m+n} + \underbrace{\frac{D-2}{12}(m^3-m)}_{\equiv A(m)}\delta_{m+n,0} \qquad (2.10)\end{aligned}$$

3

DERIVATION OF THE $[M^{-I}, M^{-J}]$ COMMUTATOR

Due to the Lorentz symmetry we find the following conserved classical Lorentz charges

$$M^{\mu\nu} = \int_0^\pi d\sigma\,(X^\mu \mathcal{P}^{\tau\nu} - X^\nu \mathcal{P}^{\tau\mu}),$$

where $\mathcal{P}^\tau$ is the momentum density (section 8.5 of [1]). Through canonical quantization, normal ordering and symmetrization to ensure hermiticity, we postulate the following quantum Lorentz generator:

$$M^{-I} \equiv \underbrace{x_0^- p^I - \frac{1}{4\alpha' p^+}(x_0^I L_0' + L_0' x_0^I)}_{\equiv l^{-I}} \underbrace{- \frac{i}{\sqrt{2\alpha'} p^+} \sum_{n=1}^\infty \frac{1}{n}(L_{-n}\alpha_n^I - a_{-n}^I L_n)}_{\equiv S^{-I}}$$

We recognize an orbital part l^{-I} and a part which is due to string oscillations S^{-I}. To compute the commutator $[M^{-I}, M^{-J}]$ we will investigate the commutator of the orbital part, the oscillatory part, and the mixed commutator separately. In the case $I = J$ the commutator is trivial. In the following we will therefore assume $I \neq J$.

3.1 ORBITAL LORENTZ GENERATORS

At first, we compute $[l^{-I}, l^{-J}]$. To reduce the number of terms we neglect normal ordering and write $L_0' x_0^I = x_0^I L_0' + i\sqrt{2\alpha'}\alpha_0^I$ but since α_0^I commutes with all α_n^J, L_0 and x_0^J, we can neglect it and hence obtain

$$[l^{-I}, l^{-J}] = \left[x_0^- p^I - \frac{1}{2\alpha' p^+} x_0^I L_0', x_0^- p^J - \frac{1}{2\alpha' p^+} x_0^J L_0'\right].$$

After distributing, we find the following commutators

$$\begin{aligned}
[x_0^- p^I, x_0^- p^J] &= 0, \\
\left[x_0^- p^I, \frac{-1}{2\alpha' p^+} x_0^J L_0'\right] &= \left[x_0^-, \frac{-1}{2\alpha' p^+}\right] x_0^J L_0' p^I + x_0^- \frac{-1}{2\alpha' p^+}[p^I, x_0^J] L_0' = \frac{-i}{2\alpha'(p^+)^2} x_0^J L_0' p^I, \\
\left[\frac{-1}{2\alpha' p^+} x_0^I L_0', x_0^- p^J\right] &= \frac{i}{2\alpha'(p^+)^2} x_0^I L_0' p^J, \\
\left[\frac{-1}{2\alpha' p^+} x_0^I L_0', \frac{-1}{2\alpha' p^+} x_0^J L_0'\right] &= \frac{1}{(2\alpha' p^+)^2}(x_0^J [x_0^I, L_0'] L_0' + x_0^I [L_0', x_0^J] L_0') = -\frac{i}{2\alpha'(p^+)^2}(x_0^J L_0' p^I - x_0^I L_0' p^J).
\end{aligned}$$

For the second line we used $[x_0^-, p^+] = i\eta^{-+} = -i$ and hence $[x_0^-, 1/p^+] = i/(p^+)^2$ as well as $[x_0^J, p^I] = i\eta^{JI} = 0$ since $I \neq J$. The third commutator follows from the second after exchanging indices and using antisymmetry of the commutator. In the last part, we used $[x_0^I, L_0'] = i\sqrt{2\alpha'}\alpha_0^I = i2\alpha' p_0^I$, which commutes with L_0'. In total, their sum vanishes.

3.2 OSCILLATOR LORENTZ GENERATOR

Next, we compute $[S^{-I}, S^{-J}]$. Since p^+ commutes with all L_n, α_n^I we can focus on

$$K^{IJ} = \sum_{n,m=1}^\infty \frac{1}{mn}[L_{-m}\alpha_m^I - \alpha_{-m}^I L_m, L_{-n}\alpha_n^J - \alpha_{-n}^J L_{-n}]$$

In order to shorten our notation, we neglect normal ordering for a moment. We standardize all terms by having the Virasoro operator on the left. With this commutation, we get an additional term of

$[\alpha^I_{-m}, L_m] = -m\alpha^I_0$. But since α^I_0 commutes with all other α^J_m we can ignore this term. Hence, we can rewrite

$$K^{IJ} = \sum_{n,m\neq 0} \frac{1}{mn}[L_{-m}\alpha^I_m, L_{-n}\alpha^J_n] \tag{3.1}$$

now we compute the commutator

$$\begin{aligned}[L_{-m}\alpha^I_m, L_{-n}\alpha^J_n] &= L_{-m}[\alpha^I_m, L_{-n}\alpha^J_n] + [L_{-m}, L_{-n}\alpha^J_n]\alpha^I_m \\ &= L_{-m}L_{-n}[\alpha^I_m, \alpha^J_n] + L_{-m}[\alpha^I_m, L_{-n}]\alpha^J_n + L_{-n}[L_{-m}, \alpha^J_n]\alpha^I_m + [L_{-m}, L_{-n}]\alpha^J_n\alpha^I_m.\end{aligned}$$

Using eqs. (2.6), (2.7) and (2.9) the commutators evaluate to

$$\begin{aligned}[\alpha^I_m, a^J_n] &= m\delta_{m+n}\eta^{IJ} = 0 \\ [\alpha^I_m, L_{-n}] &= m\alpha^I_{m-n} \\ [L_{-m}, a^J_n] &= -n\alpha^J_{n-m} \\ [L_{-m}, L_{-n}] &= (n-m)L_{-m-n} + A(-m)\delta_{m+n,0}\end{aligned}$$

which yields

$$[L_{-m}\alpha^I_m, L_{-n}\alpha^J_n] = mL_{-m}\alpha^I_{m-n}\alpha^J_n - nL_{-n}\alpha^J_{n-m}a^I_m + (n-m)L_{-m-n}\alpha^I_m\alpha^J_n + A(-m)a^I_m\alpha^J_n\delta_{m+n,0}.$$

Plugging back into eq. (3.1) and simplifying the Kronecker-delta using $m = -n$ we find

$$K^{IJ} = \sum_{n,m\neq 0}\left(\frac{1}{n}\left[L_{-m}\alpha^I_{m-n} - L_{-m-n}\alpha^I_m\right]\alpha^J_n - \frac{1}{m}\left[L_{-n}\alpha^J_{n-m} - L_{-m-n}\alpha^J_n\right]\alpha^I_m\right) - \sum_{n\neq 0}\frac{A(n)}{n^2}\alpha^I_{-n}\alpha^J_n.$$

Because of the antisymmetry of the commutator, the first and second term are identical after exchanging the summation indices $n \leftrightarrow m$ and Lorentz indices $I \leftrightarrow J$. The second order term can be simplified by splitting the sum into two parts and using $A(-n) = -A(n)$. We conclude

$$K^{IJ} = \sum_{n,m\neq 0}\left(\frac{1}{n}\left[L_{-m}\alpha^I_{m-n} - L_{-m-n}\alpha^I_m\right]\alpha^J_n - (m, I \leftrightarrow n, J)\right) - \sum_{n=1}^{\infty}\frac{A(n)}{n^2}\alpha^{[I}_{-n}\alpha^{J]}_n. \tag{3.2}$$

3.3 NORMAL ORDERING

In order to be able to split the sum and relabel indices, we have to reintroduce normal ordering, to ensure convergence of the parts. The convergence of the operator series can be understood as the convergence of the finite partial sums acting on any state in the Fock space. It suffices to check the action on the one-boson state $\alpha^J_{-n}|p^+, \vec{0}\rangle$ to see that the sum of the first term in eq. (3.2) does not converge. We therefore have to include the correction terms that arise from bringing the creation operators to the left of the Virasoro mode. Those second order terms in α take the form $\alpha^I_{-k}\alpha^J_k$:

$$\begin{aligned}: L_{-m}\alpha^I_{m-n}\alpha^J_n : &= L_{-m}\alpha^I_{m-n}\alpha^J_n + \theta(-n)[\alpha^J_n, L_{-m}]\alpha^I_{m-n} + \theta(n-m)[\alpha^I_{m-n}, L_{-m}]\alpha^J_n \\ &= L_{-m}\alpha^I_{m-n}\alpha^J_n + \theta(-n)n\alpha^I_{m-n}\alpha^J_{n-m} + \theta(n-m)(m-n)a^I_{-n}\alpha^J_n\end{aligned}$$

Here, we used the Heaviside step function $\theta(n)$. Similarly

$$: L_{-m-n}\alpha^I_m\alpha^J_n := L_{-m-n}\alpha^I_m\alpha^J_n + \theta(-n)n\alpha^J_{-m}\alpha^I_m + \theta(-m)m\alpha^I_{-n}\alpha^J_n.$$

All correction terms are already normal ordered since $I \neq J$ and therefore $[\alpha^I_{-k}, \alpha^J_k] = 0$. The sum from eq. (3.2) can be rewritten as

$$\sum_{n,m\neq 0}\left(\frac{1}{n}\left[: L_{-m}\alpha^I_{m-n}\alpha^J_n : - : L_{-m-n}\alpha^I_m\alpha^J_n :\right] - (m, I \leftrightarrow n, J) + C^{IJ}(m,n)\right), \tag{3.3}$$

with the correction term

$$C^{IJ}(m,n) = -\theta(-n)(\alpha^I_{m-n}\alpha^J_{n-m} + \alpha^J_{-m}\alpha^I_m) - \left[\theta(n-m)\left(\frac{m}{n}-1\right) + \theta(-m)\frac{m}{n}\right]\alpha^I_{-n}\alpha^J_n - (m, J \leftrightarrow n, I)$$
$$= \underbrace{[\theta(-m)-\theta(-n)]\alpha^I_{m-n}\alpha^J_{n-m}}_{=C_1^{IJ}(m,n)} + \underbrace{[\theta(n-m)-\theta(-m)]\frac{m-n}{n}\alpha^I_{-n}\alpha^J_n}_{=C_2^{IJ}(m,n)} - C_2^{JI}(n,m), \tag{3.4}$$

which consists only of second order terms in α. To arrive at eq. (3.4) we grouped terms with the same operators. Now that the individual parts are normal ordered, the sum can be split into convergent parts, which we can evaluate separately. In the first correction term, we relabel $n-m=k$, to obtain

$$\sum_{m,n\neq 0} C_1^{IJ}(m,n) = \sum_{k\in\mathbb{Z}} \alpha^I_{-k}\alpha^J_k \sum_{n\in\mathbb{Z}\setminus\{0,k\}} [\theta(k-n)-\theta(-n)] = \sum_{k=1}^{\infty} a^I_{-k}\alpha^J_k \sum_{n=1}^{k-1}[1] + \underbrace{\sum_{k\leq 0}\alpha^I_{-k}\alpha^J_k \sum_{n=k+1}^{-1}[-1]}_{k\to -k}$$
$$= \sum_{k=1}^{\infty}(k-1)a^{[I}_{-k}\alpha^{J]}_k.$$

Similarly, the second correction term yields

$$\sum_{m,n\neq 0} C_2^{IJ}(m,n) = \sum_{n\neq 0}\alpha^I_{-n}\alpha^J_n \sum_{m\neq 0}[\theta(n-m)-\theta(-m)]\frac{m-n}{n}$$
$$= \sum_{n=1}^{\infty} a^I_{-n}\alpha^J_n \sum_{m=1}^{n-1}[1]\frac{m-n}{n} + \underbrace{\sum_{n\leq -1}\alpha^I_{-n}\alpha^J_n \sum_{m=n}^{-1}[-1]\frac{m-n}{n}}_{n\to -n,\quad m-n\to k}$$
$$= \sum_{n=1}^{\infty} a^{[I}_{-n}\alpha^{J]}_n \frac{1}{n}\sum_{k=1}^{n-1} k = \sum_{n=1}^{\infty}\frac{n-1}{2}\alpha^{[I}_{-n}\alpha^{J]}_n = -\sum_{m,n\neq 0} C_2^{JI}(n,m).$$

By a symmetry argument the third one follows from the second.
We now take a closer look at the sum over m of the first term in eq. (3.3). We split it and reindex $m \to m-n$ so that it partially cancels:

$$\sum_{m\neq 0}[: L_{-m}\alpha^I_{m-n}\alpha^J_n : - : L_{-m-n}\alpha^I_m\alpha^J_n :] = \sum_{m\neq 0} : L_{-m}\alpha^I_{m-n}\alpha^J_n : - \sum_{m\neq n} : L_{-m}\alpha^I_{m-n}\alpha^J_n :$$
$$=: L_{-n}\alpha^I_0 a^J_n : - : L_0\alpha^I_{-n}\alpha^J_n : .$$

The second term of *eq.* (3.3) follows by exchanging indices. Finally by combining the previous results, we conclude

$$[S^{-I}, S^{-J}] = -\frac{1}{2\alpha'(p^+)^2}K^{IJ}, \tag{3.5}$$

where

$$K^{IJ} = \sum_{n\neq 0}\frac{1}{n}[: L_{-n}\alpha^{[I}_0 a^{J]}_n : - : L_0\alpha^{[I}_{-n}\alpha^{J]}_n :] + \sum_{n=1}^{\infty}\left(2(n-1) - \frac{A(n)}{n^2}\right)\alpha^{[I}_{-n}\alpha^{J]}_n. \tag{3.6}$$

3.4 MIXED TERM

Lastly, we compute $[S^{-I}, l^{-J}]$. Here we have to consider normal ordering, since the correction term $-i\alpha^J_0/\sqrt{2\alpha'}p^+$ does not commute with x^-_0. We find

$$[l^{-I}, S^{-J}] = \sum_{n\neq 0}\frac{1}{n}\left[x^-_0 p^I - \frac{x^I_0 L'_0}{2\alpha' p^+}, \frac{-i}{\sqrt{2\alpha'}p^+} : L_{-n}\alpha^J_n :\right]$$
$$= \sum_{n\neq 0}\frac{1}{n}\left(\frac{1}{\sqrt{2\alpha'}(p^+)^2} : L_{-n}\alpha^J_n p^I : + \frac{i}{\sqrt{2\alpha'}^3(p^+)^2}[x^I_0 L'_0, : L_{-n}\alpha^J_n :]\right) \tag{3.7}$$

In the remaining commutator, normal ordering can be neglected, since α_0^J commutes with the other factors. It hence evaluates to

$$\begin{aligned}
[x_0^I L_0', L_{-n}\alpha_n^J] &= [x_0^I, L_{-n}]L_0'\alpha_n^J + x_0^I[L_0', L_{-n}]\alpha_n^J + L_{-n}x_0^I[L_0', \alpha_n^J] \\
&= i\sqrt{2\alpha'}\alpha_{-n}^I(L_0' + n)\alpha_n^J + nx_0^I L_{-n}\alpha_n^J - nL_{-n}x_0^I\alpha_n^J \\
&= i\sqrt{2\alpha'}\alpha_{-n}^I(L_0' + n)\alpha_n^J + n[x_0^I, L_{-n}]\alpha_n^J \\
&= i\sqrt{2\alpha'}\alpha_{-n}^I(L_0' + n)\alpha_n^J = i\sqrt{2\alpha'}\alpha_n^J(L_0' - n)\alpha_{-n}^I \\
&= i\sqrt{2\alpha'} : L_0'\alpha_{-n}^I\alpha_n^J : + i\sqrt{2\alpha'}n(1 - 2\theta(-n))\alpha_{-n}^I\alpha_n^J
\end{aligned}$$

In the last line, we have reintroduced normal ordering. By using $\sqrt{2\alpha'}p^I = \alpha_0^I$ we can bring (3.7) into a similar form as (3.6)

$$[l^{-I}, S^{-J}] = \frac{1}{2\alpha'(p^+)^2}\left(\sum_{n\neq 0}\frac{1}{n}[: L_{-n}\alpha_0^I\alpha_n^J : - : L_0\alpha_{-n}^I\alpha_n^J :] + \left(2\theta(-n) - 1 - \frac{a}{n}\right)\alpha_{-n}^I\alpha_n^J\right)$$

and using the antisymmetry of the commutator we find by interchanging $I \leftrightarrow J$ the other commutator can be evaluated as well. In total, we conclude

$$[l^{-I}, S^{-J}] + [S^{-I}, l^{-J}] = \frac{1}{2\alpha'(p^+)^2}\left(\sum_{n\neq 0}\frac{1}{n}[: L_{-n}\alpha_0^{[I}\alpha_n^{J]} : - : L_0\alpha_{-n}^{[I}\alpha_n^{J]} :] - \sum_{n=1}^{\infty}\left(\frac{2a}{n} + 2\right)a_{-n}^{[I}\alpha_n^{J]}\right).$$

For the second order term we explicitly computed

$$\begin{aligned}
\sum_{n\neq 0}\alpha_{-n}^{[I}\alpha_n^{J]} &= 0 \\
\sum_{n<0}2\alpha_{-n}^{[I}\alpha_n^{J]} &= -\sum_{n=1}^{\infty}2\alpha_{-n}^{[I}\alpha_n^{J]}. \\
-\sum_{n\neq 0}\frac{a}{n}\alpha_{-n}^{[I}\alpha_n^{J]} &= -\sum_{n=1}^{\infty}\frac{a}{n}\alpha_{-n}^{[I}\alpha_n^{J]} - \sum_{n<0}\frac{a}{n}\alpha_{-n}^{[I}\alpha_n^{J]} = -\sum_{n=1}^{\infty}\frac{2a}{n}\alpha_{-n}^{[I}\alpha_n^{J]},
\end{aligned}$$

by reindexing $n \to -n$ and using the antisymmetry of $[I\ J] = -[I\ J]$.

3.5 CONCLUSION

Combining the results from previous calculations, the sum containing fourth order terms in α vanishes and only the second order remains

$$\begin{aligned}
[M^{-I}, M^{-J}] &= [l^{-I}, S^{-J}] - [l^{-J}, S^{-I}] + [S^{-I}, S^{-J}] \\
&= -\frac{1}{2\alpha'(p^+)^2}\sum_{n=1}^{\infty}\left(2\frac{a}{n} + 2 + 2(n-1) - \frac{A(n)}{n^2}\right)a_{-n}^{[I}\alpha_n^{J]} \\
&= -\frac{1}{\alpha'(p^+)^2}\sum_{n=1}^{\infty}\left(n\left[1 - \frac{D-2}{24}\right] + \frac{1}{n}\left[\frac{D-2}{24} + a\right]\right)a_{-n}^{[I}\alpha_n^{J]},
\end{aligned} \tag{3.8}$$

which agrees with eq. 12.152 from [1]. If quantization is successful, we expect the Lorentz algebra symmetry to be preserved and

$$[M^{-I}, M^{-J}] = 0$$

For the commutator to vanish for all n, we require the square brackets in eq. (3.8) to vanish. Therefore, the critical dimension and normal ordering constant of string theory are

$$\boxed{D = 26, \qquad a = -1.}$$

4

SUMMARY AND OUTLOOK

In this thesis, we successfully verified the computation of the commutator of the Lorentz generators $[M^{-I}, M^{-J}]$, which was omitted in section 12.2 of Zwiebach's textbook [1]. This meticulous computation repeatedly utilized the commutation relations of the Virasoro algebra as detailed in section 2.3. A similar but slightly different derivation can be found in section 4.2.1 of Gleb Arutyunov's lecture notes [2]. While the general steps are consistent, the solution presented here reduces computational overhead by omitting normal ordering at intermediate stages and reintroducing it later. This approach is advantageous when aiming to show that all terms of order four and six vanish, allowing section 3.3 to be skipped, as the normal ordering correction term eq. (3.4) only includes second-order terms. Alternatively, second-order terms can be derived by sandwiching the expression

$$\langle p^+, \vec{0}|\alpha^I_m[M^{-I}, M^{-J}]\alpha^J_{-m}|p^+, \vec{0}\rangle$$

as suggested in problem set 6 by Gaberdiel [3].

Our computation confirms that bosonic string theory necessitates 26 spacetime dimensions for successful quantization. Additionally, the constant a arising from the ordering ambiguity in eq. (2.8) is determined to be -1 for open strings, leading to the Hamiltonian:

$$H = L_0 - 1.$$

Despite our experience of only four spacetime dimensions, this does not invalidate string theory. The proposed resolution is the compactification of the additional dimensions, analogous to a thin cylinder that appears as a one-dimensional line from a distance. However, the experimental search for extra dimensions continues.

A

DECLARATION OF ORIGINALITY

ETH
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Swiss Federal Institute of Technology Zurich

Declaration of originality

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[1] E.g. ChatGPT, DALL E 2, Google Bard
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[3] E.g. ChatGPT, DALL E 2, Google Bard

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[1] Barton Zwiebach. *A First Course in String Theory*. 2nd ed. Cambridge University Press, 2009.

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