PROBLEM SET 9

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I adhered to the honor code on this assignment.

Additional Problem 1.

(a) *Claim.* For the space $(\mathcal{C}(K), \|.\|_\infty)$, where K is a compact subset of $\mathbb{C}^N$, closed and bounded subsets of this space need not be compact.

Proof. We will show that the set $F := \{f \in \mathcal{C}([-1,1]) : \|f\|_\infty \leq 1\}$ is not compact, despite being closed and bounded.

To begin, let (f_n) be a sequence in $\mathcal{C}([-1,1])$ defined by

$$f_n : [-1,1] \to \mathbb{R},\ f_n(x) = \begin{cases} 1 - n|x| & \text{, if } |x| \leq \frac{1}{n} \\ 0 & \text{, otherwise.} \end{cases}$$

This function can only take values in $[0,1]$, so it is bounded in sup norm by 1, so it is in F. Additionally, this sequence converges pointwise to $\chi_{\{0\}}$. As proof, take $x \in [-1,1]$.

If $x = 0$, then $f_n(x) = 1$ for all n.

If $x \neq 0$, then take $n \in \mathbb{N}$ such that $\frac{1}{n} < |x|$. Then $f_n(x) = 0$ for this n.

The pointwise convergence of (f_n) implies that any uniformly convergent (or pointwise convergent) subsequence of (f_n) must also converge to $\chi_{\{0\}}$. The function $\chi_{\{0\}}$ is not continuous, so it is not in $\mathcal{C}([-1,1])$, meaning that F is not compact. □

(b) A family $\mathcal{F}$ of functions is **Equicontinuous** if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $f \in \mathcal{F}$, one has

$$|f(x) - f(y)| < \epsilon \text{ , whenever } \|x - y\| < \delta.$$

This definition differs from uniform continuity because uniform continuity is a property of only one function f. Equicontinuity gives a value of δ that works for all $f \in \mathcal{F}$, whereas for uniform continuity, the value for δ is specific to one function.

To show for (f_n) from (a) that the family $\{f_n : n \in \mathbb{N}\}$ is not equicontinuous, first let $1 > epsilon > 0$, then fix $\delta > 0$. Then for

$$\|x - y\| < \delta \text{ , where } x = 0,$$

for f_n where $\frac{1}{n} < \|x - y\|$, we have

$$|f_n(x) - f_n(y)| = |1 - 0| = 1.$$

Therefore $\{f_n : n \in \mathbb{N}\}$ is not equicontinuous.

(c) **Theorem.** $\mathcal{F}$ is compact in $(\mathcal{C}(K), \|.\|_\infty)$ if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.

Proof. compactness implies closedness and boundedness in a metric space, so one direction of this is trivial.

For the remaining direction, it suffices to show that given an arbitrary sequence (f_n) in $\mathcal{F}$, there exists a uniformly convergent subsequence (g_n) of (f_n) with limit in $\mathcal{F}$.

(i) Since $(\mathbb{Q}+i\mathbb{Q})^N$ is dense in $\mathbb{C}^N$, K has a dense countable subset

$$D := \{x_k\ ,\ k \in \mathbb{N}\}.$$

Let (f_n) be an arbitrary sequence in $\mathcal{F}$. Then for $x_1 \in D$, the sequence $(f_n(x))$ is a bounded sequence in $\mathbf{F}$, so by Bolzano-Weierstrass, there exists a convergent subsequence $(f_{1,k}(x_1))$.

Then construct the sequence $(f_{1,k}(x_2))$, also bounded in $\mathbf{F}$. Once again, Bolzano-Weierstrass gives us a convergent subsequence $(f_{2,k}(x_2))$. Note that because each $f_k \in (f_{2,k})$ is also in $(f_{1,k})$, we know $(f_{2,k}(x_1))$ converges.

Repeating this process gives convergent subsequences $(f_{j,k}(x_j))$ for all $j \in \mathbb{N}$. Notably, $(f_{j,k})$ is a subsequence of (f_n) where $(f_{j,k}(x_i))$ converges for $i \leq j$.

We finally define the subsequence (g_n) of (f_n) by

$$g_n := f_{n,n}.$$

Then for $x_m \in D$, we know $(f_{n,k}(x_m))_{k\in\mathbb{N}}$ is a convergent sequence in $\mathbf{F}$ when $n \geq m$. Therefore $g_n(x_m) = (f_{n,n}(x_m))_{n\in\mathbb{N}}$ convergess, so g_n converges pointwise for points in D.

(ii) Our space $(\mathcal{C}(K), \|.\|_\infty)$ is a banach space, so it suffices to show (g_n) is uniformly Cauchy. To do so, begin by letting $\epsilon > 0$. Then because $\mathcal{F}$ is equicontinuous, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$,

$$|f_n(x) - f_n(y)| < \epsilon \text{ , when } \|x - y\| < \delta. \tag{1}$$

Because D is dense in K, we have

$$K \subseteq \bigcup_{x\in D} B_\delta(x).$$

This is an open cover of K, and because K is compact, there exists a finite subcover of K:

$$x_1, \dots x_j \in D \text{ such that } K \subseteq \bigcup_{i=1}^{j} B_\delta(x_i).$$

Because (g_n) is convergent on D, it is Cauchy on D. Therefore for all $x_i \in x_1, \dots, x_j$, there exists $N_i \in \mathbb{N}$ such that

$$|g_n(x_i) - g_m(x_i)| < \epsilon \text{ , for } n, m \geq N. \tag{2}$$

Because $x_1, \dots, x_j$ is a finite collection, we can define $N := \max\{N_i : 1 \leq i \leq j\}$.

Then take $x \in K$. By above, there exists $x_i \in x_1, \ldots, x_j$ such that $x \in B_\delta(x_k)$. Then for $n, m \geq N$,

$$\begin{aligned} |g_n(x) - g_m(x)| &= |g_n(x) - g_n(x_i) + g_n(x_i) - g_m(x_i) + g_m(x_i) - g_m(x)| \\ &\leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)| \\ &< 3\epsilon \end{aligned} \tag{3}$$

For above, the first and third term of (3) are each bounded by ϵ by (1), as $\|x - x_i\| < \delta$. The second term is bounded by ϵ by (2), as $x_i \in x_1, \ldots x_j$.

We conclude then that (g_n) is uniformly Cauchy, and therefore uniformly convergent in $\mathcal{F}$ □

0.1. **8A.14.** We want to prove the statement

$$\langle f, g\rangle = \frac{\|f+g\|^2 - \|f-g\|^2 + \|f+ig\|^2 - \|f-ig\|^2}{4}.$$

We will begin by splitting the right side of above into the imaginary terms and the real terms. For the real terms,

$$\begin{aligned} \|f+g\|^2 - \|f-g\|^2 &= \langle f+g, f+g\rangle - \langle f-g, f-g\rangle \\ &= \langle f, f+g\rangle + \langle g, f+g\rangle - \langle f, f-g\rangle + \langle g, f-g\rangle \quad (4) \\ &= \overline{\langle f+g, f\rangle} + \overline{\langle f+g, g\rangle} - \overline{\langle f-g, f\rangle} + \overline{\langle f-g, g\rangle} \quad (5) \\ &= \overline{\langle f, f\rangle} + \overline{\langle g, f\rangle} + \overline{\langle f, g\rangle} + \overline{\langle g, g\rangle} - \overline{\langle f, f\rangle} + \overline{\langle g, f\rangle} + \overline{\langle f, g\rangle} - \overline{\langle g, g\rangle} \quad (6) \\ &= 4Re\langle f, g\rangle. \quad (7) \end{aligned}$$

For above, (4) and (6) come from linearity in the first slot of an inner product. (5) comes from conjugate symmetry of the inner product. (7) comes from the fact that $\langle f, g\rangle + \overline{\langle f, g\rangle} = 2Re\langle f, g\rangle$, or equivalently that $\overline{\langle f, g\rangle} + \overline{\langle g, f\rangle} = 2Re\langle f, g\rangle$.

The proof of the imaginary terms is similar:

$$\begin{aligned} &\|f+ig\|^2 i - \|f-ig\|^2 i \\ &= i(\langle f+ig, f+ig\rangle - \langle f-ig, f-ig\rangle) \\ &= i\big[\langle f, f+ig\rangle + \langle ig, f+ig\rangle - \langle f, f-ig\rangle + \langle ig, f-ig\rangle\big] \\ &= i\big[\overline{\langle f+ig, f\rangle} + \overline{\langle f+ig, ig\rangle} - \overline{\langle f-ig, f\rangle} + \overline{\langle f-ig, ig\rangle}\big] \\ &= i\big[\overline{\langle f, f\rangle} + \overline{\langle ig, f\rangle} + \overline{\langle f, ig\rangle} + \overline{\langle ig, ig\rangle} - \overline{\langle f, f\rangle} + \overline{\langle ig, f\rangle} + \overline{\langle f, ig\rangle} - \overline{\langle ig, ig\rangle}\big] \\ &= i\big[-4Rei\langle f, g\rangle\big] = 4Im\langle f, g\rangle i \end{aligned}$$

The main thing to note for the imaginary terms is that the conjugate of i, $\bar{i} = -i$.

Combining both the real and imaginary parts, we have

$$4Re\langle f,g\rangle + 4Im\langle f,g\rangle i = 4\langle f,g\rangle.$$

8A.16. *Claim.* If V is a normed vector space whose norm $\|.\|$ satisfies the parallelogram equality, then there is an inner product $\langle\cdot,\cdot\rangle$ on V such that $\|f\| = \langle f,f\rangle^{\frac{1}{2}}$ for all $f\in V$.

Proof. We will define the candidate for an inner product as

$$\langle f,g\rangle := \frac{\|f+g\|^2-\|f-g\|^2}{4},$$

and prove that it satisfies the properties of an inner product.

Positivity

$\langle f,f\rangle = \frac{\|2f\|^2-\|0\|^2}{4} = \|f\|^2$, which is nonnegative by properties of the norm.

Definiteness

Again because $\langle f,f\rangle = \|f\|^2$, we know by properties of the norm that it is 0 if and only if f is 0.

Linearity

$$\begin{aligned}
&(8)\\
&\langle f+g,h\rangle = \frac{1}{4}\big[\|f+g+h\|^2 - \|f+g-h\|^2\big]\\
&(9)\qquad = \frac{1}{4}\big[(2\|f+h\|^2+2\|g\|^2-\|f+h-g\|^2) - (2\|f-h\|^2+2\|g\|^2-\|f-h-g\|^2)\big]\\
&\qquad\quad = \frac{1}{4}\big[2\|f+h\|^2-2\|f-h\|^2+\|f-h-g\|^2-\|f+h-g\|^2\big]\\
&(10)\qquad = \frac{1}{4}\big[2\|f+h\|^2-2\|f-h\|^2+(2\|g-h\|^2+2\|f\|^2-\|g-h+f\|^2)\\
&\qquad\qquad\qquad - (2\|g+h\|^2+2\|f\|^2-\|f+g+h\|)\big]\\
&\qquad\quad = \frac{1}{4}\big[2\|f+h\|^2-2\|f-h\|^2+2\|g+h\|^2-2\|g-h\|^2\\
&\qquad\qquad\qquad - (\|f+g+h\|^2-\|f+g-h\|^2)\big]\\
&(11)\qquad = 2\langle f,h\rangle + 2\langle g,h\rangle - (2\|g+h\|^2+2\|f\|^2-\|f+g+h\|)
\end{aligned}$$

For above, the parallelogram equality is used in (9) and (10). The important part of this expansion is that in the parentheses of (11) the first expansion of the inner product in (8). Therefore we can add it to both sides to get

$$2\langle f+g,h\rangle = \frac{2}{4}\big[\|f+g+h\|^2-\|f+g-h\|^2\big] = 2\langle f,h\rangle + 2\langle g,h\rangle$$

Homogeneity

To begin, assume $\alpha\in\mathbb{Z}$. Then $\langle\alpha f,g\rangle = \alpha\langle f,g\rangle$ because you can use linearity to split the inner product into α terms of $\langle f,g\rangle$.

Then assume $\alpha = \frac{1}{n}$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}\langle \frac{1}{n} f, g\rangle &= \frac{\|\frac{1}{n}f+g\|^2 - \|\frac{1}{n}f-g\|^2}{4} \\ &= \frac{\frac{1}{n^2}\|f+ng\|^2 - \frac{1}{n^2}\|f-ng\|^2}{4} \\ &= \frac{1}{n^2}\langle f, ng\rangle = \frac{1}{n}\langle f, g\rangle.\end{aligned}$$

This shows homogeneity for any $\alpha \in \mathbb{Q}$, as a number in $\mathbb{Q}$ can be represented by a number in $\mathbb{Z}$ plus $\frac{1}{n}$ for some $n \in \mathbb{N}$.

Now pick $\alpha \in \mathbb{R}$. Then there exists (a_n), a sequence in $\mathbb{Q}$ such that $(a_n) \to \alpha$.

$$\begin{aligned}\langle \alpha f, g\rangle &= \frac{\|\alpha f+g\|^2 - \|\alpha f-g\|^2}{4} \\ &= \frac{\|\lim a_n f+g\|^2 - \|\lim a_n f-g\|^2}{4} \\ &= \lim_{n\to\infty} \frac{\|a_n f+g\|^2 - \|a_n f-g\|^2}{4} \qquad (12) \\ &= \lim_{n\to\infty} \frac{a_n}{4}\left[\|f+g\|^2 - \|f-g\|^2\right] \qquad (13) \\ &= \lim_{n\to\infty} a_n\langle f, g\rangle \\ &= \alpha\langle f, g\rangle.\end{aligned}$$

For above, we can move the limit outside the norm in (12) by continuity of the norm. In (13), we are able to move a_n out of the norm because $a_n \in \mathbb{Q}$. □

Additional Problem 3.

(a) *Claim.* For all $x \in [0, 1]$, we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = 1.$$

Proof. By the binomial theorem, we know

$$\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = (x + (1-x))^n = 1^n = 1 \text{ , when } x \in [0, 1].$$

□

(b) *Claim.* For $f \in \mathcal{C}([0, 1])$, let (p_n) be defined as

$$p_n(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k} \text{ , } x \in [0, 1].$$

Then $\|p_n - f\|_\infty \to 0$ as $n \to \infty$.

This proof will use the identity

$$\sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^k (1-x)^{n-k} = nx(1-x) \text{ , for all } x \in [0,1]. \tag{14}$$

Proof. Let $\epsilon > 0$. Because $f \in \mathcal{C}([0,1])$, f is continuous on a compact domain, so f is uniformly convergent. Then let $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ , for all } x, y \in [0,1] \text{ with } |x-y| < \delta. \tag{15}$$

By the result of (a), for all $x \in [0,1]$, we have

$$\begin{aligned}
\|p_n(x) - f(x)| &= |p_n(x) - f(x) \cdot 1| \\
&= \left| \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} - f(x) \left(\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \right) \right| \\
&= \left| \sum_{k=0}^{n} \binom{n}{k} \left(f\left(\frac{k}{n}\right) - f(x) \right) x^k (1-x)^{n-k} \right|. \qquad (16)
\end{aligned}$$

We will now partition the sum over k by distinguishing whether or not k satisfies $|\frac{k}{n} - x| \geq \delta$. We will define

$$B_\delta := \{k \in \{0, \ldots, n\} : |\frac{k}{n} - x| \geq \delta\}.$$

Note that for $k \in B_\delta$, we have

$$1 \leq \frac{1}{\delta^2} \left(x - \frac{k}{n} \right)^2. \tag{17}$$

Now we will examine the sum in (16) for values of k in B_δ. In this case,

$$\begin{aligned}
(16) &= \sum_{k \in B_\delta} 1 \cdot \binom{n}{k} \left| f\left(\frac{k}{n}\right) - f(x) \right| x^k (1-x)^{n-k} && (18) \\
&\leq \sum_{k \in B_\delta} \frac{1}{\delta^2} \left(x - \frac{k}{n} \right)^2 \cdot \binom{n}{k} \left| f\left(\frac{k}{n}\right) - f(x) \right| x^k (1-x)^{n-k} && (19) \\
&\leq \sum_{k \in B_\delta} \frac{1}{\delta^2} \left(x - \frac{k}{n} \right)^2 \cdot \binom{n}{k} 2\|f\|_\infty \cdot x^k (1-x)^{n-k} \\
&= \frac{1}{\delta^2} \cdot \frac{1}{n^2} \|f\|_\infty \cdot \sum_{k \in B_\delta} \binom{n}{k} (k-nx)^2 x^k (1-x)^{n-k} && (20) \\
&= \frac{1}{\delta^2} \cdot \frac{1}{n^2} \|f\|_\infty \cdot nx(1-x) && (21) \\
&= \frac{1}{\delta^2} \cdot \frac{1}{n} \|f\|_\infty \cdot x(1-x) && (22)
\end{aligned}$$

For above, the equality in (18) holds because everything outside the absolute value is already nonnegative. (19) comes from (17). (20) involves pulling a $\frac{1}{n^2}$ out of $\left(x - \frac{k}{n}\right)^2$. (21) comes from using the identity (14). Finally, (22) converges uniformly over all $x \in [0,1]$ because $x(1-x)$ is bounded for $x \in [0,1]$, so there exists an $N \in \mathbb{N}$ such that

$$(22) < \epsilon \text{ , for all } n \geq \mathbb{N} \text{ and all } x \in [0,1].$$

This handles (16) for terms of the sum where $k \in B_\delta$. For $k \notin B_\delta$, note that inside the inner parentheses of (16), we have $f(\frac{k}{n}) - f(x)$, and because $k \notin B_\delta$, we know $|\frac{k}{n} - x| < \delta$. Then by (15), the quantity in the parentheses is bounded by ϵ. The rest of (16) inside the sum is finite when $x \in [0,1]$ by the results of (a), and this is a finite sum, so the terms of the sum in (16) where $k \notin B_\delta$ are bounded by a factor of ϵ.

Therefore the whole sum is bounded by a factor of ϵ, meaning $\|p_n(x) - f(x)\|_\infty < \epsilon$, for sufficiently large n. We can bound the supremum norm here because we can take n large enough to be a bound for all $x \in [0,1]$, as previously described. Therefore $\|p_n - f\| \to 0$. □