Problem Set 4

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Aditional Problem 1

(a) For $\emptyset \neq E \subseteq \mathbb{R}$ and $y \in \mathbb{R}$, define the distance of y to E as

$$\operatorname{dist}(y; E) := \inf_{x \in E} |x - y| \in [0, +\infty)$$

Claim. For all $y, z \in \mathbb{R}$, one has

$$|\operatorname{dist}(y; E) - \operatorname{dist}(z; E)| \leq |y - z|. \tag{1}$$

Furthermore, this implies that the map $y \mapsto \operatorname{dist}(y; E)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

Proof. For the proof of (1), let $\epsilon > 0$. Then there exists $w \in E$ such that

$$|w - z| - \epsilon < \operatorname{dist}(y; E). \tag{2}$$

We can use this inequality to estimate the left side of (1):

$$\begin{aligned}
|\operatorname{dist}(y; E) - \operatorname{dist}(z; E)| &= |\inf_{x \in E} |x - y| - \inf_{v \in E} |v - z|| \\
&< |\inf_{x \in E} |x - y| - (|w - z| - \epsilon)| \qquad (3) \\
&\leq ||w - y| - |w - z|| + \epsilon \qquad (4) \\
&\leq |w - y - w + z + \epsilon| \\
&\leq |y - z| + \epsilon
\end{aligned}$$

where (3) holds because of (2) and (4) holds by the definition of the infimum. Because the inequality in (3) is strict, we can remove the ϵ and our proof of (1) is complete.

To show continuity of the map $y \mapsto \operatorname{dist}(y; E)$, let $\epsilon > 0$. Then let $\delta = \epsilon$. Then for all $x \in E$ where $|x - y| < \epsilon$, we know by (1) that

$$|\operatorname{dist}(y; E) - \operatorname{dist}(x, E)| \leq |y - z| < \delta = \epsilon.$$

Therefore this map is continuous. $\square$

(b) *Claim.* Assume $\epsilon \neq \emptyset$ is closed. Then $\operatorname{dist}(y; E) = 0$ if and only if $y \in E$.

Proof. For the forward direction of this claim, assume that $\operatorname{dist}(y; E) = 0$. Then $\inf_{x \in E} |x - y| = 0$, and for all $\epsilon > 0$, there exists $w \in E$ such that

$$\left| \inf_{x \in E} |x - y| - |w - y| \right| < \epsilon.$$

Then let $\epsilon = \frac{1}{n}$, and use this statement to construct the sequence $(W_n)_{n \in \mathbb{N}}$ where for all $n \in \mathbb{N}$, the statement holds for w_n. Then $\lim_{n \to \infty} w_n = y$. But E is a closed set, so it contains its limit points, and $w_n \in E$ for all $n \in \mathbb{N}$, so $y \in E$.

For the backward direction of the claim, assume that $y \in E$. Then $\operatorname{dist}(y; E) = \inf_{x \in E} |x - y|$, but $\operatorname{dist}(y; E) \geq 0$, and $|y - y| = 0$, so $\inf_{x \in E} |x - y| = 0$. □

For a counterexample when E is not closed, let $E = (0, 1)$, and let $y = 0$. Then for $n \in \mathbb{N}$, let $\epsilon = \frac{1}{2n}$. We know that for all $n \in \mathbb{N}$, we have

$$y + \frac{1}{2n} = \frac{1}{2n} \in (0, 1) = E.$$

Therefore $\inf_{x \in E} |x - y| \leq \frac{1}{2n}$ for all $n \in \mathbb{N}$, so $\inf_{x \in E} |x - y| = 0$, meaning $\operatorname{dist}(y; E) = 0$ for $y \notin E$, completing the counterexample.

(c) *Urysohn's Lemma.* Let E, F be two *non-empty, disjoint, and closed* sets in $\mathbb{R}$. Then, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $0 \leq g \leq 1$ such that $g = 0$ on E and $g = 1$ on F.

Proof. Let E, F be as described in the lemma and let

$$g(y) := \frac{\operatorname{dist}(y; E)}{\operatorname{dist}(y; E) + \operatorname{dist}(y; F)}, \quad y \in \mathbb{R}.$$

First, assume $y \in E$. Then $\operatorname{dist}(y; E) = 0$ by (b), and because E and F are disjoint, $y \notin F$, so again by (b) we know $\operatorname{dist}(y; F) > 0$. Therefore $g(y) = 0$.

Then assume $y \in F$. Then $\operatorname{dist}(y; F) = 0$ and $\operatorname{dist}(y; E) > 0$, so

$$g(y) = \frac{\operatorname{dist}(y; E)}{\operatorname{dist}(y; E)} = 1.$$

Finally, assume $y \notin E$ and $y \notin F$. Then both $\operatorname{dist}(y; E)$ and $\operatorname{dist}(y; F)$ are positive, so $0 < g(x) < 1$. □

Additional Problem 2

For $\mu_{\mathbb{N}}$, the counting measure of the natural numbers, let $(\mathbb{R}, \mathcal{P}, \mu_{\mathbb{N}})$ be a measure space. Additionally, let $0 \leq f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) *Claim.* The statement

$$\int f d\mu_{\mathbb{N}} = \sum_{n=1}^{\infty} f(n) \tag{5}$$

holds for all finitely supported, non-negative, simple functions.

Proof. Because f is a simple function, it can be written as $\sum_{k=1}^{N} c_k \chi_{E_k}$ for some $c_1, \ldots c_N \in \mathbb{R}$ and $E_1, \ldots E_N \subseteq \mathbb{R}$. Then by the definition of the integral of simple functions, we know

$$\int f d\mu_{\mathbb{N}} = \sum_{k=1}^{N} c_k \mu_{\mathbb{N}}(E_k).$$

Now looking at the right side of (5), observe that for any $n \in N$,

$$f(n) = \begin{cases} c_k & \text{if } x \in E_k \\ 0 & \text{if } x \notin E_k \text{ for all k} \end{cases}$$

Therefore for any c_k, the sum on the right side of (5) adds that c_k term for each $n \in E_k$, where $n \in N$. But this is precisely $c_k \mu_{\mathbb{N}}(E_k)$, so

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{N} c_k \mu_{\mathbb{N}}(E_k) = \int f d\mu_{\mathbb{N}}.$$

□

(b) *Lemma. Given a measure space $(Y, \mathcal{S}, \mu)$, suppose that $(Y_n)_{n \in N}$ is an $\mathcal{S}$-valued sequence of nested increasing sets such that*

$$Y = \bigcup_{n \in \mathbb{N}} Y_n.$$

Then for every measurable function $0 \leq f : Y \to \mathbb{R}$, one has

$$\lim_{n \to \infty} \int \chi_{Y_n} f d\mu = \int f d\mu$$

(c) *Proof of Lemma.* Let $(f_n)_{n\in N}$ be a series of functions defined by $f_n = \chi_{Y_n} f$. Then $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of measurable functions. We also know $\lim_{n\to\infty} f_n = f$, so we can apply the monotone convergence theorem to get

$$\lim_{n\to\infty} \int \chi_{Y_n} f d\mu = \int f d\mu.$$