Problem Set 5

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I adhered to the honor code on this assignment.

3A.10

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of nonnegative $\mathcal{S}$-measurable functions. Define $f : X \to [0, \infty]$ by $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Prove that

$$\int f \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Proof. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of functions defined by

$$g_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then g_n is monotone increasing and converges to f pointwise. Therefore by the monotone convergence theorem, we have

$$\begin{aligned}
\int f \, d\mu &= \lim_{n \to \infty} \int g_n \, d\mu \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} f_k \, d\mu \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} \int f_k \, d\mu \\
&= \sum_{k=1}^{\infty} \int f_k \, d\mu.
\end{aligned}$$

The third inequality falls from linearity of the integral over finite sums. □

3A.15

Suppose λ is Lebesque measure on $\mathbb{R}$ and $f : \mathbb{R} \to [-\infty, \infty]$ is a Borel measurable function such that $\int f \, d\lambda$ is defined.

(a) For $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to [-\infty, \infty]$ by $f_t(x) = f(x - t)$. Prove that $\int f_t \, d\lambda = \int f \, d\lambda$ for all $t \in \mathbb{R}$.

Proof. Both f and f_t are Borel measurable functions, so there exists $(\varphi_n)_{nin\mathbb{N}}$ and $(\varphi_n^t)_{n \in \mathbb{N}}$, sequences of simple functions such that $\varphi_n \to f$ and $\varphi_n^t \to f_t$ pointwise. Then by the monotone convergence theorem, we have

$$\int f \, d\lambda = \lim_{n \to \infty} \int \varphi_n \, d\lambda \quad \text{and} \quad \int f_t \, d\lambda = \lim_{n \to \infty} \int \varphi_n^t \, d\lambda. \tag{1}$$

But because φ_n and φ_n^t are simple functions for all $n \in \mathbb{N}$, then for all $n \in \mathbb{N}$, there exists $c_1, \ldots, c_K \in \mathbb{R}$ and $E_1, \ldots, E_k \in \mathcal{B}(\mathbb{R})$ such that

$$\begin{aligned}
\lim_{n\to\infty} \int \varphi_n \, d\lambda &= \lim_{n\to\infty} \int \sum_{k=1}^{K} c_k \chi_{E_k} \, d\lambda \\
&= \lim_{n\to\infty} \sum_{k=1}^{K} c_k \lambda(E_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{K} c_k \lambda(E_k + t) \\
&= \lim_{n\to\infty} \int \varphi_n^t \, d\lambda.
\end{aligned}$$

Therefore, by (1), we have $\int f \, d\mu = \int f_t \, d\mu$. □

(b) For $t \in \mathbb{R} \setminus \{0\}$, define $f_t : \mathbb{R} \to [-\infty, \infty]$ by $f_t(x) = f(tx)$. Prove that $\int f_t \, d\lambda = \frac{1}{|t|} \int f \, d\lambda$ for all $t \in \mathbb{R} \setminus \{0\}$.

Proof. As in part (a), there exist borel measurable functions φ_n and φ_n^t such that $\varphi_n \to f$ and $\varphi_n^t \to f_t$ pointwise. Then by the monotone convergence theorem, we have

$$\int f \, d\lambda = \lim_{n\to\infty} \int \varphi_n \, d\lambda \quad \text{and} \quad \int f_t \, d\lambda = \lim_{n\to\infty} \int \varphi_n^t \, d\lambda. \tag{2}$$

But because φ_n and φ_n^t are simple functions for all $n \in \mathbb{N}$, then for all $n \in \mathbb{N}$, there exists $c_1, \ldots, c_K$ and $E_1, \ldots, E_k \in \mathcal{B}(\mathbb{R})$ such that

$$\begin{aligned}
\frac{1}{|t|} \lim_{n\to\infty} \int \varphi_n \, d\lambda &= \frac{1}{|t|} \lim_{n\to\infty} \int \sum_{k=1}^{K} c_k \chi_{E_k} \, d\lambda \\
&= \frac{1}{|t|} \lim_{n\to\infty} \sum_{k=1}^{K} c_k \lambda(E_k) \\
&= \frac{1}{|t|} \lim_{n\to\infty} \sum_{k=1}^{K} c_k |t| \lambda(E_k \cdot \frac{1}{|t|}) \\
&= \lim_{n\to\infty} \sum_{k=1}^{K} c_k \lambda(E_k \cdot \frac{1}{|t|}) \\
&= \lim_{n\to\infty} \int \varphi_n^t \, d\lambda.
\end{aligned}$$

□

Therefore, by (2), we have $\int f \, d\lambda = \frac{1}{|t|} \int f_t \, d\lambda$.

3A.17

Suppose that $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of non-negative $\mathcal{S}$-measurable functions on X. Define a function $f : X \to [0, \infty]$ by $f(x) = \liminf_{k\to\infty} f_k(x)$.

(b) Prove that

$$\int f \, d\mu \leq \liminf_{k\to\infty} \int f \, d\mu.$$

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions defined by

$$g_n(x) = \inf\{f_n(x), f_{n+1}(x), \ldots\}.$$

Then $g_n \to f$ pointwise, and g_n is monotone increasing. Therefore by the monotone convergence theorem, we have

$$\begin{aligned} \int f \, d\mu &= \lim_{n\to\infty} \int g_n \, d\mu \\ &\leq \liminf_{k\to\infty} \int f_k \, d\mu. \end{aligned}$$

□

(c) Give an example showing that the inequality in (b) can be a strict inequality even when $\mu(X) < \infty$ and the family of functions $\{f_k\}_{k\in\mathbb{N}}$ is uniformly bounded.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined by $f_n = \chi_{[0,\frac{1}{2}]}$ if n is odd and $f_n = \chi_{[\frac{1}{2},1]}$ if n is even. Then $X = [0, 1]$ has finite measure, and $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded. Also, let $f : X \to [0, \infty]$ be defined by $f(x) = \liminf_{k\to\infty} f_k(x)$. Then

$$\int f \, d\mu = \int \lim_{k\to\infty} \inf_{j\geq k} f_j \, d\mu.$$

But this is 0 because the limit in the right integral is 0. However, $\liminf_{k\to\infty} \int f \, d\mu = \frac{1}{2}$. □

Additional Problem 1

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. For $n \in \mathbb{N}$, let P_n denote the partition that divides $[a, b]$ into 2^n intervals of equal size. Prove that

$$L(f, [a, b]) = \lim_{n\to\infty} L(f, P_n, [a, b]) \text{ and } U(f, [a, b]) = \lim_{n\to\infty} U(f, P_n, [a, b]).$$

Proof. Without loss of generality, let $[a, b] = [0, 1]$. Then because the dyadic rationals are dense in $[0, 1]$ the upper and lower riemann sums are the same if taken over the infimum and supremum respectively over only partitions of dyadic rational endpoints. Then for any partition P with only dyadic rational endpoints, we know that eventually in $\mathbb{N}$, P_n is a finer partition than P. Therefore eventually in $\mathbb{N}$,

$$L(f, P, [0, 1]) \leq L(f, P_n, [0, 1]) \text{ and } U(f, P, [0, 1]) \geq U(f, P_n, [0, 1]).$$

Because this holds true for all partitions P, we know

$$L(f, [0, 1]) \leq \lim_{n\to\infty} L(f, P_n, [0, 1]) \text{ and } U(f, [0, 1]) \geq \lim_{n\to\infty} U(f, P_n, [0, 1]).$$

The other side of the inequality holds by definition, so the claim is proven. □

Additional Problem 2

Let $(X, \mathcal{S}, \mu)$ be a measure space and suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions on $(X, \mathcal{S}, \mu)$ with values in $[0, +\infty]$. Suppose that $f := \lim_{n\to\infty} f_n$ exists pointwise with $\int f \, d\mu < +\infty$, and that one also has

$$0 \leq f_n \leq f, \text{ for all } n \in \mathbb{N}.$$

Then, the limit

$$\lim_{n\to\infty} \int f_n \, d\mu \text{ exists}$$

and one can interchange limits and integral, i.e.,

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Proof. The sequence $(\int f_n \, d\mu)_{n\in\mathbb{N}}$ converges if and only if

$$\limsup_{n\to\infty} \int f_n \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu.$$

We know by definition that $\limsup_{n\to\infty} f_n \geq \liminf_{n\to\infty} f_n$. For the proof of the other direction of the inequality, first observe that $f_n \leq f$ for all $n \in \mathbb{N}$. Then $\int f_n \, d\mu \leq \int f_n \, d\mu$, so $\sup \int f_n \, d\mu \leq \int f \, d\mu$. Therefore $\limsup_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$. Now we can apply Fatou's lemma to say $\limsup_{n\to\infty} \int f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$. Therefore $\lim_{n\to\infty} \int f_n \, d\mu$ exists.

For the second part of this proof, observe that $f_1, f_2, \ldots$ are measurable functions, and $f_n \to f$, so f is measurable. Also, $\int f \, d\mu < \infty$ and $|f_n| \leq f$ for all $n \in \mathbb{N}$. Therefore, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

$\square$