PROBLEM SET 10

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Additional Problem 1.

(a) The collection $\{x_n\ ,\ n \in \mathbb{N}_0\}$ is linearly independent in $L^2([-1,1];dx)$.

Proof. First note that for $f, g \in L^2([-1,1];dx)$, we have $f = g$ Lebesgue a.e. implies that $f = g$ on some dense subset of $[-1,1]$. This is because if this was false, there would be an interval in $[-1,1]$ where $f \neq g$. This interval would have positive measure, meaning that $f \neq g$ Lebesgue a.e.

Then let $a_0x^0 + a_1x^1 + \cdots + a_nx^n$ be a linear combination of elements of our collection. Setting this equal to 0, we have the polynomial

$$0 = a_0x^0 + a_1x^1 + \cdots + a_nx^n$$

which we know can have at most n roots if one of it's coefficients is nonzero. However, for $f := a_0x^0 + a_1x^1 + \cdots + a_nx^n$ to be equal to 0, it would need to equal 0 on a dense subset of its possible inputs. Because it can be 0 at at most n inputs, our collection is linearly independent. $\square$

(b) To construct an ONS from $\{1, x, x^2, x^3\}$ in $L^2([-1,1];dx)$, first let $v_1 = 1$. Then

$$v_2 = \frac{x - P_{\text{span}\{1\}}(x)}{\|x - P_{\text{span}\{1\}}(x)\|}.$$

To compute this, first compute

$$P_{\text{span}\{1\}(x)} = \langle x, 1\rangle \cdot 1 = \int_{-1}^{1} x \cdot 1\, dx = 0$$

Then compute

$$\|x\| = (\langle x, x\rangle)^{\frac{1}{2}} = \left(\int_{-1}^{1} x \cdot x\, dx\right) = \left(\frac{2}{3}\right)^{\frac{1}{2}}$$

therefore

$$v_2 = \frac{x}{\sqrt{\frac{2}{3}}}.$$

To compute v_3, first observe

$$v_3 = \frac{x^2 - P_{\text{span}\{1,x\}}(x^2)}{\|x^2 - P_{\text{span}\{1,x\}}(x^2)\|}.$$

Then compute

$$P_{\text{span}\{1,x\}}(x^2) = \langle x^2, 1\rangle 1 + \langle x^2, x\rangle x = \frac{2}{3} + 0.$$

Then compute

$$\|x^2 - \frac{2}{3}\| = \left(\langle x^2 - \frac{2}{3}, x^2 - \frac{2}{3}\rangle\right)^{\frac{1}{2}} = \left(\int_{-1}^{1} (x^2 - \frac{2}{3})^2\, dx\right)^{\frac{1}{2}} = \sqrt{\frac{26}{9}}.$$

Therefore

$$v_3 = \frac{x^2 - \frac{2}{3}}{\sqrt{\frac{26}{9}}}.$$

To compute v_4, first observe

$$v_4 = \frac{x^3 - P_{\text{span}\{1,x,x^2\}}(x^3)}{\|x^3 - P_{\text{span}\{1,x,x^2\}}(x^3)\|}.$$

Then compute

$$P_{\text{span}\{1,x,x^2\}}(x^3) = \langle x^3, 1\rangle 1 + \langle x^3, x\rangle x + \langle x^3, x^2\rangle x^2 = \int_{-1}^{1} x^4 \, dx = \frac{2}{5}.$$

Then compute

$$\|x^3 - \frac{2}{5}\| = \left(\int_{-1}^{1} x^6 - \frac{4}{5}x^3 + \frac{4}{25} \, dx\right)^{\frac{1}{2}} = \left(\frac{2}{7} + \frac{8}{25}\right)^{\frac{1}{2}} = \sqrt{\frac{116}{125}}.$$

Therefore

$$v_4 = \frac{x^3 - \frac{2}{5}}{\sqrt{\frac{116}{175}}}.$$

(c) Applying Gram-Schmidt on $\{x^n : n \in \mathbb{N}\}$ yields an ONS called the **Legendre polynomials**. These polynomials form an ONB of $L^2([-1,1]; dx)$.

Proof. To begin, we know that the classical Weierstrass theorem states that for $f \in \mathcal{C}([-1,1])$, there exists a sequence of polynomial p_n such that $p_n \to f$ uniformly. Because Gram-Schmidt preserves the span of a collection, and our collection $\{x^n : n \in \mathbb{N}\}$ spans all polynomial, the Legendre polynomials span all polynomials. Therefore we can approximate any continuous f in $[-1,1]$ by Legendre polynomials.

Furthermore, we know that we can approximate any function $g \in L^2([-1,1]; dx)$ by a continuous function in $[-1,1]$. Therefore by first approximating g with a continuous function, then approximating that continuous function with our Legendre polynomials, we can approximate any element of $L^2([-1,1]; dx)$ with a sequence of these polynomials. This means

$$\overline{\text{span }\{\text{legendre polynomials}\}} = L^2([-1,1]; dx).$$

We already know the Legendre polynomials are an ONS, so they are then an ONB. $\square$

8B.16.

Suppose that V is a Hilbert space and $P : V \to V$ is a linear map such that $P^2 = P$ and $\|Pf\| \leq \|f\|$ for every $f \in V$. Then there exists a closed subspace U of V such that $P = P_U$.

Proof. Our candidate for U is Ran$P$. Then Ran$P$ is a subspace because for $x_1, x_2 \in \text{Ran } P$, there exists y_1, y_2 such that $P_{y_1} = x_1$, and $P_{y_2} = x_2$. Then

$$ax_1 + x_2 = aP_{y_1} + P_{y_2} = P(ay_1 + y_2) \in \text{Ran}P.$$

Then we want to show that for $v \in V$, we can represent v as $v = P_v + (v - P_v) = P_U v$, where $Pv \in \text{Ran}P$ and $(v - Pv) \in (\text{Ran}P)^{\perp}$. To do this, observe first define $w := v - Pv$ and take $u \in \text{Ran}P$. Then

$$\begin{aligned} \|w - tu\|^2 &= \|w\|^2 - 2tRe\langle w, u\rangle + t^2\|u\|^2, \\ \|P(w - tu)\|^2 &= t^2\|u\|^2 \\ \implies t^2\|u\|^2 &\le \|w\|^2 - 2tRe\langle w, u\rangle + t^2\|u\|^2 \\ \implies 0 &\le \|w\|^2 - 2tRe\langle w, u\rangle\ ,\ t \in \mathbb{R}. \\ \implies Re\langle w, u\rangle &= 0. \end{aligned}$$

A similar argument follows for $Im\langle w, u\rangle$. This tells us that $\langle w, u\rangle = 0$ for all $u \in U$. Therefore $v - Pv \in (\text{Ran } P)^{\perp}$. □

8C.13.

(c) The Banach space ℓ^∞ is not separable.

Proof. Assuming that this space is separable, then there exists a countable dense subset $\{x_n : n \in \mathbb{N}\}$ of the space. Then for $\epsilon > 0$, we can define an ϵ-ball around each point as

$$B_\epsilon := \{y \in \ell_\infty : \|x_n - y\|_\infty < \epsilon\}.$$

Furthermore, density of these x_n gives us

$$\ell^\infty = \bigcup_{n \in \mathbb{N}} = B_\epsilon(x_n).$$

Now examine sequences in ℓ^∞ made of $0's$ and $1's$. We know there are uncountably many of these sequences, so there must be at least two different ones of them in at least one of our countably many ϵ-balls. But if these sequences differ, then the sup norm of their difference is 1, as they are sequences over 1 and 0. But then if $\epsilon < \frac{1}{2}$, then we have a contradiction because of the way we defined our ϵ-balls. □

8C.16. Find the polynomial of degree at most 4 that minimizes $\int_0^1 |x^5 - g(x)|^2\, dx$.

Proof. First define

$$U := \text{span}\{1, x, x^2, x^3, x^4\}.$$

Then

$$\|P_U(x^5)\| = \inf_{g \in U} \|x^5 - g\|$$

This tells us that we need to calculate $P_U(x^5)$.

$$P_U(x^5) = \sum_{n=0}^{4} \langle x^5, x^n \rangle x^n = \sum_{n=0}^{4} \int_0^1 x^5 \cdot x^n \, dx = \frac{1}{6} + \frac{x}{7} + \frac{x^2}{8} + \frac{x^3}{9} + \frac{x^4}{10}.$$

$\square$

Additional Problem 2. Let $\mathscr{H}$ be a Hilbert space. Suppose that $A \in \mathcal{B}(\mathscr{H})$. Then we have

$$\|A\| = \sup_{x,y \in \mathscr{H}: \|x\|=\|y\|=1} |\langle Ax, y \rangle|.$$

Proof. We will prove this using double inequalities. First,

$$\sup_{x,y \in \mathscr{H}: \|x\|=\|y\|=1} |\langle Ax, y \rangle| \leq \sup_{\|x\|=\|y\|=1} \|Ax\| \cdot \|y\| \leq \sup_{\|x\|=\|y\|=1} \|A\| \cdot \|x\| \cdot \|y\| = \|A\|.$$

For the other direction,

$$\sup_{\|x\|=\|y\|=1} \||\langle Ax, y \rangle| \geq \sup_{\|x\|=1} |\langle Ax, \frac{Ax}{\|Ax\|} \rangle|$$

$\square$

Additional Problem 3. For the Hilbert space $\mathscr{H}$, a map $B : \mathscr{H} \times \mathscr{H} \to \mathcal{C}$ is a **Sesquilinear form** if for all $x \in \mathscr{H}$, $B(., x)$ is linear and $B(x, .)$ is conjugate linear. A Sesquilinear form B is bounded if there exists $C > 0$ such that for all $x, y \in \mathscr{H}$, one has

$$|B(x, y)| \leq C\|x\|\|y\|. \tag{1}$$

(a) Every bounded Sesquilinear form B on $\mathscr{H}$ is represented by a unique bounded operator, i.e. there exists unique $A \in \mathcal{B}(\mathscr{H})$ such that

$$B(x, y) = \langle x, Ay \rangle \text{ , for all } x, y \in \mathscr{H}.$$

Furthermore,

$$\|A\| = \sup_{\|x\|=\|y\|=1} |B(x, y)| = \inf\{C > 0 : (1) \text{ holds.}\}$$

Proof. Given a bounded Sesquilinear form B, first fix $y \in \mathscr{H}$. Then the map $B(x, y)$ is a bounded linear functional. Then by the Riesz representation theorem, we know that there exists unique h_y such that $B(x, y) = \langle x, h_y \rangle$.

Now we want to show that the map $A : y \to h_y$ is bounded and linear. To show boundedness,

$$\|A_y\| = \|h_y\| = \|B(x, y)\| = \sup_{\|x\|=1} \|B(x, y)\| \leq \sup_{\|x\|=1} C\|x\|\|y\| = C\|y\|.$$

To show linearity,

$$\langle x, h_{y_1} + h_{y_2} \rangle = B(x, h_{y_1} + h_{y_2}) = B(x, h_{y_1}) + B(x, h_{y_2}) = \langle x, h_{y_1} \rangle + \langle x, h_{y_2} \rangle.$$

For the second part of the claim,

$$\|A\| = \sup_{\|y\|=1} |Ay| = \sup_{\|y\|=\|x\|=1} |\langle x, h_y \rangle| = \sup_{\|y\|=\|x\|=1} |B(x, y)|$$

□

(b) For $A \in \mathcal{B}\mathscr{H}$, consider the Sesquilinear form

$$B_A(x, y) := \langle Ax, y \rangle, x, y \in \mathscr{H}.$$

This is a bounded Sesquilinear form, and therefore can be represented by a unique bounded operator A^*. Additionally,

$$\|A\| = \|A^*\|.$$

Proof. To show the claim, we have

$$\|A\| = \sup_{\|x\|=1} \|A_x\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle| = \sup_{\|x\|=\|y\|=1} \langle x, A^* y \rangle| = \sup_{\|x\|=1} \|A^* x\| = \|A^*\|.$$

□

(c) Let $A, B \in \mathscr{H}$, then for all $\alpha \in \mathcal{C}$,

$$(A + B)^* = A^* + B^*, (\alpha A^*) = \overline{\alpha} A^*, [A^*]^* = A.$$

Proof. For the first part of this claim, $(A + B)^*$ has the defining property

$$\langle (A + B)x, y \rangle = \langle x, (A + B)^* y \rangle, \text{ for all } x, y \in \mathscr{H}.$$

Then by linearity of the inner product, we know that

$$A\langle x, y \rangle + B\langle x, y \rangle = \langle x, A^* y \rangle + \langle x, B^* y \rangle.$$

This implies that $(A + B)^* = A^* + B^*$. □

(d)

$$\mathrm{Ker}(A^*) = [\overline{\mathrm{Ran}(A)}]^{\perp}.$$

Proof.

$$\mathrm{Ker}(A^*) =$$

□