Homework 2

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2B.12

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

a. For $k \in \mathbb{Z}^+$, let

$$G_k = \{a \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } |f(b) - f(c)| < \frac{1}{k} \text{ for all } b, c \in (a - \delta, a + \delta)\}$$

Claim. G_k is an open subset of $\mathbb{R}$ for each $k \in \mathbb{Z}^+$.

Proof. Fix $k \in \mathbb{Z}^+$, then take $g \in G_k$. Then $g \in \mathbb{R}$ and there exists $\delta > 0$ such that

$$|f(b) - f(c)| < \frac{1}{k}$$

for all $b, c \in (g - \delta, g + \delta)$. To show G_k is open, take $g' \in (g - \delta, g + \delta)$. Then there exists δ' such that

$$(g' - \delta', g' + \delta') \subseteq (g - \delta, g + \delta).$$

Then $|f(b) - f(c)| < \frac{1}{k}$ for all $b, c \in (g' - \delta', g' + \delta')$, so $g' \in G_k$. Thus G_k is open. □

b. *Claim.* The set of points at which f is continuous equals $\bigcap_{k=1}^{\infty} G_k$.

Proof. To show one side of the equality, take $x \in \bigcap_{k=1}^{\infty} G_k$. Then for any $\epsilon > 0$, there exists $k \in \mathbb{Z}^+$ such that $\frac{1}{k} < \epsilon$. Then there exists $\delta > 0$ such that

$$|f(b) - f(c)| < \frac{1}{k} < \epsilon$$

for all $b, c \in (x - \delta, x + \delta)$. Thus f is continuous at x.

To show the other side of the equality, take $x \in \mathbb{R}$ such that f is continuous at x. Then for any $k \in \mathbb{Z}^+$, there exists $\delta > 0$ such that

$$|f(b) - f(c)| < \frac{1}{k}$$

for all $b, c \in (x - \delta, x + \delta)$. Therefore for all $k \in \mathbb{Z}^+$, $x \in G_k$, so $x \in \bigcup_{k=1}^{\infty} G_k$. This completes the other side of our equality, so the set of points at which f is continuous equals $\bigcap_{k=1}^{\infty} G_k$. □

c. *Claim.* The set of points at which f is continuous is a boreal set.

Proof. Each G_k is open, so each G_k is a boreal set. Also, the set of boreal sets is a σ-algebra. Because σ-algebras are closed under countable intersections, $\bigcap_{k=1}^{\infty} G_k$ is a boreal set. Therefore the set of points at which f is continuous is a boreal set. $\square$

2B.14

a. Let $f_1, f_2, \ldots$ be a sequence of functions from a set X to $\mathbb{R}$.

Claim.

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \ldots \text{ has a limit in } \mathbb{R}\}$$
$$= \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n})).$$

To see why this is true, first fix $n \in \mathbb{N}$, and then take $j \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \geq j$. Then observe that $(f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n}))$ is the set of $x \in X$ such that $|f_j - f_k| < \frac{1}{n}$. Call this property P. Then, take the intersection of that set over all $k \geq j$, leaving n and j fixed. This results in all $x \in X$ that have property P for all $k \geq j$.Then take the union of this new set over all $j \in \mathbb{N}$. This results in all $x \in X$ that, for any j, have property P for all $k \geq j$. Finally, take the intersection of this most recent set over all $n \in \mathbb{N}$. This results in the set of all $x \in X$ such that for all n, there exists some j such that for all $k \geq j$, we know $|f_j - f_k| < \frac{1}{n}$. But this is equivalent to the cauchy criterion for convergence of a sequence, so this is the set of all $x \in X$ such that the sequence $f_1(x), f_2(x), \ldots$ has a limit in $\mathbb{R}$.

b. We know from part a. that

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \ldots \text{ has a limit in } \mathbb{R}\}$$
$$= \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n})).$$

Then because each f_i is σ-measurable for all $i \in \mathbb{N}$, we know that $f_j - f_k$ is σ-measurable for all k and j. Then because $(-\frac{1}{n}, \frac{1}{n})$ is a boreal set for all $n \in \mathbb{N}$, we know that $(f_j - f_k)^{-1}(-\frac{1}{n}, \frac{1}{n})$ is σ-measurable. Then because the σ-algebra is closed under countable unions and intersections,

$$\bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}(f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n})).$$

is σ-measurable.

Additional Problem 1

theorem. Let $O \subseteq \mathbb{R}$ be a non-empty, open set. Then, there exists a countable collection of pairwise disjoint open intervals I_n, $n \in \mathbb{N}$, such that

$$O = \bigsqcup_{n=1}^{\infty} I_n$$

a. For $q \in \mathbb{Q} \cap O$, define

$$\alpha(Q) := \inf\{x \in \mathbb{R} : (x, q] \subseteq O\}$$
$$\beta(q) := \sup\{x \in \mathbb{R} : [q, x) \subseteq O\}$$

We know there exists $q \in \mathbb{Q} \cap O$ because O is open, and nonempty, so it contains infinitely many rationals. Then, also because O is open, we know that there exists $\epsilon > 0$ such that $(q - \epsilon, q + \epsilon) \subseteq O$. Then, by definition of the infimum and supremum,

$$\inf\{x \in \mathbb{R} : (x, q] \subseteq O\} \leq q - \epsilon < q + \epsilon \leq \sup\{x \in \mathbb{R} : [q, x) \subseteq O\}$$

b. For $q \in \mathbb{Q} \cap O$, if $I_q := (\alpha(q), \beta(q))$, then $I_q \subseteq O$.

Proof. Take $x \in I_q$. Then $\alpha(q) < x < \beta(q)$. If $x \leq q$, then $x \in (\alpha(q), q] \subseteq O$. If $x > q$, then $x \in [q, \beta(q)) \subseteq O$. Therefore $I_q \subseteq O$. $\square$

c. *Claim.* $\bigcup_{q \in \mathbb{Q} \cap O} I_q = O$ and for all $q, s \in \mathbb{Q} \cap O$, either $I_q = I_s$ or $I_q \cap I_s = \emptyset$.

Proof. To show the first part of the claim, first note that we have half of the equality from the fact that each I_q is a subset of O. Then take $x \in O$. If x is irrational, then because O is open we can take a rational from the open neighborhood of x that will be in O. Therefore we can assume $x \in \mathbb{Q}$. Then there exists $I_x = (\alpha(x), \beta(x))$ such that $x \in I_x$. Then because $I_x \subseteq \bigcup_{q \in \mathbb{Q} \cap O} I_q$, we have $x \in \bigcup_{q \in \mathbb{Q} \cap O} I_q$. Therefore the first part of the claim is true.

For the second part, take $q, s \in \mathbb{Q} \cap O$. Then if $I_q \cap I_s \neq \emptyset$, there exists $x \in I_q \cap I_s$. Thus both $\alpha(q) < x < \beta(q)$ and $\alpha(s) < x < \beta(s)$. But then by the definition of α and β, we know that $\alpha(q) = \alpha(s)$ and $\beta(q) = \beta(s)$. Therefore $I_q = I_s$. This completes the proof of the second part of the claim. □

d. These parts together prove our theorem. To see this, form a sub-collection of our I_qs's where $I_q \cap I_s = \emptyset$ for any two elements I_q and I_s in our sub-collection. The union of these I_q's is still O, and they are now all disjoint from each other. Therefore we have found a countable collection of pairwise disjoint open intervals that cover O.

Additional Problem 2

Let $(X, \mathcal{S}, \mu)$ be a measure space. μ is a finite measure if $\mu(X) \leq +\infty$. μ is a σ-finite measure if there exists a countable collection $\{X_n, n \in \mathbb{N}\}$ of measurable sets such that

$$X = \bigcup_{n \in \mathbb{N}} X_n, \text{ and } \mu(X_n) \leq +\infty, \text{ for all } n \in \mathbb{N} \tag{1}$$

a. Let $(X, \mathcal{S}, \mu)$ be σ-finite.

   claim. Without loss of generality, one may assume the collection $\{X_n, n \in \mathbb{N}\}$ of measurable sets in (1) to be mutually disjoint.

   Proof. For the collection $\{X_n, n \in \mathbb{N}\}$, inductively define a new collection $\{Y_n, n \in \mathbb{N}\}$, where $Y_1 = X_1$ and $Y_n = X_n \setminus Y_{n-1}$. It follows from this definition that all the Y_n's are mutually disjoint, and that

$$X = \bigcup_{n \in \mathbb{N}} Y_n, \text{ and } Y_n = X_n \cap (X \setminus \bigcup_{k=1}^{n-1} X_k)$$

   The alternative definition of Y_n above shows that Y_n is σ-measurable for all $n \in \mathbb{N}$ because σ-measurability is closed with respect to countable union, intersection, and complementation. Therefore we can assume the collection $\{X_n, n \in \mathbb{N}\}$ of measurable sets in (1) to be mutually disjoint because if it is not, we can form a new collection that is that satisfies all of our desired properties. □

b. Assume that μ is σ-finite, and $\mathcal{C}$ is a collection of pairwise disjoint measurable sets which have strictly positive measure.

   claim. The collection $\mathcal{C}$ is at most countable.

Proof. First, assume that μ is a finite measure (not just σ-finite). Then, assume for contradiction that C is uncountable and consider the collection

$$\mathcal{A}_n := \{A \in \mathcal{C} : \mu(A) \geq \frac{1}{n}\}.$$

Then, because $\mathcal{C}$ is uncountable, there exists some $\mathcal{A}_N$ that is uncountable. Take a countably infinite number of A_i's from $\mathcal{A}_N$. We know that for this $N \in \mathbb{N}$, each A_i has measure at least $\frac{1}{N}$. Using countable aditivity and the definition of finite measure, we can say

$$+\infty = \sum_{i=1}^{\infty} \frac{1}{N} \leq \sum_{i=1}^{\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu(x) \leq +\infty.$$

This contradicts our assumption that $\mathcal{C}$ is uncountable.

Now, assume that μ is σ-finite but not necessarily finite. Then there exists a countable collection $\{X_n, n \in \mathbb{N}\}$ of pairwise disjoint, measurable sets as defined in (1). Then, for each $n \in \mathbb{N}$, form the measure space $(X, \mathcal{S}_n, \mu_n)$ where $\mathcal{S}_n$ is the σ-algebra generated by $\mathcal{S}$ on X_n and μ_n is the restriction of μ to $\mathcal{S}_n$. Then μ_n is a finite measure for all $n \in \mathbb{N}$ by the definition of σ-finiteness.

Next, we must separate each set in $\mathcal{C}$ into some X_n. To do this, we will have to modifyt $\mathcal{C}$, as any given set in $\mathcal{C}$ may not fit entirely into any X_n. If this is the case for some $C \in \mathcal{C}$, then we can split C into the parts of C that are in each X_n. Because all X_n's are disjoint, no parts of C will be in more than one of them. This modification of $\mathcal{C}$ can split each $C \in \mathcal{C}$ into at most countably many parts, so the whole collection $\mathcal{C}$ cannot become uncountable because of this modification.

Finally, for each X_n, the subset of $\mathcal{C}$ that is in X_n is a collection of pairwise disjoint measurable sets with strictly positive measure and μ_n is a finite measure, so by the previous part of this proof, this subset of $\mathcal{C}$ is at most countable. Because there are countably many X_n, the whole collection $\mathcal{C}$ is at most countable.

c. For a measure space$(X, \mathcal{S}, \mu)$, a point $x \in X$ is an atom if $\{x\} \in \mathcal{S}$ and $\mu(\{x\}) > 0$.

claim. If μ is σ-finite, then there can be at most countably many atoms.

Proof. This claim follows from part (b), as the collection of atoms is clearly a collection of pairwise disjoint measurable sets with strictly positive measures. Therefore this collection is at most countable, so there can be at most countably many atoms. □

□

2C.2

Let μ be a measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$.

claim. There exists a sequence $w_1, w_2 \dots$ in $[0, 1]$ such that

$$\mu(E) = \sum_{k \in E} w_k$$

for every set $E \subseteq \mathbb{Z}^+$.

Proof. Let $E = \{x_1, x_2, \dots\}$ for $x_i \in \mathbb{Z}^+$. Then countable aditivity tells us that

$$\mu(E) = \mu\left(\bigcup_{n \in \mathbb{N}} x_n\right) = \sum_{n \in \mathbb{N}} \mu(\{x_k\}).$$

Letting $w_i = x_i$ for all $i \in \mathbb{N}$, we have that

$$\mu(E) = \sum_{n \in \mathbb{N}} w_k.$$

$\square$

2C.3

An example of a measure μ on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ such that

$$\{\mu(E) : E \subseteq \mathbb{Z}^+\} = [0, 1]$$

is the measure defined by

$$\mu(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

where χ_E is the characteristic function of E. This function attains all points in $[0, 1]$ because for any $x \in [0, 1]$, x can be thought of as a percentage of Z^+ that is in E for some $E \subseteq \mathbb{Z}^+$. We can form subsets E that are every possible percentage of $\mathbb{Z}^+$, so we can attain all points in $[0, 1]$. This is indeed a measure because $\mu(\emptyset) = 0$, and μ is countably additive because for disjoint E_1 and E_2, their characteristic functions are never both 1, so the sum of their characteristic functions is the characteristic function of their union.

2C.10

An example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supseteq E_2 \supseteq \ldots$ of sets in $\mathcal{S}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k)$$

is as follows. Let $\mathcal{S} = \mathcal{B}$ and $X = \mathbb{R}$. For $A \subseteq R$, let $\mu(A) = \infty$ if $|A| = \infty$ and $\mu(A) = n$ if $|A| = n$. Then

$$\lim_{n\to\infty} \mu((-\frac{1}{n}, \frac{1}{n})) = \infty$$

because for all $n \in \mathbb{N}$, $|(-\frac{1}{n}, \frac{1}{n})| = \infty$. However,

$$\mu\left(\bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n})\right) = \mu(\{0\}) = 0.$$