Homework 1

Theo McGlashan

Problem 1

Define $f : [0,1] \to \mathbb{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is rational} \\ \dfrac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive} \\ & \text{integer such that } a = \dfrac{m}{n} \text{ for some integer } m. \end{cases} \text{vspace1em}$$

Claim: f is Riemann integrable and $\int_0^1 f = 0$

Proof. For $n \in \mathbb{N}$, define

$$A_n := \{x \in \mathbb{Q} \cap [0,1] : x = \frac{p}{q} \text{ for some } 1 \leq q \leq n \text{ and } 0 \leq p \leq q\}$$

Then $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$ and

$$\mathbb{Q} \cap [0,1] = \bigcup_{n \in \mathbb{N}} A_n$$

Now for $n \in \mathbb{N}$, define

$$f_n := f \cdot \chi_{A_n}$$

Then observe that for all $n \in \mathbb{N}$, for $a \in [0,1]$ where $0 < f(a) < \frac{1}{n}$, the indicator function $\chi_{A_n} = 0$. This is because there exists no q and p with $1 \leq q \leq n$ and $0 \leq p \leq q$ such that $a = \frac{p}{q}$.

However, when $f(a) \geq \frac{1}{n}$, then $a \in A_n$, so $\chi_{A_n} = 1$. This means that f_n can be thought of as all values of f that are greater than or equal to $\frac{1}{n}$.

This means that as $n \to \infty$, f_n approaches f in some sense. More precisely,

$$||f_n - f||_\infty = \frac{1}{n+1}$$

Because $\lim_{n\to\infty} \frac{1}{n+1} = 0$, we know that $f_n \to f$ uniformly. Now, observe that each f_n differs from 0 on a finite set of points, also because $f_n = 0$ when $f < \frac{1}{n}$. This means that for all $n \in \mathbb{N}$, f_n is Riemann integrable with integral 0. Therefore f is Riemann integrable with integral 0. $\square$

Problem 2

(a) Let $f : [a, b] \to \mathbb{R}$ be a non-negative, bounded function satisfying $0 \leq f \leq M$ for some $M > 0$. For $n \in \mathbb{N}$, consider the sets

$$E_{n,k} := \left\{ \frac{kM}{2^n} \leq f < \frac{(k+1)M}{2^n} \right\}, \ 0 \leq k \leq 2^n - 1,$$

and use them to define the simple function

$$\phi_n := \sum_{k=0}^{2^n-1} \frac{kM}{2^n} \chi_{E_{n,k}}.$$

Claim: for all $n \in \mathbb{N}$, one has

$$\begin{aligned} 0 &\leq \phi_n \leq \phi_{n+1} \leq f \\ 0 &\leq f - \phi_n \leq \frac{M}{2^n} \end{aligned}$$

and $\phi_n \to f$ uniformly on $[a, b]$.

Proof. For $n \in \mathbb{N}$ and $x_0 \in [a, b]$, there exists a unique k_0 such that $x_0 \in E_{n,k}$. This is because f is non-negative and bounded, and the sets $E_{n,k}$ are disjoint over different values of k. Thus for all $n \in \mathbb{N}$,

$$\phi_n(x_0) = \frac{k_0 M}{2^n} \leq f(x_0)$$

To describe ϕ_{n+1}, observe that

$$\begin{aligned} E_{n+1,2k} \cup E_{n+1,2k+1} &= \left\{ \frac{2kM}{2(2^n)} \leq f < \frac{(2k+1)M}{2(2^n)} \right\} \cup \left\{ \frac{(2k+1)M}{2(2^n)} \leq f < \frac{(2k+2)M}{2(2^n)} \right\} \\ &= \left\{ \frac{kM}{2^n} \leq f < \frac{(2k+1)M}{2(2^n)} \right\} \cup \left\{ \frac{(2k+1)M}{2(2^n)} \leq f < \frac{(k+1)M}{2^n} \right\} \\ &= \left\{ \frac{kM}{2^n} \leq f < \frac{(k+1)M}{2^n} \right\} = E_{n,k} \end{aligned}$$

This means that for $x_0 \in E_{n,k_0}$, x_0 is either in $E_{n+1,2k_0}$ or $E_{n+1,2k_0+1}$. If $x_0 \in E_{n+1,2k_0}$, then

$$\phi_{n+1}(x_0) = \frac{2k_0 M}{2(2^n)} = \frac{k_0 M}{2^n} = \phi_n(x_0)$$

If $x_0 \in E_{n+1,2k_0+1}$, then

$$\phi_{n+1}(x_0) = \frac{(2k_0+1)M}{2(2^n)} \geq \frac{k_0 M}{2^n} = \phi_n(x_0)$$

Therefore

$$0 \leq \phi_n \leq \phi_{n+1} \leq f$$

Additionally, $f(x_0) \leq \frac{(k_0+1)M}{2^n}$, so

$$f(x_0) - \phi_n(x_0) \leq \frac{(k+1)M}{2^n} - \frac{kM}{2^n} = \frac{M}{2^n}$$

Therefore $\phi_n \to f$ uniformly on $[a, b]$. □

(a) Claim: Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof. Without loss of generality, assume that f is increasing and non-negative. Now, observe that each $E_{n,k}$ must be an open interval because f is monotone. This means that for all $n \in \mathbb{N}$, ϕ_n is a step function, meaning ϕ_n is Riemann integrable. Finally, $\phi_n \to f$ uniformly on $[a, b]$, so f is Riemann integrable. □

Problem 3

Claim: If A and B are subsets of $\mathbb{R}$ and $|B| = 0$, then $|A \cup B| = |A|$.

Proof. Beacuse $|B| = 0$, for all $\epsilon > 0$, there exists open intervals $I_1, ..., I_n$ such that

$$B \subseteq \bigcup_{k=1}^{n} I_k \qquad\qquad \sum_{k=1}^{n} \ell(I_k) < \epsilon$$

It follows that $A \cup B \subseteq A \cup \bigcup_{k=1}^{n} I_k$. Also utilizing countable subaditivity, we have

$$|A \cup B| \leq |A \cup \bigcup_{k=1}^{n} I_k| \leq |A| + \sum_{k=1}^{n} \ell(I_k) < |A| + \epsilon$$

Therefore $|A \cup B| \leq |A|$, and clearly $|A| \leq |A \cup B|$, so $|A \cup B| = |A|$. □

Problem 4

For $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, let $tA = \{ta : a \in A\}$. Claim: $|tA| = |t||A|$.

Proof. Suppose $I_1, I_2, ...$ is a sequence of open intervals whose union contains A. Then $tI_1, tI_2, ...$ is a sequence of open intervals whose union contains tA. Thus

$$|tA| \leq \sum_{k=1}^{\infty} \ell(tI_k) = |T| \sum_{k=1}^{\infty} \ell(i_k)$$

Taking the infimum of the last term over all sequences $I_1, I_2, ...$ of open intervals whose union contains A, we have $|tA| \leq |T||A|$. For the other direction of inequality, note that $A = \frac{1}{t}tA$. Applying the above inequality replacing A with tA and t with $\frac{1}{t}$, we have $|A| = |\frac{1}{t}tA| \leq |tA|$. Therefore $|tA| = |t||A|$. □

Problem 5

(a) An example of a step function on $[a, b]$ with at least 3 steps is

$$f(x) = \begin{cases} -1 & a \leq x < \frac{a+b}{3}, \\ 0 & \frac{a+b}{3} \leq x < \frac{2(a+b)}{3}, \\ 1 & \frac{2(a+b)}{3} \leq x \leq b. \end{cases}$$

This function is Riemann integrable because it is on a closed interval and is continuous except for a finite number of discontinuities. The same is true for all step functions on closed intervals.

(b) For $f \in \mathcal{R}([a, b])$ and partition $P = x_0, x_1, ..., x_n$ of $[a, b]$, the upper and lower Riemann sums are:

$$L(f, P, [a, b]) = \int_a^b \left(\sum_{j=1}^{n-1} \chi_{[x_{j-1}, x_j)} \inf_{x \in [x_{j-1}, x_j]} \right) + \chi_{[x_{n-1}, x_n]} \inf_{x \in [x_{n-1}, x_n]}$$

$$U(f, P, [a, b]) = \int_a^b \left(\sum_{j=1}^{n-1} \chi_{[x_{j-1}, x_j)} \sup_{x \in [x_{j-1}, x_j]} \right) + \chi_{[x_{n-1}, x_n]} \sup_{x \in [x_{n-1}, x_n]}$$

(c) Claim: for $f \in \mathcal{R}([a, b])$ and for all $\epsilon > 0$, there exists step functions s_- and s_+ on $[a, b]$ with

$$\inf_{x \in [a,b]} f(x) \leq s_- \leq f \leq s_+ \leq \sup_{x \in [a,b]} f(x)$$

so that

$$\int_a^b |f - s_\pm| < \epsilon.$$

Proof. Because $f \in \mathcal{R}([a,b])$, the cauchy criterion for Riemann integrability states that for every $\epsilon > 0$, there exists a partition P such that

$$U(f,P) - L(f,P) < \epsilon.$$

We will use this partition $P = x_1, x_2, ..., x_n$ and the step functions defined for part (b).

$$s_- = \left(\sum_{j=1}^{n-1} \chi_{[x_{j-1},x_j)} \inf_{x\in[x_{j-1},x_j]}\right) + \chi_{[x_{n-1},x_n]} \inf_{x\in[x_{n-1},x_n]}$$

$$s_+ = \left(\sum_{j=1}^{n-1} \chi_{[x_{j-1},x_j)} \sup_{x\in[x_{j-1},x_j]}\right) + \chi_{[x_{n-1},x_n]} \sup_{x\in[x_{n-1},x_n]}$$

These functions evaluate to the infimum and supremum of f on subsets of f. The infimum on a subset is at least equal to the infimum on the set, and the supremum on a subset is at most equal to the supremum on the set. Also, the infimum of f is at most equal to f and the supremum is at least equal to f. This gives us that

$$\inf_{x\in[a,b]} f(x) \le s_{\le} f \le s_+ \le \sup_{x\in[a,b]} f(x)$$

Because we defined s_- and s_+ as in part (b), we know that

$$L(f,P) = \int_a^b s_-$$

$$U(f,P) = \int_a^b s_+$$

Therefore

$$\int_a^b |f - s_\pm| \le U(f,P) - L(f,P) < \epsilon$$

$\square$