Problem Set 6

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I adhered to the honor code on this assignment.

Additional Problem 1

Claim. For every measurable function f supported in $[-R, R]$ for some $0 < R < +\infty$, we have

$$f^*(x) \geq \frac{1}{2(|x| + R)} \|f\|_1, \text{ for every } x \in \mathbb{R}.$$

Proof. Begin by fixing $0 < R < +\infty$, and a measurable function f supported on $[-R, R]$. Then take $x \in \mathbb{R}$. Then,

$$\begin{aligned} f^*(x) &= \sup_{h>0} \frac{1}{2h} \int_{[x-h,x+h]} |f(t)| \, dt \\ &\geq \frac{1}{2(|x| + R)} \int_{[x-|x|-R,x+|x|+R]} |f(t)| \, dt && (1) \\ &\geq \frac{1}{2(|x| - R)} \int_{[-R,R]} |f(t)| \, dt \\ &= \frac{1}{2(|x| - R)} \|f\|_1. && (2) \end{aligned}$$

For above, the inequality in (1) comes from substituting the supremum over all h for the value $h = 2(|x| + R)$. The equality in (2) comes from the fact that f is 0 outside of $[-R, R]$.

This proves the claim. Furthermore, this leads to logarithmic divergence of the $\mathcal{L}^1$-norm for f^*. To see this, use the above inequality:

$$\int |f^*(x)| \, dx \geq \int \frac{1}{2(|x| + R)} \|f\|_1 \, dx \geq \frac{1}{2} \|f\|_1 \int \frac{1}{|x| + R} \, dx.$$

As R is a constant, the final integral above evaluates to a logarithmic function, so $\|f^*\|_1$ is not finite. $\square$

Additional Problem 2

Let $(X, \mathcal{S}, \mu)$ be a measure space.

Theorem. Let (f_n) be a sequence of measurable functions. Then, (f_n) is Cauchy in measure if and only if (f_n) converges in measure.

We know that convergence in measure implies Cauchy in measure, so we need only to prove the other implication.

(a) We will first show that for a sequence (f_n) that is cauchy in measure, (f_n) converges in measure if there exists a subsequence (f_{n_k}) that converges in measure.

Proof. Let $\epsilon, \eta > 0$. Then for our sequence (f_n), we assume that there exists a subsequence (f_{n_k}) that converges in measure. This tells us that there exists $N_1 \in \mathbb{N}$ such that for all $n_k \geq N_1$, we have

$$\mu(\{|f_{n_k} - f| > \eta\}) < \epsilon. \tag{3}$$

Additionally, we know from the fact that (f_n) is Cauchy in measure that there exists $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$, we have

$$\mu(\{|f_n - f_m| > \eta\}) < \epsilon. \tag{4}$$

Now we let $N = \max\{N_1, N_2\}$. Then, for all $n, m_k \geq N$, we have

$$\begin{aligned}
\mu(\{|f_n - f| > \eta\}) &= \mu(\{|f_n - f_{m_k} + f_{m_k} - f| > \eta\}) \\
&\leq \mu(\{|f_n - f_{m_k}| + |f_{m_k} - f| > \eta\}) \\
&\leq \mu(\{|f_n - f_{m_k}| > \eta\}) + \mu(\{|f_{m_k} - f| > \eta\}). \qquad (5)
\end{aligned}$$

Examining (5) we see that the first term is bounded by ϵ because of (4), and the second term is bounded by ϵ because of (3). Therefore the whole inequality is bounded by 2ϵ, so we have convergence of (f_n) in measure. $\square$

(b) We know that because (f_n) is Cauchy in measure, there exists a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ μ-a.e., for some function f. We will now prove that $f_{n_k} \to f$ in measure, which completes the proof of the theorem.

Proof. Let $\epsilon, \eta > 0$. We will begin by looking at $n_k \geq N$, where $N \in \mathbb{N}$ is taken such that for all $n, m \geq N$, because (f_n) is Cauchy in measure, we have

$$\mu(\{|f_n - f_m| > \eta\}) < \epsilon. \tag{6}$$

Then because $f_{n_k} \to f$ μ-a.e., we know that up to a set of 0 measure, we have

$$|f_{n_k} - f| > \eta \implies |f_{n_k} - f_{n_l}| > \eta, \text{ eventually in } l. \tag{7}$$

Therefore all $f_{n_k}(x)$ satisfying the left side also satisfy the right of (7), so

$$\begin{aligned}\{|f_{n_k} - f| > \eta\} &\overset{\mu-a.e.}{\subseteq} \{|f_{n_k} - f_{n_l}| > \eta\}, \text{ eventually in } l \\ &\overset{\mu-a.e.}{\subseteq} \bigcup_{L\in\mathbb{N}} \bigcap_{l\geq L} \{|f_{n_k} - f_{n_l}| > \eta\}. \end{aligned} \tag{8}$$

For above, (8) comes from the definition of a property holding eventually. But as L increases, we are taking an intersection over less sets, so (8) is a union of nested increasing sets. This lets us use our continuity properties to say

$$\begin{aligned}\mu(\{|f_{n_k} - f| > \eta\}) &\leq \lim_{L\to\infty} \mu\left(\bigcap_{l\geq L} \{|f_{n_k} - f_{n_l}| > \eta\}\right) \\ &\leq \liminf_{L\to\infty} \mu(\{|f_{n_k} - f_{n_L}| > \eta\}) < \epsilon. \end{aligned} \tag{9}$$

The final bound by ϵ in (9) comes from our original choice of n_k, which lets us use (6). $\square$

Additional Problem 3

Let (M, d) be a fixed metric space.

(a) *Claim.* (x_n) is Cauchy in (M, d) if and only if

$$\operatorname{diam}(\{x_k, k \geq n\}) \to 0 \text{ , as } n \to \infty.$$

Here given $\emptyset \neq A \subseteq M$, we define the diameter of A as

$$\operatorname{diam}(A) := \sup_{x,y\in A} d(x, y) \in [0, +\infty].$$

$(\implies)$ For the forward direction, assume that (x_n) is Cauchy in (M, d). Then let $\epsilon > 0$. Because of our assumption, we know that there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have

$$d(x_n, x_m) < \epsilon. \tag{10}$$

Therefore, letting $A_n := \{x_k : k \geq n\}$, we have that for $n \geq N$,

$$\operatorname{diam}(A_n) = \sup_{x,y\in A_n} d(x, y) \leq \epsilon.$$

The final inequality above follows from (10). This completes the forward direction.

($\Longleftarrow$) For the backward direction, assume $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$. Then let $\epsilon > 0$. We know from our assumption that there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\operatorname{diam}(A_n) = \sup_{x,y \in A_n} d(x, y) < \epsilon.$$

Therefore for $n, m \geq N$, we know

$$d(x_n, x_m) \leq \sup_{x,y \in A} d(x, y) \leq \epsilon.$$

(b) *Claim.* If (x_n) is Cauchy, then there exists a subsequence (x_{n_k}) of (x_n) such that

$$d(x_{n_{k+1}}, x_{n_k}) < \frac{1}{2^k} \text{ , for all } k \in \mathbb{N}.$$

Proof. We will prove the existence of this subsequence by inductively constructing it.

Base Case: Assume that $k = 1$. Then because (x_n) is Cauchy, part (a) lets us chose $n_1, n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that

$$d(x_{n_2}, x_{n_1}) \leq \sup_{x,y \in A_{n_1}} d(x, y) < \frac{1}{2^k} = \frac{1}{2}.$$

Then we let x_{n_1}, x_{n_2} be the first two elements of our sequence.

Inductive Case: Assume that for all $k < q \in \mathbb{N}$, we have

$$d(x_{n_k+1}, x_{n_k}) < \frac{1}{2^k}.$$

Then we have constructed elements $x_{n_1}, \dots x_{n_{q-1}}$ of our sequence. To add x_{n_q}, we know from (a) that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\operatorname{diam}(A_n) < \frac{1}{2^q}.$$

Then if we let $n_{q+1} = \max\{N, n_q + 1\}$, we know

$$d(x_{n_{q+1}}, x_{n_q}) \leq \sup_{x,y \in A_{n_q}} d(x, y) = \operatorname{diam}(A_q) < \frac{1}{2^q}.$$

Therefore x_{n_q} is the next element in our sequence. $\square$

Additional Problem 4

Let $f \in \mathcal{L}^1(\mathbb{R}, dx)$ be fixed. We then know that for all $t \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} f(x)\ dx = \int_{-\infty}^{+\infty} f(x+t)\ dx.$$

Claim. For a bounded, measurable function g, we know

$$\lim_{t \to 0} \int_{-\infty}^{+\infty} |g(x)[f(x) - f(x+t)]|\ dx = 0. \tag{11}$$

Proof. We will first prove (11) in the case that our function $f \in \mathcal{C}_c(\mathbb{R})$ meaning f is compactly supported and continuous, then extend this result to all $\mathcal{L}^1$ functions. Assume that f is supported on $[-R, R]$ for some $R \in \mathbb{R}$.

Then, let $\epsilon > 0$. Because f is continuous on a compact domain, it is uniformly continuous. Therefore there exists $1 > \delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

We assume $\delta < 1$ here for use later in the proof. Then for $|t| < \delta$, we have

$$|f(x) - f(x+t)| < \epsilon. \tag{12}$$

Now examining the integral in (11) because g is bounded, we know $|g| < M$ for some $M \in \mathbb{R}$. Therefore again for $|t| < \delta$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} |g(x)[f(x) - f(x-t)]|\ dx &\leq M \int_{-\infty}^{+\infty} |f(x) - f(x+t)|\ dx \\
&\leq M \int_{-R-1}^{R+1} |f(x) - f(x+t)|\ dx && (13) \\
&\leq M \int_{-R-1}^{R+1} \epsilon\ dx && (14) \\
&\leq M\epsilon(2R+2).
\end{aligned}$$

For above, the change of the integration bounds in (13) comes from the finite support of f on $[-R, R]$. Here we expand the interval by 1 on either side, as $|t| < \delta < 1$, so $f(t)$ can be supported on $(-R-1, R+1)$. The inequality in (14) comes from (12).

This completes the proof for the case when $f \in \mathcal{C}_c(\mathbb{R})$. To extend to the general case of $f \in \mathcal{L}^1(\mathbb{R}, dx)$ first let $\epsilon > 0$. Then, use the fact that for $f \in \mathcal{L}^1$, there exists $\phi \in \mathcal{C}_c(\mathbb{R})$ such that

$$\|f - \phi\| < \epsilon. \tag{15}$$

Now examining our integral from (11), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |g(x)[f(x) - f(x-t)]| \, dt \\
&= \int_{-\infty}^{+\infty} |g(x)[f(x) - \phi(x) + \phi(x) - \phi(x-t) + \phi(x-t) - f(x-t)]| \, dt \\
&\leq \int_{-\infty}^{+\infty} |g(x)(f(x) - \phi(x))| \, dx + \int_{-\infty}^{+\infty} |g(x)(\phi(x) - \phi(x-t))| \, dx \qquad (16) \\
&\qquad + \int_{-\infty}^{+\infty} |g(x)(\phi(x-t) - f(x-t))| \, dt \\
&\leq M \int_{-\infty}^{+\infty} |f(x) - \phi(x)| \, dx + M \int_{-\infty}^{+\infty} |\phi(x) - \phi(x-t)| \, dx \qquad (17) \\
&\qquad + M \int_{-\infty}^{+\infty} |\phi(x-t) - f(x-t)| \, dt
\end{aligned}$$

For the inequality beginning in (16), we have distributed $g(x)$, then used the triangle inequality and linearity of integrals to group terms and separate into multiple integrals. For (17), observe that the first and last integrals are bounded by $M\epsilon$ because of (15). The second term is exactly our case where $\phi \in \mathcal{C}_c(\mathbb{R})$, so by the first part of this proof, for sufficiently small $|t|$, we have a bound by $M\epsilon(2R+2)$ for $R \in \mathbb{R}$ dependent on the support of ϕ. Therefore our whole expression is bounded by $2M\epsilon + M^2\epsilon(2R+2)$, so (11) holds for $f \in \mathcal{L}^1$. $\square$