Problem Set 4

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Aditional Problem 1

(a) For $\emptyset \neq E \subseteq \mathbb{R}$ and $y \in \mathbb{R}$, define the distance of y to E as

$$\operatorname{dist}(y; E) := \inf_{x \in E} |x - y| \in [0, +\infty)$$

Claim. For all $y, z \in \mathbb{R}$, one has

$$|\operatorname{dist}(y; E) - \operatorname{dist}(z; E)| \leq |y - z|. \tag{1}$$

Furthermore, this implies that the map $y \mapsto \operatorname{dist}(y; E)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

Proof. For the proof of (1), let $\epsilon > 0$. Then there exists $w \in E$ such that

$$|w - z| - \epsilon < \operatorname{dist}(y; E). \tag{2}$$

We can use this inequality to estimate the left side of (1):

$$\begin{aligned}
|\operatorname{dist}(y; E) - \operatorname{dist}(z; E)| &= |\inf_{x \in E} |x - y| - \inf_{v \in E} |v - z|| \\
&< |\inf_{x \in E} |x - y| - (|w - z| - \epsilon)| \qquad (3) \\
&\leq ||w - y| - |w - z|| + \epsilon \qquad (4) \\
&\leq |w - y - w + z + \epsilon| \\
&\leq |y - z| + \epsilon
\end{aligned}$$

where (3) holds because of (2) and (4) holds by the definition of the infimum. Because the inequality in (3) is strict, we can remove the ϵ and our proof of (1) is complete.

To show continuity of the map $y \mapsto \operatorname{dist}(y; E)$, let $\epsilon > 0$. Then let $\delta = \epsilon$. Then for all $x \in E$ where $|x - y| < \epsilon$, we know by (1) that

$$|\operatorname{dist}(y; E) - \operatorname{dist}(x, E)| \leq |y - z| < \delta = \epsilon.$$

Therefore this map is continuous. $\square$

(b) *Claim.* Assume $\epsilon \neq \emptyset$ is closed. Then $\operatorname{dist}(y; E) = 0$ if and only if $y \in E$.

Proof. For the forward direction of this claim, assume that $\operatorname{dist}(y; E) = 0$. Then $\inf_{x\in E} |x - y| = 0$, and for all $\epsilon > 0$, there exists $w \in E$ such that

$$\left| \inf_{x\in E} |x - y| - |w - y| \right| < \epsilon.$$

Then let $\epsilon = \frac{1}{n}$, and use this statement to construct the sequence $(W_n)_{n\in\mathbb{N}}$ where for all $n \in \mathbb{N}$, the statement holds for w_n. Then $\lim_{n\to\infty} w_n = y$. But E is a closed set, so it contains its limit points, and $w_n \in E$ for all $n \in \mathbb{N}$, so $y \in E$.

For the backward direction of the claim, assume that $y \in E$. Then $\operatorname{dist}(y; E) = \inf_{x\in E} |x - y|$, but $\operatorname{dist}(y; E) \geq 0$, and $|y - y| = 0$, so $\inf_{x\in E} |x - y| = 0$. □

For a counterexample when E is not closed, let $E = (0, 1)$, and let $y = 0$. Then for $n \in \mathbb{N}$, let $\epsilon = \frac{1}{2n}$. We know that for all $n \in \mathbb{N}$, we have

$$y + \frac{1}{2n} = \frac{1}{2n} \in (0, 1) = E.$$

Therefore $\inf_{x\in E} |x-y| \leq \frac{1}{2n}$ for all $n \in \mathbb{N}$, so $\inf_{x\in E} |x-y| = 0$, meaning $\operatorname{dist}(y; E) = 0$ for $y \notin E$, completing the counterexample.

(c) *Urysohn's Lemma*. Let E, F be two *non-empty, disjoint, and closed* sets in $\mathbb{R}$. Then, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $0 \leq g \leq 1$ such that $g = 0$ on E and $g = 1$ on F.

Proof. Let E, F be as described in the lemma and let

$$g(y) := \frac{\operatorname{dist}(y; E)}{\operatorname{dist}(y; E) + \operatorname{dist}(y; F)}, \ y \in \mathbb{R}.$$

First, assume $y \in E$. Then $\operatorname{dist}(y; E) = 0$ by (b), and because E and F are disjoint, $y \notin F$, so again by (b) we know $\operatorname{dist}(y; F) > 0$. Therefore $g(y) = 0$.

Then assume $y \in F$. Then $\operatorname{dist}(y; F) = 0$ and $\operatorname{dist}(y; E) > 0$, so

$$g(y) = \frac{\operatorname{dist}(y; E)}{\operatorname{dist}(y; E)} = 1.$$

Finally, assume $y \notin E$ and $y \notin F$. Then both $\operatorname{dist}(y; E)$ and $\operatorname{dist}(y; F)$ are positive, so $0 < g(x) < 1$. □

Additional Problem 2

For $\mu_{\mathbb{N}}$, the counting measure of the natural numbers, let $(\mathbb{R}, \mathcal{P}, \mu_{\mathbb{N}})$ be a measure space. Additionally, let $0 \leq f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) *Claim.* The statement

$$\int f d\mu_{\mathbb{N}} = \sum_{n=1}^{\infty} f(n) \tag{5}$$

holds for all finitely supported, non-negative, simple functions.

Proof. Because f is a simple function, it can be written as $\sum_{k=1}^{N} c_k \chi_{E_k}$ for some $c_1, \ldots c_N \in \mathbb{R}$ and $E_1, \ldots E_N \subseteq \mathbb{R}$. Then by the definition of the integral of simple functions, we know

$$\int f d\mu_{\mathbb{N}} = \sum_{k=1}^{N} c_k \mu_{\mathbb{N}}(E_k).$$

Now looking at the right side of (5), observe that for any $n \in N$,

$$f(n) = \begin{cases} c_k & \text{if } x \in E_k \\ 0 & \text{if } x \notin E_k \text{ for all k} \end{cases}$$

Therefore for any c_k, the sum on the right side of (5) adds that c_k term for each $n \in E_k$, where $n \in N$. But this is precisely $c_k \mu_{\mathbb{N}}(E_k)$, so

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{N} c_k \mu_{\mathbb{N}}(E_k) = \int f d\mu_{\mathbb{N}}.$$

$\square$

(b) *Lemma. Given a measure space $(Y, \mathcal{S}, \mu)$, suppose that $(Y_n)_{n \in N}$ is an $\mathcal{S}$-valued sequence of nested increasing sets such that*

$$Y = \bigcup_{n \in \mathbb{N}} Y_n.$$

Then for every measurable function $0 \leq f : Y \to \mathbb{R}$, one has

$$\lim_{n \to \infty} \int \chi_{Y_n} f d\mu = \int f d\mu$$

Claim. Equation (5) holds for an arbitrary function $0 \leq f : \mathbb{R} \to \mathbb{R}$.

Proof. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of nested increasing sets defined by $Y_n = [-n, n]$. Then observe that $\mu_{\mathbb{N}}$-a.e. , we know that

$$\chi_{Y_n} f = \sum_{k=1}^{n} f(k)\chi_{\{k\}} =: \varphi_n.$$

Because these differ on a set of finite measure only, we know

$$\int \chi_{y_n} f d\mu_{\mathbb{N}} = \int \sum_{k=1}^{n} f(k)\chi_{\{k\}} d\mu_{\mathbb{N}}.$$

But φ_n is a simple function, so by (a), we have

$$\int f\chi_{Y_n} d\mu_{\mathbb{N}} = \int \varphi d\mu_{\mathbb{N}} = \sum_{k=1}^{\infty} \varphi_n(k).$$

Then because $\lim_{n\to\infty} Y_n = Y$, we can say

$$\int f d\mu_{\mathbb{N}} = \lim_{n\to\infty} \int f\chi_{Y_n} d\mu_{\mathbb{N}} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \varphi_n(k) = \sum_{k=1}^{\infty} f(k)$$

□

(c) *Proof of Lemma.* Let $(f_n)_{n\in N}$ be a series of functions defined by $f_n = \chi_{Y_n} f$. Then $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of measurable functions. We also know $\lim_{n\to\infty} f_n = f$, so we can apply the monotone convergence theorem to get

$$\lim_{n\to\infty} \int \chi_{Y_n} f d\mu = \int f d\mu.$$

Additional Problem 3

A Borel measure μ on $X \subseteq \mathbb{R}$ is regular if for every Borel set $A \subseteq X$, there exists a closed set $f \subseteq A$ and open set $O \supseteq A$ such that

$$\mu(A \setminus F) < \epsilon \text{ and } \mu(O \setminus A) < \epsilon$$

(a) Let μ be a finite regular Borel measure on $X = [a, b]$, and $f : [a, b] \to \mathbb{R}$ be a measurable function.

Claim. For all $\epsilon, \eta > 0$, there exists a step function $s : [a, b] \to \mathbb{R}$ such that

$$\mu(\{|f - s| > \eta\}) < \epsilon.$$

Proof. Let $\epsilon, \eta > 0$. Then because f is measurable, we know that there exists $A \subseteq [a, b]$ where $\mu([a, b] \setminus A) < \epsilon$ and $f|_A$ is bounded. Then construct a sequence of simple functions

$$(\varphi_n)_{n\in\mathbb{N}} \text{ such that } \varphi_n \nearrow f \text{ on } A.$$

Then because f is bounded on A, this convergence is uniform on A. Then there exists $N \in \mathbb{N}$ such that

$$|\varphi_N - f| < \eta. \tag{6}$$

By definition of simple functions, we know

$$\varphi_n = \sum_{k=1}^{K_n} c_k \chi_{E_k} \text{ for } c_k \in \mathbb{R} \text{ and } E_k \subseteq \mathbb{R}.$$

Then because f is defined on $[a, b]$ and $\mu([a, b]) < \infty$ because μ is a finite measure, we know $\mu(E_k) < \infty$ for all E_k. Then by Littlewood's first principle, we have

$$\mu\left(E_k \triangle \left(\bigsqcup_{m=1}^{M} I_{k,m}\right)\right) < \frac{\epsilon}{2^k} \text{ for intervals } I_{k,m}.$$

Then define a sequence of functions $(s_n)_{n\in N}$ where

$$s_n = \sum_{k=1}^{K} c_k \sum_{m=1}^{M} \chi_{I_{k,m}}$$

assuming without loss of generality that all $I_{k,m}$ are disjoint. Then by this definition,

$$\{|\varphi_n - s_n| \neq 0\} \subseteq \bigcup_{k=1}^{K} \left(E_k \triangle \bigsqcup_{m=1}^{M} I_{k,m}\right).$$

But the right side of the above has measure $\leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$, so the left side has measure $\leq \epsilon$, which is the same as measure $< \epsilon$ (just vary ϵ at the start of the proof).

Then for s_N, where N is defined near the beginning of the proof, we have

$$\mu(\{|f - s_N| > \eta\}) < \mu(\{|f - \varphi_N| > \eta\}) + \epsilon = \epsilon.$$

Here the first inequality holds because the measure of the points where φ_N and s_N are different is less than ϵ, and the second equality follows from (6) telling us that there are no points where $|f - \varphi_N| > \eta$. Then because S_N is a step function, we have our claim. □

(b) *Claim.* For every measurable function $f : [a, b] \to \mathbb{R}$ and every $\eta, \mu > 0$, there exists a continuous function $h[a, b] \to \mathbb{R}$ such that

$$\mu(\{|f - h| > \eta\}) < \epsilon.$$

Proof. Let $\epsilon, \eta > 0$. Then by (a), there exists step function $s : [a, b] \to \mathbb{R}$ such that $\mu(\{|f - s|\eta\}) < \epsilon$. Then by definition,

$$s = \sum_{k=1}^{n} c_k \chi_{I_k} \quad \text{where } I_k \text{ are intervals and } c_k \in \mathbb{R}.$$

Assume without loss of generality that each I_k is disjoint. Then for each I_k, we do not know if this is a closed or open interval. However, μ is regular, so we have a closed interval I'_k such that $\mu(I_k\ I'_k) < \epsilon$. Then for I'_j, I'_{j+1}, we know by Urysohn's Lemma that there exists $g : \mathbb{R} \to \mathbb{R}$, a continuous function such that $0 \leq g \leq 1$ and $g|_{I'_j} = 0,\ g|_{I'_{j+1}} = 1$. Then for any $a \in I'_j$ and $b \in I'_{j+1}$, construct the function

$$h_j := (s(a) - s(b))g + s(a)$$

which will be the same regardless of our choice of x and y. This is a continuous function valued at $s(a)$ on I'_j and $s(b)$ on I'_{j+1}, meaning it is valued at s on these intervals. Then $h_j = h_{j+1}$ on I'_{j+1}, so we can define a function h that takes the value of each h_j for I'_j to I'_{j+1}, then each h_j will overlap with the next, but they will both have the same constant value where they overlap.

Because of the construction of our intervals I'_k, we know that h differs from s at most on a set of measure $< n\epsilon$, which can still be made arbitrarily small, so our claim holds. □

(c) *Claim.* The counting measure $\mu_{\mathbb{N}}$ is sigma finite and regular on $\mathbb{R}$.

Proof. $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$, and $\mu_{\mathbb{N}}(-n, n)$ is finite for all $n \in \mathbb{N}$, so $\mu_{\mathbb{N}}$ is sigma finite.

For any Borel set B, construct an open set O such that for each connected component of B, O is locally the smallest open interval (a, b) containing the component such that $a, b \in \mathbb{N}$. Then $\mu_{\mathbb{N}}(a, b)$ equals the measure of this component of B, so $\mu(O) = \mu(B)$.

Then construct a closed set F such that for each connected component of B, F is locally the largest open interval $[c, d]$ contained in the component such that $c, d \in \mathbb{N}$. Then $\mu_{\mathbb{N}}[a, b]$ equals the measure of this component of B, so $\mu(F) = \mu(B)$. Therefore $\mu_{\mathbb{N}}$ is regular. □

Claim. The Dirac delta measure δ_c for every fixed $c \in \mathbb{R}$ is a regular Borel measure on $\mathbb{R}$.

Proof. Take $c \in \mathbb{R}$. Then for a Borel set B, construct an open set O. If $c \notin B$, then let O be the intersection of all open sets containing B. Then if $c \in O$, then c is in every open set containing B. But then $c \in B$, which is a contradiction, so $c \notin O$. Therefore B and O have measure O. If $c \in B$, then any open set containing B has measure 1, as does B.

Now construct a closed set F. If $c \in B$, then let F be the union of all closed sets contained in B. Then if $c \notin F$, then c is not in any closed set containing B. But then $c \notin B$, which is a contradiction, so $c \in F$. Therefore B and F both have measure 1. If $c \notin B$, then any closed set contained in B has measure 0, as does B. Therefore this measure is regular on $\mathbb{R}$. □

3A.3

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f : X \to [0, \infty]$ is an $\mathcal{S}$-measurable function.

Claim.

$$\int f d\mu > 0 \text{ if and only if } \mu(\{x \in X : f(x) > 0\}) > 0.$$

Proof. ($\Longrightarrow$) For the forward direction, we prove the contrapositive by showing that

$$\mu(\{x \in X : f(x) > 0\}) = 0 \implies \int f d\mu = 0.$$

To do this, first define $B := \{x \in X : f(x) > 0\}$. Then let $A_1, \ldots A_n$ be a partition of $\mathcal{S}$. Then

$$\mathcal{L}(f; p) = \sum_{k=1}^{n} \mu(A_k) \inf_{A_k} f.$$

Then for any A_k where $A_k \cap B \neq A_k$, there exists $x \in A_k$ such that $f(x) = 0$, so $\inf_{A_k} = 0$. Therefore only those A_k's where $A_k \subseteq B$ are relevant for this sum. But $\mu(B) = 0$ by hypothesis, so $\mu(A_k) = 0$ for all $A_k \subseteq B$, so the whole sum is 0. Then because we chose a partition of $\mathcal{S}$ arbitrarily, we know $\int f d\mu = 0$.

($\Longleftarrow$) For the backwards direction, let $(B_n)_{n\in\mathbb{N}}$ be a sequence of sets where

$$B_n := \{x \in X : f(x) > \frac{1}{n}\}.$$

Then this sequence is nested increasing. so by our continuity properties we have

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n) > 0$$

where the last inequality follows from our hypothesis. Then assume that this limit is $c > 0$ for some $c \in \mathbb{R}$. Then by the definition of convergence, we know that there exists an $N \in \mathbb{N}$ such that $|\mu(A_N) - c| < c$. Then $\mu(A_n) > 0$. Now let $P = A_N, X \setminus A_N$ be a partition of X. Note $A_N \in \mathcal{S}$ because $A_N = f^{-1}(\frac{1}{N}, \infty)$. But then, noting $\inf_{A_N} f = \frac{1}{N}$, we have

$$\int f d\mu \geq \mathcal{L}(f; P) = \mu(A_n) \inf_{A_N} f + \mu(X \setminus A_N) \inf_{X\setminus A_N} f \geq \mu(A_n)\frac{1}{N} > 0.$$

$\square$