Keyword Document 2

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Readings for this week: Chapter 2, p. 25-37, p. 41-25.

Key Result 1

Nonexistence of extension of length to all subsets of $\mathbb{R}$.

This theorem states that there does not exist a function μ with the following properties:

(a) μ is a function from the set of subsets of $\mathbb{R}$ to $[0, \infty]$.

(b) $\mu(I) = \ell(I)$ for every open interval I of $\mathbb{R}$.

(c) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence of sets $A_1, A_2, \ldots$ of subsets of $\mathbb{R}$.

(d) $\mu(t + A) = \mu(A)$ for every $A \subseteq \mathbb{R}$ and every $t \in \mathbb{R}$.

This is an important theorem because it describes a significant limitation of the outer measure. If we lose any of the properties (b), (c), or (d), then our outer measure loses a good deal of it's usefulness. (b) and (d) need to stay in order for our measure to make any sense as a representation of length, and (c) is incredibly useful in proving theorems. Therefore we conclude tht we must abandon (a), and find some subset of $\mathbb{R}$ on which we can define our measure.

Key Result 2

Definiton of a *measure*.

If X is a set and $\mathcal{S}$ is a σ-algebra on X, then a *measure* on $(X, \mathcal{S})$ is a function $\mu\ ;\ \mathcal{S} \to [0, \infty]$ such that: $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every disjoint sequence $E_1, E_2, \ldots$ of sets in $\mathcal{S}$.

This is an important definition as it allows us to extend our notion of measure outside of just $\mathbb{R}$. We can now define a measure on any set that has a possible σ-algebra. It is also worth noting that this definition lost key properties that we desired for our measure on $\mathbb{R}$, namely that $\mu(I) = \ell(I)$ for all open intervals I and that measures are translation invariant. It makes sense that these conditions are gone, as we are no longer necessarilly working in spaces that have a notion of length as we commonly understand it.

Key Strategy

Several times throughout this reading it was necessary to change one of the sides of an interval in $\mathbb{R}$ from open to closed or vice versa. This was done with the equality

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}).$$

This was first used to show that half-open intervals are Borel sets because the half-open interval can be rewritten as a countable intersection of open intervals. It is used more throughout the reading, mostly when dealing with Borel sets in cases where intervals need be open.