Problem Set 8

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I adhered to the honor code on this assignment.

Additional Problem 1

Let μ be a regular Borel measure on $\mathbb{R}$ which is finite on all compact sets. Then for $0 < p < +\infty$, a measurable function $f : \mathbb{R} \to \mathbb{C}$ is locally $\mathcal{L}^p$ if for each $x \in \mathbb{R}$, there exists $r > 0$ such that

$$f\chi_{(x-r,x+r)} \in \mathcal{L}^p(\mu). \tag{1}$$

(a) **Claim.** $f \in \mathcal{L}^p_{\text{loc}}(\mu)$ if and only if

$$f\chi_K \in \mathcal{L}^p(\mu) \text{ , for every compact } K \subseteq \mathbb{R}. \tag{2}$$

Proof. ($\Longleftarrow$) Assume (2) holds for some function f. Then for $x \in \mathbb{R}$ and some $r(x) > 0$, we let K be the compact set $[x-r, x+r]$. Then by our assumption,

$$\int |f\chi_{(x-r,x+r)}|^p \, d\mu \leq \int |f\chi_K| \, d\mu < +\infty.$$

Therefore $f \in \mathcal{L}^p_{\text{loc}}(\mu)$.

($\Longrightarrow$) Assume that $f \in \mathcal{L}^p_{\text{loc}}(\mu)$, and take $K \subseteq \mathbb{R}$ to be compact. Then for all $x \in K$, there exists $r(x) > 0$ satisfying (1). K then has the open cover

$$K \subseteq \bigcup_{x \in K} (x - r(x), x + r(x)).$$

Because k is closed and bounded, there exists a finite collection $x_1, \ldots x_n =: A$ such that

$$K \subseteq \bigcup_{x_i \in A} (x_i - r(x_i), x_i + r(x_i)) =: B.$$

Using this finite open cover, we have

$$\int |f\chi_K|^p \, d\mu \leq \int |f\chi_B|^p \, d\mu \leq \sum_{i=1}^{n} \int |f\chi_{(x_i - r(x_i), x_i + r(x_i))}|^p \, d\mu < +\infty.$$

Therefore $f\chi_K \in \mathcal{L}^p(\mu)$ for all compact $K \subseteq \mathbb{R}$. $\square$

(b) The function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = 1$ is in $\mathcal{L}^p_{\text{loc}}(\mu)$ but not in $\mathcal{L}^p(\mu)$. It is in $\mathcal{L}^p_{\text{loc}}(\mu)$ because for any compact $K \subseteq \mathbb{R}$, we have

$$\int |f\chi_K|^p \, d\mu = \int \chi_K \, d\mu = \mu(K) < +\infty,$$

where the final inequality above is true by our hypothesis that μ is finite on all compact sets. Then by (a), $f \in \mathcal{L}^p_{\text{loc}}(\mu)$. However, $f \notin \mathcal{L}^p(\mu)$, because

$$\|f\|_p = \left(\int 1 \, d\mu\right)^{\frac{1}{p}} \to +\infty \text{ , for all } 0 < p < +\infty.$$

(c) Let $\mathcal{C}_c(\mathbb{R})$, be the $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ with compact support, equipped with the sup norm $\|.\|_\infty$. Then for $1 \leq p < \infty$, fix $f \in \mathcal{L}^p_{\text{loc}}(\mu)$, and define

$$\ell_f(\phi) := \int f\phi \, d\mu \text{ , for } \phi \in \mathcal{C}_c(\mathbb{R}). \tag{3}$$

Claim. The integral in (3) is well defined and yields a linear functional on $\mathcal{C}_c(\mathbb{R})$.

Proof. To show that ℓ_f is well defined, we must show that $\ell_f(\phi) < +\infty$ for all f and ϕ. To do this, for $1 \leq q < +\infty$ such that $1 = \frac{1}{p} + \frac{1}{q}$, observe that

$$\ell_f(\phi) \leq \int |f\phi| \, d\mu = \int_K |f\phi| \, d\mu \leq \|f\chi_K\|_p \|\phi\|_q < +\infty.$$

For the above, K is the compact support of ϕ, meaning $|f\phi|$ is supported on K, so the equality holds. The second inequality comes from holder. For the final inequality, the first factor is finite because $f \in \mathcal{L}^p_{\text{loc}}(\mu)$, which by (a) implies $f\chi_K \in \mathcal{L}^p(\mu)$. The second factor is finite because $\phi \in \mathcal{C}_c(\mathbb{R})$. This shows well-definedness of ℓ_f.

Additionally, we know ℓ_f maps to $\mathbb{R}$ and not $\overline{\mathbb{R}}$ by the above, so ℓ_f is a linear functional if it is linear. But linearity of ℓ_f follows quite simply from linearity of the integral. □

(d) **Claim.**

$$\ell_f(\phi) = 0 \text{ , for all } \phi \in \mathcal{C}_c(\mathbb{R}) \tag{4}$$

implies that $f = 0$, μ-a.e.

Proof. We will prove this by contradiction, assuming that $\mu(\{f(x) \neq 0\}) > 0$. Furthermore, assume without loss of generality that $\mu(\{f(x) > 0\}) > 0$, and that this measure is finite, as if not, then clearly (4) is false. Then take $R > 0$ such that $\{f > 0\} \subseteq [-R, R] =: A$.

Using the polar representation of $\mathbb{C}$-valued measurable functions, we know that there exists $h : \mathbb{R} \to \mathbb{C}$ with $|h| = 1$ such that $f = |f| \cdot h$, or equivalently, $|f| = \frac{1}{h} f$. Then define $\phi = \frac{1}{h}\chi_A$, and $0 < q < +\infty$ such that $1 = \frac{1}{p} + \frac{1}{q}$. Then $\phi \in \mathcal{L}^q(\mu)$, and we know $\mathcal{C}_c(\mathbb{R})$ is dense in $\mathcal{L}^q(\mu)$, so there exists $\phi' \in \mathcal{C}_c(\mathbb{R})$ such that for $\epsilon > 0$,

$$\|\phi - \phi'\|_q < \epsilon. \tag{5}$$

From here, we have for the same $\epsilon > 0$, that

$$\begin{aligned} 0 < \int f \, d\mu \leq \int f\chi_A \, d\mu &\leq \int |f|\chi_A \, d\mu && (6) \\ &\leq \int |f\phi| \, d\mu && (7) \\ &\leq \int |f(\phi - \phi' + \phi')| \, d\mu \\ &\leq \int |f(\phi - \phi')| \, d\mu + \int |f\phi'| \, d\mu && (8) \\ &\leq \|f\|_p \|\phi - \phi'\|_q < \|f\|_p \cdot \epsilon. && (9) \end{aligned}$$

For above, the second inequality in (6) follows because f is positive only on A. (7) follows from the definition of ϕ and that $|h| = 1$. The first inequality in (9) comes from holder and that the second term of (8) is 0 by (4). The final inequality comes from (5).

Overall, this is a contradiction, because if $\int f \, d\mu < \|f\|_p \cdot \epsilon$ for all $\epsilon > 0$, then $\int f \, d\mu = 0$, but $\int f \, d\mu > 0$. $\square$

Additional Problem 2

(a) **Claim.** Let $K \subseteq \mathbb{R}$ be compact. Then $\mathcal{C}_c(K)$, the space of continuous complex valued functions on K, is a Banach space with respect to the supremum norm.

Proof. We must first verify that the supremum norm acts as a norm on this space. Positive definiteness, homogeneity, and the triangle inequality all follow easily from the definition of the supremum norm. Also, for all $f \in \mathcal{C}_c(K)$, because f is continuous and defined on a compact set, the extreme value theorem tells us that f has a max and min. Therefore $\|f\|_\infty < +\infty$, so $\|.\|_\infty$ is indeed a norm on our space.

To see that this is a Banach space, recall that a sequence being uniformly Cauchy implies that it is uniformly convergent. Therefore for (f_n), where $f_n \in \mathcal{C}_c(K)$, if (f_n) is cauchy in $\|.\|_\infty$, then it is uniformly cauchy. Therefore it is uniformly convergent to some function f. Recall as well that the uniform limit of a sequence of continuous functions is continuous, so f is continuous. Therefore $f \in \mathcal{C}_c(K)$, so this is indeed a Banach space. $\square$

(b) **Claim.** $\mathcal{C}_c(\mathbb{R})$ is not a Banach space with respect to the supremum norm.

Proof. Define the sequence of functions (f_n) by

$$f_n(x) = e^{-x^2}\chi_{[-n,n]} + (1 - n(x-n))\chi_{(n,n+\frac{1}{n}]} + (1 + n(x+n))\chi_{[-n-\frac{1}{n},-n)}.$$

While it is not immediately obvious that these functions are continuous, the second and third terms serve to remove the discontinuity in the first term at n and $-n$ by creating a "steep" line from 1 to 0 at these points.

Then for $\epsilon > 0$, there exists $n \geq m \in \mathbb{N}$ such that $e^{-m^2} < \epsilon$ because $e^{-m^2} \to 0$. Then

$$\begin{aligned} \|f_m - f_n\|_\infty &= \|f_m - f_n\|_{\infty;[-m,m]^c} && (10) \\ &\leq \|f_m\|_{\infty;[-m,m]^c} + \|f_n\|_{\infty;[-m,m]^c} \\ &\leq 2\epsilon. \end{aligned}$$

For above, (10) holds because on $[-m, m]$, both f_m and f_n evaluate to e^{-x^2}.

This means that (f_n) is Cauchy in $\|.\|_\infty$, but $(f_n) \to e^{-x^2}$, which does not have compact support, so $\lim_{n\to\infty} f_n \notin \mathcal{C}_c(\mathbb{R})$. Therefore $\mathcal{C}_c(\mathbb{R})$ is not a banach space with respect to the supremum norm. $\square$

Additional Problem 3

(a) **Claim.** If V and W are both finite dimensional $\mathbb{C}$-vector spaces, then every linear map between V and W is bounded.

Proof. For linear map $T : V \to W$, because all norms on finite dimensional vector spaces are equivalent, it suffices to show that T is bounded in $\|.\|_1$. Then for $x \in V$, we know that

$$x = \sum_{i=1}^{N} c_i v_i \text{ , where } c_i \in \mathbb{C} \text{ , and } \mathcal{B} := \{v_1, \ldots v_N\} \text{ form a basis of } V.$$

From here, if we define $M := \max_{v_j \in \mathcal{B}} \|Tv_j\|_1$, then

$$\|Tx\|_1 \leq \sum_{i=1}^{N} |c_k| \|Tv_k\|_1 \leq NM \sum_{i=1}^{N} |c_i| = NM\|x\|_1.$$

Notably, both N and M are independent of our choice of x, so T is bounded. $\square$

(b) Suppose μ is a finite Borel measure on $\mathbb{R}$, and take $f \in \mathcal{L}^\infty(\mu)$. Define the multiplication operator

$$T_f : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu) \ , \ T_f(\phi) = f \cdot \phi.$$

We know already that T_f is a bounded linear operator with

$$\|T_f \phi\|_2 \leq \|f\|_\infty \|\phi\|_2 \ , \text{ for all } \phi \in \mathcal{L}^2(\mu). \tag{11}$$

Claim. The operator norm of T_f has the value

$$\|T_f\| = \|f\|_\infty.$$

Proof. Using the definition of the operator norm, we have

$$\|T_f\| = \sup_{\|\phi\|_2 \leq 1} \|T_f \phi\|_2 \leq \|f\|_\infty \|\phi\|_2 \leq \|f\|_\infty.$$

For above, the first inequality follows from (11), and the second from the above supremum being over $\|\phi\|_2 \leq 1$.

It remains to show $\|T_f\| \geq \|f\|_\infty$. To do so, for $\epsilon > 0$, define

$$A := \{f(x) > \|f\|_\infty - \epsilon\}.$$

Then A is measurable, and $\mu(A) > 0$ as shown in set 7. Then define

$$\phi = \frac{\chi_A}{\sqrt{\mu(A)}}.$$

By this definition, $\phi \in \mathcal{L}^2(\mu)$ with $\|\phi\|_2 = 1$. Thus

$$\begin{aligned}
\|T_f\| = \sup_{\|\varphi\|_2 \leq 1} \|T_f\varphi\|_2 \geq \|T_f\phi\|_2 &= \|f\phi\|_2 \\
&= \left(\int |f\phi|^2 \, d\mu\right)^{\frac{1}{2}} \\
&\geq \left(\int_A |f\phi|^2 \, d\mu\right)^{\frac{1}{2}} \\
&\geq \left((\|f\|_\infty - \epsilon)^2 \int_A |\phi|^2 \, d\mu\right)^{\frac{1}{2}} \qquad (12) \\
&= \|f\|_\infty - \epsilon. \qquad (13)
\end{aligned}$$

For above, (12) holds because our definition of A, and (13) holds because $\|\phi\|_2 = 1$, and ϕ is supported only on A.

Therefore we conclude $\|T_f\| = \|f\|_\infty$. □

Additional Problem 4

For the Banach space $(X, \|.\|_X)$, we denote $\mathcal{B}(x) := \mathcal{B}(X, X)$, the vector space of bounded linear operators on X with respect to the operator norm. Then the identity operator $I : X \to X$, $I(x) = x$ is a bounded linear operator which is invertible.

(a) **Claim.** For $T, S \in \mathcal{B}(X)$, one has the composition $TS \in \mathcal{B}(x)$ with

$$\|TS\| \leq \|T\|\|S\|.$$

Proof. Because both T and S are bounded operators, we have

$$\|Tx\|_x \leq \|T\|\|x\|_x \text{ , and } \|S_x\|_x \leq \|S\|\|x\|_x.$$

With this, we can say

$$\begin{aligned}
\|TS\| &= \sup_{\|x\| \leq 1} \|TSx\|_X \\
&\leq \sup_{\|x\| \leq 1} \|T\|\|Sx\|_X \\
&\leq \sup_{\|x\| \leq 1} \|T\|\|S\|\|x\|_X \\
&\leq \|T\|\|S\|
\end{aligned}$$

□

(c) Let $P \in \mathcal{B}(X)$ be given with $\|P\| < 1$. Then defined

$$T := I - P \in \mathcal{B}(X).$$

Claim. T is invertible, or equivalently, there exists $S \in \mathcal{B}(X)$ such that $TS = ST = I$.

Proof. To begin, define

$$S := \sum_{n=0}^{\infty} P^n.$$

We know $\|P\| < 1$, so we have

$$\sum_{n=0}^{\infty} \|P\|^n < +\infty.$$

Then by result 6.41, we have $S < +\infty$, so $S \in \mathcal{B}(X)$. Computing TS, with the computation of ST being very similar, we have

$$TS = \sum_{n=0}^{\infty} P^n - P \sum_{n=0}^{\infty} P^n = P^0 = I.$$

□