Problem Set 3

Theo McGlashan

Additional Problem 1

A measure μ on $(\mathbb{R}, \mathcal{S})$ is called **Outer Regular** if for every $A \in \mathcal{S}$, we have for all $\epsilon > 0$, there exists open set $O \in \mathbb{S}$ with $A \subseteq O$ such that $\mu(O \setminus A) < \epsilon$.

Claim. For every $A \in \mathbb{S}$ with $\mu(A) < +\infty$ and for all $\epsilon > 0$, there exists a finite collection of pairwise disjoint open intervals $I_1, \ldots, I_N$ for some $N \in \mathbb{N}$ such that

$$\mu\left(A \triangle \left(\bigsqcup_{k=1}^{N} I_k\right)\right) < \epsilon.$$

Proof. Because μ is outer regular, we know that for all $A \in \mathcal{S}$ and $\epsilon > 0$, there exists an open set $O \in \mathcal{S}$ with $A \subseteq (O)$ and $\mu(O \setminus A) < \epsilon$. As O is an open set, we know that

$$O = \bigsqcup_{n \in \mathbb{N}} I_n \text{ for some collection of open intervals } (I_n)_{n \in \mathbb{N}}.$$

Because $\mu(O) < +\infty$ and because of additivity, we know

$$\mu\left(\bigsqcup_{n \in \mathbb{N}} I_n\right) = \sum_{n \in \mathbb{N}} \mu(I_n) < +\infty.$$

Using the cauchy criterion for series, we can construct a finite collection $I_1, \ldots I_N$ such that

$$\mu(O) - \sum_{k=1}^{N} \mu(I_k) < \epsilon, \text{ or equivalently, } \mu\left(O \setminus \bigsqcup_{k=1}^{N} I_k\right) < \epsilon.$$

Therefore

$$\begin{aligned}
\mu\left(A \triangle \left(\bigsqcup_{k=1}^{N} I_k\right)\right) &= \mu\left(\left(A \cup \bigsqcup_{k=1}^{N} I_k\right) \setminus \left(A \cap \bigsqcup_{k=1}^{N} I_k\right)\right) \\
&\leq \mu\left(O \setminus \left(A \cap \bigsqcup_{k=1}^{N} I_k\right)\right) & (1) \\
&= \mu\left((O \setminus A) \cup \left(O \setminus \bigsqcup_{k=1}^{N} I_k\right)\right) \\
&\leq \mu(O \setminus A) + \mu\left(O \setminus \bigsqcup_{k=1}^{N} I_k\right) < 2\epsilon & (2)
\end{aligned}$$

Note that (1) follows from the fact that $A \cup \bigsqcup_{n \in \mathbb{N}} I_n \subseteq O$ and (2) follows from subaditivity. $\square$

Additional Problem 2

(b) Let $(X, \mathcal{S}, \mu)$ be a measure space.

Claim. Let $\{A_n, \in \mathbb{N}\} \subseteq \mathcal{S}$. If $\sum_{n\in\mathbb{N}} \mu(A_n) < +\infty$, then

$$\mu(\{x \in A_n\ ,\ i.o.\}) = 0.$$

Proof. The sequence $(\bigcup_{k=n}^{\infty} A_k)_{n\in\mathbb{N}}$ is a nested decreasing sequence of sets. Also, by subaditivity and our hypothesis,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k) < +\infty.$$

Therefore by continuity properties and the result of part (a), we know that

$$\mu(\{x \in A_n\ ,\ i.o.\}) = \mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right).$$

Then because $\sum_{k=1}^{\infty} \mu(A_k) < +\infty$, we know by the cauchy criterion for series that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\sum_{k=n}^{\infty} \mu(A_k) < \epsilon$. But this means the above limit is zero, and so $\mu(\{x \in A_n\ ,\ i.o.\}) = 0$. □

(c) A number $\alpha \in [0,1] \setminus \mathbb{Q}$ is called *Liouville* if there exists $c \in \mathbb{R}$ such that $|a - \frac{p}{q}| < e^{-cq}$ for infinitely many $\frac{p}{q} \in \mathbb{Q}$.

Claim. The set of Liouville numbers in $[0,1]$ has zero Lebesgue measure.

Proof. First, fix $c \in \mathbb{R}$. Then define

$$A_q := \bigcup_{p=0}^{q} \left(\frac{p}{q} - e^{-cq}, \frac{p}{q} + e^{-cq}\right)$$

Observe that if and only if $\alpha \in A_n$ infinitely often in $\mathbb{N}$, then α is Liouville. Next, observe that

$$\mu(A_q) \leq \sum_{p=0}^{q} 2e^{-cq} \leq 2(q+1)e^{-cq}$$

Therefore we know that

$$\sum_{q=1}^{\infty} \mu(A_q) \leq \sum_{q=1}^{\infty} \frac{(2q+1)}{e^{cq}} < +\infty$$

This means we can apply the results of part (b) to say that the set of Liouville numbers has zero Lebesgue measure. $\square$

Additional Problem 3

For the cantor function Λ, define $g : [0,1] \to [0,2]$ to be the function with expression

$$g(x) = \Lambda(x) + x$$

(a) *Claim.* g is continuous, bijective, and has a continuous inverse $h := g^{-1}$.

Proof. We know that Λ is continuous, so g is the pointwise sum of two continuous functions, so it is continuous.

To see g is surjective, observe that $g(0) = 0$ and $g(1) = 2$. Then because g is continuous, we know by the intermittent value theorem that g must attain all values between 0 and 2, so it is surjective.

To see g is injective, take $x, y \in [0,1]$ such that $g(x) = g(y)$. Therefore $\Lambda(x) + x = \Lambda(y) + y$, so $\lambda(x) - \lambda(y) = y - x$. Now assume without loss of generality that $x > y$. Then because Λ is monotone increasing, we know $\Lambda(x) > \Lambda(y)$, so $\Lambda(x) - \Lambda(y) > 0$. But $y - x < 0$, so we have a contradiction. Therefore $x = y$ and g is injective.

To see g^{-1} is continuous, take $a, b \in [0,2]$ with $a \geq b$. Then there exists $x, y \in [0,1]$ such that $g(x) = a$ and $g(y) = b$ by surjectivity. Because g is monotone increasing, we know that $x \geq y$. Then $g^{-1}(a) = x$ and $g^{-1}(b) = y$, so g^{-1} is monotone increasing. g^{-1} is also surjective, because g is a bijection. Because g^{-1} is bounded, surjective, and monotone, it is continuous. $\square$

(b) *Claim.* $g(C)$ is measurable with $|g(C)| = 1$.

Proof. Observe that on C^c, Λ is constant. Therefore for $x \in C^c$, $g(x) = x + k$ where k is the constant value of the cantor function locally around x. Then for an interval $I \subseteq C^c$, we know that $g(I) = I + k$, so $|g(I)| = |I|$. In the construction of the cantor set, we remove finitely many open intervals from $[0,1]$ in each iteration, and there are finitely many iterations. Therefore the compliment of the cantor set is a countable (also disjoint) union of open intervals. Therefore by additivity,

$$|g(C^c)| = \sum_{n \in \mathbb{N}} |g(I_n)| = \sum_{n \in \mathbb{N}} |I_n| = |C^c| = 1.$$

This along with the fact that $g(C^c) \subseteq [0,2]$ means that

$$|[0,2] \setminus g(C^c)| = |[0,2]| - |g(C^c)| = 2 - 1 = 1.$$

And because we can write $[0,2]$ as the disjoint union $g(C) \sqcup g(C^c)$, we know that $|g(C)| = 1$. $\square$

Claim. $g(C)$ contains a non-measurable set A.

Proof. Define the relation on [0,2] by $x \sim x' \iff (x - x') \in \mathbb{Q}$. Then construct the set of equivalence classes $\{[x] : [x] \cap g(C) \neq \emptyset\}$ For each element of this set, chose a representative in $g(C)$ using the axiom of choice. Let A be the set of these representatives. If we assume that A is measurable, we can say that

$$g(c) \subseteq \bigsqcup_{r \in \mathbb{Q} \cup [-2,2]} (A + r) \subseteq (-2, 4).$$

We know that $|g(C)| = 1$, and for all $r \in \mathbb{Q} \cap [-2,2]$, we know $|A| = |A + r|$. Therefore

$$1 = |g(c)| \leq \sum_{r \in \mathbb{Q} \cap [-2,2]} |A| \leq |(-2,4)| = 6.$$

But then $|A|$ must be nonzero for the sum to be at least 1, but then the sum diverges because it is an infinite sum of nonzero numbers. Therefore we have a contradiction, and A is not measurable. $\square$

(c) *Claim.* g maps some measurable set surjectively onto a non-measurable set.

Proof. We know from (b) that $A \subseteq g(C)$ for non-measurable set A. Therefore $g^{-1}(A) \subseteq C$. Then define $D = g^{-1}(A) \cap C$. Then $g(D) = A$, and because $|C| = 0$, we know that $|D| = 0$, so D is measurable. Because $g(D) = A$, this map is surjective. $\square$

(d) *Claim.* $D := g^{-1}(A)$ is a Lebesque measurable set but not a Borel set.

Proof. We know that g^{-1} is continuous and $[0,2]$ is Borel-measurable, so by 2.41 we know g^{-1} is a Borel-measurable function. Then assume that $g^{-1}(A) = D$ is Borel-measurable. We know by 2.51 that $g(g^{-1}(A))$ is Borel-measurable. But $g(g^{-1}(A)) = A$, which is not Borel-measurable, so we have a contradiction. Therefore D is not Borel-measurable. Note also that we can interchange the preimage and inverse here because g is a bijection. $\square$

Additional Problem 4

(b) For a fixed $n \in \mathbb{N}$, let $I_{n,k}$ for $k = 1, \dots 2^n$ be the component subintervals of the n-th level Cantor set

$$I_n := \bigsqcup_{k=1}^{2^n} I_{n,k} = [0,1] \setminus \left(\bigsqcup_{k=1}^{n} G_k \right).$$

Where G_n are the gaps removed at the n-th stage of the construction of C.

Claim. If $x, y \in C$ with $|x - y| < 3^{-n}$, then x and y are in the same component $I_{n,k}$, and $|\Lambda(x) - \Lambda(y)| \leq 2^{-n}$.

Proof. If $|x-y| < 3^{-n}$, then x and y written in base 3 must have the same first n digits. Therefore $\Lambda(x)$ and $\Lambda(y)$ are base 2 numbers with the same first n digits, so they differ at most by 2^{-n}. Because x and y are in C, we know they have a base 3 representation with only 0's and 2's. Because each level of the cantor set removes the middle third of all remaining intervals, splitting these intervals in 2, the base 3 representation of x and y containing only 0's and 2's can be thought of as an index to what remaining interval of the cantor set x and y are in, where for the k-th digit of x and y, the value of k decides which of the intervals formed at the kth level of the cantor set x and y go in. Therefore if x and y share the same first n digits, they are in the same interval at least up to the nth level of the cantor set. □

Claim. the above result implies the cantor function Λ is continuous on C.

Proof. Let $\epsilon > 0$. Then take $n \in \mathbb{N}$ where $2^{-n} < \epsilon$. Then let $\delta = 3^{-n}$. By the previous claim, we know that for $x, y \in C$ with $|x - y| < \delta$, we have

$$|\Lambda(x) - \Lambda(y)| \leq 2^{-n} < \epsilon.$$

Therefore Λ is continuous on C. □