MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

# Dictionary of Inequalities

## Second Edition

Peter Bullen

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*This book is dedicated to my wife*

*Georgina Bullen*

# Contents

# INTRODUCTION

The object of this dictionary is to provide an easy way for researchers to locate an inequality either by name or by subject. Proofs will not be given, although the methods used may be indicated. Instead, references will be given where such details and further information can be found.

The bibliography is not intended to be complete, although an attempt has been made to include all books on inequalities. References mentioned in these books are not repeated; only papers more recent than the books or which seem to have been missed by the other references are included.

Inequalities are not necessarily given in their most general forms; in some cases the intuitive physical version is given rather than the more abstract mathematical formulation that may appear elsewhere under a different guise. Usually the most common version is given, and the more general results are stated later as extensions or variants. In addition, the name of the inequality may be historically that of a variant of the one stated; when this is done it will be made clear.

The history of most inequalities is often obscure, therefore the correct attribution is not always easy to make. Little attempt is made to trace the history of particular inequalities; this topic is covered by many excellent articles by the late D.S. Mitrinović and members of his school. Some of these articles are summarized in the book *Analytic Inequalities* by Mitrinović, [AI][1].

The field of inequalities is vast and some restrictions have had to be made. The whole area of elementary geometric inequalities is, with few exceptions, omitted because such inequalities are rarely used except within the field itself. Those interested are referred to the monographs *Geometric Inequalities* by Bottema, Đordjević, Janić, Mitrinović & Vasić, [Bot], and *Recent Advances in Geometric Inequalities* by Mitrinović, Pečarić & Volenec, [MPV]; also *Inequalities: Theory of Majorization and its Applications,* Chapter 8, by Marshall, Olkin & Arnold, [MOA]. In addition, very few results from number theory are given; again, those interested are referred to the literature, for instance, *Inequalities in Number Theory* by Mitrinović & Popadić, [MP].

[1]A list of abbreviations for the titles of certain standard references can be found at the beginning of the Bibliography on p. 321.

In recent years the number of papers on inequalities, as well as the number of journals devoted to this subject, has considerably increased. Most of the inequalities in this book have been subjected to various generalizations and in particular have often been put into an abstract form. Although these abstractions often enable many inequalities to be placed under one roof, so to speak, and are therefore of interest, it is felt that often the beauty and importance of the original is lost. In any case this direction has not been pursued. Those interested in this development are referred to the literature; for example, the books *Classical and New Inequalities in Analysis* by Mitrinović, Pečarić & Fink, [MPF], and *"Convex Functions, Partial Orderings and Statistical Applications,"* by Pečarić, Proschan & Tong, [PPT].

Many of the basic inequalities exist at various levels of generality, as statements about real numbers, complex numbers, vectors, and so on. In such cases the inequality will be presented in its simplest form with the other forms as extensions, or under a different heading.

Basic concepts will be assumed to be known although less basic ideas will be defined in the appropriate entry.

Most notations taken from real analysis are standard but some that are peculiar to this monograph are in the section, **Notations 1**, p. xv; full details can be found in the books within the reference sections.

Transliteration of names from Cyrillic and Chinese causes some difficulties as the actual spelling has varied from time to time and from journal to journal. In the case of Cyrillic the spelling used is given in the old Mathematical Reviews, old Zentralblatt, form. This is very close to that of the British Standards Institute used in the *Encyclopedia of Mathematics*, [EM]; it has the advantage of being less oriented to the English language. However, this is not insisted upon when reference to the literature is made. Other Slavic languages employ various diacritical letters and that spelling has been used. In the case of Chinese, the spelling chosen is that used in the book or paper where the inequality occurs, but other versions used will be given if known; in addition the family name is given first as is usual, except for Chinese who are living abroad and who have adopted the standard Western convention. Cross references are given between the various transliterations that have been given.

It is hoped that this dictionary will have the attraction attributed to most successful dictionaries — that on looking up a given item one is led by interest to others, and perhaps never gets to the original object being researched.

An attempt has been made to be complete, accurate, and up to date. However, new inequalities, variants, and generalizations of known inequalities come in every set of new journals that arrive in the library. Readers wishing to keep up with the latest inequalities are urged to go to the latest issues of the reference journals — *Mathematical Reviews, Zentralblatt für Mathematik und ihre Grenzegebiete, Referativniĭ Žurnal Matematika.*[2] Also there are journals devoted to inequalities: *Univerzitet u Beogradu Publikacije Elektrotehničkog*

[2] Реферативный Журнал Математика.

*Fakulteta. Serija Matematika i Fizika*,[3] all of whose volumes, from the first in the early fifties, are full of papers on inequalities; the recent *Journal for Inequalities and Applications* and *Mathematical Inequalities and Applications*; the *American Mathematical Monthly*, the *Journal of Mathematical Analysis and Applications*, the *Mathematics Gazette*, and the *Mathematics Magazine*, as well as most of the various journals appearing in the countries of the Balkans; and there are the earlier issues of the *Journal, and Proceedings of the London Mathematical Society*, and the *Quarterly Journal of Mathematics, Oxford*.

Some may find a favorite inequality omitted, or misquoted. Please let the author know of these errors and omissions; if there is another edition the corrections and additions will be included. Accuracy is difficult as the original papers are often difficult to obtain, and most books on inequalities contain errors, a notable exception being *Inequalities* by Hardy, Littlewood & Polyá, [HLP], the field's classic.

## INTRODUCTION TO SECOND EDITION

The new edition is a rewriting of the first, removing all errors found, or pointed out, and adding new results that have appeared in the last fifteen years.

The scene has changed in that time in several ways. Certain journals have ceased to publish but have been replaced by several new and very active publications. The biggest loss was *Univerzitet u Beogradu Publikacije Elektrotehničkog Fakulteta. Serija Matematika* ceased publication in 2007. It has been replaced by *Applicable Analysis and Discrete Mathematics*, a journal with less emphasis on inequalities. However there are several new journals that must be mentioned: *Journal of Mathematical Inequalities, Mathematical Inequalities and Applications, Journal of Inequalities and Applications*, and *Journal of Inequalities in Pure and Applied Mathematics*. There are no doubt others but this is indicative of the explosive growth of research in this area. There is not a part of the world that does not include mathematicians working on inequalities. Also mention should be made of the Research Group in Mathematical Inequalities and Applications, RGMIA, that can be found at http://rgmia.org/index.php.

This brings me to the other way in which the scene has changed: the rise and universal use of the Internet. Where possible I have included URLs for important references and a list of useful URLs is given below although such references are very likely to change. A case in point is the RGMIA that suffered a loss in support from its home university resulting in many references becoming out of date and unavailable. The references given here were up to date at the time of going to press. An example of this use of the Internet is one of the basic references: [EM] *Encyclopædia of Mathematics* that is now available online at

[3] The *"i Fizika"* was dropped after 1982.

http://www.encyclopediaofmath.org. In addition of course there is Wikipedia and Google that between them seem to be able to answer almost any question.

One point of some interest is the existence of a few mathematicians that are extremely prolific in the field of inequalities, and further devote all of their research to this field. In the previous generation the doyen of this group was certainly Professor D. S. Mitrinović. Presently he has several worthy successors; to mention just five, H. Alzer, S. S. Dragomir, B. G. Pachpatte, J. E. Pečarić, and Qi Feng.

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# Notations

## 1 General Notations

**1.1** $\mathbb{R}$ is the set of real numbers. $\mathbb{Q}$ the set of rational numbers.
$\mathbb{R}^+ = \{x : x \in \mathbb{R} \ \wedge x \geq 0\}$; the set of non-negative real numbers.
$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$.
$\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of non-negative integers.
$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$, the set of all integers.
$\mathbb{C}$ is the set of complex numbers. $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
$\mathbb{S}^* = \mathbb{S} \setminus \{0\}$, where $\mathbb{S}$ is any of the above.

If $q \in \mathbb{Q}$ then $q = a/b, a \in \mathbb{Z}, b \in \mathbb{N}^*; a, b$ are not unique but are if we require them to be co-prime, have no common factors. This is called the *reduced form* of $q$.

$[a, b]$, $]a, b[$, $[a, b[$, $]a, b]$ denote the open, closed(compact), half-open (-closed) intervals in $\mathbb{R}$, or in any of the above that are appropriate.[4]

In general we will use $n, m, i, j, k$ to be integers, and $x, y$, to be real, and $z = re^{i\theta}$ to be complex.

However, in a situation where complex numbers are involved $i = \sqrt{-1}$.

$D = \{z;\ |z| < 1\}$, $\overline{D} = \{z;\ |z| \leq 1\}$ are, respectively, the *open unit disk (ball) and closed unit disk (ball) in* $\mathbb{C}$.

If $z \in \mathbb{C}$ to write $z \geq r$ means that $z, r$ are real and the inequality holds.

**1.2** If $p, q \in \mathbb{R}^* \setminus \{1\}$ then *$p$ is the conjugate of $q$, $p, q$ are conjugate indices*, if

$$\frac{1}{p} + \frac{1}{q} = 1; \qquad \text{or} \qquad q = \frac{p}{p-1}.$$

From time to time $p = 1, q = \infty$ is allowed.[5]

**1.3** If (I) denotes the inequality $P < Q$, or $P \leq Q$, then ($\sim$I) will denote the *reverse inequality* $P > Q$, respectively, $P \geq Q$. For more details on this topic see: **Reverse, Inverse, and Converse inequalities**.

[4]The usage $(a,b)$ is restricted to mean a point in $\mathbb{R}^2$; see Section 2.1 below.
[5]Sometimes the word *dual* is used instead of conjugate.

**1.4** A *domain* is an open simply connected set; the closure of such a set is called a *closed domain.*

If $A$ is a set then its *boundary, or frontier*, is written $\partial A$.

If $A \subseteq X$, $f : A \to Y$ is a *function* with *domain* $A$ and *co-domain* $Y$.

The *image of* $A$ *by* $f$ is $f[A] = \{y;\ y = f(x), x \in A\}$.

If $B \subseteq Y$ then the *pre-image of* $B$ *by* $f$ is $f^{-1}[B] = \{x;\ f(x) \in B\}$.

In general the word function will mean a real valued function of a real variable; that is, in the above notation: $X = Y = \mathbb{R}$.

**1.5** Partial derivatives are denoted in various ways: thus $\partial^2 f/\partial x \partial y$, $f''_{12}$, etc.

# 2 Sequences and n-tuples

**2.1** If $a_i \in \mathbb{R}$, $i \in \mathbb{N}^*$, then the *real sequence* $\underline{a}$ is: $\underline{a} = (a_1, a_2, \ldots)$. If $n \in \mathbb{N}^*$ then $\underline{a}$ will denote the *real* $n$*-tuple*, $\underline{a} = (a_1, a_2, \ldots, a_n)$.

The $a_i, i = 1, \ldots$ are called the *elements, entries, or terms of the sequence,* $n$*-tuple* $\underline{a}$.

A sequence or $n$-tuple can have other kinds of elements, in particular complex numbers, but unless otherwise specified the term sequence, $n$-tuple, will mean real sequence, $n$-tuple.

If $\underline{a}$ is an $n$-tuple its *length* is, $|\underline{a}| = \sqrt{|a_1|^2 + \cdots + |a_n|^2}$.

If $\underline{b}$ is another $n$-tuple and $\lambda \in \mathbb{R}$ then:

$$\lambda\underline{a} = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n), \quad \underline{a} + \underline{b} = (a_1 + b_1, \ldots, a_n + b_n), \quad \underline{a}.\underline{b} = \sum_{i=1}^{n} a_i b_i.$$

If $\underline{a}.\underline{b} = 0$ we say that $\underline{a}$ and $\underline{b}$ are *orthogonal.*

$A_k = \sum_{i=1}^{k} a_i,\ k = 1, 2, \ldots$ denotes the $k$-th partial sum of the sequence or $n$-tuple $\underline{a}$; if needed we will put $A_0 = 0$.

These notations extend to sequences providing the relevant series converge.

$\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n\ factors}$ is set of all real $n$-tuples; clearly $\mathbb{R}^1 = \mathbb{R}$.

The plane $\mathbb{R}^2$ can be identified with the set $\mathbb{C}, z = x + iy$ when it is called the *complex plane.*

$B_n = \{\underline{a};\ |\underline{a}| < 1\}$, $\overline{B}_n = \{\underline{a};\ |\underline{a}| \leq 1\}$ are, respectively, the *unit ball, closed unit ball in* $\mathbb{R}^n$; if $n = 1$, or the value of $n$ is obvious, the suffix may be omitted.

The surface of $B_3$ can be identified with $\overline{\mathbb{C}}$ by the stereographic from $(0, 0, 1)$, the point identified with $\infty$; then the surface of $B_3$ is called the *Riemann sphere.*

If $U, V \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}$ then:

$$U + V = \{\underline{x} : \underline{x} = \underline{u} + \underline{v}, \underline{u} \in U, \underline{v} \in V\}; \qquad tU = \{\underline{x} : \underline{x} = t\underline{u}, \underline{u} \in U\}.$$

$\underline{e}_1 = (1, 0, \ldots, 0), \underline{e}_2 = (0, 1, 0, \ldots, 0), \ldots \underline{e}_n = (0, \ldots, 0, 1)$, is the *standard basis* of $\mathbb{R}^n$.

In the cases of $n = 2, 3$ it is usual to write: $\underline{e}_1 = \underline{i}, \underline{e}_2 = \underline{j}$ and $\underline{e}_1 = \underline{i}, \underline{e}_2 = \underline{j}, \underline{e}_3 = \underline{k}$, respectively.

If $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ then $\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i$;

$\underline{0} = (0, 0, 0, \dots, 0) = \sum_{i=1}^{n} 0\underline{e}_i$ will denote the *zero, or null* $n$-tuple.

$\underline{e} = (1, 1, 1, \dots, 1) = \sum_{i=1}^{n} \underline{e}_i$

If $\lambda \in \mathbb{R}$ then $\lambda\underline{e} = (\lambda, \lambda, \lambda, \dots, \lambda)$ is called a *constant $n$-tuple.*

$\underline{a}_i' = (a_1, a_2, \dots a_{i-1}, a_{i+1}, \dots, a_n),\ n \geq 2$, the $(n-1)$-tuple obtained from $\underline{a}$ by omitting the $i$-th element.

**2.2** The *differences of* $\underline{a}$ are defined inductively as follows:

$$\begin{aligned} \Delta^0 a_i = a_i; \Delta^1 a_i &= \Delta a_i = a_i - a_{i+1}; \Delta^2 a_i = a_i + a_{i+2} - 2a_{i+1}, \\ \Delta^k a_i &= \Delta(\Delta^{k-1} a_i),\ k = 2, 3, \dots . \end{aligned}$$

and we write $\Delta\underline{a}, \Delta^k\underline{a}, k > 1$, for the sequences of these differences.

We will also write $\tilde{\Delta} a_i = -\Delta^1 a_i$, and then $\tilde{\Delta}^k a_i = (-1)^k \Delta^k a_i, k \geq 2$.

**2.3** If $f, g$ are suitable functions then:

$$f(\underline{a}) = \big(f(a_1), f(a_2), \dots\big); \quad g(\underline{a}, \underline{b}) = \big(g(a_1, b_1), g(a_2, b_2), \dots\big).$$

For example: $\underline{a}^p = (a_1^p, a_2^p, \dots), \underline{a}\,\underline{b} = (a_1b_1, a_2b_2, \dots)$.

$\sum_k! f(a_{i_1}, \dots, a_{i_k})$ means that the sum is taken over all permutation of $k$ elements from $a_1, \dots, a_n$; in the case $k = n$, we write $\sum! f(\underline{a})$.

The above notations can easily be modified to allow for complex $n$-tuples and sequences. In addition many of the concepts apply in much more abstract situations when the term *vector* is used.

**2.4** $\underline{a} \sim \underline{b}$ means that for some $\lambda, \mu \in \mathbb{R}$, not both zero, $\lambda\underline{a} + \mu\underline{b} = \underline{0}$

$\underline{a} \sim^+ \underline{b}$ means that for some $\lambda, \mu \in \mathbb{R}^+$, not both zero, $\lambda\underline{a} + \mu\underline{b} = \underline{0}$

Geometrically $\underline{a} \sim \underline{b}$ means that $\underline{a}, \underline{b}$ are on the same line through the origin, while $\underline{a} \sim^+ \underline{b}$ means that they are on the same ray, or half-line, with vertex at the origin.

**2.5** $\underline{a} \leq \underline{b}$ means that for all $i$, $a_i \leq b_i$; the definitions of other inequalities can be made analogously.

In particular if $\underline{a} > \underline{0}, (\geq \underline{0})$ then $\underline{a}$ is said to be *positive, (non-negative)*; similar definitions can be made for *negative, (non-positive)* sequences, $n$-tuples.

$\underline{a} > M$ means that for all $i, a_i > M$, that is $\underline{a} > M\underline{e}$, and similarly for the other inequalities.

If $\Delta\underline{a}, > \underline{0}, (\geq \underline{0})$ then $\underline{a}$ is said to be *strictly decreasing,(decreasing)*; in a similar way we can define *strictly increasing, (increasing)* $\underline{a}$.

**2.6** If $p \in \mathbb{R}^*$, $\|\underline{a}\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$ or $\left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}$.
In particular $\|\underline{a}\|_2 = |\underline{a}|$.
As usual when $p = 1$ the suffix is omitted.
Of course for negative $p$ we assume that $\underline{a}$ is positive.
$\ell_p$ denotes the set of all sequences $\underline{a}$ with finite $\|\underline{a}\|_p$.

$$\begin{aligned} \|\underline{a}\|_\infty &= \max \underline{a} = \max\{a_1, \ldots, a_n\} \text{ or } \sup\{a_1, \ldots\}; \\ \|\underline{a}\|_{-\infty} &= \min \underline{a} = \min\{a_1, \ldots, a_n\} \text{ or } \inf\{a_1, \ldots\}. \end{aligned}$$

$\ell_\infty$ is the set of all sequences $\underline{a}$ with finite $\|\underline{a}\|_\infty$.

**2.7** $\underline{a} \star \underline{b}$ is the *convolution* of the positive sequences $\underline{a}$, $\underline{b}$, the sequence $\underline{c}$ with terms

$$c_n = \sum_{i+j=n} a_i b_j, n \in \mathbb{N}.$$

# 3 Means

**3.1** Means[6] are an important source of inequalities and the more common means are listed first, followed by a few means that are defined for two numbers only. Other means are defined in the entries dealing with their inequalities. See: **Arithmetico-geometric Mean Inequalities, Counter Harmonic Mean Inequalities, Difference Means of Gini, Gini-Dresher Mean Inequalities, Hamy Mean Inequalities, Heronian Mean Inequalities, Logarithmic Mean Inequalities, Mixed Mean Inequalities, Muirhead Symmetric Function and Mean Inequalities, Nanjundiah's Inverse Mean Inequalities, Quasi-arithmetic Mean Inequalities, and Whiteley Mean Inequalities**.

If $p \in \overline{\mathbb{R}}$ the *p power mean, of order n*, of the positive $n$-tuple $\underline{a}$ with positive weight $\underline{w}$ is

$$\mathfrak{M}_n^{[p]}(\underline{a}; \underline{w}) = \begin{cases} \left(\dfrac{1}{W_n} \displaystyle\sum_{i=1}^{n} w_i a_i^p\right)^{1/p}, & \text{if } p \in \mathbb{R}^*, \\ \mathfrak{G}_n(\underline{a}; \underline{w}), & \text{if } p = 0, \\ \max \underline{a}, & \text{if } p = \infty, \\ \min \underline{a}, & \text{if } p = -\infty. \end{cases}$$

The definitions for $p = 0, \pm\infty$ are justified as limits; see [H, p. 176]. In particular, if $p = 1, -1$ the means are the *arithmetic, harmonic mean, of order n*, of the positive sequence $\underline{a}$ with positive weight $\underline{w}$, written $\mathfrak{A}_n(\underline{a}; \underline{w}), \mathfrak{H}_n(\underline{a}; \underline{w})$, respectively. The mean with $p = 0$ is called the *geometric mean, of order n*, of the positive sequence $\underline{a}$ with positive weight $\underline{w}$.

[6] A definition of this term is given in the entry **Mean Inequalities**.

In the case of equal weights, reference to the weights will be omitted from the notation, thus $\mathfrak{M}_n^{[p]}(\underline{a})$. The $n$-tuples in the above notations may be written explicitly, $\mathfrak{M}_n^{[p]}(a_1,\ldots,a_n;w_1,\ldots,w_n)$, or $\mathfrak{M}_n^{[p]}(a_i;w_i;1 \le i \le n)$. Most of these mean definitions are well-defined for more general sequences and will be used in that way when needed.

If the weights are allowed $k$ zero elements, $1 \le k < n$, this is equivalent to reducing $n$ by k.

Given two sequences $\underline{a}, \underline{w}$, the sequence $\{\mathfrak{A}_1(\underline{a};\underline{w}), \mathfrak{A}_2(\underline{a};\underline{w}),\ldots\}$ has an obvious meaning and will be written $\underline{\mathfrak{A}}(\underline{a};\underline{w})$, although the reference to $\underline{a}, \underline{w}$ will be omitted when convenient. Similar notations will be used for other means.

**3.2** The various means introduced above can be regarded, for given sequences $\underline{a}, \underline{w}$, as functions of $n \in \mathbb{N}^*$. They can then, by extension, be regarded as functions on non-empty finite subsets of $\mathbb{N}^*$, the *index sets*. For instance if $\mathcal{I}$ is such a subset of $\mathbb{N}^*$

$$\mathfrak{A}_{\mathcal{I}}(\underline{a};\underline{w}) = \frac{1}{W_{\mathcal{I}}}\sum_{i\in\mathcal{I}} w_i a_i, \text{ where } W_{\mathcal{I}} = \sum_{i\in\mathcal{I}} w_i.$$

This notation will be used in analogous situations without explanation.

A real valued function $\phi$ on the index sets is, respectively, *positive, increasing, superadditive*, if:

$$\phi(\mathcal{I}) > 0, \quad \mathcal{I} \subseteq \mathcal{J} \Longrightarrow \phi(\mathcal{I}) \le \phi(\mathcal{J}), \quad \mathcal{I}\cap\mathcal{J} = \emptyset \Longrightarrow \phi(\mathcal{I}\cup\mathcal{J}) \ge \phi(\mathcal{I}) + \phi(\mathcal{J}),$$

where $\mathcal{I}, \mathcal{J}$ are index sets; a definition of convexity of such functions can be found in **Hlwaka-Type Inequalities** COMMENTS (iii).

**3.3** If $\underline{a}$ is a positive $n$-tuple, $r \in \mathbb{N}^*$, then:

**3.3.1** the $r$-th *elementary symmetric functions of* $\underline{a}$ are

$$e_n^{[r]}(\underline{a}) = \frac{1}{r!}\sum_r ! \prod_{j=1}^r a_{i_j}, \qquad p_n^{[r]}(\underline{a}) = \frac{e_n^{[r]}(\underline{a})}{\binom{n}{r}}, \quad 1 \le r \le n;$$

$$e_n^{[0]}(u\underline{a}) = p_n^{[0]}(\underline{a}) = 1; \qquad e_n^{[r]}(\underline{a}) = p_n^{[r]}(\underline{a}) = 0, \qquad r > n,\ r < 0;$$

**3.3.2** the $r$-th *symmetric mean of* $\underline{a}$ is[7]

$$\mathfrak{P}_n^{[r]}(\underline{a}) = \left(p_n^{[r]}(\underline{a})\right)^{1/r}, 1 \le r \le n; \quad \mathfrak{P}_n^{[0]}(\underline{a}) = 1; \quad \mathfrak{P}_n^{[r]}(\underline{a}) = 0,\ r > n,\ r < 0.$$

It is easily seen that

$$\mathfrak{P}_n^{[1]}(\underline{a}) = \mathfrak{A}_n(\underline{a}), \quad \mathfrak{P}_n^{[n]}(\underline{a}) = \mathfrak{G}_n(\underline{a}), \quad \mathfrak{P}_n^{[n-1]}(\underline{a}) = \left(\frac{\mathfrak{G}_n^n(\underline{a})}{\mathfrak{H}_n(\underline{a})}\right)^{1/(n-1)}.$$

[7]In many references $p_n^{[r]}$ is called the symmetric mean.

For other relations see: **Hamy Mean Inequalities, Mixed Mean Inequalities, Muirhead Symmetric Function, and Mean Inequalities**.

The symmetric functions can be generated as follows:

$$\prod_{i=1}^{n}(1+a_i x) = \sum_{i=1}^{n} e_n^{[i]}(\underline{a})x^i = \sum_{i=1}^{n}\binom{n}{i} p_n^{[i]}(\underline{a})x^i; \tag{1}$$

*or*

$$e_n^{[r]}(\underline{a}) = \sum\Big(\prod_{i=1}^{n} a_i^{i_j}\Big),$$

where the sum is over the $\binom{n}{r}$ $n$-tuples $(i_1,\ldots,i_n)$ with $i_j = 0$ or $1$, $1 \le j \le n$, and $\sum_{j=1}^{n} = r$.

**3.3.3** The *complete symmetric functions* are defined for $r \neq 0$ by

$$c_n^{[r]}(\underline{a}) = \sum\Big(\prod_{i=1}^{n} a_i^{i_j}\Big) \quad \text{and} \quad q_n^{[r]}(\underline{a}) = \frac{c_n^{[r]}(\underline{a})}{\binom{n+r-1}{r}},$$

where the sum is over the $\binom{n+r-1}{r}$ $n$-tuples $(i_1,\ldots,i_n)$ with $\sum_{j=1}^{n} = r$.

In addition $c_n^{[0]}(\underline{a})$ is defined to be 1.

Equivalently

$$\prod_{i=1}^{n}(1-a_i x)^{-1} = \sum_{i=1}^{n} c_n^{[i]}(\underline{a})x^i = \sum_{i=1}^{n}\binom{n}{i} q_n^{[i]}(\underline{a})x^i.$$

**3.3.4** The $r$-th *complete symmetric mean of* $\underline{a}$ is

$$\mathfrak{Q}_n^{[k]}(\underline{a}) = \big(q_n^{[r]}(\underline{a})\big)^{1/r}.$$

A common extension of these symmetric functions and means can be seen under the entry **Whitely Symmetric Inequalities**.

**3.4** If $p \in \mathbb{R}$ and $a, b > 0$, $a \neq b$, the *(generalized) logarithmic mean of* $a, b$, is

$$\mathfrak{L}^{[p]}(a,b) = \begin{cases} \left(\dfrac{(b^{p+1}-a^{p+1})}{(p+1)(b-a)}\right)^{1/p}, & \text{if } p \neq, -1, 0, \\[2ex] \dfrac{(b-a)}{(\log b - \log a)}, & \text{if } p = -1, \\[2ex] \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/b-a}, & \text{if } p = 0. \end{cases}$$

The case $p = -1$ is called the *logarithmic mean of* $a, b$, and will be written $\mathfrak{L}(a,b)$, while the case $p = 0$ is the *identric mean of* $a, b$, written $\mathfrak{I}(a,b)$.

The definition is completed by putting $\mathfrak{L}^{[p]}(a,a) = a$.

There are several easy identities involving these means:

$$\mathfrak{L}^{[-2]}(a,b) = \mathfrak{G}_2(a,b), \quad \mathfrak{L}^{[-1/2]}(a,b) = \mathfrak{M}_2^{[1/2]}(a,b) = \frac{\mathfrak{A}_2(a,b) + \mathfrak{G}_2(a,b)}{2},$$

$$\mathfrak{L}^{[1]}(a,b) = \mathfrak{A}_2(a,b), \quad \mathfrak{L}^{[-3]}(a,b) = \sqrt[3]{\mathfrak{H}_2(a,b)\mathfrak{G}_2^2(a,b)}.$$

# 4 Continuous Variable Analogues

Many of the above concepts can be extended to functions

**4.1** If $(X,\mu)$, or $(X,\mathcal{M},\mu)$, is a measure space we will also write $\mu$ for the outer measure defined on all the subsets of $X$ that extends the measure on $\mathcal{M}$. If then $f : X \to \mathbb{R}$, $p > 0$,

$$||f||_{p,\mu,X} = \left( \int_X |f|^p \,\mathrm{d}\mu \right)^{1/p},$$

and we write $\mathcal{L}^p_\mu(X)$ for the class of real functions on $X$ with $||f||_{p,\mu,X} < \infty$;

As usual, if $p = 1$ it is omitted from the notations.

In addition if $p = \infty$:

$$||f||_{\infty,\mu,X} = \mu\text{-essential upper bound of } |f| = \inf_{N\in\mathcal{N}} \sup_{x\in X\setminus N} |f(x)|,$$

where $\mathcal{N} = \{N; \mu(N) = 0\}$.

Then the set of functions for which this quantity is finite is written $\mathcal{L}^\infty_\mu(X)$.

In all cases if the measure or the space is clear it may be omitted from the notation; thus $||f||_{p,X}$, or just $||f||_p$.

We will write $\lambda_n$ for Lebesgue measure in $\mathbb{R}^n$ and if $E \subseteq \mathbb{R}^n$ write $|E|_n$ for $\lambda_n(E)$; further we will write $\int_E f(\underline{x})\,\mathrm{d}\lambda_n$ as $\int_E f(\underline{x})\,\mathrm{d}\underline{x}$.

In the case $n = 1$, or if its value is clear, the suffix is omitted.

When using Lebesgue measure it is omitted from the notation, thus $\mathcal{L}^p(X) = \mathcal{L}^p_{\lambda_n}(X)$.

REFERENCES *Hewitt & Stromberg* [HS, pp. 188–189, 347], *Rudin* [R76, p. 250].

**4.2** If $f \in \mathcal{L}^r(\mathbb{R}^p), g \in \mathcal{L}^s(\mathbb{R}^p)$, where $1 \le r, s \le \infty$ and then the *convolution function*

$$f \star g(x) = \int_{\mathbb{R}^p} f(x-t)g(t)\,\mathrm{d}t$$

is defined almost everywhere and $f \star g \in \mathcal{L}^t(\mathbb{R}^p)$ where $1/r + 1/s = 1 + 1/t$.

REFERENCE *Zygmund* [Z, vol. I pp. 36,38, vol. II p. 252].

**4.3** Various of the means defined above have integral analogues. If $f \geq 0, f < \infty$ almost everywhere, $0 < \mu([a,b]) < \infty$, the various integrals exist, and in case of negative $p$ are not zero;[8]

$$\mathfrak{M}^{[p]}_{[a,b]}(f;\mu) = \begin{cases} \left(\dfrac{\int_a^b f^p \,\mathrm{d}\mu}{\mu([a,b])}\right)^{1/p}, & \text{if } p \in \mathbb{R}^*; \\ \exp\left(\dfrac{\int_a^b \log\circ f \,\mathrm{d}\mu}{\mu([a,b])}\right), & \text{if } p = 0; \\ \mu\text{-essential upper bound of } f, & \text{if } p = \infty; \\ \mu\text{-essential lower bound of } f, & \text{if } p = -\infty. \end{cases}$$

As usual we write $\mathfrak{H}_{[a,b]}(f;\mu), \mathfrak{G}_{[a,b]}(f;\mu), \mathfrak{A}_{[a,b]}(f;\mu)$ if $p = -1, 0, 1$, respectively. In the case that $\mu = \lambda$ reference to the measure in the notation will be omitted. Further if $\mu$ is absolutely continuous, when $f\mathrm{d}\mu$ can be written $f(x)w(x)\mathrm{d}x$ say, the notation will be $\mathfrak{M}^{[p]}_{[a,b]}(f;w)$.

The interval $[a,b]$ can be replaced by a more general $\mu$- measurable set, of positive $\mu$-measure.

It is useful to note that if $i(x) = x, a \leq x \leq b$ then

$$\mathfrak{L}^{[p]}(a,b) = \mathfrak{M}^{[p]}_{[a,b]}(i),\ p \in \mathbb{R}.$$

# 5 Rearrangements and Orders

Other important sources of inequalities, and of proofs of inequalities, are the concepts of rearrangements and orders.

**5.1** $X$, or $(X, \preceq)$, is an *ordered, or partially ordered, set* if $\preceq$ is an *order on* $X$; that is for all $x, y, z \in X$:

(i) $x \preceq x$,
(ii) $x \preceq y$ and $y \preceq z$ implies that $x \preceq z$,
(iii) $x \preceq y$ and $y \preceq x$ implies that $x = y$;
if (iii) does not hold then $\preceq$ is called a *pre-order on* $X$.

If in addition $x \vee y = \sup\{x,y\}$ and $x \wedge y = \inf\{x,y\}$ always exist then $X$ is called a *lattice*.

If $(x \vee y) \wedge z = (x \wedge z) \vee (x \wedge z)$ then $X$ is a *distributive lattice*.

The previous notation is extended to:
$A \vee B = \{z;\ z = x \vee y, x \in A, y \in B\}$,
$A \wedge B = \{z;\ z = x \wedge y, x \in A, y \in B\}$, where of course $A, B \subseteq X$.
See: various entries in [EM].

[8] The case of $p = 0$ needs more care. See: [H, pp. 136–139].

The two real $n$-tuples $\underline{a}, \underline{b}$ are said to be *similarly ordered* when

$$(a_i - a_j)(b_i - b_j) \geq 0 \quad \text{for all} \quad i, j,\, 1 \leq i, j \leq n. \tag{1}$$

If ($\sim$1) holds then we say the $n$-tuples are *oppositely ordered.*

Equivalently $\underline{a}, \underline{b}$ are similarly ordered when under a simultaneous permutation both become increasing.

Given a real $n$-tuple $\underline{a}$ then $\underline{a}^*, \underline{a}_*$ will denote the $n$-tuple derived from $\underline{a}$ by rearranging the elements in decreasing, increasing order, respectively. We then will write

$$\underline{a}^* = (a_{[1]}, \ldots, a_{[n]}), \qquad \underline{a}_* = (a_{(1)}, \ldots, a_{(n)}).$$

$\underline{b} \prec \underline{a}$, read as $\underline{b}$ *precedes* $\underline{a}$, means:

$$\sum_{1=1}^{k} b_{[i]} \leq \sum_{1=1}^{k} a_{[i]},\ 1 \leq k < n, \quad \text{and} \quad \sum_{1=1}^{n} b_i = \sum_{1=1}^{n} a_i. \tag{2}$$

This is a pre-order on the set of $n$-tuples, and is an order on the set of decreasing, or on the set of increasing, $n$-tuples.

$\underline{b} \prec^w \underline{a}$, read as $\underline{b}$ *weakly precedes* $\underline{a}$, if (2) holds with the last equality weakened to $\leq$.

Suppose that $\underline{a} = (a_{-n}, \ldots, a_n)$ then two rearrangements of $\underline{a}$, $\underline{a}^+ = (a^+_{-n}, \ldots, a^+_n)$, ${}^+\underline{a} = ({}^+a_{-n}, \ldots, {}^+a_n)$ are defined by

$$a^+_0 \geq a^+_1 \geq a^+_{-1} \geq a^+_2 \geq a^+_{-2} \geq \cdots; \quad {}^+a_0 \geq {}^+a_{-1} \geq {}^+a_1 \geq {}^+a_{-2} \geq {}^+a_2 \geq \cdots.$$

If every value of the elements of $\underline{a}$, except the largest, occurs an even number of times, and the largest value occurs an odd number of times then $\underline{a}^+ = {}^+\underline{a}$. We then say that $\underline{a}$ is *symmetrical*, and write $\underline{a}^{(*)}$ for either of the rearrangements just defined; that is $\underline{a}^{(*)} = (a^{(*)}_{-n}, \ldots, a^{(*)}_n)$ where

$$a^{(*)}_0 \geq a_1{}^{(*)} = a^{(*)}_{-1} \geq a^{(*)}_2 = a^{(*)}_{-2} \geq \cdots$$

So $\underline{a}^{(*)}$ is a symmetrically decreasing rearrangement of $\underline{a}$; while the other rearrangements are as symmetrical as possible.

**5.2** Two functions $f, g : I \to \mathbb{R}$ are *similarly ordered* if

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq 0, \qquad \text{for all} \quad x, y \in I. \tag{3}$$

If ($\sim$3) holds we say the functions are *oppositely ordered.*

This can easily be extended to functions of several variables.

Reference *Pólya & Szegő* [PS51, p. 151].

$f \prec g$ on $[a, b]$, read as $f$ *precedes* $g$ *on* $[a, b]$, if $f, g$ are two decreasing functions defined on $[a, b]$ and if

$$\int_a^x f \leq \int_a^x g, \quad a \leq x < b, \text{ and } \quad \int_a^b f = \int_a^b g.$$

Given a non-negative measurable function $f$ on $[0,a]$ that is finite almost everywhere then $m_f(y) = |E_y| = \left|\{x;\ f(x) > y\}\right|$, defines the *distribution function $m_f$ of $f$.*

Two functions, $f, g$ having $m_f = m_g$ are said to be *equidistributed or equimeasurable.*

Every function $f$ has a decreasing right continuous function $f^*$, and an increasing right continuous function $f_*$, that are equidistributed with $f$, called the *increasing, decreasing, rearrangement of $f$*. If $f : \mathbb{R}^n \to \mathbb{R}$ then the function $f^{(*)}$ defined by

$$f^{(*)}(\underline{x}) = \sup\{y;\ |E_y| > v_n|\underline{x}|^n\},$$

is called the *spherical, or symmetrical, decreasing rearrangement of $f$*; $v_n$ is the volume of the unit ball defined in **Volume of the Unit Ball**.

REFERENCES [HLP], *Kawohl* [Ka], *Lieb & Loss* [LL].

# 6 Some Functions and Classes of Functions

**6.1** Various types of *convex functions* are defined as part of the entries for their inequalities: **Convex Function Inequalities, Convex Matrix Function Inequalities, n-Convex Function Inequalities, Log-convex Function Inequalities, Q-class Function Inequalities, Quasi-convex Function Inequalities, Schur Convex Function Inequalities, Strongly Convex Function Inequalities, Subadditive Function Inequalities, and Subharmonic Function Inequalities**.

$V(f; a, b)$ is the *variation of $f$ on $[a,b]$*; functions of *bounded variation* are defined in *Rudin* [R76, p. 117].

Function properties can hold on a set or at a point, as for instance continuity, differentiability; a property is said to hold *locally* if at each point of its domain there is a neighborhood in which the property holds. If the domain is compact then it holds everywhere but not in general; thus $f(x) = x : \mathbb{R} \to \mathbb{R}$ is locally bounded but not bounded.

**6.2** $\mathcal{C}(E)$ is the set of all functions continuous on the set $E$.

$\mathcal{C}_0(E)$ the set of all functions continuous on the set $E$ that is unbounded above such that $\lim_{x\to\infty, x\in E} f(x) = 0$; these functions are said to be *"zero at infinity."*

If $f$ is continuous and $X$ is compact, or if $f$ has compact support, or if $f \in \mathcal{C}_0(E)$, then

$$||f||_{\infty,X} = ||f||_\infty = \max_{x\in X} |f(x)|.$$

$\mathcal{C}^n(E), n \in \mathbb{N}^*$, is the set of functions $f$ having continuous derivatives of order up to and including $n$; we also write $\mathcal{C}^0(E) = \mathcal{C}(E)$.

$\mathcal{C}^\infty(E)$ is the class of functions having derivatives of all orders on a set $E$.

Further classes of functions are defined under the entry **Sobolev's Inequalities**.

**6.3** $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$, and differentiable is said to be *analytic in* $\Omega$.[9] An injective analytic function is said to be *univalent*, otherwise it is *multivalent*; specifically *$m$-valent* if it assumes each value in its range at most $m$ times.

If a univalent function has as its domain $\mathbb{C}$ it is said to be an *entire function*.

If $f$ is analytic in $D$ and if $0 < r < 1, 0 \le p \le \infty$, then

$$M_p(f;r) = \begin{cases} \exp\left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, \mathrm{d}\theta\right), & \text{if } p = 0, \\ \left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, \mathrm{d}\theta\right)^{1/p}, & \text{if } 0 < p < \infty, \\ \sup_{-\pi \le \theta \le \pi} |f(re^{i\theta})|, & \text{if } p = \infty. \end{cases}$$

$\mathcal{H}^p(D)$ is H the class of functions with $\sup_{0 \le r < 1} M_p(h;r) < \infty$; see **Analytic Function Inequalities** COMMENTS (i).

This notation can easily be extended to other classes of functions, and to functions with domain any disk centered at the origin, or to an annulus such as $\{z;\, a < |z| < b\}$.

Related functions are defined in **Conjugate Harmonic Function Inequalities, Harmonic Function Inequalities, Quasi-conformal Function Inequalities, and Subharmonic Function Inequalities**.

**6.4** If the sequence of functions $\phi_n : [a,b] \to \mathbb{C}, n \in \mathbb{N}$ has the properties

$$\int_a^b \phi_n \overline{\phi}_m = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

for $n, m \in \mathbb{N}$, then $\phi_n, n \in \mathbb{N}$ is called an *orthonormal sequence of complex valued functions defined on an interval* $[a,b]$. Given such a sequence of functions the sequence $\underline{c}$ defined by

$$c_n = \int_a^b f \overline{\phi}_n, \quad n \in \mathbb{N},$$

is the *sequence of Fourier coefficients of $f$ with respect to $\phi_n, n \in \mathbb{N}$*. If, in addition, $\sup_{a \le x \le b} |\phi_n(x)| \le M, n \in \mathbb{N}$, the orthonormal sequence is said to be *uniformly bounded.*

Other index sets, such as $\mathbb{Z}$ are possible, and real orthonormal sequences can be considered. See: *Zygmund* [Z, vol. I, pp. 5–7].

**6.5** $[\cdot]$ denotes the *greatest integer function*,

$$[x] = \max\{n;\, n \le x < n+1, n \in \mathbb{Z}\}.$$

[9]The terms *holomorphic, regular* are also used in the literature.

**6.6** If $x > 0$,

$$\log^+ x = \max\{0, \log x\} = \begin{cases} \log x, & \text{if } x \geq 1, \\ 0, & \text{if } 0 < x \leq 1. \end{cases}$$

**6.7** The *indicator function of a set A* is

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

**6.8** The *factorial or gamma function* is $x!$, or $\Gamma(x) = (x-1)!$; $0! = 1! = 1$, if $n = 2, 3, \ldots$ $n! = n(n-1). \ldots .2.1$; and $(1/2)! = \sqrt{\pi}/2$.

The domain of this function is $\mathbb{C} \setminus \{-1, -2, \ldots\}$, and if both $z$ and $z-1$ are in this domain $z! = z(z-1)!$.

Related functions are defined in **Digamma Function Inequalities and Vietoris's Inequality**.

**6.9** The *(Riemann) Zeta Function*, $\zeta(s)$, is a function of the complex variable $s = \sigma + it$[10] defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.$$

Its definition is extended by analytic continuation to all $s \in \mathbb{C} \setminus \{1\}$. It is analytic at all ponts of $\mathbb{C}$ except for a pole at $s = 1$.

**6.11** Other functions are defined in the following entries: **Bernšteĭn Polynomial Inequalities, Beta Function Inequalities, Bieberbach's Conjecture, Čebišev Polynomial Inequalities, Copula Inequalities, Internal Function Inequalities, Laguerre Function Inequalities, Lipschitz Function Inequalities, N-function Inequalities, Segre's Inequalities, Semi-continuous Function Inequalities, Starshaped Function Inequalities, Totally Positive Function Inequalities, Trigonometric Polynomial Inequalities, and Zeta Function Inequalities**.

# 7 Matrices

$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ denotes an $m \times n$ *matrix*; the $a_{ij}$ are the *entries* and if they are real, complex then $A$ is said to be *real, complex*. If $m = n$ then $A$ is called a *square matrix.*

$A^T = (a_{ji})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is the *transpose of A*.

$A^* = (\overline{a}_{ji})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is the *conjugate transpose of A*.

[10]This is the traditional notation for this subject.

If $A = A^T$ then $A$ is *symmetric*; if $A = A^*$ then $A$ is *Hermitian.*

$\det A = \det(A)$, $\mathrm{tr}A = \mathrm{tr}(A), \mathrm{rank}A = \mathrm{rank}(A), \mathrm{per}A = \mathrm{per}(A)$ denote the *determinant, trace, rank, and permanent of* $A$, respectively.

$I_n$ denotes $n \times n$ *unit matrix*; just written $I$ if the value of $n$ is obvious.

If $A$ is an $n \times n$ matrix, $n \geq 2$, and if $1 \leq i_1 < i_2 < \cdots < i_m \leq n, 1 \leq m \leq n$, then $A_{i_1,\ldots,i_m}$ denotes the principal submatrix of $A$ made up of the intersection of the rows and columns $i_1, \ldots, i_m$ of $A$.

$A_i'$ denotes the sub-matrix of $A$ obtained by deleting the $i$-th row and column.

If $A = (a_{ij})_{\substack{1\leq i\leq m\\1\leq j\leq n}}, B = (b_{ij})_{\substack{1\leq i\leq m\\1\leq j\leq n}}$ then their *Hadamard product* is the matrix $A * B = *AB = (a_{ij}b_{ij})_{\substack{1\leq i\leq m\\1\leq j\leq n}}$.

The eigenvalues of a square matrix $A$ will be written $\lambda_s(A)$, and when real, by the notation in **5**, $\lambda_{[s]}(A), \lambda_{(s)}(A)$ will denote them in descending, ascending, order; in particular $\lambda_{[1]}(A)$ is the largest, or dominant, eigen-value. $|\lambda_{[s]}(A)|, |\lambda_{(s)}(A)|$ will denote their absolute values in descending, ascending, order.

A square matrix with non-negative entries with all row and column sums equal to 1 is called *doubly stochastic.*

Various concepts of order and norm will be defined in the relevant entries.

Full details can be found in *Horn & Johnson* [HJ] and *Marcus & Minc* [MM].

# 8 Probability and Statistics

**8.1** $(\Omega, \mathcal{A}, P)$ with $\Omega$ is a non-empty set, $\mathcal{A}$ a $\sigma$-field of subsets of $\Omega$, called *events*, and $P$ a probability on $\mathcal{A}$ is a *probability space* and an $\mathcal{A}$-measurable function $X : \Omega \to \mathbb{R}$ is called a *random variable.*

$EX = EX(\Omega) = \int_\Omega X \,\mathrm{d}P$ is the *(mathematical) expectation of* $X$; this is also called the *mean* and written $\mu$.

$\sigma^2 X = \sigma^2 X(\Omega) = E(X - EX)^2$ is the *variance of* $X$;

$\sigma X$ is the *standard deviation.*

$CV(X) = \sigma X / EX$ is the *coefficient of variation.*

The probability of the *event* $\{\omega;\ X(\omega) > r\}$ will be written $P\big(X > r\big)$.

$mX$ defined by $P\big(X \geq m\big) = P\big(X \leq m\big)$ is the *median of* $X$.

$EX^r$, $E|X|^r$ are the *k-th moment, absolute moment, of* $X$, respectively.

This terminology is used more generally and if $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $k \in \mathbb{R}^+$ then

$$\mu_k(f) = \mu_k = \int_{\mathbb{R}^+} x^k f(x) \,\mathrm{d}x$$

is called the *k-th moment of* $f$.

If the $\sigma$-field $\mathcal{A}$ is finite then $\Omega$ is called a *finite probability space.*

If $\mathcal{B} \subseteq \mathcal{A}$ is another $\sigma$-field then $E(X|\mathcal{B})$ denotes the *conditional expectation* of $X$ with respect to B.

**8.2** If $T = \mathbb{N}$, or $[0, \infty[$ then *stochastic process* $\mathcal{X} = (X_t, \mathcal{F}_t, t \in T)$ defined on the probability space $(\Omega, \mathcal{F}, P)$ is a *martingale* if $\mathcal{F}_t$ is a $\sigma$-field, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, s \le t,\ s, t \in T,\ EX_t < \infty, t \in T,\ X_t$ is $\mathcal{F}_t$-measurable, $t \in T$, and if almost surely, that is with probability one,

$$E(X_t|\mathcal{F}_s) = X_s, s \le t,\ s, t \in T. \tag{1}$$

If in (1) $=$ is replaced by $\ge, (\le)$, then we have a *sub-martingale, (super-martingale).*

**8.3** There are various functions involved with the *Gaussian, or normal distribution.*

$$E(x) = \frac{2}{\sqrt{\pi}} e^{-x^2},\ \mathrm{erf}(x) = \int_0^x E,\ \ \mathrm{erfc}(x) = \int_x^\infty E,\ \ \mathrm{mr}(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2}\,\mathrm{d}t,$$

where $x \in \mathbb{R}$.

The functions erf, erfc are the *error function* and the *complementary error function* respectively; and the last function is usually called the *Mills's Ratio.*[11]

There are some simple relations between these functions:

$$\mathrm{erf}(0) = 0;\ \mathrm{erf}(\infty) = 1;\ \mathrm{erf} + \mathrm{erfc} = 1;\ \mathrm{erf}(-x) = -\,\mathrm{erf}(x);\ \mathrm{mr} = \frac{\sqrt{2}\,\mathrm{erfc}}{E}.$$

The *Gaussian measure* is the probability measure on $\mathbb{R}$ defined by:

$$\gamma(A) = \frac{1}{2} \int_A E\,\mathrm{d}\lambda,\ A \subseteq \mathbb{R}.$$

Note that $\mathrm{erf}(x) = \gamma\big([-x, x]\big)$.

This measure has an extension to $\mathbb{R}^n, n > 1$, that is denoted by $\gamma_n$.

In addition there is the *normal distribution function.*

$$\Phi(x) = \frac{1}{2}\Big(1 + \mathrm{erf}(x/\sqrt{2})\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\mathrm{d}t.$$

Reference *Loève* [L].

# 9 Symbols for Certain Inequalities

Certain inequalities will be referred to by a symbol to make referencing easier. They are listed here.

1. Bernoulli's Inequality .............................................. (B)
2. Cauchy's Inequality ................................................ (C)

[11] Although the study of this function goes back to Laplace; see *Gasul & Utzet* [127].

3. Čebyšev's Inequality . . . . . . (Č)
4. Geometric-Arithmetic Mean Inequality . . . . . . (GA)
5. Harmonic-Arithmetic Mean Inequality . . . . . . (HA)
6. Harmonic-Geometric Mean Inequality . . . . . . (HG)
7. Hölder's Inequality . . . . . . (H)
8. Jensen's Inequality . . . . . . (J)
9. Minkowski's Inequality . . . . . . (M)
10. Power Mean Inequality . . . . . . (r;s)
11. Symmetric Mean Inequality . . . . . . S(r;s)
12. Triangle Inequality . . . . . . (T)

# 10 Bibliographic References

As is seen below, the Bibliography is in four sections. References are given in the order of these sections. Each section of references is ended with a semi-colon and in each set of references the individual items are separated by commas. See for instance the references for **Abel's Inequalities**.

# 11 Transliteration of Cyrillic

There are several diacritical letters in various Slavic languages that use the Latin alphabet; for instance č, š, ž pronounced ch, sh, zh. These are used in the transliteration of Cyrillic following the old Mathematical Reviews (and Zentralblatt) practice.[12] The advantage is that they are less English or Western European oriented as is the more recent usage of the Mathematical Reviews.[13]

ж = ž, Ж = Ž; х=h, Х=H; ц =c, Ц =C; ч = č, Ч = Č; ш=š, Ш =Š; щ =šč. Щ =Šč; ю =yu, Ю= Yu; я=ya, Я= Ya.

# 12 Other Alphabets

## 12.1 Gothic

$\mathfrak{a}, \mathfrak{A}, \mathfrak{b}, \mathfrak{B}, \mathfrak{c}, \mathfrak{C}, \mathfrak{d}, \mathfrak{D}, \mathfrak{e}, \mathfrak{E}, \mathfrak{f}, \mathfrak{F}, \mathfrak{g}, \mathfrak{G}, \mathfrak{h}, \mathfrak{H}, \mathfrak{i}, \mathfrak{I}, \mathfrak{j}, \mathfrak{J}, \mathfrak{k}, \mathfrak{K}, \mathfrak{l}, \mathfrak{L}, \mathfrak{m}, \mathfrak{M}, \mathfrak{n}, \mathfrak{N}, \mathfrak{o}, \mathfrak{O}, \mathfrak{p}, \mathfrak{P}, \mathfrak{q}, \mathfrak{Q}, \mathfrak{r}, \mathfrak{R}, \mathfrak{s}, \mathfrak{S}, \mathfrak{t}, \mathfrak{T}, \mathfrak{u}, \mathfrak{U}, \mathfrak{v}, \mathfrak{V}, \mathfrak{w}, \mathfrak{W}, \mathfrak{x}, \mathfrak{X}, \mathfrak{y}, \mathfrak{Y}, \mathfrak{z}, \mathfrak{Z}$.

## 12.2 Greek

$\alpha, \beta, \gamma, \Gamma, \delta, \Delta, \epsilon, \varepsilon, \zeta, \eta, \theta, \Theta, \iota, \kappa, \lambda, \Lambda, \mu, \nu, \xi, \Xi, \omega, \pi, \Pi, \varpi, \rho, \varrho, \sigma, \Sigma, \tau, \upsilon, \Upsilon, \phi, \Phi, \varphi, \chi, \psi, \Psi, \omega, \Omega$.

[12] There are some exceptions caused by the vagaries of LaTeX.

[13] Consider the case of Меншьов who first published in the West in French and was transliterated as Menchoff; this leads to a mispronunciation in English. No such problem arises writing Menšov.

**12.3 Calligraphic** $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$.

**12.4** Finally apologies may be necessary to the many authors with Chinese or Japanese names. I have tried to use the names correctly and can only hope that I have succeeded.

## 13 Some Useful URLs

The following were correct but cannot be guaranteed. Further, some may need a subscription; in that connection it is useful to find out if you are attached to a library that supports these contacts as many universities do. In any case it is worth exploring a little as some items are available both as open access and not open access:

*Acta Applicandæ Mathematicæ*
http://link.springer.com/journal/10440

*American Mathematical Monthly*
http://www.maa.org/publications/periodicals/american-mathematical-monthly *CRC Concise Encyclopedia of Mathematics*
https://archive.org/details/CrcEncyclopediaOfMathematics

*Computers & Mathematics with Applications*
http://www.sciencedirect.com/science/journal/08981221/32/2

*Crux Mathematicorum*
http://cms.math.ca/crux/

*Discrete Mathematics*
http://www.sciencedirect.com/science/journal/0012365X/open-access

*Elemente der Mathematik*
http://www.ems-ph.org/home.php

*Encyclopædia of Mathematics*
http://www.encyclopediaofmath.org

*Project Euclid - mathematics and statistics resources online*
http://projecteuclid.org

*General Inequalities 1: Proceedings of First International Conference on General Inequalities, Oberwolfach, 1976; Birkhaüser Verlag, Basel, 1978*[14]
http://link.springer.com/book/10.1007/978-3-0348-5563-1/page/1

*General Inequalities 2: Proceedings of Second International Conference on General Inequalities, Oberwolfach, 1978; Birkhuser Verlag, 1980*
http://link.springer.com/book/10.1007%2F978-3-0348-6324-7

*General Inequalities 3: Proceedings of Third International Conference on General Inequalities, Oberwolfach. 1981; Basel-Boston-Stuttgart, Birkhuser Verlag, 1983*
http://link.springer.com/book/10.1007%2F978-3-0348-6290-5

[14] All these proceedings are now published by Springer.

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*General Inequalities 4: Proceedings of Fourth International Conference on General Inequalities, Oberwolfach. 1983; Basel-Boston-Stuttgart, Birkhuser Verlag, 1984*

http://link.springer.com/book/10.1007%2F978-3-0348-6259-2

*General Inequalities 5: Proceedings of Fifth International Conference on General Inequalities, Oberwolfach. 1986; Basel-Boston-Stuttgart, Birkhuser Verlag, 1987*

http://www.springer.com/new+%26+forthcoming+titles+(default)/book/978-3-7643-1799-7

*General Inequalities 6: Proceedings of Sixth International Conference on General Inequalities, Oberwolfach. 1990; Basel-Boston-Stuttgart, Birkhuser Verlag, 1992*

http://www.springer.com/new+%26+forthcoming+titles+(default)/book/978-3-7643-2737-8

*General Inequalities 7: Proceedings of Seventh International Conference on General Inequalities, Oberwolfach. 1995; Basel-Boston-Stuttgart, Birkhuser Verlag, 1997*

http://link.springer.com/book/10.1007%2F978-3-0348-8942-1

*Glasgow Mathematical Journal*

http://journals.cambridge.org/action/displayJournal?jid=GMJ

*International Journal of Mathematics and Mathematical Sciences*

http://www.hindawi.com/journals/ijmms/contents/

*James Cook Mathematical Notes*

http://www.maths.ed.ac.uk/cook/

*Journal of Inequalities and Applications*

http://www.journalofinequalitiesandapplications.com/

*Journal of Inequalities in Pure and Applied Mathematics*

http://www.emis.de/journals/JIPAM/index-4.html

*Journal of Mathematical Analysis and Applications*

http://www.sciencedirect.com/science/journal/0022247X

*Journal of Mathematical Inequalities*

http://ele-math.com/

*Linear Algebra and its Applications*

http://www.sciencedirect.com/science/journal/00243795

*Lists of mathematicans*

http://en.wikipedia.org/wiki/Special:Search/Lists

http://www-groups.dcs.st-andrews.ac.uk/%7Ehistory/index.html

https://en.wikipedia.org/wiki/Category:Russian_mathematicians

*Mathematica Bohemica*

http://mb.math.cas.cz/MBtoc.html

*Mathematical Intelligencer*

http://www.springer.com/mathematics/journal/283

*Mathematical Inequalities & Applications*

http://ele-math.com/

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*Mathematical Reviews*
http://www.ams.org/mathscinet/
*Mathematics Magazine*
http://www.maa.org/publications/periodicals/mathematics-magazine
*Mathematika Balkanica*
http://www.mathbalkanica.info
*The MacTutor History of Mathematics Archive*
http://www-groups.dcs.st-andrews.ac.uk/%7Ehistory/index.html
http://turnbull.mcs.st-and.ac.uk/ history/Mathematicians/
*Nonlinear Analysis: Theory, Methods, and Applications*
http://www.sciencedirect.com/science/journal/0362546X
*Proceedings of the American Mathematical Society*
http://www.jstor.org/journals/00029939.html
*Referativniĭ Žurnal Matematika* [Реферативный Журнал Математика.]
http://www.viniti.ru/pro_ref_el.html
*RGMIA Monographs*
http://rgmia.org/monographs.php
*Titchmarsh: The Theory of Functions*
https://archive.org/details/TheTheoryOfFunctions
*Univerzitet u Beogradu Publikacije Elektrotehničkog Fakulteta. Serija Matematika*
http://pefmath2.etf.rs/pages/main.php?referrer=2
*Wolfram MathWorld*
http://mathworld.wolfram.com/
*Zentralblatt für Mathematik und ihre Grenzgebiete (Zentralblatt MATH)*
https://www.zbmath.org/

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# 1 Abel–Arithmetic

**Abel's Inequalities** *(a) If $\underline{w}, \underline{a}$ are $n$-tuples and $\underline{a}$ is monotonic then*

$$\left|\sum_{i=1}^{n} w_i a_i \le\right| \le \max_{1\le i\le n}\{|W_i|\}\big(|a_1| + 2|a_n|\big).$$

*(b) Let $\underline{a}$ be an $n$-tuple with $A_i > 0, 1 \le i \le n$, then*

$$\frac{A_n}{A_1} \le \exp\left(\sum_{i=2}^{n} \frac{a_i}{A_{i-1}}\right).$$

*(c) If $p, q$ are conjugate indices with $p < 0$, and $\underline{a}$ is a positive sequence, then*

$$\sum_{i=2}^{\infty} \frac{a_i}{A_i^{1/q}} < -pa_1^{1/p} \tag{1}$$

*(d) If $\underline{z}$ is a complex $n$-tuple and $\underline{a}$ a decreasing $n$-tuple of non-negative numbers then*

$$\left|\sum_{j=1}^{n} a_j z_j\right| \le a_1 \max_{1\le k\le n} |Z_k|;$$

*if $\underline{b}$ is an increasing sequence of non-negative numbers,*

$$\left|\sum_{j=1}^{n} b_j z_j\right| \le 2b_n \max_{1\le k\le n} |Z_k|.$$

COMMENTS (i) (a), known as *Abel's lemma*, is a simple consequence of *Abel's transformation or summation formula*, an "integration by parts formula" for sequences:

$$\sum_{i=1}^{n} w_i a_i = W_n a_n + \sum_{i=1}^{n-1} W_i \Delta a_i. \tag{2}$$



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(ii) If $\underline{a}$ is non-negative and decreasing then we have the following result, simpler than (a), known as *Abel's inequality*:

$$a_1 \min_{1\le i\le n} W_i \le \sum_{i=1}^n w_i a_i \le a_1 \max_{1\le i\le n} W_i. \tag{3}$$

Extensions *(a)* [Bromwich] *If $\underline{w}, \underline{a}$ are $n$-tuples with $\underline{a}$ decreasing, and if*

$$\overline{W}_k = \max\{W_i,\, 1\le i\le k-1\}, \qquad \overline{W}'_k = \max\{W_i,\, k\le i\le n\},$$
$$r\underline{W}_k = \min\{W_i,\, 1\le i\le k-1\}, \qquad \underline{W}'_k = \min\{W_i,\, k\le i\le n\},$$

*for $1\le k\le n$ then*

$$\underline{W}_k(a_1-a_k)+\underline{W}'_k a_k \le \sum_{i=1}^n w_i a_i \le \overline{W}_k(a_1-a_k)+\overline{W}'_k a_k. \tag{4}$$

*(b)* [Redheffer] *(i) Under the hypotheses of (b) above*

$$\frac{A_n}{A_1} \le \left(1+\frac{1}{n-1}\sum_{i=2}^n \frac{a_i}{A_{i-1}}\right)^{n-1},$$

*with equality if and only if for some $\lambda>0$ $A_i=\lambda^{i-1}A_1,\ 1\le i\le n$.*
*(ii) Under the hypotheses of (b) above and $\underline{w}$ a positive sequence with $w_1=1$*

$$\sum_{i=2}^\infty w_i A_i^{1/p} < q\sum_{i=1}^\infty w_i^{1/q} a_i^{1/p}.$$

Comments (iii) The last inequality reduces to (1) on putting $w_i = a_i/A_i$ for all $i$.

(iv) See also: **Integral Inequalities** Discrete Analogue.

Related Results *(a)* [Kalajdžić] *If $\underline{a}$, $\underline{w}$ are $n$-tuples with $W_n=0$ and if* sgn $W_k = (-1)^{k-1}$sgn $w_1$, *$1\le k\le n$, then*

$$\left|\sum_{i=1}^n w_i a_i\right| \le \frac12 \max_{1\le k\le n-1} |\Delta a_{k-1}| \sum_{i=1}^{n-1} |w_i|.$$

*(b)* [Bennett] *If $\underline{a}, \underline{b}, \underline{c}$ are non-negative sequences, with $\underline{c}$ decreasing then*

$$A_n \le B_n,\ n=1,2,\ldots \quad\Longrightarrow\quad \sum_{i=1}^n a_i c_i \le \sum_{i=1}^n b_i c_i,\ n=1,2,\ldots.$$

Comments (v) This last result follows by two applications of (2), and is related to **Steffensen's Inequalities** (b).

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(vi) Integral analogues of (3) and (4) are given by Bromwich; see **Integral Mean Value Theorems** (b), COMMENTS (ii).

(vii) For an application of this result see **Steffensen's Inequalities** COMMENTS (iii).

REFERENCES [AI, pp. 32–33], [EM, vol. 1, pp. 5, 8],[MPF, pp. 333–337]; *Bennett* [Be, p. 9], *Bromwich* [Br, pp. 57–58, 473–475], *Steele* [S, pp. 208–210, 221, 279], *Zygmund* [Z, vol. I, pp. 3–4]; *Kalajdžić* [150], *Redheffer* [279, pp. 690, 696].

**Abi-Khuzam's Inequality** *If $A, B, C$ are the angles of a triangle then*

$$\sin A \sin B \sin C \le \left(\frac{2\sqrt{3}}{2\pi}\right)^3 ABC,$$

*with equality if and only if the triangle is equilateral.*

REFERENCE *Klamkin* [157].

**Absolutely Monotonic Function Inequalities** *(a) If $f, f \in \mathcal{C}^\infty\big([a,b]\big)$, is absolutely monotonic then*

$$f^{(k)} \ge 0, \quad a < x < b,\ k \in \mathbb{N}.$$

*(b) If $f$ is absolutely monotonic on $]-\infty, 0\,]$ then*

$$\sqrt{f^{(k)} f^{(k+2)}} \ge f^{(k+1)}, \quad k \in \mathbb{N}.$$

*(c) If $f$ is absolutely monotonic on $]-\infty, 0\,[$ then*

$$\det\left(\big(f^{(i+j-2)}\big)_{\substack{1\le i\le n\\ 1\le j\le n}}\right) \ge 0 \quad \textit{and} \quad \det\left(\big(f^{(i+j-1)}\big)_{\substack{1\le i\le n\\ 1\le j\le n}}\right) \ge 0, \qquad n \ge 1.$$

*(d)* [FINK] *If $\underline{a}, \underline{b}$ are non-negative $n$-tuples of integers with $\underline{a} \prec \underline{b}$ and if $f$ is absolutely monotonic on $[0, \infty\,[$,*

$$\prod_{i=1}^n f^{(a_i)} \le \prod_{i=1}^n f^{(b_i)},$$

*with equality if for some $a > 0,\ f(x) = e^{ax}$.*

COMMENTS (i) (a) is just the definition of *absolutely monotonic.*

(ii) (b) is a simple consequence of the integral analogue of (Č), or (GA), and a well known integral representation for the class of absolutely monotonic functions.

(iii) The first inequality in (c) generalizes that in (d).

(iv) See also: **Completely Monotonic Function Inequalities**.

REFERENCES [MPF, pp. 365–377]; *Widder* [W, pp. 167–168].

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**Absolute Value Inequalities** *(a) If $a \in \mathbb{R}$ then*

$$-|a| \le a \le |a|,$$

*with equality on the left-hand side if and only if $a \le 0$, and on the right-hand side if and only if $a \ge 0$.*

*(b) If $b \ge 0$ then:* $\quad |a| \le b \iff -b \le a \le b.$

*(c) If $a, b \in \mathbb{R}$ then:*

*(i)*

$$ab \le |a|\,|b|,$$

*with equality if and only if $ab \ge 0$;*

*(ii)*

$$|a+b| = \begin{cases} |a| + |b|, & \text{if } ab \ge 0, \\ \big||a| - |b|\big|, & \text{if } ab \le 0; \end{cases}$$

*(iii)*

$$|a+b| \le |a| + |b|, \tag{1}$$

$$|a-b| \ge \big||a| - |b|\big|, \tag{2}$$

*with equality if and only if $ab \ge 0$.*

*(d) If $a, b, c \in \mathbb{R}$ then*

$$|a-c| \le |a-b| + |b-c|, \tag{3}$$

*with equality if and only if $b$ is between $a$ and $c$.*

Comments (i) The proofs are by examining the cases in the definition of absolute value:

$$|a| = \begin{cases} a, & \text{if } a \ge 0, \\ -a, & \text{if } a \le 0. \end{cases}$$

Alternatively (c)(iii) follows from (c)(i) by considering $(a \pm b)^2$, using the definition

$$|a| = \sqrt{a^2};$$

and (d), which is a special case of (T), follows from (1).

(ii) The condition $ab \ge 0$ can be written as $a \sim^+ b$. See: **Triangle Inequality** (a),(b).

Extensions and Other Results *(a) If $a_j \in \mathbb{R}, 1 \le j \le n$, then*

$$|a_1 + \cdots + a_n| \le |a_1| + \cdots + |a_n|,$$

*with equality, if and only if all, the non-zero $a_j$ have the same sign.*
*(b) If $a_j \in \mathbb{R}$ with $|a_j| \le 1, 1 \le j \le 4$, then*

$$\big|a_1(a_3 + a_4) + a_2(a_3 - a_4)\big| \le 2. \tag{4}$$

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COMMENTS (iii) The right-hand side of (4) is attained when $a_j = 1, 1 \leq j \leq 4$.

(iv) If the numbers in (b) are taken to be complex, the upper bound in (4) is $2\sqrt{2}$, an upper bound that is attained when $a_1 = i, a_2 = 1, a_3 = (1+i)/\sqrt{2}, a_4 = (1-i)/\sqrt{2}$.

(v) See also: **Complex Number Inequalities** (A), (B). EXTENSION (A), **Triangle Inequality**.

REFERENCES [EM, vol. 1, p. 22]; *Apostol* [A67, pp. 31–33], *Halmos* [Ha, p. 15], *Hewitt & Stromberg* [HS, pp. 36–37].

**Aczél & Varga's Inequality** *If $\underline{a}, \underline{b}$ are $n$-tuples with $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ then*

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \leq \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2,$$

*with equality if and only if $\underline{a} \sim \underline{b}$.*

COMMENTS (i) This result can be considered as an analogue of (C) in a non-Euclidean geometry. As this space is called a Lorentz space this inequality is sometimes called a *Lorentz inequality*.

(ii) The Aczél inequality was used to prove the **Aleksandrov-Fenchel Inequality**.

EXTENSIONS *(a)* [POPOVICIU] *If $p, q > 1$ are conjugate indices and $\underline{a}, \underline{b}$ are $n$-tuples with $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/q} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{1}$$

*If $0 < p < 1$ then (~1) holds.*

*(b)* [BELLMAN] *If $p \geq 1$, or $p < 0$ and $\underline{a}, \underline{b}$ are $n$-tuples with $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^p - \sum_{i=2}^n b_i^p > 0$ then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq \left((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\right)^{1/p}. \tag{2}$$

*If $0 < p < 1$ then (~2) holds.*

COMMENTS (iii) The results of Aczél and Popoviciu can be deduced from **Jensen-Pečarić Inequalities** REVERSE INEQUALITY; for similar results see **Alzer's Inequalities** (E) PSEUDO-ARITHMETIC AND PSEUDO-GEOMETIC MEANS.

(iv) Bellman's result is a deduction from (M).

(v) A unified approach to all of these inequalities can be found in the references.

REFERENCES [AI, pp. 57–59], [BB, pp. 38–39], [MPF, pp. 117–121], [PPT, pp. 124–126]; *Bullen* [79], *Losonczi & Páles* [183].

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**Adamović's Inequality** *The following inequality holds if all the factors are positive:*

$$\prod_{i=1}^{n} a_i \geq \prod_{i=1}^{n} \big(A_n - (n-1)a_i\big),$$

*with equality if and only if $\underline{a}$ is constant.*

COMMENTS (i) This inequality has been generalized by Klamkin.
(ii) If $a, b, c$ are the sides of a triangle then the case $n = 3$ gives

$$abc \geq (a+b-c)(b+c-a)(c+a-b);$$

this is known as *Padoa's inequality.*

REFERENCES [AI, pp. 208–209]; *Bottema, Đorđević, Janić, Mitrinović & Vasić* [Bot, p. 12], *Mitrinović, Pečarić & Volenec* [MPV]; *Klamkin* [156], [158].

**Agarwal's Inequality** *If $R = [0,a] \times [0,b]$ and $f \in \mathcal{C}^2(R)$, with $f(x,0) = f(0,y) = 0, 0 \leq x \leq a, 0 \leq y \leq b$, then*

$$\int_R |f||f''_{12}| \leq \frac{ab}{2\sqrt{2}} \int_R |f''_{12}|^2.$$

COMMENT The best value of the numerical constant on the right-hand side is not known; but it is greater than $\big(3 + \sqrt{13}\big)/24$.

REFERENCE [GI3, pp. 501–503].

**Ahlswede-Daykin Inequality** *Let $f_i : X \to [0, \infty[$, $1 \leq i \leq 4$, where $X$ is a distributive lattice. If for all $a, b \in X$*

$$f_1(a)f_2(b) \leq f_3(a \vee b)f_4(a \wedge b), \tag{1}$$

*then for all $A, B \subseteq X$,*

$$\sum_{a\in A} f(a) \sum_{b\in B} f(b) \leq \sum_{x\in A\vee B} f(x) \sum_{y\in A\wedge B} f(y).$$

COMMENT This is also known as *the Four Functions Inequality.* It is an example of a *correlation inequality*; see also **Holley's Inequality,** COMMENTS.

REFERENCE [EM, Supp. 201].

**Aleksandrov-Fenchel Inequality**[15] See: **Mixed-volume Inequalities** EXTENSIONS, **Permanent Inequalities,** (D).

**Almost Symmetric Function Inequalities** See: **Segre's Inequalities.**

[15] This is D. A. Aleksandrov.

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**Alternating Sum Inequalities** *If $f$ is a non-negative function on $n$-tuples of non-negative integers that is increasing in each variable, and if $\underline{a}, \underline{b}$ are two such $n$-tuples with $A_n \le B_n$ then*

$$f(\underline{a}) \ge (-1)^{A_n} \sum (-1)^{C_n} f(\underline{c}),$$

*where the sum is over all $n$-tuples of non-negative integers, $\underline{c}$, with $A_n \le C_n \le B_n$.*

COMMENT See also: **Opial's Inequalities** (B), **Szegő's Inequality**.

REFERENCES *Borwein & Borwein* [66].

**Alzer's Inequalities** *(a) If $f$ is convex on $[a,b]$ with a bounded second derivative then*

$$\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{(b-a)^2}{8} \sup_{a \le x \le b} f''(x).$$

*(b) If $\underline{n} = \{1, 2, \ldots, n\}, n > 1$, then*

$$\frac{1}{e^2} < \mathfrak{G}_n^2(\underline{n}) - \mathfrak{G}_{n-1}(\underline{n-1})\mathfrak{G}_{n+1}(\underline{n+1}).$$

*The constant on the left-hand side is best possible.*
*(c) If $\underline{a}, \underline{b}$ are real sequences satisfying $b_i \le \min\{a_i, a_{i+1}\}, i = 1, 2, \ldots,$ then with $p \ge 1$,*

$$\sum_{i=0}^{n} a_i^p - \sum_{i=0}^{n-1} b_i^p \le \left(\sum_{i=0}^{n} a_i - \sum_{i=0}^{n-1} b_i\right)^p.$$

*(d) If $a_1 \le \dfrac{a_2}{2} \le \cdots \le \dfrac{a_n}{n}$ then*

$$A_n \le \frac{(n+1)a_n}{2}.$$

*(e)* [PSEUDO-ARITHMETIC AND PSEUDO-GEOMETRIC MEANS] *If $\underline{a}, \underline{w}$ are positive $n$-tuples define*

$$\mathfrak{a}_n(\underline{a};\underline{w}) = \frac{W_n}{w_1}a_1 - \frac{1}{w_1}\sum_{i=2}^{n} w_i a_i, \quad \mathfrak{g}_n(\underline{a};\underline{w}) = \frac{a_1^{W_n/w_1}}{\prod_{i=2}^{n} a_i^{w_i/w_1}}; \tag{1}$$

*then*

$$\mathfrak{a}_n(\underline{a};\underline{w}) \le \mathfrak{g}_n(\underline{a};\underline{w}), \tag{2}$$

*with equality if and only if $\underline{a}$ is constant.*

COMMENTS (i) (a) is an easy consequence of

$$[a, \frac{a+b}{2}, b; f] = \frac{f''(c)}{2}$$

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for some $c, a < c < b$; this notation is defined in **n-Convex Function Inequalities** (1).

(ii) (c) implies **Székely, Clark & Entringer Inequality** (a).

(iii) Inequality (d) was used by Alzer in very interesting extension of (C).

(iv) The quantities in (1) are known as the the *pseudo-arithmetic, and pseudo-geometric means of order $n$ of $\underline{a}$ with weight $\underline{w}$*, respectively.

(v) Inequality (2) is an easy deduction from the **Jensen-Pečarić Inequality** Reverse Inequality.

(vi) Inequality (2) is an analogue for these pseudo means of (GA), and has extension of both Popoviciu- and Rado-type; there is also an analogue of **Fan's Inequality**. The inequality of **Aczél & Varga** is an analogue of (C).

Extension [Kivinukk] *If $f \in \mathcal{L}^p([a,b]), 1 \le p \le \infty$, is convex, and if $q$ is the conjugate index*

$$\overline{1-\lambda}\, f(a)+\lambda f(b)-f\big(\overline{1-\lambda}\, a+\lambda b\big) \le 2^{1/p}(q+1)^{-1/q}\lambda(1-\lambda)(b-a)^{1+1/q}||f''||_{p,[a,b]}.$$

Comments (iv) The case $p=\infty, \lambda = 1/2$ is (a) above.

(v) See also **Minc-Sathre Inequality** Comment (ii).

(vi) There are numerous other inequalities by Alzer.

References [GI6, pp. 5–16] [H, pp. 171–173]; *Alzer* [18],[24], *Bullen* [79], *Kivinukk* [154].

**Analytic Function Inequalities** *(a) If $f$ is analytic in $D$ and if $0 \le r_1 \le r_2 < 1$ then,*

$$M_p(f;r_1) \le M_p(f;r_2), \quad 0 \le p \le \infty. \tag{1}$$

*(b) If $f, g$ are analytic in $D$ and if $1 \le p \le \infty$, then*

$$M_p(f+g;r) \le M_p(f;r) + M_p(g;r),\ 0 \le r < 1.$$

*(c) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is bounded and analytic in $D$ with $|f(z)| < M,\ |z| < 1$, and if*

$$s_n(z) = \sum_{k=0}^{n} a_k z^k,\ n \in \mathbb{N}; \quad \sigma_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} s_k(z), n \ge 1,$$

*then:*

$$\begin{aligned} |\sigma_n(z)| &\le M, && \text{if } |z| < 1;\\ |s_n(z)| &\le KM\log n, && \text{if } |z| < 1;\\ |s_n(z)| &\le M, && \text{if } |z| < 1/2; \end{aligned}$$

*where $K$ is an absolute constant.*

Comments (i) From (1) if $f \in \mathcal{H}^p(D)$ then $||f||_p = \lim_{r\to 1} M_p(f;r)$ exists and is finite. If $1 \le p \le \infty$, $||f||_p$ is a norm on the space $\mathcal{H}^p(D)$, as (b) readily

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proves; (b) is an easy consequence of (M). Further properties of $M_p(f;r)$ are given in **Hardy's Analytic Function Inequality**.

(ii) The results in (a) and (b) readily extend to harmonic functions.

(iii) The converse of the first inequality in (c) holds, in the sense that if $\sigma_n(z) \le M, |z| < 1$ then $|f(z)| < M, |z| < 1$.

(iv) See also: **Area Theorems, Bieberbach's Conjecture, Bloch's Constant, Borel-Carathéodory Inequality, Cauchy-Hadamard Inequality, Cauchy Transform Inequality, Distortion Theorems, Entire Function Inequalities, Fejér-Riesz Theorem, Gabriel's Problem, Hadamard's Three Circles Theorem, Landau's Constant, Lebedev-Milin Inequalities, Littlewood-Paley Inequalities, Maximum-Modulus Principle, Phragmén-Lindelöf Inequality, Picard-Schottky Theorem, Rotation Theorems, and Subordination Inequalities**.

References *Rudin* [R87, pp. 330–331], *Titchmarsh* [T75, pp. 235–238].

**Andersson's Inequality** *If $f_k : [a,b] \mapsto \mathbb{R}_+$, $1 \le k \le n$, are convex and increasing, $f_k(a) = 0$, $1 \le k \le n$, then*

$$\mathfrak{A}_{[a,b]}\Big(\prod_{k=1}^{n} f_k\Big) \ge \frac{2^n}{n+1}\Big(\prod_{k=1}^{n} \mathfrak{A}_{[a,b]}(f_i)\Big), \tag{1}$$

*with equality if and only if for some $\lambda_i \in \mathbb{R}_+$, $f_i(x) = \lambda_i(x-a)$, $1 \le i \le n$.*

Comment (i) The condition $f_k(a) = 0$, $1 \le k \le n$, is essential as is shown by the case $n = 2, a = 0, b = 1$, and $f_1(x) = 1 + x^2$, $f_2(x) = 1 + x^3$.

Extension *Inequality (1) holds if the conditions on the functions are replaced by $f_i \in \mathcal{M}, 1 \le i \le n$, where*

$$\mathcal{M} = \{f;\ f : [a,b] \mapsto \mathbb{R}, f \in \mathcal{C}^1, f(a) = 0 \ \textit{with}\, f(x)/(x-a) \ \textit{increasing on}\, [a,b]\}.$$

Comment (ii) If $a = 0, b = 1$, and $g(x) = x^2/(1+x)$, then $g \in \mathcal{M}$, but $g$ is not convex. On the other hand if $h$ satisfies the condition of the main result then $h(x)/(x-a)$ is increasing since $h(x)/(x-a) = \dfrac{1}{x-a}\displaystyle\int_a^t h' = \int_0^1 h'\big((x-a)t + a\big)\,\mathrm{d}t$; so $h \in \mathcal{M}$.

References *Andersson* [43], *Fink* [116].

**Arc Length Inequality** *If $f, g$ are continuous and of bounded variation on $[a,b]$ and $L$ is the arc length of the curve $\{(x,y);\ x = f(t), y = g(t), a \le t \le b\}$, then*

$$\int_a^b \sqrt{f'^2 + g'^2} \le L \le V(f;[a,b]) + V(g;[a,b]).$$

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*There is equality on the left if and only if $f, g$ are absolutely continuous; there is equality on the right if and only if $f'g' = 0$ almost everywhere, and for some sets $A, B$ with $A \cup B = \{x; f'(x) = \infty$, and $g'(x) = \infty\}$ we have $\big|f[A]\big| = \big|g[B]\big| = 0$.*

References *Saks* [Sa, pp. 121–125]; *Cater* [82].

**Area Theorems** [Grönwall, Bieberbach] *If $f$ is univalent in $\{z; |z| > 1\}$ with $f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Comments (i) The name of this result follows from the proof which computes the area outside $f\big[\{z; |z| > 1\}\big]$

Extension [Grunsky, Goluzin] *If $f$ is analytic and $m$-valent for $|z| > 1$ with $f(z) = \sum_{n=-m}^{\infty} b_n z^{-n}, b_{-m} \neq 0$ then*

$$\begin{aligned} \sum_{n=1}^{\infty} n|b_n|^2 &\leq \sum_{n=1}^{m} n|b_{-n}|^2; \\ \sum_{n=1}^{m} n|b_n||b_{-n}|^2 &\leq \sum_{n=1}^{m} n|b_{-n}|^2. \end{aligned}$$

Comment (ii) See also: **Bieberbach's Conjecture**.

References [EM, vol. 1, pp. 245–246]; *Itô* [I, vol. III, p. 1666]; *Ahlfors* [Ah73, pp. 82–87], *Conway* [C, vol. II, pp. 56–58].

**Arithmetic's Basic Inequalities** *(a) If $a, b \in \mathbb{R}$ then:*

$$\begin{aligned} a > b &\qquad \text{if and only if } a - b \text{ is positive;} \\ a < b &\qquad \text{if and only if } b - a \text{ is positive.} \end{aligned}$$

*(b) If $a, b, c \in \mathbb{R}$ and if $a > b$ and $b > c$ then $a > c$, and we write $a > b > c$.*

*(c) If $a, b, c \in \mathbb{R}$ and if $a > b$ then $a + c > b + c$,*

$$ac \begin{cases} > bc & \text{if } c > 0, \\ < bc & \text{if } c < 0. \end{cases}$$

*(d) If $a, b, c, d \in \mathbb{R}$ and if $a > b, c > d$ then*

$$a + c > b + d.$$

*(e) If $a, b, c, d \in (\mathbb{R}^+)^*$ and if $a > b, c > d$ then*

$$ac > bd, \quad \frac{a}{d} > \frac{b}{c}.$$

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*(f) If $a, b \in \mathbb{R}$ then*

$$\begin{aligned} a > b > 0 \quad or \quad 0 > a > b \qquad &\Longrightarrow \qquad \frac{1}{a} < \frac{1}{b}; \\ a > 0 > b \qquad &\Longrightarrow \qquad \frac{1}{a} > \frac{1}{b}. \end{aligned}$$

*(g) If $a, b \in \mathbb{R}$ then*

$$a > b > 0 \Longrightarrow \begin{cases} a^r > b^r, & \text{if } r > 0, \\ a^r < b^r, & \text{if } r < 0. \end{cases}$$

Comments (i) (a) is just the definition of inequality, and all of the other results follow from it.

(ii) These inequalities are called *strict inequalities*, and we also have the following.

Extensions [Weak Inequalities] *(a) If $a, b \in \mathbb{R}$ then:*

*$a \geq b$ if and only if $a - b$ is positive or zero;*

*$a \leq b$ if and only if $b - a$ is positive or zero.*

*(b) If $a, b, c \in \mathbb{R}$ and if $a \geq b$ then $a + c \geq b + c$ and*

$$ac \begin{cases} \geq bc, & \text{if } c \geq 0, \\ \leq bc, & \text{if } c \leq 0, \end{cases}$$

*with equality if either $c = 0$ or $a = b$.*
*(c) If $a \geq b$ and if $b > c$, or if $a > b$ and $b \geq c$ then $a > c$.*
*(d) If $a, b \in \mathbb{R}$ then*

$$a \geq b > 0 \Longrightarrow \begin{cases} a^r \geq b^r, & \text{if } r > 0, \\ a^r \leq b^r, & \text{if } r < 0. \end{cases}$$

*with equality if and only if $a = b$.*

Comment (iii) See also: **Exponential Function Inequalities** (a). Most of these inequalities can be extended by induction.

Extensions continued *(e) If $a_i \geq b_i, 1 \leq i \leq n$ then*

$$a_1 + \cdots + a_n \geq b_1 + \cdots + b_n,$$

*and this inequality is strict if at least one of the inequalities $a_i \geq b_i, 1 \leq i \leq n$ is strict.*
*(f) If $a_i \geq b_i \geq 0, 1 \leq i \leq n$ then*

$$a_1 a_2 \ldots a_n \geq b_1 b_2 \ldots b_n.$$

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COMMENT (iv) See also: **Mediant Inequalities** (a).

REFERENCES *Apostol* [A67, pp. 15–17], *Bulajich Manfrino, Ortega & Delgado* [BOD, pp. 1–2], *Herman, Kučera & Šimša* [HKS, pp. 90–94].

**Arithmetic-Geometric Mean Inequality** See: **Geometric-Arithmetic Mean Inequality**.

**Arithmetic Mean Inequalities** *(a) If $\underline{a}, \underline{w}$ are positive n-tuples then*

$$\min \underline{a} \leq \mathfrak{A}_n(\underline{a};\underline{w}) \leq \max \underline{a}, \tag{1}$$

*with equality if and only if $\underline{a}$ is constant.*
*(b)* [DIANANDA] *If $\underline{a}, \underline{w}$ are positive n-tuples such that for some positive integer $k$, $\underline{a} \geq k$ then*

$$\mathfrak{A}_n(\underline{a};\underline{w}) \leq \left(\prod_{i=1}^{n} \frac{a_i}{[a_i]}\right)\mathfrak{A}_n([\underline{a}];\underline{w}) < \left(1+\frac{1}{k}\right)^{n-1}\mathfrak{A}_n(\underline{a};\underline{w}). \tag{2}$$

*In particular if $\underline{a} \geq n-1$ then*

$$\left(\prod_{i=1}^{n} \frac{a_i}{[a_i]}\right)\mathfrak{A}_n([\underline{a}];\underline{w}) < e\,\mathfrak{A}_n(\underline{a};\underline{w}).$$

*Further there is equality on the left of (2) if and only if all the entries in $\underline{a}$ are integers.*
*(c)* [PREŠIĆ] *If $\underline{a}$ is a real n-tuple then*

$$\min_{1\leq i<j\leq n}(a_i-a_j)^2 \leq \frac{12}{n^2-1}\left(\mathfrak{A}_n(\underline{a}^2)-\big(\mathfrak{A}_n(\underline{a})\big)^2\right).$$

*(d)* [REDHEFFER] *If $\underline{a}$ is an increasing positive sequence and if $p<0$, $q$ the conjugate index, then*

$$\sum_{i=1}^{\infty}\frac{a_i}{\mathfrak{A}_i^{1/q}(\underline{a})} < (1-p)\sum_{i=1}^{\infty}\mathfrak{A}_i^{1/p}(\underline{a}).$$

*(e) If $\mathfrak{A}_n(\underline{a};\underline{w}) = 1$ then*

$$\mathfrak{A}_n(\underline{a}\log\underline{a};\underline{w}) \geq 0,$$

*with equality if and only if $\underline{a}$ is constant.*
*(f)* [KLAMKIN] *If $\underline{a}$ is a positive n-tuple,*

$$r\sum_{r}!\frac{a_1\cdots a_r}{a_1+\cdots+a_r} \leq \binom{n}{r}\left(\mathfrak{A}_n(\underline{a})\right)^{r-1}.$$

COMMENTS (i) Variants of (c) have been given by Lupaş.
(ii) Inequality (d) is a recurrent inequality; see **Recurrent Inequalities**.
(iii) Inequality (f) is proved using Schur convexity; see **Schur Convex Function Inequalities**.

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Extension [Cauchy] *If $\underline{a}, \underline{b}, \underline{w}$ are positive n-tuples*

$$\min\left(\underline{a}\,\underline{b}^{-1}\right) \le \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{A}_n(\underline{b};\underline{w})} \le \max\left(\underline{a}\,\underline{b}^{-1}\right).$$

Comment (iv) Inequality (1) and its extension above, are properties common to most means; see **Mean Inequalities** (1), Extensions.

Integral Analogue [Favard] *If $f \ge 0$ on $[a,b]$ is concave then*

$$\mathfrak{A}_{[a,b]}(f) \ge \frac{1}{2}\max_{a\le x\le b} f.$$

Comments (v) The last inequality is a limiting case of **Favard's Inequalities** (b).

(vi) For an integral analogue of the basic inequality (1) see **Integral Mean Value Theorems** (a).

(vii) See also: **Binomial Function Inequalities** (j), **Čebyšev's Inequality, Convex Sequence Inequalities** (d), **Geometric-Arithmetic Mean Inequality** (5), **Grüsses' Inequalities** (a), Discrete Analogues, **Harmonic Mean Inequalities** (c) and Comments (v), **Increasing Function Inequalities** (2) and Extension, **Kantorović's Inequality, Karamata's Inequality, Levinson's Inequality, Logarithmic Mean Inequalities** Corollaries (b), (c), Extensions (a), **Mitrinović & Đoković's Inequality** Extension, **Mixed Mean Inequalities** Special Cases (b), **Muirhead Symmetric Function and Mean Inequalities** Comments (ii), **Nanson's Inequality, n-convex Sequence Inequalities** (b), Extension, **Statistical Inequalities** (c).

References [AI, pp. 204,340–341], [BB, p. 44], [H, pp. 62,63,165–166], [MOA, p. 104]; *Lupaş* [189], *Redheffer* [279, p. 689].

## Arithmetico-Geometric Mean Inequalities

If $0 < a = a_0 \le b = b_0$ define the sequences $\underline{a}, \underline{b}$ by

$$a_n = \mathfrak{G}_2(a_{n-1}, b_{n-1}), \quad b_n = \mathfrak{A}_2(a_{n-1}, b_{n-1}),\ n \ge 1.$$

The *arithmetico-geometric mean* of $a$ and $b$ is

$$\mathfrak{A}\otimes\mathfrak{G}(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{1}$$

This is also called the *arithmetic-geometric compound mean.*

*If $0 < a \le b$ then*

$$a < \mathfrak{G}_2(a,b) < \mathfrak{L}(a,b) < \mathfrak{A}\otimes\mathfrak{G}(a,b) < \mathfrak{A}_2(a,b) < b.$$

Comments (i) In an analogous way we can define the *geometric-harmonic compound mean*, $\mathfrak{G}\otimes\mathfrak{H}(a,b)$, and the *arithmetic-harmonic compound mean*, $\mathfrak{A}\otimes\mathfrak{H}(a,b)$.

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(ii) In general any two means can be *compounded* in this way provided the associated limits in (1) exist.

(iii) Not all the means obtained in this way are new; in particular $\mathfrak{A}\otimes\mathfrak{H}(a,b) = \mathfrak{G}_2(a,b)$.

REFERENCES [H, pp. 68–69, 413–420], [MPF, pp. 47–48]; *Borwein & Borwein* [BB, pp. 1–5]; *Almkvist & Berndt* [4], *Borwein & Borwein* [67].

**Askey-Karlin Inequalities** *If* $f(x) = \sum_{n\in\mathbb{N}} a_n x^n$, $0 \le x < 1$, $A_{n+1} = \sum_{k=0}^{n} a_k$, $\Phi : [0,\infty[\to [0,\infty[$ *increasing and convex, and* $\beta > -2$ *then*

$$\begin{aligned}\int_0^1 \Phi\circ|f|(x)(1-x)^\beta\,\mathrm{d}x &\le (\beta+1)!\sum_{n\in\mathbb{N}}\Phi(|A_{n+1}|)\frac{n!}{(n+\beta+2)!};\\ \int_0^1 \frac{\Phi\circ|f|(x)}{(1-x)^2}\,\mathrm{d}x &\le \sum_{n\in\mathbb{N}}\Phi\left(\frac{|A_{n+1}|}{n+1}\right).\end{aligned}$$

COMMENT These results have been generalized by Pachpatte.

REFERENCE *Pachpatte* [244].

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# 2 Backward–Bushell

**Backward Hölder Inequality** See: **Young's Convolution Inequality** COMMENTS (iii).

**Banach Algebra Inequalities** *If $X$ is a Banach algebra then for all $x, y \in B$*

$$||xy|| \le ||x|| \, ||y||.$$

COMMENTS (i) This is the defining inequality for a topological algebra that is a Banach space; for a definition see: **Norm Inequalities** COMMENTS (i).

(ii) For particular cases see: **Bounded Variation Function Inequalities** (b), **Matrix Norm Inequalities**.

REFERENCES [EM, vol. 1, pp. 329–332]; *Hewitt & Stromberg* [HS, pp. 83–84].

**Banach Space Inequalities** See: **Kallman-Rota Inequality, von Neumann & Jordan Inequality**.

**Barnes's Inequalities** *(a) If $\underline{a}, \underline{b}$ are non-negative concave $n$-tuples, $\underline{a}$ increasing and $\underline{b}$ decreasing, then*

$$\sum_{i=1}^{n} a_i b_i \ge \frac{n-2}{2n-1} \left( \sum_{i=1} a_i^2 \right)^{1/2} \left( \sum_{i=1} b_i^2 \right)^{1/2}. \tag{1}$$

*The inequality is sharp, and is strict unless $a_i = n - i, b_i = i - 1, 1 \le i \le n$.*
*(b) Let $\underline{a}, \underline{b}$ be non-negative non-null $n$-tuples with $p, q$ conjugate indices and $||\underline{a}||_p \ne 0, ||\underline{b}||_q \ne 0$, define*

$$H_{p,q}(\underline{a}, \underline{b}) = \frac{\underline{a}.\underline{b}}{||\underline{a}||_p ||\underline{b}||_q}.$$

*If $\underline{c}$ is an increasing $n$-tuple with $\underline{c} \prec \underline{a}$, and if $\underline{d}$ is a decreasing $n$-tuple with $\underline{b} \prec \underline{d}$, and if $p, q \ge 1$*

$$H_{p,q}(\underline{a}, \underline{b}) \ge H_{p,q}(\underline{c}, \underline{d}). \tag{2}$$

*If $\underline{c}$ and $\underline{d}$ are increasing with $\underline{c} \prec \underline{a}$ and $\underline{d} \prec \underline{b}$, and if $p, q \le 1$ inequality (~2) holds.*



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Comments (i) Inequality (1) is an inverse inequality of (C) and (2) is an inverse inequality (H). A definition of an inverse inequality is given in **Notations** 1(c). See also: **Reverse, Inverse, and Converse inequalities**.

(ii) Extensions of (2) to more than two $n$-tuples, and other situations have been given; see the references.

(iii) For another inequality of Barnes see **Inverse Hölder's Inequality**.

References [AI, p. 386], [MPF, pp. 148–156].

**Beckenbach's Inequalities** *(a) If $p, q > 1$ are conjugate indices, $\underline{a}, \underline{b}$ positive $n$-tuples, and if $1 \leq m < n$ define,*

$$\tilde{a}_i = \begin{cases} a_i, & \text{if } 1 \leq i \leq m, \\ \left(\dfrac{b_i \sum_{j=1}^m a_j^p}{\sum_{j=1}^m a_j b_j}\right)^{q/p}, & \text{if } m+1 \leq i \leq n. \end{cases}$$

*Then given $a_j, 1 \leq j \leq m$, and $\underline{b}$*

$$\frac{\left(\sum_{1=1}^n a_i^p\right)^{1/p}}{\sum_{i=1}^n a_i b_i} \geq \frac{\left(\sum_{1=1}^n \tilde{a}_i^p\right)^{1/p}}{\sum_{i=1}^n \tilde{a}_i b_i},$$

*for all choices of $a_i, m < i \leq n$, with equality if and only if $a_i = \tilde{a}_i, m < i \leq n$.*
*(b) If $p, q > 1$ are conjugate indices, $\alpha, \beta, \gamma$ positive real numbers, $f \in \mathcal{L}^p([a,b])$, $g \in \mathcal{L}^q([a,b])$, then*

$$\frac{\left(\alpha + \gamma \int_a^b f^p\right)^{1/p}}{\beta + \gamma \int_a^b fg} \geq \frac{\left(\alpha + \gamma \int_a^b h^p\right)^{1/p}}{\beta + \gamma \int_a^b hg},$$

*where $h = (ag/b)^{q/p}$. There is equality if and only if $f = h$ almost everywhere.*

Comments (i) The case $m = 1$ of (a) is just (H).

(ii) Integral analogues of (a) have been given.

(iii) A discrete analogue of (b) is easily stated; and if the $+$ signs are replaced by $-$ signs in all four places the inequality is reversed. This is called the *Beckenbach-Lorentz inequality.*

(iv) For another inequality of Beckenbach see **Counter-Harmonic Mean Inequalities** (d).

References [AI, p. 52], [H, pp. 196–198], [MPF, pp. 156–163], [PPT, pp. 122–124].

**Beckenbach-Lorentz Inequality** See: **Beckenbach's Inequalities** Comments (iii).

**Bellman-Bihari Inequalities** See: **Grönwall's Inequality** Comments (iii).

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**Bendixson's Inequalities** See: **Hirsch's Inequalities**.

**Bennett's Inequalities** *(a) If $n > 0$ then for all real $r > 0$,*

$$1 \le \left(\frac{(n+1)\sum_{i=1}^{n} i^{-1}}{n\sum_{i=1}^{n+1} i^{-1}}\right)^{1/r} \le \frac{\big((n+1)!\big)^{1/(n+1)}}{(n!)^{1/n}}; \tag{1}$$

*and*

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^{r} < \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{n+2-i}{i}\right)^{r}. \tag{2}$$

*(b) If $0 < r < 1$ and $\underline{a}$ is a non-negative $n$-tuple then, (i):*

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=i}^{n} a_j\right)^{r} \le a_n(r)\sum_{i=1}^{n}\max_{1\le i\le n} a_i^r, \tag{3}$$

*where $a_n(r)$ is the left-hand side of (2), and (ii):*

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{j=i}^{\infty} a_j\right)^{r} \le \frac{\pi r}{\sin \pi r}\sum_{i=1}^{\infty}\sup_{k\ge i} a_k^r;$$

*the constant on the right-hand side is best possible, and the inequality strict unless $\underline{a} = \underline{0}$.*
*(c) If $\underline{a}$ is a non-negative sequence then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{p} \begin{cases} \le \zeta(p)\displaystyle\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{a_i}{i}\right)^{p}, & \text{if } p \ge 2,\\[2ex] \ge (p-1)^{-1/p}\displaystyle\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{a_i}{i}\right)^{p}, & \text{if } 1 \le p \le 2. \end{cases}$$

*The first inequality is strict unless $a_2 = a_3 = \cdots = 0$, the second inequality is strict unless $\underline{a} = \underline{0}$. The constants on the right-hand sides are best possible.*
*(d) If $\underline{a}$ is a non-negative sequence and $1 < p < \infty$ then,*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{p} \le \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{|a_j|}{i+j-1}\right)^{p} \le \left(\frac{\pi}{q\sin \pi/p}\right)^{p}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{p},$$

*where $q$ is the conjugate index. The constants are best possible. The inequality on the left is strict unless $\underline{a}$ has at most one non-zero entry; the inequality on*

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*the right is strict unless* $\underline{a} = \underline{0}$.
*(e) If* $\underline{a}$ *is a non-negative sequence and* $1 < p < \infty$ *then,*

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{a_i}{i}\right)^p \le (p-1)\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{|a_j|}{i+j-1}\right)^p.$$

*The constant is best possible, and the inequality is strict unless* $\underline{a} = \underline{0}$.

COMMENTS (i) (1) is a refinement of Bennett's result due to Alzer, and the constants are best possible.

(ii) The simple proof of (2) given by Alzer depends on the strict convexity of the function $f(x) = x^{-r}(1-x)^r + x^r(1-x)^{-r}$. Alzer has also given an inequality converse to (1).

(iii) The proof of (3) is quite complex.

(iv) There is a left-hand side for the inequality in (d) but the constant is not known; see *Bennett*, [Be].

(v) For other inequalities due to Bennett see: **Abel's Inequalities** RELATED RESULTS (B), **Copson's Inequality** EXTENSIONS, **Hardy's Inequality** EXTENSIONS (C), **Hilbert's Inequalities** EXTENSIONS (B), **Littlewood's Conjectures, Zeta Function Inequalities**.

REFERENCES *Bennett* [Be, pp. 47–56]; *Alzer* [20, 22, 23], *Bennett* [57].

**Benson's Inequalities** *(a) Given three functions* $u(x), P(u,x) > 0, G(u,x)$ *write* $p(x) = P\big(u(x),x\big)$, $g(x) = G\big(u(x),x\big)$. *If* $u,p,g \in \mathcal{C}^1\left([a,b]\right)$, *and* $n \ge 1$ *then*

$$\int_a^b \left\{p(x){u'}^{2n}(x) + (2n-1)\quad p^{\frac{-1}{2n-1}}(x)\big(G_1'\big)^{\frac{2n}{2n-1}}\big(u(x),x\big) + 2nG_2'\big(u(x),x\big)\right\}\mathrm{d}x \ge 2n\big[g(b)-g(a)\big].$$

*There is equality if and only if* $u' = \big(G_1'/P\big)^{1/(2n-1)}$.
*(b) Given functions* $u(x), P(u',u,x) > 0, G(u',u,x)$ *write* $p(x) = P\big(u'(x),u(x),x\big)$, $g(x) = G\big(u'(x),u(x),x\big), g_i'(x) = G_i'\big(u'(x),u(x),x\big), i = 1, 2.3$. *If* $u \in \mathcal{C}^2([a,b])$ *and* $p, g \in \mathcal{C}^1([a,b])$ *then*

$$\int_a^b \left\{p(x)u''(x)^2 + \frac{{g_1'}^2(x)}{p(x)} + 2u'(x)g_2'(x) + 2g_3'(x)\right\}\mathrm{d}x \ge 2\left[g(b)-g(a)\right].$$

*There is equality if and only if* $u'' = \big(G_1'/P\big)$.

COMMENT These results give many important special cases, including the **Heisenberg-Weyl Inequality**, and **Wirtinger's Inequality**.

REFERENCE [AI, pp. 126–129].

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**Bergh's Inequality** *If* $f : [0, \infty[ \to [0, \infty[$ *satisfies*

$$0 \le f(x) \le \max\{1, xy^{-1}\} f(y), \qquad \text{for all } 0 < x, y < \infty, \tag{1}$$

*and if* $0 < p < q \le \infty$, $0 < r < 1$ *then*

$$\left(\int_0^\infty \frac{1}{x}\left(\frac{f(x)}{x^r}\right)^q \mathrm{d}x\right)^{1/q} \le \left(\frac{1-r}{qr}\right)^{1/q}\left(\frac{pr}{1-r}\right)^{1/p}\left(\int_0^\infty \frac{2}{x}\left(\frac{f(x)}{x^r}\right)^p \mathrm{d}x\right)^{1/p}.$$

*The constant is best possible, and there is equality only if* $f(x) = \min\{1, x\}$.

COMMENTS (i) Functions satisfying (1) have been called *quasi-concave functions.*

(ii) This inequality has been extended to other classes of functions; see the reference.

REFERENCES *Pečarić & Persson* [265].

**Bergström's Inequality** *(a) If* $A, B$ *are positive definite matrices then*

$$\frac{\det(A+B)}{\det(A_i' + B_i')} \ge \frac{\det A}{\det A_i'} + \frac{\det B}{\det B_i'}.$$

*(b) If* $x_i, a_i \in \mathbb{R}, a_i > 0, 1 \le i \le n$, *then*

$$\sum_{i=1}^n \frac{x_i^2}{a_i} \ge \frac{\left(\sum_{i=1}^n x_i\right)^2}{\sum_{i=1}^n a_i},$$

*with equality if and only if* $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

COMMENT The result of Bergström appears in several different forms as the references below show.

REFERENCES [AI, p. 315], [BB, pp. 67–69], [MOA, p. 688], [MPF, p. 214]; *Fong & Bresler* [117], *Pop* [272].

**Bernoulli's Inequality** *If* $x \ge -1, x \ne 0$, *and if* $\alpha > 1$ *or if* $\alpha < 0$ *then*

$$(1+x)^\alpha > 1 + \alpha x; \tag{B}$$

*if* $0 < \alpha < 1$ *(~B) holds.*

COMMENTS (i) The inequality (B) in the case $\alpha = 2, 3, \ldots$, is *Bernoulli's inequality.* It is a standard elementary example used when teaching mathematical induction. Further (B) also holds if $-2 \le x < -1$.

(ii) When $\alpha < 0$, when of course $x \ne -1$, the inequality follows from the case $\alpha > 0$.

(iii) The case $\alpha = 1/(n+1), n = 1, 2, \ldots$ of (~B) follows from properties a certain polynomial; see: **Polynomial Inequalities** COMMENTS (i).

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(iv) In general the result follows by a simple application of Taylor's Theorem.

VARIANTS *(a) If $a, b > 0, a \neq b,$ and $\alpha > 1$ or $\alpha < 0$,*

$$\frac{a^\alpha}{b^{\alpha-1}} > \alpha a - (\alpha - 1)b; \tag{1}$$

*and ($\sim$1) holds when $0 < \alpha < 1$.*
*(b) If $a, b$ and $\alpha$ are as in (a) and $\alpha + \beta = 1$ then*

$$a^\alpha b^\beta > \alpha a + \beta b; \tag{2}$$

*and if $\alpha > 0, \beta > 0$ and $\alpha + \beta = 1$ then ($\sim$2) holds.*
*(c) If $x \geq 0$, $x \neq 1$ and if $\alpha > 1$ or if $\alpha < 0$ then*

$$x^\alpha - 1 > \alpha(x - 1), \tag{3}$$

*while if $0 < \alpha < 1$ then ($\sim$3) holds.*

COMMENTS (v) Putting $x = \frac{a}{b} - 1$ in (B) gives (1), a symmetric form of (B).
(vi) (2) is an immediate consequence of (1).
(vii) Replacing $1 + x$ by $x$ in (B), gives (3).
(viii) (B) is equivalent to (GA). In particular ($\sim$2) is the case $n = 2$ of (GA).

EXTENSIONS *(a) If $a_i > -1, 1 \leq i \leq n$, and are all positive or all negative then*

$$\prod_{i=1}^{n}(1 + a_i) > 1 + A_n.$$

*(b) If $x > 1$, $0 < \alpha < 1$ then*

$$\frac{1}{2}\alpha(1-\alpha)\frac{x}{(1+x)^2} < 1 + \alpha x - (1+x)^\alpha < \frac{1}{2}\alpha(1-\alpha)x^2(1+x).$$

INVERSE INEQUALITY *If $\underline{a}$ is a positive $n$-tuple and $n > 1$ then*

$$\prod_{i=1}^{n}(1 + a_i) < \sum_{i=0}^{n}\frac{A_n^i}{i!}.$$

COMMENTS (ix) A discussion of the history of Bernoulli's inequality can be found in the paper by Mitrinović & Pečarić.
(x) Other extensions are given in **Binomial Function Inequalities** (c), **Favard's Inequalities** COMMENT(IV), **Gerber's Inequality** COMMENTS (II), **Kaczmarz & Steinhaus's Inequalities, Leindler's Inequality, Weierstrass's Inequalities**.

REFERENCES [AI, pp. 34–36], [H, pp. 4–6, 213], [HLP, pp. 39–43, 60, 103, 107], [MPF, pp. 65–81]; Mitrinović & Pečarić [218].

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**Bernšteĭn's Inequalities** See: **Bernšteĭn's Inequality; Bernšteĭn Polynomial Inequalities; Bernšteĭn's Probability Inequality**.

**Bernšteĭn's Inequality** *(a)* [Trigonometric Polynomial Case] *If $T_n$ is a trigonometric polynomial of degree at most $n$ then*

$$||T_n^{(r)}||_{\infty,[-\pi,\pi]} \le n^r ||T_n||_{\infty,[-\pi,\pi]}, \quad r = 1, 2, \ldots .$$

*(b)* [Polynomial Case] *If $p_n$ is a polynomial of degree at most $n$, then*

$$\left|p_n'(x)\right| \le \frac{n||p||_{\infty,[a,b]}}{\sqrt{(x-a)(b-x)}}, \quad a < x < b.$$

*(c)* [Complex Polynomial Case] *If $p_n(z)$ is a complex polynomial of degree at most $n$ then*

$$||p_n'||_{\infty,|z|=1} \le n||p_n||_{\infty,|z|=1},$$

*with equality if and only if all the zeros of $p_n$ are at the origin.*

Comments (i) For a definition of *trigonometric polynomial of degree at most $n$* see: **Trigonometric Polynomial Inequalities**.

(ii) The results in (a) cannot be improved as the case $T_n(x) = \cos nx$ shows.

(iii) (a) is a particular case of the **Entire Function Inequalities** (b).

(iv) (b) is a consequence of a proof of **Markov's Inequality**.

(v) (c) should be compared with **Erdős's Inequalities** (b).

(vi) Extension to other norms have been made; see [AI].

References [AI, pp. 228, 260], [EM, vol. 1, pp. 365–366]; Zygmund,[Z, p. 11]; Govil & Nyuydinkong [134].

**Bernšteĭn Polynomial Inequalities** If $f : [0,1] \to \mathbb{R}, n \in \mathbb{N}$ then

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k},$$

is called the *$n$-th Bernšteĭn polynomial of $f$*.

*(a) For all $x \in [0,1]$*

$$B_n^2(f,x) \le B_n(f^2,x).$$

*(b) A continuous function $f : [0,1] \to \mathbb{R}$ is convex if and only if either of the following holds:*

*(i)* [Temple] $\quad B_{n+1}(f,x) \le B_n(f,x), \quad n \in \mathbb{N}, 0 \le x \le 1;$

*(ii)* [Pólya & Schoenberg] $\quad f(x) \le B_n(f,x), \quad n \in \mathbb{N}, 0 \le x \le 1;$

The following inequality was used to obtain basic results for these polynomials. *If $x \in [0,1]$ and $n \ge 1$ then*

$$0 \le \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} \le \frac{1}{4n}. \tag{1}$$

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COMMENT See also: **Ultraspherical Poynomial Inequalities** COMMENTS (iii).

REFERENCES [PPT, pp. 292–294]; *Kazarinoff* [K, pp. 55–56].

**Bernšteĭn's Probability Inequality** *If $X_i, 1 \le i \le n$, are independent random variables with*

$$EX_i = 0, \quad \sigma^2 X_i = b_i,\ 1 \le i \le n,$$

*and if for $j > 2$,*

$$E|X_i|^j \le \frac{b_i}{2} H^{j-2} j!\,,$$

*then*

$$P\big(|X_1 + \cdots + X_n| > r\big) \le 2\exp\left(-\frac{r^2}{2\big(B_n + Hr\big)}\right). \tag{1}$$

COMMENTS (i) This is a sharpening of the **Čebišev Probability Inequality**.

(ii) In particular if the random variables are identically distributed and bounded, for instance let $EX_i = 0$, $EX_i^2 = \sigma^2$, $|X_i| \le M$, $1 \le i \le n$, then (1) reduces to

$$P\big(|X_1 + \cdots + X_n| > r\sigma\sqrt{n}\big) \le 2\exp\bigg(-\frac{r^2}{2(1+\alpha)}\bigg),$$

where $\alpha = Mr/3\sigma\sqrt{n}$.

(iii) A lower estimate for the left-hand side of (1) has been given by Kolmogorov.

REFERENCES [EM, vol. 1, p. 365, vol. 2, p. 120]

**Berry-Esseen Inequality** *Let $X_i, 1 \le i \le n$, be equally distributed random variables with*

$$EX_i = 0,\ EX_i^2 = \sigma^2,\ E|X_i|^3 < \infty,\ 1 \le i \le n,$$

*then*

$$\sup_x \left| P\left\{ \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^n X_j \le x \right\} - \mathfrak{p}(x) \right| \le \frac{33}{4} \frac{E|X_j|^3}{\sigma^3\sqrt{n}}\,.$$

REFERENCES [EM, vol. 1, pp. 369–370]; *Feller* [F, vol. II, pp. 542–546], *Loève* [L, pp. 282–288].

**Berwald's Inequality** See: **Favard's Inequalities** EXTENSION (A), **Geometric-Arithmetic Mean Inequality** INTEGRAL ANALOGUES (B).

**Bessel Function Inequalities** *(a)* [GIORDANO & LAFORGIA] *If $J_\nu$ is the Bessel function of first kind, $\nu > -1$, and if $\underline{a}$ is a positive $n$-tuple with $\underline{a} \le j_{\nu_1}$, the smallest positive zero of $J_\nu$, then*

$$\mathfrak{G}_n\big(J_\nu(\underline{a})\big) \le J_\nu\big(\mathfrak{M}_n^{[2]}(\underline{a})\big).$$

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*(b)* [Elbert & Laforgia; Makai] *If $\nu > 0$ and $j(\nu) = j_{\nu_k}$ is the $k$-th positive zero of $J_\nu$ then*

$$\big(\nu + j(\nu)\big)j''(\nu) > \frac{\nu j'(\nu)^2}{j(\nu)} - j'(\nu), \text{ and } \left(\frac{j(\nu)}{\nu}\right)' \le 0,$$

*(c)* [Ross D. K.] *If $x \in \mathbb{R}$ then*

$$J_{\nu+1}^2(x) \ge J_\nu(x)J_{\nu+2}(x).$$

Comment (i) Inequality (c) is related to the similar inequalities **Legendre Polynomial Inequalities** (b), **Ultraspherical Polynomial Inequalities** (b).

Corollaries *(a) If $0 < a_i < \pi, 1 \le i \le n$ then*

$$\mathfrak{G}_n\big(\sin \underline{a}\big) \le \sin\big(\mathfrak{M}_n^{[2]}(\underline{a})\big).$$

*(b) With the above notation the function $f(\nu) = \nu/j(\nu)$ is concave.*

Comments (ii) Corollary (a) follows by taking $\nu = 1/2$; it should be compared with **Trigonometric Function Inequalities** (r). The result implies geometric inequalities for triangles and convex polygons.

(iii) See also: **Enveloping Series Inequalities** Comments (iii), **Mahajan's Inequality, Oppenheim's Problem** Comments (ii).

References [GI1, pp. 35–38], [GI5, pp. 139–150].

**Bessel's Inequality** *If $f \in \mathcal{L}^2([a,b])$ and if $\phi_n, n \in \mathbb{N}$ is an orthonormal sequence of complex valued functions defined on $[a,b]$ with $\underline{c}$ the sequence of Fourier coefficients of $f$ with respect to $\phi_n, n \in \mathbb{N}$, then*

$$||\underline{c}||_2 \le ||f||_2. \tag{1}$$

Comments (i) This result remains valid, with the same proof if $\mathcal{L}^2([a,b])$ is replaced by any Hilbert space $H$, the orthonormal sequence by any orthonormal $n$-tuple, or sequence in $H$, $\underline{x}_k, k \in I$ say, and the sequence $\underline{c}$ by $c_k = \underline{x}.\underline{x}_k, k \in I$. Then (1) becomes

$$\sum_{k\in I} |\underline{x}.\underline{x}_k|^2 \le \underline{x}.\underline{x} = ||\underline{x}||^2 \tag{2}$$

(ii) In the notation of (i) the proof of (2) depends on the identity

$$\left|\left|\underline{x} - \sum_{k\in I}\lambda_k\underline{x}_k\right|\right|^2 = ||\underline{x}||^2 - \sum_{k\in I}|(\underline{x}.\underline{x}_k)|^2 + \sum_{k\in I}|\lambda_k - (\underline{x}.\underline{x}_k)|^2,$$

where $\lambda_k, k \in I$, is any set of complex numbers.

(iii) If the orthonormal sequence is complete then (1), and (2), becomes an equality known as *Parseval's equality*.

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Extension [Bombieri] *If* $\underline{a}, \underline{b}_k, 1 \le k \le m$, *are complex* $n$*-tuples, then*

$$\sum_{k=1}^{m} |\underline{a}.\underline{b}_k|^2 \le |\underline{a}|^2 \max_{1\le k\le m} \sum_{j=1}^{m} |\underline{b}_k.\underline{b}_j|, \tag{3}$$

*with equality if and only if* $\underline{a}, \underline{b}_k, 1 \le k \le m$, *are linearly dependent.*

Comments (iv) The case $m = 1$ of (3) is just (C).

(v) For a further extension see: **Ostrowski's Inequalities** Comments (i), **Hausdorff-Young Inequalities, Paley's Inequalities**.

References [EM, vol. 1, pp. 373–374; vol. 7, pp. 93–94], [MPF, pp. 391–405]; *Courant & Hilbert* [CH, pp. 51–52], *Lieb & Loss* [LL, pp. 66–67], *Zygmund* [Z, vol. I, p. 13].

## Beta Function Inequalities

If $\Re x > 0, \Re y > 0$ then the *Beta function* is defined by

$$B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

(In the results given below we will always have $x > 0, y > 0$.)
*(a) If* $x, y, z > 0$ *then*

$$(x+y)B(z, x+y) \ge xB(z,x)yB(z,y). \tag{1}$$

*(b) If* $m, n, p, q$ *are positive real numbers with* $p - m$ *and* $q - n$ *the same sign then*

$$B(p,q)B(m,n) \le B(p,n)B(m,q); \tag{2}$$

*in particular* $B(m,n) \ge \sqrt{B(m,m)B(n,n)}$.
*If* $p - m$ *and* $q - n$ *are of opposite sign then* $(\sim 2)$ *holds.*
*(c) The beta function is log-convex in both variables on* $]0,\infty[\times]0,\infty[$, *that is if* $p, q, m, n > 0$, *and* $0 \le t \le 1$, *then*

$$B(\overline{1-t}\,p + tm, \overline{1-t}\,q + tn) \le B(p,q)^{1-t}B(m,n)^t.$$

Comments (i) Inequality (1) is a deduction from **Stolarsky's Inequality**.

(ii) Inequality (2) can be deduced from a weighted form of **Čebisev's Inequality** Integral Analogue.

(iii) The log-convexity is a consequence of the integral analogue of (H).

(iv) See also: **Vietoris's Inequality**.

Extensions *If* $\underline{m}$ *is an* $n-$*tuple,* $n \ge 2$, *with* $\Re m_k > 0, 1 \le k \le n$, *then the* generalized Beta function of order $n$ *is*

$$B_n(\underline{m}) = \frac{(m_1-1)!\ldots(m_n-1)!}{(m_1+\cdots+m_n-1)!} = \frac{\Gamma(m_1)\ldots\Gamma(m_n)}{\Gamma(m_1+\cdots+m_n)}.$$

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*(a) If* $\underline{u} < \underline{v}$ *then* $B_n(\underline{u}) > B_n(\underline{v})$.
*(b)* $B_n$ *is log-convex; that is if* $0 \le t \le 1$ *then*

$$B_n\big((1-t)\underline{u} + t\underline{v}\big) \le B_n^{(1-t)}(\underline{u})B_n^t(\underline{v}).$$

Comment (v) Clearly $B_2(m,n) = B(m,n)$.

Reference *Alzer* [35], *Dedić, Matić & Pečarić* [98], *Dragomir, Agarwal & Barnett* [104], *Sasvári* [293], *Stolarsky* [304].

**Beth-van der Corput Inequality** *If* $z, w \in \mathbb{C}$ *and* $p \ge 2$ *then*

$$|w+z|^p + |w-z|^p \ge 2\left(|w|^p + |z|^p\right). \tag{1}$$

Comment (i) This is an inverse of **Clarkson's Inequalities** (1).

Extension [Klamkin] *If* $\underline{a}_j, 1 \le j \le m$, *are real n-tuples and* $p > 2$ *or* $p < 0$ *then*

$$\sum |\pm \underline{a}_1 \pm \cdots \pm \underline{a}_m|^p \ge 2^m \left(\sum_{j=1}^m |\underline{a}_j|^2\right)^{p/2} \ge 2^m \sum_{j=1}^m |\underline{a}_j|^p, \tag{2}$$

*where the sum on the left-hand side is over all* $2^n$ *permutations of the* $\pm$ *signs. If* $0 < p < 2$ *then* $(\sim 2)$ *holds; and for* $p = 0, 2$ *(2) is an identity.*

Comments (ii) If $n = 2$ and $p \ge 2$ an application of (r;s) shows that (2) generalizes (1).

(iii) In the case $p = 1$ (2) has the following geometric interpretation. *Amongst all paralleletopes of given edge lengths, the rectangular one has the greatest sum of lengths of diagonals.*

(iv) Again in the case $p = 1$, and with $m = 3$, (2) can be regarded as an inverse of **Hlawka's Inequality**; put $2\underline{a}_1 = \underline{a} + \underline{b}$, $2\underline{a}_2 = \underline{b} + \underline{c}$, $2\underline{a}_3 = \underline{c} + \underline{a}$;

$$|\underline{a}| + |\underline{b}| + |\underline{c}| + |\underline{a} + \underline{b} + \underline{c}| \le 2\left(|\underline{a} + \underline{b}|^2 + |\underline{b} + \underline{c}|^2 + |\underline{c} + \underline{a}|^2\right)^{1/2}.$$

(v) An extension of the (1) to inner product spaces has been made by Dragomir & Sándor.

References [AI, p. 322], [MPF, pp. 523, 544–551], [PPT, pp. 134–135].

**Bieberbach's Conjecture** *If* $f(z) = z + \sum_{n\ge 2} a_n z^n$ *is univalent in D then*

$$|a_n| \le n.$$

*There is equality if and only if f is a rotation of the Koebe function.*

Comments (i) This famous conjecture was proved by de Branges in 1984.

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(ii) The hypothesis of injectivity cannot be omitted as the function $z+3z^2$ shows.

(iii) The Koebe function is defined in **Distortion Theorems** COMMENTS (I).

REFERENCES [GI5, pp. 3–16]; *Ahlfors* [Ah73, pp. 82–9], *Conway* [C, vol. II, pp. 63–64], *Gelbaum & Olmsted* [GO, pp. 184–185], *Gong* [GS]; *Pommerenke* [271].

## Bienaymé-Čebišev Inequality

See: **Čebišev Probability Inequality** COMMENTS (I).

## Bihari-Bellman Inequalities

See: **Bellman-Bihari Inequalities**.

## Bilinear Form Inequalities of M. Riesz

*Let $p, q \geq 1$, $p', q'$ the conjugate indices of $p, q$ respectively. If $\underline{a} = \{a_i,\ i \in \mathbb{N}^*\}$, $\underline{b} = \{a_j,\ j \in \mathbb{N}^*\}$, $\underline{c} = \{c_{ij}, i, j \in \mathbb{N}^*\}$ are non-negative, and $\underline{\alpha} = \{\sum_{i\in\mathbb{N}^*} c_{ij}a_i, j \in \mathbb{N}^*\}$, $\underline{\beta} = \{\sum_{j\in\mathbb{N}^*} c_{ij}b_j, i \in \mathbb{N}^*\}$ then the following three statements are equivalent:*

$$\begin{aligned} \sum_{i,j\in\mathbb{N}^*} c_{ij}a_ib_j &\leq C||\underline{a}||_p||\underline{b}||_q, \textit{ for all } \underline{a}, \underline{b};\\ ||\underline{\alpha}||_{q'} &\leq C||\underline{a}||_p, \textit{ for all } \underline{a};\\ ||\underline{\beta}||_{q'} &\leq C||\underline{b}||_p, \textit{ for all } \underline{b}. \end{aligned}$$

COMMENTS (i) These equivalencies can be stated in terms of strict inequalities, and they also hold for finite sequences; see [HLP].

(ii) See also: **Aczél & Varga Inequality, Grothendieck's Inequality, Hardy-Littlewood-Pólya-Schur Inequalities, Hilbert's Inequalities, Multilinear Form Inequalities, and Quadratic Form Inequalities**.

REFERENCE [HLP, pp. 204–225].

## Binomial Coefficient Inequalities

*(a) If $n > 2$ then*

$$\frac{2^{2n}}{n+1} < \binom{2n}{n} < \frac{(2n+2)^n}{(n+1)!}\,.$$

*(b) If $n > 1$*

$$\binom{2n}{n} > \frac{4^n}{2\sqrt{n}}\,.$$

*(c)* [ÅSLUND] *If $x > n$ and $y = (1+1/n)^n$ then*

$$\binom{x}{n} \leq \frac{x^x}{yn^n(x-n)^{x-n}}\,.$$

*(d) If $x^n + y^n = z^n$ then*

$$\binom{x}{n} + \binom{y}{n} < \binom{z}{n} < \binom{x}{n} + \binom{y}{n} + \binom{z-1}{n-1}.$$

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COMMENT Many other inequalities can be found in the first reference.

REFERENCES [AI, pp. 194–196], [H, p. 363].

**Binomial Function Inequalities** *(a) If* $-1 < u < v$ *then,*

$$(1+v)^{1/v} \quad < \quad (1+u)^{1/u}, \tag{1}$$

$$(1+v)^{1+\frac{1}{v}} \quad > \quad (1+u)^{1+\frac{1}{u}}. \tag{2}$$

*(b) If* $x > 0,\ 0 \le k < n$ *then:* $(1+x)^n > \sum_{i=0}^{k} \binom{n}{i} x^i$. (3)

*(c) If* $n > 1$ *and* $-1 < x < 1/(n-1), x \neq 0$*, then,*

$$(1+x)^n < 1 + \frac{nx}{1-(n-1)x}\,.$$

*(d)* [KARANICOLOFF] *If* $0 < q < p$*,* $0 < m < n$ *and* $0 < x < 1$ *then,*

$$(1-x^p)^m > (1-x^q)^n. \qquad (4).$$

*If* $0 < p < q$ *and* $0 < x < x_0$*,* $x_0$ *being the unique root of* $(1-x^p)^m = (1-x^q)^n$ *in* $]0,1[$*, then (∼4) also holds, while if* $x_0 < x < 1$ *(4) holds.*
*(e) If* $x + y = 1, 0 < x < 1$ *and if* $r, s > 1$ *then*

$$(1-x^r)^s + (1-y^s)^r > 1.$$

*(f)* [BENNETT] *If* $x \in \mathbb{R}$*,* $m, n \in \mathbb{N}$ *with* $m, n > x$ *then*

$$\left(1+\frac{x}{m}\right)^m \left(1-\frac{x}{n}\right)^n < 1.$$

*(g)* [MALEŠEVIĆ] *If* $1 \le k \le n$ *then*

$$\left(1+\frac{1}{n}\right)^k \le 1 + \frac{k}{n} + \frac{(k-1)^2}{n^2}\,,$$

*while if* $0 \le k \le n, n \ge 2$*,*

$$\left(1+\frac{1}{n}\right)^k \le \frac{n+1}{n+1-k}\,.$$

*(h) If* $x, y > 0, x \neq y$ *and either* $r > 1$ *or* $r < 0$ *then*

$$rx^{r-1}(x-y) > x^r - y^r > ry^{r-1}(x-y); \tag{5}$$

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*if* $0 < r < 1$ *then* $(\sim 5)$ *holds.*
*(j) If* $x, q > 0, q \in \mathbb{Q}$, *and if* $m = \min\{q, qx^{q-1}\}, M = \max\{q, qx^{q-1}\}$ *then*

$$\begin{aligned}
\mathfrak{H}_2(m,M) &< \frac{x^q-1}{x-1} < \mathfrak{G}_2(m,M \quad \text{if} \quad 0<q<\frac{1}{2};\\
\mathfrak{G}_2(m,M) &< \frac{x^q-1}{x-1} < \mathfrak{A}_2(m,M) \ \text{ if } q>2;\\
\mathfrak{A}_2(m,M) &< \frac{x^q-1}{x-1} < M \qquad\qquad \text{if } 1<q<2.
\end{aligned}$$

*(k)* [Madevski] *If* $x > 1, p \geq 1$ *then*

$$(x-1)^p \leq x^p - 1.$$

*(ℓ)* [Lyons] *If* $p \geq 1$ *then*

$$\Big(\frac{n}{p}\Big)!\sum_{i=1}^{n}\frac{x^{i/p}(1-x)^{(n-i)/p}}{\Big(\dfrac{i}{p}\Big)!\Big(\dfrac{n-i}{p}\Big)!} \leq p^2$$

Comments (i) (a) is an easy consequence of the strict concavity or convexity of the functions on the left-hand sides.

(ii) The proof of $(\ell)$ is quite complicated and it is conjectured that the right-hand side can be replaced by $p$. This conjecture was confirmed for certain values of $p$ by Love.

Special Cases *(a) If* $n = 1, 2, \ldots,$

$$\Big(1+\frac{1}{n+1}\Big)^{n+1} > \Big(1+\frac{1}{n}\Big)^{n}, \tag{6}$$

$$\Big(1+\frac{1}{n+1}\Big)^{n+2} < \Big(1+\frac{1}{n}\Big)^{n+1}, \tag{7}$$

$$\Big(1-\frac{1}{n}\Big)^{n} < \Big(1-\frac{1}{n+1}\Big)^{n+1}. \tag{8}$$

*(b) If* $x > 0, 0 < p < q$, *then*

$$\Big(1+\frac{x}{p}\Big)^{p} < \Big(1+\frac{x}{q}\Big)^{q}.$$

Comments (iii) Inequalities (6) and (7), particular cases of (1), (2) respectively can be obtained using (GA) and (HG); and (8) is an easy consequence of (6).

(iv) Inequalities (6), (7) are important in the theory of the exponential function; see **Exponential Function Inequalities** (1), (2).

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(v) Inequalities (6), (7) can be extended by inserting a variable $x$. Thus: $(1+x/n)^n$ increases, as a function of $n$, and $(1+x/n)^{n+1}$ decreases as a function of $n$. An approach to the first can be made by using **Chong's Inequalities** (2) and the fact that

$$\Big(1+\frac{x}{n},1+\frac{x}{n},\cdots,1+\frac{x}{n}\Big) \prec \Big(1+\frac{x}{n-1},1+\frac{x}{n-1},\cdots,1+\frac{x}{n-1},1\Big).$$

(vi) (b) is obtained from (1) by putting $v=x/p, u=x/q$, with $x>0, 0<p<q$. This inequality holds under the above conditions for $x$ satisfying $0>x>-p$. In addition the inequality is valid if either $0>q>p$ and $x<-q$, or if $q<0<p, -p<x<-q$.

(vii) See also: **Bernoulli's Inequality, Brown's Inequalities, Gerber's Inequality, Kacmarz & Steinhaus's Inequalities, Leindler's Inequality, Polynomial Inequalities, Series Inequalities** (d).

References [AI, pp. 34–35, 278, 280, 356, 365, 384], [H, pp. 8, 24], [HLP, pp. 39–42, 102–103], [MPF, pp. 68, 95]; *Apostol, Mugler, Scott, Sterrett & Watkins* [A92, p. 444]); *Melzak* [M, pp. 64–65]; *Liu Z.* [176]; *Love* [186], *Malešević* [192], *Savov* [294].

**Biplanar Mean Inequalities** See: **Gini-Dresher Mean Inequalities** Comments (i).

**Blaschke-Santaló Inequality** *If $K \subset \mathbb{R}^n$ is convex, compact with $\overset{\circ}{K} \neq \emptyset$, and centroid the origin, then*

$$|K|\,|K^*| \le v_n^2,$$

*with equality if and only if $K$ is the unit ball.*

Comments (i) For a definition of $v_n$, the volume of the unit ball, see **Notations** 4(b).

(ii) $K^*$ is the *polar of $K$*, that is

$$K^* = \{\underline{a};\ \underline{a}.\underline{b} \le 1,\ \underline{b} \in K\}.$$

(ii) The cases $n=2,3$ are due to Blaschke; the other cases were given, much later, by Santaló.

(iii) For an inverse inequality see **Mahler's Inequalities**.

References [EM, Supp., pp. 129–130].

**Bloch's Constant** *If $f$ is analytic in the $D$ with $|f'(0)|=1$ and $B_f$ is the radius of the largest open disk contained in a sheet of the Riemann surface of $f$ and if $B=\inf B_f$, where the* inf *is taken over all such $f$, then*

$$\frac{\sqrt{3}}{4} \le B \le 0.472, \qquad \textit{and} \qquad B \le L,$$

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*where $L$ is Landau's constant.*

COMMENTS (i) $B$ is called *Bloch's constant.*
(ii) See also: **Landau's Constant**.

REFERENCES [EM, vol. 1, p. 406]; *Ahlfors* [Ah73, pp. 14–15], *Conway* [C, vol. I, pp. 297–298].

**Block Type Inequalities** *If $f \in \mathcal{C}^1(\mathbb{R})$, not identically zero, with $f, f' \in \mathcal{L}^2(\mathbb{R})$ then*

$$f(x) < \left(\int_{\mathbb{R}} f'^2\right)^{1/4} \left(\int_{\mathbb{R}} f^2\right)^{1/4}, \quad x \in \mathbb{R}.$$

COMMENT This is a typical example of a class of inequalities introduced by Block. It gives a bound on the function in terms of its integral and the integral of its derivative.

REFERENCES [I2, pp. 262, 282–283].

**Boas's Inequality** *If $f : [-\pi, \pi] \to \mathbb{R}$; $\tilde{f}(x) = f(x)\mathrm{sign}(x), -\pi \le x \le \pi$; and if $f, \tilde{f}$ have absolutely convergent Fourier series with coefficients $a_n, n \in \mathbb{N}, b_n, n \ge 1$ and $\tilde{a}_n, n \in \mathbb{N}, \tilde{b}_n, n \ge 1$, respectively; then,*

$$\int_{-\pi}^{\pi} \frac{|f(x)|}{x}\,\mathrm{d}x \le \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(|a_n| + |b_n|\big) + \frac{1}{2}\tilde{a}_0 + \sum_{n=1}^{\infty}\big(|\tilde{a}_n| + |\tilde{b}_n|\big).$$

COMMENTS (i) This has been extended to higher dimensions by Zadereĭ.
(ii) For another inequality by Boas see **Function Inequalities** (a).
(ii) See **Wirtinger's Inequality** EXTENSIONS (b) for another inequality involving Fourier coefficients.

REFERENCES *Boas* [Bo]; *Zadereĭ* [332].

**Bobkov's Inequality** See: **Gaussian Measure Inequalities** (b).

**Bohr-Favard Inequality** *If $f$ is a function of period $2\pi$ with absolutely continuous $(r-1)$-st derivative, then for each $n$ there is a trigonometric polynomial $T_n$ of order $n$ such that*

$$||f - T_n||_{\infty,[0,2\pi]} \le C_{n,r}||f^{(r)}||_{\infty,[0,2\pi]},$$

*where*

$$C_{n,r} = \frac{4}{\pi(n+1)^r}\sum_{k\in\mathbb{N}}\frac{(-1)^{k(r-1)}}{(2k+1)^{r+1}}.$$

COMMENT This result is best possible; the case $r = 1$ is due to Bohr. It is part of a large theory on trigonometric approximation.

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References [EM, vol. 1, p. 415, vol.3, p. 480]; [*Zygmund*], vol. I, p. 377.

**Bohr's Inequality** *If $z_1, z_2 \in \mathbb{C}$ and if $c > 0$ then*

$$|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + \frac{1}{c})|z_2|^2,$$

*with equality if and only if $z_2 = cz_1$.*

Comment This is a deduction from (C).

Extensions *(a)* [Archbold] *If $z_i \in \mathbb{C}, 1 \le i \le n$, and if $\underline{w}$ is a positive $n$-tuple such that $W_n = 1$ then*

$$\left|\sum_{i=1}^{n} z_i\right|^2 \le \sum_{i=1}^{n} \frac{|z_i|^2}{w_i}.$$

*(b)* [Vasić & Kečkić] *If $z_i \in \mathbb{C}, 1 \le i \le n$, $\underline{w}$ a positive $n$-tuple and $r > 1$ then*

$$\left|\sum_{i=1}^{n} z_i\right|^r \le \left(\sum_{i=1}^{n} w_i^{1/(r-1)}\right)^{r-1} \sum_{i=1}^{n} \frac{|z_i|^r}{w_i},$$

*with equality if and only if $|z_1|^{r-1}/w_1 = \cdots = |z_n|^{r-1}/w_n$, and $z_k\overline{z}_j \ge 0$, $k, j = 1, \ldots, n$*

References [AI, pp. 312–315, 338–339], [HLP, p. 61], [MPF, pp. 499–505], [PPT, p. 131].

**Bonferroni's Inequalities** *Let $A_i, 1 \le i \le n$, be events in the probability space $(\Omega, \mathcal{A}, P)$; denote by $S_j$ the probability that $j, j \ge 1$, events occur simultaneously, and put $S_0 = 0$; and if $0 \le m \le n$, let $P_m$ be the probability that at least $m$ events occur, and $P_m^e$ the probability that exactly $m$ events occur. Then for $p, q$ satisfying $0 \le 2p + 1 \le n - m, 0 \le 2q \le n - m$*

$$\sum_{j=0}^{2p+1} (-1)^j \binom{m+j-1}{j} S_{m+j} \quad \le \quad P_m \le \sum_{j=0}^{2q} (-1)^j \binom{m+j-1}{j} S_{m+j},$$
$$\sum_{j=0}^{2p+1} (-1)^j \binom{m+j}{j} S_{m+j} \quad \le \quad P_m^e \le \sum_{j=0}^{2q} (-1)^j \binom{m+j}{j} S_{m+j}.$$

Comment Clearly $S_j = \sum_{1 \le i_1 < i_2 \cdots < i_n \le n} P\big(A_{i_1} \cap \ldots \cap A_{i_j}\big)$.

References [EM, Supp. pp. 142–143]; *Feller* [F, vol. I, pp. 88–101], *Galambos & Simonelli* [GaS].

**Bonnesen's Inequality** *If $A$ is the area of a convex domain in $\mathbb{R}^2$, with $L$ the length of its boundary then,*

$$L^2 - 4\pi A \ge \pi^2 (R - r)^2,$$

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*where $r$ is the inner radius of the domain, and $R$ is the outer radius of the domain. The left-hand side is always positive except when the domain is a disk, and then $R = r$.*

COMMENTS (i) Definitions of inner and outer radius can be found in **Isodiametric Inequality** COMMENTS (ii).
(ii) See also: **Gale's Inequality**.

REFERENCES [EM, vol. 1, p. 420].

**Borel-Carathéodory Inequality** *If $f$ is analytic on $\{z; |z| \le R\}$ and if $A(r) = \max_{|z|=r} \Re f(z)$, then for $r < R$*

$$M_\infty(r, f) \le \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|. \tag{1}$$

*In particular if $A(R) \ge 0$ then*

$$M_\infty(r, f) \le \frac{R+r}{R-r} \big(A(R) + |f(0)|\big).$$

EXTENSION *Under the above conditions and with $A(R) \ge 0$, and $n \ge 1$,*

$$M_\infty\big(r, f^{(n)}\big) \le \frac{2^{n+2} n! R}{(R-r)^{n+1}} \Big(A(R) + \big|f(0)\big|\Big).$$

REFERENCES *Boas* [Bo, p. 135], *Conway* [C, p. 129], *Titchmarsh* [T86, pp. 174–176].

**Bounded Variation Function Inequalities** *(a) If $f, g$ are of bounded variation on $[a, b]$ then so is $f + g$ and*

$$V(f+g; a, b) \le V(f; a, b) + V(g; a, b).$$

*(b) If in addition $f(a) = g(a) = 0$ then*

$$V(fg; a, b) \le V(f; a, b) V(g; a, b).$$

*(c) If on $[a, b]$, $f$ is continuous, $g$ of bounded variation, or $f, g$ are both of bounded variation and $g$ is continuous, then*

$$\left| \int_a^b f \,\mathrm{d}g \right| \le \int_a^b |f(x)| \,\mathrm{d}V(g; a, x) \le \sup_{a \le x \le b} |f(x)|\, V(g; a, b).$$

*(d) If on $[a, b]$, $f$ is continuous, $g$ of bounded variation and if $F(x) = \int_a^x f \,\mathrm{d}g$, $a \le x \le b$ then*

$$V(F; a, x) \le \int_a^x |f|\, |\mathrm{d}g|,\ a \le x \le b.$$

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*(e) If $f$ is of bounded variation on* $[0,1]$ *then*

$$\left|\int_0^1 f - \frac{1}{n}\sum_{i=1}^n f\left(\frac{i}{n}\right)\right| \le \frac{V(f;0,1)}{n}, \quad n = 1, 2, \ldots.$$

COMMENTS (i) Results (a) and (b) have been extended by A. M. Russell to functions of bounded higher order variation. The integrals above are Riemann-Stieltjes integrals.

(ii) Of course under the hypotheses of (a) $fg$ is also of bounded variation but the inequality in (b) need not hold; we can only get

$$V(fg;a,b) \le ||f||_{\infty,[a,b]}V(g;a,b) + ||g||_{\infty,[a,b]}V(f;a,b).$$

(iii) (c) follows from **Integral Inequalities** EXTENSIONS (a).

EXTENSION [KARAMATA] *If $f$ is of bounded variation on $[a,b]$, $g$ bounded on $[a,b]$, with $f, g \in \mathcal{L}_\mu([a,b]), \mu$ bounded, then*

$$\begin{aligned}\left|\int_a^b fg\,\mathrm{d}\mu\right| &\le \Big(|f(b)| + V(f;a,b)\Big)\sup_{a\le x\le b}\left|\int_a^x g\,\mathrm{d}\mu\right|;\\ \left|\int_a^b fg\,\mathrm{d}\mu\right| &\le \Big(|f(a)| + V(f;a,b)\Big)\sup_{a\le x\le b}\left|\int_x^b g\,\mathrm{d}\mu\right|.\end{aligned}$$

COMMENT (iv) See also **Arc Length Inequality, Variation Inequalities**.

REFERENCES [MPF, p. 337]; *Pólya & Szegő* [PS, p. 49], *Rudin* [R76, pp. 118–119, 122], *Widder* [W, pp. 8–10]; *Bullen* [75], *Russell, A.* [284].

## Brascamp-Lieb-Luttinger Inequality See: **Rearrangement Inequalities** INTEGRAL ANALOGUES.

## Brown's Inequalities[16] *If $0 \le x \le 1$ and $s, t \ge 1$ then*

$$(1 + x + x^2) \ge (1 + x^s)^{1/s}(1 + x^t)^{1/t} \quad \text{if} \quad \frac{1}{s} + \frac{1}{t} = \frac{\log 3}{\log 2}, \quad \text{and} \quad s + t \le \frac{8}{3}.$$

COMMENT This inequality has applications in measure theory.

EXTENSIONS *(a) If $0 \le x \le 1$ and $s, t \ge 1$ then*

$$(1 + x + x^2) \ge (1 + x^s)^{1/s}(1 + x^t + x^{2t})^{1/t}, \quad \text{if} \quad \frac{1}{s}\frac{\log 3}{\log 2} + \frac{1}{t} = 1.$$

[16] This is Gavin Brown.

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*(b) If* $0 \le x \le 1$ *and* $s, t \ge 1$ *then*

$$(1+x+x^2+x^3) \ge \begin{cases} (1+x^s)^{1/s}(1+x^t+x^{2t}+x^{3t})^{1/t}, & \text{if } \dfrac{1}{2s}+\dfrac{1}{t}=1,\\[2ex] (1+x^s+x^{2s})^{1/s}(1+x^t+x^{2t})^{1/t}, & \text{if } \dfrac{1}{s}+\dfrac{1}{t}=\dfrac{1}{\log_4 3},\\[2ex] (1+x^s+x^{2s})^{1/s}(1+x^t+x^{2t}+x^{3t})^{1/t}, & \text{if } \dfrac{\log_4 3}{s}+\dfrac{1}{t}=1 \end{cases}$$

*(c)* [ALZER] *If* $0 \le x \le 1$ *and* $s, t \ge 1$ *then*

$$(1+x+x^2) \ge \left(1+\frac{tx^s+sx^t}{s+t}\right)^{\frac{1}{s}+\frac{1}{t}}, \quad \text{if} \quad \frac{1}{s}+\frac{1}{t}=\frac{\log 3}{\log 2} \quad \text{and} \quad s+t \le \frac{4}{3}+\frac{\log 4}{\log 3}.$$

REFERENCES *Alzer* [36], *Brown, G.* [72], *Kemp* [153].

**Brunn-Minkowski Inequalities** *(a) If* $A, B$ *are bounded measurable sets in* $\mathbb{R}^n$ *such that for* $0 < t < 1$ *the set* $(1-t)A + tB$ *is also measurable then:*

$$|(1-t)A+tB| \ge |A|^{1-t}|B|^t.$$

*(b) if* $A, B$ *are non-empty sets in* $\mathbb{R}^n$ *that satisfy the conditions in (a) then:*

$$|(1-t)A+tB|^{1/n} \ge (1-t)|A|^{1/n}+t|B|^{1/n}.$$

*(c) If* $A, B$ *are non-empty bounded measurable sets in* $\mathbb{R}^n$ *such that* $sA+tB$ *is measurable then:*

$$|sA+tB|^{1/n} \ge s|A|^{1/n}+t|B|^{1/n}.$$

COMMENTS (i) It should be noted that $sA+tB$ need not be measurable even if both $A$ and $B$ are; a classical result of Sierpinski.

(ii) Extensive generalizations and discussions of the cases of equality can be found in the references.

(iii) See also: **Gaussian Measure Inequalities, Mixed-volume Inequalities, Isoperimetric Inequalities,** COMMENTS (I), **Prékopa-Leindler Inequalities**.

REFERENCES [EM, vol. 1, p. 484; Supp., p. 81], [MPF, pp. 174–178]; *Gardner* [126].

**Brunn-Minkowski-Lusternik Inequality** See: **Brunn-Minkowski Inequality**(B).

**Burkholder-Davis-Gundy Inequality**[17] See: **Martingale Inequalities** COMMENTS (III).

[17] This is Burgess Davis.

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**Burkill's Inequality**[18] See: **Hlawka-Type Inequalities** EXTENSIONS.

**Bushell-Okrasiński's Inequality** *If* $0 < b \leq 1, p \geq 1$ *and if* $f \in \mathcal{C}([0,b])$ *is non-negative and increasing then*

$$\int_0^x (x-t)^{p-1} f^p(t)\,\mathrm{d}t \leq \left(\int_0^x f\right)^p, \quad 0 \leq x \leq b$$

REFERENCES [GI6, pp. 495–496].

[18]This is J. C. Burkill.

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# 3 Čakalov–Cyclic

**Čakalov's Inequality** *If $\underline{a}$ is an increasing positive non-constant n-tuple, $\underline{w}$ a positive n-tuple, $n > 2$, then*

$$\lambda_n\big\{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\big\} \ge \lambda_{n-1}\big\{\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})\big\},$$

*where*

$$\lambda_n = \frac{W_n^2}{(W_{n-1} - w_1)}.$$

Comments (i) The proof, due to Čakalov, is the same as that of the more general result in the reference.

(ii) Since it is easily seen that $\tilde{\lambda}_n = \lambda_{n-1}/\lambda_n > W_{n-1}/W_n$, this inequality generalizes **Rado's Geometric-Arithmetic Mean Inequality Extension** for this class of sequences; there is no analogous extension of **Popoviciu's Geometric-Arithmetic Mean Inequality Extension**.

(iii) In general $\tilde{\lambda}_n$ is an unattained lower bound, unlike the lower bound $W_n/W_{n-1}$ in Rado's geometric-arithmetic mean inequality extension. However $\tilde{\lambda}_n$ is best possible in that for any $\tilde{\lambda}_n' > \tilde{\lambda}_n, n = 1, 2, \ldots$ there are sequences $\underline{a}$ for which the inequality would fail.

(iv) The geometric mean can be replaced by a large class of quasi-arithmetic $\mathfrak{M}$-means; those for which $M^{-1}$ is increasing, convex, and 3- convex; for the definition of these terms see **Quasi-arithmetic Mean Inequalities, n-Convex Sequence Inequalities**.

Reference [H, pp. 286–288].

**Capacity Inequalities** *(a)* [Poincaré] *If $V$ is the volume of a domain in $\mathbb{R}^3$ and $C$ its capacity then*

$$C^3 \ge \frac{3V}{4\pi},$$

*with equality only when the domain is spherical.*
*If $A$ is the area of a domain in $\mathbb{R}^2$ and $C$ its capacity then*

$$C^2 \ge \frac{4A}{\pi^3},$$



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*with equality only when the domain is spherical.*
*(b) If* $\Omega_1, \Omega_2$ *are two domains in* $\mathbb{R}^3$ *and* $C$ *is a capacity, then*

$$C(\Omega_1 \cup \Omega_2) + C(\Omega_1 \cap \Omega_2) \leq C(\Omega_1) + C(\Omega_2); \tag{1}$$

*and if* $\Omega_1 \subseteq \Omega_2$ *then* $C(\Omega_1) \leq C(\Omega_2)$.

COMMENTS (i) In $\mathbb{R}^3$ *capacity* can be defined by

$$C = C(\partial\Omega) = \frac{1}{4\pi} \inf \int_{\mathbb{R}^3 \setminus \Omega} |\nabla f|^2,$$

where the inf is over all $f$ such that $f(\underline{x}) = 1, \underline{x} \in \partial\Omega$, and $f(\underline{x}) \to 0$ as $|\underline{x}| \to \infty$.

An analogous definition can be given in $\mathbb{R}^n, n > 3$. The case $n = 2$ is a little different. In addition the whole theory of capacity can be given in a very abstract setting.

(ii) The set function property in (1) is called *strong sub-additivity*; it is a property of abstract capacities.

(iii) The inequalities in (a) are examples of **Symmetrization Inequalities**.

(iv) See also: **Logarithmic Capacity Inequalities**.

REFERENCES [EM, vol. 2, pp. 14–17]; *Conway* [C, vol. 2, pp. 331–336], *Pólya & Szegö* [PS51, pp. 1, 8–13, 42–44], *Protter & Weinberger* [PW, pp. 121–128].

**Cardinal Number Inequalities** *(a) If* $\alpha$ *is any infinite cardinal then*

$$\alpha \geq \aleph_0.$$

*(b) If* $\alpha$ *is any cardinal number then*

$$2^\alpha > \alpha.$$

*(c) If* $\alpha, \beta$ *are two ordinal numbers then*

$$\alpha < \beta \implies \aleph_\alpha < \aleph_\beta.$$

COMMENT The *generalized continuum hypothesis* states that for all ordinals $\alpha$, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$; in particular the case $\alpha = 0$, that is $2^{\aleph_0} = \aleph_1$, is the *continuum hypothesis*.

REFERENCES [EM, vol. 1, p. 69; vol. 3, pp. 23–24, 390–391]; *Hewitt & Stromberg* [HS, pp. 22–24].

**Carleman's Inequality** *If* $\underline{a}$ *is a non-null convergent sequence,*

$$\sum_{i=1}^{\infty} \mathfrak{G}_i(\underline{a}) < e \sum_{i=1}^{\infty} a_i. \tag{1}$$

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*The constant is best possible.*

EXTENSIONS *(a) If $0 < p < 1$, and $\underline{a}$ is a non-negative non-null sequence*

$$\sum_{i=1}^{\infty} \mathfrak{M}_i^{[p]}(\underline{a}) < \left(\frac{1}{1-p}\right)^{1/p} \sum_{i=1}^{\infty} a_i. \tag{2}$$

*(b)* [REDHEFFER] *If $\underline{a}, \underline{b}$ are non-negative $n$-tuples then*

$$\sum_{i=1}^{n} i(b_i - 1)\mathfrak{G}_i(\underline{a}) + n\mathfrak{G}_n(\underline{a}) \le \sum_{i=1}^{n} a_i b_i^i .$$

*There is equality if and only if $a_i b_i^i = \mathfrak{G}_{i-1}(\underline{a}), 2 \le i \le n$.*
*(c)* [ALZER] *If $\underline{a}, \underline{w}$ are positive sequences then*

$$\sum_{i=1}^{\infty} w_i \mathfrak{G}_i(\underline{a};\underline{w}) + \frac{1}{2}\sum_{i=1}^{\infty} \frac{w_i^2}{W_i}\mathfrak{G}_i(\underline{a};\underline{w}) < e\sum_{i=1}^{\infty} w_i a_i .$$

*(d)* [LONG & LINH] *Inequality (2) is valid if $-1 \le p < 1,\ p \ne 0$, and if $p < -1$ if the constant on the right-hand side is replaced by $\dfrac{p}{p-1}2^{(p-1)/p}$.*

COMMENTS (i) Inequality (2) is just **Hardy's Inequality** (1) with a change of notation. Letting $p \to 0$ in this extension gives the weaker form of (1), with $<$ replaced by $\le$.

(ii) The finite case of inequality (1) follows from Redheffer's extension by taking $b_i = 1 + 1/i, i = 1, 2, \ldots$ and using **Exponential Function Inequalities** (1). This extension is one of Redheffer's **Recurrent Inequalities**.

(iii) Alzer's extension implies a weighted version of (1), due in a weaker form to Pólya.

(iv) There are other extensions of the finite sum case; see: **Redheffer's Inequalities** (1), (2), **Kaluza-Szegő Inequality, Nanjundiah's Mixed Mean Inequalities** COMMENTS (ii). The finite sum cases do not, of course, always imply the strict inequality in the series form.

INTEGRAL ANALOGUE [PÓLYA-KNOPP] *Unless $f \ge 0$ is zero almost everywhere*

$$\int_0^{\infty} \mathcal{G}_{[0,x]}(f)\,\mathrm{d}x = \int_0^{\infty} \exp\left(\frac{1}{x}\int_0^{x} \log\circ f(t)\,\mathrm{d}t\right)\mathrm{d}x < e\int_0^{\infty} f.$$

COMMENT (v) See also: **Heinig's Inequality** COMMENTS (iii).

REFERENCES [AI, p. 131], [EM, vol. 2, p. 25], [GI3, pp. 123–140], [H, pp. 140–141, 147, 289], [HLP, pp. 249–250], [PPT, pp. 231, 234]; *Bennett* [Be, pp. 39–40]; *Alzer* [19], *Bullen* [77], *Duncan & McGregor* [109], *Johansson, Persson & Wedestig* [149], *Long & Linh* [178].

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**Carleman's Integral Inequality** *If* $B = \{\underline{x};\, 0 < |\underline{x}| < 1\} \subset \mathbb{R}^2$, $f \in \mathcal{C}^2(B)$, $p \in \mathbb{R}$ *then*

$$\int_B |\underline{x}|^p |f(\underline{x})|^2 \,\mathrm{d}\underline{x} \le C \int_B |\underline{x}|^p |\nabla^2 f(\underline{x})|^2 \,\mathrm{d}\underline{x}.$$

Comment This has been extended to higher dimensions by Meškov.

Reference *Meškov* [209].

**Carlson's Inequalities**[19] See: **Mixed Mean Inequalities, Muirhead Symmetric Function and Mean Inequalities** (d)

**Carlson's Inequality**[20] *If* $\underline{a}$ *is a non-negative sequence that is not identically zero then*

$$\left(\sum_{i=1}^{\infty} a_i\right)^4 < \pi^2 \left(\sum_{i=1}^{\infty} a_i^2\right)\left(\sum_{i=1}^{\infty} i^2 a^2\right).$$

*The constant* $\pi^2$ *is best possible.*

Integral Analogues *If* $f > 0$ *and* $xf(x) \in \mathcal{L}^2([0,\infty[)$ *then*

$$\left(\int_0^\infty f\right)^4 \le \pi^2 \left(\int_0^\infty f^2\right)\left(\int_0^\infty x^2 f^2(x)\,\mathrm{d}x\right).$$

*The constant* $\pi^2$ *is best possible.*

Extension *If* $f : [0,\infty[ \to ]0,\infty[$ *and if* $p, q, \lambda, \mu$ *are positive real numbers with* $\lambda < p+1, \mu < q+1$ *then*

$$\left(\int_O^\infty f\right)^{\mu(p+1)+\lambda(q+1)} \le C_{p,q,\lambda,\mu}\left(\int_0^\infty x^{p-\lambda} f^{p+1}(x)\,\mathrm{d}x\right)^{\mu}\left(\int_0^\infty x^{q+\mu} f^{q+1}(x)\,\mathrm{d}x\right)^{\lambda}.$$

Comments (i) The best value of the constant has been found by Levin & Stečkin.

(ii) Many other extensions are given in the first two references.

References [AI, pp. 370–372], [BB, pp. 175–177], [EM, vol. 2, p. 27]; *Larsson, Maligranda, Pečarić & Persson* [LMP]; *Bărză* [52].

[19] This is B. C. Carlson.
[20] This is F. D. Carlson.

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**Cauchy-Hadamard Inequality** *(a) If $f$ is analytic in $\{z; |z-z_0| \le r\}$ and if $M(r)$ is the maximum of $|f(z)|$ on the circle $|z-z_0| = r$ then*

$$|f^{(k)}(z_0)| \le k!\frac{M(r)}{r^k},\ k \in \mathbb{N}. \tag{1}$$

*(b) If $f$ is analytic in the domain $\Omega$ then*

$$\limsup_{k\to\infty}\left(\frac{f^{(k)}(z)}{k!}\right)^{1/k} \le \frac{1}{\rho(z,\partial\Omega)},$$

*where $\rho(z,\partial\Omega)$ denotes the distance of $z$ from the boundary of $\Omega$.*

COMMENTS (i) The first inequality is known as *Cauchy's Inequality or Estimate.*

(ii) Another form of (1) is

$$|c_k| \le \frac{M(r)}{r^k},\ k \in \mathbb{N},$$

where $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$.

REFERENCES [EM, vol.2, p. 62]; *Ahlfors* [Ah78, p. 122], *Conway* [C, vol. I, p. 73], *Levin & Stečkin* [LS], *Rudin* [R87, pp. 213–214], *Titchmarsh* [T75, pp. 84–85].

**Cauchy's Inequality** If $\underline{a}, \underline{b}$ are positive $n$-tuples then

$$\sum_{i=1}^{n} a_ib_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}, \tag{C}$$

with equality if and only if $\underline{a} \sim \underline{b}$.

COMMENTS (i) This is a special case of (H), to which it is equivalent. However it can be proved independently as any book on linear algebra will show. In particular (C) is implied by either the identity

$$\sum_{i=1}^{n}(a_ix+b_i)^2 = x^2\big(\sum_{i=1}^{n} a_i^2\big) + 2x\sum_{i=1}^{n} a_ib_i + \sum_{i=1}^{n} b_i^2,$$

or by the *Lagrange Identity*,

$$\big(\sum_{i=1}^{n} a_i^2\big)\big(\sum_{i=1}^{n} b_i^2\big) - \big(\sum_{i=1}^{n} a_ib_i\big)^2 = \sum_{i,j=1}^{n}(a_ib_j - a_jb_i)^2.$$

(ii) (C) can be written

$$\underline{a}.\underline{b} \le |\underline{a}||\underline{b}|.$$

(iii) In proving (C) there would be no loss of generality in assuming that $|\underline{a}| = |\underline{b}| = 1$.

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(iv) (C) is the case $m = 1$ of **Bessel's Inequality** (3).

Extensions *(a) If $\underline{a}, \underline{b}$ are complex $n$-tuples then*

$$\left|\sum_{i=1}^{n} a_i\overline{b}_i\right| \le \sum_{i=1}^{n} |a_i||b_i| \le \big(\sum_{i=1}^{n} |a_i|^2\big)^{1/2}\big(\sum_{i=1}^{n} |b_i|^2\big)^{1/2}, \tag{1}$$

*with equality if and only if $\underline{a} \sim \overline{\underline{b}}$.*
*(b) If $\underline{a}, \underline{b}$ are real $n$-tuples, $\underline{w}$ a positive $n$-tuple then*

$$\sum_{i=1}^{n} w_i|a_i||b_i| \le \big(\sum_{i=1}^{n} w_i a_i^2\big)^{1/2}\big(\sum_{i=1}^{n} w_i b_i^2\big)^{1/2}. \tag{2}$$

*(c)* [Wagner] *If $\underline{a}, \underline{b}$ are positive $n$-tuples and $0 \le x \le 1$, then*

$$\left(\sum_{i=1}^{n} a_i b_i \quad + \quad \big(x \sum_{i\ne j; i,j=1}^{n} a_i b_j\big)\right)^2 \\ \le \left(\sum_{i=1}^{n} a_i^2 + \big(2x \sum_{i<j; i,j=1}^{n} a_i a_j\big)\right)\left(\sum_{i=1}^{n} b_i^2 + \big(2x \sum_{i<j; i,j=1}^{n} b_i b_j\big)\right).$$

*(d)* [Klamkin] *If $\underline{a}, \underline{b}$ are positive $n$-tuples,*

$$\begin{aligned}\left(\sum_{i=1}^{n} a_i b_i\right)^2 \quad &\le \quad \left(\sum_{i=1}^{n} a_i^{2-\frac{n-1}{n}} b_i^{\frac{n-1}{n}}\right)\left(\sum_{i=1}^{n} a_i^{\frac{n-1}{n}} b_i^{2-\frac{n-1}{n}}\right) \\ &\le \quad \cdots\cdots \le \left(\sum_{i=1}^{n} a_i^{2-\frac{k}{n}} b_i^{\frac{k}{n}}\right)\left(\sum_{i=1}^{n} a_i^{\frac{k}{n}} b_i^{2-\frac{k}{n}}\right) \\ &\le \quad \cdots \le \left(\sum_{i=1}^{n} a_i^{2-\frac{1}{n}} b_i^{\frac{1}{n}}\right)\left(\sum_{i=1}^{n} a_i^{\frac{1}{n}} b_i^{2-\frac{1}{n}}\right) \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).\end{aligned}$$

*(e)* [Callebaut] *If either $1 \le z \le y \le 2$ or $0 \le y \le z \le 1$, and if $\underline{a}, \underline{b}$ are positive $n$-tuples then*

$$\left(\sum_{i=1}^{n} a_i^z b_i^{2-z}\right)\left(\sum_{i=1}^{n} a_i^{2-z} b_i^{z}\right) \le \left(\sum_{i=1}^{n} a_i^y b_i^{2-y}\right)\left(\sum_{i=1}^{n} a_i^{2-y} b_i^{y}\right). \tag{3}$$

*(f)* [McLaughlin] *If $\underline{a}, \underline{b}$ are real $2n$-tuples then*

$$\big(\sum_{i=1}^{2n} a_i b_i\big)^2 \le \big(\sum_{i=1}^{2n} a_i^2\big)\big(\sum_{i=1}^{2n} b_i^2\big) - \left(\sum_{i=1}^{n} a_{2i}b_{2i-1} - a_{2i-1}b_{2i}\right)^2.$$

*(g)* [Dragomir] *If $\underline{a}, \underline{b}, \underline{c}$ are real $n$-tuples with $|\underline{c}| = 1$ then*

$$|\underline{a}.\underline{b}| \le |\underline{a}.\underline{b} - (\underline{a}.\underline{c})(\underline{c}.\underline{b})| + |(\underline{a}.\underline{c})(\underline{c}.\underline{b})| \le |\underline{a}|\,|\underline{b}|.$$

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*(h)* [Alzer] *If $\underline{a}, \underline{b}$ are decreasing positive $n$-tuples then*

$$\sum_{i=1}^{n} a_i b_i \le \min\left\{ \sum_{i=1}^{n} a_i,\ \sum_{i=1}^{n} b_i,\ \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2} \right\}.$$

*(j)* [Ryser] *If $\underline{a}, \underline{b}$ are real $n$-tuples and if $\underline{m} = \min\{\underline{a}, \underline{b}\}$, $\underline{M} = \max\{\underline{a}, \underline{b}\}$ then*

$$|\underline{a}.\underline{b}| \le |\underline{M}|\,|\underline{m}| \le |\underline{a}|\,|\underline{b}|.$$

Comments (v) It is easy to see that (2) and (C) are the same.

(vi) The inequality of Callebaut, (3) above, interpolates (C) in the sense that if $z = 1, y = 2$, or $y = 0, z = 1$ it reduces to (C). It says that if $f(z)$ is the left-hand side of (2) then $f$ is decreasing on $[0, 1]$, and increasing on $[1, 2]$, and takes as value the left-hand side of (C) when $z = 1$, and the right-hand side of (C) if $z = 0$ or 2. For a similar interpolation of (GA) see **Chong's Inequalities** (c).

Integral Analogue *If $f, g \in \mathcal{L}^2([a, b])$ then $fg \in \mathcal{L}([a, b])$ and*

$$||fg|| \le ||f||_2||g||_2.$$

*There is equality if and only if $Af = Bg$ almost everywhere, where not both of the constants $A, B$ are zero.*

Inverse Integral Analogues [Bellman] *If $f, g$ are non-negative and concave on $[0, 1]$ then*

$$\int_0^1 fg \ge \frac{1}{2}\left(\int_0^1 f^2\right)^{1/2}\left(\int_0^1 g^2\right)^{1/2}.$$

Comments (vii) There are many converse inequalities for (C) that are discussed elswhere; see for instance **Barnes's Inequalities** (1), **Pólya & Szegő's Inequality, Zagier's Inequality**.

(viii) The result of Bellman has been generalized by Alzer.

(ix) For another inequality also known as Cauchy's inequality see: **Cauchy-Hadamard Inequality** (1).

(x) See also: **Aczél's Inequality** Comments (i), **Complex Number Inequalities,** Extensions, **Determinant Inequalities** (d), **Gram Determinant Inequalities** (3), **Inner Product Inequalities** (2), **Milne's Inequality, Ostrowski's Inequalities** (a), Extensions (a), **Quaternion Inequalities, Seitz's Inequality, Trace Inequalities** (b).

References [AI, pp. 30–32, 41–44], [EM, vol. 1, p. 485, vol. 2, p. 6], [H, pp. 183–185, 196, 200], [HLP, pp. 16,132–134], [MPF, pp. 488–491], [PPT, p. 118]; *Ahlfors* [Ah78, pp. 10–11], *Dragomir* [D04], *Herman, Kučera & Šimša* [HKS, pp. 127–134], *Hewitt & Stromberg* [HS, p. 190], *Pólya & Szegö* [PS, p. 68]; *Alzer* [12], *Dragomir* [102, 103].

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**Cauchy-Schwarz-Bunyakovskiĭ Inequality** See: **Cauchy's Inequality**.

**Cauchy-Schwarz Inequality** See: **Cauchy's Inequality.**

**Cauchy Transform Inequality** [AHLFORS & BEURLING] *If $K \subset \mathbb{C}$ is compact with $|K| > 0$ then*

$$\left| \int_K \frac{1}{z-w}\,\mathrm{d}w \right| \le \sqrt{\pi|K|}.$$

COMMENT The integral is with respect to Lebesgue measure in the plane. It is a special case of the *Cauchy transform of a measure*; in general the Lebesgue measure on $K$ is replaced by any measure on $\mathbb{C}$ having compact support.

REFERENCE [*Conway,* vol. II, pp. 192–196].

**Čebišev's Inequality** *If $\underline{a}, \underline{b}$ are similarly ordered $n$-tuples then*

$$\mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w}) \le \mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}), \tag{$\hat{C}$}$$

*with equality if and only if $\underline{a}$, or $\underline{b}$ is constant.*

COMMENTS (i) This famous inequality has been given many proofs; the most elementary depends on the identity:

$$n\sum_{i=1}^n a_ib_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i = \frac{1}{2}\sum_{i,j=1}^n (a_i-a_j)(b_i-b_j). \tag{1}$$

(ii) The relation $\mathfrak{A}_n(\underline{a}) + \mathfrak{A}_n(\underline{b}) = \mathfrak{A}_n(\underline{a}+\underline{b})$ is trivial and (Č) is the non-trivial multiplicative analogue of this. Similarly $\mathfrak{G}_n(\underline{a})\mathfrak{G}_n(\underline{b}) = \mathfrak{G}_n(\underline{a}\,\underline{b})$, the case $r = 0$ of (2) below, is trivial; its additive analogue is given by **Power Mean Inequalities** (4), COMMENTS (iv).

(iii) While the conditions given are sufficient for (Č) they are not necessary; a set of necessary and sufficient conditions has been given by Sasser & Slater.

EXTENSIONS *(a) If $0 < r < \infty$ and $\underline{a}, \underline{b}$ are similarly ordered positive $n$-tuples then*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\mathfrak{M}_n^{[r]}(\underline{b};\underline{w}) \le \mathfrak{M}_n^{[r]}(\underline{a}\,\underline{b};\underline{w}), \tag{2}$$

*with equality if and only if either $r = 0$, or $r \ne 0$ and $\underline{a}$ or $\underline{b}$ is constant.*
*If $-\infty < r < 0$ and $\underline{a}, \underline{b}$ are oppositely ordered then (2) also holds.*
*If $0 < r < \infty, (-\infty < r < 0)$, and $\underline{a}, \underline{b}$ are oppositely, (similarly) ordered then* $(\sim 2)$ *holds.*
*(b) If the $k$ non-negative $n$-tuples, $\underline{a}_i, 1 \le i \le k$, are similarly ordered*

$$\prod_{i=1}^k \mathfrak{A}_n(\underline{a}_i, \underline{w}) \le \mathfrak{A}\left(\prod_{i=1}^k \underline{a}_i; \underline{w}\right).$$

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*(c)* [Alzer] *If $\underline{a}, \underline{w}$ are both strictly increasing then*

$$\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}) - \mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w}) \ge K(\underline{w}) \min_{1\le i,j\le n-1} \{\Delta a_i, \Delta b_i\},$$

*where*

$$K(\underline{w}) = \frac{1}{W_n}\sum_{i=1}^n i^2 w_i - \left(\frac{1}{W_n}\sum_{i=1}^n i w_i\right)^2.$$

*(d)* [Vasić & Đorđević] *If $\underline{a}, \underline{b}$ are non-negative increasing convex $n$-tuples and if $\underline{n} = \{0, 1, \ldots, n-1\}$ then*

$$\mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{b};\underline{w}) \le \frac{\mathfrak{A}_n(\underline{n};\underline{w})^2}{\mathfrak{A}_n(\underline{n}^2,\underline{w})}\mathfrak{A}_n(\underline{a}\,\underline{b};\underline{w}).$$

*(e)* [Vasić & Pečarić] *Under the hypothese of (b),*

$$0 \le \mathfrak{A}_2\left(\prod_{i=1}^k \underline{a}_i;\underline{w}\right) - \prod_{i=1}^k \mathfrak{A}_2(\underline{a}_i;\underline{w}) \le \cdots \le \mathfrak{A}_n\left(\prod_{i=1}^k \underline{a}_i;\underline{w}\right) - \prod_{i=1}^k \mathfrak{A}_2(\underline{a}_i;\underline{w}).$$

Comments (iv) For an extension of (b) see **Mean Monotonic Sequence Inequalities**.

(v) Extension (d) of (Č) is an example of various results using $n$-convex sequences.

(vi) Better lower bounds for the difference between the right-hand side and left-hand side of (Č) have been obtained by Alzer.

(vii) Further extensions can be found in Pečarić, and extension to functions of index sets have been given by Vasić & Pečarić; see also [H], **Seitz's Inequality**.

Integral Analogue *If $f, g \in \mathcal{L}([a,b])$ and are both monotonic in the same sense then*

$$\int_a^b f \int_a^b g \le (b-a)\int_a^b fg,$$

*with equality if and only if one of the functions is constant almost everywhere.*

Comments (viii) There are extensions of this last result to weighted integral means and to products of more than two functions; see: [AI]. In addition the concept of similarly ordered functions can be used; see: [HLP].

(ix) A proof of the integral analogue using an identity similar to (1) has been given by Andreev; see: [MPF, p. 243, (7.1)]. This allows the conditions on the functions to be weakened to their being *synchronous*; that is $\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \ge 0$ almost everywhere. This condition has been further weakened by Niculescu & Rovenţa to requiring that the averages of the two functions be synchronous.

(x) A detailed review of the history and development of Čebišev's inequality has been given by Mitrinović & Vasić; see also [MPF].

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(xi) For inverse inequalities see: **Grüsses' Inequalities** (A), **Karamata's Inequalities, Ostrowski's Inequalities** (B).

References [AI, pp. 36–41], [EM, vol.2, p. 119], [H, pp. 161–165, 215–216], [HLP, pp. 43–44, 168], [MPF, pp. 239–293, 351–358], [PPT, pp. 197–228]; *Herman, Kućera & Šimša* [HKS, pp. 145–151]; *Alzer* [15], *Niculescu & Rovenţa* [235].

**Čebišev Polynomial Inequalities** The *Čebišev polynomial of degree* $n$ is defined by

$$T_n(x) = \cos(n \arccos x), \qquad n \geq 1, \quad -1 \leq x \leq 1.$$

The *monic Čebišev polynomial of degree* $n$ is $\tilde{T}_n = 2^{1-n}T_n$. *If* $p_n$ *is a monic polynomial of degree* $n$ *that is not* $\tilde{T}_n$ *then*

$$||p_n||_{\infty,[-1,1]} > ||\tilde{T}_n||_{\infty,[-1,1]} = \frac{1}{2^{n-1}}.$$

Comments (i) A monic polynomial has the term of highest degree with coefficient 1.

(ii) This inequality says:

*of all monic polynomials of degree* $n$ *on the interval* $[-1, 1]$ *the one that deviates least from zero is the monic Čebišev polynomial of degree* $n$.

(iii) See also: **Markov's Inequality** Comments (i), Extensions (b).

References [EM, vol. 2, pp. 123–124], [I1, pp. 321–328].

**Čebišev's Probability Inequality** *If* $X$ *is a random variable with finite expectation then*

$$P\big(|X - EX| \geq r\big) \leq \frac{\sigma^2 X}{r^2}.$$

Comments (i) This is also known as the *Bienaymé-Čebišev inequality*.

(i) Refinements of this result can be found in **Bernšteĭn's Probability Inequality, Kolmogorov's Probability Inequality, Markov's Probability Inequality**.

(ii) See also: **Markov's Probability Inequality** Comments (ii).

References [EM, vol. 2, pp. 119–120, vol.5, pp. 295–296]; *Feller* [F, vol. I, pp. 219–221], *Loève* [L, pp. 234–236].

**Chassan's Inequality** See: **Ostrowski's Inequality** ,Comments (iii).

**Chebyshev's Inequalities** See: **Čebišev's Inequality, Čebišev Polynomial Inequalities, Čebišev's Probability Inequality**.

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**Chernoff Bounds** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $f : X^2 \to \mathbb{R}$, $\mu_n$ the product measure on $X^n$, and $f_n$ the function $f_n(\underline{u}, \underline{v}) = \frac{1}{n}\sum_1^n f(u_i, v_i), \underline{u}, \underline{v} \in X^n$. Then for $r \in \mathbb{R}, s \geq 0$,*

$$\mu_n\left(\left\{\underline{u};\ f_n(\underline{u}, \underline{v}) < r\right\}\right) \leq \left(e^{sr} \sup_{v \in X} \int_X e^{-sf(u,v)}\,\mathrm{d}\mu(u)\right)^n.$$

Reference [GI1, pp. 131–132].

**Chi Inequality** *If $\underline{a}$ is an A-sequence then*

$$(m+1)a_m + a_n \geq n(m+1);$$

*and*

$$2 < \sum_{n=1}^{\infty} \frac{1}{a_n} < 4.$$

Comment An *A-sequence* $\underline{a}$ is a positive strictly increasing sequence with $a_1 \geq 1$ and such that no element is the sum of two or more distinct earlier elements. They are also called *sum free sets.*

Reference [CE, pp. 5, 240].

**Chong's Inequalities** *(a) If $\underline{b}$ is a rearrangement of $\underline{a}$, both positive n-tuples, then*

$$\sum_{i=1}^n \frac{b_i}{a_i} \geq n; \qquad \prod_{i=1}^n a_,^{a_i} \geq \prod_{i=1}^n a_,^{b_i} \tag{1}$$

*The inequalities are strict unless $\underline{a} = \underline{b}$. (b) If $\underline{a}, \underline{b}$ are positive n-tuples and $\underline{a} \prec \underline{b}$ then*

$$\prod_{i=1}^n a_i \leq \prod_{i=1}^n b_i. \tag{2}$$

*(c) If $0 \leq x < y \leq 1$ and if the positive n-tuple $\underline{a}$ is not constant,*

$$\begin{aligned} \mathfrak{A}_n\left(\mathfrak{G}_n^{1-x}(\underline{a}; \underline{w})\underline{a}^x; \underline{w}\right) &< \mathfrak{A}_n\left(\mathfrak{G}_n^{1-y}(\underline{a}; \underline{w})\underline{a}^y; \underline{w}\right); \\ \mathfrak{G}_n\left(x\mathfrak{A}_n(\underline{a}; \underline{w}) + (1-x)\underline{a}; \underline{w}\right)\right) &< \mathfrak{G}_n\left(y\mathfrak{A}_n(\underline{a}; \underline{w}) + (1-y)\underline{a}; \underline{w}\right)\right). \end{aligned} \tag{3}$$

Comments (i) The first inequality in (1) can be used to prove (GA).

(ii) (b) follows by first proving the case $\underline{a} = \lambda\underline{b} + (1-\lambda)\underline{b}'$ where $\underline{b}'$ is a permutation of $\underline{b}$. The general case follows from properties of the order relation $\prec$.

(iii) Applications of (2) occur in **Bernoulli's Inequality**, Extensions (a), and also in **Binomial Function Inequalities** Comments (iv).

It can be used to prove the equal weight case of (GA) by noting that $\mathfrak{A}_n(\underline{a}) \prec \underline{a}$.

(iv) Both inequalities (3) reduce to (GA) if $x = 0,\ y = 1$.

A similar interpolation result for (C) is due to Callebaut, see **Cauchy's Inequality** Extensions (e).

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REFERENCES [H, pp. 23–24, 105, 152], [MPF, pp. 37–40].

**Choquet's Theorem** *Let $X$ be a locally convex Hausdorff space, $K \subseteq X$, a metrizable compact set; further let $\mu$ and $\lambda$ be Borel probability measures on $K$. Further assume that: (a) $\mu$ and $\lambda$ have the same barycenter, $x_\mu$ say, (b) $\lambda(K \setminus extK) = 0$, (c) $\lambda \succ \mu$.*
*If then $f : K \mapsto \mathbb{R}$ is continuous and convex:*

$$f(x_\mu) \leq \int_K f \,\mathrm{d}\mu \leq \int_{extK} f \,\mathrm{d}\lambda.$$

COMMENTS (i) The fact that for each such $\mu$ there is a $\lambda$ with the stated properties is *Choquet's theorem*, one of the more important theorems of modern analysis. The inequality is an extreme extension of the **Hermite-Hadamard Inequality**.

(ii) The *barycenter* of a probability measure $\mu$ on $K$ is the unique point $x_\mu$ such that $\ell(x_\mu) = \int_K \ell \,\mathrm{d}\mu$ for all continuous linear functionals $\ell$ on $X$. In the case $X = \mathbb{R}^n$ this is the same as the first moment of $\mu$.

(iii) (b) says that the support of $\lambda$ is the set of extreme points of $K$.

(iv) (c) says that $\lambda$ majorizes $\mu$ in the sense that $\int_K f \,\mathrm{d}\lambda \geq \int_K f \,\mathrm{d}\mu$ for all continuous convex $f : K \mapsto \mathbb{R}$. This is a partial order on the set of all probability measures on $K$; $\lambda$ is an extremal element under this partial order; it is not necessarily unique.

REFERENCES *Niculescu & Persson* [NP, pp. 177–202]; *Niculescu* [233].

**Circulant Matrix Inequalities** Given an $n$-tuple $\underline{a}$ then the *circulant matrix of $\underline{a}$* is

$$C(\underline{a}) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.$$

[BECKENBACH & BELLMAN] *(a) If $n$ is odd, and $\sum_{i=1}^n a_i \geq 0$ then*

$$\det C(\underline{a}) \geq 0. \tag{1}$$

*(b) If $n = 2m$ is even and if $\left|\sum_{k=1}^m a_{2k-1}\right| \geq \left|\sum_{k=1}^m a_{2k}\right|$ then (1) holds.*

COMMENTS (i) The cases of equality are discussed in the reference.

(ii) In addition the reference gives analogous results for the *skew-circulant matrix*, obtained by changing the sign of each element on one side of the diagonal of $C(\underline{a})$.

REFERENCE [GI1, pp. 39–48].

**Clarkson's Inequalities** *(a) If $p \geq 2$, $q$ the conjugate index, $\underline{a}, \underline{b}$ complex sequences in $\ell_p$, then*

$$\begin{aligned} ||\underline{a}+\underline{b}||_p^p + ||\underline{a}-\underline{b}||_p^p &\leq 2^{p-1}\big(||\underline{a}||_p^p + ||\underline{b}||_p^p\big); \\ ||\underline{a}+\underline{b}||_p^p + ||\underline{a}-\underline{b}||_p^p &\leq 2\big(||\underline{a}||_p^q + ||\underline{b}||_p^q\big)^{p-1}. \end{aligned}$$

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*(b) If* $1 < p < 2$, *q the conjugate index,* $\underline{a}, \underline{b}$ *complex sequences in* $\ell_p$, *then*

$$||\underline{a}+\underline{b}||_p^q + ||\underline{a}-\underline{b}||_p^q \leq 2\big(||\underline{a}||_p^p + ||\underline{b}||_p^p\big)^{q-1}.$$

Comments (i) These inequalities were used by Clarkson to prove that the spaces $\ell_p, \mathcal{L}^p$ are uniformly convex; see for instance *Hewitt & Stromberg*, [HS].

(ii) These inequalities have easily stated integral analogues for functions in $\mathcal{L}^p$.

The proofs of the above results depend on two inequalities between complex numbers that are of some interest.

Two Complex Number Inequalities *(a) If* $z, w \in \mathbb{C}, p \geq 2$ *then*

$$|w+z|^p + |w-z|^p \leq 2^{p-1}\big(|w|^p + |z|^p\big). (1)$$

*(b) If* $z, w \in \mathbb{C}, 1 < p \leq 2$, *q the conjugate index, then*

$$|w+z|^q + |w-z|^q \leq 2\big(|w|^p + |z|^p\big)^{q-1}.$$

Comments (iii) This result is derived from the real case, the proof of which uses elementary calculus.

(iv) (1) is an inverse of the **Beth & van der Corput Inequality**.

(v) These inequalities, and the following, have been extended to unitary spaces.

Extension [Koskela] *If* $w, z \in \mathbb{C}$, $r, s > 0$, $r'$ *the conjugate index of* $r$, *and*

$$t = \begin{cases} \min\{2, s\} & \textit{if } r \leq 2, \\ \min\{r', s\} & \textit{if } r > 2; \end{cases}$$

*then*

$$|w+z|^r + |w-z|^r \leq C\left(|w|^s + |z|^s\right)^{r/s}, \tag{2}$$

*where*

$$C = 2^{1-\frac{r}{s}+\frac{r}{t}}.$$

*There is equality in (2): (i) for all* $w, z$ *if* $r = s = 2$; *(ii) if* $wz = 0$ *and* $0 < s < r' \leq 2$, *or* $0 < r < 2$ *and* $0 < s < 2$; *(iii)* $\Re(w\overline{z}) = 0$ *and* $0 < r < 2$; *(iv)* $w = \pm iz$ *and* $0 < r < 2 < s$; *(v)* $|w| = |z|$ *and* $r = 2 < s$; *(vi)* $w = \pm z$ *and* $2 < r$ *and* $r' < s$; *(vii)* $wz = 0$ *or* $w = \pm z$ *and* $2 < r = s'$.

Comments (vi) This result follows using (H), or (r;s), when $0 < r \leq 2$, and (J) when $2 < r$ and $s \leq r'$.

(vii) The case of equality (i), $r = s = 2$, is just **Parallelogram Inequality**, Comments (i).

(vii) Koskela used the above to extend the Clarkson inequalities.

(viii) See also **Hanner's Inequalities, von Neumann & Jordan Inequality**.

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REFERENCES [MPF, pp. 534–558], [PPT, p. 135]; *Hewitt & Stromberg* [HS, pp. 225–227].

**Clausius-Duhem Inequality** *Let $\Omega_t \subset \mathbb{R}^3$ be a domain depending on time, $t$, with a smooth boundary, $\theta = \theta(\underline{x}, t)$ the temperature at $\underline{x} \in \Omega_t$, $q$ the heat flux per unit area though $\partial\Omega_t$. $\rho = \rho(\underline{x}, t)$ the density at time $t$, and $\underline{x} \in \Omega_t$, $h = h(\underline{x}, t)$ the mass density of radiation heat, and $S(\overline{\Omega}_t)$ the total entropy of $\overline{\Omega}_t$ then*

$$\frac{\partial S(\overline{\Omega}_t)}{\partial t} \geq \int_{\Omega_t} \frac{h}{\theta}\rho + \oint_{\partial\Omega_t} \frac{q}{\theta}.$$

COMMENT For other entropy inequalities see **Entropy Inequalities, Shannon's Inequality**.

REFERENCE [EM, Supp., pp. 185–186].

**Cohn-Vossen Inequality** *If $M$ is a non-compact Riemannian manifold with no boundary then*

$$\int_M K \,\mathrm{d}S \leq 2\pi\chi,$$

*where $K$ is the Gaussian curvature, and $\chi$ the Euler characteristic.*

COMMENT In case $M$ is compact and closed, or with a smooth boundary, the inequality becomes the *Gauss-Bonnet theorem*:

$$\int_M K \,\mathrm{d}S + \int_{\partial M} k_g \,\mathrm{d}\ell = 2\pi\chi,$$

where $k_g$ is the geodesic curvature.

REFERENCE [EM, vol. 4, p. 196].

**Complementary Error Function Inequalities** See: **Error Function Inequalities**

**Complete Symmetric Function Inequalities** *(a) If $1 \leq r \leq n-1$ and $\underline{a}$ is a positive $n$-tuple, then*

$$q_n^{[r-1]}(\underline{a})q_n^{[r+1]}(\underline{a}) \geq \left(q_n^{[r]}(\underline{a})\right)^2, \tag{1}$$

*with equality if and only if $\underline{a}$ is constant.*
*(b)* [MCLEOD, BASTON] *If $\underline{a}, \underline{b}$ are a positive $n$-tuples, $r, s$ are integers, $1 \leq r \leq s \leq n$, and either $s = r$ or $s = r + 1$ then*

$$\left(\frac{c_n^{[s]}(\underline{a}+\underline{b})}{c_n^{[s-r]}(\underline{a}+\underline{b})}\right)^{1/r} \leq \left(\frac{c_n^{[s]}(\underline{a})}{c_n^{[s-r]}(\underline{a})}\right)^{1/r} + \left(\frac{c_n^{[s]}(\underline{b})}{c_n^{[s-r]}(\underline{b})}\right)^{1/r}. \tag{2}$$

*(c)* [ŌZEKI] *If $\underline{a}$ is a positive log-convex sequence so is $q_n^{[r]}(\underline{a})$; that is,*

$$q_{n-1}^{[r]}(\underline{a})q_{n+1}^{[r]}(\underline{a}) \geq \left(q_n^{[r]}(\underline{a})\right)^2, \ 1 \leq r \leq n-1.$$

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COMMENTS (i) (1) is the analogue of **Elementary Symmetric Function Inequalities** (2).

(ii) Nothing is known about the other cases of (2), it would be of interest to complete the result and so obtain an analogue of **Marcus & Lopes's Inequality**. **Au: "but" or "so"?**

(iii) $q_n^{[r]}$ is Schur concave, strictly if $r > 1$; for a definition see: **Schur Convex Function Inequalities**.

REFERENCES [H, pp. 342, 343], [MOA, pp. 81–82], [MPF, p. 165].

**Complete Symmetric Mean Inequalities** *(a) If $\underline{a}$ is a positive $n$-tuple, and if $1 \le r \le n$ then*

$$\min \underline{a} \le \mathfrak{Q}_n^{[r]}(\underline{a}) \le \max \underline{a},$$

*with equality if and only if $\underline{a}$ is constant.*
*(b) If $\underline{a}$ is a positive $n$-tuple, $1 \le r < s \le n$ then*

$$\mathfrak{Q}_n^{[r]}(\underline{a}) \le \mathfrak{Q}_n^{[s]}(\underline{a}),$$

*with equality if and only if $\underline{a}$ is constant.*
*(c) If $\underline{a}, \underline{b}$ are positive $n$-tuples, $r$ is an integer and $1 \le r \le n$ then*

$$\mathfrak{Q}_n^{[r]}(\underline{a} + \underline{b}) \le \mathfrak{Q}_n^{[r]}(\underline{a}) + \mathfrak{Q}_n^{[r]}(\underline{b}).$$

*(d) If $\underline{a}$ is a positive $n$-tuple, $r$ is an integer and $1 \le r \le n$ then*

$$\mathfrak{P}_n^{[r]}(\underline{a}) \le \mathfrak{Q}_n^{[r]}(\underline{a}),$$

*with equality if and only if either $r = 1$ or $\underline{a}$ is constant.*

COMMENTS (i) The inequality in (b) is the analogue of S(r;s).

(ii) The inequality in (c) follows from the case $s = r$ of **Complete Symmetric Function Inequalities** (2).

REFERENCES [H, pp. 342–343, 349, 362], [MPF, pp. 16–17].

**Completely Monotonic Function Inequalities** (a) *If $f \in \mathcal{C}^\infty(I)$ is completely monotonic then $(-1)^k f^{(k)} \ge 0$, $k \in \mathbb{N}$.*
(b) [FINK] *If $\underline{a}, \underline{b}$ are non-negative $n$-tuples of integers with $\underline{a} \prec \underline{b}$ and if $f$ is completely monotonic on $[0, \infty[$,*

$$\prod_{i=1}^n (-1)^{a_i} f^{(a_i)} \le \prod_{i=1}^n (-1)^{b_i} f^{(b_i)},$$

*with equality if for some $a > 0$, $f(x) = e^{-ax}$.*

COMMENTS (i) This is just the definition of completely monotonic.

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(ii) For examples see: **Absolutely Monotonic Function Inequalities, Mills's Ratio Inequalities**.

REFERENCES [MPF, pp. 365–377]; *Widder* [W, pp. 167–168].

## Complex Function Inequalities

*(a) If $f$ is a continuous complex-valued function defined on $[a,b]$ then*

$$\Re\left(e^{i\theta}\int_a^b f\right) \le \int_a^b |f|, \tag{1}$$

*and*

$$\left|\int_a^b f\right| \le \int_a^b |f|. \tag{2}$$

*(b) If $f$ is a complex-valued function continuous on the piecewise differentiable arc $\gamma$ in $\mathbb{C}$ then*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_\gamma |f(z)|\,|\,\mathrm{d}z|.$$

COMMENTS (i) (1) follows from the right-hand side of the first inequality in **Complex Number Inequalities** (a). Then (2) follows from (1) by a suitable choice of $\theta$.

(ii) A discrete analogue of (2) is **Complex Number Inequalities** EXTENSIONS (a). A converse can be found in **Wilf's Inequality** INTEGRAL ANALOGUES.

(iii) Most inequalities covered by this heading can be found elsewhere; in particular in the many entries for special functions: **Binomial, Factorial, Conjugate Harmonic, Hyperbolic, Laguerre, Logarithmic, Polynomial, Trigonometric**, etc. See also references in **Analytic Function Inequalities**.

REFERENCE [Ah78, pp. 101–104].

## Complex Number Inequalities

*(a) For all $z \in \mathbb{C}$,*

$$-|z| \le \Re z \le |z|; \qquad -|z| \le \Im z \le |z|,$$

*with equality on the right-hand side of first case if and only if $z \ge 0$.*
*(b) If $z_1, z_2, z_3$ are complex numbers then*

$$\begin{array}{lrcl} (i) & |z_1+z_2| & \le & |z_1|+|z_2|, \\ (ii) & |z_1-z_2| & \ge & \left||z_1|-|z_2|\right|, \\ (iii) & |z_1-z_3| & \le & |z_1-z_2|+|z_2-z_3|. \end{array}$$

*There is equality: in (i) and (ii) if and only if $z_1\overline{z}_2 \ge 0$, equivalently $z_1, z_2$ are on the same ray from the origin: in (iii) if and only if $z_2$ is between $z_1$ and $z_3$.*
*(c) If $|z|, |w| < 1$ then*

$$\frac{|z|-|w|}{1-|z||w|} \le \left|\frac{z-w}{z\overline{w}-1}\right| \le \frac{|z|+|w|}{1+|z||w|} < 1.$$

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*(d) If* $\Re z \geq 1$ *then*

$$|z^{n+1} - 1| > |z|^n |z - 1|.$$

*(e) If* $0 < \theta < \pi/2$, $z = e^{i\theta}\cos\theta$ *then*

$$|1 - z| < |1 - z^n|.$$

*(f)* [Bergström] *If* $p, q \in \mathbb{R}$, *and if* $pq(p+q) > 0$ *then*

$$\frac{|z+w|^2}{p+q} \leq \frac{|z|^2}{p} + \frac{|w|^2}{q}; \tag{4}$$

*while if* $pq(p+q) < 0$ *then* ($\sim$*2) holds.*
*(g)* [Bourbaki-Janous] *(i) If* $\underline{z}$ *is a complex* $n$*-tuple with* $\sum_{k=1}^{n} |z_k| \leq h < 1$ *then*

$$\left| \prod_{k=1}^{n} (1 + z_k) - 1 - \sum_{k=1}^{n} z_k \right| \leq \frac{h^2}{1-h}.$$

*(ii) If* $0 < \lambda \leq 1/(2\pi + 1)$ *and for all* $\mathcal{J} \subseteq \{1, \ldots, n\}$ *we have that*

$$\left| \prod_{k \in \mathcal{J}} (1 + z_k) - 1 \right| \leq \lambda,$$

*then*

$$\sum_{k=1}^{n} |z_k| \leq \frac{2\pi\lambda}{\alpha(\lambda)}, \quad \text{where } \alpha(\lambda) = 1 + \sqrt{\frac{1 - \lambda(2\pi+1)^2}{1-\lambda}}.$$

*(h)* [Redheffer & C. Smith] *Let* $a, b$ *be non-zero complex numbers, with* $|a+b| = \sigma$, $|a-b| = \delta$, $\sigma\delta \neq 0$ *; then if* $z \in \overline{D}$,

$$\max\left\{|az+b|, |a+bz|\right\} \geq \frac{\sigma\delta}{\sqrt{\sigma^2+\delta^2}};$$

*further given* $\sigma, \delta$ *there are* $a, b, z$ *for which equality holds.*

Comments (i) (a) is immediate from the definitions.
(ii) For (1) just apply (a) to the identity

$$|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\Re z_1 \overline{z}_2.$$

(iii) Of course (3) is the complex number version of (T).
(iv) (f) follows from the identity

$$\frac{|z|^2}{p} + \frac{|w|^2}{q} - \frac{|z+w|^2}{p+q} = \frac{|qz - pw|^2}{pq(p+q)}.$$

It is generalized in **Norm Inequalities** (4).

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(v) A slightly weaker particular case of (g)(ii), $\lambda = 1/81$, and $\frac{2\pi\lambda}{\alpha(\lambda)}$ replaced by 8/81, was used by Bourbaki to prove that if a family of complex numbers was multipliable then it was summable.

(vi) The inequality in (h) is related to the **Goldberg-Straus Inequality**.

Extensions *(a) If $\underline{z}$ is complex $n$-tuple,*

$$\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{m} |z_k|,$$

*with equality if and only if for all non-zero terms $z_j/z_k > 0$.*
*(b) For all $z_1, z_2, z_3 \in \mathbb{C}$,*

$$1 \le |1 + z_1| + |z_1 + z_2| + |z_2 + z_3| + |z_3|$$

*(c)* [de Bruijn] *If $\underline{w}$ is a real $n$-tuple and $\underline{z}$ a complex $n$-tuple then*

$$\left|\sum_{k=1}^{n} w_k z_k\right| \le \frac{1}{\sqrt{2}}\Big(\sum_{k=1}^{n} w_k^2\Big)^{1/2}\left(\sum_{k=1}^{n} |z_k|^2 + \Big|\sum_{k=1}^{n} z_k^2\Big|\right)^{1/2},$$

*with equality if and only if for some complex $\lambda$, $w_k = \Re\lambda z_k$, $1 \le k \le n$, and $\sum_{k=1}^{n} \lambda^2 z_k^2 \ge 0$.*
*(d) If $z, w \in \mathbb{C}, r \ge 0$ then*

$$|z + w|^r \le \begin{cases} |z|^r + |w|^r, & \text{if } r \le 1,\\ 2^{r-1}(|z|^r + |w|^r), & \text{if } r > 1.r \end{cases}$$

Comment (vi) An integral analogue of (a) is **Complex Function Inequalities** (2).

Inverse Inequalities *(a) If $\underline{z}$ is a complex $n$-tuple, $\underline{z} \ne \underline{0}$, then for some index set $\mathcal{I} \subseteq \{1, \ldots, n\}$,*

$$\left|\sum_{k\in\mathcal{I}} z_k\right| > \frac{1}{\pi}\sum_{k\in\mathcal{I}} |z_k|.$$

*(b) If $|\arg z - \arg w| \le \theta \le \pi$ and if $n \ge 1$ then*

$$|z - w|^n \le (|z|^n + |w|^n) \max\left\{1,\ 2^{n-1}\sin^n \theta/2\right\};$$

*while if $w, z \ne 0$*

$$|z - w| \ge \frac{1}{2}\,(|z| + |w|)\left|\frac{z}{|z|} - \frac{w}{|w|}\right|.$$

Comments (vii) The constant $1/\pi$ in (a) is best possible.

(viii) See also: **Abel's Inequalities** (d), **Absolute Value Inequalities** (a) Comments (iv), **Beth & van der Corput's Inequality, Bohr's Inequality,**

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**Cauchy's Inequality,** Extension (a), **Clarkson's Inequality** Two Complex Number Inequalities, Extensions, **Leindler's Inequality, Wilf's Inequality, van der Corput's Inequality**.

References [AI, pp. 310–336], [GI2, pp. 47–51], [MPF, pp. 78, 89–90, 499–500]; *Bourbaki* [B55, Livre III, p. 113], *Steele* [S, pp. 224, 283]; *Janous* [146].

**Compound Mean Inequalities** See: **Arithmetico-geometric Compound Mean Inequalities**.

**Conjugate Convex Function Inequalities** If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ define $f^{\#} : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$f^{\#}(\underline{a}) = \sup\left\{t;\ t = \underline{a}.\underline{b} - f(\underline{b}),\ \underline{b} \in \mathbb{R}^n\right\}.$$

The function $f^{\#}$ is called the *conjugate, or Legendre transform, of* $f$. It is, in a sense, the best function for an inequality of type (1) below.

The conjugate function is convex and its domain can be generalized to any convex set.

*(a) If $f, g$ have conjugates $f^{\#}$, $g^{\#}$, respectively, then*

$$f \geq g \quad \Longrightarrow \quad f^{\#} \geq g^{\#}.$$

*(b)* [Young's Inequality] *If $f > -\infty$ and is convex on a convex domain, and if $f^{\#}$ is its conjugate*

$$\underline{a}.\underline{b} \leq f(\underline{a}) + f^{\#}(\underline{b}). \tag{1}$$

Comments (i) In particular if $f(x) = |x|^p/p, p > 1$ then $f^{\#}(x) = |x|^q/q, q$ the conjugate index. In this case (1) reduces to **Geometric-Arithmetic Mean Inequality** (2). If $f(x) = e^x$ then $f^{\#}(x) = x \log x - x, x > 0, = 0, x = 0, = -\infty, x < 0$; and then **Young's Inequalities** (1), and **Logarithmic Function Inequalities** (e) are particular cases of (1).

(ii) Inequality (1) is called *Fenchel's inequality* by Rockafellar.

References [EM, vol. 2, pp. 336–337], [PPT, p. 241]; *Roberts & Varberg* [RV, pp. 21, 28–36, 110–111], *Rockafellar* [R, pp. 102–106], *Steele* [S, pp. 150, 263].

**Conjugate Function Inequalities** See: **Conjugate Harmonic Function Inequalities** Comments (iv).

**Conjugate Harmonic Function Inequalities** If $f$ is analytic in $D$ and if $f = u + iv$, the two real functions $u, v$ are harmonic in $D$ and $v$ is called the *harmonic conjugate of* $u$.

[M.Riesz] *(a) If $u$ is harmonic in $D, u \in \mathcal{H}_p(D), 1 < p < \infty$, and if $v$ is the harmonic conjugate, then $v \in \mathcal{H}_p(D)$ and there is a constant $C_p$ such that*

$$||v||_p \leq C_p ||u||_p.$$

*In the case $p = 1$,*

$$||v||_1 \leq A \sup_{0<r<1} \frac{1}{2} \int_0^{2\pi} |u(re^{i\theta})| \log^+ |u(re^{i\theta})| \, \mathrm{d}\theta.$$

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*(b) If $\Delta$ is any diameter of $D$ and $u \in \mathcal{H}(D)$ then*

$$\int_\Delta |v(z)|\,|\mathrm{d}z| \le \frac{1}{2}||u||.$$

COMMENTS (i) The norm is defined in **Analytic Function Inequalities** COMMENTS (i).

(ii) If $p, q$ are conjugate indices then $C_p = C_q$. The exact value of the constant is known.

(iii) An extension of (b) is given in **Fejér-Riesz Theorem**.

(iv) These results have real function analogues. In this case if $f \in \mathcal{L}(-\pi, \pi)$ the *conjugate function* is

$$\hat{f}(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0+} \int_\epsilon^\pi \frac{f(x+t) - f(x-t)}{2\tan t/2}\,\mathrm{d}t.$$

With this definition of conjugate functions, the above theorem of M. Riesz holds.

This real result extends to higher dimensions.

REFERENCES [EM, vol. 2, pp. 336, 338–339; vol. 4, pp. 366–369; vol. 6, pp. 131–140]; *Hirschman* [Hir, pp. 164–167]; *Rudin* [R87, pp. 330–331, 345–347], *Zygmund* [Z, vol. I, pp. 51, 253–258].

## Conte's Inequality

*If $x > 0$ then*

$$\left(x + \frac{x^2}{24} + \frac{x^3}{12}\right) e^{-3x^2/4} < e^{-x^2} \int_0^x e^{t^2}\,\mathrm{d}t \le \frac{\pi^2}{8x}\left(1 - e^{-x^2}\right).$$

COMMENTS (i) This is said to be related to the Mills's ratio.

(ii) The right-hand side has been refined in the reference.

REFERENCES [AI, p. 181]; *Qi, Cui & Xu, S. L.* [278].

## Continued Fraction Inequalities

The *continued fraction*

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cdots}}}} = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_n|}{|a_n} + \cdots$$

will be written $C = [a_0; a_1, a_2 \ldots : b_1, \ldots]$; and the finite continued fraction

$$C_n = [a_0; a_1, a_2 \ldots, a_n : b_1, \ldots, b_n] = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_n|}{|a_n}$$

is called the *n-th convergent* of the infinite continued fraction $C$.

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In the case $b_n = 1, n \geq 1$ they are omitted from the bracket notation, and the continued fraction is said to be *regular or simple*.

*(a) If $a_n > 0, b_n \geq 0, n \in \mathbb{N}$ then*

$$C_{2n-1} > C_{2n+1} > C_{2n+2} > C_{2n}, n \geq 1.$$

*(b) If $C$ is a regular continued fraction with $a_n$ a positive integer, $n \geq 1$, then $C_n = P_n/Q_n$, with $P_n, Q_n$ integers, $n \geq 1$, $C_n \to C$ and*

$$\begin{array}{rcll} C_{2n-1} > C_{2n+1} > & C & > C_{2n+2} > C_{2n}, & n \geq 1; \\ Q_n & \geq & 2^{(n-1)/2}, & n \geq 2; \end{array}$$

*further*

$$|C - C_n| \leq \frac{1}{Q_n Q_{n+1}}. \tag{1}$$

Comment An inverse of (1) can be found in **Mediant Inequalities** (1).

References [EM, vol. 2, pp. 375–377], [MPF, pp. 661–665]; *Hinčin* [HiA].

**Converse Inequalities** See: **Reverse, Inverse, and Converse Inequalities**.

**Convex Functions of Higher Order Inequalities** See: **n-Convex Function Inequalities**.

**Convex Function Inequalities** *(a) If $f$ is a convex function defined on the compact interval $[a, b]$ then for all $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y); \tag{1}$$

*if $f$ is strictly convex then (1) is strict unless $x = y$, $\lambda = 0$ or $\lambda = 1$.*
*(b) If $a \leq x_1 < x_2 < x_3 \leq b$ then*

$$f(x_2) \leq \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3), \tag{2}$$

*or equivalently*

$$f(x_2)(x_3 - x_1) \leq f(x_3)(x_1 - x_2) + f(x_1)(x_2 - x_3),$$

*or*

$$0 \leq \begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix}.$$

*(c) With the same notation as in (b), but only requiring $x_1 < x_3$,*

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3}. \tag{3}$$

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*(d) With the same notation as in (b), but no order restriction on* $x_1, x_2, x_3$,

$$\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)} + \frac{f(x_2)}{(x_2-x_3)(x_2-x_1)} + \frac{f(x_3)}{(x_3-x_1)(x_3-x_2)} \geq 0. \tag{4}$$

*(e) If* $x_1, x_2, y_1, y_2$ *are points of* $[a,b]$ *with* $x_1 \leq y_1$, $x_2 \leq y_2$, *and* $x_1 \neq x_2$, $y_1 \neq y_2$ *then*

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}. \tag{5}$$

*(f)* [Three chords lemma] *If* $a < x < b$ *then*

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

Comments (i) Since (a) is the definition of convexity, and of strict convexity, no proof is required. In addition the definition of a concave function, strictly concave function, is one for which ($\sim$1) holds, strictly, under the conditions stated.

(ii) If (1) is only required for $\lambda = \frac{1}{2}$ then the function is said to be *Jensen, J-, or mid-point convex*. Such a function satisfies (1) with $\lambda \in \mathbb{Q}$. If $f$ is bounded above then mid-point convexity implies convexity; the same is true if we only require that $f$ is bounded above by a Lebesgue integrable function.

(iii) The geometric interpretation of (1) is:

*the graph of a convex $f$ lies below its chords; if $f$, strictly convex, is equivalent to being convex and the graph containing no straight line segments.*

(iv) (b)–(f) follow from (a) by rewriting and notation changes.

(v) The second equivalent form of (2) says that the area of the triangle $P_1P_2P_3$ is non-negative, where $P_i = (x_i, f(x_i)), 1 \leq i \leq 3$.

(vi) The interpretation of (5) is that for convex functions the chords to the graph have slopes that increase to the right; see **Star-shaped Function Inequalities** Comments (ii).

(vii) It is important to note that if $f$ has a second derivative then f is convex if and only if $f'' \geq 0$, and if $f'' > 0$, except possibly at a finite number of points, then $f$ is strictly convex. So the exponential function is strictly convex, as is $x^\alpha$ if $\alpha > 1$ or $\alpha < 0$; while the logarithmic function is strictly concave, as is $x^\alpha$ if $0 < \alpha < 1$.

In any case, using (vi), if $f$ is convex both $f'_\pm$ exist.

(viii) An important special case of (5) is

$$f(x+z) - f(x) \leq f(y+z) - f(y), \tag{6}$$

when $x \leq y$ and $x, y, y+z \in [a,b]$.

Inequality (6) is sometimes called the *property of equal increasing increments*. A function that has this property is said to be *Wright convex*.[21]

[21] This is E. M. Wright.

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Derivative Inequalities *(a) If $f$ is convex on $[a,b]$ and if $a \le x \le y < b$ then $f'_+(x) \le f'_+(y)$; if $f$ is strictly convex and $x \ne y$ this inequality is strict.*
*(b) If $f$ is convex on $[a,b]$ and if $a < x \le y \le b$ then $f'_-(x) \le f'_-(y)$; if $f$ is strictly convex and $x \ne y$ this inequality is strict.*
*(c) If $f$ is convex on $[a,b]$ then on $]a,b[$ we have $f'_- \le f'_+$ ; this inequality is strict if $f$ is strictly convex.*
*(d) If $f$ is convex on $[a,b]$ and if $x,y$ are points of $[a,b[$ then*

$$f(y) \ge f(x) + (y-x)f'_+(x),$$

*the inequality being strict if $f$ is strictly convex.*
*(e) If $f$ is convex on $[a,b]$ and if $a < x < y < b$ then*

$$f'_-(x) \le f'_+(x) \le \frac{f(y)-f(x)}{y-x} \le f'_-(y) \le f'_+(y).$$

The definition of a convex function of several variables, or even of a function on a vector space, is just (1) with the obvious vector interpretation of the notation; in addition now the domain must be a convex set. If $f$ is twice differentiable then it is convex if for all $\underline{x} \in G$, and all real $u_i, u_j, 1 \le i,j \le n$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(\underline{x})}{\partial x_i \partial x_j} u_i u_j \ge 0.$$

Higher Dimensions *(a) If $f$ is continuously differentiable on the convex domain $G$ in $\mathbb{R}^n$ and if $\underline{a}, \underline{b} \in G$ then*

$$f(\underline{b}) - f(\underline{a}) \ge \sum_{i=1}^{n} \frac{\partial f(\underline{a})}{\partial a_i}(b_i - a_i).$$

*(b) If $f$ is convex on $\mathbb{R}^n$, symmetric and homogeneous of the first degree then*

$$f(\underline{a}) \ge f(1,\ldots,1)\mathfrak{A}_n(\underline{a}).$$

Comments (ix) Definitions of the terms used in the last result are to be found in **Segre's Inequalities**.

(x) An important extension of (1), obtained by an induction argument, is (J). See: **Jensen's Inequality**.

(xi) A very important generalization of the whole concept of convexity is *higher order convexity*; see $n$**-Convex Function Inequalities**.

(xii) See also: **Alzer's Inequalities** (a), Extensions**, Arithmetic Mean Inequalities** Integral Analogues**, Askey-Karlin Inequalities, Bennett's Inequalities** Comments (i), **Bernšteĭn Polynomial Inequalities** (b), **Binomial Function Inequalities** Comments (i), **Brunn-Minkowski Inequalities** (a), **Cauchy's Inequality** Converse Integral Analogues, **Conjugate Convex Function Inequalities, Convex Function Integral Inequalities, Convex Matrix Function**

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**Inequalities, Digamma Function Inequalities** (A), **Factorial Function Inequalities, Favard's Inequalities, Hermite-Hadamard Inequality, Hua's Inequality** EXTENSION, **Jensen's Inequality, Jensen-Pečarić Inequality, Jensen-Steffensen Inequality, Log-convex Function Inequalities, Order Inequalities** (B), **Petrović's Inequality, Popoviciui's Convex Function Inequality, Power Mean Inequalities** INTEGRAL ANALOGUES (C), **Q-class Function Inequalities** COMMENTS (II), **Quasi-arithmetic Mean Inequalities** COMMENT **(i), Quasi-convex Function Inequalities, Rearrangement Inequalities** EXTENSIONS (A), **Schur Convex Function Inequalities, s-Convex Function Inequalities, Sequentially Convex Function Inequalities, Strongly Convex Function Inequalities, Szegő's Inequality, Thunsdorff's Inequality, Ting's Inequality**.

REFERENCES [AI, pp. 10–26], [EM, vol. 2, pp. 415–416], [H, pp. 25–59], [HLP, pp. 70–81, 91–96],[MOA, pp. 445–462], [MPF, pp. 1–19], [PPT, pp. 1–14]; *Popviciu* [PT], *Roberts & Varberg* [RV], *Steele* [S, pp. 87–104].

**Convex Function Integral Inequalities** [SENDOV] *If $f$ is non-negative and convex on $[0,1]$ and $p \geq 0$ then*

$$\frac{1}{(p+1)(p+2)}\int_0^1 f \leq \int_0^1 t^p f(t)\,\mathrm{d}t \leq \frac{2}{(p+2)}\int_0^1 f. \tag{1}$$

If $-1 < p \leq 0$ then $(\sim 1)$ holds.

COMMENTS (i) This inequality, with $p \in \mathbb{N}$, results from a problem of Sendov that was solved by Dočev and Skordev. This generalization is due to Mitrinović & Pečarić.

(ii) See also: **Hermite-Hadamard Inequality, Petschke's Inequality, Rado's Inequality, Rahmail's Inequality, Thunsdorff's Inequality, Ting's Inequality**.

REFERENCE *Mitrinović & Pečarić* [217].

**Convex Matrix Function Inequalities** If $I$ is a interval in $\mathbb{R}$ then a function $f : I \to \mathbb{R}$ is said to be a *convex matrix function of order $n$* if for all $n \times n$ Hermitian matrices $A, B$ with eigenvalues in $I$, and all $\lambda, 0 \leq \lambda \leq 1$,

$$\overline{1-\lambda}\, f(A) + \lambda f(B) \geq f(\overline{1-\lambda}\, A + \lambda B);$$

where $A \geq B$ means that $A - B$ is positive semi-definite.

COMMENTS (i) A convex matrix function of all orders on $I$ is said to be *operator convex on $I$*.

(ii) Few explicit examples of convex matrix functions are known; $x^2, x^{-1}$, $1/\sqrt{x}$ are operator convex and $\sqrt{x}$ is operator concave. If $\alpha > 2$ or $\alpha < -1$ then $x^\alpha$ is not operator convex; nothing is known about the remaining exponents except $\alpha = 2, \alpha = -1$ mentioned above. Using the convexity of $1/\sqrt{x}$ it can be shown that $1/x^{1/2^k}$, $k \in \mathbb{N}^*$ are all operator convex; similarly using the concavity of $\sqrt{x}$ all of $x^{1/2^k}$, $k \in \mathbb{N}^*$ are operator concave.

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[Davis, P. J.] *If $f$ is a convex matrix function of order $n$ on $I$ and if $A$ is an $n \times n$ Hermitian matrix with eigenvalues in $I$ then*

$$f(A)_{i_1,\ldots,i_m} \geq f(A_{i_1,\ldots,i_m}).$$

References [MOA, pp. 467–474]; *Roberts & Varberg* [RV, pp. 259–261]; *Chollet* [88].

## Convex Sequences of Higher Order Inequalities

See: **n-Convex Sequence Inequalities**.

## Convex Sequence Inequalities

*(a) If $\underline{a}$ is a real convex sequence then*

$$\Delta^2 \underline{a} \geq 0, \tag{1}$$

*and $\Delta\underline{a}$ decreases.*
*(b) If $\underline{a}$ is a real bounded convex sequence then*

$$\Delta\underline{a} \geq 0, \text{ and } \lim_{n\to\infty} n\Delta a_n = 0.$$

*(c) If $\underline{a}$ is a real convex sequence and $\underline{b} = \left\{\frac{a_2}{1}, \frac{a_3}{2}, \ldots\right\}$ then*

$$\Delta\underline{b} \geq 0.$$

*(d)* [Ōzeki] *If $\underline{a}$ is real convex sequence so is $\mathfrak{A}(\underline{a})$.*
*(e) If $\underline{a}, \underline{b}$ are positive convex sequences with $a_2 \geq a_1, b_2 \geq b_1$ then $\underline{c}$ is a convex where*

$$c_n = \frac{1}{n}\sum_{i=1}^{n} a_i b_{n+1-i}.$$

Comments (i) (1) is just the definition of a *convex sequence.*

(ii) If $f$ is a mid-point convex function and $a_i = f(i)$, $i = 1, 2, \ldots$ then $\underline{a}$ is a convex sequence.

(iii) If (1) holds strictly we say that the sequence is *strictly convex*; while if ($\sim$1) hold, strictly, we say that the *sequence is concave, strictly concave.*

(iv) A definition of *higher order convex sequences* can be given; see **n-Convex Sequence Inequalities**, where a generalization of (d) can be found.

(v) See also: **Barnes's Inequalities** (a), **Čebišev's Inequality** Extensions (d), **Log-convex Sequence Inequalities, Haber's Inequality** Extensions, **Nanson's Inequality, Sequentially Convex Function Inequalities, Thunsdorff's Inequality** Discrete Analogue.

References [AI, p. 202], [EM, vol. 2, pp. 419–420], [H, pp. 11–16], [PPT, pp. 6, 277, 289]; *Pečarić* [P87, p. 165].

## Convolution Inequalities

See: **Young's Convolution Inequality**.

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**Copson's Inequality** *If $\underline{a}$ is a non-negative sequence and if $p > 1$ then,*

$$\sum_{n=1}^{\infty}\left(\sum_{k\geq n}\frac{a_k}{k}\right)^p \leq p^p\sum_{n=1}^{\infty}a_n^p. \tag{1}$$

*There is equality if and only if $\underline{a} = \underline{0}$.*
*If $0 < p < 1$ then $(\sim 1)$ holds.*

Comments (i) This result can be deduced, using **Hölder's Inequality** Other Forms (c) and **Hardy's Inequality** (1).

Extensions [Bennett] *(a) If $\underline{a}$ is a real sequence, and if $p > 1$ then*

$$!\underline{a}!_p \leq \left(\sum_{n=1}^{\infty}\left(\sum_{k\geq n}\frac{a_k}{k}\right)^p\right)^{1/p} \leq p\,!\underline{a}!_p.$$

*There is equality on the left if $\underline{a}$ has at most one non-zero entry, and on the right only when $\underline{a} = \underline{0}$.*
*(b) If $\underline{a}$ is a positive sequence, and if $\alpha > 0, 0 < p < 1$ then*

$$\sum_{n\in\mathbb{N}}\left(\sum_{i=1}^{n}\frac{\binom{i+\alpha-1-n}{i-n}a_i}{\binom{i+\alpha}{i}}\right)^p \geq \left(\frac{\alpha!(p^{-1}-1)!}{(p^{-1}+\alpha-1)!}\right)^p\sum_{n\in\mathbb{N}}a_n^p.$$

Comments (ii) The notation in (a) is explained before **Hardy's Inequality** Extensions (c).

(iii) The inequality in (b) reduces to $(\sim 1)$ on putting $\alpha = 1$. The result should be compared with **Knopp's Inequalities** (a).

(iv) See also **Bennett's Inequalities** (d) **Hardy-Littlewood-Pólya Inequalities** Discrete Analogues; and for other inequalities by Copson see **Hardy's Inequalities** Comments (iv).

References [CE, p. 330], [HLP, pp. 246–247]; *Bennett* [Be, pp. 25–28, 38–39], *Grosse-Erdmann* [GE].

**Copula Inequalities** *If $c : S = [0,1]\times[0,1] \to [0,1]$ is a copula then:*
*(a) if $\underline{x}_i = (x_i, y_i) \in S, i = 1, 2$ with $0 \leq x_1 \leq x_2 \leq 1, 0 \leq y_1 \leq y_2 \leq 1$,*

$$c(x_1,y_1) - c(x_1,y_2) - c(x_2,y_1) + c(x_2,y_2) \geq 0; \tag{1}$$

*(b) for all $\underline{x}_i \in S, i = 1, 2$,*

$$|c(\underline{x}_1) - c(\underline{x}_2)| \leq |x_1 - x_2| - |y_1 - y_2|;$$

*(c) for all $\underline{x} = (x, y) \in S$,*

$$\max\{x + y - 1, 0\} \leq c(\underline{x}) \leq \min\{x, y\}.$$

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COMMENTS (i) A *copula* is a function such as $c$ that satisfies (1) and

$$c(0,a) = c(a,0), \qquad c(1,a) = c(a,1) = a, \qquad 0 \le a \le 1. \tag{2}$$

(ii) The inequalities in (b) and (c) are easily deduced from (1) and (2).

(iii) For a multiplicative analogue of this see **Totally Positive Function Inequalities**.

REFERENCES [CE, p. 330], [EM, Supp., pp. 199–201], [GI1, pp. 133–149], [GI4, p. 397]; *Nelsen* [Ne].

**Cordes's Inequality** *If $A, B$ are positive bounded linear operators on the Hilbert space $X$ and if $0 \le \alpha \le 1$ then*

$$||A^\alpha B^\alpha|| \le ||AB||^\alpha.$$

COMMENT This inequality is equivalent to the **Heinz-Kato Inequality**, and to the **Löwner-Heinz Inequality**.

REFERENCE [EM, Supp., p. 289].

**Correlation Inequalities** See: **Ahlswede-Daykin Inequality, FKG Inequality, Holley's Inequality**.

**Counter Harmonic Mean Inequalities** If $-\infty < p < \infty$ then the *$p$ counter-harmonic mean, of order $n$, of the positive sequence $\underline{a}$ with positive weight $\underline{w}$* is

$$\mathfrak{H}_n^{[p]}(\underline{a};\underline{w}) = \frac{\sum_{i=1}^n w_i a_1^p}{\sum_{i=1}^n w_i a_i^{p-1}}.$$

This definition is completed by defining

$$\mathfrak{H}_n^{[-\infty]}(\underline{a};\underline{w}) = \min \underline{a}; \quad \mathfrak{H}_n^{[\infty]}(\underline{a};\underline{w}) = \max \underline{a}.$$

These are justified as limits of the previous definition.

In particular,

$$\mathfrak{H}_n^{[0]}(\underline{a};\underline{w}) = \mathfrak{H}_n(\underline{a};\underline{w}); \qquad \mathfrak{H}_n^{[1]}(\underline{a};\underline{w}) = \mathfrak{A}_n(\underline{a};\underline{w}).$$

*(a) If $1 \le r \le \infty$ then*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) \ge \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}), \tag{1}$$

*while if $-\infty \le r \le 1$ inequality (~1) holds. The inequalities are strict unless $r = \pm\infty, 1$ or $\underline{a}$ is constant.*
*(b) If $-\infty \le r \le 0$ then*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) \le \mathfrak{M}_n^{[r+1]}(\underline{a};\underline{w}). \tag{2}$$

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*Inequality (2) is strict unless $r = -\infty$ or $\underline{a}$ is constant.*
*(c) If $-\infty \le r < s \le \infty$ then*

$$\mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) \ge \mathfrak{H}_n^{[s]}(\underline{a};\underline{w}), \tag{3}$$

*with equality only if $\underline{a}$ is constant.*
*(d)* [BECKENBACH] *If $1 \le r \le 2$ then*

$$\mathfrak{H}_n^{[r]}(\underline{a}+\underline{b};\underline{w}) \le \mathfrak{H}_n^{[r]}(\underline{a};\underline{w}) + \mathfrak{H}_n^{[r]}(\underline{b};\underline{w}). \tag{4}$$

*If $0 \le r \le 1$ inequality ($\sim$4) holds. Inequality (4) is strict unless $r = 1$ or $\underline{a} \sim \underline{b}$.*

COMMENTS (i) In (a) and (b) the extreme values of the parameter $r$ are trivial and in the other cases follow by an application of simple algebra and (r;s).

(ii) In (c) the inequalities for the extreme values of the parameters are trivial; the other cases follow from the convexity properties of the power means; see **Power Mean Inequalities** (d), (e).

(iii) The result in (d) follows by an application of **Radon's Inequality** and (M). An extensive generalization has been given by Liu & Chen.

(iv) The counter harmonic means have been generalized by both Gini and Bonferroni; see **Gini-Dresher Mean Inequalities**.

REFERENCES [BB, pp. 27–28], [H, pp. 245–251], [MPF, pp. 156–163], [PPT, pp. 122–124]; *Liu Q. M. & Chen J.* [175].

**Cutler-Olsen Inequality** *If $E \subseteq \mathbb{R}^n$ is a Borel set then $\dim E \le \sup_\mu \underline{R}(\mu)$, where $\dim E$ is the Hausdorff dimension of $E$, $\underline{R}(\mu)$ is the lower Renyi dimension of the probability measure $\mu$ on $E$, and the sup is over all such $\mu$.*

REFERENCE *Zindulka* [340].

**Cyclic Inequalities** *(a)* [DAYKIN] *If the positive $n$-tuple $\underline{a}$ is extended to a sequence by defining $a_{n+r} = a_r, r \in \mathbb{N}$, then*

$$\sum_{i=1}^n \frac{a_i + a_{i+2}}{a_i + a_{i+1}} \ge n\gamma(n),$$

*where*

$$\gamma(n) > \frac{n+1}{2n}.$$

*(b)* [ALZER] *If $\underline{a}$ is a positive $n$-tuple and if $a_{n+1} = a_1$ then*

$$\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})} \ge \frac{1}{n}\sum_{i=1}^n \frac{a_i}{a_{i+1}} \ge \left(\sum_{i=1}^n \frac{a_i}{a_{i+1}} - (n-1)\right)^{1/n} \ge 1,$$

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*with equality if and only if* $a_1 = \cdots = a_n$.

Comment (i) It has been shown by Elbert that $\lim_{n\to\infty} \gamma(n)$ exists, $\gamma$, say; and if $n \ge 3$,

$$\gamma(n) \ge \gamma = .978912\ldots.$$

Extension [Kovačeć] *Let* $s : I^2 \to \mathbb{R}$, *where* $I \subseteq \mathbb{R}$ *is an interval, have the following properties: (i)* $\sigma(x) = s(x,x)$ *is increasing on* $I$, *(ii)* $\tau(x) = s(a,x) + s(x,b)$ *is increasing on* $[\max\{a,b\}, \infty[ \cap I$. *If then* $\underline{a}$ *is an* $n$*-tuple with elements in* $I$, *and if* $a = \min \underline{a}$, *then*

$$\sum_{i=1}^{n} s(a_i, a_{i+1}) \ge ns(a,a),$$

*where* $a_{n+1} = a_1$ *and* $a = \min \underline{a}$.

Comments (ii) A particular case of this last result, $I = ]0, \infty[, s(x,y) = x/y$, is **Korovkin's Inequality**.

(iii) See also: **Elementary Symmetric Function Inequalities** Comments (vi), **Nesbitt's Inequality, Schur's Inequality, Shapiro's Inequality**. In addition many cyclic inequalities are given in the second reference.

References [AI, pp. 132–138], [GI4, p. 411], [MPF, pp. 407–471]; *Mitrinović & Pečarić* [MP91a]; *Alzer* [6].

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# 4 Davies–Dunkl

**Davies & Petersen Inequality** *If $\underline{a}$ is a non-negative sequence, and $p \geq 1$ then*

$$A_n^p \leq p \sum_{k=1}^{n} a_k A_k^{p-1}.$$

Comment See also: **Pachpatte's Series Inequalities**.

Reference *Davies & Petersen* [96].

**Davis's Inequality**[22] See: **Burkholder-Davis-Gundy Inequality**.

**de la Vallée Poussin's Inequality** *If on the interval $[a, b]$ we have that*

$$y'' + gy' + fy = 0, \quad y(a) = y(b) = 0$$

*then*

$$1 < 2||g||_{\infty,[a,b]}(b-a) + ||f||_{\infty,[a,b]}\frac{(b-a)^2}{2}.$$

Comment This result has been extended by many writers.

References [MPF]; *Agarwal & Pang* [AP]; *Brown, R. C., Fink & Hinton* [73]

**Derivative Inequalities** *(a) If $u, v$ are differentiable on $]a, b[$ and such that for any $c, a < c < b, u(c) = v(c)$ implies $u'(c) < v'(c)$, and if $u(a+) < v(a+)$ then*

$$u < v.$$

*(b) If $f$ is differentiable on $[a, b]$ and if for all points $x \in A, A \subseteq [a, b], |f'(x)| \leq M$ then*

$$\big|f[A]\big| \leq M|A|.$$

Comments (i) (a) has many extensions and important applications; see *Walter* [WW]. In particular we can drop the condition of differentiability, and replace the first inequality condition by either $u'_+(c) < v'_+(c)$, or $u'_-(c) < v'_-(c)$.

[22]This is Philip J. Davis.



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(ii) The result in (b) can be generalized by dropping the differentiability condition and replacing *"for all points* $x \in A, A \subseteq [a,b], |f'(x)| \leq M$*"* by *"for all points* $x \in A, A \subseteq [a,b], |\overline{f}'_+(x)| \leq M$*, and* $|\underline{f}'_-(x)| \geq -M$*"*.

A completely different generalization is in *Zygmund* [Z].

EXTENSION *If* $u, v$ *are continuous on* $[a,b]$*, differentiable on* $]a,b[$*, and such that for some function* $f$*,*

$$u'(x) - f\big(x, (u(x)\big) < v'(x) - f\big(x, v(x)\big), \quad a < x < b \quad \textit{and if} \quad u(a) < v(a)$$

*then*

$$u(x) < v(x), \quad a < x < b.$$

COMMEMT (iii) There are many other inequalities involving derivatives; see in particular **Alzer's Inequalities** (A), EXTENSION, **Hardy-Littlewood-Landau Derivative Inequalities, Integral Inequalities** (D),(E), **Mean Value Theorem of Differential Calculus**.

REFERENCES *Saks* [Sa, pp. 226–227], *Walter* [WW, pp. 54–57], *Zygmund* [Z, vol. II, pp. 88–89].

**Descartes Rule of Signs** *If* $r$ *is the number of positive roots of the polynomial* $p(x) = a_0 + \cdots a_n x^n$*, where the coefficients are all real, and if* $v$ *the number of sign variations in* $\{a_n, \ldots, a_0\}$ *then*

$$0 \leq r \leq v;$$

*more precisely* $v - r$ *is always an even non-negative integer.*

COMMEMT In calculating $v$ all zeros in the coefficients are ignored, and in calculating $r$ multiple roots are counted according to their muliplicity.

REFERENCES [EM, vol.3, p. 59].

**Determinant Inequalities** *(a)* [FAN] *If* $A, B$ *are real positive definite matrices and* $0 \leq \lambda \leq 1$*, then*

$$\det\big((1-\lambda)A + \lambda B\big) \geq (\det A)^{1-\lambda} \det B^{\lambda}.$$

*(b)* [MINKOWSKI] *If* $A, B$ *are non-negative* $n \times n$ *Hermitian matrices then*

$$(\det A)^{1/n} + (\det B)^{1/n} \leq (\det(A+B))^{1/n}; \tag{1}$$

*and hence,*

$$\det A + \det B \leq \det(A+B).$$

*(c) If* $A$ *is a real positive definite* $n \times n$ *matrix then*

$$|\det A| \leq |\det A_{1,2,\ldots k}||\det A_{k+1,\ldots,n}|.$$

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*In particular*

$$|\det A| \le \left|\prod_{i=1}^{n} a_{ii}\right|.$$

*(d) If* $C = (c_{ij})_{1\le i,j\le n} = (a_i^{r_j})_{1\le i,j\le n}$, *where* $\underline{a}, \underline{r}$ *are strictly decreasing n-tuples,* $\underline{a}$ *positive, then* $\det C > 0$.
*Further if for some* $k$, $1 < k \le n$, $c'_{ij} = c_{ij}$, $i \ne k-1, c'_{k-1j} = r_k c_{kj}$ *and* $C' = (c'_{ij})_{1\le i,j\le n}$ *then* $\det C' > 0$.

COMMENTS (i) In (d) $\det C'$ is just $a_k \dfrac{\partial \det C}{\partial a_k}\Big|_{a_{k-1}=a_k}$.

In other words the $(k-1)$th column of $C$ is replaced by the column $(r_j a_i^{r_j})_{1\le j\le n}$. This covers the case when $\underline{a}$ is strictly decreasing except that $a_{k-1} = a_k$. Further extensions to general decreasing positive $n$-tuples can easily be made.

(ii) The inequality (d) is due to Good but Ursell's proof allows for extension to general decreasing $n$-tuples.

EXAMPLE If $a > b > 0$ then

$$\begin{vmatrix} \pi^2 a^\pi & \pi a^\pi & a^\pi & \pi b^\pi & b^\pi \\ e^2 a^e & e a^e & a^e i & e b^e & b^e \\ 4a^2 & 2a^2 & a^2 & 2b^2 & b^2 \\ 0 & 0 & 1 & 0 & 1 \\ 2a^{-\sqrt{2}} & -\sqrt{2}a^{-\sqrt{2}} & a^{-\sqrt{2}} & -\sqrt{2}b^{-\sqrt{2}} & b^{-\sqrt{2}} \end{vmatrix} > 0.$$

EXTENSIONS *(a)* [FAN] *(i) If* $A, B$ *are positive definite* $n \times n$*-matrices and if* $|A|_k$ *is the product of the first* $k$ *smallest eigenvalues and if* $0 \le \lambda \le 1$ *then*

$$|\lambda A + (1-\lambda)B|_k \le |A|_k^\lambda |B|_k^{1-\lambda}.$$

*(ii) If* $A, B$ *are real symmetric* $n \times n$*-matrices and if* ${}_k|A|$ *is the sum of the first* $k$ *largest eigenvalues and if* $0 \le \lambda \le 1$ *then*

$${}_k\,|\lambda A + (1-\lambda)B| \ge {}_k|A|^\lambda\,{}_k|B|^{1-\lambda}.$$

*(b)* [OPPENHEIM] *Under same assumptions as in (a),*

$$|A+B|_k^{1/k} \le |A|_k^{1/k} + |B|_k^{1/k}.$$

COMMENTS (iii) Many variants and extensions can be found in [MPF].

(iv) See also: **Bergström's Inequality, Circulant Matrix Inequalities, Gram Determinant Inequalities, Hadamard's Determinant Inequality, Permanent Inequalities**.

REFERENCES [BB, pp. 63–64, 74–75], [EM, vol.6, p. 248], [HLP, p. 16], [MPF, pp. 211–238]; *Marcus & Minc* [MM, pp. 115,117]; *Good* [133], *Ursell* [317].

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**Díaz & Metcalf Inequality** See: **Polyá & Szegő's Inequality** Extensions (b).

**Difference Means of Gini** If $m > 1$, and $\underline{a}$ is a positive $m$-tuple, the *Gini difference mean of* $\underline{a}$ is,

$$\mathfrak{D}_m(\underline{a}) = \frac{2}{m(m-1)} \sum_{\substack{i,j=1\\1\le i<j\le m}}^{m} |a_i - a_j|.$$

If then $\underline{b}$ is a positive $n$-tuple, $n > 1$, the *Gini mixed difference mean of* $\underline{a}$ and $\underline{b}$ is,

$$\mathfrak{D}_{m,n}(\underline{a}, \underline{b}) = \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} |a_i - b_j|.$$

Anand-Zagier Inequality *If* $\underline{a}$ *is a positive* $m$*-tuple,* $\underline{b}$ *is a positive* $n$*-tuple, with* $m, n > 1$*, then,*

$$\frac{\mathfrak{D}_{m,n}(\underline{a}, \underline{b})}{\mathfrak{A}_m(\underline{a})\mathfrak{A}_n(\underline{b})} \ge \frac{1}{4}\left(\frac{(m-1)\mathfrak{D}_m(\underline{a})}{m\mathfrak{A}_m^2(\underline{a})} + \frac{(n-1)\mathfrak{D}_n(\underline{b})}{n\mathfrak{A}_n^2(\underline{b})}\right). \tag{1}$$

Comments (i) This inequality was conjectured by Anand and proved by Zagier.

(ii) The quantity $\phi_{11}(\underline{a}) = (n-1)\mathfrak{D}_n(\underline{a})/4n\mathfrak{A}(\underline{a})$ was suggested by Gini as a measure of inequality, the *Gini coefficient*.

(iii) It is easy to see that

$$\phi_{11}(\underline{a}) = 1 - \frac{\sum_{i,j=1}^{n} \min\{a_i, a_j\}}{n^2\mathfrak{A}_n(\underline{a})} = 1 + \frac{1}{n} - \frac{2\sum_{i=1}^{n} i a_{[i]}}{n^2\mathfrak{A}_n(\underline{a})}.$$

(iv) As a result of the previous comment it is seen that (1) is equivalent to

$$2\frac{\sum_{i,j=1}^{n} \min\{a_i, b_j\}}{\mathfrak{A}_n(\underline{a})\mathfrak{A}_n(\underline{b})} \le \frac{\sum_{i,j=1}^{n} \min\{a_i, a_j\}}{\mathfrak{A}_n(\underline{a})} + \frac{\sum_{i,j=1}^{n} \min\{b_i, b_j\}}{\mathfrak{A}_n(\underline{b})}.$$

(v) An integral analogue for these means can be found in **Mulholland's Inequality**.

References [EM, vol. 4, p. 277], [MOA, p. 563], [MPF, pp. 584–586].

**Digamma Function Inequalities** The *digamma or psi function* is defined as:

$$\mathcal{F}(z) = \Psi(z+1) = (\log z!)'.$$

The derivatives of this function are called the the *multigamma*, or *polygamma, functions*; in particular the *trigamma function, tetragamma function*, etc.

*(a) The digamma function is increasing and concave on* $]0, \infty[$*; that is if* $x, y, u, v$ *are positive with* $x < y$*, and if* $0 \le t \le 1$*, then*

$$\Psi(x) \le \Psi(y), \quad \Psi(\overline{1-t}\,u + tv) \ge (1-t)\Psi(u) + t\Psi(v),$$

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*respectively.*
*(b) The reciprocal of the trigamma function is convex; equivalently*
*(i)*

$$\Psi''^2 \geq \frac{\Psi'\Psi'''}{2}.$$

*or*

$$(ii) \qquad \sum_{k\in\mathbb{N}} \frac{1}{(z+k)^3} > \frac{\sqrt{3}}{2}\left(\sum_{k\in\mathbb{N}} \frac{1}{(z+k)^2}\right)^{1/2}\left(\sum_{k\in\mathbb{N}} \frac{1}{(z+k)^4}\right)^{1/2}.$$

Comments (i) Inequality (b)(ii) shows that convexity property is a form of (∼C).

(ii) An interesting collection of inequalities for the multigamma functions have been given by Alzer.

References *Abramowitz & Stegun* [AS, pp. 258-260]; *Jeffreys & Jeffreys* [JJ, pp. 465–466]; *Alzer* [34], *Dragomir,Agarwal & Barnett* [104], *Trimble,Wells & F. T. Wright* [314].

**Dirichlet Kernel Inequalities** The *Dirichlet kernel*, or *Dirichlet summation kernel* is defined by:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{1}{2}\sum_{k=-n}^{n} e^{ikx} = \frac{\sin(n+1/2)x}{2\sin x/2},\ n \in \mathbb{N};$$

The *modified Dirichlet kernel* is :

$$D_n^*(x) = D_n(x) - \frac{1}{2}\cos nx = \frac{\sin nx}{\tan x/2},\ n \in \mathbb{N},$$

*(a) If $n \in \mathbb{N}$ then*

$$|D_n^*(x)| \leq n.$$

*(b) If $n \geq 1$ and $0 < t \leq \pi$ then*

$$|D_n^*(x)| \leq \frac{1}{t}.$$

*(c) If $0 \leq x \leq \pi$ then*

$$|D_n^{(r)}(x)| \leq \frac{Cn^{r+1}}{1+nx}.$$

*(d)* [Makai] *If $n \geq 1$ and $k, m \in \mathbb{N}$ then*

$$\sum_{i=0}^{m} D_k\left(\frac{2\pi}{n}i\right) > 0;$$

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*in particular if* $0 \le k < n, 1 \le m < n$,

$$\sum_{i=1}^{m} D_k\left(\frac{2\pi}{n}i\right) < \min\left\{\frac{n}{2}, \frac{n-k+m}{2}\right\}.$$

COMMENTS (i) The Dirichlet kernel has analogues in more general forms of convergence. These more general kernels have inequalities that are extensions of the inequalities above. For the case of $(C,1)$-summability see: **Fejér Kernel Inequalities**; and for Abel summability see: **Poisson Kernel Inequalities**.

(ii) See also: **Lebesgue Constants**.

REFERENCESS [AI, p. 248], [MPF, p. 584]; *Zygmund* [Z, vol. I, pp. 49–51, 94–95, vol. II, p. 60].

**Distortion Theorems** [KOEBE] *If $f$ is univalent in $D$, with $f(0) = 0, f'(0) = 1$, then*

$$\begin{aligned}\frac{|z|}{(1+|z|)^2} \le |f(z)| &\le \frac{|z|}{(1-|z|)^2};\\ \frac{1-|z|}{(1+|z|)^3} \le |f'(z)| &\le \frac{1+|z|}{(1-|z|)^3}.\end{aligned}$$

*There is equality if and only if $f$ is a rotation of the Koebe function.*

COMMENTS (i) The *Koebe function* is $f(z) = \sum_{n=1}^{\infty} nz^n = z\big/(1-z)^2$, and if $\alpha \in \mathbb{R}$, $a(z) = z\big/(1-e^{i\alpha}z)^2$ is a *rotation of the Koebe function.*

(ii) These results give estimates for the distortion at a point under a conformal map.

EXTENSION *If $K$ is a compact subset of the region $\Omega \subseteq \mathbb{C}$, then there is a constant $M$, depending on $K$, such that for all $f$, univalent on $\Omega$, and all $z, w \in K$,*

$$\frac{1}{M} \le \frac{|f'(z)|}{|f'(w)|} \le M.$$

COMMENT (iii) See also: **Rotation Theorems**.

REFERENCES [EM, vol.3, pp. 269–271]; *Ahlfors* [Ah73, pp. 84–85], *Conway* [C, vol. II, pp. 65–70], *Gong* [GS].

**Dočev's Inequality** See: **Geometric-Arithmetic Mean Inequality**, CONVERSE INEQUALITIES (b).

**Doob's Upcrossing Inequality** *If $\mathcal{X} = (X_n, \mathcal{F}_n, n \in \mathbb{N})$ is a submartingale, and if $\beta_m(a,b)$ is the number of upcrossings of $[a,b]$ by $\mathcal{X}$ in $m$ steps then*

$$E\beta_m(a,b) \le \frac{E|X_m| + |a|}{b-a}.$$

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COMMENT For another inequality by Doob see: **Martingale Inequalities** (A).

REFERENCES [EM, vol.6, p. 110]; *Loève* [L, p. 532].

**Doubly Stochastic Inequalities** See: **Order Inequalities** (A), **van der Waerden's Conjecture**.

**Dresher's Inequality** See: **Gini-Dresher Mean Inequalities** (C).

**Duff's Inequality** *If* $p > 0$ *and* $f$ *differentiable almost everywhere on* $[0, a]$ *then*

$$\int_0^a |(f')^*|^p \leq \int_0^a |f'|^p.$$

COMMENT For the definition of decreasing rearrangement of a function see: **Notations 5**.

EXTENSION [MITRINOVIĆ & PEČARIĆ] *If* $p > 0$ *and* $f$ *differentiable almost everywhere on* $[0, a]$ *and* $H$ *a non-decreasing function then*

$$\int_0^a H \circ |(f')^*| \leq \int_0^a H \circ |f'|.$$

REFERENCES *Duff* [108], *Mitrinović & Pečarić* [215].

**Dunkl & Williams Inequality** *If* $X$ *is a unitary space,* $x \neq 0, y \neq 0$ *then*

$$||x - y|| \geq \left(\frac{||x|| + ||y||}{2}\right) \left|\left|\frac{x}{||x||} - \frac{y}{||y||}\right|\right|,$$

*with equality if and only if either* $||x|| = ||y||$*, or* $||x|| + ||y|| = ||x - y||$*.*

COMMENTS (i) For the definition of a unitary space see: **Inner Product Space Inequalities** COMMENTS (i).

(ii) This is related to (T).

(iii) It appears to be an open question as to whether this inequality characterizes inner product space.

REFERENCE [MPF, pp. 515–519].

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# 5 Efron–Extended

**Efron-Stein Inequality**[23] *If $X_i, 1 \le i \le n+1$, are independent identically distributed random variables and $S$ a symmetric function then,*

$$\sigma^2 S(X_1, \ldots, X_n) \le EQ;$$

*where, writing* $\tilde{S} = \frac{1}{n+1}\sum_{i=1}^{n+1} S(X_, \ldots, X_i, X_{i+1}, \ldots, X_{n+1})$,

$$Q = \sum_{i=1}^{n+1}\big(S(X_, \ldots, X_i, X_{i+1}, \ldots, X_{n+1}) - \tilde{S}\big)^2.$$

Comment A definition of symmetric functions is given in **Segre's Inequalities**.

References *Tong, ed.* [T, pp. 112–114].

**Eigenvalue Inequalities** *(a)* [Schur] *If $A = \{a_{ij}\}$ is an $n \times n$ complex matrix then*

$$\sum_{i=1}^{n} |\lambda_i(A)|^2 \le \sum_{i,j=1}^{n} |a_{ij}|^2, \tag{1}$$

*with equality if and only if $A$ is normal.*
*(b) If $A, B$ are two non-negative $n \times n$ matrices, such that $B - A$ is also non-negative then,*

$$|\lambda_{[1]}(A)| \le |\lambda_{[1]}(B)|.$$

*(c)* [Cauchy] *If $A$ is a Hermitian $n \times n$ matrix then for $1 \le s \le n - r$,*

$$\lambda_{[s+r]}(A) \le \lambda_{[s]}(A_{i_1,\ldots,i_{n-r}}) \le \lambda_{[s]}(A).$$

*(d) If $A, B$ are non-negative $n \times n$ matrices and if $C$ is their Hadamard product then*

$$\lambda_{[1]}(C) \le \lambda_{[1]}(A)\lambda_{[1]}(B).$$

*(e) If $A$ is an $n \times n$ complex matrix then*

$$|\lambda_{[1]}(I + A)| \le 1 + \lambda_{[1]}(A);$$

[23]This is C. Stein.



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*there is equality if $A$ is non-negative.*
*(f)* [Komaroff] *If $A, B$ are Hermitian matrices, $A$ positive semi-definite, then for each $m, 1 \leq m \leq n$*

$$\begin{aligned}\sum_{k=1}^{m} \lambda_{(n-k+1)}(AB) &\leq \sum_{k=1}^{m} \lambda_{(k)}(A)\lambda_{(k)}(B);\\ \sum_{k=1}^{m} \lambda_{(k)}(A)\lambda_{(n-k+1)}(B) &\leq \sum_{k=1}^{m} \lambda_{(k)}(AB).\end{aligned}$$

Comments (i) Schur's result has been used to prove (GA). $A$ is a *normal matrix* if it is Hermitian and $AA^* = A^*A$.
(ii) $\rho(A) = |\lambda_{[1]}(A)|$ is called the *spectral radius of $A$*.
(iii) See also: **Determinant Inequalities** Extensions, **Hirsch's Inequalities, Rayleigh-Ritz Ratio, Weyl's Inequality**.

References [H, p. 102]; *Horn & Johnson* [HJ, pp. 491, 507], *Marcus & Minc* [MM, pp. 119, 126, 142]; *Komaroff* [161].

**Elementary Symmetric Function Inequalities** *(a) If $1 \leq r \leq n-1$ then*

$$e_n^{[r-1]}(\underline{a})e_n^{[r+1]}(\underline{a}) < \left(e_n^{[r]}(\underline{a})\right)^2; \tag{1}$$
$$p_n^{[r-1]}(\underline{a})p_n^{[r+1]}(\underline{a}) \leq \left(p_n^{[r]}(\underline{a})\right)^2; \tag{2}$$

*with equality in (2) if and only if $\underline{a}$ is a constant.*
*(b) If $1 \leq r < s \leq n$ then,*

$$\begin{aligned}e_n^{[r-1]}(\underline{a})e_n^{[s]}(\underline{a}) &< e_n^{[r]}(\underline{a})e_n^{[s-1]}(\underline{a});\\ p_n^{[r-1]}(\underline{a})p_n^{[s]}(\underline{a}) &\leq p_n^{[r]}(\underline{a})p_n^{[s-1]}(\underline{a});\end{aligned}$$

*with equality in the second if and only if $\underline{a}$ is constant.*
*(c) If $1 \leq r \leq n-1$ then,*

$$\begin{aligned}e_n^{[r-1]}(\underline{a}) > e_n^{[r]}(\underline{a}) &\implies e_n^{[r]}(\underline{a}) > e_n^{[r+1]}(\underline{a});\\ p_n^{[r-1]}(\underline{a}) > p_n^{[r]}(\underline{a}) &\implies p_n^{[r]}(\underline{a}) > p_n^{[r+1]}(\underline{a}).\end{aligned}$$

*(d) If $1 \leq r+s \leq n$ then,*

$$p_n^{[r+s]}(\underline{a}) \leq p_n^{[r]}(\underline{a})p_n^{[s]}(\underline{a}),$$

*with equality if and only if $\underline{a}$ is constant.*

Comments (i) Inequality (1) is an easy consequence of (2). In fact (2) gives more,

$$e_n^{[r-1]}(\underline{a})e_n^{[r+1]}(\underline{a}) \leq \frac{r(n-r)}{(r+1)(n-r+1)}\left(e_n^{[r]}(\underline{a})\right)^2.$$

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A similar result to this last inequality is EXTENSION (e) below.

An inductive proof of (1) can be given; both (1) and (2) follow from **Newton's Inequalities** (1). Note that (2) in the case $n = 2$ is just (GA).

(ii) (b), (c) are immediate consequences of (1) and (2), and (d) follows from (2).

(iii) Extensions of these "quadratic" inequalities to "cubic" inequalities have been given by Rosset.

EXTENSIONS *(a)* [LOG-CONCAVITY] *If* $1 \le t < r < s \le n$ *then*

$$p_n^{[r]}(\underline{a}) \ge \left(p_n^{[t]}(\underline{a})\right)^{(s-r)/(s-t)} \left(p_n^{[s]}(\underline{a})\right)^{(r-t)/(s-t)}.$$

*There is equality if and only if* $\underline{a}$ *is constant.*

*(b)* [POPOVICIU-TYPE] *If* $1 \le r < s \le n$*, and if* $0 < p \le q$ *then*

$$\left(\frac{e_n^{[r]}(\underline{a})}{e_{n+1}^{[r]}(\underline{a})}\right)^p \ge \left(\frac{e_n^{[s]}(\underline{a})}{e_{n+1}^{[s]}(\underline{a})}\right)^q.$$

*(c)* [OBREŠKOV]

$$\frac{\left(3e_n^{[3]}(\underline{a}) - e_n^{[1]}(\underline{a})e_n^{[2]}(\underline{a})\right)^2}{2\left((e_n^{[1]}(\underline{a}))^2 - 2e_n^{[2]}(\underline{a})\right)\left(2(e_n^{[2]}(\underline{a}))^2 - 3e_n^{[1]}(\underline{a})e_n^{[3]}(\underline{a})\right)} \le \frac{n-1}{n},$$

*with equality if and only if* $\underline{a}$ *is constant.*

*(d)*

$$e_n^{[r]} > 0, 1 \le r \le n \iff a_r > 0, 1 \le r \le n.$$

*(e)* [JECKLIN] *If* $1 \le 2r \le n$ *then*

$$\sum_{i=1}^{n}(-1)^{i+1}e_{n-i}^{[r-1]}(\underline{a})e_{n+i}^{[r+1]}(\underline{a}) \le \frac{\binom{n}{r}-1}{2\binom{n}{r}}\left(e_n^{[r]}(\underline{a})\right)^2.$$

*(f)* [ŌZEKI] *If* $\underline{a}$ *is a positive log-convex sequence then so is* $p_n^{[r]}(\underline{a})$*, that is*

$$p_{n-1}^{[r]}(\underline{a})p_{n+1}^{[r]}(\underline{a}) \ge \left(p_n^{[r]}(\underline{a})\right)^2, 1 \le k \le n-1.$$

*(g)* [LIN MI & TRUDINGER]

$$e_n^{[r-1]}(\underline{a}_i') \le C_{n,r}e_n^{[r-1]}(\underline{a}).$$

COMMENTS (iv) The log-concavity result is an easy induction from (2). See: [*Ku, Ku & Zhang*].

(v) A series of elementary inequalities in the case $n = 3$ have been given by Klamkin and can be found in [MPF].

(vi) Elementary symmetric function inequalities can be regarded as examples of **Cyclic Inequalities**.

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(vii) $e_n^{[r]}$ is Schur concave, strictly so if $r > 1$; for the definition see: **Schur Convex Function Inequalities**.

(viii) The constant in the inequality in (g) is estimated in the reference.

(ix) See also: **Complete Symmetric Function Inequalities, Marcus & Lopes Inequality, Symmetric Mean Inequalities**.

References [AI, pp. 95–107, 211–212, 336–337], [H, pp. 321–338], [HLP, pp. 51–55, 104–105], [MOA, pp. 104 –118], [MPF, pp. 163–164]; *Ku, Ku & Zhang* [163], *Lin & Trudinger* [174], *Rosset* [283].

**Elliptic Integral Inequalities** The integral

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - mt^2}} \,\mathrm{d}t$$

is called the *complete elliptic integral of the first kind*; $m = \sin^2 \alpha$ is called the *parameter*, $\alpha$ the *modular angle*, in particular, then $0 \le m \le 1$. The usage *complete* refers to the upper limit, or *amplitude*, being $\pi/2$; other amplitudes, or upper limits, give what is called the *incomplete elliptic integral of the first kind*. The elliptic integrals of the second and third kind are also defined. These integrals occur in consideration of the arc-length of ellipses, and surface areas of ellipsoids. In addition they are the basis for the definition of the Jacobian elliptic functions.

$$1 + \left(\frac{\pi}{\log 16} - 1\right)(1 - m) < \frac{K(m)}{\log(4/\sqrt{1-m})} < 1 + \frac{1}{4}(1 - m);$$

*the constants* $\dfrac{\pi}{\log 16}$ *and* $\dfrac{1}{4}$ *are sharp.*

Comment See also: **Symmetric Elliptic Integral Inequalities**.

References *Abramowitz & Stegun* [AS, pp. 589–590]; *Alzer* [29].

**Enflo's Inequality** *Let* $\underline{u}_i, 1 \le i \le m$, $\underline{v}_j, 1 \le j \le n$, *be vectors in* $\mathbb{R}^k$ *then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} |\underline{u}_i.\underline{v}_j|^2 \le \left(\sum_{i=1}^{m}\sum_{i'=1}^{m} |\underline{u}_i.\underline{u}_{i'}|^2\right)^{1/2} \left(\sum_{j=1}^{n}\sum_{j'=1}^{n} |\underline{v}_j.\underline{v}_{j'}|^2\right)^{1/2}.$$

Comment This result can be extended to general inner product spaces.

Reference *Steele* [S, pp. 225, 284].

**Entire Function Inequalities** *(a)* [Hadamard] *If $f$ is an entire function of finite order $\lambda$, and if $\gamma$ is the genus of $f$, then,*

$$\gamma \le \lambda \le \gamma + 1.$$

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*(b) If $f$ is an entire function of order at most $\sigma$ then*

$$\sup_{x\in\mathbb{R}}|f^{(r)}(x)| \le \sigma^r \sup_{x\in\mathbb{R}}|f(x)|,\ r\in\mathbb{N}.$$

*(c) If $f$ is an entire function of finite order $\lambda$ and if $n(r)$ is the number of zeros of $f$ in $\{z;\ |z|\le r\}$ then for all $\epsilon>0$ there is a constant $K$ such that*

$$n(r) < Kr^{\lambda+\epsilon}.$$

COMMENT (i) The *order, $\lambda$, of an entire function $f$* is the infimum of the $\mu$ such that

$$\max_{|z|\le r}|f(z)| < e^{r^\mu}, \quad r>r_0.$$

(ii) The *genus of an entire function $f$* is the smallest integer $\gamma$ such that $f$ can be represented in the form

$$f(z) = z^m e^{g(z)} \prod_n \left(1-\frac{z}{a_n}\right) e^{z/a_n+(1/2)(z/a_n)^2+\cdots+(1/\gamma)(z/a_n)^\gamma},$$

where $g$ is a polynomial of degree at most $\gamma$; if there is no such representation then $\gamma=\infty$.

(iii) A special case of (b) is **Bernšteĭn's Inequality** (a).

REFERENCES [EM, vol. 1, p. 366; vol.3, p. 385–387]; *Ahlfors* [Ah78, pp. 194, 206–210], *Titchmarsh*, [T75, 1975, p. 249].

**Entropy Inequalities** *If $(X,\mathcal{A},P)$ is a finite probability space with $\mathcal{B}\subseteq\mathcal{A}$ and $H$ the entropy function, then*

$$\begin{aligned} H(\mathcal{A}|\mathcal{B}) &\le H(\mathcal{A}); \qquad (1)\\ H(\mathcal{A}\mathcal{B}) &\le H(\mathcal{A})+H(\mathcal{B}). \end{aligned}$$

COMMENTS (i) The entropy function $H$ is defined in **Shannon's Inequality** COMMENTS (i). In the present context

$$H(\mathcal{A}) = -\sum_{A\in\mathcal{A}} p(A)\log_2 p(A).$$

(ii) The main results in this area are due to Hinčin who called inequality (1) *Shannon's fundamental inequality*.

(iii) In the case that $X$ is a two element space with probabilities $p,q,\ p+q=1$ we write $H(p.q) = -p\log_2 p - q\log_2 q$ and the following bounds have been given

$$\begin{aligned} \log 2 &\le \frac{H(p,q)}{\log_2 p \log_2 q} \le 1,\\ 4pq &\le H(p,q) \le (4pq)^{1/2\log 2}. \end{aligned}$$

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(iv) See also: **Shannon's Inequality, Rényi's Inequalities**.

REFERENCES [EM, vol.3, pp. 387–388], [GI2, pp. 435–445], [GI5, pp. 411–417]; [MPF, pp. 646–648]; *Tong, ed.* [T, pp. 68–77]; *Sándor* [288], *Topsøe* [313].

## Enveloping Series Inequalities

*If $a \in \mathbb{C}$ and if the series $\sum_{n\in\mathbb{N}} a_n$ envelopes $a$, then for all $n \in \mathbb{N}$,*

$$\left| a - \sum_{k=0}^{n} a_k \right| < |a_{n+1}|.$$

COMMENTS (i) This is just the definition of *a series that envelopes a number*. In particular this is the case in the real situation if for all $n \in \mathbb{N}$, $a - \sum_{k=0}^{n} a_k = \theta_n a_{n+1}$, for some $\theta_n, 0 < \theta_n < 1$.

(ii) **Gerber's Inequality** shows that the Taylor series of the function $(1+x)^\alpha$ envelopes the function; and the **Trigonometric Function Inequalities** (m) show the same for sine and cosine functions.

(iii) The first reference gives many such results for various special functions.

REFERENCES [GI2, pp. 161–175]; *Pólya & Szegö*, [PS, pp. 32–36].

## Equimeasurable Function Inequalities

See: **Duff's Inequality, Hardy-Littlewood Maximal Inequalities, Modulus of Continuity Inequalities** (A)**, Spherical Rearrangement Inequalities, Variation Inequalities**.

## Equivalent Inequalities

Many very different pairs of inequalities are equivalent in the sense that given the one inequality the other can be proved and conversely. This is pointed out in various entries.

See: **Bernoulli's Inequality** COMMENTS (VIII), **Bilinear Form Inequalities of M. Riesz, Cauchy's Inequality** COMMENTS (I), **Cordes's Inequality** COMMENT, **Geometric-Arithmetic Mean Inequality** COMMENTS (III), **Geometric Mean Inequalities** COMMENTS (I), **Harmonic Mean Inequalities** COMMENTS (I), (V) **Hausdorff-Young Inequalities** COMMENTS (I), **Heinz-Kato-Furuta Inequality** COMMENTS (IV), **Hölder's Inequality** COMMENTS (II), **Isoperimetric Inequalities** COMMENTS (I), **Kantorovič's Inequality** COMMENTS (I), **Löwner-Heinz Inequality** COMMENTS (IV), **Paley's Inequalities** COMMENTS (III), **Polyá & Szegő's Inequality** COMMENTS (I),(IV), **Popoviciu's Geometric-Arithmetic Mean Inequality Extension** COMMENTS (II), **Power Mean Inequalities** COMMENTS (II), **Q-class Function Inequalities** COMMENTS (I), **Rado's Geometric-Arithmetic Mean Inequality Extension** COMMENTS (I), **Rennie's Inequality** COMMENTS (I), **Sobolev's Inequalities** COMMENTS (II), **Steffensen's Inequality** COMMENTS (II), **Walsh's Inequality** COMMENTS (II).

A detailed discussion can be found in the references.

REFERENCES [H, pp. 127, 183, 213, 271–273], [MOA, pp. 657–670], [MPF, pp. 191–209].

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**Erdős's Inequalities** *(a) Let $p$ be a polynomial of degree $n$ with real zeros all of which lie outside the interval* $]-1,1[$ *then*

$$||p'||_{\infty,[-1,1]} \leq \frac{en}{2}||p||_{\infty,[-1,1]}.$$

*(b) If $p_n(z)$ is a complex polynomial that does not vanish in $D$ then*

$$||p_n'||_{\infty,|z|=1} \leq \frac{1}{2}n||p_n||_{\infty,|z|=1},$$

*with equality if and only if $p_n(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.*
*(c) If $-1 \leq a_1 < a_2 < \cdots < a_n \leq 1$ then*

$$\sum_{1 \leq i < j \leq n} \frac{1}{a_j - a_i} \geq \frac{n^2 \log n}{8}.$$

COMMENTS (i) (a) should be compared with **Markov's Inequality**,[24] and (b) with **Bernšteĭn's Inequality** (c).

(ii) Another inequality where the zeros are restricted, the polynomial being required to be zero at $\pm 1$, has been given by Schur.

EXTENSION [MÁTÉ] *Let $p$ be a polynomial of degree $n$ with $n-k$ real roots all of which lie outside the interval* $]-1,1[$ *then*

$$||p'||_{\infty,[-1,1]} \leq 6ne^{\pi\sqrt{k}}||p||_{\infty,[-1,1]}.$$

REFERENCES *Milovanović, Mitrinović & Rassias* [MMR, p. 628]; *Steele* [S, pp. 134, 260–261]; *Govil & Nyuydinkong* [134], *Máté* [201], *Nikolov* [236].

**Erdős & Grünwald Inequality** *If $p$ is a real polynomial with $p(a) = p(b) = 0, a < b$ and if $p(x) > 0, a < x < b$, then:*

$$\frac{(b-a)^2}{3}\frac{p'(a)p'(b)}{p'(a)-p'(b)} \leq \int_a^b p \leq \frac{2(b-a)}{3} \max_{a \leq x \leq b} p(x).$$

COMMENTS (i) This can be interpreted as:

*the area under the graph of the polynomial $p$ lies between two-thirds of the area of the containing rectangle and two-thirds of the area of the tangential triangle.*

(ii) This result has been extended to higher dimensions by *René*.

REFERENCE *René* [281].

[24] That is A. A. Markov.

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**Erdős-Mordell Inequality** *If $P$ is interior to a triangle then the sum of its distances from the sides of the triangle is at most one-half of the sum of its distances from the vertices of the triangle.*

EXTENSION *If $P$ is interior to a convex $n$-gon then the sum of its distances from the sides of the polygon is at most $\cos \pi/n$ times the sum of its distances from the vertices of the polygon*

COMMENTS (i) The Erdős-Mordell inequality is the case $n = 3$ of the extension.
(ii) The extension is obtained using the **Fejes Tóth Inequality**.

REFERENCES *Abi-Khuzam* [1].

**Ergodic Inequality** *If $T : \mathcal{L}_\mu(X) \to \mathcal{L}_\mu(X)$ is a linear operator with $|T|_1 \leq 1$, $|T|_\infty \leq 1$, and if $f \in \mathcal{L}^p_\mu(X)$ write*

$$\tilde{f} = \sup_{n\geq 1} \left| \frac{1}{n} \sum_{k=0}^{n-1} T^k f \right|.$$

*Then, with $q$ the conjugate index,*

$$||\tilde{f}||_{p,\mu} \leq \begin{cases} 2q^{1/p}||f||_{p,\mu}, & \text{if } 1 < p < \infty, \\ 2\left(\mu(X) + \int_X |f| \log^+ |f| \,\mathrm{d}\mu\right) & \text{if } p = 1. \end{cases}$$

COMMENT $|T|_p = \sup ||Tf||_{p,\mu}$ where the supremum is taken over all functions $f$ with $f \in \mathcal{L}_\mu(X)$ with $||f||_{p,\mu} < \infty$.

REFERENCES *Dunford & Schwarz* [DS, pp. 669, 678–679]; *Îto* [I, vol. I, pp. 531–533].

**Erhard's Inequality** See: **Gaussian Measure Inequalities** COMMENT (I).

**Error Function Inequalities** *(a) If $x, y \geq 0$,*

$$\mathrm{erf}(x)\,\mathrm{erf}(y) \geq \mathrm{erf}(x) + \mathrm{erf}(y) - \mathrm{erf}(x+y).$$

*(b) If $x > 0$,*

$$\mathrm{erfc}\, x < \min\left\{ e^{-x^2}, \frac{e^{-x^2}}{x\sqrt{\pi}} \right\}.$$

*(c)* [CHU]

$$\sqrt{1 - e^{-ax^2}} \leq \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} \,\mathrm{d}t \leq \sqrt{1 - e^{-bx^2}} \iff 0 \leq a \leq 1/2, \text{ and } \quad b \geq 2/\pi.$$

*(d)* [GORDON] *If $x > 0$,*

$$\frac{x}{x^2+1} \leq \mathrm{mr}(x) \leq \frac{1}{x}.$$

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*(e) The function*$(\mathrm{mr})^{-1}$*is strictly convex; equivalently,*

$$2xe^{x^2} < \int_x^\infty e^{-t^2}\,\mathrm{d}t < \frac{4x}{3x+\sqrt{4+x^2}}.$$

COMMENT (i) The lower bound in (d) has been improved by Birnbaum to $(\sqrt{4+x^2}-x)/2$. Other extensions and improvements can be found in the first reference.

EXTENSION *Let $p>0$, $x\geq 0$ and define*

$$\mathrm{mr}_p(x) = e^{x^p}\int_x^\infty e^{-t^p}\,\mathrm{d}t,$$

*which is just the Mills ratio when $p=2$. If $p\geq 2$ then $(\mathrm{mr}_p)^{-1}$ is strictly convex, equivalently,*

$$px^{p-1}e^{x^p} < \int_x^\infty e^{-t^p}\,\mathrm{d}t < \frac{4}{3+\sqrt{1+8(1-\frac{1}{p})x^{-p}}}.$$

COMMENT (ii) See also: **Conte's Inequality**.

REFERENCES [AI, pp. 177–181, 385]; *Mascioni* [200], *Chu* [89], *Qi,Cui & Xu* [278].

**Euler's Constant Inequalities** The real number

$$\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^n 1/i - \log n\right) = -(x!)'_{x=0} \approx 0.57721566490\cdots,$$

is called *Euler's constant*. It is not known whether it is irrational or not.
*(a)* [RAO]

$$\frac{1}{2n+1} < \frac{1}{2n} - \frac{1}{8n^2} < \left(\sum_{i=1}^n 1/i - \log n\right) - \gamma < \frac{1}{2n}.$$

*(b)* [SANDHAM]

$$\gamma < \sum_{i=1}^p \frac{1}{i} + \sum_{i=1}^q \frac{1}{i} - \sum_{i=1}^{pq}\frac{1}{i} \leq 1.$$

COMMENTS (i) The outer inequalities in (a) are due to various authors the stronger left-hand side is due to Rao.

(ii) The constant $\gamma$ is aso called the *Euler-Mascheroni* constant.

REFERENCES [AI, pp. 187–188], [EM, vol. 3, p. 424]; *Apostol, Chrestenson, Ogilvy, Richmond & Schoonmaker,eds.*[A69, pp. 389–390]; *Lu* [187].

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**Euler's Inequality** *If $r$ and $R$ are, respectively, the radii of the in-circle and the circum-circle of a triangle then $R \geq 2r$ with equality if and only if the triangle is equilateral.*

REFERENCES *Bulajich Manfrino, Ortega & Delgado* [BOD, p. 67]

## Exponential Function Inequalities

REAL INEQUALITIES *(a) If $a \geq b > 0$ then*

$$a^x \geq b^x,$$

*with equality if and only if either $a = b$ or $x = 0$.*

*(b) If $n \in \mathbb{N}^*, n \neq 3$ then* $$n^{1/n} < 3^{1/3}.$$

*(c)* [WANG C. L.] *If $x \neq 0$,*

$$e^x > \left(1 + \frac{x}{n}\right)^n > e^x \left(1 + \frac{x}{n}\right)^{-x}, \quad n = 1, 2, \ldots; \tag{1}$$

*and if $0 < x \leq 1$,*

$$e^x < \left(1 + \frac{x}{n}\right)^{n+1}, \quad n = 1, 2, \ldots. \tag{2}$$

*In particular, if $x \neq 0$,*

$$e^x > 1 + x. \tag{3}$$

*(d) If $x \neq e$*

$$e^x > x^e. \tag{4}$$

*(e) If $x > 0$ then*

$$x^x \geq e^{x-1}.$$

*(f) If $0 < x < e$ then*

$$(e + x)^{e-x} > (e - x)^{e+x}.$$

*(g) If $x, y > 0$ then*

$$x^y + y^x > 1.$$

*(h)* [CIBULIS] *If $1 < y < x$ then*

$$yx^y\big(y^x - (y - 1)^x\big) > xy^x\big(x^y - (x - 1)^y\big).$$

*(j) If $a, x > 0$ then*

$$e^x > \left(\frac{ex}{a}\right)^a.$$

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*(k) If* $x<1, x\neq 0$ *then*

$$e^x<\frac{1}{1-x}.$$

*(ℓ) If* $0\le x\le y$ *then*

$$1\le\frac{e^{-x}-xe^{-1/x}}{1-x}\le\frac{3}{e},$$

*where both bounds are best possible. Further the function in the center is strictly increasing.*
*(m) If* $a>b>d$, $a>c>d$ *and* $s<t, s,t\neq 0$, *then*

$$\frac{e^{as}-e^{bs}}{e^{cs}-e^{ds}}<\frac{e^{at}-e^{bt}}{e^{ct}-e^{dt}}.$$

COMMENTS (i) Inequalities (1), (2) are basic in the theory of the exponential function, the right-hand sides tending to the left-hand side as $n\to\infty$. See: **Binomial Function Inequalities** COMMENTS (iv).

If $x>0$ the right-hand side of (1) can be replaced by $e^x(1+x/n)^{-x/2}$.

(ii) Inequality (3) is an immediate consequence of Taylor's Theorem. In a certain sense (3) characterizes the number $e$ in that if $a^x>1+x, x\neq 0$ then $a=e$; see [Ha].

(iii) Inequality (4) follows from the strict concavity of the logarithmic function at $x=e$.

As we see from (d), the graphs of $e^x$ and $x^e$ touch at $(e,e^e)$. However the graphs of $a^x$ and $x^a$, $a>1, a\neq e$ cross at $x=a$, and at another point; call these points $\alpha,\beta$ with $\alpha<\beta$. Then $a^x<x^a$ if $\alpha<x<\beta$, while the opposite inequality holds if either $x<\alpha$ or $x>\beta$. [If $0<a<1$ then we can take $\alpha=a,\beta=\infty$]. In particular we have that $e^\pi>\pi^e$.

(iv) The inequality in (j) is important for large values of $a$.

(v) The inequality (ℓ) is due to Alzer and the final comment is in the second reference.

(vi) (m) is a consequence of the mean value theorem of differentiation.

COMPLEX INEQUALITIES *(a) If* $z\in\mathbb{C}$ *with* $0<|z|<1$ *then*

$$\frac{|z|}{4}<|e^z-1|<\frac{7|z|}{4}.$$

*(b) If* $z\in\mathbb{C}$

$$|e^z-1|\le e^{|z|}-1\le|z|e^{|z|}.$$

*(c)* [KLOOSTERMAN] *If* $z\in\mathbb{C}$ *then*

$$\left|e^z-\Big(1+\frac{z}{n}\Big)^n\right|<\left|e^{|z|}-\Big(1+\frac{|z|}{n}\Big)^n\right|<\frac{|z|^2}{2n}e^{|z|}.$$

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INEQUALITIES INVOLVING THE REMAINDER OF THE TAYLOR SERIES *Let the function $I_n$ denote the $(n+1)$-th remainder of the Taylor series for $e^x$; that is,*

$$I_n(x) = e^x - \sum_{i=0}^{n} \frac{x^i}{i!} = \sum_{i=n+1}^{\infty} \frac{x^i}{i!}, \quad x \in \mathbb{R}, \text{ or } \mathbb{C}.$$

*(a)* [SEWELL] *If $x \geq 0$ then*

$$I_n(x) \leq \frac{xe^x}{n}.$$

*(b) If $|z| \leq 1$ then*

$$|I_n(z)| \leq \frac{1}{(n+1)!}\left(1 + \frac{2}{n+1}\right).$$

*(c)* [ALZER] *if $n \geq 1, x > 0$ then*

$$I_n(x)I_{n+1}(x) > \frac{n+1}{n+2} I_n^2(x).$$

*The constant is best possible.*

COMMENT (vii) This last result has been extended in *Alzer, Brenner & Ruehr*.

EXTENSIONS *(a) If $x \neq 0$ and if $n$ is odd then*

$$e^x > 1 + x + \cdots + \frac{x^n}{n!}.$$

*If $n$ is even this inequality holds if $x > 0$, while the opposite holds if $x < 0$.*
*(b)* [NANJUNDIAH]

$$\frac{2n+2}{2n+1}\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}.$$

*(c) If $x, n > 0$ then*

$$e^x < \left(1 + \frac{x}{n}\right)^{n+x/2}.$$

*In particular if $n \geq 1,\ 0 \leq x \leq n$,*

$$e^x < \left(1 + \frac{x}{n}\right)^n + \frac{x^2e^x}{n}.$$

*(d)*

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}.$$

*(e) If $\alpha \geq 1,\ 0 \leq \beta \leq e-2$, and $-1 < x < 1$, then*

$$\frac{e}{\alpha+2}\left(\alpha + \frac{2}{\sqrt{1-x^2}}\right) \leq \left(\frac{1+x}{1-x}\right)^{1/x} \leq \frac{e}{\beta+2}\left(\beta + \frac{2}{\sqrt{1-x^2}}\right)$$

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COMMENTS (viii) Inequality in (b) follows from the particular case of the **Logarithmic Mean Inequalities** (1), $\mathfrak{L}_1(a,b) > \mathfrak{L}_0(a,b) > \mathfrak{L}_{-2}(a,b)$.

(ix) See also: **Binomial Function Inequalities** COMMENTS (III),(IV), **Conte's Inequality, Series Inequalities** (D).

REFERENCES [AI, pp. 266–269, 279–281, 323–324], [H, p. 167], [HLP, pp. 103–104, 106]; *Abramowicz & Stegun* [AS, p. 70], *Apostol, Mugler, Scott, Sterrett,& Watkins, eds.*, [A92, pp. 445–452]; *Borwein & Borwein* [Bs, p. 317], *Dienes* [D, p. 135], *Halmos* [H, p. 19], *Melzak* [M, pp. 64–67]; *Alzer,Brenner & Ruehr* [39], *Cibulis* [91], *Good* [133], *Janous* [146], *Mond & Pečarić* [222], *Qi* [276], *Wang* [321].

## Extended Mean Inequalities

See: **Stolarsky Mean Inequalities**.

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# 6 Factorial–Furuta

**Factorial Function Inequalities** *(a)* [Gautschi] *(i) If* $0 \leq y \leq 1$ *then*

$$(x+1)^y \geq \frac{x!}{(x+y)!} \geq x^y;$$

*(ii)*

$$x!\left(\frac{1}{x}\right)! \geq 1.$$

*(iii) There is a constant* $\alpha, -1 < \alpha < 0$ *such that if* $\underline{a} > \alpha$ *then*

$$\mathfrak{G}_n(\underline{a}!; \underline{w}) \geq \big(\mathfrak{G}_n(\underline{a}; w)\big)!,$$

*with equality if and only if* $\underline{a}$ *is constant. The inequality is reversed if* $-1 < \underline{a} < \alpha$.

*(b)* [Watson] *If* $x \geq 0$ *then*

$$\sqrt{x + \frac{1}{4}} \leq \frac{x!}{(x - \frac{1}{2})!} \leq \sqrt{x + \frac{1}{\pi}}\,.$$

*(c)* [Gurland] *If* $n \in \mathbb{N}$

$$\sqrt{n + \frac{1}{4}} \leq \frac{n!}{(n - \frac{1}{2})!} \leq \sqrt{n + \frac{1}{4} + \frac{1}{16n+12}}\,.$$

*(d)* [Hinčin] *If* $n_i \in \mathbb{N}, 1 \leq i \leq k, n = n_1 + \cdots n_k$ *then*

$$\frac{n_1! \cdots n_k!}{(2n_i)! \cdots (2n_k)!} \leq \frac{1}{2^n}\,.$$

*(e) If* $n > 2$ *then*

$$n^{n/2} < n! < \left(\frac{n+1}{2}\right)^n.$$

*(f) Writing* $(2n)!! = \prod_{i=1}^n (2i), (2n-1)!! = \prod_{i=1}^n (2i-1)$ *we have if* $n \geq 2$ *that*

$$n^n > (2n-1)!!; \qquad (n+1)^n > 2n!! > \big((n+1)!\big)^n.$$



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*(g) If $n \ge 2$ then*

$$(n!)^4 < \left(\frac{n(n+1)^3}{8}\right)^n.$$

*(h) If $n \ge 3$ then*

$$n! < n^n < (n!)! < n^{n^n} < \big((n!)!\big)!\,.$$

*(j)* [Ostrowski] *If $n, p \in \mathbb{N}$, $0 \le x \le n$ then*

$$n! \ge \left| x^{(p-1)/p} \prod_{i=1}^{n} (x-i) \right|.$$

*(k)* [Whittaker & Watson] *If $\underline{a}$ is a positive $n$-tuple*

$$\big(\mathfrak{A}_n(\underline{a})\big)! \le \mathfrak{G}_n(\underline{a}!).$$

*(ℓ)* [Eliezer] *If $x, y > 0$,*

$$\mathfrak{G}_2\left(\frac{x!}{x^{x+1}}, \frac{y!}{y^{y+1}}\right) \ge \frac{\big((x+y)/2\big)!}{\big((x+y)/2\big)^{1/2(x+y)+1}}\,.$$

*(m) If $z \in \mathbb{C}$ then*

$$\big|z!\big| \le \big|\Re z!\big|.$$

*(n) If $m, n, p, q$ are positive real numbers with $p-m$ and $q-n$ the same sign then*

$$(p+n)!(q+m)! \le (p+q)!(m+n)!.$$

*The opposite inequality holds if $p-m$ and $q-n$ are of opposite sign.*
*In particular*

$$(p+m)! \ge \sqrt{(2p)!(2m)!}.$$

*(o) If $p > -1, |q| < p+1$ then*

$$(p!)^2 \le (p-q)!(p+q)!$$

*(p) The function $\log x!$ is superadditive if $x > 0$, and convex if $x > -1$; that is if $0 \le t \le 1$*

$$\log(x+y)! \ge \log x! + \log y!, \qquad (\overline{1-t}\,x + ty)! \le (x!)^{1-t}(y!)^t,$$

*respectively.*
*(q)*

$$e\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n$$

*(r) If $\alpha = 0.21609\ldots$, with $\underline{a}, \underline{w}$ positive $n$-tuples, $n \ge 2$, with $\underline{a} \le \alpha$, and $W_n = 1$ then*

$$(\alpha!)^{\min \underline{w}} \le \frac{\mathfrak{G}_n\big((\underline{a}-1)!; \underline{w}\big)}{\big(\mathfrak{G}_n(\underline{a}; \underline{w}) - 1\big)!} \le 1$$

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*(s)* [Alsina & Tomás] *If* $0 \le x \le 1$ *then:* $\displaystyle \frac{1}{n!} \le \frac{(x!)^n}{(nx)!} \le 1.$

*(t)* [Ivády] *If* $0 \le x \le 1$ *then:* $\displaystyle \frac{x^2+1}{x+1} \le x! \le \frac{x^2+2}{x+2}.$

Comments (i) Inequalities (a)–(d) and other similar inequalities have been discussed by Slavić.

(ii) Inequality (a) (i) was proved by Gautschi for $x \in \mathbb{N}$ and with a weaker left-hand side; in this form it is due to Wendel.

(iii) Inequality (a) (iii) is due to Lucht; the case of equal weights is due to Gautschi. The inequality is a special case of a more general inequality based on the fact that $\log\big((e^x)!\big)$ is convex. The value of $\alpha$ is approximately $-.7831\ldots$. This result is an analogue of the same inequality with the geometric means replaced by arithmetic means which is just a statement that $x!$ is strictly convex. Alzer has even further extended this result by replacing the geometric means by certain power means

(iv) The inequalities in (k), ($\ell$) use **Log-convex Function Inequalities** Comments (ii).

(v) Inequalities (n) and (o) can be deduced from a weighted form of **Čebisev's Inequality** Integral Analogue.

(vi) Inequality (p) follows from the integral analogue of (H).

(vii) (q) is a very simple form of the inequality in **Stirling's Formula**.

(viii) The right-hand side of (r) is due to Lucht and the left-hand side to Alzer.

(ix) See also: **Alzer's Inequalities** (b), **Beta Function Inequalities, Binomial Coefficient Inequalities, Digamma Function Inequalities, Minc-Sathre Inequality, Stirling's Formula, Wallis's Inequality**.

References [AI, pp. 192–194, 285–286], [GI3, pp. 277–280], [MOA, p. 104]; *Hájos, Neukomm & Surányi eds.*[HNS, pp. 13,55]; *Cloud & Drachman* [CD, p. 77], *Jeffreys & Jeffreys* [JJ, pp. 462–473], *Klambauer* [Kl, p. 410]; *Alzer* [21, 26, 31, 33], *Dragomir, Agarwal & Barnett* [104], *Elezović, Giordano & Pečarić* [111], *García-Caballero & Moreno* [125], *Jameson* [145], *Kupán & Szász* [168], *Lucht* [188], *Merkle* [208], *Slavić* [301].

## Faĭziev's Inequality

*If* $\underline{a}$ *is a non-negative* $n$*-tuple then*

$$\big(a_1^{n+1} + \cdots + a_n^{n+1}\big) \ge \big(a_1 \cdots a_n\big)\big(a_1 + \cdots + a_n\big).$$

Reference *Faĭziev* [114].

## Fan's Inequality

*If* $\underline{w}$ *is a positive* $n$*-tuple and if* $0 \le a_i \le 1/2,\ 1 \le i \le n,$ *then*

$$\frac{\mathfrak{H}_n(\underline{a},\underline{w})}{\mathfrak{H}_n(\underline{e}-\underline{a},\underline{w})} \le \frac{\mathfrak{G}_n(\underline{a},\underline{w})}{\mathfrak{G}_n(\underline{e}-\underline{a},\underline{w})} \le \frac{\mathfrak{A}_n(\underline{a},\underline{w})}{\mathfrak{A}_n(\underline{e}-\underline{a},\underline{w})}$$

*with equality if and only if* $\underline{a}$ *is constant.*

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COMMENT (i) The equal weight case of the right-hand inequality is due to Fan. The left inequality was added by Wang W. L. & Wang P. F. These two inequalities have been the subject of much research and there are many generalizations.

EXTENSIONS *(a)* [WANG Z., CHEN JI & LI G. X.] *If $\underline{w}$ is a positive $n$-tuple and if $0 \le a_i \le 1/2,\ 1 \le i \le n$, and if $r < s$ then*

$$\frac{\mathfrak{M}_n^{[r]}(\underline{a}, \underline{w})}{\mathfrak{M}_n^{[r]}(\underline{e} - \underline{a}, \underline{w})} \le \frac{\mathfrak{M}_n^{[s]}(\underline{a}, \underline{w})}{\mathfrak{M}_n^{[s]}(\underline{e} - \underline{a}, \underline{w})},$$

*if and only if $|r + s| \le 3$, and $2^r/r \ge 2^s/s$ when $r > 0$, and $r2^r < s2^s$ when $s < 0$.*
*(b)* [WANG W. L. & WANG P. F.] *If $0 \le a_i \le 1/2,\ 1 \le i \le n$, and if $1 \le r < s \le n$ then*

$$\frac{\mathfrak{P}_n^{[s]}(\underline{a})}{\mathfrak{P}_n^{[s]}(\underline{e} - \underline{a})} \le \frac{\mathfrak{P}_n^{[r]}(\underline{a})}{\mathfrak{P}_n^{[r]}(\underline{e} - \underline{a})}.$$

*(c)* [ALZER] *If $\underline{w}$ is a positive $n$-tuple and if $0 \le a_i \le 1/2,\ 1 \le i \le n$, then*

$$\frac{\min \underline{a}}{1 - \min \underline{a}} \le \frac{\mathfrak{A}_n(\underline{e} - \underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{e} - \underline{a}; \underline{w})}{\mathfrak{A}_n(\underline{a}; \underline{w}) - \mathfrak{G}_n(\underline{a}; \underline{w})} \le \frac{\max \underline{a}}{1 - \max \underline{a}},$$

*with equality if and only if $\underline{a}$ is constant.*
*(d) With the notation of the main result*

$$\frac{\mathfrak{G}_2(a, b)}{\mathfrak{G}_2(1 - a, 1 - b)} \le \frac{\mathfrak{I}(a, b)}{\mathfrak{I}(1 - a, 1 - b)} \le \frac{\mathfrak{A}_2(a, b)}{\mathfrak{A}_2(1 - a, 1 - b)},$$

COMMENTS (ii) Some restrictions on $r, s$ are necessary in (a), as Chan, Goldberg & Gonek showed.

(iii) The original proof of (c) by Alzer has been considerably shortened by Mercer by making an ingenious use of a result of Cartwright & Field; see **Geometric- Arithmetic Mean Inequality** INVERSE INEQUALITIES (C).

(iv) Inequalities (c) imply the original Fan inequality.

(v) (d) had been extended to $n$-variables.

(vi) In the literature it is common to write $\mathfrak{A}'_n(\underline{a}; \underline{w}) = \mathfrak{A}_n(\underline{e} - \underline{a}; \underline{w})$, and similarly for the other means.

(vii) A further of extension is in **Levinson's Inequality**; and an inverse inequality has been given by Alzer. The same author has written a survey article on this inequality.

(viii) See also: **Logarithmic Mean Inequalities** EXTENSIONS(B), **Weierstrass's Inequalities** RELATED INEQUALITIES (A).

(ix) For other inequalities of Fan see: **Determinant Inequalities** EXTENSIONS (A), **Minimax Theorems** (B), **Trace Inequalities** (D), **Weyl's Inequalities** EXTENSIONS (B).

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REFERENCES [AI, p. 363], [BB, p. 5], [H, pp. 295–298], [MPF, pp. 25–32]; *Alzer* [25], *Mercer, P.*[207]; *Sándor & Trif* [292].

**Fan-Taussky-Todd Inequalities** *If $\underline{a}$ is a real $n$-tuple and if $a_0, a_{n+1}$ are defined to be zero then*

$$4\sin^2\left(\frac{\pi}{2(n+1)}\right)\sum_{i=1}^{n} a_i^2 \le \sum_{i=0}^{n}(\Delta a_i)^2,$$

*with equality if and only if $a_i = c\sin i\pi/(n+1),\ 1 \le i \le n$; also*

$$4\sin^2\left(\frac{\pi}{2(2n+1)}\right)\sum_{i=1}^{n} a_i^2 \le \sum_{i=0}^{n-1}(\Delta a_i)^2,$$

*with equality if and only if $a_i = c\sin i\pi/(2n+1),\ 1 \le i \le n$.*

COMMENT (i) This is a discrete analogue of **Wirtinger's Inequality**.

EXTENSIONS *(a) With $\underline{a}$ as above*

$$16\sin^4\left(\frac{\pi}{2(n+1)}\right)\sum_{i=1}^{n} a_i^2 \le \sum_{i=0}^{n}(\Delta^2 a_i)^2,$$

*with equality if and only if $a_i = c\sin i\pi/(n+1),\ 1 \le i \le n$.*
*(b)* [REDHEFFER] *If $\underline{a}$ is a real $n$-tuple and if for some $\theta$, $0 \le \theta < \dfrac{\pi}{n}$,*

$$\mu \le 4\sin^2\theta/2, \qquad \lambda \ge 1 - \frac{\sin(n+1)\theta}{\sin n\theta},$$

*then*

$$\mu\sum_{i=1}^{n} a_i^2 \le a_1^2 + \sum_{i=1}^{n-1}\Delta a_i^2 + \lambda a_n^2.$$

*(c)* [ALZER] *If $\underline{a}$ is a complex $n$-tuple with $A_n = 0$ then*

$$\max_{1\le k\le n}|a_k|^2 \le \frac{n^2-1}{2n}\sum_{k=1}^{n}(\Delta a_k)^2,$$

*where $a_{n+1} = a_n$.*

COMMENTS (ii) (a) has been extended to higher order differences, and converse inequalities have been given. See: *Milovanović & Milovanović*.

(iii) Redheffer's result is a recurrent inequality; see **Recurrent Inequalities**.

INVERSE INEQUALITIES [ALZER] *With the above notations,*

$$\sum_{i=0}^{n}(\Delta a_i)^2 \le 2\Big(1+\cos\frac{\pi}{n+1}\Big)\sum_{i=1}^{n} a_i^2; \qquad \sum_{i=0}^{n-1}(\Delta a_i)^2 \le 2\Big(1+\cos\frac{2\pi}{2n+1}\Big)\sum_{i=1}^{n} a_i^2.$$

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*The constants on the right-hand sides are best possible.*

REFERENCES [AI, pp. 131,150], [BB, p. 183]; *Alzer* [14, 16], *Milovanović & Milovanović* [211].

**Farwig & Zwick's Lemma** *If $f$ is $n$-convex and has a continuous $n$-th derivative on $[a,b]$; and if for some points $a \le y_0 \le \cdots y_{n-1} \le b$, $[y_k,\ldots,y_{n-1};f] \ge 0, 0 \le k \le n-1$, then*

$$f^k(b) \ge 0, \quad 0 \le k \le n-1.$$

COMMENT The strong condition on the n-th derivative can be relaxed.

REFERENCE [PPT, pp. 30–32].

**Fatou's Lemma** *If $f_n, n \in \mathbb{N}$ is a sequence of non-negative measurable functions defined on the measurable set $E, E \subseteq \mathbb{R}$, then*

$$\int_E \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n.$$

COMMENTS (i) Strict inequality is possible in this basic result of measure theory; consider for instance $E = \mathbb{R}, f_n = 1_{[n,n+1]}, n \in \mathbb{N}$.

(ii) The result extends to general measure spaces.

REFERENCES *Hewitt & Stromberg* [HS, p. 172], *Rudin* [R91, p. 246].

**Favard's Inequalities** *(a) Let $f$ be a non-negative, continuous, concave function not identically zero on $[a,b]$ and put $\overline{f} = \mathfrak{A}_{[a,b]}(f)$. If $g$ is convex on $[0,2\overline{f}]$ then*

$$\mathfrak{A}_{[a,b]}(g \circ f) \le \mathfrak{A}_{[0,2\overline{f}]}(g).$$

*(b) If $f$ is a non-negative concave function on $[0,1]$, and $1 \le p \le \infty$ then*

$$\int_0^1 f \ge \begin{cases} \dfrac{(p+1)^{1/p}}{2}||f||_p & \text{if } 1 \le p < \infty, \\ \dfrac{\max f}{2} & \text{if } p = \infty. \end{cases}$$

*The constant on the right-hand side is best possible.*

COMMENTS (i) In (b) the case $p = \infty$ follows from the other case by letting $p \to \infty$.

(ii) The result (b) is connected with **Grüss-Barnes Inequality**, and is a particular case of **Thunsdorff's Inequality**. See also: **Arithmetic Mean Inequalities** INTEGRAL ANALOGUES, **Geometric-Arithmetic Mean Inequalities** INTEGRAL ANALOGUES (B).

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EXTENSIONS (a) [BERWALD] *Let $f$ be a non-negative, continuous, concave function not identically zero on $[a,b]$ and let $h$ be continuous and strictly monotonic on $[0,\ell]$, where $z=\ell$ is the unique positive solution of the equation*

$$\mathfrak{A}_{[0,z]}(h)=\mathfrak{A}_{[a,b]}(h\circ f).$$

*If $g$ is convex with respect to $h$ on $[0,\ell]$ then*

$$\mathfrak{A}_{[a,b]}(g\circ f)\le \mathfrak{A}_{[0,\ell]}(g).$$

*(b) If $f$ is a non-negative concave function on $[0,1]$ and $-1<r\le s$, then*

$$(r+1)^{1/r}||f||_r\ge (s+1)^{1/s}||f||_s.$$

COMMENTS (iii) For a definition of "$g$ is convex with respect to $h$" see: **Quasi-arithmetic Mean Inequalities** COMMENTS (i).

(iv) Extension (b) has been used to extend (B):

$$1+\alpha x\le \left(\frac{||f||_x}{||f||_{\alpha x}}\right)^{\alpha x}(1+x)^\alpha,$$

where $f$ is as in (b), and $x,\alpha$ are as in (B). If the conditions of ($\sim$B) hold then this last inequality is reversed.

REFERENCES [BB, pp. 43–44], [PPT, pp. 212–217]; *Alzer* [13], *Brenner & Alzer* [71], *Maligranda, Pečarić & Persson* [194].

**Fejér-Jackson-Grönwall Inequality** See: **Fejér-Jackson Inequality** COMMENT (i).

**Fejér-Jackson Inequality** *If $0<x<\pi$ then*

$$0<\sum_{i=1}^{n}\frac{\sin ix}{i}<\pi-x. \tag{1}$$

COMMENT (i) It is the left-hand side of this that is known as the *Fejér-Jackson inequality*, or sometimes the *Fejér-Jackson-Grönwall inequality*.

EXTENSIONS (a) *If $0<x<\pi$, $-1\le\epsilon\le 1$, $p-q=0$ or $1$ then*

$$\sum_{i=1}^{p}\frac{\sin(2i-1)x}{2i-1}>\epsilon\sum_{i=1}^{q}\frac{\sin 2ix}{2i}; \tag{2}$$

*in particular*

$$0<\sum_{i=1}^{n}(-1)^{n-1}\frac{\sin ix}{i}.$$

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*(b)* [Askey-Steinig] *If* $0 \le x < y \le \pi$ *then*

$$\frac{1}{\sin x/2}\sum_{i=1}^{n}\frac{\sin ix}{i} > \frac{1}{\sin y/2}\sum_{i=1}^{n}\frac{\sin iy}{i}\,.$$

*(c)* [Tudor] *If* $0 < x < \pi/(2m-1)$ *then*

$$\sum_{i=m}^{m+n-1}\frac{\sin ix}{i} > 0, \quad 0 < x < \pi.$$

*(d)* [Szegő, Schweitzer, M.] *If* $0 \le x \le 2\pi/3$ *then*

$$\sum_{i=1}^{n}\binom{n+2-i}{2} i\sin ix > 0.$$

Comments (ii) If we put $\epsilon = -1$ in (2) we get (1).

The case $\epsilon = 0$ gives an analogous result, due to Fejér:

$$\sum_{i=1}^{n}\frac{\sin(2i-1)x}{2i-1} > 0.$$

(iii) See also: **Rogosinski-Szegő Inequality, Sine Integral Inequalities, Trigonometric Polynomial Inequalities** (b), (c), **Young's Inequalities** (d).

References [AI, pp. 249, 251, 255], [MPF, pp. 611–627]; *Gluchoff & Hartmann* [132], *Tudor* [315].

**Fejér Kernel Inequalities** The quantity

$$K_k(x) = \frac{1}{k+1}\sum_{i=0}^{k}\frac{\sin(i+1/2)x}{2\sin x/2} = \frac{2}{k+1}\left(\frac{\sin(k+1/2)x}{2\sin x/2}\right)^2,$$

is called the *Fejér kernel.*

In terms of the Dirichlet kernel, see **Dirichlet Kernel Inequalities**,

$$K_k(x) = \frac{1}{k+1}\sum_{i=0}^{k} D_i(x).$$

*(a) If* $k \in \mathbb{N}$ *then*

$$k+1 > K_k(x) \ge 0.$$

*(b) If* $k \ge 1, 0 < x \le \pi$ *then for some constant* $C$,

$$K_k(x) \le \frac{C}{(k+1)x^2}\,.$$

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*(c) If $k \geq 1, t \leq x \leq \pi$ then*

$$K_k(x) \leq \frac{1}{2(k+1)\sin^2 t/2}\,.$$

COMMENTS (i) Extensions of these results to other kernels can be found in the references.

(ii) See also: **Dirichlet Kernel Inequalities, Poisson Kernel Inequalities**.

REFERENCES *Hewitt & Stromberg* [HS, p. 292], *Zygmund* [Z, vol. I, pp. 88–89, 94–95; vol. II, p. 60].

**Fejér-Riesz Theorem**[25] *If $f$ is analytic in $\overline{D}$ and if $p > 0$, then for all $\theta$,*

$$\int_{-1}^{1} |f(re^{i\theta})|^p \,\mathrm{d}r \leq \frac{1}{2}\int_{-\pi}^{\pi} |f(e^{it})|^p \,\mathrm{d}t.$$

*The constant is best possible.*

COMMENT The case $p = 1$ is given in **Conjugate Harmonic Function Inequalities** (b).

REFERENCES [AI, p. 334]; *Heins* [He, pp. 122, 347], *Zygmund* [Z, vol. I, p. 258].

**Fejes Tóth Inequality** *If $x_i, \delta_i$, $1 \leq i \leq n$, are two sets of positive numbers, and if $\sum_{i=1}^{n} \delta_i = \pi$,, then*

$$\sum_{i=1}^{n} x_i x_{i+1} \cos\delta_i \leq \cos\frac{\pi}{n}\sum_{i=1}^{n} x_i^2$$

*where $x_{n+1} = x_1$.*

COMMENTS (i) This was conjectured by Fejes Tóth and proved for general $n$ by Lenhard.

(ii) The interest of this inequality is that it gave an extension of the **Erdős-Mordell Inequality**.

REFERENCE *Abi-Khuzam* [1].

**Fenchel's Inequality** See: **Conjugate Convex Function Inequalities** COMMENTS (II).

**Fibonacci Number Inequalities** The elements of sequence $F_1, F_2, F_3, \ldots$ of positive integers defined by

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2},\ n \geq 3,$$

are called *Fibonacci numbers.*

[25] This is M. Riesz.

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*If $n$ is even*

$$F_n < \frac{1}{\sqrt{5}}\Big(\frac{\sqrt{5}+1}{2}\Big)^n < F_n + \frac{1}{3};$$

*if $n$ is odd*

$$F_n - \frac{1}{3} < \frac{1}{\sqrt{5}}\Big(\frac{\sqrt{5}+1}{2}\Big)^n < F_n.$$

COMMENTS (i) More precisely we have *Binet's formula*; for all $n \ge 1$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - (-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^n\right). \tag{1}$$

(ii) It follows from the above results that $F_n$ is the closest integer to $\left(\dfrac{1+\sqrt{5}}{2}\right)^n$.

(iii) The second fraction inside the brackets in (1) is called the *Golden mean, or section*; the first is its reciprocal.

REFERENCES [EM, vol. 4, pp. 1–3]; *Körner* [Kor, pp. 250–257].

**Fink's Inequalities** See: **Shafer-Fink Inequalities**.

**Fischer's Inequalities**[26] *If $\underline{p}, \underline{q}$ are positive $n$-tuples, $n \ge 3$, with $P_n = Q_n = 1$ and if $c \le -1$ or $c \ge 0$ then*

$$\sum_{i=1}^n p_i\big(\frac{p_i}{q_i}\big)^c \ge 1. \tag{1}$$

*If $0 < c < 1$ then (~1) holds.*

COMMENT The power function is essentially the only function for which an inequality of this type holds. Any function $f : ]0,1[ \to ]0,\infty[$ for which

$$\sum_{i=1}^n p_i \frac{f(p_i)}{f(q_i)} \ge 1, n \ge 3,$$

is internal on $]0, 1/2]$; for a definition of internal function see: **Internal function Inequalities**.

REFERENCE [MPF, pp. 641–645].

**FKG Inequality** *If $X$ is a distributive lattice, let $\ell : X \to [0,\infty[$ satisfy*

$$\ell(a)\ell(b) \le \ell(a\vee b)\ell(a\wedge b), \qquad \textit{for all } a, b \in X.$$

*If $f, g : X \to \mathbb{R}$ are increasing then*

$$\sum_{x\in X}\ell(x)f(x)\sum_{x\in X}\ell(x)g(x) \le \sum_{x\in X}\ell(x)\sum_{x\in X}\ell(x)f(x)g(x).$$

[26] This is P. Fischer.

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Comments (i) This is also known as *the Fortuin-Kasteleyn-Ginibre inequality*; it is an example of a *correlation inequality*.

(ii) A function such as $\ell$ is called *log supermodular*.

Reference [EM, Supp., pp. 201–202, 253–254].

**Fortuin-Kasteleyn-Ginibre Inequality** See: **FKG Inequality**.

**Four Functions Inequality** See: **Ahlswede-Daykin Inequality**.

**Fourier Transform Inequalities** If $f \in \mathcal{L}(\mathbb{R})$ the *Fourier transform of* $f$ is

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ixt}\,\mathrm{d}x.$$

In the case of other assumptions on $f$ the existence of the transform is more delicate; see the references.

We can also define the *Fourier sine, and Fourier cosine transforms*,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin t\,\mathrm{d}t, \qquad \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos t\,\mathrm{d}t.$$

*(a) If $f \in \mathcal{L}(\mathbb{R}^+)$ with $\lim_{t\to\infty} f(t) = 0$ then*

$$\sum_{i=1}^{\infty}(-1)^k f(k\pi) \quad < \quad \int_0^\infty f(t)\cos t\,\mathrm{d}t < \sum_{i=0}^{\infty}(-1)^k f(k\pi);$$

$$\sum_{i=0}^{\infty}(-1)^k f\left((k+\frac{1}{2})\pi\right) \quad < \quad \int_0^\infty f(t)\sin t\,\mathrm{d}t < f(0) + \sum_{i=0}^{\infty}(-1)^k f\left((k+\frac{1}{2})\pi\right).$$

*(b)* [Titchmarsh's Theorem] *If $f \in \mathcal{L}^p(\mathbb{R})$, $1 < p \le 2$, and if $q$ is the conjugate index, then $\hat{f} \in \mathcal{L}^q(\mathbb{R})$ and*

$$\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\hat{f}|^q\right)^{1/q} \le \frac{p^{1/p}}{q^{1/q}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|f|^p\right)^{1/p}, \tag{1}$$

*with equality if and only if $f(x) = ae^{-\alpha x^2}, \alpha > 0$, almost everywhere.*

Comments (i) (a) is a deduction from **Steffensen's Inequalities** (1).

(ii) Titchmarsh obtained (1) with the constant on the right-hand side equal to one as an integral analogue of **Hausdorff-Young Inequalities** (a). The correct value of the constant and the cases of equality were given by Beckner. The similarity of the constant to that in **Young's Convolution Inequality** is more than a coincidence; see the references. In addition the result can be extended to $\mathbb{R}^n, n \ge 2$

Extensions *(a)* [Paley-Titchmarsh] *If $|f(x)|^q x^{q-2} \in \mathcal{L}(\mathbb{R}), q > 2$, then $\hat{f} \in \mathcal{L}^q(\mathbb{R})$ and*

$$\int_{\mathbb{R}}|\hat{f}|^q \le C\int_{\mathbb{R}}|f(x)|^q|x|^{q-2}\,\mathrm{d}x,$$

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*where $C$ depends only on $q$.*
*If $f \in \mathcal{L}^p(\mathbb{R}), 1 < p < 2$, then $|\hat{f}(x)|^p x^{p-2} \in \mathcal{L}^p(\mathbb{R})$ and*

$$\int_{\mathbb{R}} |\hat{f}(x)|^p |x|^{p-2}\,\mathrm{d}x \leq C \int_{\mathbb{R}} |f|^p,$$

*where $C$ depends only on $p$.*
*(b)* [Riemann-Lebesgue Lemma] *If $f \in \mathcal{L}(\mathbb{R})$ then $\hat{f} \in \mathcal{C}_0(\mathbb{R})$ and*

$$||\hat{f}(x)||_\infty \leq ||f||.$$

Comment (iii) Extension (a) is an integral analogue of **Paley's Inequalities**. These results have been further extended to allow weight functions.

References [AI, pp. 109–110], [EM, vol. 4, pp. 80–83], [GI5, pp. 217–232]; *Lieb & Loss* [LL, pp. 120–122], *Rudin* [R87, pp. 184–185], [R91, p. 169], *Titchmarsh* [T86, pp. 3–4, 96–113], *Zygmund* [Z, vol. II, pp. 246–248, 254–255].

**Frank-Pick Inequality** See: **Thunsdorff's Inequality** Comments (ii).

**Frequency Inequalities** *If $A$ is a plane membrane and $\Lambda$ is its principal frequency then*

$$j_{0_1}\sqrt{\frac{K}{2|A|}} \geq \Lambda \geq \frac{j_{0_1}\pi}{\sqrt{|A|}},$$

*where $j_{0_1}$ is the smallest positive root of the Bessel function of the first kind $J_0$, and $K$ is a constant depending on $A$. Equality occurs when the membrane is circular.*

Comments (i) The *principal frequency of a plane membrane $A$* can be defined as

$$\Lambda^2 = \inf \frac{\int_A |\nabla f|^2}{\int_A |f|^2},$$

where the inf is over all $f$ zero on $\partial A$.

(ii) Precisely: let $h(\underline{x}, s)$ denote the distance of $\underline{x}, \underline{x} \in \mathring{A}$, from the tangent to $\partial A$ at the point $s$, where $s$ is the arc length on $\partial A$, $0 \leq s \leq \ell$; then

$$K = \inf\left\{\underline{x} \in \mathring{A};\ \oint_{\partial A} \frac{ds}{h(\underline{x}, s)}\right\}.$$

(iii) $j_{0_1} \approx 2.4048$.
(iv) This is an example of **Symmetrization Inequalities**.

Reference *Abramowicz & Stegun* [AS, p. 409]; *Pólya & Szegő* [PS51, pp. 2, 8, 9, 87–94, 195–196].

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**Friederichs's Inequality** *If $\Omega \subset \mathbb{R}^n$ is a simply connected bounded domain with a locally Lipschitz boundary and if $f \in \mathcal{W}^{1,2}(\Omega)$ then*

$$\int_\Omega f^2 \le C\Big(\int_\Omega |\nabla f|^2 + \oint_{\partial\Omega} f^2\Big)$$

*where $C$ is a constant that only depends on $\Omega$ and $n$.*

COMMENTS (i) A definition of $\mathcal{W}^{1,2}(\Omega)$ is given in **Sobolev's Inequalities**.

(ii) This inequality has been the subject of much study and generalization; see the references. It is sometimes called a *Poincaré inequality, or a Nirenberg inequality*; see **Poincaré's Inequalities**. The case $n = 2, f \in \mathcal{C}^2(\overline{\Omega})$ is due to Friederichs.

(iii) If the condition $f = 0$ holds on $\partial R$ this can be regarded as a generalization of **Wirtinger's Inequality**.

REFERENCES [EM, vol. 4, p. 115]; *Opic & Kufner* [OK, p. 2]; *Dostanić* [101].

**Fuchs Inequality** See: **Order Inequalities** COMMENT (VI).

**Function Inequalities** *(a)* [BOAS] *If $A = [0,1[$ or $[0,1]$ and $f,g$ are strictly star-shaped functions, with $f \in \mathcal{C}(A)$, $f(0) = 0$, $1 < f(1) \le \infty$, $0 < x < 1, f(x) \ne x$; and $g \in \mathcal{C}\big(f[A]\big)$, $g(1) \le 1$, then*

$$f \circ g \le g \circ f.$$

*In particular*

$$\arcsin\big(\sinh\frac{x}{2}\big) \le \sinh\big(\frac{1}{2}\arcsin x\big).$$

*(b)* [KUBO,T.] *If $f, g \in \mathcal{C}\big([a,b]\big)$, $f$ piecewise monotonic with $g[I] \subseteq I$ if $f$ is monotonic on the closed interval $I$, and if $f$ is increasing, decreasing, on $I$ then $g(x) \ge x$, $g(x) \le x$, respectively; then*

$$f \circ g \ge f.$$

*(c)* [SÁNDOR]*If $f : I \mapsto J$, where $I, J$ are nonempty subsets of $\mathbb{R}_+^*$ is a bijection, and suppose that $x_1 < x_2 \Rightarrow f(x_1)/x_1 < f(x_2)/x_2,\ x_1, x_2 \in I$:*

$$\textit{if}\quad x \in I, y \in J, \quad \textit{and if} \quad f(x) \ge (\le) y \quad \textit{then} \quad f(x)f^{-1}(y) \ge (\le) xy,$$

*with equality if and only if $y = f(x)$.*

COMMENTS (i) For a definition of strictly star-shaped see: **Star-shaped Function Inequalities** COMMENTS (I).

(ii) For a particular case of (b) see **Trigonometric Function Inequalities** (D).

(iii) See also under particular functions: **Absolutely and Completely Monotonic, Almost Symmetric, Analytic, Bernšteĭn Polynomial, Bessel, Beta, Binomial, Bounded Variation, Čebišev Polynomial, Complete Symmetric,**

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**Conjugate Convex, Conjugate Harmonic, Convex, Complex, Digamma, Elementary Symmetric, Entire, Equimeasurable, Error, Exponential, Factorial, Harmonic, Hurwitz Zeta, Hyperbolic, Incomplete Beta, Incomplete Factorial, Increasing, Internal, Laguerre, Legendre, Lipschitz, Logarithmic, Maximal, Monotonic, n-Convex, N-Function, Polynomial, Q-class, Quasi-conformal, Schur Convex, s-Convex, Sine Integral, Subharmonic, Star-shaped, Subadditive, Symmetric, Totally Positive, Trigonometric, Trigonometric Polynomial Ultraspherical Polynomials, Wright Convex, Univalent, Zeta**.

References *Boas* [63], *Kubo, T* [165], *Sándor* [290].

**Furuta's Inequality** *If $A, B$ are positive bounded linear operators on the Hilbert space $X$ such that $A \geq B$, and if $r \geq 0, p \geq 0, q \geq 1$ with $(1+r)q \geq p+r$ then*

$$\left(B^{r/2}A^pB^{r/2}\right)^{1/q} \geq \left(B^{r/2}B^pB^{r/2}\right)^{1/q},$$

*and*

$$\left(A^{r/2}A^pA^{r/2}\right)^{1/q} \geq \left(A^{r/2}B^pA^{r/2}\right)^{1/q}.$$

Comment This reduces to the **Löwner-Heinz Inequality** on putting $r = 0, \alpha = p/q$.

References [EM, Supp., pp. 260–262]; *Furuta* [Fu, pp. 129–136].

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# 7 Gabriel–Guha

**Gabriel's Problem** *If $\Lambda$ is a closed convex curve lying inside the closed convex curve $\Gamma$ and if $f$ is analytic inside $\Gamma$ then:*

$$\int_\Lambda |f(z)|\,|\,\mathrm{d}z| < A\int_\Gamma |f(z)|\,|\,\mathrm{d}z|,$$

*where $A$ is some absolute constant.*

COMMENT Gabriel conjectured that $A = 2$ but while this is correct when $\Gamma$ is a circle it is not true in general; the best known value is $A = 3.6$.

REFERENCE [AI, p. 336].

**Gagliardo-Nirenberg Inequality** See: **Sobolev's Inequalities** (B).

**Gale's Inequality** *If $d$ is the diameter of a bounded domain in $\mathbb{R}^n, n \geq 2$ and $\ell$ the edge of the smallest circumscribing regular simplex then:*

$$\ell \leq d\sqrt{\frac{n(n+1)}{2}}\,.$$

COMMENT See also: **Bonnensens Inequality, Isoperimetric Inequalities, Young's Inequalities** (E).

REFERENCE [EM, vol. 5, p. 204].

**Gamma Function Inequalities** See: **Factorial Function Inequalities**.

**Gauss's Inequality** *If $f : ]0,\infty[ \to \mathbb{R}$ is decreasing and if $x > 0$ then:*

$$x^2\int_x^\infty f \leq \frac{4}{9}\int_0^\infty t^2 f(t)\,\mathrm{d}t.$$

EXTENSION [VOLKOV] *If $f, G : [0,\infty[ \to \mathbb{R}$ with $f$ decreasing, $G$ increasing, and differentiable with $G(t) \geq t$, $t > 0$ then:*

$$\int_{G(0)}^\infty f \leq \int_0^\infty fG'.$$



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COMMENT (i) Taking $G(t) = x + 4t^3/27x^2$ gives the original result.

INVERSE INEQUALITY [ALZER] *If* $f : ]0, \infty[ \to [0, \infty[$ *and if* $x > 0$ *then,*

$$x^2 \int_x^\infty f \geq 3 \int_0^x t^2 f(t+x)\,\mathrm{d}t.$$

*The constant on the right-hand side is best possible.*

COMMENT (ii) See also: **Moment Inequalities, Pólya's Inequality, Steffensen's Inequalities** (A).

REFERENCES [AI, p. 300], [MPF, pp. 313, 319]; *Alzer* [11], *Pečarić* [262], *Varošanec & Pečarić* [318], *Varošanec, Pečarić & Šunde* [319].

**Gauss-Winkler Inequality** *If* $v_r, r \geq 0$ *denotes the r-th absolute moment of a probability distribution* $P$ *on* $\mathbb{R}^+$ *that has a continuous decreasing derivative, and if* $r \leq s$ *then:*

$$\big((r+1)v_r\big)^{1/r} \leq \big((s+1)v_s\big)^{1/s}.$$

COMMENTS (i) Without the constant factors the inequality is weaker, and is a particular case of (r;s).

(ii) The original Gauss result was the case $r = 2, s = 4$ of this:
*if* $f \geq 0$ *is decreasing, then provided the integrals exist,*

$$\left(\int_0^\infty x^2 f(x)\,\mathrm{d}x\right)^2 \leq \frac{5}{9}\int_0^\infty f(x)\,\mathrm{d}x \int_0^\infty x^4 f(x)\,\mathrm{d}x.$$

(iii) The argument used by Winckler to obtain this result was found to be wrong; a correct proof was given later by Faber.

EXTENSIONS [BEESACK] *If* $P$ *is a probability distribution on* $\mathbb{R}$, *with* $P(x) = 0, -\infty \leq x \leq 0, P(\infty) = 1$ *and* $(-1)^k P^{(k)}$ *positive, continuous, and decreasing,* $1 \leq k \leq n_0$, *then with the above notation, if* $r \leq s$,

$$\left(\binom{r+k}{k} v_r\right)^{1/r} \leq \left(\binom{s+k}{k} v_s\right)^{1/s}, \quad 1 \leq k \leq n_0;$$

*and if* $m < n < r$,

$$\binom{n+k}{k} v_n \leq \left(\binom{m+k}{k} v_m\right)^{(r-n)/(r-m)} \left(\binom{r+k}{k} v_r\right)^{(n-m)/(r-m)}.$$

COMMENT (iv) Compare these results with **Klamkin & Newman Inequalities**.

REFERENCES [GI7, pp. 27–37], [MPF, pp. 53–56], [PPT, pp. 217–228]; *Pečarić & Varošanec* [267].

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**Gaussian Error Function Inequalities** See: **Error Function Inequalities.**

**Gaussian Measure Inequalities** *(a) If $A, B$ are Borel sets in $\mathbb{R}^n$ then*

$$\gamma_n\Big((1-t)A+tB\Big) \geq \gamma_n^{1-t}(A)\gamma_n^t(B).$$

*(b)*[Bobkov's Inequality] *If $f:\mathbb{R}^n \mapsto [0,1]$ is locally Lipschitz then*

$$\Phi'\circ\Phi\Big(\int_{\mathbb{R}^n} f\,\mathrm{d}\gamma_n\Big) \leq \int_{\mathbb{R}^n}\sqrt{(\Phi'\circ\Phi)^2(f)+|\nabla f|^2}\,\mathrm{d}\gamma_n$$

Comments (i) Inequality (a), to be compared with **Brunn-Minkowski Inequalities** (a), has been considerably generalized to a result known as *Erhard's inequality*; see the reference.

(ii) The function $\Phi'\circ\Phi$ is called the *Gaussian Isoperimetric function*. Inequality (b) is to be compared with the **Sobelev's Inequalities** (b); see the reference.

Reference *Latała* [169].

**Gaussian Probability** See: **Error Function Inequalities.**

**Gautschi's Inequality** See: **Factorial Function Inequalities** (a), Comments (ii), **Incomplete Factorial Function Inequalities** Comments (ii).

**Gegenbauer Polynomial Inequalities** See: **Ultraspherical Polynomial Inequalities.**

**Generalized Logarithmic Mean Inequalities** See **Logarithmic and Generalized Logarithmic Mean Inequalities.**

**Geometric-Arithmetic Mean Inequality** *If $\underline{a}$ is a positive $n$-tuple, $n \geq 1$, then:*

$$\mathfrak{G}_n(\underline{a};\underline{w}) \leq \mathfrak{A}_n(\underline{a};\underline{w}), \tag{GA}$$

*with equality if and only if $\underline{a}$ is constant.*

Comment (i) The most direct proof is first to obtain the case $n = 2$, and then make an induction on $n$. However there are many proofs available, a total of 74 are given in [H]; and this does not count the indirect proofs where (GA) is a special case of a more general result. It is surprising that attacking (GA) via the equal weight case is more bothersome than a direct attack; for, although the $n = 2$ case is now trivial, the induction is more difficult, and the move to general weights needs some care.

Some Special Cases *(a)* [Equal Weight Case] *If $\underline{a}$ is a positive $n$-tuple*

$$(a_1\cdots a_n)^{1/n} \leq \frac{a_1+\cdots+a_n}{n}, \tag{1}$$

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*with equality if and only if $\underline{a}$ is constant.*
*(b)* [CASE $n = 2$] *If $x, y, p$ are positive, $x = u^p, y = v^q$, $q$ the conjugate index of $p$, then*

$$x^{1/p}y^{1/q} \le \frac{x}{p} + \frac{y}{q}, \quad \textit{or} \quad uv \le \frac{u^p}{p} + \frac{v^q}{q}, \tag{2}$$

*with equality if and only if $x = y$, or $u = v$.*
*(c)* [EQUAL WEIGHT CASE; $n = 2$] *If $a, b$ are positive*

$$\sqrt{ab} \le \frac{a+b}{2}, \tag{3}$$

*with equality if and only if $a = b$.*

COMMENTS (ii) (1) has many proofs, for instance it is a particular case of **Newton's Inequalities** (1), (2). See: **Newton's Inequalities** COMMENTS (iv), and [H].

(iii) (2) is just **Bernoulli's Inequality** (2). In fact (GA) is equivalent to (B). See **Bernoulli's Inequality** COMMENTS (viii).

(iv) (3) is easily seen to be equivalent to $(c - d)^2 \ge 0$; put $c = \sqrt{a}, d = \sqrt{b}$.

(v) Inequalities (1), (2) have some simple geometric interpretations. See: **Geometric Inequalities** (A).

There are many extensions that will be discussed in their own rights. See: **Chong's Inequalities** (C), **Cyclic Inequalities** (B), **Geometric Mean Inequalities** (D), **Kober & Dianada Inequalities, Nanjundiah's Mixed Mean Inequalities, O'Shea's Inequality, Popoviciu's Geometric-Arithmetic Mean Inequality Extension, Rado's Geometric-Arithmetic Mean Inequality Extension, Redheffer's Inequalities, Sierpinski's Inequalities**.

Here we will mention a few of the others.

EXTENSIONS OF GA *(a)*[DIFFERENT WEIGHTS]

$$\mathfrak{G}_n(\underline{a}; \underline{u}) \le \frac{V_n \mathfrak{G}_n(\underline{u}; \underline{u})}{U_n \mathfrak{G}_n(\underline{v}; \underline{u})} \mathfrak{A}_n(\underline{a}; \underline{v}),$$

*with equality if and only if $a_1 v_1 u_i^{-1} = \cdots = a_n v_n u_n^{-1}$.*
*(b)* [LUPAŞ & MITROVIĆ Ž.]

$$1 + \mathfrak{G}_n(\underline{a}) \le \mathfrak{G}_n(1 + \underline{a}) \le 1 + \mathfrak{A}_n(\underline{a}),$$

*with equality if and only if $\underline{a}$ is constant.*
*(c)* [SIEGEL] *If $n \ge 2$ and $\underline{a}$ has no two terms equal then:*

$$\mathfrak{G}_n(\underline{a})\mathfrak{G}_n(\underline{b}) \le \mathfrak{A}_n(\underline{a}),$$

*where $b_i = 1 + (n - i)/(t + i - 1)$, $i = 1, \ldots, n$, and $t$ is the unique positive root of the equation*

$$\frac{1}{n!} \prod_{i=0}^{n-2} \left( \frac{t+i}{t-i} \right)^{n+i-1} \prod_{1 \le i < j \le n} (a_i - a_j)^2 = \left( \prod_{i=1}^{n} a_i \right)^{n-1}.$$

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*(d)* [Hunter] *If $n \ge 2$ and $\underline{a}$ has no two terms equal then:*

$$\mathfrak{G}_n(\underline{a}) \le \left\{(1+(n-1)t)(1-t)^{n-1}\right\}^{1/n}\mathfrak{A}_n(\underline{a}),$$

*where $t$, $0 < t < 1$, is a root of the equation*

$$\sum_{1\le i<j\le n}(a_i - a_j)^2 = t^2(n-1)A_n^2.$$

*(e)* [Fink & Jodheit] *If $0 < a_1 \le \cdots \le a_n$ then:*

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{A}_n(\underline{a};\underline{w}),$$

*where $\underline{w}$ is defined by*

$$w_i = \frac{W_n}{n}\prod_{k<i}\left\{1-\left(1-(\frac{a_k}{a_i})^{1/n}\right)^n\right\}, \quad 1\le i\le n.$$

*(f)* [Mijalković & Keller]

$$\mathfrak{A}_n(\underline{a};\underline{w}) \le \mathfrak{G}_n(\underline{a};\underline{a}\,\underline{w}).$$

*(g)* [Zaciu]

$$\left(\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{A}_n(\underline{b})}\right)^{\mathfrak{A}_n(\underline{a})} \le \left(\frac{\mathfrak{G}_n(\underline{a}^{\underline{a}})}{\mathfrak{G}_n(\underline{b}^{\underline{b}})}\right).$$

*(h)* [Åkerberg]

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(i!)^{1/i}}{i+1}\mathfrak{G}_i(\underline{a}) \le \mathfrak{A}_n(\underline{a}).$$

*(j)* [Alzer]

$$\frac{3}{e}\left(\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\right) \ge \mathfrak{A}_n(\underline{a};\underline{w})e^{-\mathfrak{G}_n/\mathfrak{A}_n} - \mathfrak{G}_n(\underline{a};\underline{w})e^{-\mathfrak{A}_n/\mathfrak{G}_n} \ge 0,$$

*with equality if and only if $\underline{a}$ is constant.*
*(k)* [Schaumberger]

$$\frac{(b-a)^2}{8\mathfrak{A}_2(a,b)} \le \mathfrak{A}_2(a,b) - \mathfrak{G}_2(a,b) \le \frac{(b-a)^2}{8\mathfrak{G}_2(a,b)}.$$

*($\ell$) If $u, p > 0, v > -1/p$ then:*

$$uv \le u\frac{u^p-1}{p} + \left(\frac{1+pv}{1+p}\right)^{(1+p)/p}.$$

*(m)* [(GA) with General Weights] *If $0 < a_1 \le \cdots \le a_n$ and $\underline{w}$ is a real $n$-tuple with*

$$W_n \ne 0, \qquad 0 \le \frac{W_i}{W_n},\ 1\le i\le n, \tag{4}$$

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*then (GA) holds, with equality if and only if* $\underline{a}$ *is constant.*
*(n)* [Complex Case] *If* $\underline{z}$ *is a complex n-tuple with* $z_k = r_k e^{i\theta_k}, 0 \le |\theta_k| < \psi < \pi/2$, *1*$\le k \le n$. *then*

$$\cos\psi\ \mathfrak{G}_n(|\underline{z}|) \le \big|\mathfrak{A}_n(\underline{z})\big|.$$

Comments (vi) (b) is related to **Weierstrass's Inequalities**; see also **Geometric Mean Inequalities** (f), **Harmonic Mean Inequalities** (d). The left hand inequality follows using (GA) and **Schur Convex Function Inequalities** (b), or by using **Geometric Mean Inequalities** (1). The right-hand side is an immediate consequence of (GA).

(vii) The constant in (j), cannot be improved.

(viii) Inequality ($\ell$) is obtained by a simple change of variable in (2).

(ix) (m) can be considered as a special case of the **Jensen-Steffensen Inequality** or it can be proved directly by the method used to prove that result. A particular case of (m) is **Bernoulli's Inequality** (1).

The restriction (4) on weights is, in a way, quite natural in that we have the following: *if* $0 < a_1 \le \cdots \le a_n$ *and* $\underline{w}$ *is a real n-tuple then:*

$$\min \underline{a} \le \mathfrak{A}_n(\underline{a};\underline{w}) \le \max \underline{a} \tag{5}$$

*if and only if (1) holds.*

There are many converse forms for (GA) of which we give a few of the most important.

Inverse Inequalities *(a)* [Mond & Shisha] *Assume that* $W_n = 1$, *and that* $0 < m \le \underline{a} \le M$ *then:*

$$0 \le \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \le \mathfrak{A}_2(m, M; 1-\theta, \theta) - \mathfrak{G}_2(m, M; 1-\theta, \theta), \tag{6}$$

*where if* $\mu = M/m$

$$\theta = \frac{\log(\mu - 1) - \log\log\mu}{\log\mu}.$$

*There is equality on the right of (6) if and only if for some subset* $J$ *of* $1, \ldots, n$ *we have*

$$\sum_{i\in J} w_i = \theta \qquad a_i = M,\ i \in J, \quad a_i = m,\ i \notin J.$$

*(b)* [Dočev] *If* $\underline{a}$ *and* $\mu$ *are as in the previous result then:*

$$1 \le \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{G}_n(\underline{a};\underline{w})} \le \frac{(\mu-1)\mu^{1/\mu-1}}{e\log\mu}.$$

*Further, the right-hand side of the last expression is increasing as a function of* $M$, *and decreasing as a function of* $m$.
*(c)* [Cartwright & Field] *If* $\underline{a}$ *and* $\underline{w}$ *are as in (a),* $\underline{a}$ *is not constant, then,*

$$\frac{1}{2M}\sum_{k=1}^n w_k\left(a_k - \mathfrak{A}_n(\underline{a};\underline{w})\right)^2 \le \mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \le \frac{1}{2m}\sum_{k=1}^n w_k\left(a_k - \mathfrak{A}_n(\underline{a};\underline{w})\right)^2$$

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*(d)* [ZHUANG] *If* $0 < \alpha \le a \le A$, $0 < \beta \le b \le B$, $p > 1$, *and* $q$ *the conjugate index then*

$$\frac{a^p}{p} + \frac{b^q}{q} \le Kab,$$

*where*

$$K = \max\left\{\frac{\alpha^p/p + B^q/q}{\alpha B}, \frac{A^p/p + \beta^q/q}{A\beta}\right\}.$$

*(e)* [KOBER-DIANANDA] *If* $\underline{a}$ *is a non-constant non-negative* $n$*-tuple and* $\underline{w}$ *is an* $n$*-tuple with* $W_n = 1$ *then*

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \le \frac{1}{2w}\sum_{i,j=1}^{n} w_i w_j \left(\sqrt{a_i} - \sqrt{a_j}\right)^2,$$

*with equality if and only if either* $n = 2$ *or* $n > 2$ *and the* $a_i$ *with minimum weight is zero and all the other* $a_i$ *are equal and positive.*

COMMENTS (x) The inequality of Cartwright & Field has been generalized by Alzer, replacing the arithmetic mean on the left-hand side by the geometric mean.

(xi) This same inequality has been used to give a simple proof of Alzer's additive analogue of **Fan's Inequality**.

(xii) A proof of the result of Dočev can be obtained using the method given in [H, Theorem 25, pp. 45– 46]. Further results of this form have been given by Clausing; see [GI4].

INTEGRAL ANALOGUES (a) If $f \in \mathcal{L}_\mu([a,b]), f \ge 0$ then:

$$\mathfrak{G}_{[a,b]}(f;\mu) \le \mathfrak{A}_{[a,b]}(f;\mu).$$

(b) [BERWALD] If $f \ge 0$ on $[a,b]$ is concave then:

$$\mathfrak{G}_{[a,b]}(f;\mu) \ge \frac{2}{e}\mathfrak{A}_{[a,b]}(f;\mu).$$

COMMENTS (xiii) Inequality (b) is a particular case of Berwald's inequality, see **Favard's Inequalities** EXTENSIONS (a). The sequence form of this result was also given by Alzer.

(xiv) See also: **Conjugate Convex Function Inequalities** COMMENTS (I), **Factorial Function Inequalities** (K), **Fan's Inequality, Henrici's Inequality, Kalajdžić Inequality** COMMENTS (II), **Korovkin's Inequality** COMMENT (II),(III), **Levinson's Inequality, Log-convex Functions Inequalities** (2), **Matrix Inequalities** (C), **Mixed Mean Inequalities** SPECIAL CASES (B), **Muirhead Symmetric Function and Mean Inequalities, Order Inequalities** COMMENTS (IV), **O'Shea's Inequality** COMMENT, **Power mean Inequalities, Rearrangement Inequalities, Sierpinski's Inequalities, Symmetric Mean Inequalities** COMMENTS (II), **Weierstrass's Inequalities** RELATED INEQUALITIES (B).

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REFERENCES [AI, pp. 84–85, 208, 310–311, 344], [BB, pp. 3–15, 44], [GI3, pp. 61–62], [H, pp. 75–124, 149–160], [HLP, pp. 17–21, 61, 136–139], [MOA, pp. 125–126], [MPF, pp. 36–40]; *Hájos, Neukomm & Surányi eds.*[HNS, pp. 70–72]; *Pólya & Szegö* [PS, pp. 62–65, 67], *Steele* [S, pp. 36, 235]; *Alzer* [pp. 6, 9, 28], *Bullen* [78], *Dianada* [99], *Zhuang* [339].

In addition all books on inequalities, as well as many others, contain discussions of (GA).

**Geometric Inequalities** (a) DERIVED FROM (GA). The equal weight case of (GA) and of (HG) have some simple geometric interpretations; in this section we will call these special cases (GA) and (HG), respectively.

(i) *The case* $n = 2$. Let ABCD be a trapezium with parallel sides AB, DC, with $AB = b > DC = a$. Let IJ and GH be parallel to the AB with the first bisecting AD and the second dividing the trapezium into two similar trapezia. If K is the point of intersection of AC and BD, let the line EKF be parallel to AB. Then since $EF = \mathfrak{H}_2(a,b)$, $GH = \mathfrak{G}_2(a,b)$, $IJ = \mathfrak{A}_2(a,b)$ it follows from (HG) and (GA) that EF is nearer to DC than GH, and GH is nearer to DC than IJ.

(ii) *The case* $n = 3$. A simple deduction from (GA) in this case is:

*Of all the triangles of given perimeter the equilateral has greatest area.*

(iii) It is easy to see that (GA) is equivalent to either of the following:

*if* $\prod_{i=1}^n a_i = 1$ *then* $\sum_{i=1}^n a_i \ge n$;

*if* $\sum_{i=1}^n a_i = 1$ *then* $\prod_{i=1}^n a_i \le n^{-n}$;

*with equality in either case if and only if* $\underline{a}$ *is constant.*

These forms give (GA) simple geometric interpretations:
*($\alpha$) Of all n-parallelepipeds of given volume the one of least perimeter is the n-cube.*
*($\beta$) Of all n-parallelepipeds of given perimeter the one of greatest volume is the n-cube.*

*($\gamma$) Of all the partitions of the interval* $[0,1]$ *into n sub-intervals, the partition into equal sub-intervals is the one for which the product of the sub-interval lengths is the greatest.*

(b) DERIVED FROM $S(r;s)$. If $a_1, a_2, a_3$, are the sides of a parallelepiped then $\mathfrak{P}_3^{[1]}(a_1,a_2,a_3)$, $\mathfrak{P}_3^{[2]}(a_1,a_2,a_3)$ and $\mathfrak{P}_3^{[3]}(a_1,a_2,a_3)$ are, respectively, the side of the cube of the same perimeter, $P$ say, the side of the cube of the same area, $A$ say, and the side of the cube of the same volume, $V$ say. The inequality S(r;s) says that unless $a_1 = a_2 = a_3$,

$$P > A > V.$$

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(c) Derived from (C). If $a, b, c$ are the lengths of the semi-axes of an ellipsoid of surface area $S$ then

$$\frac{4\pi}{3}(ab+bc+ca) \le S \le \frac{4\pi}{3}(a^2+b^2+c^2),$$

and

$$S \ge 4\pi(abc)^{2/3}.$$

Both of these inequalities are strict unless $a = b = c$.

(d) Derived from Karamata's Order Inequalities See: **Order Inequalities**(b). If $ABC, A'B'C'$ are two triangles with the angles of the second lying between the smallest and largest angle of the first then if $f$ is continuous and convex,

$$f(A') + f(B') + f(C') \le f(A) + f(B) + f(C);$$

further if $f$ is strictly convex then there is equality if and only if $A = A', B = B', C = C'$. In particular if $0 < kA \le \pi$ then

$$\sin kA + \sin kB + \sin kC \le \sin kA' + \sin kB' + \sin kC'.$$

(e) Others. $(\alpha)$ If $s$ is the semi-perimeter of a triangle, $R$ the radius of its circum-circle, and $r$ the radius of its in-circle, the so-called *elements of the triangle*, then:

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + s(4R+r)^3 \le 0.$$

Conversely, if the positive numbers $s, r, R$ satisfy this inequality then they are the elements of a triangle.

$(\beta)$ If $a, b, c$ are the lengths of the sides of a right-angled triangle, with $c$ the hypotenuse, then if $r \in \mathbb{R}$:

$$a^r + b^r \begin{cases} > c^r & \text{if } r < 2, \\ = c^r & \text{if } r = 2, \\ < c^r & \text{if } r > 2, \end{cases} \tag{1}$$

Comments (i) In $(\beta)$ above the case $r = 2$ is Pythagoras's Theorem, the case $r = 1$ is the **Triangle Inequality** Comment (iv), and the case $r = 0$ is completely trivial. See also **Trigonometric Function Inequalities** (u).

(ii) See also: **Abi-Khuzam's Inequality, Adamović's Inequality** Comments (ii), **Bessel Function Inequalities** Comments (ii), **Beth-van der Corput Inequality** Comments (iii), **Blaschke-Santaló Inequality, Bonnenesen's Inequality, Brunn-Minkowski Inequalities, Erdős & Grünwald Inequality** Comment (i), **Erdős-Mordell Inequality, Euler's Inequality, Gale's Inequality, Hadamard's Determinant Inequality** Comments (ii), **Isodiametric Inequality, Isoperimetric Inequalities, Jarníck's Inequality, Jensen's Inequality**

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COMMENTS (ii), (viii), **Mahler's Inequalities, Mixed-volume Inequalities, n-Simplex Inequality, Parallelogram Inequality** COMMENTS (ii), **Ptolemy's Inequality, Schwarz's Lemma** COMMENTS (ii), **Symmetrization Inequalities, Triangle Inequality** COMMENTS (iv), **Young's Inequalities** (e).

REFERENCES [H, pp. 66–67, 82–84, 330], [HLP, pp. 36–37], [MOA, pp. 289–296]; *Bottema, Đordjević, Janić, Mitrinović & Vasić* [Bot], *Herman, Kučera & Šimša* [HKS, p. 97], *Mitrinović, Pečarić & Volenec* [MPV], *Pólya & Szegö* [PS, pp. 72–73]; *Frucht & Klamkin* [118], *Oppenheim* [238].

## Geometric Mean Inequalities

*(a) If $\underline{a}$ is a positive $n$-tuple,*

$$\min \underline{a} \leq \mathfrak{G}_n(\underline{a};\underline{w}) \leq \max \underline{a},$$

*with equality if and only if $\underline{a}$ is constant.*
*(b) If $\underline{a}, \underline{b}$ are positive $n$-tuples,*

$$\mathfrak{G}_n(\underline{a};\underline{w}) + \mathfrak{G}_n(\underline{b};\underline{w}) \leq \mathfrak{G}_n(\underline{a}+\underline{b};\underline{w}); \tag{1}$$

*with equality if and only if $\underline{a} \sim \underline{b}$.*
*(c)* [DAYKIN & SCHMEICHEL] *If $\underline{a}$ is a positive $n$-tuple, define $a_{n+k} = a_k$, and the $n$-tuple $\underline{b}$ by $b_k = A_{k+n} - A_k$, $1 \leq k \leq n$, then:*

$$n\mathfrak{G}_n(\underline{a}) \leq \mathfrak{G}_n(\underline{b}).$$

*(d) If $\underline{a}$ is a positive $n$-tuple and the $n$-tuples $\underline{b}, \underline{c}$ are defined by:*
$b_k = \mathfrak{A}_n(\underline{a}) - (n-1)a_k \geq 0$, $c_k = \left(\mathfrak{A}_n(\underline{a}) - a_k\right)/(n-1) \geq 0$, $1 \leq k \leq n$,

*then:*

$$\mathfrak{G}_n(\underline{b}) \leq \mathfrak{G}_n(\underline{a}) \leq \mathfrak{G}_n(\underline{c}).$$

*(e)* [REDHEFFER & VOIGT] *Let $\underline{a}$ be a positive sequence, with $\sum_{i=1}^{\infty} a_{2i-1} < \infty$, then:*

$$\sum_{i=1}^{\infty}(-1)^{i-1} i\mathfrak{G}(\underline{a}) \leq \sum_{i=1}^{\infty} a_{2i-1},$$

*whenever the left-hand side converges.*
*If $a_{2i-1} \neq o(1/i)$ the inequality is strict, but if $a_{2i-1} = o(1/i)$ there is exactly one choice of $\{a_{2i}\}$ for which equality holds.*
*(f)* [KLAMKIN & NEWMAN] *If $\underline{a}$ is a non-negative $n$-tuple with $\sum_{i=1}^{n} a_i = 1$,*

$$\mathfrak{G}_n(1 \pm \underline{a}) \geq (n \pm 1)\mathfrak{G}_n(\underline{a}),$$

*and*

$$\frac{\mathfrak{G}_n(1+\underline{a})}{\mathfrak{G}_n(1-\underline{a})} \geq \frac{n+1}{n-1}. \tag{2}$$

COMMENTS (i) Inequality (1) is sometimes called *Hölder's Inequality* and is just **Power Mean Inequalities** (4). It is easily seen to be equivalent to (H).

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(ii) The inequality in (e) can be deduced from **Rado's Geometric-Arithmetic Mean Inequality Extension** (2); various extensions are given in *Redheffer & Voigt* [280].

(iii) The inequalities in (f) are related to **Weierstrass's Inequalities**, and to **Geometric-Arithmetic Mean Inequalities** EXTENSIONS (b). In particular (2) is the equal weight case of **Weierstrass's Inequalities** RELATED INEQUALITIES (b). In addition they can be deduced using Schur convexity.

(iv) See also: **Alzer's Inequalities** (B), **Bessel Function Inequalities** (A), **Binomial Function Inequalities** (J), **Geometric-Arithmetic Mean Inequality, Harmonic Mean Inequalities** (B),(D), **Hölder's Inequality** SPECIAL CASES (A), **Kaluza-Szegő Inequality, Levinson's Inequality** SPECIAL CASES, **Logarithmic Mean Inequalities** (B),(C),(D), COROLLARIES,EXTENSIONS, **Mitrinović & Đoković Inequality, Mixed Mean Inequalities** SPECIAL CASES (B), **Muirhead Symmetric Function and Mean Inequalities** COMMENTS (II).

INTEGRAL ANALOGUE *If $f, g \in \mathcal{L}[a,b]$,*

$$\mathfrak{G}_{[a,b]}(f) + \mathfrak{G}_{[a,b]}(g) \le \mathfrak{G}_{[a,b]}(f+g),$$

*with equality if and only if the right-hand side is zero or $Af = Bg$ almost everywhere, where the real $A, B$ are not both zero.*

REFERENCES [H, pp. 93–94, 123, 214], [HLP, pp. 21–24,138]; *Pólya & Szegö* [PS, pp. 69–70]; *Redheffer & Voigt* [280].

## Gerber's Inequality

*If $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and $x > -1$,*

$$\binom{\alpha}{n+1} x^{n+1} \begin{cases} \ge 0 \\ = 0 \\ \le 0 \end{cases} \implies (1+x)^\alpha \begin{cases} \ge \sum_{i=0}^{n} \binom{\alpha}{i} x^i, \\ = \sum_{i=0}^{n} \binom{\alpha}{i} x^i, \\ \le \sum_{i=0}^{n} \binom{\alpha}{i} x^i. \end{cases} \tag{1}$$

COMMENTS (i) This is a simple consequence of Taylor's Theorem for the function $a(x) = (1+x)^\alpha$.

(ii) (B) is the case $n = 1$ of (1).

(iii) See also: **Kaczmarz & Steinhaus Inequalities**.

EXTENSION *If $a < 0 < b$ and $f : ]a,b[ \to \mathbb{R}$ has derivatives of order $(n+1)$ on $]a,b[$, let $T_n(x)$ be the $n$-th order Taylor polynomial of $f$ centred at $0$, that is*

$$T_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!} f^{(i)}(x).$$

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*Then:*

$$x^{n+1}f^{(n+1)}(x)\begin{cases}\geq 0,\\ =0,\\ \leq 0,\end{cases}\quad a<x<b,\quad\Longrightarrow\quad f(x)\begin{cases}\geq T_n(x),\\ r=T_n(x),\\ \leq T_n(x).\end{cases}$$

*Further if $x^{n+1}f^{(n+1)}(x)$ has only a finite number of zeros in $]a,b[$ the first and last inequalities are strict.*

COMMENTS (iv) This is an immediate consequence of Taylor's theorem; for another such inequality see the last reference.

(v) See also: **Enveloping Series Inequalities** COMMENTS (ii), **Trigonometric Function Inequalities** COMMENTS (vi).

REFERENCES [AI, pp. 35–36], [GI2, pp. 143–144], [MPF, p. 66].

**Gini-Dresher Mean Inequalities** If $\underline{a}$ is a positive $n$-tuple, and $p,q\in\mathbb{R}$, with $p\geq q$, define

$$\mathfrak{B}_n^{p,q}(\underline{a};\underline{w})=\begin{cases}\left(\dfrac{\sum_{i=1}^n w_ia_i^p}{\sum_{i=1}^n w_ia_i^q}\right)^{1/(p-q)}, & \text{if } p\neq q,\\[2ex] \left(\prod_{i=1}^n a_i^{w_ia_i^p}\right)^{1/(\sum_{i=1}^n w_ia_i^p)}, & \text{if } p=q.\end{cases}$$

These quantities are known as *Gini or Gini-Dresher Means of* $\underline{a}$, and have been studied and generalized by various authors. The definition of $\mathfrak{B}_n^{p,p}(\underline{a},\underline{w})$ is justified by limits. Particular cases are other well-known means:

$$\mathfrak{B}_n^{p,0}(\underline{a};\underline{w})=\mathfrak{M}_n^{[p]}(\underline{a};\underline{w}),\ p\geq 0;\qquad \mathfrak{B}_n^{0,q}(\underline{a};\underline{w})=\mathfrak{M}_n^{[q]}(\underline{a};\underline{w}),\ q\leq 0;$$
$$\mathfrak{B}_n^{p,p-1}(\underline{a};\underline{w})\;=\;\mathfrak{H}_n^{[p]}(\underline{a};\underline{w}).$$

A special case of these means is the *Lehmer family of means*: if $a,b$ are positive and $r$ real

$$\mathfrak{Le}^r(a,b)=\frac{a^{r+1}+b^{r+1}}{a^r+b^r}.$$

Clearly:

$$\mathfrak{Le}^r(a,b)=\mathfrak{B}_2^{r+1,r}(a,b);$$

and in particular

$$\mathfrak{Le}^0(a,b)=\mathfrak{A}_2(a,b),\ \mathfrak{Le}^{-1/2}(a,b)=\mathfrak{G}_2(a,b),\ \mathfrak{Le}^{-1}(a,b)=\mathfrak{H}_2(a,b).$$

COMMENT (i) These means have been extended to what are known as *Biplanar means*; see [H]. In addition it is easy to define integral analogues.

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*(a) Unless* $\underline{a}$ *is constant*

$$\min \underline{a} < \mathfrak{B}_n^{p,q}(\underline{a};\underline{w}) < \max \underline{a}. \tag{1}$$

*(b) If* $p_1 \le p_2$, $q_1 \le q_2$ *then*

$$\mathfrak{B}_n^{p_1,q_1}(\underline{a};\underline{w}) \le \mathfrak{B}_n^{p_2,q_2}(\underline{a};\underline{w}). \tag{2}$$

*(c)* [Dresher] *If* $\underline{a}, \underline{b}, \underline{w}$ *are positive* $n$*-tuples and if* $p \ge 1 \ge q \ge 0$ *then*

$$\mathfrak{B}_n^{p,q}(\underline{a}+\underline{b};\underline{w}) \le \mathfrak{B}_n^{p,q}(\underline{a};\underline{w}) + \mathfrak{B}_n^{p,q}(\underline{b};\underline{w}).$$

Comments (ii) This last inequality, (c), generalizes **Counter Harmonic Mean Inequalities** (4). The conditions on $p, q$ are necessary as well as sufficient.

(iii) See also: **Lyapunov's Inequality** Comments (ii) for a particular case of (2);

(iv) For another kind of mean due to Gini see: **Difference Means of Gini**.

References [GI3, pp. 107–122], [H, pp. 248–351, 316–317, 366–367], [MPF, pp. 156–163, 491], [PPT, pp. 119–121]; *Alzer* [5], *Losonczi & Páles* [182], *Páles* [250].

**Goldberg-Straus Inequality** *If* $z_j \in \mathbb{C}, 1 \le j \le n$, *with* $\sigma = \left|\sum_{j=1}^n z_j\right|$, $\delta = \max_{1\le j,k\le n}|z_j - z_k|$, *and* $\sigma\delta > 0$, *then for all* $m$*-tuples* $\underline{a}_j, 1 \le j \le n$,

$$\max\left|\sum_{j=1}^n z_j \underline{a}_{k_j}\right| \ge \frac{\sigma\delta}{2\sigma+\delta}\max_{1\le j\le n}|\underline{a}_j|,$$

*where the maximum on the left-hand side is over all permutations of the indices,* $k_1, \ldots, k_n$.

Comments (i) The value of the constant on the right-hand side is not optimal for any $n$, but is the best that can be chosen independently of $n$.

(ii) A related inequality is: **Complex Number Inequalities** (h).

References [GI2, pp. 37–51], [GI3, pp. 195–204].

**Godunova & Levin's Inequality** See: **Opial's Inequality** Comment (i).

**Goldstein's Inequality** *If* $\underline{a}_j, |\underline{a}_j| = 1, 1 \le j \le m$, *are real* $n$*-tuples that do not lie in the same half-space then:*

$$\left|\sum_{j=1}^m \underline{a}_j\right| < m - 2.$$

Comments (i) This is due to Klamkin & Newman; the case $n = 2$ is due to Goldstein.

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(ii) The result is best possible in the sense that we can get as close to $m-2$ by considering a sequence of convex polytopes that converge to a line segment.

REFERENCE *Klamkin & Newman* [159].

**Gorny's Inequality** *If $f:[a,b]\to\mathbb{R}$ has bounded derivatives of all orders $k$, $0\le k\le n$, define*

$$M_k=||f^{(k)}||_{\infty,[a,b]},\quad M_n'=\max\{M_n,M_0n!(b-a)^{-n}\},$$

*then*

$$M_k\le 4e^{2k}\binom{n}{k}^k M_0^{1-k/n}M_n^{k/n},\quad 0<k<n;$$

*and*

$$f^{(k)}\big((a+b)/2\big)\le 16(2e)^kM_0^{1-k/n}M_n^{k/n},\quad 0<k<n.$$

COMMENT These are extensions of the **Hardy-Littlewood-Landau Derivative Inequalities**.

REFERENCE [AI, pp. 138–139].

**Gram Determinant Inequalities** *(a) If $\underline{a}_j$, $1\le j\le m$, are real $n$-tuples define*

$$G=G(\underline{a}_1,\ldots,\underline{a}_m)=\begin{pmatrix}\underline{a}_1\cdot\underline{a}_1&\cdots&\underline{a}_1\cdot\underline{a}_m\\ \vdots&\ddots&\vdots\\ \underline{a}_m\cdot\underline{a}_1&\cdots&\underline{a}_m\cdot\underline{a}_m\end{pmatrix};$$

*then,*

$$\prod_{j=1}^m|\underline{a}_j|^2\ge G(\underline{a}_1,\ldots,\underline{a}_m)\ge 0,\tag{1}$$

*with equality on the right-hand side if and only if the $\underline{a}_j, 1\le j\le m$, are linearly dependent, and equality on the left-hand side if and only if the $\underline{a}_j, 1\le j\le m$, are mutually orthogonal.*
*(b) With the above notation,*

$$\frac{G(\underline{a}_1,\ldots,\underline{a}_m)}{G(\underline{a}_1,\ldots,\underline{a}_k)}\le\frac{G(\underline{a}_2,\ldots,\underline{a}_m)}{G(\underline{a}_2,\ldots,\underline{a}_k)}\le\cdots\le G(\underline{a}_{k+1},\ldots,\underline{a}_m).\tag{2}$$

*In particular*

$$G(\underline{a}_1,\ldots,\underline{a}_m)\le G(\underline{a}_1,\ldots,\underline{a}_k)G(\underline{a}_{k+1},\ldots,\underline{a}_m).$$

*(c) If $\underline{a}_j,\underline{b}_j$, $1\le k\le m$, are real $n$-tuples then*

$$G(\underline{a}_1,\ldots,\underline{a}_m)G(\underline{b}_1,\ldots,\underline{b}_m)\ge\left|\det\begin{pmatrix}\underline{a}_1\cdot\underline{b}_1&\cdots&\underline{a}_1\cdot\underline{b}_m\\ \vdots&\ddots&\vdots\\ \underline{a}_m\cdot\underline{b}_1&\cdots&\underline{a}_m\cdot\underline{b}_m\end{pmatrix}\right|^2,\tag{3}$$

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*with equality if and only if the vectors $\underline{a}_j, 1 \le j \le m$, and $\underline{b}_j, 1 \le j \le m$, span the same space.*

COMMENTS (i) This result holds for $m$ vectors in a unitary space.

(ii) The determinant $G$ is called *the Gram Determinant*, and the inequality on the right in (1) is called *Gram's Inequality*.

(iii) The inequality (3) can be regarded as a generalization of (C), to which it reduces when $m = 1$.

(iv) For an integral analogue see: **Mitrinović & Pečarić's Inequality**.

REFERENCES [AI, pp. 45–48], [EM, vol. 4, p. 293], [MPF, pp. 595–609], [PPT, p. 201]; *Courant & Hilbert* [CH, pp. 34–36], *Mitrinović & Pečarić* [MP91b, pp. 46–51];*Dragomir & Mond* [107], *Sinnadurai* [299].

**Greub & Rheinboldt's Inequality** See: **Polyá & Szegő's Inequality** COMMENTS (iii).

**Grönwall-Bellman Inequalities** See: **Grönwall's Inequality** COMMENTS (iii).

**Grönwall's Inequality** *Let $f, u, v$ be continuous on $[a, b]$ with $v \ge 0$ then*

$$f(x) \le u(x) + \int_a^x vf, \quad a \le x \le b,$$

*implies that*

$$f(x) \le u(x) + \int_a^x u(y)v(y)e^{\int_y^x v}\,\mathrm{d}y, \quad a \le x \le b. \tag{1}$$

COMMENTS (i) The significance of this result is that the right-hand side of (1) no longer involves $f$.

(ii) If $u$ is absolutely continuous the right-hand side of (1) can be written

$$e^{\int_a^x v}\left(u(a) + \int_a^x u'(y)e^{\int_y^a v}\,\mathrm{d}y\right).$$

(iii) This inequality is of importance in the theory of differential equations, as a result there have been many extensions both in the conditions under which it applies, and to similar inequalities for other differential equations. Related results are due to Bellman, Bihari and Henry; so that the reference is made to inequalities of *Grönwall-Bellman, Bellman-Bihari, Henry-Grönwall, and Bihari-Bellman type.*

REFERENCES [AI, pp. 374–375], [BB, pp. 131–136], [EM, vol. 3, pp. 170–171, Supp. II, pp. 51–52], [I3, pp. 333–340]; *Walter* [WW, pp. 11–41]; *Medved'* [204].

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**Grothendieck's Inequality** *If $n \geq 2$ and $(a_{ij})_{1\le i,j\le n}$ is a real or complex square matrix with the property that*

$$\left|\sum_{i,j=1}^{n} a_{ij}x_iy_j\right| \leq \sup_{1\le i,j\le n} |x_i|\,|y_j|,$$

*then there is a constant $K_G(n)$ such that for any elements $\underline{u}, \underline{v}$ of an $n$-dimensional Hilbert space,*

$$\left|\sum_{i,j=1}^{n} a_{ij}\langle \underline{u}_i, \underline{v}_j\rangle\right| \leq K_G(n) \sup_{1\le i,j\le n} ||\underline{u}_i||\,||\underline{v}_j||.$$

COMMENTS (i) The constant $K_G(n)$ increases with $n$ and the limit, as $n \to \infty$, $K_G$ say, is called *Grothendieck's Constant.*

(ii) In the real case $\pi/2 < K_G \leq \pi/2\log(1+\sqrt{2})$.

(iii) The value in the complex case is smaller since then $K_G < \pi/2$.

REFERENCES [GI6, pp. 201–206, 481], [MPF, pp. 569–571].

**Grunsky's Inequalities** *If $f$ is univalent in $D$ with $f(0) = f'(0)$ then for all complex $n$-tuples $\underline{z}$ with $z_r \in D$, $1 \leq r \leq n$, and all complex $n$-tuples $\underline{w}$*

$$\sum_{r,s=1}^{n} w_r\overline{w}_s \log\frac{1}{1-z_r\overline{z}_s} \geq \left|\sum_{r,s=1}^{n} w_r\overline{w}_s \log\left\{\frac{z_rz_s}{f(z_r)f(z_s)}\frac{f(z_r)-f(z_s)}{z_r-z_s}\right\}\right|.$$

COMMENTS (i) Whenever $z_r = z_s$ the last term on the right-hand side is interpreted as $f'(z_s)$.

(ii) Conversely if $f$ is analytic on $D$ with $f(0) = f'(0)$ and the inequality holds for all such $\underline{z}, \underline{w}$ then $f$ is univalent.

(iii) This inequality can be written as $\underline{w}^*A\underline{w} \geq |\underline{w}^TB\underline{w}|$, where $\underline{w}^T$ is the transpose of $\underline{w}$ and $A$, $B$ are the matrices with entries the log term on the left-hand side and right-hand side, respectively.

(iv) For other forms of this inequality see [EM], and the recent book by Gong

REFERENCES [EM, vol. 1, p. 245; vol. 3, p. 269; vol.9, p. 338]; *Gong* [GS], *Horn & Johnson* [HJ, p. 202].

**Grüss-Barnes Inequality** *If $f, g : [0,1] \to [0,\infty[$ are concave, and if $p, q \geq 1$ then*

$$\int_0^1 fg \geq \frac{(p+1)^{1/p}(q+1)^{1/q}}{6}||f||_p||g||_q. \tag{1}$$

*If $0 < p, q \leq 1$ (~1) holds with 6 replaced by 3.*

COMMENTS (i) The case $p = q = 1$ is due to G. Grüss, the case $p, q \geq 1$ and conjugate indices are by Bellman and the general result is by Barnes.

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(ii) The general case follows from Grüss's result and **Favard's Inequalities** (b).

EXTENSIONS [BORELL] *(a) Under the same conditions as above*

$$\int_0^1 fg \geq \frac{(p+1)^{1/p}(q+1)^{1/q}}{6}||f||_p||g||_q + \frac{f(0)g(0)+f(1)g(1)}{6}.$$

*(b) If $f_i, 1 \leq i \leq n$ satisfy the conditions above then*

$$\int_0^1 \prod_{i=1}^n f_i \geq \frac{[(n+1)/2]![n/2]!}{(n+1)!} \prod_{i=1}^n (p_i+1)^{1/p_1}||f||_{p_i}.$$

COMMENT (iii) (b) was published independently by Godunova & Levin.

REFERENCES [PPT, pp. 223–224]; *Maligranda, Pečarić & Persson* [194].

**Grüsses' Inequalities**[27] *(a) If $f, g$ are integrable on $[0,1]$, with $\alpha \leq f \leq A$, and $\beta \leq g \leq B$, then*

$$\left| \int_0^1 fg - \int_0^1 f \int_0^1 g \right| \leq \frac{1}{4}(A-\alpha)(B-\beta).$$

*(b)* [G. GRÜSS] *If $m, n \in \mathbb{N}$ then*

$$\frac{1}{m+n+1} - \frac{1}{(m+1)(n+1)} \leq \frac{4}{45}.$$

COMMENTS (i) (a) is an inverse of the integral analogue of (Č). The result was conjectured by H. Grüss and proved by G. Grüss. For another such result see **Karamata's Inequalities**.

(ii) The constant on the right-hand side of (a) is best possible as can be seen by taking $f(x) = g(x) = \operatorname{sgn}\bigl(x - (a+b)/2\bigr)$.

DISCRETE ANALOGUE [BIERNACKI, PIDEK & RYLL-NARDZEWSKI] *If $\underline{a}, \underline{b}$ are real $n$-tuples with $\alpha \leq \underline{a} \leq A$, and $\beta \leq \underline{b} \leq B$, then*

$$|\mathfrak{A}(\underline{a}\,\underline{b}) - \mathfrak{A}(\underline{a})\mathfrak{A}(\underline{b})| \leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(A-\alpha)(B-\beta).$$

COMMENT (iii) There are many generalizations in the references; see also **Ostrowski's Inequalities** (b).

REFERENCES [AI, pp. 70–74, 190–191], [MPF, pp. 295–310], [PPT, pp. 206–212]; *Rassias & Srivastava* [RS, pp. 93–113].

[27] This refers to both H. and G. Grüss.

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**Guha's Inequality** *If* $p \geq q > 0,\ x \geq y > 0$ *then*

$$(px + y + a)(x + qy + a) \geq [(p+1)x + a]\,[(q+1)y + a]\,.$$

Comments (i) A proof of this inequality follows by noting that the difference between the left-hand side and right-hand side is just $(px - qy)(x - y)$.
(ii) This inequality is the basis of a proof of (GA).

Reference [H, pp. 101–102].

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# 8 Haber–Hyperbolic

**Haber's Inequality** *If $a+b>0$, $a \neq b$ then*

$$\left(\frac{a+b}{2}\right)^n < \frac{a^n + a^{n-1}b + \cdots + b^n}{n+1}, \; n = 2, 3, \ldots; \qquad (1)$$

*if $a+b<0$ then (1) holds if $n$ is even, but if $n$ is odd then $(\sim 1)$ holds.*

COMMENTS (i) This can be proved by induction.

(ii) A special case of (1) is **Polynomial Inequalities** (4).

EXTENSION [MERCER, A.] *If $\underline{a}$ is convex then*

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}a_i \leq \frac{a_0 + \cdots + a_n}{n+1}.$$

REFERENCES [GI2, pp. 143–144]; *Mercer, A.*[205].

**Hadamard's Determinant Inequality** *If $A$ is an $m \times n$ complex matrix with rows the complex $n$-tuples $\underline{r}_k, 1 \leq k \leq m$, then:*

$$\det(AA^*) \leq \prod_{k=1}^{m} |\underline{r}_k|^2.$$

*There is equality if one of $\underline{r}_k$ is zero, or, if no $\underline{r}_k$ is zero and the $\underline{r}_k, 1 \leq k \leq m$, are orthogonal.*

COMMENTS (i) This inequality is often referred to as *Hadamard's inequality*; as is Corollary (b) below.

(ii) In the real case the geometric meaning of the Hadamard inequality is:

*in $\mathbb{R}^n$ the volume of a polyhedron formed by $n$ vectors is largest when the vectors are orthogonal.*

COROLLARIES *(a) If $A$ is an $n \times n$ complex matrix with rows the complex $n$-tuples $\underline{r}_k, 1 \leq k \leq n$, and columns the complex $n$-tuples $\underline{c}_k, 1 \leq k \leq n$, then*

$$|\det A| \leq \min\left\{\prod_{k=1}^{n}|\underline{r}_k|, \prod_{k=1}^{n}|\underline{c}_k|\right\}.$$



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*(b) If $A$ is a positive semi-definite $n \times n$ matrix then*

$$\det A \le \prod_{i=1}^{n} a_{ii},$$

*with equality if and only if $A$ is diagonal.*

COMMENT (iii) The cases of equality for the corollaries follow easily from those in the main result.

REFERENCES [BB, p. 64], [EM, vol. 4, pp. 350–351],[HLP, p. 34–36],[MOA, p. 306], [MPF, p. 213]; *Courant & Hilbert* [CH, pp. 36–37], *Horn & Johnson* [HJ, pp. 476–482], *Marcus & Minc* [MM, p. 114], *Price* [Pr, pp. 607–610].

**Hadamard's Integral Inequality** See: **Hermite-Hadamard Inequality**.

**Hadamard Product Inequalities** *(a) If $A_i, B_i$, $1 \le i \le m$, are $n \times n$ Hermitian matrices then*

$$A_i \le B_i,\ 1 \le i \le m \implies \prod_{i=1}^{m} {*A_i} \le \prod_{i=1}^{m} {*B_i}.$$

*(b) If $A_i$, $1 \le i \le m$ are $n \times n$, Hermitian matrices and if $p \ge 1$,*

$$\prod_{i=1}^{m} {*A_i} \le \big(\prod_{i=1}^{m} {*A_i^p}\big)^{1/p}.$$

*(c) If $A_i, B_i$, $1 \le i \le m$, are $n \times n$ Hermitian matrices then for $p \ge 1$ and $q$ the conjugate index,*

$$\sum_{i=1}^{m} A_i * B_i \le \big(\sum_{i=1}^{m} A_i^p\big)^{1/p}\big(\sum_{i=1}^{m} B_i^q\big)^{1/q}.$$

*(d)If $A$ and $B$ are $n \times n$ Hermitian matrices then*

$$A * B \ge \mathfrak{G}_2(A,B) * \mathfrak{G}_2(A,B); \qquad A * B \ge \mathfrak{H}_2(A,B) * \mathfrak{H}_2(A,B).$$

*(e)* [OPPENHEIM] *If $A$ and $B$ are $n \times n$ positive definite matrices then:*

$$\det(A * B) \ge \prod_{i=1}^{n} a_{ii} \det B,$$

*with equality if and only if $B$ is diagonal.*

COMMENTS (i) To say that $A \ge B$ is to say that $A - B$ is positive semi-definite.
(ii) The definitions of the geometric and harmonic means are given in **Matrix Inequalities** COMMENT (II).

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REFERENCES *Ando* [44, 45], *Markham* [198].

**Hadamard's Three Circles Theorem** *If $f$ is analytic on the closed annulus,*
$a \leq |z| \leq b$*, then:*

$$M_\infty(f;r) \leq M_\infty(f;a)^{\log(b/r)\big/\log(b/a)} M_\infty(f;b)^{\log(r/a)\big/\log(b/a)}, \quad a \leq r \leq b.$$

COMMENTS (i) This result says that $M_\infty(f;r)$ is a log-convex function of $\log r$.

(ii) A similar result holds for harmonic functions, and can be extended to the solutions of more general elliptic partial differential equations; see **Harnack's Inequalities** COMMENT.

(iii) See also: **Hardy's Analytic Function Inequality** and **Phragmén-Lindelöf Inequality**.

REFERENCES [AI, p. 19], [EM, vol. 4, p. 351]; *Ahlfors* [Ah78, p. 165],*Conway* [C, vol. I, p. 133], *Pólya & Szegö* [PS, pp. 164–166], *Protter & Weinberger* [PW, pp. 128– 137].

**Hajela's Inequality** *Let $\mu$ be a probability measure on $X$, $1 \leq \sqrt[3]{3}p < q < \infty$, $f \in \mathcal{L}^p_\mu(X)$, not constant $\mu$-almost everywhere, then*

$$\begin{aligned} 0 \quad < \quad & \frac{q-p}{q}\big(||f||^p_{q,\mu} - ||f||^p_{p,\mu} - ||f||^p_{p,\mu}\log||f||^p_{q,\mu} + \int_X |f|^p \log|f|^p \,\mathrm{d}\mu\big) \\ & \leq ||f||^p_{q,\mu} - ||f||^p_{p,\mu}. \end{aligned}$$

COMMENT The left inequality holds under the wider condition $1 \leq p < q$.

REFERENCE *Hajela* [138].

**Hammer's Inequalities** See: **Quadrature Inequalities**.

**Halmos's Inequalities** *(a) If $f$ is absolutely continuous on $[0,1]$ with $f(0) = f(1) = 1$ and $\int_0^1 |f'| = 1$ then $|f| \leq 1/2$.*
*(b) If $A$ and $B$ are bounded linear operators on a Hilbert space such that $A$, or $B$, commutes with $AB - BA$ then*

$$||I - (AB - BA)|| \geq 1.$$

REFERENCES *Halmos* [Ha, p. 62]; *Maher* [191].

**Hamy Mean Inequalities** If $\underline{a}$ is a positive $n$-tuple, and if $1 \leq r \leq n, r \in \mathbb{N}$, define

$$\mathfrak{S}^{[r]}_n(\underline{a}) = \frac{1}{r!\binom{n}{r}} \sum_r ! \left(\prod_{j=1}^r a_{i_j}\right)^{1/r}.$$

This quantity is called the *Hamy Mean of* $\underline{a}$.

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It reduces to well-known means in special cases:

$$\mathfrak{S}_n^{[n]}(\underline{a}) = \mathfrak{P}_n^{[n]}(\underline{a}) = \mathfrak{G}_n(\underline{a}), \text{ and } \mathfrak{S}_n^{[1]}(\underline{a}) = \mathfrak{P}_n^{[1]}(\underline{a}) = \mathfrak{A}_n(\underline{a}).$$

*(a) With the above notation, if* $r, s \in \mathbb{N}, 1 \le r < s \le n$, *then:*

$$\mathfrak{S}_n^{[s]}(\underline{a}) \le \mathfrak{S}_n^{[r]}(\underline{a}),$$

*with equality if and only if* $\underline{a}$ *is constant.*
*(b) If* $1 \le r \le n$,

$$\mathfrak{S}_n^{[r]}(\underline{a}) \le \mathfrak{P}_n^{[r]}(\underline{a}),$$

*with equality if and only if either* $r = 1$, $r = n$ *or* $\underline{a}$ *is a constant.*

Comment (b) is an easy consequence of (r;s).

Reference [H, pp. 364–366].

**Hanner's Inequalities** *If* $f, g \in \mathcal{L}_\mu^p(X)$ *and* $2 \le p < \infty$ *then*

$$\begin{aligned} ||f+g||_p^p + ||f-g||_p^p &\le \big|||f||_p - ||g||_p\big| + \big(||f||_p + ||g||_p\big)^p; &(1)\\ 2^p\big(||f||_p^p + ||g||_p^p\big) &\le \big(||f+g||_p + ||f-g||_p\big)^p + \big|||f+g||_p - ||f-g||_p\big|^p. &(2)\end{aligned}$$

*If* $1 \le p \le 2$ *then* $(\sim 1), (\sim 2)$ *hold.*

Comment These inequalities are connected with the uniform convexity of the $\mathcal{L}^p$ spaces. See also: **Clarkson's Inequalities**.

Reference *Lieb & Loss* [LL, pp. 42–44, 69].

**Hansen's Inequality** *If* $A$, *and* $B$ *are operators on a Hilbert space, with* $A \ge 0$ *and* $||B|| \le 1$ *then for all* $t$, $0 < t \le 1$,

$$(B^*AB)^t \ge B^*A^tB.$$

Comment The power function can be replaced by any operator monotone function.

Reference *Uchiyama* [316].

**Hardy's Analytic Function Inequality** *If* $f$ *is analytic in the open annulus* $a < |z| < b$ *and* $1 \le p < \infty$ *then,* $M_p(f; r)$ *is a log-convex function of* $\log r, a < r < b$.

Comment This result, which also holds for harmonic functions, should be compared to **Hadamard's Three Circle Theorem** and the **Phragmén-Lindelöf Inequality**.

References [AI, p. 19], [I3, pp. 9–21]; *Conway* [C, vol. I, p. 134].

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**Hardy's Inequality** *If $p > 1$ and $\underline{a}$ is a non-negative, non-null $\ell_p$ sequence,*

$$\sum_{i=1}^{\infty} \mathfrak{A}_i^p(\underline{a}) < \left(\frac{p}{p-1}\right)^p \sum_{i=1}^{\infty} a_i^p; \tag{1}$$

*or*

$$||\underline{\mathfrak{A}}||_p < \frac{p}{p-1}||\underline{a}||_p.$$

*The constant is best possible.*

Comments (i) Both inequality (1) and its integral analogue, (5) below, are known as *Hardy's Inequality*, or sometimes as the *Hardy-Landau Inequality*, as it was Landau who determined the value of the constant.

(ii) Putting $a_i$ for $a_i^p, i \geq 1$, in (1) and letting $p \to \infty$, gives a weak form of **Carleman's Inequality.**

Extensions *(a)* [Redheffer] *With the hypotheses as above,*

$$\sum_{i=1}^{n} \mathfrak{A}_i^p(\underline{a}) + \frac{np}{p-1}\mathfrak{A}_n^p(\underline{a}) < \left(\frac{p}{p-1}\right)^p \sum_{i=1}^{n} a_i^p. \tag{2}$$

*(b)* [Copson] *If $p > 1, a_i \geq 0, w_i > 0, i = 1, \ldots,$ and not all the $a_i$ are zero then*

$$\sum_{i=1}^{\infty} w_n \mathfrak{A}_i^p(\underline{a}; \underline{w}) < \left(\frac{p}{p-1}\right)^p \sum_{i=1}^{\infty} w_n a_i^p. \tag{3}$$

Comments (iii) Inequality (2) is an example of a recurrent inequality. See: **Recurrent Inequalities**.

(iv) Inequality (3) can be deduced from (1) and **Hölder's Inequality** Other Forms (c). It is known as *Copson's Inequality*, it is the weighted form of (1). Another inequality of the same name is discussed in **Copson's Inequality**.

Some extra notation is needed for a converse and an important extension of Hardy's Inequality (1).

If $p > 1$ and $\underline{a}$ is a real sequence such that $\sum_{i=1}^{n} |a_i|^p = O(n)$ then we say that $\underline{a} \in \mathfrak{g}(p)$.

The collection of all such sequences is a Banach space with norm defined by

$$||\underline{a}||_{\mathfrak{g}(p)} = \sup_n \left(\frac{1}{n}\sum_{i=1}^{n} |a_i|^p\right)^{1/p}.$$

Consider all factorizations of a real sequence $\underline{a}$ as $\underline{a} = \underline{u}\,\underline{v}, \underline{u} \in \ell_p, \underline{v} \in \mathfrak{g}(q), q$ the conjugate index, and define

$$!\underline{a}!_p = \inf ||\underline{u}||_p||\underline{v}||_{\mathfrak{g}(q)},$$

where the inf is taken over all such factorizations of $\underline{a}$.

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Extension *(c)* [Bennett] *If $p > 1$,*

$$(p-1)^{-1/p}!\underline{a}!_p \le ||\mathfrak{A}_i^p(\underline{a})||_p \le q\,!\underline{a}!_p, \tag{4}$$

*$q$ being the conjugate index. The constants are best possible, and the right-hand inequality is strict unless $\underline{a} = \underline{0}$.*

Comments (v) Bennett's result actually gives more: *the center term in the inequalities is finite if and only if the sequence admits a factorization of the type above.*

(vi) (1) follows from (4) by taking $\underline{u} = \underline{a}, \underline{v} = \underline{e}$.

(vii) Other inverse inequalities have been studied by Neugebauer.

Integral Analogues *If $p > 1$ and, $f \ge 0$,*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p; \tag{5}$$

$$\int_0^\infty \left(\int_x^\infty f\right)^p \mathrm{d}x \le p^p \int_0^\infty x^p f^p(x)\,\mathrm{d}x. \tag{6}$$

*The inequalities are strict unless $f = 0$ almost everywhere, and the constants are best possible.*

Comments (viii) A change of variable, $y = x^{1/q}$, shows that (5) is equivalent to:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g\right)^p \frac{1}{x}\,\mathrm{d}x \le \int_0^\infty g^p(x)\frac{1}{x}\,\mathrm{d}x, \tag{7}$$

where $xf(x) = yg(y)$, and $q$ is the conjugate index. A proof of (7) follows by an application of the integral analogue of (J) and a reverse in the order of integration.

(ix) Inequalities (5) and (6) can be rewritten by putting $F' = f$ to give an inequality between a function and its derivative. In this and the original form they have been the subject of much study and generalization. These extensions include introducing weights, varying the exponents on each side, using fractional integrals, replacing the Cesàro mean on the left-hand side by a more general mean, and going to higher dimensions, amongst many others. See: **Bennett's Inequalities** (b), **HELP Inequalities**, **Sobolev's Inequalities**.

Extensions (a) If $f \ge 0$, and $p > 1, m > 1$ or $p < 0, m < 1$, then:

$$\int_0^\infty x^{-m}\left(\int_0^x f\right)^p \mathrm{d}x \le \left|\frac{p}{m-1}\right|^p \int_0^\infty x^{p-m} f^p(x)\,\mathrm{d}x. \tag{8}$$

If $0 < p < 1, m \ne 1$ then ($\sim$8) holds.

(b) [Izumi & Izumi] If $p > 1, m > 1, f > 0$ then

$$\int_0^\pi x^{-m}\left(\int_{x/2}^x f\right)^p \mathrm{d}x < \left(\frac{p}{m-1}\right)^p \int_0^\pi x^{-m}\,|f(x/2) - f(x)|^p\,\mathrm{d}x.$$

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COMMENT (x) The research on this inequality is extensive as the more recent references show.

REFERENCES [AI, p. 131], [EM, vol. 4, p. 369], [GI2, pp. 55–80], [GI3, pp. 205–218], [GI4, pp. 47–57], [GI6, pp. 33–58], [GI7, pp. 3–16], [HLP, pp. 239–249], [PPT, pp. 229–239]; *Bennett* [Be, pp. 13–18], *Grosse-Erdmann* [GE], *Opic & Kufner* [OK], *Zwillinger* [Zw, p. 206]; *Chu, Xu & Zhang* [90], *Kufner, Maligranda & Persson* [167], *Leindler* [173]; *Marcus, Mizel & Pinchover* [197],[28] *Mohupatra & Vajravelu* [220], *Neugebauer* [229], *Pachpatte* [243], *Pachpatte & Love* [254], *Persson & Samko* [268], *Ross* [282], *Sobolevskiĭ* [303].

**Hardy-Landau Inequality** See: **Hardy's Inequality** COMMENTS (i).

**Hardy-Littlewood-Landau Derivative Inequalities** *If* $f : [0, \infty[ \to \mathbb{R}$ *has bounded derivatives of orders* $k, 0 \leq k \leq n$, *let* $M_k = ||f^{(k)}||_\infty$ *then:*

$$M_k^n \leq C_{n,k}^n M_0^{n-k} M_n^k, \quad 0 < k < n; \tag{1}$$

*where*

$$C_{n,1} \leq 2^{n-1}. \tag{2}$$

*In particular:*

$$C_{2,1}^2 = 2,\ C_{3,1}^3 = 9/8,\ C_{3,2}^3 = 3,\ C_{4,1}^4 = 512/375,\ C_{4,2}^4 = 36/25,\ C_{5/2}^5 = 125/72.$$

COMMENTS (i) Inequality (2), and the value of $C_{2,1}$ are due to Hadamard, and the other values were obtained by Šilov, and the values are best possible.

(ii) The study of such inequalities was emphasized by Hardy, Littlewood, and Landau.

(iii) It is natural to try to extend (1) to other $\mathcal{L}^p$ norms. See: **Hardy-Littlewood-Pólya Inequalities** (a), **Kolmogorov's Inequalities**. Another kind of inequality between derivatives can be found in **Sobolev's Inequalities**.

(iv) For another extension see **Gorny's Inequality**, and for ones where ordinary derivatives are replaced by Sturm-Liouville expressions see **HELP Inequalities**.

(v) The above inequalities have also been studied in their discrete forms. See: [GI5, GI6].

(vi) See also: **Halmos's Inequalities** (A), **Opial's Inequalities, Wirtinger's Inequality**.

REFERENCES [AI, pp. 138–140], [EM, vol. 5, p. 295], [GI5, pp. 367–379], [GI6, pp. 459–462], [HLP, pp. 187–193]; *Benyon, Brown & Evans* [59].

**Hardy-Littlewood Maximal Inequalities** *(a) If* $f$ *is a non-negative measurable function on* $]0, a[$ *that is finite almost everywhere and if* $E \subseteq [0, a]$ *is measurable then*

$$\int_E f \leq \int_0^{|E|} f^*.$$

[28] This is Moshe Marcus. In all other cases it is Marvin Marcus.

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*(b)* [The Maximal Theorem] *If $f \in \mathcal{L}(0,a), f \geq 0$ define*

$$\theta(x) = \theta_f(x) = \sup_{0 \leq y < x} \frac{1}{x-y} \int_y^x f, \quad 0 < x \leq a.$$

*If then $\chi \geq 0$ is an increasing function,*

$$\int_0^a \chi \circ \theta_f \leq \int_0^a \chi \circ \theta_{f^*}.$$

Comments (i) Since $f^*$ is decreasing the function $\theta_{f^*}$ is given by

$$\theta_{f^*}(x) = \frac{1}{x} \int_0^x f^*.$$

(ii) If we replace $f^*$ by $f_*$ and $\theta_f$ by

$$\theta^\blacktriangle(x) = \theta_f^\blacktriangle(x) = \sup_{x < y \leq a} \frac{1}{y-x} \int_x^y f, \quad 0 \leq x < a$$

the above result remains valid.

(iii) Hence if

$$\Theta_f(x) = \max\{\theta_f(x), \theta_f^\blacktriangle(x)\} = \sup_{\substack{0 \leq y \leq a \\ y \neq x}} \frac{1}{y-x} \int_x^y f, \quad 0 \leq x \leq a.$$

we get that

$$\int_0^a \chi \circ \Theta_f \leq 2 \int_0^a \chi \circ \theta_{f^*}.$$

Taking particular functions $\chi$ we get the following important results.

Corollaries *(a) If $f \in \mathcal{L}^p(0,a), p > 1, f \geq 0$ then $\Theta_f \in \mathcal{L}_p(0,a)$ and*

$$\int_0^a \Theta_f^p \leq 2 \left( \frac{p}{p-1} \right)^p \int_0^a f^p.$$

*(b) If $f \in \mathcal{L}(0,a), f \geq 0$ then $\Theta_f \in \mathcal{L}^p(0,a), 0 < p < 1$, and*

$$\int_0^a \Theta_f^p \leq \frac{2a^{1-p}}{1-p} \int_0^a f^p.$$

*(c) If $f \log^+ f \in \mathcal{L}(0,a), f \geq 0$ then $\Theta_f \in \mathcal{L}(0,a)$ and*

$$\int_0^a \Theta_f \leq 4 \int_0^a f \log^+ f \; + \; C.$$

Comment (iv) The above can easily be extended to functions non-negative, measurable, and finite almost everywhere on any bounded interval $[a,b]$.

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*If $f$ is measurable, finite almost everywhere, and periodic, with period $2\pi$, define*

$$M(x) = M_f(x) = \sup_{0<|t|<\pi} \frac{1}{t}\int_x^{x+t} |f|;$$

*and let $\theta_{|f|}$ be defined using the interval $[-2\pi, 2\pi]$, then:*

$$M_f \le \theta_{|f|}.$$

Comments (v) The above results imply integral inequalities for the function $M_f$, the *Hardy-Littlewood Maximal Function.*

(vi) See also: **Kakeya's Maximal Function Inequality, Rearrangement Inequalities, Spherical Rearrangement Inequalities**.

References [EM, vol. 4, pp. 370–371]; *Hewitt & Stromberg* [HS, pp. 422–429], *Zygmund* [Z, vol. I, pp. 30–33].

**Hardy-Littlewood-Pólya Inequalities** *(a) If $f, f'' \in \mathcal{L}^2([0,\infty[)$ then:*

$$||f'||^2_{2,[0,\infty[} < 2||f||_{2,[0,\infty[}||f''||_{2,[0,\infty[}, \tag{1}$$

*unless $f(x) = Ae^{-BX/2}\sin(Bx\sqrt{3}/2 - \pi/3)$ almost everywhere.*
*If $f, f'' \in \mathcal{L}^2(\mathbb{R})$ then*

$$||f'||^2_{2,\mathbb{R}} < ||f||_{2,\mathbb{R}}||f''||_{2,\mathbb{R}}, \tag{2}$$

*unless $f(x) = 0$ almost everywhere.*
*(b) If $\underline{a}, \underline{b}$ are non-negative, non-null sequences, $p, q$ conjugate indices then:*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{a_i b_j}{\max\{i,j\}} < pq\left(\sum_{i=1}^{\infty} a_i^p\right)^{1/p}\left(\sum_{j=1}^{\infty} b_j^q\right)^{1/q}. \tag{3}$$

*The constant is best possible.*

Comments (i) (1) is often called the *Hardy-Littlewood-Pólya Inequality* and has been the object of much research; the same name is also given to the much easier inequality (2). It is natural to see what can be said for the other $\mathcal{L}^p$ classes. See: **Hardy-Littlewood-Landau Derivative Inequalities, Kolmogorov's Inequalities**.

In addition, the possibility of weights and more general derivative operators has been explored. See: [GI], **HELP Inequalities**.

(ii) An integral analogue of (3) is easily stated.

Discrete Analogues [Copson] *If $\underline{a}$ is a real sequence,*

$$\left(\sum_{n=0}^{\infty}(\Delta a_n)^2\right)^2 \le 4\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty}(\Delta^2 a_n)^2;$$

$$\left(\sum_{n=-\infty}^{\infty}(\Delta a_n)^2\right)^2 \le \sum_{n=-\infty}^{\infty} a_n^2 \sum_{n=-\infty}^{\infty}(\Delta^2 a_n)^2;$$

*with equality if and only if $\underline{a} = \underline{0}$.*

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Comments (iii) As with the integral originals these discrete inequalities have been extended, to higher order differences, for instance.

(iv) See also: **Multilinear Form Inequalities, Rearrangement Inequalities** (a), **Sobolev's Inequalities**.

References [GI5, pp. 29–63], [HLP, pp. 187–193, 254], [PPT, p. 234]; *Borogovac & Arslanagić* [65].

**Hardy-Littlewood-Pólya-Schur Inequalities** *Let $p > 1$, $q$ the conjugate index, and $K : [0,\infty[\times[0,\infty[\to [0,\infty[$ , homogeneous of degree $-1$ with*

$$\int_0^\infty \frac{K(x,1)}{x^{1/p}}\,\mathrm{d}x = \int_0^\infty \frac{K(1,y)}{y^{1/q}}\,\mathrm{d}x = C, \tag{1}$$

*where both of the integrands are strictly decreasing then:*

$$\sum_{m,n\in\mathbb{N}} K(m,n)a_m b_n \le C||\underline{a}||_p||\underline{b}||_q, \tag{2}$$

*with equality if and only if $\underline{a} = \underline{b} = \underline{0}$;*
*and*

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}}\left(\sum_{m\in\mathbb{N}} K(m,n)a_m\right)^p\right)^{1/p} &\le C||\underline{a}||_p, \quad \left(\sum_{m\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} K(m,n)b_n\right)^q\right)^{1/q} \\ &\le C||\underline{b}||_q,\end{aligned}$$

*with equality if and only if $\underline{a} = \underline{0}$ in the first case and $\underline{b} = \underline{0}$ in the second. In all cases the constant in the right-hand side is best possible.*

Comments (i) The case $p = 2$ is due to Schur, and the result has been the object of much study and generalization. Homogeneous functions are defined in **Segre's Inequalities**.

(ii) A particular case of (2) is **Hilbert's Inequalities** (2).

Integral Analogues *Let $p > 1$, $q$ the conjugate index, and $K : [0,\infty[^2\to [0,\infty[$, homogeneous of degree $-1$ with (1) holding, then:*

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \le C||f||_{p,[0,\infty[}||g||_{q,[0,\infty[}; \tag{3}$$

$$\left(\int_0^\infty\left(\int_0^\infty K(x,y)f(x)\,\mathrm{d}x\right)^p\mathrm{d}y\right)^{1/p} \le C||f||_{p,[0,\infty[}; \tag{4}$$

$$\left(\int_0^\infty\left(\int_0^\infty K(x,y)g(y)\,\mathrm{d}y\right)^q\mathrm{d}x\right)^{1/q} \le C||g||_{q,[0,\infty[}. \tag{5}$$

*If $K$ is positive there is equality in (3) if and only if $f = 0$ almost everywhere, and $g = 0$ almost everywhere, in (4) if and only if $f = 0$ almost everywhere, and in (5) if and only if $g = 0$ almost everywhere.*

Comments (iii) A particular case of (3) is **Hilbert's Inequalities** (3).

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(iv) The various forms of these inequalities are equivalent. See: **Bilinear Form Inequalities of M. Riesz**.

References [GI2, pp. 277–286, 458–461], [GI3, p. 207], [HLP, pp. 227–232, 243–246].

**Hardy-Littlewood-Sobolev Inequalities** *(a) If $f_1, \ldots, f_m, h$ are non-negative,*

$$\int_{\mathbb{R}} (f_1 \star \cdots \star f_m) h \leq \int_{\mathbb{R}} (f_1^{(*)} \star \cdots \star f_m^{(*)}) h^{(*)}.$$

*(b) If $f \in \mathcal{L}^p(\mathbb{R}^n), g \in \mathcal{L}^q(\mathbb{R}^n)$, where $p, q > 1$ and $1/p + 1/q + \lambda/n = 2, 0 < \lambda < n$, then*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} h(y) \, \mathrm{d}x \, \mathrm{d}y \right| \leq C||f||_p ||g||_q,$$

*where $C$ depends on $n, \lambda, p$ only.*

Comments (i) The definition of convolution given in **Notations 4** is easily extended to the convolution of more than two functions.

(ii) The case $m = 2$ of (a) is by F. Riesz, based on an argument of Hardy-Littlewood; the extension to general $m$ is due to Sobolev.

(iii) Inequality (a) has been further generalized in the reference.

(iv) In the case $p = q$ of (b) the exact value of the constant as well as the cases of equality are known. The inequality is sometimes called the *Weak Young Inequality*.

References [HLP, pp. 279–288]; *Lieb & Loss* [LL, pp. 98–102].

**Harker-Kasper Inequality** *If $\underline{p}$ is a non-negative $n$-tuple with $P_n = 1$ and if $g(x) = \sum_{i=1}^n p_i \cos \alpha_i x, x \in \mathbb{R}$ then*

$$g^2(x) \leq \frac{1}{2}\bigl(1 + g(2x)\bigr).$$

Comments (i) If $n = 1$ this reduces to the simple identity $\cos^2 \alpha x = \frac{1}{2}(1 + \cos 2\alpha x)$.

(ii) This inequality is of importance in crystallography.

Reference *Steele* [S, pp. 13–14, 227]; *Harker & Kasper* [139].

**Harmonic Function Inequalities** See: **Analytic Function Inequalities** Comments (ii), **Conjugate Harmonic Function Inequalities, Hadamard's Three Circles Theorem** Comments (ii), **Hardy's Analytic Function Inequality** Comment, **Harnack's Inequalities, Maximum-Modulus Principle** Comments (i).

**Harmonic Mean Inequalities** Results concerning this mean are usually subsumed under the results involving the arithmetic mean, to which it is closely

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related; see **Notations 3**. However there are some results that are worth noting.
*If $\underline{a}$ is a positive $n$-tuple:*
*(a)*

$$\min \underline{a} \leq \mathfrak{H}_n(\underline{a};\underline{w}) \leq \max \underline{a},$$

*with equality if and only if $\underline{a}$ is constant.*
*(b)* [Harmonic-Geometric Mean Inequality]

$$\mathfrak{H}_n(\underline{a};\underline{w}) \leq \mathfrak{G}_n(\underline{a};\underline{w}), \tag{HG}$$

*with equality if and only if $\underline{a}$ is constant.*
*(c)* [Klamkin] *If $n \geq 2$ then*

$$W_n\sqrt{\frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{H}_n(\underline{a};\underline{w})}} \geq w_n + W_{n-1}\sqrt{\frac{\mathfrak{A}_{n-1}(\underline{a};\underline{w})}{\mathfrak{H}_{n-1}(\underline{a};\underline{w})}}, \tag{1}$$

*with equality only if $a_n = \sqrt{\mathfrak{A}_{n-1}(\underline{a};\underline{w})\mathfrak{H}_{n-1}(\underline{a};\underline{w})}$.*
*(d)*

$$1 + \mathfrak{G}_n(\underline{a}) \leq \mathfrak{G}_n(1+\underline{a}^{-1}) \leq 1 + \frac{1}{\mathfrak{H}_n(\underline{a})},$$

*with equality if and only if $\underline{a}$ is constant.*

Comments (i) (HG) is equivalent to (GA); and it follows easily from (GA) using the definition of the harmonic mean.

(ii) As (HG) is so closely related to (GA) it has many extensions similar to those discussed for that inequality.

(iii) In addition (HG) is a special case of (r;s), and so again has many of the extensions that are discussed for that inequality.

(iv) Some interpretations of this inequality, in the equal weight case, are discussed in **Geometric Inequalities** (a).

(v) The inequality

$$\mathfrak{H}_n(\underline{a};\underline{w}) \leq \mathfrak{A}_n(\underline{a};\underline{w}), \tag{HA}$$

follows by combining (HG) and (GA).

While (HA) is apparently weaker than (GA) it is actually equivalent to it; see **Walsh's Inequality**.

(vi) Inequality (1) is a Rado-Popoviciu type extension of (HA).

(vii) Integral analogues of (HG) and (HA) are easily stated as particular cases of the integral analogues of (r;s).

(viii) See also: **Binomial Function Inequalities** (j), **Kantorovič's Inequality**, **Knopp's Inequalities** Corollary, **Sierpinski's Inequalities**.

References [H, pp. 72, 128–129], [MPF, p. 73].

**Harnack's Inequalities** *Let $h$ be a non-negative harmonic function in the domain $\Omega \subseteq \mathbb{R}^p$, $B = B(\underline{u}_0, r) = \{\underline{u}; |\underline{u} - \underline{u}_0| < r\}$, with $\overline{B} \subset \Omega$ then for all $\rho, 0 \leq \rho < r$.*

$$\max_{\underline{u}\in B} h(\underline{u}) \leq \left(\frac{r+\rho}{r-\rho}\right)^n \min_{\underline{u}\in B} h(\underline{u}).$$

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*Further if $K \subset \Omega$ is compact there is a constant $M$ depending only on $\Omega$ and $K$ such that for all $\underline{u}, \underline{v} \in K$*

$$\frac{1}{M} h(\underline{v}) \le h(\underline{u}) \le M h(\underline{v}).$$

COMMENT Harmonic functions are solutions of *Laplace's Equation* $\nabla^2 h = 0$ and the important Harnack inequalities have been extended to solutions of more general elliptic partial differential equations; see *Protter & Weinberger*, [PW]. Further extensions make these inequalities a subject of much research.

SPECIAL CASES *With the above notation for all $\underline{u}, \underline{v} \in B(\underline{u}_0, r/3)$,*

$$\left(\frac{2}{3}\right)^n h(\underline{u}) \le h(\underline{u}_0) \le 2^n h(\underline{v}).$$

REFERENCES [EM, vol. 4, pp. 387–388l], [GI3, pp. 333-339]; *Ahlfors* [Ah78, pp. 235–236], *Lieb & Loss* [LL, p. 209], *Mitrović & Žubrinić* [MZ, p. 238], *Protter & Weinberger* [PW, pp. 106–121, 157, 194].

**Hausdorff-Young Inequalities** *(a) If $1 < p \le 2$, $q$ the conjugate index, and $f \in \mathcal{L}^p\big([-\pi, \pi]\big)$ and if*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\,\mathrm{d}t, \quad n \in \mathbb{Z}, \tag{1}$$

*then*

$$||\underline{c}||_q \le \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^p\right)^{1/p}.$$

*(b) If $1 < p \le 2$, $q$ the conjugate index, and $||\underline{c}||_p < \infty$, $\underline{c} = \{c_n, n \in \mathbb{Z}\}$ a complex sequence, then there is a function $f \in \mathcal{L}^q\big([-\pi, \pi]\big)$ satisfying (1) and*

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^q\right)^{1/q} \le ||\underline{c}||_p.$$

EXTENSIONS [F. & M. RIESZ] *(a) If $1 < p \le 2$, $q$ the conjugate index, $f \in \mathcal{L}^p\big([a, b]\big)$ and if $\phi_n, n \in \mathbb{N}$ is a uniformly bounded orthonormal sequence of complex valued functions defined on an interval $[a, b]$, with $||\phi_n||_{\infty,[a,b]} \le M, n \in \mathbb{N}$, then if $\underline{c}$ is the sequence of Fourier coefficients of $f$ with respect to $\phi_n, n \in \mathbb{N}$,*

$$||\underline{c}||_q \le M^{(2/p)-1} ||f||_{p,[a,b]}.$$

*(b) If $\phi_n, n \in \mathbb{N}$ is as in (a), $1 < p \le 2$, $q$ the conjugate index, and $||\underline{c}||_p < \infty$, $\underline{c} = \{c_n, n \in \mathbb{N}\}$ a complex sequence, then there is a function $f \in \mathcal{L}^q\big([a, b]\big)$ having $\underline{c}$ as its sequence of Fourier coefficients with respect to $\phi_n, n \in \mathbb{N}$, and*

$$||f||_{q,[a,b]} \le M^{(2/p)-1} ||\underline{c}||_p.$$

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COMMENTS (i) The two parts of all these results are equivalent in that either implies the other.

(ii) The case $p = 2$ of part (a) of the above results is Bessel's Inequality. See: **Bessel's Inequality** (1).

(iii) For an integral analogue of the Hausdorff-Young result see: **Fourier Transform Inequalities** (b).

REFERENCES [EM, vol. 4, pp. 394–395, vol.8, pp. 154–155], [MPF, pp. 400–401]; *Rudin* [R87, pp. 247–249], *Zygmund* [Z, vol. II, pp. 101–105].

**Hayashi's Inequality** A minor modification of Steffensen's inequality. See: **Steffensen's Inequalities** COMMENTS (i).

**Heinig's Inequality** *If $\gamma, p \in \mathbb{R}$, with $p > 0$, and if $f : [0, \infty[ \to [0, \infty[$ is measurable, then*

$$\int_0^\infty x^\gamma \exp\left(\frac{p}{x^p}\int_0^x t^{p-1}\log f(t)\,\mathrm{d}t\right)\mathrm{d}x \le e^{(\gamma+1)/p}\int_0^\infty x^\gamma f(x)\,\mathrm{d}x.$$

COMMENTS (i) The constant is best possible and is due to Cochrane & Lee, C.S.

(ii) An extension has been given by Love.

(iii) These inequalities generalize those of Carleman and Knopp. See: **Carleman's Inequality** (1), INTEGRAL ANALOGUE.

DISCRETE ANALOGUE *Let $p, \gamma$ be real numbers, $p \ge 1, \gamma \ge 0$. If $\underline{a}$ is a sequence with $0 \le \underline{a} \le 1$ and $\sum_{i=1}^\infty i^\gamma a_i < \infty$ then*

$$\sum_{n=1}^\infty n^\gamma \big(\prod_{i=1}^n a_i^{i^{p-1}}\big)^{p/n^p} \le e^{(\gamma+1)/p}\sum_{i=1}^\infty i^\gamma a_i.$$

REFERENCES [GI5, pp. 87–93]; *Cochran & Lee* [93], *Love* [184].

**Heinz Inequality** *If $A, B$ are bounded positive linear operators on a Hilbert space $X$, if $L$ is a bounded linear operator on $X$, and if $0 \le \alpha \le 1$, then:*

$$||AL + LB|| \ge ||A^\alpha LB^{1-\alpha} + A^{1-\alpha}LB^\alpha||. \tag{1}$$

COMMENTS (i) An operator $A$ on the Hilbert space $X$ is *positive* if, $\langle Ax, x\rangle \ge 0$ for all $x \in X$; we then write $A \ge 0$.

(ii) Inequality (1) implies the *Heinz-Kato Inequality*. See: **Heinz-Kato-Furuta Inequality** (1).

(iii) See also: **Löwner-Heinz Inequality**.

REFERENCES [EM, Supp. pp. 288–289].

**Heinz-Kato-Furuta Inequality** *Let $X$ be a complex Hilbert space, $A, B$ two positive bounded linear operators in $X$, $L$ a bounded linear operator*

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*in $X$ such that for all $x, y \in X$, $||Lx|| \le ||Ax||$ and $||L^*y|| \le ||By||$, then for all $x, y \in X$ and $0 \le \alpha, \beta \le 1, \alpha+\beta \ge 1$*

$$\left|\left\langle L|L|^{\alpha+\beta}x, y\right\rangle\right| \le ||A^\alpha x||\,||B^\beta y||.$$

COMMENTS (i) The definition of a positive operator is in **Heinz Inequality** COMMENTS (i).

(ii) $L^*$ denotes the *adjoint* of $L$, defined by $\langle Lx, y\rangle = \langle x, L^*y\rangle$.

(iii) If $\alpha+\beta=1$ we get the *Heinz-Kato inequality*,

$$\left|\left\langle Lx, y\right\rangle\right| \le ||A^\alpha x||\,||B^{1-\alpha} y||. \tag{1}$$

(iv) Inequality (1) is equivalent to the Löwner-Heinz Inequality. See: **Löwner-Heinz Inequality**.

REFERENCES [EM, Supp. pp. 288–289].

**Heinz-Kato Inequality** See: **Heinz-Kato-Furuta Inequality** (1).

**Heisenberg-Weyl Inequality** *If $f$ is continuously differentiable on $[0, \infty[$, $p > 1$, $q$ the conjugate index, and $r > -1$, then:*

$$\int_0^\infty x^r|f(x)|^p\,\mathrm{d}x \le \frac{p}{r+1}\left(\int_0^\infty x^{q(r+1)}|f(x)|^p\,\mathrm{d}x\right)^{1/q}\left(\int_0^\infty |f'|^p\right)^{1/p}.$$

*There is equality if and only if $f(x) = Ae^{-Bx^{q+r(q-1)}}$, for some positive $B$ and non-negative $A$.*

COMMENTS (i) In the case $p=2, r=0$, due to Weyl, this is the *Heisenberg Uncertainty Principle* of Quantum Mechanics.

(ii) This inequality can be deduced from **Benson's Inequalities**.

REFERENCES [AI, p. 128], [HLP, pp. 165–166]; *George* [G, pp. 297–299], *Zwillinger* [Zw, p. 209]; *Gao* [123].

**HELP Inequalities** These inequalities generalize the **Hardy-Littlewood-Pólya Inequalities** (1), (2) in asking whether, for some finite constant $K$,

$$\left(\int_a^b u|f'|^2 + v|f|^2\right)^2 \le K^2\left(\int_a^b w|f|^2\right)\left(\int_a^b w\left|\frac{-(uf')'+vf}{w}\right|^2\right)$$

holds for any function making the right-hand side finite. As might be expected the detailed answer depends on properties of the differential equation

$$-(uf')' + vf = \lambda wf.$$

The name refers to Hardy, Everitt-Evans, Littlewood, and Pólya, because the above extension was introduced by Evans & Everitt.

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References [GI4, pp. 15–23], [GI5, pp. 337–346], [GI6, pp. 269–305], [GI7, pp. 179–192]; *Zwillinger* [Zw, p. 214]; *Evans, Everitt, Hayman & Jones* [113].

**Henrici's Inequality** *If* $\underline{a} \ge 1$,

$$\mathfrak{A}_n\left(\frac{1}{\underline{e}+\underline{a}}\right) \ge \frac{1}{1+\mathfrak{G}_n(\underline{a})},$$

*with equality if and only if* $\underline{a}$ *is constant.*

Extensions *(a) If* $1/(1+f)$ *is strictly convex on* $[c,d]$ *and if* $c \le \underline{a} \le d$ *then:*

$$\mathfrak{A}_n\left(\frac{1}{\underline{e}+f(\underline{a})};\underline{w}\right) \ge \frac{1}{1+f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big)},$$

*with equality if and only if* $\underline{a}$ *is constant.*
*(b) If* $0 \le r < s < \infty$ *and if* $\underline{a} \ge \big((s-r)/(s+r)\big)^{r/s}$,

$$\mathfrak{A}_n\left(\frac{1}{\underline{e}+\underline{a}^s};\underline{w}\right) \ge \frac{1}{1+\Big(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\Big)^s},$$

*with equality if and only if* $\underline{a}$ *is constant.*
*(c)* [Rado-type] *If* $\mathfrak{G}_{n-1}(\underline{a}) \ge 1$ *and* $a_n^{(n+1)/n}\mathfrak{G}_{n-1}(\underline{a}) \ge 1$ *then*

$$n\left(\mathfrak{A}_n\left(\frac{1}{\underline{e}+\underline{a}}\right) - \frac{1}{1+\mathfrak{G}_n(\underline{a})}\right) \\ \ge (n-1)\left(\mathfrak{A}_{n-1}\left(\frac{1}{\underline{e}+\underline{a}}\right) - \frac{1}{1+\mathfrak{G}_{n-1}(\underline{a})}\right).$$

*(d)* [Borwein, D.] *If* $0 < a_i \le \cdots \le a_n \le 1/\mathfrak{G}_{n-1}(\underline{a})$, *then:*

$$\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{A}_n(\underline{a})+\mathfrak{G}_n(\underline{a})^n} \ge \mathfrak{A}_n\left(\frac{1}{\underline{e}+\underline{a}}\right).$$

*(e)* [Kalajdžić] *If* $1-1/n \le \alpha \le n$ *then*

$$\mathfrak{A}_n\left(\frac{1}{\underline{e}+\underline{a}}\right) \ge \alpha \implies \frac{1}{1+\mathfrak{G}_n(\underline{a})} \ge \alpha.$$

References [AI, pp. 212–213], [GI6, pp. 437–440], [H, pp. 274–275, 284], [MPF, p. 73].

**Henry-Grönwall Inequalities** See: **Grönwall's Inequality** Comments (iii).

**Heronian Mean Inequalities** A particular case of an Extended Mean, see **Logarithmic Mean Inequalities**, is the *Heronian Mean*:

$$\mathfrak{H}_e(a,b) = \mathfrak{E}_{1/2,3/2}(a,b) = \frac{a+\sqrt{ab}+b}{3}, \quad 0 < a < b.$$

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[JANOUS]

$$\mathfrak{M}_2^{[\log 2/\log 3]}(a,b) < \mathfrak{H}_e(a,b) < \mathfrak{M}_2^{[2/3]}(a,b),$$

*and the values* $\log 2/\log 3, 2/3$ *are best possible in that the first cannot be increased and the second cannot be decreased.*

Noting that $\mathfrak{H}_e(a,b) = \frac{2}{3}\mathfrak{A}_2(a,b) + \frac{1}{3}\mathfrak{G}_2(a,b)$ it is natural to define the *generalized Heronian mean* as

$$\mathfrak{H}_e^{[t]}(a,b) = (1-t)\mathfrak{G}_2(a,b) + t\,\mathfrak{A}_2(a,b), \quad 0 \le t \le 1.$$

So that $\mathfrak{H}_e^{[2/3]}(a,b) = \mathfrak{H}_e(a,b)$, and $\mathfrak{H}_e^{[1/2]}(a,b) = \mathfrak{M}_2^{[1/2]}(a,b)$.

COMMENT (i) It is easy to see that $\mathfrak{H}_e^{[t]}$ is an increasing function of $t$.

[JANOUS] *(a) If* $0 \le t \le 1/2$ *then*

$$\mathfrak{M}_2^{[(\log 2/(\log 2 - \log t)]}(a,b) > \mathfrak{H}_e^{[t]}(a,b) > \mathfrak{M}_2^{[t]}(a,b); \tag{1}$$

*if* $1/2 \le t < 1$ *then* $(\sim 1)$ *holds. In both cases the exponents are, as in the previous result, best possible*
*(b)*

$$\mathfrak{H}_e^{[0]}(a,b) < \mathfrak{L}(a,b) < \mathfrak{H}_e^{[1/3]}(a,b), \tag{2}$$

*the value* $0$ *cannot be decreased, and the value* $1/3$ *cannot be increased.*
*(c)*

$$\mathfrak{H}_e^{[2/3]}(a,b) < \mathfrak{I}(a,b) < \mathfrak{H}_e^{[2/e]}(a,b);$$

*again the number* $1/3$ *cannot be decreased, nor the value* $2/e$ *increased.*

COMMENTS (ii) Inequality (2) should be compared with **Logarithmic Mean Inequalities** (2).

(iii) While all these results are found in the paper by Janous they had been proved earlier by Leach & Sholander and Sándor.

REFERENCES [H, p. 399–401]; *Janous* [147], *Leach & Sholander* [171], *Sándor* [287].

**Hermite-Hadamard Inequality** *If $f$ is convex on $[a,b]$, and if $a \le c < d \le b$, then*

$$\frac{f(c)+f(d)}{2} \ge \frac{1}{d-c}\int_c^d f \ge f\left(\frac{c+d}{2}\right), \tag{1}$$

*and*

$$\frac{1}{b-a}\int_a^b f \ge \frac{1}{n}\sum_{i=0}^{n-1} f\left(a + \frac{i}{n-1}(b-a)\right). \tag{2}$$

COMMENTS (i) The left inequality in (1) just says:

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*the area under a convex curve is less than the area under the trapezoid formed by the lines:*

$$x = c, x = d, y = 0, \big(y - f(c)\big)(d - c) = \big(f(d) - f(c)\big)(x - c).$$

Improvements can be obtained by applying this inequality to the two trapezoids obtained by using the lines:

$$\begin{aligned} x = c, x = c' = \frac{c+d}{2}, \quad y = 0, \quad \big(y - f(c)\big)(c' - c) = \big(f(c') - f(c)\big)(x - c),\\ x = c', x = d, \quad y = 0, \quad \big(y - f(c')\big)(d - c') = \big(f(d) - f(c')\big)(x - c'). \end{aligned}$$

(ii) Inequality (1) is capable of considerable generalisation. See: **Choquet's Theorem**.

(iii) The right-hand side of (2) actually tends to the left-hand side as $n \to \infty$.

Extensions *(a)* [Lupaş] *If $f$ is convex on $[a, b]$ then:*

$$\mathfrak{A}_2\big(f(a), f(b); p, q\big) \ge \frac{1}{b-a}\int_a^b f \ge f\big(\mathfrak{A}_2(a, b; p, q)\big)$$

*provided*

$$\left|\frac{c+d}{2}\right| \le \frac{b-a}{p+q}\min\{p, q\}.$$

*(b)* [Féjer] *If $f$ is convex on $[a, b]$ and $g \ge 0$ is symmetric with respect to the mid-point $(a+b)/2$ then:*

$$\frac{f(a) + f(b)}{2}\int_a^b g \ge \frac{1}{b-a}\int_a^b fg \ge f\left(\frac{a+b}{2}\right)\int_a^b g.$$

Comments (iv) The Lupaş style extension can also be made to Féjer's result.

(v) Other extensions have been made to $n$-convex functions and to arithmetic means of order $n$.

References [H, pp. 29–30], [HLP, p. 98], [MPF, p. 10], [PPT, pp. 137–151]; *Niculescu & Persson* [NP, Chapter 4], *Roberts & Varberg* [RV, p. 15]; *Alzer & Brenner* [37], *Neuman* [231].

**Higher Order Convex Function Inequalities** See: **n-Convex Function Inequalities**.

**Higher Order Convex Sequence Inequalities** See: **n-Convex Sequence Inequalities**.

**Hilbert's Inequalities** *(a) If $\underline{a}$ is a non-negative $n$-tuple, then*

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{a_i a_j}{i+j+1} \le \pi\sum_{i=0}^{n} a_i^2. \tag{1}$$

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*(b) If $\underline{a}, \underline{b}$ are non-negative sequences, $p > 1$, $q$ the conjugate index then*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a_i b_j}{i+j+1} \le \frac{\pi}{\sin(\pi/p)}\left(\sum_{i=0}^{\infty} a_i^p\right)^{1/p}\left(\sum_{i=0}^{\infty} b_i^q\right)^{1/q}, \tag{2}$$

*with equality only if either $\underline{a}$ or $\underline{b}$ has all elements zero. The constant on the right-hand side is best possible.*

COMMENTS (i) Inequalities (1) and (2) are both known as *Hilbert's inequality*.

(ii) The constant $\pi$ on the right-hand side of (1) is not best possible; it can be replaced by $(n+1)\sin\pi/(n+1)$. An asymptotic form for the best possible constant has been given. The constant on the right-hand side of (2) is best possible.

EXTENSIONS *(a)* [WIDDER] *If $\underline{a}$ is a non-negative $n$-tuple, then:*

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{a_i a_j}{i+j+1} \le \pi\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(i+j)!}{i!j!}\frac{a_i a_j}{2^{i+j+1}}.$$

*(b)* [BENNETT] *If $\underline{a}$ is a real sequence and if $p > 1$ then:*

$$\frac{1}{(p-1)^{1/p}}!\underline{a}!_p \le \left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{|a_j|}{i+j-1}\right)^p\right)^{1/p} \le \frac{\pi}{\sin\pi/p}!\underline{a}!_p.$$

*The constants are best possible and both inequalities are strict unless $\underline{a} = \underline{0}$.*
*(c)* [KUBO F.] *If $\underline{a}$ is a real $n$-tuple then:*

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{a_i a_j}{\sin\big((i+j)\pi/n\big)} \le (n \pm 1)\sum_{i=0}^{m}|a_i|^2,\ 1 \le m \le [n/2].$$

COMMENT (iii) The notations in (b) are defined above **Hardy's Inequalities** EXTENSIONS (c).

INTEGRAL ANALOGUE *If $f \in \mathcal{L}^p([0,\infty[)$, $g \in \mathcal{L}^q([0,\infty[)$, $f \ge 0, g \ge 0$, $p, q$ conjugate indices then*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,\mathrm{d}x\mathrm{d}y \le \frac{\pi}{\sin(\pi/p)}||f||_p\,||g||_q, \tag{3}$$

*with equality if and only if either $f = 0$ or $g = 0$.*

COMMENTS (iv) There are extensions in which the function $K(x, y) = 1/(x+y)$ is replaced by a more general function; see [EM].

(v) See also: **Bennett's Inequalities** (c), (d).

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REFERENCES [AI, pp. 357–358], [EM, vol. 4, pp. 417–418], [HLP, pp. 212–214, 227–259], [PPT, p. 234]; *Bennett* [Be, pp. 53–54], *Krnić, Pečarić, Perić & Vuković* [KPPV];*Gao & Yang* [124], *Kubo, F* [164].

**Hilbert's Transform Inequality** *If $f \in \mathcal{L}^p(\mathbb{R})$ and if*

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t) - f(x-t)}{t}\,\mathrm{d}t,$$

*then $\tilde{f} \in \mathcal{L}^p(\mathbb{R})$ and*

$$||\tilde{f}||_{p,\mathbb{R}} \le C||f||_{p,\mathbb{R}}\,,$$

*where $C$ depends only on $p$.*

COMMENT $\tilde{f}$ is called the *Hilbert transform of $f$*.

REFERENCES [EM, vol. 4, pp. 433-434]; *Hirschman* [Hir, pp. 168–169]; *Titchmarsh* [T75, pp. 132–138], *Zygmund* [Z, vol. II, pp. 256–257].

**Hinčin-Kahane Inequality** *If $f(x) = \sum_{n=1}^\infty c_n r_n(x)$, $0 \le x \le 1$, where $r_n(x) = \operatorname{sign}\circ\sin(2^n\pi x)$, $0 \le x \le 1$, $n = 1, 2\ldots$, and $||\underline{c}||_2 < \infty$, then for any $p > 0$*

$$A_p||\underline{c}||_2 \le ||f||_{p,[0.1]} \le B_p||\underline{c}||_2,$$

*where $B_p = O(\sqrt{p})$ as $p \to \infty$.*

COMMENT The functions $r_n$, $n = 1, 2, \ldots$ are called *Rademacher functions*; they form an orthonormal system called the *Rademacher system*, and the series for $f$ in the above inequality is called its *Rademacher series*.

REFERENCES [EM, vol. 5, pp. 267–268]; *Zygmund* [Z, vol. I, p. 213].

**Hinčin-Littlewood Inequality** *If $r_n, n \in \mathbb{N}$, are the Rademacher functions, and $f(x) = \sum_{n\in\mathbb{N}} c_n r_n(x), 0 \le x \le 1$, with $\sum_{n\in\mathbb{N}} |c_n|^2 = C^2 < \infty$, then there are constants $\lambda, \Lambda$ depending only on $r$, such that*

$$\lambda C \le ||f||_{r,[0,1]} \le \Lambda C.$$

COMMENTS (i) The Rademacher functions are defined in the previous entry.
(ii) If $\nu$ is the least even integer not less than $r$, then $\Lambda \le \sqrt{2\nu}$.

REFERENCES [MPF, pp. 566–568]; *Zygmund* [Z, vol. I, pp. 213–214].

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**Hirsch's Inequalities** *If $A$ is a complex $n \times n$ matrix then*

$$\begin{aligned}
|\lambda_s(A)| &\leq n \max\Big\{|a_{ij}|,\, 1 \leq i \leq n, 1 \leq j \leq n\Big\};\\
\big|\Re(\lambda_s)\big| &\leq n \max\Big\{\Big|\frac{a_{ij}+\overline{a}_{ji}}{2}\Big|,\, 1 \leq i \leq n, 1 \leq j \leq n\Big\};\\
\Big|\Im(\lambda_s)\Big| &\leq n \max\Big\{\Big|\frac{a_{ij}-\overline{a}_{ji}}{2i}\Big|,\, 1 \leq i \leq n, 1 \leq j \leq n\Big\}.\\
\lambda_{[n]}\Big(\frac{A+A^*}{2}\Big) &\leq \Re\big(\lambda_s(A)\big) \leq \lambda_{[1]}\Big(\frac{A+A^*}{2}\Big);\\
\lambda_{[n]}\Big(\frac{A-A^*}{2i}\Big) &\leq \Im\big(\lambda_s(A)\big) \leq \lambda_{[1]}\Big(\frac{A-A^*}{2i}\Big);
\end{aligned}$$

COMMENT The third inequality above, in the real case, is due to Bendixson, who also gave the following result.

[BENDIXSON] *If $A$ is a real $n \times n$ matrix then*

$$\big|\Im(\lambda_s)\big| \leq \max\Big\{\Big|\frac{a_{ij}-\overline{a}_{ji}}{2i}\Big|,\, 1 \leq i \leq n, 1 \leq j \leq n\Big\}\sqrt{\frac{n(n-1)}{2}}.$$

REFERENCE *Marcus & Minc* [MM, pp. 140–142].

**Hlawka's Inequality** *If $\underline{a}, \underline{b}, \underline{c}$ are real $n$-tuples,*

$$|\underline{a}+\underline{b}+\underline{c}| + |\underline{a}| + |\underline{b}| + |\underline{c}| \geq |\underline{a}+\underline{b}| + |\underline{b}+\underline{c}| + |\underline{c}+\underline{a}|. \tag{1}$$

COMMENTS (i) This result follows from (T) and the following identity:

$$\begin{aligned}
(|a| + |b| + |c| &- |a+b| - |b+c| - |c+a| + |a+b+c|)\\
&\times (|a| + |b| + |c| + |a+b+c|)\\
&= (|a| + |b| - |a+b|)(|c| - |a+b| + |a+b+c|)\\
&\quad + (|b| + |c| - |b+c|)(|a| - |b+c| + |a+b+c|)\\
&\quad + (|c| + |a| - |c+a|)(|b| - |c+a| + |a+b+c|).
\end{aligned} \tag{2}$$

This identity is called the *Identity of Hlwaka* and follows by using the simple identity

$$|a|^2 + |b|^2 + |c|^2 + |a+b+c|^2 = |a+b|^2 + |b+c|^2 + |c+a|^2.$$

(ii) Both Hlawka's identity and inequality hold in unitary spaces; for a definition see **Inner Product Inequalities**.

EXTENSIONS *(a)* [ADAMOVIĆ] *If $\underline{a}_k, 1 \leq k \leq p$, are real $n$-tuples , $p \geq 2$, then*

$$(p-2)\sum_{k=1}^{p} |\underline{a}_k| + \Big|\sum_{k=1}^{p} \underline{a}_k\Big| \geq \sum_{1 \leq k < j \leq p} |\underline{a}_k + \underline{a}_j|.$$

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*(b)* [Đoković] *With the notation of (a)*

$$\sum_{k=1}^{p} |\underline{a}_k| + (p-2)\left|\sum_{k=1}^{p} \underline{a}_k\right| \ge \sum_{i=1}^{p} |\underline{a}_1 + \cdots \underline{a}_{i-1} + \underline{a}_{i+1} + \cdots + \underline{a}_p|.$$

Comments (iii) Extension (a) reduces to (1) if $p=3$; its proof is based on a generalization of (2).

(iv) An inequality that can be considered an inverse to (1) is given in **Beth & van der Corput Inequality**, Comments (iv).

(v) All these results have been placed in a very general setting. See: [*Takahasi, Takahashi & Honda*].

(vi) See also: **Hlawka-Type Inequalities, Hornich's Inequality, Popoviciu's Convex Function Inequality, Quadrilateral Inequalities**.

References [AI, pp. 171–176], [MPF, pp. 521–534]; *Hlawka* [Hl], *Mitrinović & Pečarić* [MP91b, pp. 100–144]; *Jiang & Cheng* [148], *Takahasi, Takahashi & Honda* [307], *Takahasi, Takahashi & Wada* [308].

**Hlawka's Integral Inequality** *If* $f(0+) \le 0$ *and* $f'^2(x)e^{-2x} \in \mathcal{L}\big([0,t]\big)$, $f$ *not zero almost everywhere, then*

$$\frac{1}{2}f^2(t)e^{-2t} + \int_0^t e^{-2x}\big|f(x)f'(x)\big|\,\mathrm{d}x < \int_0^t e^{-2x}f'^2(x)\,\mathrm{d}x.$$

Comment This has been extended to higher dimensions by Redheffer.

Reference *Hlawka* [Hl]; [I2, pp. 273–276].

**Hlawka-Type Inequalities** [J. C. Burkill] *If* $\underline{a}, \underline{w}$ *are positive triples with* $W_3 = 1$ *then*

$$\begin{aligned} a_1^{w_1}a_2^{w_2}a_3^{w_3} \quad &+ \quad (w_1a_1 + w_2a_2 + w_3a_3) \\ &\ge \quad (w_1+w_2)a_1^{w_1/(w_1+w_2)}a_2^{w_2/(w_1+w_2)} + (w_2+w_3)a_2^{w_2/(w_2+w_3)}a_3^{w_3/(w_2+w_3)} \\ &\qquad\qquad + (w_3+w_1)a_3^{w_3/(w_3+w_1)}a_1^{w_1/(w_3+w_1)}. \end{aligned}$$

Comments (i) The methods of proof for this result are elementary.

(ii) The reason for the name given to this kind of inequality by Burkill is clear on comparing (1) with the **Hlawka Inequality** (1).

Extension [Burkill] *If* $f : [a,b] \to \mathbb{R}$ *is convex, and if* $a_i \in [a,b], 1 \le i \le 3$, $\underline{w}$ *a positive triple, and defining* $a_4 = a_1$, $w_4 = w_1$, *then*

$$f\big(\mathfrak{A}_3(\underline{a};\underline{w})\big) + \mathfrak{A}_3\big(f(\underline{a});\underline{w}\big) \ge \frac{1}{W_3}\left(\sum_{j=1}^{3}(w_j + w_{j+1})f\left(\frac{w_ja_j + w_{j+1}a_{j+1}}{w_j + w_{j+1}}\right)\right).$$

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COMMENTS (iii) Many of the inequalities discussed have been concerned with the superadditivity of functions of index sets, see for instance **Rado's Geometric-Arithmetic Mean Inequality Extension** EXTENSION.

(iv) Hlawka-type inequalities are concerned with convexity of such functions,

$$\sigma(\mathcal{I}\cup\mathcal{J}\cup\mathcal{K})+\sigma(\mathcal{I})+\sigma(\mathcal{J})+\sigma(\mathcal{K})\geq\sigma(\mathcal{I}\cup\mathcal{J})+\sigma(\mathcal{J}\cup\mathcal{K})+\sigma(\mathcal{K}\cup\mathcal{K}).$$

For a further discussion and elaboration of this point see [MPF].

REFERENCES [H, pp. 258–260],[MPF, pp. 527–534], [PPT, pp. 171–181].

**Hölder Function Inequalities** See **Lipschitz Function Inequalities** COMMENTS (ii).

**Hölder's Inequality** *If $\underline{a},\underline{b}$ are positive $n$-tuples and if $p,q$ are conjugate indices with $p>0$ and $q>0$, equivalently $p>1$ or $q>1$, then*

$$\sum_{i=1}^n a_ib_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q}; \tag{H}$$

*or*

$$||\underline{a}\,\underline{b}|| \leq ||\underline{a}||_p||\underline{b}||_q. \tag{$H_N$}$$

*If either $p<0$, or $q<0$ the inequality ($\sim$H) holds. The inequality is strict unless $\underline{a}^p \sim \underline{b}^q$.*

COMMENTS (i) This famous inequality has many proofs and perhaps the simplest is to rewrite (H) as

$$\sum_{i=1}^n \left(\frac{a_i^p}{\sum_{j=1}^n a_j^p}\right)^{1/p}\left(\frac{b_i^p}{\sum_{j=1}^n b_j^p}\right)^{1/q} \leq 1$$

and then apply **Geometric-Arithmetic Mean Inequality** (2) to the terms on the left-hand side.

The case of ($\sim$H) follows by an algebraic argument from (H).

(ii) The case $p=q=2$ of (H) is just (C). It is of some interest that (C) is equivalent to (H).

(iii) The inequality ($H_N$) is valid if $p=1$, when $q=\infty$, and there is equality if and only if $\underline{b}$ is constant.

EXTENSIONS *(a) If $\underline{a},\underline{b}$ are complex $n$-tuples, and $p,q$ are as in (H),*

$$\left|\sum_{i=1}^n a_i\overline{b}_i\right| \leq \sum_{i=1}^n |a_i||b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}\left(\sum_{i=1}^n |b_i|^q\right)^{1/q}; \tag{1}$$

*There is equality if and only if $|\underline{a}|^p \sim |\underline{b}|^q$, and $\arg a_ib_i$ does not depend on $i$.*
*(b) If $\underline{a},\underline{b},\underline{w}$ are positive $n$-tuples, and $p,q$ are as in (H) then*

$$\sum_{i=1}^n w_ia_1b_i \leq \left(\sum_{i=1}^n w_ia_1^p\right)^{1/p}\left(\sum_{i=1}^n w_ib_i^q\right)^{1/q}. \tag{2}$$

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*(c)* [GENERALIZED HÖLDER INEQUALITY; JENSEN] *Suppose that* $r_i > 0, 1 \le i \le m$, *and* $a_{ij} > 0, 1 \le i \le m, 1 \le j \le n$, *and define*

$$\frac{1}{\rho_m} = \sum_{i=1}^{m} \frac{1}{r_i},$$

*then*

$$\left( \sum_{j=1}^{n} \Big\{ \prod_{i=1}^{m} a_{ij} \Big\}^{\rho_m} \right)^{1/\rho_m} \le \prod_{i=1}^{m} \Big\{ \sum_{j=1}^{n} a_{ij}^{r_i} \Big\}^{1/r_i}. \tag{3}$$

*There is equality in (3) only if the n-tuples* $(a_{i1}^{r_i}, \ldots, a_{in}^{r_i})$, $1 \le i \le m$, *are linearly dependent. If* $r_1 > 0, r_i < 0$, $2 \le i \le m$, *then (∼2) holds.*

*(d)* [POPOVICIU-TYPE] *Write* $H_m^{1/\rho_m}(\underline{a})$ *for the ratio of the left-hand side of (3) to its right-hand side, then if* $m \ge 2$,

$$\rho_m H_m(\underline{a}) \le \rho_{m-1} H_{m-1}(\underline{a}) + \frac{1}{r_m}.$$

*(e)* [FUNCTIONS OF INDEX SETS] *If* $\mathcal{I}$ *is an index set define*

$$\chi(\mathcal{I}) = \left( \sum_{i \in \mathcal{I}}^{n} a_i^p \right)^{1/p} \left( \sum_{i \in \mathcal{I}}^{n} b_i^q \right)^{1/q} - \sum_{i \in \mathcal{I}} a_i b_i,$$

*where* $\underline{a}, \underline{b}, p, q$ *are as in (H), then* $\chi \ge 0$ , *and if* $\mathcal{I} \cap \mathcal{J} = \emptyset$,

$$\chi(\mathcal{I}) + \chi(\mathcal{J}) \le \chi(\mathcal{I} \cup \mathcal{J}), \tag{4}$$

*with equality if and only if* $\left( \sum_{i \in \mathcal{I}} a_i^p, \sum_{i \in \mathcal{J}} a_i^p, \right)$ *is proportional to* $\left( \sum_{i \in \mathcal{I}} b_i^q, \sum_{i \in \mathcal{J}} b_i^q, \right)$.

COMMENTS (iii) It is easily seen that (2) and (H) are identical.

(iv) In the notation of (d), (3) is just $H_m(\underline{a}) \le 1$.

(v) Inequality (3) holds even if $\sum_{i=1}^{m} 1/r_i > 1/\rho_m$, and it is then strict. This follows from **Power Sums Inequalities** (1).

(vi) Inequality (4) just says that $\chi$ is superadditive, while (H) says it is non-negative.

SPECIAL CASES *(a) If* $\underline{a}$ *is a positive m-tuple then*

$$\prod_{i=1}^{m} (1 + a_i) \ge \left(1 + \mathfrak{G}_m(\underline{a})\right)^m.$$

*(b) If* $\underline{a}, \underline{b}$ *are positive n-tuples and if*

$$p > 0, \quad q > 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

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*then*

$$\left(\sum_{i=1}^{n} a_i^r b_i^r\right)^{1/r} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}. \tag{5}$$

*If $p < 0$, or $q < 0$ and $r > 0$, or if all three parameters are negative the ($\sim$5) holds; while if $p < 0$, or $q < 0$ and $r < 0$ then (5) holds.*
*(c) If $\underline{a}, \underline{b}, \underline{c}$ are positive $n$-tuples such that $\underline{a}\,\underline{b}\,\underline{c} = \underline{e}$ and if*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

*then*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \left(\sum_{i=1}^{n} c_i^r\right)^{1/r} \geq 1, \tag{6}$$

*if all but one of the $p, q, r$ are positive, while if all but one are negative ($\sim$ 6) holds.*

COMMENTS (vii) (a) is (3) with $n = 2$, $a_{i1} = 1, a_{i2} = a_i$, $r_i = 1, 1 \leq i \leq m$. The result is related to **Weierstrass Inequalities**.

(viii) (b) is just the case $m = 2$ of (3), with a change of notation; and (c) is just a symmetric form for (b).

The generality of (H) has meant that many inequalities are special cases often in almost impenetrable disguises. See: **Radon's Inequality, Lyapunov's Inequality**.

OTHER FORMS *(a)* [ROGERS'S INEQUALITY] *If $\underline{a}, \underline{b}$ are positive $n$-tuples and $0 < s < 1$ then*

$$\sum_{i=1}^{n} a_i^s b_i^{1-s} \leq \left(\sum_{i=1}^{n} a_i\right)^{s} \left(\sum_{i=1}^{n} b_i\right)^{1-s}. \tag{7}$$

*(b) If $\underline{a}, \underline{b}$ are positive $n$-tuples with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$ then (H) is equivalent to*

$$\sum_{i=1}^{n} a_i^{1/p} b_i^{1/q} \leq 1$$

*with equality if and only if $\underline{a} = \underline{b}$.*
*(c) If $f$ $\underline{a}, \underline{b}$ are positive $n$-tuples, $p > 1$ and $q$ is the conjugate index, and if $B > 0$ then a necessary and sufficient condition that $\left(\sum_{i=1}^{n} a^p\right)^{1/p} \leq A$ is that $\sum_{i=1}^{n} a_i b_i \leq AB$ for all $\underline{b}$ for which $\left(\sum_{i=1}^{n} b_i^q\right)^{1/q} < B$.*

INTEGRAL ANALOGUE *If $p, q$ are conjugate indices, $p \geq 1$, and if $f \in \mathcal{L}^p([a,b])$, $g \in \mathcal{L}^q([a,b])$, $f, g \geq 0$ then $fg \in \mathcal{L}([a,b])$ and*

$$||fg|| \leq ||f||_p \, ||g||_q.$$

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*with equality if and only if for some $A, B$, not both zero $A|f|^{p-1} = B|g|$ almost everywhere.*

Comments (ix) The integral analogue given above extends to general measure spaces; and other forms of (H) can also be given integral analogues; see [HLP].

(x) For another integral analogue see **Young's Inequalities** (c);

(xi) For inverse inequalities see: **Barnes's Inequalities** (b), **Inverse Hölder Inequalities, Petschke's Inequality**.

(xii) Maligranda has pointed out that there seems a good reason for calling (H) *Rogers's Inequality*.

(xiii) There is another equivalent inequality that is sometimes called Hölder's Inequality see: **Geometric Mean Inequalities** Comments (i), and **Power Mean Inequalities** Comments (iv).

(xiv) See also: **Backward Hölder Inequality**, **Beckenbach's Inequalities** Comments (i), **Hadamard Product Inequalities** (d), **Young's Convolution Inequality** Comments (iii).

References [AI, pp. 50–54], [BB, pp. 41–42], [EM, vol. 4, pp. 438–439], [H, pp. 178–183, 193–194, 211–212], [HLP, pp. 21–26, 61, 139–143], [MOA, pp. 657–665], [MPF, pp. 99–107, 111–117, 135–209], [PPT, pp. 112–114, 126–128]; *Lieb & Loss* [LL, pp. 39–40], *Mitrinović & Pečarić* [MP90b], *Pólya & Szegő* [PS, pp. 68–69]; *Maligranda* [193], *Sun* [306].

**Hölder-McCarthy Inequality** *If $A$ is a positive linear operator on a Hilbert space, and if $0 \le t \le 1$, then for all unit vectors $\underline{x}$*

$$\langle A\underline{x}, \underline{x}\rangle^t \ge \langle A^t\underline{x}, \underline{x}\rangle$$

Comment This is equivalent to the **Young's Inequalities** Extensions (c).

Reference *Furuta* [Fu].

**Holley's Inequality** *If $X$ is a distributive lattice, let $\ell_i : X \to [0, \infty[, i = 1, 2$, satisfy*

$$\ell_1(a)\ell_2(b) \le \ell_1(a \vee b)\ell_2(a \wedge b), \qquad \textit{for all } a, b \in X. \tag{1}$$

*If then $f : X \to \mathbb{R}$ is increasing*

$$\sum_{x\in X} \ell_2(x)f(x) \le \sum_{x\in X} \ell_1(x)f(x).$$

Comment Condition (1) generalizes the log modularity condition of the **FKG Inequality**, and is in turn generalized by condition (1) of the **Ahlswede-Daykin Inequality**.

Reference [EM, Supp., pp. 202, 292–293].

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**Holomorphic Function Inequalities** See: **Analytic Function Inequalities.**

**Horn's Inequalities** See: **Weyl's Inequalities** COMMENTS (v).

**Hornich's Inequality** *If $\underline{a}_k, 0 \le k \le p$, are real $n$-tuples such that for some $\lambda \ge 1$*

$$\lambda \underline{a}_0 + \sum_{k=1}^{p} \underline{a}_k = \underline{0},$$

*then*

$$\sum_{k=1}^{p} \big( |\underline{a}_k + \underline{a}_0| - |\underline{a}_k| \big) \le (n-2)|\underline{a}_0|.$$

COMMENTS (i) If $\lambda < 1$ the inequality need not hold.

(ii) This inequality can be deduced from **Hlawka's Inequality**.

REFERENCES [AI, pp. 172–173], [MPF, pp. 521–522].

**Hua's Inequality** *If $\delta > 0, \alpha > 0$ and $\underline{a}$ is a real $n$-tuple then*

$$\left( \delta - \sum_{i=1}^{n} a_i \right)^2 + \alpha \left( \sum_{i=1}^{n} a_i^2 \right) \ge \frac{\alpha}{n+\alpha} \delta^2,$$

*with equality if and only if $a_1 = \cdots = a_n = \delta/(n+\alpha)$.*

COMMENT (i) This has applications in the additive theory of primes.

EXTENSIONS [PEARCE & PEČARIĆ] *If $f$ is convex then, with the above notation,*

$$f\left( \delta - \sum_{i=1}^{n} a_i \right) + \alpha \left( \sum_{i=1}^{n} f(a_i) \right) \ge \frac{n+\alpha}{\alpha} f\left( \frac{\alpha\delta}{n+\alpha} \right),$$

*and if $f$ is strictly convex there is equality if and only if $a_1 = \cdots = a_n = \delta/(n+\alpha)$.*

COMMENT (ii) This is deduced from (J), as is the following.

INTEGRAL ANALOGUE *If $\delta, \alpha > 0$, $g \in \mathcal{L}([0,x]), g \ge 0$, and $f; [a,b]] \to \mathbb{R}$, convex and such that $g\big[[0,x]\big] \subseteq [a/\alpha, b/\alpha]$, and $\big(\delta - \int_0^x g\big) \in [a,b]$ then*

$$f\left( \delta - \int_0^x g \right) + \frac{1}{\alpha} \int_0^x f\big(\alpha g(t)\big)\, \mathrm{d}t \ge \frac{x+\alpha}{\alpha} f\left( \frac{\alpha\delta}{x+\alpha} \right).$$

*If $f$ is stricly convex there is equality if and only if $g(t) = \delta/(x+\alpha)$.*

REFERENCES *Hua* [Hua, p. 104]; *Pearce & Pečarić* [255, 257].

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**Hunter's Inequality** See: **Geometric-Arithmetic Mean Inequality**, Extensions (d).

**Hurwitz Zeta Function Inequalities** See: **Zeta Function Inequalities**.

**Huygens's Inequalities** *If* $0 < |x| < \pi/2$ *then*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \qquad \text{and} \qquad \frac{2x}{\sin x} + \frac{x}{\tan x} > 3$$

Comment In fact the second of these inequalities was given by de Cusa and both were given obscure proofs by Snell.

Extension [Neumann & Sándor] *If* $x \in \mathbb{R}^*$ *then*

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3.$$

Comment See also: **Hyperbolic function Inequalities.**

Reference *Sándor* [289], *Sándor & Bencze* [291].

**Hyperbolic Function Inequalities** *(a) If* $x \in \mathbb{R}$ *then*

$$\sinh x < \cosh x; \qquad \tanh x \leq x \leq \sinh x;$$

*with equality in the second expression if and only if* $x = 0$.
*(b)* [Sándor] *(i) If* $x \in \mathbb{R}$ *then*

$$\frac{x}{\operatorname{arcsinh} x} \leq \frac{\sinh x}{x},$$

*with equality if and only if* $x = 0$.
*(ii) If* $-1 < x < 1$ *then*

$$\frac{x}{\operatorname{arctanh} x} \leq \frac{\tanh x}{x},$$

*with equality if and only if* $x = 0$.
*(ii) If* $0 < x < \pi/2$ *then:*

$$\tan x + \tanh x > 2x, \qquad \frac{2}{\cosh x} < 1 + \cos x.$$

*(c)* [van der Corput] *If* $x, y \in \mathbb{R}^+$ *then*

$$|\cosh x - \cosh y| \geq |x - y|\sqrt{\sinh x \sinh y}.$$

*(d)* [Lazarević, Páles] *If* $x \in \mathbb{R}^*$ *then*

$$1 < \frac{\sinh x}{x} < \cosh x < \left(\frac{\sinh x}{x}\right)^3$$

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*(e) if* $0 < |x| < \pi$ *then*

$$\frac{\sinh x}{x} \le \frac{\pi^2 + x^2}{\pi^2 - x^2}.$$

*(f) If* $-\infty < r < \infty$ *and*

$$r_1 = \begin{cases} \min\big\{(r+2)/3, (r\log 2)/(\log r + 1)\big\}, & \text{if } r > -1,\ r \ne 0, \\ \min\big\{2/3, \log 2\big\}, & \text{if } r = 0, \\ \min\big\{(r+2)/3, 0\big\} & \text{if } r \le -1, \end{cases}$$

*and* $r_2$ *is defined as* $r_1$ *but with min replaced by max, then for* $t > 0$*,*

$$(\cosh r_1 t)^{1/r_1} \le \left(\frac{\sinh(r+1)t}{r\sinh t}\right)^{1/r} \le (\cosh r_2 t)^{1/r_2},$$

*where the cases* $r = 0, -1$ *are taken to be the limits as* $r \to 0$ *of these values.*
*(g) If* $z \in \mathbb{C}$ *then*

$$|\sinh \Im z| \le |\sin z| \le \cosh \Im z; \qquad |\sinh \Im z| \le |\cos z| \le \cosh \Im z;$$
$$|\mathrm{cosec} z| \le \mathrm{cosech}|\Im z|; \qquad |\cos z| \le \cosh|z|; \quad |\sin z| \le \sinh|z|;$$

COMMENTS (i) These inequalities can be compared with results in **Trigonometric Function Inequalities**. Some have been extended to generalized trigonometric and hyperbolic functions by Yang C. Y.

(ii) The result of Sándor is a consequence of **Function Inequalities** (c).

(iii) The exponent 3 in (d) is best possible.

(iv) The inequality in (f) follows from **Pittenger's Inequalities** on putting $e^{2t} = b/a$.

(v) Compare (e) with **Jordan's Inequality** EXTENSIONS (a).

(vi) See also: **Function Inequalities** (a), **Huygens's Inequalities, Lochs's Inequality, Shafer-Fink Inequality** COMMENTS (ii).

REFERENCES [AI, pp. 270, 323], [H, p. 389]; *Cloud & Drachman* [CD, p. 14]; *Chen, Zhao & Qi* [86], *Páles* [251], *Sándor* [290], *Yang, C. Y.* [330]

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# 9 Incomplete–Iyengar

**Incomplete Beta Function Inequalities** See: **Vietoris's Inequality**.

**Incomplete Factorial Function Inequalities** The *incomplete factorial, or Gamma, function* is defined as:

$$\Gamma(a+1, x) = \int_x^\infty u^a e^{-u}\,\mathrm{d}u, \quad a, x > 0$$

COMMENT (i) Slightly different definitions can be found in various references.

*(a) If* $a > a + c^{-1}$ *then*

$$\Gamma(a, x) < cx^a e^{-x}.$$

*(b)* [ALZER] *If* $p > 1, x > 0$ *, and if* $\alpha = \big((p^{-1})!\big)^{-p}$ *then*

$$(p^{-1})!(1 - e^{-x^p})^{1/p} < \int_0^x e^{-t^p}\,\mathrm{d}t < (p^{-1})!(1 - e^{-\alpha x^p})^{1/p},$$

*(c)* [ELBERT & LAFORGIA] *If* $x \ge 0$ *and* $p > p^* \approx 1.87705\ldots$ *then*

$$\int_0^x e^{-t^p}\,\mathrm{d}t \int_x^\infty e^{-t^p}\,\mathrm{d}t < \frac{1}{4}. \tag{1}$$

COMMENTS (ii) The integral in (b) can be expressed as $p^{-1}\big((\Gamma(p^{-1}) - \Gamma(p^{-1}, x^p)\big)$. The inequality generalizes one due to Gautschi.

(iii) $p^*$ is the unique soution of the equation $\phi(p) = 1/4$, where $\phi(p)$ is the maximum, for $x > 0$, of the left-hand side of (1). It can be shown that $\phi(1) = 1$, and $\lim_{p\to\infty} \phi(p) = 1/4$.

REFERENCES [CE, pp. 696–700], [EM, vol. 5, pp. 32–33]; *Elbert & Laforgia* [110], *Natalini & Palumbo* [228].

**Incomplete Gamma Function Inequalities** See: **Incomplete Factorial Function Inequalities**.



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**Increasing Function Inequalities** *(a) If $f$ is an increasing function on $[a,b]$ then if $a \le x < y \le b$,*

$$f(x) \le f(y); \tag{1}$$

*if $f$ is strictly increasing (1) is strict if $x \ne y$.*
*(b)* [Jensen] *If $\underline{a}, \underline{w}$ are positive $n$-tuples, and if $f: ]0,\infty[ \to \mathbb{R}$ is increasing then*

$$\mathfrak{A}_n(\underline{a}, \underline{w}) \le f\left(\sum_{i=1}^{n} a_i\right). \tag{2}$$

*If $f$ is strictly increasing and $n > 1$ then (2) is strict. If $f$ is decreasing then ($\sim$2) holds.*
*(c) If $\mu$ is an increasing set function defined on the sets in the collection $\mathcal{A}$ then*

$$A, B \in \mathcal{A},\ A \subseteq B \quad \Longrightarrow \quad \mu(A) \le \mu(B). \tag{3}$$

*If $\mu$ is strictly increasing then (3) is strict if $A \subset B$.*

Comments (i) (a) is just the definition of *increasing and strictly increasing functions*. In addition, by definition a *decreasing function* is one for which ($\sim$1) holds. Similar comments hold for the set function in (c).

(ii) Inequality (2) characterizes increasing functions just as (J) characterizes convex functions.

Extension [Vasić & Pečarić] *If $f: [0,a] \to \mathbb{R}$ is increasing and $\underline{a}, \underline{v}, \underline{w}$ are non-negative $n$-tuples satisfying*

$$a_j \in [0,a]; \quad \sum_{i=1}^{n} v_i a_i \ge a_j, \quad 1 \le j \le n; \quad \sum_{i=1}^{n} v_i a_i \in [0,a];$$

*then*

$$\mathfrak{A}_n\big(f(\underline{a}); \underline{w}\big) \le f\left(\sum_{i=1}^{n} v_i a_i\right).$$

Comments (iii) This property also characterizes increasing functions. It has applications in **Star-shaped Function Inequalities**.

(iv) See also: **Integral Inequalities** (d), **Integral Test Inequality**.

References [HLP, pp. 83–84], [PPT, pp. 151–152]; *Mitrinović & Pečarić* [MP90a].

**Induction Inequality** *If the functions $M_n, N_n$ are defined for all $n \in \mathbb{N}, n \ge n_0$ with $M_{n_0} \ge N_{n_0}$ and if one of (R) or (P) holds:*

$$M_n - N_n \ge M_{n-1} - N_{n-1} \quad \forall n \ge n_0 + 1; \tag{R}$$

$M_n, N_n, M_{n-1}, N_{n-1}$ *are all positive and*

$$\frac{M_n}{N_n} \ge \frac{M_{n-1}}{N_{n-1}} \qquad \forall n \ge n_0 + 1; \tag{P}$$

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*then* $M_n \geq N_n, n \geq n_0$. *Furthermore if* $M_{n_1} > N_{n_1}$ *for some* $n_1 \geq n_0$ *then* $M_n \geq N_n, n \geq n_1$.

COMMENTS (i) To prove that $M_n \geq N_n, n_1 \geq n \geq n_0$ it suffices to require that (R) or (P) hold for all $n, n_0 + 1 \leq n \leq n_1$.

(ii) For applications of this result see: **Popoviciu's Geometric-Arithmetic Mean Inequality Extension** COMMENT(III), **Rado's Geometric-Arithmetic Mean Inequality Extension** COMMENT(II).

REFERENCE *Herman, Kučera & Šimša* [HKS, pp. 136–144].

**Inf and Sup Inequalities** *(a) If* $\underline{a}, \underline{b}$ *are real sequences then*

$$\sup \underline{a} + \inf \underline{b} \leq \sup(\underline{a} + \underline{b}) \leq \sup \underline{a} + \sup \underline{b}, \tag{1}$$

*provided the terms in the inequalities are defined.*
*(b) If* $\underline{a}, \underline{b}$ *are non-negative sequences then*

$$\sup \underline{a} \inf \underline{b} \leq \sup \underline{a}\, \underline{b} \leq \sup \underline{a} \sup \underline{b},$$

*provided the terms in the inequalities are defined.*

COMMENTS (i) The operations on these inequalities are understood to be in $\overline{\mathbb{R}}$. In addition we could assume the terms of the sequences to be taking values in $\overline{\mathbb{R}}$.

(ii) Similar results can be stated for functions with values in $\overline{\mathbb{R}}$.

(iii) For an application of these results see: **Upper and Lower Limit Inequalities**.

(iv) (1) shows that $||\underline{a}||_\infty$ is a norm. See: **Norm Inequalities** COMMENT (III).

(iv) See aso: **Minkowski's Inequality** COMMENT (III).

REFERENCE *Bourbaki* [B60, pp. 162–163].

**Ingham–Jessen Inequality** See: **Jessen's Inequality** COMMENT(II).

**Inner Product Inequalities** *If* $X$ *is any inner product space and* $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ *is the inner product on* $X$ *then for any* $x, y, z \in X, \lambda \in \mathbb{R}$

$$\begin{aligned} \langle x, x \rangle > &\quad 0, x \neq 0; &(1)\\ |\langle x, y \rangle| \leq &\quad ||x||\,||y||; &(2)\\ 2\big(||x||^2 + ||y||^2\big) = &\quad ||x+y||^2 + ||x-y||^2. &(3) \end{aligned}$$

COMMENTS (i) (1), (2) and

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle, \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \langle x, y \rangle = \langle y, x \rangle; \tag{4}$$

just give the definition of an *inner product*; and then $X$ together with $\langle \cdot, \cdot \rangle$ is called an *inner product space, or pre-Hilbert space.*

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If in the above we replace $\mathbb{R}$ by $\mathbb{C}$, and in (3) make the change $\langle x,y\rangle = \overline{\langle y,x\rangle}$, we get a *complex inner product*, and $X$ is a *complex inner product space, or a unitary space*. A complete inner product or unitary space is called *Hilbert space*.

(ii) Inequality (2) is just (C) in this setting, and remains valid for unitary spaces.

(iii) Inequality (3) is the *parallelogram identity*. See: **Parallelogram Inequality** COMMENTS (I), **Norm Inequalities** COMMENTS (VI), **von Neumann & Jordan Inequality** COMMENTS (II).

(iv) $\mathbb{R}^n$ is an inner product space with $\langle \underline{a},\underline{b}\rangle = \underline{a}.\underline{b}$. More generally $\ell_2$ is an inner product space with $\langle \underline{a},\underline{b}\rangle = \sum_{i=1}^{\infty} a_i b_i$; this series converges by (C).

(v) The space $\mathcal{L}^2([a,b])$ can also be taken to be an inner product space, with $\langle f,g\rangle = \int_a^b fg$, provided we identify functions that are equal almost everywhere.

(vi) It is easy to see that in both the real and complex case $||x|| = \sqrt{\langle x,x\rangle}$ is a norm. See: **Norm Inequalities**.

(vii) See also: **Bessel's Inequality** COMMENTS (I), **Beth & van der Corput's Inequality** COMMENTS (V), **Clarkson's Inequalities** COMMENTS (V), **Cordes's Inequality**, **Dunkl & Williams Inequality, Gram Determinant Inequalities** COMMENTS (I), **Grothendiek's Inequality, Heinz-Kato-Furuta Inequality, Hlawka's Inequality** COMMENTS (II), **Löwner-Heinz Inequality**.

REFERENCE [EM, vol. 5, p. 89; vol. 9, p. 337].

## Integral Inequalities

*(a) If $f \in \mathcal{C}([a,b]$, $f \geq 0$ and positive somewhere then $\int_a^b f > 0$.*

*(b) If $f$ is real, or complex, integrable function on $[a,b]$, with $|f|$ also integrable, then*

$$\left|\int_a^b f\right| \leq \int_a^b |f|,$$

*with equality if and only if $\arg f$ is constant almost everywhere.*

*(c)* [OSTROWSKI] *If $f,g$ are integrable on $[a,b]$ with $f$ monotonic, $f(a)f(b) \geq 0$ and $|f(a)| \geq |f(b)|$ then*

$$\left|\int_a^b fg\right| \leq |f(a)| \max_{a\leq x\leq b}\left|\int_a^x g\right|.$$

*(d) If $f$ is an increasing function on the interval $[a,b]$ then*

$$\int_a^b f' \leq f(b) - f(a). \tag{1}$$

*(e) If $f$ is differentiable at all points of a measurable set $D$ then*

$$\lambda\big(f[D]\big) \leq \int_D |f'|.$$

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*(f)* [Freimer & Mudholkar] *Let $f : [0,\infty[\to \mathbb{R}$ be positive, decreasing and integrable, and let $X > 0$; then there is a $x, 0 < x < X$ such that*

$$f(x)(X-x) \le \int_x^\infty f.$$

*(g)* [Karamata] *Let $h, g, fg, fh \in \mathcal{L}_\mu([a,b])$, $f \ge 0$, decreasing and $\int_a^b fh\,\mathrm{d}\mu > 0$; write $G(x) = \int_a^x g\,\mathrm{d}\mu, H(x) = \int_a^x h\,\mathrm{d}\mu$, $a \le x \le b$ and assume that $H > 0$; then*

$$\inf_{a\le x\le b}\frac{G(x)}{H(x)} \le \frac{\int_a^b fg\,\mathrm{d}\mu}{\int_a^b fh\,\mathrm{d}\mu} \le \sup_{a\le x\le b}\frac{G(x)}{H(x)}.$$

*(h) If either $0 < m < M$ or $m > M < 0$, and $f : [0,1] \mapsto [m,M]$ is continuous and not a constant then*

$$\frac{1}{M^3} \le \frac{\left(\int_0^1 f\right)^{-1} - \int_0^1 (f)^{-1}}{\left(\int_0^1 f\right)^2 - \int_0^1 (f)^2} \le \frac{1}{m^3}. \tag{2}$$

*(j)* [Beesack] *Let $f : [a,b] \mapsto \mathbb{R}$ be nonnegative and integrable then*

$$\int_a^x f(t)\Big(\int_a^t f(x)\,\mathrm{d}x\Big)^{n-1}\mathrm{d}t \le \frac{1}{n}\Big(\int_a^x f(t)\,\mathrm{d}t\Big)^n,\ a \le x \le b,\ n = 1,2,\ldots.$$

*(k)* [Bougoffa] *If $f : [a,b] \mapsto \mathbb{R}^+$is continuous and differentiable on$]a,b[$ with $f' \le 1$, and if $\alpha, \beta > 0$ then:*

$$\int_t^b f \ge \int_t^d (x-a)\,\mathrm{d}x, a \le t \le b,\implies \int_a^b f^{\alpha+\beta} \ge \int_a^b (x-a)^\alpha f^\beta(x)\,\mathrm{d}x$$

Comments (i) (b) is an integral analogue of (T). An inverse inequality can be found in **Wilf's Inequality** Integral Analogues.

(ii) It is known that (1) can be strict even if $f$ is continuous; if $f$ is absolutely continuous there is always equality.

(iii) (k) solves the problem posed by Ngô, Thang, Dat & Tuan.

Discrete Analogue [Karamata] *If $\underline{a}, \underline{u}, \underline{v}$ are positive $n$-tuples with $\underline{a}$ decreasing then*

$$\min_{1\le i\le n}\frac{U_i}{V_i} \le \frac{\sum_{i=1}^n a_iu_i}{\sum_{i=1}^n a_iv_i} \le \max_{1\le i\le n}\frac{U_i}{V_i}.$$

Extensions *(a) If on $[a,b]$, $f$ is continuous, $g$ of bounded variation , or $f, g$ are both of bounded variation and $g$ is continuous, then*

$$\left|\int_a^b f\,\mathrm{d}g\right| \le \int_a^b |f|\,|\mathrm{d}g|.$$

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*(b)* [Ostrowski] *let $f$ be a monotone integrable function on $[a,b]$, and $g$ a real or complex function integrable on $[a,b]$ then*

$$\left|\int_a^b fg\right| \le |f(a)| \max_{a\le x\le b}\left|\int_a^x g\right| + |f(b)| \max_{a\le x\le b}\left|\int_x^b g\right|.$$

*(c) Let: $p:[a,b]\mapsto [0,\infty[$ be continuous and $\int_a^b p = 1$; $f:[a,b]\mapsto [c,d]$be continuous and not a constant ; $F,G:[c,d]\to\mathbb{R}$ be twice differentiable with $0\le k\le F\le K$ and $0\le m\le G\le M$. Then*

$$\frac{k}{M} \le \frac{F\big(\int_a^b fp\big) - \int_a^b F(g)p}{G\big(\int_a^b fp\big) - \int_a^b G(f)p} \le \frac{K}{m}.$$

Comments (iv) Putting $F(x)=x^{-1}, G(x)=x^2$ in (c) leads to (2).

(v) The integrals in (a) are Riemann-Stieltjes integrals.

(vi) See also: **Agarwal's Inequality, Bounded Variation Function Inequalities**(c),(d), Extension, **Complex Function Inequalities**, **Erdős & Grünwald Inequality, Gauss's Inequality, Integral Mean Value Theorems, Persistence of Inequalities** Integral Analogue, **Steffensen's Inequalities**,

(vii) Of course many of the inequalities given in this book are integral inequalities, as well as many being integral analogues of discrete inequalities, but here we mean inequalities that are properties of integration.

References [AI, pp. 301–302], [MPF, p. 116, 337, 476–481]; *Hewitt & Stromberg* [HS, p. 284], *Titchmarsh* [T75, pp. 361–362, 373], *Widder* [W, pp. 8–10]; *Alzer* [8], *Bougoffa* [69], *Horváth* [142].

**Integral Mean Value Theorems** *(a) If $m\le f(x)\le M, a\le x\le b$, then*

$$m \le \frac{1}{b-a}\int_a^b f \le M. \tag{1}$$

*(b)* [Bonnet] *If $f\ge 0$, continuous and increasing, and $g$ is of bounded variation then*

$$f(b)\inf_{a\le x\le b}\int_x^b \mathrm{d}g \le \int_a^b f\,\mathrm{d}g \le f(b)\sup_{a\le x\le b}\int_x^b \mathrm{d}g,$$

*and if $f$ is decreasing,*

$$f(a)\inf_{a\le x\le b}\int_a^x \mathrm{d}g \le \int_a^b f\,\mathrm{d}g \le f(a)\sup_{a\le x\le b}\int_a^x \mathrm{d}g.$$

Comments (i) inequality (1) is an integral analogue of **Arithmetic Mean Inequalities** (1).

(ii) The integrals in (b) are Riemann-Stieltjes integrals.

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EXTENSIONS *(a) Let $f$ be $\mu$-measurable on $E$, $m = \inf_{x\in E} f(x), M = \sup_{x\in E} f(x)$, $g, fg \in \mathcal{L}_\mu(E)$ then*

$$m\int_E |g|\,\mathrm{d}\mu \le \int_E f|g|\,\mathrm{d}\mu \le M\int_E |g|\,\mathrm{d}\mu.$$

*(b)* [FEJÉR] *If $T_n$ is a trigonometric polynomial of degree $n$, $m = \min_{0\le x\le 2\pi} T_n(x)$, $M = \max_{0\le x\le 2\pi} T_n(x)$ then*

$$m + \frac{M-m}{n+1} \le \frac{1}{2\pi}\int_0^{2\pi} T_n \le M - \frac{M-m}{n+1}.$$

*(c)* [LUKÁCS, F.] *If $p_n$ is a polynomial of degree $n$, $m = \min_{a\le x\le b} p_n(x)$, and $M = \max_{a\le x\le b} p_n(x)$ then*

$$m + \frac{M-m}{\tau_n} \le \frac{1}{2\pi}\int_0^{2\pi} p_n \le M - \frac{M-m}{\tau_n},$$

*where*

$$\tau_n = \begin{cases} (m+1)^2, & \text{if } n = 2m,\\ (m+1)(m+2), & \text{if } n = 2m+1. \end{cases}$$

*The constant $\tau_n$ is best possible.*

COMMENTS (iii) An extension of Bonnet's results, analogous to **Abel's Inequalities** (4), has been given by Bromwich.

(iv) A definition of trigonometric polynomial of degree $n$ is given in **Trigonometric Polynomial Inequalities**.

(v) See also: **Integral Inequalities**, **Mean Value Theorem of Differential Calculus, Napier's Inequality**.

REFERENCES *Apostol* [A67, vol. 1, p. 358], *Bromwich* [Br, pp. 473–475], *Saks* [Sa, p. 26], *Szegö* [Sz, pp. 178–181], *Widder* [W, pp. 16–20].

**Integral Test Inequality** *If $f$ is a non-negative strictly decreasing function defined on the interval $[m-1, n]$, $m, n \in \mathbb{N}$ then*

$$\sum_{m}^{n} f(i) < \int_{m-1}^{n} f < \sum_{m-1}^{n-1} f(i). \tag{1}$$

COMMENTS (i) This follows by considering the areas the terms in the inequalities represent.

(ii) The name is given since this inequality is the basis of the convergence test of the same name; the series $\sum_{i=m}^{\infty} f(i)$ and the improper integral $\int_m^\infty f$ either both converge or both diverge.

REFERENCE *Knopp* [Kn, pp. 294–295].

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**Internal Function Inequalities** *If $f : [a, b] \to \mathbb{R}$ is an internal function then for all $x, y$, $a \le x, y \le b$*

$$\min\{f(x), f(y)\} \le f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\}.$$

Comments (i) This is just the definition of an *internal function on* $[a, b]$.
(ii) See: **Fischer's Inequalities** Comment.

Reference [MPF, p. 641].

**Interpolation Inequalities** See: **Riesz-Thorin Theorem**.

**Inverse Hölder Inequalities** *If $f, g : [0, 1] \to [0, \infty[$ are concave, and if $p, q \ge 1$ then*

$$\int_0^1 fg \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{6} ||f||_p||g||_q. \tag{1}$$

*If $-1 < p, q \le 1$ (~1) holds with 6 replaced by 3.*

Comments (i) The case $p = q = 1$ is due to G. Grüss, the case $p, q \ge 1$ and conjugate indices is by Bellman and the general result is by Barnes.
(ii) The general case follows from the Grüss result and **Favard's Inequalities** (b).

Extensions *(a)* [Borell] *Under the same conditions as above*

$$\int_0^1 fg \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{6} ||f||_p||g||_q + \frac{f(0)g(0) + f(1)g(1)}{6}.$$

*(b)* [Nehari] *If $f_i, 1 \le i \le n$, satisfy the conditions above, and if $p_i \ge 1, 1 \le i \le n$, then*

$$\int_0^1 \prod_{i=1}^n f_i \ge \frac{[(n+1)/2]![n/2]!}{(n+1)!} \prod_{i=1}^n (p_i + 1)^{1/p_1} ||f_i||_{p_i}.$$

*There is equality if and only if $\left[\frac{n}{2}\right]$ of the functions are equal to $x$ and the rest are equal to $1 - x$.*

Comments (iii) (b) was published independently by Godunova & Levin.
(iv) (a) is a particular case of (b) and it might be noted that the maximum of the right-hand side occurs when $p = q = 2$.
(v) An error in Nehari's original proof is corrected by Choi K. P.
(v) There are many converses discussed in the references.
(v) See also: **Grüsses' Inequalities** (a), **Petschke's Inequality**.
It might be noted that $[(n+1)/2]n = n - \left[\frac{n}{2}\right]$.

References [AI, pp. 73, 385-386], [BB, pp. 39–42], [MPF, pp. 148–156], [PPT, pp. 223–225]; *Barnes* [51], *Choi* [87], *Maligranda, Pečarić & Persson* [194].

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**Inverse Inequalities** See: **Reverse, Inverse, and Converse Inequalities**.

**Irrationality Measure Inequalities** *If $p, q$ are positive integers, $p \geq 2$ and $q$ large enough, then*

$$|\pi - \frac{p}{q}| > q^{-23.72}.$$

REFERENCE *Borwein & Borwein* [Bs, pp. 362–386].

**Isodiametric Inequality** [CLASSICAL RESULT] *If $A$ is the area of a plane region with inner and outer radii $R, r$, respectively, then*

$$R^2 \geq \frac{A}{\pi} \geq r^2,$$

*with equality when the region is circular.*

COMMENTS (i) The *inner radius of a set* is the radius of the largest disk covered by the set; the *outer radius of a set* is the radius of the smallest disk that covers the set.

(ii) This is an example of **Symmetrization Inequalities**.

EXTENSION *For all sets $A \subseteq \mathbb{R}^n$ if $d(A)$ the diameter of $A$ then*

$$|A|_n \leq v_n \left(\frac{d(A)}{2}\right)^n.$$

COMMENTS (iii) This inequality is of interest since $A$ need not be contained in a ball of diameter $d(A)$.

(iv) See also **Isoperimetric Inequalities, n-Simplex Inequality**.

REFERENCES *Evans & Gariepy* [EG, p. 69], *Pólya & Szegö* [PS51, p. 8].

**Isoperimetric Inequalities** *(a)* [THE CLASSICAL ISOPERIMETRIC INEQUALITIES] *If $A$ is the area, and $\ell$ the perimeter of a plane region then*

$$A \leq \frac{\ell^2}{4\pi},$$

*with equality only when the region is a circular.*
*(b) If $V$ is the volume, $S$ the surface area of a domain in $\mathbb{R}^3$ then*

$$V^2 \leq \frac{S^3}{36\pi},$$

*with equality only if the domain is spherical.*

EXTENSION *If $V$ is the $n$-dimensional volume of a domain in $\mathbb{R}^n, n \geq 2$, with $A$ the $n-1$-dimensional area of its bounding surface then*

$$\frac{V^{n-1}}{A^n} \leq \frac{v_n^{n-1}}{a_n^n}, \quad \text{or} \quad V_n^{n-1} \leq \frac{A_n^n}{n^n v_n}. \tag{1}$$

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COMMENTS (i) Inequality (1) is a consequence of the **Brunn-Minkowski Inequalities** (c) and, surprisingly, is, in the case of a $\mathcal{C}^1$ boundary, equivalent to **Sobolev's Inequalities** (b).

(ii) See also: **Blaschke-Santaló Inequality, Bonnesen's Inequality, Gale's Inequality, Geometric Inequalities, Isodiametric Inequality, Mahler's Inequalities, Mixed-Volume Inequalities, Symmetrization Inequalities, Young's Inequalities** (e). This is an area of much research.

REFERENCES [EM, vol. 5, pp. 203–208]; *Bandle* [Ba], *Bobkov & Houdré* [BH, p. 1], *Chavel* [CI], *Pólya & Szegö* [PS51, p. 8]; *Osserman* [239], *Xu, Zeng & Zhou* [328].

## Iyengar's Inequality

*If $f$ is differentiable on $[0,1]$ with $|f'| \le 1$ then*

$$\left|\int_0^1 f - \frac{f(0)+f(1)}{2}\right| \le \frac{1-\big(f(1)-f(0)\big)^2}{4}.$$

COMMENT (i) This inequality was proved analytically by Iyengar and almost immdiately given a simple geometric proof by Mahajani and so is sometimes called the *Iyengar-Mahajani Inequality*,

EXTENSION *If $f$ is differentiable on $[a,b]$ with $|f'| \le M$ then*

$$\frac{1}{b-a}\int_a^b f - \frac{f(a)+f(b)}{2} \le \frac{M(b-a)}{4} - \frac{\big(f(b)-f(a)\big)^2}{4M(b-a)}$$

*With equality if and only if $f(x) = f(a) + M(x-a)$.*

COMMENTS (ii) This extension is due to Qi and, independently, Agarwal & Dragomir. They used different methods of proof.

(iii) Much work has been done extending this result that is connected to the *Trapezoidal Inequality*, see **Quadrature Inequalities** (a).

REFERENCES [AI, pp. 207–298]; *Agarwal & Dragomir* [2], *Čuljak & Elezović* [95], *Qi* [275], *Wang, Xie & Yang* [324].

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# 10 Jackson-Jordan

**Jackson's Inequality** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and of period $2\pi$ then*

$$\inf_{T_n \in \mathcal{T}_n} ||f - T_n||_\infty \le C\omega(f; n^{-1}), \tag{1}$$

*where $\mathcal{T}_n$ is the set of all trigonometric polynomials of degree $n$, $\omega$ is the modulus of continuity, and $C$ is an absolute constant.*

Comment (i) The definition of $\omega$ is given in **Modulus of Continuity Inequalities**, *(i)*.

Extension *If $f : \mathbb{R} \to \mathbb{R}$ has continuous derivative of order $k, k \ge 1$, and of period $2\pi$ then, with the above notation,*

$$\inf_{T_n \in \mathcal{T}_n} ||f - T_n||_\infty \le \frac{C_k}{n^k}\omega(f^{(k)}; n^{-1}).$$

Comment (ii) A similar result can be stated with trigonometric polynomials replaced by polynomials; see: [EM].

References [EM, vol. 5, p. 219], [GI1, p. 85–114].

**Jarníck's Inequality** *(a) If a closed convex region has area $A$, and perimeter $L$ then*

$$|n - A| < L,$$

*where $n$ is the number of lattice points in the region.*
*(b)* [Nosarzewska] *Under the same conditions,*

$$-\frac{L}{2} < n - A \le 1 + \frac{L}{2}.$$

Comment A point is a *lattice point* if it has integer coordinates.

Reference [CE, pp. 952, 1248].

**Jensen's Inequality** *If $I$ is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is convex, $\underline{a} \in I^n$, $n \ge 2$ and $\underline{w}$ a positive $n$-tuple, then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i a_i\right) \le \frac{1}{W_n}\sum_{i=1}^n w_i f(a_i); \tag{J}$$



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*or equivalently,*

$$f\left(\mathfrak{A}_n(\underline{a};\underline{w})\right) \le \mathfrak{A}_n(f(\underline{a});\underline{w}). \tag{1}$$

*If $f$ is strictly convex then (J) is strict unless $\underline{a}$ is constant.*

COMMENTS (i) The case $n=2$ of (J) is just the definition of convexity. See: **Convex Function Inequalities** (1). The general case follows by induction. See also: **Order Inequalities** COMMENTS (i).

(ii) (J) can be interpreted as saying:

*If $f$ is convex then the centroid of the points $P_i = \big(a_i, f(a_i)\big)$ $1 \le i \le n$, lies above the graph of $f$.*

(iii) As one of the most important general inequalities it is worth noting other obviously equivalent formulations.

Putting $p_i = w_i/Wn, 1 \le i \le n$, when of course $0 < p_i < 1, 1 \le i \le n$ and $P_n = 1$, (J) becomes

$$f\left(\sum_{i=1}^{n} p_i a_i\right) \le \sum_{i=1}^{n} p_i f(a_i); \tag{$J_P$}$$

called $(J_P)$ because the weights can be considered as probabilities.

Alternatively putting $t_i = p_i, l \le i \le n-1, t_n = 1 - \sum_{i=1}^{n-1} t_i$ then (J) can be written

$$D(t_1, \ldots, t_{n-1}) = \sum_{i=1}^{n} t_i f(a_i) - f\left(\sum_{i=1}^{n} t_i a_i\right) \ge 0.$$

In addition for clarity we will, when useful, name (J) as $(J_n)$.

EXTENSIONS *(a)* [FUNCTIONS OF INDEX SETS; VASIĆ & MIJALKOVIĆ] *Let $I, f, \underline{a}, \underline{w}$ be as in (J) and define $F$ on the index sets by*

$$D(\mathcal{I}) = W_{\mathcal{I}}\left\{\mathfrak{A}_{\mathcal{I}}\big(f(\underline{a});\underline{w}\big) - f\big(\mathfrak{A}_{\mathcal{I}}(\underline{a};\underline{w})\big)\right\}. \tag{2}$$

*Then $D \ge 0$, and if $\mathcal{I} \cap \mathcal{J} = \emptyset$,*

$$D(\mathcal{I}) + D(\mathcal{J}) \le D(\mathcal{I} \cup \mathcal{J}). \tag{3}$$

*Further, if $f$ is strictly convex (3) is strict unless $A_{\mathcal{I}}(\underline{a};\underline{w}) = A_{\mathcal{J}}(\underline{a};\underline{w})$.*
*(b)* [DRAGOMIR & CRSTICI] *Let $I, f, \underline{a}, \underline{w}$ be as in (J) and if $\underline{v}$ is a $k$-tuple, with $1 \le k \le n$ then*

$$\begin{aligned} f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big) \quad &\le \tfrac{1}{W_n^n} \textstyle\sum_{i_1,\ldots,i_k=1}^{n} \big(\prod_{j=1}^{k} w_{i_j}\big) f\big(\tfrac{1}{V_k}\sum_{j=1}^{k} v_j a_{i_j}\big) \\ &\le \mathfrak{A}_n\big(f(\underline{a});\underline{w}\big). \end{aligned}$$

COMMENT (iii) (3) is a simple deduction from (J). While (J) says that the function $D$ of (2) is non-negative, the refinement (3) says that $D$ is super-additive.

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Corollary [Rado Type] *If* $\mathcal{I}_i = \{1, 2, \ldots, i\}$, $1 \le i \le n$ *then, with* $D_{\mathcal{I}}$ *as in (2),*

$$D_{\mathcal{I}_n} \ge \cdots \ge D_{\mathcal{I}_2},$$

*and in particular*

$$D_{\mathcal{I}_n} \ge \sup_{1\le i,j\le n, i\ne j} \left\{ w_i f(a_i) + w_j f(a_j) - (w_i + w_j) f\left(\frac{w_i a_i + w_j a_j}{w_i + w_j}\right)\right\}.$$

Comments (iv) The first part of the corollary is an immediate consequence of (3) and the second is just $D_{\mathcal{I}_n} \ge D_{\mathcal{I}_2}$, noting that the terms in the sum in $D_{\mathcal{I}_n}$ can be rearranged.

(v) Another series of inequalities interpolating the two sides of (J) can be found in **Sequentially Convex Function Inequalities** Comments (iii).

Inverse Inequalities *(a)* [Lah & Ribarić] *Let* $I, f, \underline{a}, \underline{w}$ *be as in (J), and let*

$$a = \min \underline{a}; \qquad A = \max \underline{a}; \qquad \overline{a} = \mathfrak{A}_n(\underline{a}; \underline{w}), \tag{4}$$

*then*

$$\mathfrak{A}_n(f(\underline{a}); \underline{w}) \le \frac{A - \overline{a}}{A - a} f(a) + \frac{\overline{a} - a}{A - a} f(A),$$

*with equality if and only if for each* $i$*, either* $a_i = A$ *or* $a_i = a$*.*

*(b)* [Mitrinović & Vasić] *If* $I$ *is an interval in* $\mathbb{R}$ *and* $f : I \to \mathbb{R}$ *is positive, strictly convex and twice differentiable,* $\underline{a} \in I^n$, $n \ge 2$*, then*

$$\frac{1}{W_n}\sum_{i=1}^n w_i f(a_i) \le \lambda f\left(\frac{1}{W_n}\sum_{i=1}^n w_i a_i\right),$$

*where, if* $\phi = (f')^{-1}$ *and, using the notation of (4), with*

$$\mu = \frac{f(A) - f(a)}{A - a}, \qquad \nu = \frac{Af(a) - af(A)}{A - a},$$

$\lambda$ *is the unique solution of*

$$f \circ \phi\left(\frac{\mu}{\lambda}\right) = \frac{\mu}{\lambda}\phi\left(\frac{\mu}{\lambda}\right) + \frac{\nu}{\lambda}.$$

Comments (vii) The result of Lah & Ribarić is a simple consequence of **Convex Function Inequalities**, (2). It extends Comments (ii) to:

*the centroid of the points* $\big(a_i, f(a_i)\big)$, $1 \le i \le n$ *lies below the chord joining* $(a, f(a))$ *to* $(A, f(A))$*;* $a, A$ *as in (4) above.*

(viii) The proof of the Mitrinović & Vasić result (b) is based on the geometrical remarks in Comments (ii) and (vii); they called this the *centroid method* of proving inequalities.

(ix) For another form of (J) see: **Quasi-Arithmetic Mean Inequalities** (1).

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Integral Analogue *If $w\phi \in \mathcal{L}(a,b)$, $w \geq 0, \int_a^b w > 0$, $f$ convex on the interval $I$, $I$ containing the range of $\phi$, and if $wf \circ \phi \in \mathcal{L}(a,b)$ then*

$$f\left(\frac{\int_a^b w\phi}{\int_a^b w}\right) \leq \frac{\int_a^b wf \circ \phi}{\int_a^b w}.$$

*If $f$ is strictly convex then this inequality is strict unless $f$ is constant.*

Comments (xi) This can be extended to general measure spaces; see, for instance: *Hewitt & Stromberg* [HS] or *Lieb & Loss* [LL].

(x) See also **Slater's Inequality**, and for another inequality also called Jensen's inequality see **Power Sum Inequalities** Comments (i).

(xi) This important inequality has been the subject of much research in recent years see in particular [*MPF, PPT*].

(xii) For extensions that allow real weights see: **Jensen-Pečarić Inequalities, Jensen-Steffensen Inequality**. Clearly if all the weights are negative the above results still hold. If the weights are non-negative, that us if zero weights are allowed, then this is equivalent to considering $(J_k), 1 \leq k \leq\ n$, in the same result. However if weights of variable sign occur then both (J) and ($\sim$ J) can hold as the following simple result shows.

Lemma *If $I$ is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is convex, $x, y \in I$ then*

$$f\big((1-t)x+ty\big)\begin{cases}\leq (1-t)f(x)+tf(y) & \text{if } 0<t<1, & (J_2)\\ \geq (1-t)f(x)+tf(y) & \text{if } t<0 \text{ or } t>1. & (\sim J_2)\end{cases}$$

(xiii) Since many inequalities between means are special cases of (J) any extensions of (J) that allow real weights will give similar extensions to these mean inequalities. These extensions are easily made and will not be given explicitly.

References [EM, vol. 5, pp. 234–235], [H, pp. 5–48], [MOA, pp. 654–655], [MPF, pp. 1–19, 681–695], [PPT, pp. 43–57, 83–87, 105, 133–134]; *Hájos, Neukomm & Surányi* [HNS, pp. 73–77]; *Conway* [C, vol. II, p. 225], *Hewitt & Stromberg* [HS, p. 202], *Lieb & Loss*[LL, pp. 38–39], *Pólya & Szegö* [PS, pp. 65–67], 1972, *Roberts & Varberg* [RV, pp. 89,189–190]; *Dragomir & Crstici* [106].

**Jensen-Pečarić Inequalities** *Let $n \geq 3, I$ an interval in $\mathbb{R}$ and $\underline{a}$ and $n$-tuple with elements in $I$: further let $\underline{p}$ be an $n$-tuple of weights with elements in $\mathbb{R}^*$ and $P_n = 1$. Assume that:*
*$(\Pi_1)$ the maximum and minimum elements of $\underline{a}$ have positive weights;*
*$(\Pi_2)$ each positive weight dominates the sum of all the negative weights.*
*If then $f : I \to \mathbb{R}$ is convex $(J_P)$ holds.*

Comment (i) The conditions $(\Pi_1), (\Pi_2)$ imply that $\mathfrak{A}_n(\underline{a}; \underline{p}) \in I$ as is required for $(J_P)$. See: **Jensen-Steffensen Inequality** Comments (ii), (iii).

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Extension *Let $U$ be an open convex set in $\mathbb{R}^k$, $\mathbf{a}_i \in U, 1 \le i \le n$, and let $p_i, 1 \le i \le n$, be non-zero real numbers with $P_n = 1$ and $I_- = \{i; 1 \le i \le n \wedge p_i < 0\}, I_+ = \{i; 1 \le i \le n \wedge p_i > 0\}$. Further assume that $\forall i,\ i \in I_-$, $\mathbf{a_i}$ lies in the convex hull of the set $\{\mathbf{a}_i; i \in I_+\}$ and that $\forall j,\ j \in I_+,\ p_j + \sum_{i\in I_-} p_i \ge 0$. If $f : U \mapsto \mathbb{R}$ is convex then $(J_P)$ holds.*

Reverse Inequality [Vasić & Pečarić] *If $I$ is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is convex, $\underline{a} \in I^n$, $n \ge 2$ and $\underline{w}$ a real $n$-tuple with $W_n > 0, w_1 > 0$ and $w_i \le 0$, $2 \le i \le n$, and $\mathfrak{A}_n(\underline{a}; \underline{w}) \in I$, then ($\sim$J) holds.*

Comments (ii) This can be proved by applying (J) to the $n$-tuple $\big(\mathfrak{A}_n(\underline{a}; \underline{w}), a_2, \dots, a_n\big)$ with the weights $(W_n, -w_2, \dots, -w_n)$.

(iii) The case $n = 2$ of this result is given in **Jensen's Inequality** Lemma.

Reference *Bullen* [79].

**Jensen-Steffensen Inequality** *If $I$ is some interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ convex, $\underline{a}$ a monotone $n$-tuple with $a_i \in I, 1 \le i \le n, n \ge 3$, and $\underline{w}$ is a real $n$-tuple with*

$$W_n \ne 0, \qquad 0 \le \frac{W_i}{W_n} \le 1,\ 1 \le i \le n, \tag{$S$}$$

*then (J) holds, and holds strictly unless $\underline{a}$ is constant.*

Comments (i) A proof of (a), based on **Steffensen's Inequalities** (1), was given by Steffensen. A simple inductive proof has been given; see: *Bullen* [78].

(ii) A discussion of condition (S) can be found in **Geometric-Arithmetic Mean Inequality** Comments (ix); in particular inequality (5) in that reference shows that (S) implies that $\mathfrak{A}_n(\underline{a}; \underline{w}) \in I$.

(iii) The conditions $(\Pi_1), (\Pi_2)$ of **Jensen-Pečarić Inequalities** imply (S). Although the conditions required for Pečarić's result are stronger the result has the advantage of not requiring monotonicity and so can be extended to higher dimensions; see **Jensen-Pečarić Inequalities** extension.

(iv) Many of the inequalities can be derived from (J), and so usually they will have generalizations that allow weights satisfying (1); see for instance **Geometric-Arithmetic Mean Inequality**, Extensions (m).

Special Case [Mercer, A.] *If $f$ is convex on an interval containing the terms of the increasing $n$-tuple $\underline{a}$, and if $\underline{p}$ is a positive $n$-tuple with $P_n = 1$ then*

$$f\Big(a_1 + a_n - \sum_{i=1}^n p_i(a_i)\Big) \le f(a_1) + f(a_n) - \sum_{i=1}^n p_i f(a_i).$$

Extension [Ciesielski] *Let $\underline{w}$ be a real $n$-tuple with*

$$W_k \ge 0, 1 \le k \le n, \qquad \textit{and} \qquad \sum_{i=1}^n |w_i| > 0.$$

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*If $f$, $f'$ are both convex functions on $[0,\ell]$ with $f(0)\le 0$ then for any decreasing $n$-tuple, $\underline{a}$ with terms in $[0,\ell]$*

$$f\left(\frac{\sum_{i=1}^n w_i a_i}{\sum_{i=1}^n |w_i|}\right) \le \frac{\sum_{i=1}^n w_i f(a_i)}{\sum_{i=1}^n |w_i|}.$$

Integral Analogue [Steffensen] *If $f$ is convex on an interval containing the range of $\phi$, an increasing function, and if*

$$0 \le \int_a^x w \le \int_a^b w,\ a \le x \le b; \quad \textit{and} \quad \int_a^b w > 0,$$

*then*

$$f\left(\frac{\int_a^b w\phi}{\int_a^b w}\right) \le \frac{\int_a^b wf\circ\phi}{\int_a^b w}.$$

Reverse Inequality [Pečarić] *Let $I, f, \underline{a}$ be as in (J) and $\underline{w}$ a real $n$-tuple with*

$$W_n > 0, \quad \mathfrak{A}_n(\underline{a};\underline{w}) \in I;$$

*and for some $m, 1 \le m \le n$,*

$$W_k \le 0,\ 1 \le k < m, \qquad W_n - W_{k-1} \le 0,\ m < k \le n;$$

*then ($\sim$J) holds.*

References [AI, pp. 109–110, 115–116],[GI4, pp. 87–92], [H, pp. 37–43], [MPF, pp. 6–7], [PPT, pp. 57–63, 83–105, 161]; *Bullen* [78], *Mercer, A.* [206].

**Jessen's Inequality** *Let $\underline{a}^{(j)} = (a_{1j},\dots,a_{mj}), 1 \le j \le n$, and $\underline{u}$ be positive $m$-tuples, $\underline{a}_{(i)} = (a_{i1},\dots,a_{1n}), 1 \le i \le m$, and $\underline{v}$ positive $n$-tuples. If $-\infty \le r < s \le \infty$ then:*

$$\mathfrak{M}_n^{[s]}\left(\mathfrak{M}_m^{[r]}(\underline{a}^{(j)};\underline{u});\underline{v}\right) \le \mathfrak{M}_m^{[r]}\left(\mathfrak{M}_n^{[s]}(\underline{a}_{(i)};\underline{v});\underline{u}\right).$$

*There is equality if and only if $a_{ij} = b_i c_j, 1 \le i \le m, 1 \le j \le n$.*

Comments (i) This is a form of (M); see **Minkowski's Inequality** (2). See also: **Hölder's Inequality** Extensions (c).

(ii) This inequality is sometimes called the *Ingham-Jessen Inequality*.

(iii) An inverse inequality has been given by Tôyama.

Integral Analogue *If $[a,b]$, $[c,d]$ are intervals in $\overline{\mathbb{R}}$ and if $-\infty < r < s < \infty$ then:*

$$\left(\int_c^d \left(\int_a^b |f(x,y)|^r\,\mathrm{d}x\right)^{s/r}\mathrm{d}y\right)^{1/s} \le \left(\int_a^b \left(\int_c^d |f(x,y)|^s\,\mathrm{d}y\right)^{r/s}\mathrm{d}x\right)^{1/r}$$

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*with equality if and only if $f(x,y)=\phi(x)\psi(y)$ almost everywhere.*

COMMENT (iv) This has been generalized by Kalman to allow the indices $r, s$ to be functions of $x, y$, respectively.

REFERENCES [AI, p. 285], [H, p. 214], [HLP, pp. 31–32], [MPF, p. 109, 181]; *Kalman* [151].

**Jordan's Inequality** *If* $0<|x|\le \pi/2$ *then*

$$\frac{2}{\pi}\le\frac{\sin x}{x}<1.$$

COMMENTS (i) This is an immediate consequence of the strict concavity of the sine function on the interval $[0,\pi/2]$.

EXTENSIONS (a) [REDHEFFER] For all $x\neq 0$

$$\frac{\sin x}{x}\ge\frac{\pi^2-x^2}{\pi^2+x^2}.$$

(b) [EVERITT] If $0<x\le\pi$ then

$$\frac{\sin x}{x}<\min\left\{\frac{2(1-\cos x)}{x^2},\ \frac{2+\cos x}{3}\right\};$$

and

$$\frac{\sin x}{x}>\max\left\{\frac{4-4\cos x-x^2}{x^2},\ \frac{3\cos x+12-x^3}{15}\right\}.$$

(c) [BECKER & STARK] If $0<\alpha<1$ and $0\le x\le 2\alpha\pi$ then

$$1-\cos x\ge\left(\frac{\sin\alpha\pi}{\alpha\pi}\right)^2\frac{x^2}{2}.$$

COMMENTS (ii) See also: **Trigonometric Function Inequalities** (a).

(iii) For an analogue of Redheffer's result see: **Hyperbolic Function Inequalities** (e),

REFERENCES [AI, pp. 33, 354]; *Becker & Stark* [54].

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# 11 Kaczmarz–Ky Fan

**Kaczmarz & Steinhaus Inequalities** *If $p > 2$ then for some constant $\alpha$ depending only on $p$,*

$$|1+x|^p \le 1 + px + \sum_{i=2}^{[p]} \binom{p}{i} x^i + \alpha |x|^p, \quad x \in \mathbb{R}.$$

Corollary *If $p > 2$ and $f, g \in \mathcal{L}^p([a,b])$ then there are constants $\alpha, \beta$ depending only on $p$ such that*

$$\int_a^b |f+g|^p \le \int_a^b |f|^p + p\int_a^b |f|^{p-2} fg + \alpha \int_a^b |g|^p + \beta \sum_{i=2}^{[p]} \int_a^b |f|^{p-1}|g|^i.$$

Comment This inequality is important in the theory of orthogonal series.

References [MPF, p. 66]; *Kacmarz & Steinhaus* [KS, p. 247].

**Kakeya's Maximal Function Inequality** *If $R$ is a rectangle of sides $a, b, a < b$, put $e(R) = \alpha = a/b, 0 < \alpha < 1$, and for $f : \mathbb{R}^2 \to \mathbb{R}$ define*

$$K_\alpha f(\underline{x}) = \sup_{x \in R, e(R)=\alpha} \frac{1}{|R|} \int_R |f|, \underline{x} \in \mathbb{R}^2.$$

*Then*

$$K_\alpha f \le \frac{C}{\alpha} M_f,$$

*and if $f \in \mathcal{L}^2(\mathbb{R}^2), \beta > 0$,*

$$\begin{aligned} ||K_\alpha f||_2 \le & \quad C_1\Big(\log \tfrac{2}{\alpha}\Big)^2 ||f||_2; \\ \big|\{\underline{x};\, K_\alpha f(\underline{x}) > \beta\}\big| \le & \quad C_2\Big(\log \tfrac{2}{\alpha}\Big)^3 \tfrac{||f||_2^2}{\beta^2}. \end{aligned}$$

Comments (i) $M_f$ is the Hardy-Littlewood maximal function, defined in **Hardy-Littlewood Maximal Inequalities** Comments (v); the extension of that definition to higher dimensions is immediate.



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(ii) The quantity $e(R)$ is called the *eccentricity of* $R$; and $K_\alpha f$ is the *Kakeya maximal function.*

(iii) These results have partial extensions to higher dimensions.

REFERENCES *Igari* [144], *Müller* [226].

**Kalajdžić's Inequality** *Let $b>1$, $\underline{a},\underline{w}$ positive $n$-tuples with $W_n=1$ then*

$$\sum! b^{a_1^{w_{i_1}}\dots a_n^{w_{i_n}}} \le (n-1)!\sum_{i=1}^n b^{a_i}, \tag{1}$$

*with equality if and only if $\underline{a}$ is constant.*

COMMENT (i) This follows from (GA) and the method of proof shows that (a) below holds.

EXTENSIONS *(a) Under the above hypotheses but allowing $\underline{w}$ to have zero values*

$$\sum! b^{a_1^{w_{i_1}}\dots a_n^{w_{i_n}}} \le (n-1)!\sum_{i=1}^n b^{a_i^{W_n}};$$

*equality can occur if all but at most one of the $w_i$ are zero, or if $\underline{a}$ is constant.*
*(b) If $\underline{a},\underline{w}$ are positive $n$-tuples, $W_n=1$, then*

$$\sum! a_1^{w_{i_1}}\cdots a_n^{w_{i_n}} \le (n-1)!\sum_{i=1}^n a_i.$$

COMMENT (ii) In the case of equal weights (b) is an extension of (GA).

REFERENCES [H, p. 122].

**Kallman-Rota Inequality** *If $A$ is the infinitesimal generator of a strongly continuous semigroup of contractions on a Banach space $X$, and if $x$ is in the domain of $A^2$ then*

$$||Ax||^2 \le 4||x||\,||A^2x||.$$

COMMENTS (i) This inequality can be used to prove **Hardy-Littlewood-Pólya Inequality** (1).

(ii) Definitions of the terms used can be found in the first reference.

REFERENCES [EM, vol. 8, p. 254]; *Biler & Witkowski* [BW, p. 123].

**Kalman's Inequality** See: **Jessen's Inequality** COMMENT (iv).

**Kaluza-Szegő Inequality** *If $\underline{a}$ is a positive sequence,*

$$\sum_{i=1}^n \mathfrak{G}_i(\underline{a}) \le \sum_{i=1}^n \left(1+\frac{1}{i}\right)^i a_i.$$

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COMMENT This is an extension of the finite form of **Carleman's Inequality**.

REFERENCE *Redheffer* [279, p. 684].

**Kantorovič's Inequality** *If* $0 < m \le \underline{a} \le M$ *then*

$$\mathfrak{A}_n(\underline{a};\underline{w})\mathfrak{A}_n(\underline{a}^{-1};\underline{w}) = \frac{\mathfrak{A}_n(\underline{a};\underline{w})}{\mathfrak{H}_n(\underline{a};\underline{w})} \le \frac{(M+m)^2}{4Mm} = \left(\frac{M+m}{2}\right)\left(\frac{M^{-1}+m^{-1}}{2}\right),$$

*with equality if and only if there is a* $\mathcal{I} \subseteq \{1,\dots,n\}$ *such that* $W_{\mathcal{I}} = 1/2$, $a_i = M, i \in \mathcal{I}$ *and* $a_i = m, i \notin \mathcal{I}$.

COMMENTS (i) This inequality, the equal weight case of which is due to P. Schweitzer, is equivalent to the **Pólya & Szegő Inequality**. It is also a special case of a result of Specht; see **Power Mean Inequalities** INVERSE INEQUALITIES (b).

(ii) In the equal weight case, and when $n$ is odd equality is impossible; a better constant has been obtained in this case by Clausing; see: [GI3].

INTEGRAL ANALOGUE [SCHWEITZER, P.] *If* $f, 1/f \in \mathcal{L}([a,b])$ *with* $0 < m \le f \le M$ *then*

$$\int_a^b f \int_a^b \frac{1}{f} \le \frac{(M+m)^2}{4Mm}(b-a)^2.$$

COMMENT (iii) For a inverse inequality see **Walsh's Inequality**; and for a matrix analogue see: **Matrix Inequalities** COMMENTS (I). For other generalizations see: **Symmetric Mean Inequalities** COMMENTS (IV), **Variance Inequalities** COMMENT (II).

REFERENCES [AI, pp. 59–66], [BB, pp. 44-45],[GI3, p. 61], [H, pp. 235–237], [MOA, pp. 102–103], [MPF, pp. 684–685].

**Karamata's Inequalities** *If* $f, g$ *are integrable on* $[0,1]$ *with* $0 < \alpha \le f \le A$*, and* $0 < \beta \le g \le B$ *then*

$$\left(\frac{\sqrt{\alpha\beta}+\sqrt{AB}}{\sqrt{\alpha B}+\sqrt{A\beta}}\right)^{-2} \le \frac{\mathfrak{A}_{[0,1]}(fg)}{\mathfrak{A}_{[0,1]}(f)\mathfrak{A}_{[0,1]}(g)} \le \left(\frac{\sqrt{\alpha\beta}+\sqrt{AB}}{\sqrt{\alpha B}+\sqrt{A\beta}}\right)^{2}.$$

COMMENTS (i) The right-hand side is greater than or equal to 1.

(ii) This result is an inverse inequality for (Č); for another see: **Grüsses' Inequalities** (a).

(iii) For a more important Karamata inequality see: **Order Inequalities** COMMENT (VI).

REFERENCE [MPF, p. 684].

**Khinchine-Littlewood Inequality** See: **Hinčin-Littlewood Inequality**.

**Klamkin's Inequality** See: **Beth-van der Corput Inequality** Extensions.

**Klamkin-Newman Inequalities** *Let $\underline{a}$ be a non-negative $n$-tuple, $a_0 = 0$ and $0 \le \tilde{\Delta} a_i \le 1, 0 \le i \le n-1$; if $r \ge 1, s+1 \ge 2(r+1)$ then*

$$\left((s+1)\sum_{i=1}^{n} a_i^s\right)^{1/(s+1)} \le \left((r+1)\sum_{i=1}^{n} a_i^r\right)^{1/(r+1)}.$$

Extensions (a) [Meir] *Let $\underline{a}$ be a non-negative $n$-tuple, $a_0 = 0$, $0 \le w_0 \le \cdots \le w_n$, and $0 \le \tilde{\Delta} a_i \le (w_i + w_{i-1})/2, 0 \le i \le n-1$; if $r \ge 1, s+1 \ge 2(r+1)$ then*

$$\left((s+1)\sum_{i=1}^{n} w_i a_i^s\right)^{1/(s+1)} \le \left((r+1)\sum_{i=1}^{n} w_i a_i^r\right)^{1/(r+1)}.$$

*(b)* [Milovanović, G.] *Let $f, g : [0,\infty[\to \mathbb{R}$ satisfy (i) $f(0) = f'(0) = g(0) = g'(0) = 0$, (ii) $f', g'$ are convex. If $\underline{a}, \underline{w}$ are as in (a) and $h = g \circ f$ then*

$$h^{-1}\left(\sum_{i=1}^{n} w_i h'(a_i)\right) \le f^{-1}\left(\sum_{i=1}^{n} w_i f'(a_i)\right).$$

Comments (i) (a) is the case $h(x) = x^{s+1}, f(x) = x^{r+1}$ of (b).

(ii) Another pair of functions satisfying the hypotheses of (b) are $f(x) = x^2, g(x) = x^3 e^x$.

(iii) Compare these to the **Gauss-Winkler Inequality**.

Reference [PPT, pp. 166–169].

**Kneser's Inequality** *If $p$ is a polynomial of degree $n$, $p = qr$, where $q, r$ are polynomials of degrees $m, n-m$, respectively, then*

$$||q||_{\infty,[-1,1]}||r||_{\infty,[-1,1]} \le \frac{1}{2} C_{n,m} C_{n,n-m} ||p||_{\infty,[-1,1]},$$

*where*

$$C_{n,m} = 2^m \prod_{k=1}^{m}\left(1 + \cos\frac{2k-1}{2n}\pi\right).$$

*In particular*

$$||q||_{\infty,[-1,1]}||r||_{\infty,[-1,1]} \le 2^{n-1} \prod_{k=1}^{[n/2]}\left(1 + \cos\frac{2k-1}{2n}\pi\right)||p||_{\infty,[-1,1]}.$$

Comments (i) The constant on the right-hand side of the last inequality is approximately $(3.20991\cdots)^n$.

(ii) This result has been extensively discussed by P. Borwein.

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REFERENCE *Borwein, P.* [68].

**Knopp's Inequalities** *(a) If $\underline{a}$ is a positive $\ell_p$ sequence, $p > 0$, and if $\alpha > 0$ then*

$$\sum_{n\in\mathbb{N}} \binom{n+\alpha}{n}^p \left(\sum_{i=0}^{n} \frac{\binom{n+\alpha-1-i}{n-i}}{a_i}\right)^{-p} \le \frac{(\alpha+p^{-1})!}{\alpha!(p^{-1})!} \sum_{n\in\mathbb{N}} a_n^p.$$

*(b) If $\underline{a}$ is a positive sequence and $\alpha > 0$ then*

$$\sum_{n\in\mathbb{N}} \left(\prod_{i=0}^{n} a_i^{\binom{n-\alpha-1-i}{n-i}}\right)^{1/\binom{n+\alpha}{n}} \le e^{\Psi(\alpha)-\Psi(0)} \sum_{n\in\mathbb{N}} a_n$$

COMMENTS (i) For a definition of $\Psi$ see: **Digamma Function Inequalities**.

(ii) These inequalities are due to Knopp, but the exact constants on the right-hand sides were given by Bennett.

COROLLARY *If $\underline{a}$ is a positive $\ell_p$ sequence, $p > 0$, then*

$$\sum_{n=1}^{\infty} (\mathfrak{H}_n(\underline{a}))^p \le \left(\frac{p+1}{p}\right)^p \sum_{n=1}^{\infty} a_n^p \qquad \text{or} \qquad ||\underline{\mathfrak{H}}(\underline{a})||_p \le \frac{p+1}{p}||\underline{a}||_p.$$

COMMENTS (iii) This is obtained from (a) by taking $\alpha = 1$.

(iv) See also: **Copson's Inequalities** EXTENSIONS (b).

REFERENCES *Bennett* [Be, pp. 37–39].

**Kober-Diananda Inequalities** *Assume that $\underline{a}$ is a non-constant positive $n$-tuple, that $\underline{w}$ is a positive $n$-tuple with $W_n = 1$ and that*

$$S_n(\underline{a}) = \frac{1}{2}\sum_{i,j=1}^{n} (\sqrt{a}_i - \sqrt{a}_j)^2, \quad S_n(\underline{a};\underline{w}) = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j (\sqrt{a}_i - \sqrt{a}_j)^2;$$

*then*

$$\frac{\min \underline{w}}{n-1} < \frac{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})}{S_n(\underline{a})} < \max \underline{w},$$

*and*

$$\frac{1}{1-\min \underline{w}} < \frac{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})}{S_n(\underline{a};\underline{w})} < \frac{1}{\min \underline{w}}.$$

COMMENTS (i) If $\underline{a}$ is allowed to be non-negative the bounds in these inequalities can be attained.

(ii) If $\underline{a} \to \alpha\underline{e} \ne \underline{0}$ then the central term in the second set of inequalities tends to 2.

(iii) These results can be used to improve both (H) and (M). See: [H].

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REFERENCES [AI, pp. 81–83], [H, pp. 141–145, 194], [MPF, pp. 113–114].

**Kolmogorov's Inequalities** *If $f \in \mathcal{L}^r(\mathbb{R}), f^{(n)} \in \mathcal{L}^p(\mathbb{R})$ then*

$$||f^{(k)}||_q \le C_{p,r,k,n}||f||_r^{\nu}||f^{(n)}||_p^{1-\nu}$$

*where*

$$\nu = \frac{n-k-p^{-1}+q^{-1}}{n-p^{-1}+r^{-1}}.$$

COMMENTS (i) Kolmogorov's result is the case $p = q = r = \infty$; the general case is due to Stein. So this inequality is sometimes called the *Kolmogorov-Stein Inequality*.

(ii) In the case $k = 1, n = p = q = r = 2$ this is **Hardy-Littlewood-Pólya Inequalities** (2), while in the case $p = q = r = \infty$ it is **Hardy-Littlewood-Landau Derivative Inequalities** (1), and the case $k = 2, n = 4, p = q = r = 2$ is due to Kurepa. See also: **Sobolev's Inequalities**.

(ii) For other inequalities due to Kolmogorov see: **Kolmogorov's Probability Inequality**.

REFERENCES [EM, vol. 5, p. 295]; *Biler & Witkowski* [BW, p. 23]; *Bang & Le* [49].

**Kolmogorov's Probability Inequality** *If $X_i$, $1 \le i \le n$, are independent random variables of zero mean and finite variances, and if $S_j = \sum_{i=1}^{j} X_i$, $1 \le j \le n$, then*

$$P\big(\max_{1\le j\le n}|S_j| > r\big) \le \frac{\sigma^2 S_n}{r^2}.$$

COMMENTS (i) This extends **Čebišev's Probability Inequality**.

(ii) An inverse inequality can be found in the last reference, and an extension is in **Lévy's Inequalities**.

(ii) See also: **Martingale Inequalities** (a), COMMENTS (i).

REFERENCES [EM, vol. 2, p. 120]; *Feller* [F, vol. II, p. 156], *Loève* [L, p. 235].

**Kolmogorov-Stein Inequality** See: **Kolmogorov's Inequality** COMMENT (I).

**König's Inequality** *If $\underline{a}, \underline{b}$ are decreasing non-negative $n$-tuples and if $p > 0$ then.*

$$\prod_{i=1}^{k} b_i \le \prod_{i=1}^{k} a_i,\ 1 \le k \le n, \quad \Longrightarrow \quad \sum_{i=1}^{k} b_i^p \le \sum_{i=1}^{k} a_i^p,\ 1 \le k \le n.$$

COMMENT This was used by König to give a proof of **Weyl's Inequalities** (c).

REFERENCE *König* [Kon, p. 24]

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**Korn's Inequality** *If $\Omega \subset \mathbb{R}^p$ is a bounded domain, $f_i : \Omega \to \mathbb{R}, 1 \le i \le p$, differentiable, then for some constant $C$*

$$\int_\Omega \left\{ \sum_{i,j=1}^p \left( \frac{\partial f_i}{\partial x_j} \right)^2 + \sum_{i=1}^p f_i^2 \right\} \mathrm{d}\underline{x} \le C \int_\Omega \left\{ \sum_{i,j=1}^p \left( \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \right)^2 + \sum_{i=1}^p f_i^2 \right\} \mathrm{d}\underline{x}\,.$$

Comment If the second term on the right-hand side is omitted this is *Korn's First Inequality*, and then the present result is *Korn's Second Inequality*.

Reference [EM, vol. 5, p. 299].

**Korovkin's Inequality** *If $n \ge 2$ and $\underline{a}$ is a positive $n$-tuple then*

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \ge n, \tag{1}$$

*with equality if and only if $\underline{a}$ is constant.*

Comments (i) The proof is by induction; the $n = 2$ case being the well-known elementary inequality,

$$x > 0 \quad \text{and} \quad x \ne 1 \quad \Longrightarrow x + \frac{1}{x} > 2. \tag{2}$$

(ii) Putting $x = \sqrt{a}/\sqrt{b}$ we see that (2) is just **Geometric-Arithmetic Inequality** (3). Inequality (1) can be used to prove the equal weight case of (GA), [H, p. 99.]

Extensions *(a)* [Janić & Vasić] *If $1 \le k < n$ and $\underline{a}$ is a positive $n$-tuple, then*

$$\frac{a_1 + \cdots + a_k}{a_{k+1} + \cdots + a_n} + \frac{a_2 + \cdots + a_{k+1}}{a_{k+2} + \cdots + a_1} + \cdots + \frac{a_n + \cdots + a_{k-1}}{a_k + \cdots + a_{n-1}} \ge \frac{nk}{n-k}.$$

*(b)* [Chong] *If $a \ge 1$ and $\alpha \ge 1$ then*

$$\alpha a + \frac{1}{a} \ge \alpha + 1.$$

*(c)* [Teng] *If $\underline{a}$ is a positive $n$-tuple, $n \ge 2$ then*

$$a_1 + \cdots + a_n + \frac{1}{a_1 \ldots a_n} \ge n + 1,$$

*with equality if and only if $\underline{a} = \underline{e}$.*

Comments (iii) In fact the Extension (c) is equivalent to the equal weight case of (GA).

(iv) A generalization of (1) is in **Cyclic Inequalities** Extension, Comments (ii).

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Reference [H, pp. 10–11, 99, 113–114, 123–124].

**Kunyeda's Inequality** *If the zeros of the complex polynomial* $z^n + \sum_{k=1}^{n-1} a_k z^k$ *lie in the disk* $\{z; |z| \le r\}$ *then for any pair of conjugate indices,* $p, q, p > 1$

$$r < \left(1 + \left(\sum_{k=1}^{n-1} |a_k|^p\right)^{q/p}\right)^{1/q}.$$

Comment (i) The quantity $r$ is called the *inclusion radius* of the polynomial.

Reference *Steele* [S, pp. 148–149, 261–262].

**Ky Fan–Taussky Todd Inequalities** See: **Fan-Taussky-Todd Inequalities**.

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# 12 Labelle–Lyons

**Labelle's Inequality** *If* $p(x) = \sum_{i=0}^{n} a_i x^i$ *then*

$$|a_n| \le \frac{(2n)!}{2^n (n!)^2} \sqrt{\frac{2n+1}{2}}\, ||p||_{\infty,[-1,1]}.$$

EXTENSION [TARIQ] *If* $p_n(x) = \sum_{i=0}^{n} a_i x^i$ *and* $p(1) = 0$, *then*

$$|a_n| \le \frac{n}{n+1} \frac{(2n)!}{2^n (n!)^2} \sqrt{\frac{2n+1}{2}} ||p_n||_{\infty,[-1,1]}.$$

REFERENCE *Tariq* [309].

**Laguerre Function Inequalities** The entire functions

$$L_\nu^\mu(z) = \binom{\mu+\nu}{\nu} \sum_{n=0}^{\infty} \frac{\nu(\nu+1)\cdots(\nu-n+1)}{(\mu+1)\cdots(\mu+n)} \frac{(-z)^n}{n!}$$

are called *Laguerre functions*, or in the case that $\nu \in \mathbb{N}$, *Laguerre polynomials*.

*If* $\mu \ge 0, \nu \in \mathbb{N}, x \ge 0$ *then*

$$\left|L_\nu^\mu(x)\right| \le \binom{\mu+\nu}{\nu} e^{x/2}.$$

EXTENSION [LOVE] *If* $\Re\mu \ge 0, \nu \in \mathbb{N}, x > 0$ *then*

$$\left|L_\nu^\mu(x)\right| \le \left|\frac{\mu}{\Re\mu}\binom{\mu+\nu}{\nu}\right| e^{x/2}.$$

COMMENT Other extensions can be found in the paper by Love; see also **Enveloping Series Inequalities** COMMENTS (iii).

REFERENCES [EM, vol. 5, p. 341]; *Abramowitz & Stegun* [AS, p. 775]; *Szegő* [Sz], *Widder* [W, pp. 168–171]; *Love* [185].



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**Laguerre Polynomial Inequalities** See: **Laguerre Function Inequalities**.

**Laguerre-Samuelson Inequality** *If $\underline{a}$ is a positive $n$-tuple write $\overline{\underline{a}} = \mathfrak{A}_n(\underline{a})$, the expectation, and $\sigma = \mathfrak{Q}_n(\underline{a} - \overline{\underline{a}})$, the standard deviation, then*

$$\overline{\underline{a}} - \sigma\sqrt{n-1} \le a_i \le \overline{\underline{a}} + \sigma\sqrt{n-1}, \quad 1 \le i \le n.$$

COMMENT This inequality was obtained in the theory of polynomials by Laguerre in 1880, and later independently in the above form, by Samuelson in 1968.

REFERENCE *Rassias & Srivastava* [RS, pp. 151–181].

**Landau's Constant** *Let $L_f$ denote the inner radius of the image of $D$ by a function $f$ analytic in $D$, and put $L = \inf L_f$ where the* inf *is taken over all functions analytic in $D$ with $f(0) = f'(0) = 0$, then*

$$B \le L, \qquad \textit{and} \qquad \frac{1}{2} \le L \le \frac{(-2/3)!(-1/6)!}{(-5/6)!} \approx 0.5432588\ldots;$$

*where $B$ is Bloch's constant.*

COMMENTS (i) $L$ is called *Landau's Constant*. it is coinjectured that the right-hand side is an equality.

(ii) A definition of inner radius can be found in **Isodiametric Inequality** COMMENTS (i).

(iii) See also: **Bloch's Constant**.

REFERENCES [EM, vol. 5, pp. 346–347]; *Conway* [C, vol. I, pp. 297–298].

**Landau's Inequality** See: **Hardy-Littlewood-Landau Derivative Inequalities**.

**Lattice Inequalities** See: **Ahlswede-Daykin Inequality, FKG Inequality, Holley's Inequality, Subadditive Function Inequalities** (b).

**Lebedev-Milin Inequalities** *If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is analytic in some neighborhood of the origin, and if $g(z) = e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n$ then:*

$$\sum_{n=0}^{\infty} |b_n|^2 \le \exp\left(\sum_{n=1}^{\infty} n|a_n|^2\right); \tag{1}$$

$$\sum_{n=0}^{N} |b_n|^2 \le (N+1)\exp\left(\frac{1}{N+1}\sum_{m=1}^{N}\sum_{n=1}^{m}\big(n|a_n|^2 - \frac{1}{n}\big)\right); \tag{2}$$

$$|b_N|^2 \le \exp\left(\sum_{n=1}^{N}\big(n|a_n|^2 - \frac{1}{n}\big)\right). \tag{3}$$

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*(a) If the right-hand side of (1) is finite there is equality if and only if for some $\gamma \in D$, and all $n \geq 1$ , $a_n = \gamma^n/n$;*
*(b) there is equality in (2), or (3), for a given $N$, if and only if for some $\gamma = e^{i\theta}, \theta \in \mathbb{R}$, and all $n, 1 \leq n \leq N$ , $a_n = \gamma^n/n$.*

COMMENT These inequalities are important in the solution of **Bieberbach's Conjecture**.

REFERENCE *Conway* [C, vol. II, pp. 149–155].

**Legendre Polynomial Inequalities** *(a) If $P_n$ is a Legendre polynomial then*

$$\sqrt{\sin\theta}\big|P_n(\cos\theta)\big| < \sqrt{\frac{2}{\pi n}}; \quad 0 \leq \theta \leq \pi.$$

*The constant is best possible.*
*(b)* [TURÁN] *If $-1 \leq x \leq 1$ then*

$$P_{n+1}^2(x) \geq P_n(x)P_{n+2}(x).$$

COMMENTS (i) For generalizations of (a) see: **Martin's Inequalities, Ultraspherical Polynomial Inequalities** (a).

(ii) (b) is an example of one of many similar inequalities satisfied by various special functions. See: **Bessel Function Inequalities** (c), **Ultraspherical Polynomials Inequalities** (b), COMMENTS (i)–(iii).

REFERENCES [GI1, pp. 35–38]; *Szegő* [Sz, p. 165].

**Lebesgue Constants** The $n$-th *Lebesgue constant* is

$$L_n = \frac{2}{\pi}\int_0^\pi \big|D_n\big|,$$

where $D_n$ is the **Dirichlet kernel**. For large values of $n$:

$$\frac{4}{\pi^2}\log n < L_n < 3 + \frac{4}{\pi^2}\log n$$

REFERENCES [CE];*Zygmund* [Z, vol. 1, p. 67]; *Chen & Choi* [84].

**Legendre Transform Inequality** See: **Conjugate Convex Function Inequalities**.

**Lehmer Mean Inequalities** See: **Gini-Dresher Mean Inequalities**.

**Leindler's Inequality** *If $z \in \mathbb{C}$ and $p \geq 2$ then*

$$1 + p\Re z + a_p|z|^2 + b_p|z|^p \leq |1+z|^p \leq 1 + p\Re z + A_p|z|^2 + B_p|z|^p,$$

*where $a_p, b_p, A_p, B_p$ are any positive real numbers that satisfy either*

$$0 < a_p < \frac{p}{2},\ 0 < b_p \leq \mu_1(p),\ 1 < B_p < \infty,\ M_1(p) \leq A_p < \infty,$$

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*where*

$$\mu_1(p) = \inf_{t\geq 2} \frac{(t-1)^p + pt - 1 - a_p t^2}{t^p};\ M_1(p) = \sup_{t>0} \frac{(t+1)^p - 1 - pt - B_p t^p}{t^2};$$

*or*

$$0 < b_p < \mu_2(p),\ 0 < a_p \leq \mu_3(p),\ \frac{p(p-1)}{2} < A_p < \infty,\ M_2(p) \leq B_p < \infty,$$

*where*

$$\mu_2(p) = \inf_{t\geq 2} \tfrac{(t-1)^p + pt - 1}{t^p};\ \mu_3(p) = \inf_{t\geq 2} \tfrac{(t-1)^p + pt - 1 - b_p t^p}{t^2};$$
$$M_2(p) = \sup_{t>0} \tfrac{(t+1)^p - 1 - pt - A_p t^2}{t^p}.$$

*The ranges* $]a_p, b_p[$, $]A_p, B_p[$ *are best possible.*

Comment The case $p = 4$, with $A_4 = 3, B_4 = 3, a_4 = 1, b_4 = 1/5$ is due to Shapiro.

Reference [MPF, pp. 66–68].

**Levinson's Inequality** *Let $f$ be 3-convex on $[a, b]$ and let $\underline{a}, \underline{b}$ be $n$-tuples with elements in $[a, b]$ such that*

$$a_1 + b_1 = \cdots = a_n + b_n; \qquad \max \underline{a} \leq \min \underline{b}. \tag{1}$$

*If $\underline{w}$ is a positive $n$-tuple then*

$$\mathfrak{A}_n\big(f(\underline{a}); \underline{w}\big) - f\big(\mathfrak{A}_n(\underline{a}; \underline{w})\big) \leq \mathfrak{A}_n\big(f(\underline{b}); \underline{w}\big) - f\big(\mathfrak{A}_n(\underline{b}; \underline{w})\big). \tag{2}$$

*Conversely if (1) holds for all $n$ and all $n$-tuples $\underline{a}, \underline{b}$ satisfying (1) then $f$ is 3-convex.*

Comment (i) Inequality (2), which generalizes the **Fan Inequality** Comment (vii), was proved by Levinson under slightly more restrictive conditions, see: [AI]. A stronger version has been proved by Pečarić.

Special Cases *Let $\underline{a}, \underline{b}$ satisfy (1) and $\underline{w}$ be a positive $n$-tuple then:*
*(i) if $s > 0, s > t$ or $t > 2s$; or $s = 0, t > 0$; or $s < 0$, $s > t > 2s$ and if $t \neq 0$*

$$\left(\left(\mathfrak{M}_n^{[t]}(\underline{a}; \underline{w})\right)^t - \left(\mathfrak{M}_n^{[s]}(\underline{a}; \underline{w})\right)^t\right)^{1/t} \leq \left(\left(\mathfrak{M}_n^{[t]}(\underline{b}; \underline{w})\right)^t - \left(\mathfrak{M}_n^{[s]}(\underline{b}; \underline{w})\right)^t\right)^{1/t};$$

*while if $t = 0$*

$$\frac{\mathfrak{G}_n(\underline{a}; \underline{w})}{\mathfrak{M}_n^{[s]}(\underline{a}; \underline{w})} \leq \frac{\mathfrak{G}_n(\underline{b}; \underline{w})}{\mathfrak{M}_n^{[s]}(\underline{b}; \underline{w})};$$

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*(ii) these inequalities are reversed if $s > 0, s < t < 2s$ or $s = 0, t < 0$ or $s < 0, t \neq 0$. These inequalities are strict unless $\underline{a}$ is constant.*

EXTENSION *Let $f$ be 3-convex on $[a,b]$, let $I$, $J$ be two disjoint index sets, $\underline{w}$ a positive sequence, and let $\underline{a}, \underline{b}$ be sequences with elements in $[a,b]$, and such that*

$$\mathfrak{A}_I(\underline{a}+\underline{b};\underline{w}) = \mathfrak{A}_J(\underline{a}+\underline{b};\underline{w}) = 2c,$$

*and*

$$\mathfrak{A}_I(\underline{a};\underline{w}) + \mathfrak{A}_I(\underline{a};\underline{w}) \leq 2c,\ \mathfrak{A}_{I\cup J}(\underline{a};\underline{w}) < c.$$

*The function, defined on the index sets,*

$$F(I) = W_I\Bigg(\Big(f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big) - \mathfrak{A}_I\big(f(\underline{a});\underline{w}\big)\Big) - \Big(f\big(\mathfrak{A}_n(\underline{b};\underline{w})\big) - \mathfrak{A}_I\big(f(\underline{b});\underline{w}\big)\Big)\Bigg)$$

*is super-additive.*

COMMENT (ii) Taking $I = \{1,2,\ldots,k-1\}$, $J = \{k\}$, $k = 1,\ldots,n$ gives a Levinson's inequality under weaker conditions than stated above. The proof of this generalization is particularly simple.

REFERENCES [AI, p. 363],[H, pp. 294–296], [MPF, pp. 32–36], [PPT, pp. 71–75]; *Pečarić & Raşa* [266].

**Lévy's Inequalities** *If $X_i, 1 \leq i \leq n$, are independent random variables and if $S_k = \sum_{i=1}^{k} X_i, 1 \leq k \leq n$, then*

$$P\left\{\max_{1\leq k\leq n}\{S_k - m(S-k-S_n)\} \geq \quad r\right\} \leq 2P\{S_n \geq r\};$$
$$P\left\{\max_{1\leq k\leq n}\{|S_k - m(S-k-S_n)|\} \geq \quad r\right\} \leq 2P\big\{|S_n| \geq r\big\}.$$

*In particular if the random variables are symmetrically distributed*

$$P\big\{\max_{1\leq k\leq n} S_k \geq x\big\} \leq \quad 2P\big\{S_n \geq r\big\};$$
$$P\big\{\max_{1\leq k\leq n} |S_k| \geq x\big\} \leq \quad 2P\big\{|S_n| \geq r\big\}.$$

COMMENT These are extensions of **Kolmogorov's Probability Inequality**.

REFERENCES [EM, vol. 5, pp. 404–405]; *Loève* [L, pp. 247–248].

**L'Hôpital's Rule** *Hypotheses: $a, b \in \overline{\mathbb{R}}$; $f, g : ]a,b[ \to \mathbb{R}$ are differentiable; $g$ is never zero; $g'$ is never zero; $c = a$* or *$b$; either (a) or (b) holds;*

$$(a) \qquad \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0,$$
$$(b) \qquad \lim_{x\to c} |g(x)| = \infty.$$

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*Conclusion:*

$$\liminf_{x\to c}\frac{f'(x)}{g'(x)} \le \liminf_{x\to c}\frac{f(x)}{g(x)} \le \limsup_{x\to c}\frac{f(x)}{g(x)} \le \limsup_{x\to c}\frac{f'(x)}{g'(x)}.$$

*In particular*

$$if \qquad \lim_{x\to c}\frac{f'(x)}{g'(x)} = \ell,\ \ell\in\overline{\mathbb{R}}, \qquad then \qquad \lim_{x\to c}\frac{f(x)}{g(x)} = \ell.$$

COMMENTS (i) A discrete analogue has been given by Agarwal.

(ii) This is a result of fundamental importance in the evaluation of many limits.

REFERENCES [GI4, pp. 95–98]; *Taylor* [310].

**Liapounoff's Inequality** See: **Lyapunov's Inequality**.

**Lidskiĭ-Wielandt Inequality** See: **Weyl's Inequalities** COMMENTS (v).

**Lieb-Thiring Inequality** *If $H = -\nabla^2 + V$, the Schrödinger operator on $\mathcal{L}^2(\mathbb{R}^n)$, has negative eigenvalues $\lambda_1 \le \lambda_2 \le \cdots < 0$ , then for suitable $\gamma \ge 0$, and constants $L(\gamma, n)$,*

$$\sum_{k\ge 1}|\lambda_k|^\gamma \le L(\gamma,n)\int_{\mathbb{R}^n}(\max\{-V(x),0\})^{\gamma+n/2}\,\mathrm{d}x.$$

COMMENT If $n = 1$ we must have $\gamma \ge 1/2$, if $n = 2$ we must have $\gamma > 0$, and if $n \ge 3$ we must have $\gamma \ge 0$

REFERENCE [EM, Supp. II, pp. 11-313].

**Lipschitz Function Inequalities** *If $f$ is a real-valued Lipschitz function of order $\alpha, 0 < \alpha \le 1$, defined on a set $E$ in $\mathbb{R}^n$ then for all $\underline{u}, \underline{v} \in E$ and some constant $C$*

$$|f(\underline{u}) - f(\underline{v})| \le C|\underline{u} - \underline{v}|^\alpha.$$

COMMENTS (i) This is just the definition of a *Lipschitz function of order $\alpha$*. The lower bound of the constants $C$ is called the *Lipschitz constant of order $\alpha$ of the function.*

(ii) In many references a Lipschitz function means a Lipschitz function of order 1, and the other classes are called *Hölder functions of order* $\alpha, 0 < \alpha < 1$.

REFERENCE [EM, vol. 5, p. 532].

**Littlewood's Conjectures** *(a) If $\underline{a}$ is a non-negative sequence and if $p, q \ge 1$ then*

$$\sum_{n=1}^{\infty} a_n^p\Big(A_n\sum_{m\ge n}a_m^{1+p/q}\Big)^q \le \Big(\frac{q(2p-1)}{p}\Big)^q\sum_{n=1}^{\infty}\big(a_n^pA_n^q\big)^2.$$

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*(b) Given a positive integer $m$, and if $n_k, 1 \leq k \leq m-1$, is any strictly increasing set of positive integers, $c_k \in \mathbb{C}, |c_k| = 1, 1 \leq k \leq m-1$, then writing $z = e^{i\theta}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |1 + c_1 z^{n_1} + \cdots + c_{m-1} z^{n_{m-1}}| \, \mathrm{d}\theta \geq \frac{1}{2\pi}\int_0^{2\pi} |1 + z + \cdots + z^{m-1}| \, \mathrm{d}\theta$$

Comments (i) In (a) the cases $p = 2, q = 1$ and $p = 1, q = 2$ were conjectured in a set of problems by Littlewood. The conjecture was solved, in the general form, by Bennett.

(ii) More precisely (b) is called the *Sharp Littlewood conjecture.*

(iii) The conjecture (b) has been verified if $c_k = \pm 1$, and $n_k = k, 1 \leq k \leq m-1$.

Extensions Conjecture (b) can be extended by using the $M_p$ norm, instead of $M_1$ as above, $0 \leq p < 2$, and with the inequality reversed if $p > 2$. In the case $p = 2$ there is equality.

Comment (iv) This extension is known under the restrictions in Comment (iii) provided that $0 \leq p \leq 4$.

(v) There is a better-known Littlewood conjecture in the theory of Diophantine approximations.

References [I1, pp. 151–162]; *Bennett* [56], *Klemeš* [160].

**Littlewood-Paley Inequalities** *Let $1 < p < \infty$, $F(z) = \sum_{n\in\mathbb{N}} c_n z^n$ be analytic $D$ and let $\sum_{n\in\mathbb{N}} c_n e^{in\theta}$ be the Fourier series of $f(\theta) = F(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta})$, and suppose that $f \in \mathcal{L}^p([-\pi,\pi])$. If $\lambda > 1$ and $n_0 = 0, n_1 = 1 < n_2 < n_3 < \cdots$ is such that $n_{k+1}/n_k > \lambda$, $k = 1, 2, \ldots$ put $\Delta_0 = c_0, \Delta_k(\theta) = \sum_{m=n_{k-1}+1}^{n_k} c_m e^{im\theta}, k = 1, 2, \ldots, -\pi \leq \theta \leq \pi$, and define*

$$\gamma(\theta) = \left(\sum_{k\in\mathbb{N}} |\Delta_k(\theta)|^2\right)^{1/2}, \quad -\pi \leq \theta \leq \pi.$$

*Then*

$$A_{p,\lambda}||f||_p \leq ||\gamma||_p \leq B_{p,\lambda}||f||_p,$$

*where $A_{p,\lambda}, B_{p,\lambda}$ are constants that depend only on $p$ and $\lambda$.*

Comments This very important result in Fourier analysis has been the subject of much research. Extensions can be found in the reference and in more recent literature.

Reference *Zygmund* [Z, vol. II, pp. 222–241].

**Littlewood's Problem** *If $n_k \in \mathbb{Z}$, $1 \leq k \leq n$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{k=1}^{n} e^{in_k t}\right| \mathrm{d}t \geq \frac{4\log n}{\pi^3}.$$

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COMMENTS (i) Littlewood conjectured the size of the right-hand side in 1948, and the conjecture was settled in 1981 by McGehee, Pigno & B. Smith, and independently by Konyagin in 1981. Of course there are several problems and conjectures by Littlewood; this is sometimes called *the Littlewood Problem for Integrals*.

(ii) The value of the constant on the right-hand side was obtained by Stegeman; the best value is not known but it is conjectured that the $\pi^3$ can be replaced by $\pi^2$.

REFERENCES [EM, vol. 5, p. 534], [GI3, pp. 141–148].

**Lochs's Inequality** *If $x > 0$ then*

$$\tanh x > \sin x \cos x.$$

REFERENCE [AI, p. 270].

**Logarithmic Capacity Inequalities** If $E \subseteq \mathbb{C}$ put

$$v(E) = \inf \big\{t;\ t = \int\!\!\int \log \frac{1}{|z-w|}\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(w)\big\},$$

where the infimum is over all measures $\mu$ of total mass 1, and having as support a compact subset of $E$; we assume that $E$ is such that this collection of measures is not empty. Then, the *logarithmic capacity of $E$* is $C(E) = e^{-v(E)}$.

*If $E$ is a Borel set then*

$$C(E) \geq \sqrt{\frac{|E|}{e\pi}}.$$

COMMENT See also: **Capacity Inequalities**.

REFERENCE *Conway* [C, vol. II, pp. 331–336].

**Logarithmic Function Inequalities**

*(a) If $n \geq 1$,* $\dfrac{1}{n+1} < \log\left(1 + \frac{1}{n}\right) < \frac{1}{n}$.

*(b) If $n \geq 2$,*

$$1 + \frac{1}{12n + 1/4} - \frac{1}{12(n+1) + 1/4} < \big(n + \frac{1}{2}\big) \log\left(1 + \frac{1}{n}\right) < 1 + \frac{1}{12n} - \frac{1}{12(n+1)}.$$

*(c) If $x > 0, x \neq 1$,*

$$\log x < x - 1; \qquad \frac{\log x}{x-1} \leq \frac{1}{\sqrt{x}}; \qquad \frac{\log x}{x-1} \leq \frac{1 + \sqrt[3]{x}}{x + \sqrt[3]{x}}.$$

*(d) If $x > -1, x \neq 0$,* $\dfrac{2|x|}{2+x} < \big|\log(1+x)\big| < \frac{|x|}{\sqrt{1+x}}$.

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*(e) If $x > 0$ then*
$$xy \le x\log x + e^{y-1},$$

*with equality if and only if $y = 1 + \log x$.*

*(f) If $a, x > 0$ then*
$$\log x \le \frac{a}{e}\sqrt[a]{x}; \qquad -\log x \le \tfrac{1}{eax^a}.$$

*(g)* [Lehmer] *If $a, b, x \ge 0$ then*
$$\log\left(1+\frac{x}{a}\right)\log\left(1+\frac{b}{x}\right) < \frac{a}{b}.$$

*(h) If $z \in \mathbb{C}$ with $|z| < 1$ then*
$$\big|\log(1+z)\big| \le -\log\big(1-|z|\big),$$

*and*
$$\frac{|z|}{1+|z|} \le \big|\log(1+z)\big| \le |z|\frac{1+|z|}{|1+z|}.$$

*(j) If $a > 1$ then*
$$\frac{\log_a(n^{1/e^n})}{\log_a e} < \sum_{i=1}^n \frac{1}{i} < \frac{\log_a(ne)}{\log_a e};$$

*in particular,*
$$\frac{1}{n} + \log n < \sum_{i=1}^n \frac{1}{i} < 1 + \log n.$$

*(k)* [Wang C. L.] *If $x \ne 1$ then*
$$\log x < n(x^{1/n} - 1) < x^{1/n}\log x,\ n = 1, 2, \cdots.$$

Comments (i) The inequalities in (b) are of importance in the deduction of **Stirling's Formula**.

(ii) For (c) just note that $1 - x + \log x$ has a unique maximum at $x = 1$; or use the strict concavity of the logarithmic function at $x = 1$. For another use of the strict concavity of the logarithmic function see **Exponential Function Inequalities** Comments (iii).

(iii) (d) follows by considering the differences between the center term and the two outside terms, or from special cases of the **Logarithmic Mean Inequalities** (1), namely:
$$\mathfrak{L}_{-2}(a,b) < \mathfrak{L}_{-1}(a,b) < \mathfrak{L}_0(a,b) < \mathfrak{L}_1(a,b).$$

(iv) (e) can be deduced from **Young's Inequalities** (b); alternatively let $p \to 0$ in **Geometric- Arithmetic Mean Inequality** Extensions ($\ell$); see also **Conjugate Convex Function Inequalities** Comments (i).

(v) Inequalities (f) are of importance for large values of $a$.

(vi) The left-hand inequality of (k) generalizes the first inequality in (c).

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(vii) See also: **Euler's Constant Inequalities** (A), **Logarithmic Mean Inequalities**, **Napier's Inequality**, **Schlömilch-Lemonnier Inequality**, **Series Inequalities** (C).

References [AI, pp. 49, 181–182, 266, 272–273, 326–328], [H, pp. 6–11, 167], [HLP, pp. 61, 107], [MPF, p. 79]; *Abramowicz & Stegun* [AS, p. 68].

## Logarithmic and Generalized Logarithmic Mean Inequalities

*(a) If $0 < a \le b$ and if $-\infty < r < s < \infty$ then*

$$a \le \mathfrak{L}^{[r]}(a,b) \le \mathfrak{L}^{[s]}(a,b) \le b, \tag{1}$$

*with equality if and only if $a = b$.*

*(b)* [Sándor]

$$\frac{3}{\mathfrak{L}(a,b)} < \frac{1}{\mathfrak{G}_2(a,b)} + \frac{2}{\mathfrak{H}_2(a,b)}.$$

*(c)* [Sándor]

$$\mathfrak{I}(a,b) > \mathfrak{G}_2^2(a,b)\exp\left(\mathfrak{G}_2(\log a, \log b)\right).$$

*(d)* [Lin T. P.]

$$\mathfrak{G}_2(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{M}_2^{[1/3]}(a,b). \tag{2}$$

*(e)* [Alzer] *If $0 < m \le a \le b \le M$ and if $r \le s$ then*

$$\frac{\mathfrak{L}_r(m,M)}{\mathfrak{L}_s(m,M)} \le \min_{a,b} \frac{\mathfrak{L}_r(a,b)}{\mathfrak{L}_s(a,b)} \le \max_{a,b} \frac{\mathfrak{L}_r(a,b)}{\mathfrak{L}_s(a,b)} \le 1.$$

Comments (i) Inequality (1) follows from the integral analogue of (r;s).

(ii) Lin's result is a particular case of **Pittenger's Inequalities**, and should be compared with **Heronian Mean Inequalities** (1).

(iii) The right-hand of Alzer's result is an immediate corollary of (1).

Corollaries *(a) If $0 < a \le b$, and if $-2 < r < -1/2$ then*

$$\mathfrak{G}_2(a,b) \le \mathfrak{L}^{[r]}(a,b) \le \mathfrak{M}_2^{[1/2]}(a,b),$$

*with equality only if $a = b$.*

*(b) If $0 < a \le b$, and if $-2 < r < 1$ then*

$$\mathfrak{G}_2(a,b) \le \mathfrak{L}^{[r]}(a,b) \le \mathfrak{A}_2(a,b),$$

*with equality only if $a = b$.*

*(c)* $\mathfrak{H}_2(a,b) \le \frac{1}{\mathfrak{I}(a,b)} \le \frac{1}{\mathfrak{L}(a,b)} \le \mathfrak{G}_2(a,b) \le \mathfrak{L}(a,b) \le \mathfrak{I}(a,b) \le \mathfrak{A}_2(a,b)$.

Comments (iv) Inequalities (a), (b) result from (1) and the simple identities between the various means.

(v) The last three inequalities in (c) are particular cases of (1), and the whole set follows on letting $n \to \infty$ in **Nanjundiah Inequalities** (1).

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(vi) An interesing extension of (2) has been given by Hästö.

EXTENSIONS [ALZER] *(a) If* $0 < a < b$ *then*

$$\sqrt{\mathfrak{A}_2(a,b)\mathfrak{G}_2(a,b)} < \sqrt{\mathfrak{L}(a,b)\mathfrak{I}(a,b)} < \mathfrak{M}_2^{[2]}(a,b);$$
$$\mathfrak{L}(a,b) + \mathfrak{I}(a,b) < \mathfrak{A}_2(a,b) + \mathfrak{G}_2(a,b)$$
$$\sqrt{\mathfrak{G}_2(a,b)\mathfrak{I}(a,b)} < \mathfrak{L}(a,b) < \frac{1}{2}\left[\mathfrak{G}_2(a,b) + \mathfrak{I}(a,b)\right];$$
$$\mathfrak{G}_2(a,b) < \sqrt{\mathfrak{L}^{[-r-1]}(a,b)\mathfrak{L}^{[r-1]}(a,b)} < \mathfrak{I}(a,b), \quad r \neq 0.$$

*(b) If* $0 < a, b \leq 1/2, a \neq b$ *then*

$$\frac{\mathfrak{G}_2(a,b)}{\mathfrak{G}_2(1-a,1-b)} \leq \frac{\mathfrak{L}(a,b)}{\mathfrak{L}(1-a,1-b)} \leq \frac{\mathfrak{M}_2^{[1/3]}(a,b)}{\mathfrak{M}_2^{[1/3]}(1-a,1-b)}.$$

COMMENTS (vi) Alzer's result (b) is a Fan extension of (2).

(vii) Pittenger and Neuman have defined the logarithmic mean of general n-tuples.

(viii) See also **Arithmetico-Geometric Mean Inequalities, Muirhead Symmetric Function and Mean Inequalities** (d), **Rado's Inequality** EXTENSIONS, **Stolarsky Mean Inequalities**.

REFERENCES [AI, p. 273], [H, pp. 167–170],[MPF, pp. 40–44]; *Alzer* [27], *Hästö* [140], *Neuman* [232], *Páles* [251], *Pittenger* [270], *Sándor* [286].

**Logarithmic Sobolev Inequalities** *(a) If* $f \in \mathcal{C}([-\pi,\pi])$ *with* $f' \in \mathcal{L}^2([-\pi,\pi])$, $\int_{-\pi}^{\pi} f'^2 \geq 0$, *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2 \log f^2 \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2\right)\log\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2\right) + \frac{1}{\pi}\int_{-\pi}^{\pi} f'^2.$$

*(b)* [GROSS] *If* $f : \mathbb{R} \to \mathbb{C}$ *and* $\int_{-\infty}^{\infty} |f'(t)|^2 \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t < \infty$, *then*

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2 \log|f(t)|^2 \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t \\ \leq \frac{1}{2}\left(\int_{-\infty}^{\infty} |f(t)|^2 \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t\right)\log\left(\int_{-\infty}^{\infty} |f(t)|^2 \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t\right) \\ + \int_{-\infty}^{\infty} |f'(t)|^2 \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t.\end{aligned}$$

COMMENTS (i) There are many extensions of these results; in particular Gross has extended his result to higher dimensions.

(ii) See also **Sobolev's Inequalities**.

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REFERENCES [EM, Supp., p. 307]; *Émery & Yukich* [112], *Gross* [135], *Pearson* [260].

**Log-convex Function Inequalities** *(a) If $f$ is log-convex and if $0 \le \lambda \le 1$ then*

$$f\big((1-\lambda)x + \lambda y\big) \le f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1}$$

*(b) The necessary and sufficient condition that*

$$f\big(\mathfrak{A}_n(\underline{a};\underline{w})\big) \le \mathfrak{G}_n\big(f(\underline{a});\underline{w}\big), \tag{2}$$

*is that $f$ be log-convex.*

COMMENTS (i) The inequality (1) is just the definition; a function $f$ is *log-convex* precisely when $\log \circ f$ is convex. Such functions are sometimes called *multiplicatively convex*, because of the form of inequality (1).

(ii) A log-convex function is convex. An interesting example of a log-convex function is $x!, x > -1$.

(iii) Inequality (2) is a particular case of **Quasi-arithmetic Mean Inequalities** (c). In turn a particular case of (2) is **Shannon's Inequality** (1).

(iv) Many inequalities say that a certain function is log-convex; see for instance **Elementary Symmetric Function Inequalities** EXTENSIONS (A), **Hadamard's Three Circles Theorem, Hardy's Analytic Function Inequality, Lyapunov's Inequality** COMMENTS (III), **Phragmen-Lindelöf's Inequality, Riesz-Thorin Theorem** COMMENTS (II), **Series' Inequalities** COMMENTS (II), **Trigonometric Function Inequalities** COMMENTS (IX).

REFERENCES [H, p. 48–49], [MPF, p. 3], [PPT, pp. 294–295]; *Roberts & Varberg* [RV, pp. 18–19].

**Log-convex Sequence Inequalities** *If $\alpha \ge 0$ and if the real sequence $\underline{a}$ is $\alpha$-log-convex then*

$$a_n^2 \le \frac{(n+1)(\alpha+n-1)}{n(\alpha+n)} a_{n+1}a_{n-1}, \quad n \ge 2. \tag{2}$$

COMMENTS (i) This is just the definition of an *$\alpha$-log-convex sequence*. When $\alpha = 1$ we say *log-convex* instead of 1-log-convex. Letting $\alpha \to \infty$ in (2) gives definition of the case $\alpha = \infty$, such a sequence is said to be *weakly log-convex*.

(ii) If instead we have the inequality ($\sim$2) we get the definitions of *$\alpha$-log-concave, log-concave*, and in the case $\alpha = \infty$, *strongly log-concave*.

(iii) The sequence $(-1)^n\binom{-\alpha}{n}$, $n \ge 1$ is $\alpha'$-log-convex, (-concave), for all $\alpha' \ge \alpha, (\le \alpha)$.

The sequence $1/n$, $n \ge 1$ is $\alpha$-log-convex if $\alpha \ge 0$. The sequence $1/n!, n \ge 1$ is $\alpha$-log-concave if $\alpha \le \infty$.

(iv) See also: **Elementary Symmetric Function Inequalities** EXTENSIONS (F), **Complete Symmetric Function Inequalities** (C).

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References [H, pp. 16–21], [PPT, pp. 288–292].

**Loomis-Whitney Inequality** *Let $A$ be a finite set of points in $\mathbb{R}^3$ and let $A_1, A_2, A_3$ denote the projections of this set on the coordinate planes then*

$$\mathrm{card}(A) \leq \prod_{i=1}^{3} \mathrm{card}(A_i)^{1/2},$$

*where* card$(A)$*, etc., denotes the cardinality of the set $A$, etc.*

Comment This inequality provides a bound on the number of elements in a set in terms of the numbers of elements in its three projections.

Reference *Steele* [S, pp. 16–17, 230–231].

**Lorentz Inequality** See: **Aczél's Inequality**.

**Love-Young Inequalities**[29] *(a) If $\underline{a}, \underline{b}$ are complex $n$-tuples and if $p, q > 0$ then for some $k$,$1 \leq k \leq n$,*

$$|a_k b_k| \leq \mathfrak{M}_n^{[p]}(\underline{a})\mathfrak{M}_n^{[q]}(\underline{b}).$$

*(b) Hypotheses: $\underline{x} = (x_1, \ldots, x_m)$ is defined from the $n$-tuple $\underline{a} = (a_1, \ldots, a_n)$, $m \leq n$, by replacing certain of the consecutive commas, ',' , by plus signs, '+'; and the $m$-tuple $\underline{y}$ is obtained in the same way from the $n$-tuple $\underline{b}$;*

$$S_{p,q}(\underline{a}, \underline{b}) = \max \left\{ (\sum_{i=1}^{m} |x_k|^p)^{1/p},\ (\sum_{i=1}^{m} |y_k|^q)^{1/q}) \right\},$$

*where the maximum is over all possible choices of $\underline{x}, \underline{y}$; $p, q > 0$; $1/p + 1/q > 1$, Conclusions:*

$$\left| \sum_{1 \leq r \leq s \leq n} a_r b_s \right| \leq \left\{ 1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right) \right\} S_{p,q}(\underline{a}, \underline{b}).$$

Comments (i) (a) can be used to prove (H).

(ii) (b) is of importance in certain work with Stieltjes integral; $\zeta$ denotes the Zeta function.

Reference *Young* [331].

**Lower and Upper Limit Inequalities** See: **Upper and Lower Limit Inequalities**.

**Lower Semi-continuous Function Inequalities** See: **Semi-continuous Function Inequalities**.

[29]This is L. C. Young. Only used for this reference; all other occurrences are W. H. Young.

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**Löwner-Heinz Inequality** *If $A, B$ are positive bounded linear operators on the Hilbert space $X$ and if $0 \le \alpha \le 1$ then*

$$A \ge B \implies A^\alpha \ge B^\alpha.$$

COMMENTS (i) The definition of a positive operator is given in **Heinz Inequality** COMMENTS (i).

(ii) This result is based on the fact that $x^\alpha, 0 \le \alpha \le 1$ is an operator monotone function; see **Monotone Matrix Function Inequalities** COMMENTS (ii).

(iii) A beautiful extension to this inequality can be found in **Furuta's Inequality**.

(iv) This inequality is equivalent to the Heinz-Kato Inequality, see **Heinz-Kato-Furuta Inequality** (1); see also **Cordes's Inequality**. Many equivalent forms of this inequality have been given by Furuta in the reference.

REFERENCES [EM, Supp. p. 359], [GI7, pp. 65–76]; *Furuta* [Fu, pp. 127–128, 226–229]; *Zhan* [334], *Furuta* [119].

**Lupaş & Mitrović Inequality**[30] See: **Geometric-Arithmetic Mean Inequality**, EXTENSIONS (b).

**Lyapunov's Inequality** *If $0 < t < s < r$ then*

$$\Big(\sum_{i=1}^n w_i a_i^s\Big)^{r-t} \le \Big(\sum_{i=1}^n w_i a_i^t\Big)^{r-s}\Big(\sum_{i=1}^n w_i a_i^r\Big)^{s-t}.$$

COMMENTS (i) This is **Hölder's Inequality** (2) with $p = (r-t)/(r-s), q = (r-t)/(s-t)$ and $\underline{a}^p, \underline{b}^q$ replaced by $\underline{a}^t, \underline{a}^r$, respectively.

(ii) Lyapounov's inequality can be written in terms of the Gini means

$$\mathfrak{B}_n^{s,t}(\underline{a};\underline{w}) \le \mathfrak{B}_n^{r,t}(\underline{a};\underline{w});$$

and so is a particular case of **Gini-Dresher Mean Inequalities** (2).

(iii) Liapunov's inequality is just the statement that a certain function is log-convex; see: [MOA].

AN INTEGRAL ANALOGUE *If $0 < t < s < r$ and $f \in \mathcal{L}^r([a,b])$ then*

$$\left(\int_a^b |f|^s\right)^{r-t} \le \left(\int_a^b |f|^t\right)^{r-s}\left(\int_a^b |f|^r\right)^{s-t},$$

*with equality if and only if either $f$ is constant on some measurable subset and zero on its complement, or $r = s$, or $s = t$, or $t(2r-s) = rs$.*

COMMENT (iv) This last can be extended to more general measure spaces.

[30]This is Žarko M. Mitrović.

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REFERENCES [AI, p. 54], [H, pp. 181–182], [MOA, pp. 107, 659–660], [MPF, p. 101].

**Lyons's Inequality** *If* $0 < \alpha \le 1$, $x, y > 0$ *then*

$$\alpha \sum_{k=0}^{n} \binom{\alpha n}{\alpha k} x^{\alpha k} y^{\alpha(n-k)} \le \frac{1}{\alpha}(x+y)^{\alpha n}.$$

COMMENTS (i) Lyons conjectures that the factor $1/\alpha$ on the right-hand side can be replaced by 1; this has been proved by Love in the cases $\alpha = 2m^{-1}$, $m = 1, 2, \ldots$.

(ii) For another inequality of Lyons see **Binomial Function Inequalities** $(\ell)$.

REFERENCE *Love* [186].

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# 13 Mahajan–Myers

**Mahajan's Inequality** *If $J_\nu$ is the Bessel function of the first kind then*

$$(x+1)^{\nu+1}J_\nu\left(\frac{\pi}{x+1}\right) - x^{\nu+1}J_\nu\left(\frac{\pi}{x}\right) > \left(\frac{\pi}{2}\right)^\nu \frac{1}{\nu!}.$$

References *Lorch & Muldoon* [180], *Mahajan* [190].

**Mahler's Inequalities** *If $K \subset \mathbb{R}^n$ is convex and compact, with $\mathring{K} \neq \emptyset$ and centroid the origin, then*

$$|K|\,|K^*| \geq \frac{(n+1)^{n+1}}{(n!)^2},$$

*with equality if and only if $K$ is the simplex with centroid the origin.*
*If $K$ is symmetric with respect to the origin, then*

$$|K|\,|K^*| \geq \frac{4^n}{n!},$$

*with equality if and only if $K$ is the $n$-cube $[-1,1]^n$.*

Comments (i) For the notation $K^*$ see: **Blaschke-Santaló Inequality** Comments (ii).

(ii) These inequalities are only known in the case $n = 2$, they are conjectured for the other cases. The special case when $K$ is a zonoid is *Reisner's Inequality*.

(iii) See also: **Minkowski-Mahler Inequality**.

References [EM, Supp., pp. 129–130].

**Marchaud's Inequality** If $f : \mathbb{R} \to \mathbb{R}$ is continuous and of period $2\pi$ the *$r$-th modulus of continuity* is

$$\omega_r(\delta; f) = \sup_{|u|<\ \delta}\left\{\left|\left|\sum_{k=0}^{r}\binom{r}{k}(-1)^k f(x+ku)\right|\right|_{\infty,[-\pi,\pi]}\right\}.$$

*(a) If $r \leq s$ then*

$$\omega_s(\delta; f) \leq 2^{s-r}\omega_r(\delta; f). \tag{1}$$



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*(b)* [Marchaud] *Under the same hypotheses as in (a) there is a constant depending only on $r, s$ such that*

$$\omega_r(\delta; f) \leq C_{r,s}\delta^r \int_\delta^1 \omega_s(y; f) y^{-(r+1)}\, \mathrm{d}y.$$

Comments (i) Inequality (1) is an easy consequence of the definition.
(ii) See also: **Modulus of Continuity Inequalities**.

References [GI4, pp. 221–238]; *Timan* [Ti, p. 104].

**Marcus & Lopes Inequality**[31] *(a) If $r, s$ are integers, $1 \leq r \leq s \leq n$, then*

$$\left(\frac{e_n^{[s]}(\underline{a}+\underline{b})}{e_n^{[s-r]}(\underline{a}+\underline{b})}\right)^{1/r} \geq \left(\frac{e_n^{[s]}(\underline{a})}{e_n^{[s-r]}(\underline{a})}\right)^{1/r} + \left(\frac{e_n^{[s]}(\underline{b})}{e_n^{[s-r]}(\underline{b})}\right)^{1/r},$$

*with equality if and only if either $\underline{a} \sim \underline{b}$, or $r = s-1$.*
*(b) If $r$ is an integer and $1 \leq r \leq n$ then*

$$\mathfrak{P}_n^{[r]}(\underline{a}+\underline{b}) \geq \mathfrak{P}_n^{[r]}(\underline{a}) + \mathfrak{P}_n^{[r]}(\underline{b}),$$

*with equality if and only if either $\underline{a} \sim \underline{b}$, or $r = 1$.*

Comments (i) (b) is just the case $s = r$ of (a).
(ii) For extensions see: **Complete Symmetric Function Inequalities** (b), Comments (ii), **Complete Symmetric Mean Inequalities** (c), **Muirhead Symmetric Function and Mean Inequalities** Extensions (b), **Whiteley Mean Inequalities** (d).

References [AI, pp. 102–104], [BB, pp. 33–35], [H, pp. 338–341], [MOA, pp. 116–117], [MPF, pp. 163–164], [PPT, p. 303].

**Markov's Inequality**[32] *If $p_n$ is a polynomial of degree at most $n$ then*

$$||p_n'||_{\infty,[a,b]} \leq \frac{2n^2}{b-a}||p_n||_{\infty,[a,b]}.$$

Comments (i) This inequality is best possible as is shown by $T_n$, the Čebišev polynomial of degree $n$. For a definition see: **Čebišev Polynomial Inequalities**.
(ii) The inequality implied for higher derivatives is not best possible; a best possible result is given in (b) below.

Extensions *(a)* [Duffin & Schaeffer] *If $p_n$ is a polynomial of degree at most $n$ on $[-1, 1]$. And if $|p_n(\cos(k\pi/n)| \leq 1, 0 \leq k \leq n$, then*

$$||p_n'||_{\infty,[-1,1]} \leq n^2.$$

[31] This is Marvin D. Marcus.
[32] This is A. A. Markov.

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*(b)* [V.A. Markov] *If $p_n$ is a polynomial of degree at most $n$ on $[-1,1]$. And if $|p_n(x)| \le 1$ then*

$$||p_n^{(k)}||_{\infty,[-1,1]} \le T_n^{(k)}(1) = \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-\overline{k-1}^2)}{(2k-1)!!}, \quad k \ge 1.$$

Comments (iv) The notation is defined in **Factorial Function Inequalities** (f).
(v) See also: **Bernšteĭn's Inequality** Comments (iv), **Erdős's Inequality**.

References [EM, vol. 6, p. 100; Supp., pp. 365–366], [GI6, pp. 161–174]; *Milovanović, Mitrinović & Rassias* [MMR, pp. 527–623].

## Markov's Probability Inequality[33]

*If $X$ is a random variable and $k > 0$ then*

$$P(X \ge r) \le \frac{E|X|^k}{r^k}. \tag{1}$$

Comments (i) (1) is a simple deduction from **Probability Inequalities**; just take $f(x) = |x|^k$. The same result also gives an inverse inequality for (1);

$$P(X \ge r) \ge \frac{E|X|^k - r^k}{||X^k||_{\infty,P}}.$$

(ii) The case $k = 2$ of (1) is **Čebišev's Probability Inequality**.

References [EM, vol. 2, pp. 119–120]; *Loève* [L, p. 158].

## Maroni's Inequality

See: **Opial's Inequality** Comment (i).

## Martin's Inequalities

*If $P_n$ is a Legendre polynomial then if $0 \le \theta \le \pi$,*

$$\begin{aligned} \sqrt{\sin\theta}P_n(\cos\theta) &\le \frac{\sqrt{2}}{\sqrt{\pi(n+1/2)}}; \\ P_n(\cos\theta) &\le \frac{1}{\sqrt[4]{1+n(n+1)\sin^2\theta}}. \end{aligned}$$

Comments (i) These have been generalized by Common.
(ii). See: **Minc-Sathre Inequality** Comment (i).

References *Common* [94].

## Martingale Inequalities

*(a)* [Doob's Inequality] *If $\mathcal{X} = (X_n, \mathcal{F}_n, n \in \mathbb{N})$ is a non-negative sub-martingale in $(\Omega, \mathcal{F}, P)$, $X_n^* = \max_{1\le i\le n} X_i, n \in \mathbb{N}$, then, writing $||\cdot||_p = \big(E|\cdot|^p\big)^{1/p}$,*

$$P\{X_n^* \ge r\} \le \frac{EX_n}{r}; \tag{1}$$

[33]This is A. A. Markov.

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*and*

$$||X_n||_p \leq \begin{cases} ||X_n^*||_p \leq \dfrac{p}{p-1}||X_n||_p, & \textit{if } p > 1, \\ \\ \dfrac{e}{e-1}\left[1 + ||X_n \log^+ X_n||\right], & \textit{if } p = 1. \end{cases}$$

*(b)* [Burkholder's Inequalities] *If* $\mathcal{X} = (X_n, \mathcal{F}_n, n \in \mathbb{N})$ *is a martingale and if* $p > 1$ *then*

$$\frac{p-1}{18p^{3/2}}\left|\left|\sqrt{\{X\}_n}\right|\right|_p \leq ||X_n||_p \leq \frac{18p^{3/2}}{\sqrt{p-1}}\left|\left|\sqrt{\{X\}_n}\right|\right|_p,$$

*where*

$$\{X\}_n = \sum_{i=1}^n (\Delta X_i)^2, \textit{ and } X_0 = 0.$$

Comments (i) Inequality (1) is just **Kolmogorov's Probability Inequality** extended to this situation.

(ii) Using the above results we can deduce that if $p > 1$,

$$\frac{p-1}{18p^{3/2}}\left|\left|\sqrt{\{X\}_n}\right|\right|_p \leq ||X_n^*||_p \leq \frac{18p^{3/2}}{\sqrt{(p-1)^3}}\left|\left|\sqrt{\{X\}_n}\right|\right|_p.$$

(iii) The inequality in (ii) has been extended to the cases $p = 1, \infty$ by Davis, and Gundy; as a result the inequality is known as the *Burkholder-Davis-Gundy Inequality*.[34]

(iv) See also: **Doob's Upcrossing Inequality**.

References [EM, vol. 6, pp. 109–110; Supp., p. 163]; *Tong ed.*[T, pp. 78–83]; *Feller* [F, vol. II, pp. 241–242], *Loève* [L, pp. 530–535].

**Mason-Stothers Inequality**[35] *If* $f, g, h = f + g$ *are relatively prime polynomials of degrees* $\lambda, \mu, \nu$*, respectively, and if* $k$ *is the number of distinct roots of* $f, g, h$ *then*

$$\max\{\lambda, \mu, \nu\} \leq k - 1.$$

Comment This result can be used to give a simple proof of Fermat's Last Theorem for polynomials.

Reference *Lang* [Ls].

**Mathieu's Inequality** *If* $c \neq 0$ *then*

$$\frac{1}{c^2 + 1/2} < \sum_{i=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} < \frac{1}{c^2}.$$

[34] This is B. Davis.
[35] This is R. Mason.

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COMMENT The inequality on the right was conjectured by Mathieu, and proved 60 years later by Berg.

EXTENSIONS [EMERSLEBEN, WANG & WANG][36] *If $c \neq 0$ then*

$$\frac{1}{c^2} - \frac{5}{16c^4} < \sum_{i=1}^{\infty} \frac{2n}{(n^2+c^2)^2} = \frac{1}{c^2} - \frac{1}{16c^4} + O\left(\frac{1}{c^6}\right).$$

REFERENCES [AI, pp. 360–362], [MPF, pp. 629–634]; *Russell, D.* [285].

**Matrix Inequalities** *(a)* [MARSHALL & OLKIN] *If $A$ is a $m \times n$ complex matrix, $B$ a $p \times q$ complex matrix of rank $p$ then*

$$AA^* \geq AB^* \left(BB^*\right)^{-1} BA^*,$$

*where by $C \geq D$ we mean that $C - D$ is positive definite.*
*(b) If the eigenvalues of the $n \times n$ complex matrix $A$ are all positive and lie in the interval $[m, M]$ and if $B$ is a $p \times n$ complex matrix with $BB^* = I$ then, with the same notation as in (i),*

$$BA^{-1}B^* \leq \frac{M+m}{Mm} I - \frac{BAB^*}{Mm} \leq \frac{(M+m)^2}{4Mm} \left(BAB^*\right)^{-1}.$$

*(c)* [SAGAE & TANABE] *If $\underline{C}$ is an $n$-tuple of $m \times m$ positive definite matrices and $\underline{w}$ is a positive $n$-tuple with $W_n = 1$ then*

$$\mathfrak{H}_n(\underline{C}; \underline{w}) \leq \mathfrak{G}_n(\underline{C}; \underline{w}) \leq \mathfrak{A}_n(\underline{C}; \underline{w}),$$

*with equality if and only if $C_1 = \cdots = C_n$.*
*(d) If $A, B$ are two $n \times n$ Hermitian positive definite matrices and if $0 \leq \lambda \leq 1$, then*

$$\left(\lambda A + (1-\lambda)B\right)^{-1} \leq \lambda A^{-1} + (1-\lambda)B^{-1}.$$

COMMENTS (i) The first inequality of Marshall & Olkin is a matrix analogue of (C), the second a matrix analogue of **Kantorovićs Inequality**; for another form see: *Furuta* [Fu].

(ii) The definition of the arithmetic mean, the harmonic mean, in fact every power mean except the geometric mean, is obvious; care must be taken in the case of the geometric mean as a product of positive definite matrices need not be positive definite: so we define $\mathfrak{G}_n(\underline{C}; \underline{w})$ as

$$C_n^{1/2}\Big(C_n^{-1/2}C_{n-1}^{1/2}\cdots \cdots\left(C_3^{-1/2}C_2^{1/2}\left(C_2^{-1/2}C_1C_2\right)^{v_1}C_2^{1/2}C_3^{-1/2}\right)^{v_2}\cdots \cdots C_{n-1}^{1/2}C_n^{-1/2}\Big)^{v_{n-1}} C_n^{-1/2},$$

[36]This is Wang Chung Lie & Wang Xing Hua.

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where $v_k = 1 - \dfrac{w_{k+1}}{W_{k+1}}, 1 \le k \le n-1$. Of course (c) is just a matrix analogue of (GA).

(iii) See also: **Circulant Matrix Inequalities, Convex Matrix Function Inequalities, Determinant Inequalities, Eigenvalue Inequalities, Gram Determinant Inequalities, Grunsky's Inequalities** COMMENTS (III), **Hadamard's Determinant Inequality, Hadamard Product Inequalities, Halmos's Inequality, Hirsch's Inequalities, Matrix Norm Inequalities, Milne's Inequality** COMMENT (IV), **Monotone Matrix Function Inequalities, Permanent Inequalities, Rank Inequalities, Rayleigh-Ritz Ratio, Trace Inequalities, Weyl's Inequalities**.

REFERENCES [GI7, pp. 77–91], [MPF, pp. 96, 225]; *Furuta* [Fu, pp. 188–197]; *Zhan* [334]; *Furuta & Yanagida* [121], *Lawson & Lim* [170].

## Matrix Norm Inequalities

If we write $C_n$ for the complex vector space of $n \times n$ complex matrices then a function $||\cdot|| : C_n \to \mathbb{R}$ is called a *matrix norm* if it is a norm in the sense of **Norm Inequalities** COMMENTS (i), and if in addition $A, B \in C_n$ then $||AB|| \le ||A||\,||B||$; see **Banach Algebra Inequalities**.

If $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ is an $m \times n$ complex matrix and if $1 \le p < \infty$, we write

$$|A|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p}.$$

[OSTROWSKI] *If $A, B$ are two complex matrices such that $AB$ exists, and if $1 \le p \le 2$, then*

$$|AB|_p \le |A|_p|B|_p.$$

*In particular $|\cdot|_p, 1 \le p \le 2$, is a matrix norm on $C_n$.*

COMMENTS (i) This result is in general false if $p \ge 2$; consider $p = \infty$ and $A = B \in C_2$, matrices all of whose entries are equal to 1.

(ii) The case $p = 2$ is called the *Euclidean, Fröbenius, Schur or Hilbert-Schmidt norm*; it is equal to $\sqrt{tr(AA^*)}$.

EXTENSIONS [GOLDBERG & STRAUS] *If $A$ is a $m \times k$ complex matrix, and $B$ a $k \times n$ complex matrix, and $p \ge 2$, then*

$$|AB|_p \le k^{(p-2)/p}|A|_p|B|_p.$$

COMMENTS (iii) These results have been further extended by Goldberg to allow different values of $p$ for $A$ and $B$.

REFERENCES [GI4, pp. 185–189], [HLP, p. 36]; *Horn & Johnson* [HJ, pp. 290–320], *Marcus & Minc* [MM, p. 18].

## Max and Min Inequalities

See: **Inf and Sup Inequalities**.

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**Maximal Function Inequalities** See: **Hardy-Littlewood Maximal Inequalities**.

**Maximin Theorem** See: **Minimax Theorems**.

**Maximum-Modulus Principle** *If $f$ is analytic, not constant, in a closed domain $\Omega$ that has a simple closed boundary then*

$$|f(z)| \leq M,\ z \in \partial\Omega \quad \Longrightarrow \quad |f(z)| < M,\ z \in \overset{\circ}{\Omega}\,.$$

COMMENTS (i) This result can be stated as:

*if $f$ is analytic and not constant in a region $\Omega$ then $|f(z)|$ has no maximum in $\Omega$.*

A similar result holds for harmonic and subharmonic functions, and is then called the *Maximum Principle*. The result has been extended to solutions of certain types of differential equations. See: **Harnack's Inequalities** COMMENT, [PW].

(ii) This result is exceedingly important in the study of analytic functions and many basic results such follow from it; for instance, those in **Hadamard's Three Circles Theorem, Harnack's Inequalities, Phragmén-Lindelöf Inequality**.

REFERENCES [EM, vol. 6, pp. 174–175]; *Conway* [C, vol. I, pp. 124–125, 255–257], *Mitrović & Žubrinić* [MZ, pp. 233–238], *Pólya & Szegö* [PS, pp. 131, 157–160, 165–166], *Protter & Weinberger* [PW, pp. 1–9, 61–67, 72–75], *Titchmarsh* [T75, pp. 165–187].

**Maximum Principle for Continuous Functions** *If $K$ is a compact set and $f$ is continuous real-valued on $K$ then for some $c, d \in K$*

$$f(d) \leq f(x) \leq f(c), \qquad x \in K.$$

COMMENTS (i) This is a basic property of continuous functions.

(ii) A more elementary maximum principle is discussed by Pan.

REFERENCES *Apostol* [A67, vol. I, p. 377], *Hewitt & Stromberg* [HS, p. 74], *Rudin* [R76, p. 77]; *Pan* [253].

**Maximum Principle for Harmonic Functions** See: **Maximum-Modulus Principle**, COMMENTS (i).

**Mean Inequalities** *If $\mathfrak{M}(\underline{a})$, a continuous function on $n$-tuples, is a mean then*

$$\min \underline{a} \leq \mathfrak{M}(\underline{a}) \leq \max \underline{a}\,, \tag{1}$$

*with equality only if $\underline{a}$ is constant.*

COMMENTS (i) This is just the definition of a mean, or what is sometimes called a *strict mean*.

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(ii) In **Notations 3** there are several examples of means, and references to others.
(iii) Most means have the properties:

$$\textit{homogeneity:}\qquad \mathfrak{M}(\lambda\underline{a}) = \lambda\mathfrak{M}(\underline{a}); \tag{2}$$
$$\textit{monotonicity:}\qquad \underline{a} \le \underline{b} \Longrightarrow \mathfrak{M}(\underline{a}) \le \mathfrak{M}(\underline{b}), \text{ with equality } \iff \underline{a} = \underline{b}. \tag{3}$$

Extensions *If a mean has the properties (1)–(3) mentioned above then,*

$$\min\left(\underline{a}\,\underline{b}^{-1}\right) \le \frac{\mathfrak{M}(\underline{a})}{\mathfrak{M}(\underline{b})} \le \max\left(\underline{a}\,\underline{b}^{-1}\right).$$

Comment (iv) See also: **Integral Mean Value Theorems**.

References [H, pp. 62,63,435–438]; *Borwein & Borwein* [Bs, pp. 230–232].

**Mean Monotonic Sequence Inequalities** Let us say that *the sequence $\underline{a}$ is mean monotonic, or monotonic in the mean*, if the sequence of arithmetic means, $\{\mathfrak{A}_1(\underline{a};\underline{w}), \mathfrak{A}_2(\underline{a};\underline{w}), \ldots\}$, is monotonic; in particular we can define *increasing, decreasing, in mean.*

[Burkill & Mirsky][37] *If the $k$ non-negative sequences $\underline{a}_i, 1 \le i \le k$, are mean monotonic in the same sense then*

$$\prod_{i=1}^{k} \mathfrak{A}_n(\underline{a}_i, \underline{w}) \le \mathfrak{A}\left(\prod_{i=1}^{k} \underline{a}_i; \underline{w}\right).$$

Comment In the case $k = 2$ the sequences can be taken to be real. This should be compared to (Č) and **Čebišev's Inequality** (2).

References [H, p. 165], [MPF, pp. 265, 271], [PPT, p. 199].

**Mean Value Theorem of Differential Calculus** *If the continuous function $f : [a,b] \to \mathbb{R}^n$ is differentiable on $]a,b[$ then for some $c, a < c < b$*

$$|f(b) - f(a)| \le (b-a)|f'(c)|;$$

*in particular*

$$\frac{|f(b) - f(a)|}{|b-a|} \le \sup_{a<x<b} |f'(x)|.$$

Comments (i) For the case $n = 1$ the inequalities above can be replaced by

$$f(b) - f(a) = (b-a)f'(c); \tag{1}$$

such a $c$ is called a *mean value point of $f$ on $[a,b]$*.

[37] This is H. Burkill.

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(ii) It is an easy deduction from (1) that if $a \le x < y \le b$,

$$\inf_{a<c<b} f'(c) \le \frac{f(y)-f(x)}{y-x} \le \sup_{a<c<b} f'(c).$$

In fact the bounds of $f'(x)$ and $\big(f(y)-f(x)\big)/(y-x)$ are the same.

(iii) This result can be extended to unilateral derivatives, and to more general situations.

(iv) For an application of (1) see: **Napier's Inequality**.

Extensions [Garsia, Fichera, Sneider] *If* $f' \in \mathcal{L}^p([0,1]), p \ge 1$, *then*

$$\int_0^1 \int_0^1 \left|\frac{f(x)-f(y)}{y-x}\right|^p \mathrm{d}x\,\mathrm{d}y \le C_p \int_0^1 |f'|^p,$$

*where* $C_p \le C_1 = \log 4$.

Comment Ostrowski posed the problem of finding the smallest value of $C_p$.

References [GI1, p. 319]; *Bourbaki* [B49, pp. 18–31], *Mitrović & Žubrinić* [MZ, p. 352], *Rudin* [R76, pp. 92–93, 99].

**Measure Inequalities** *(a) If* $\mu$ *is a measure on* $X$ *and if* $A \subseteq \bigcup_{n\in\mathbb{N}} B_n \subseteq X$ *then*

$$\mu A \le \sum_{n\in\mathbb{N}} \mu B_n. \tag{1}$$

*(b)* [Kaufman & Rickert] *(i) If* $\mu$ *is a measure with values in* $\mathbb{R}^n$ *of total variation* 1*, then there is a measurable set* $E$ *such that*

$$|\mu(E)| \ge \frac{\big((n-2)/2\big)!}{2\sqrt{n}\big((n-1)/2\big)!}$$

*(ii) If* $\mu$ *is a complex measure of total variation* 1 *there is a measurable set* $E$ *such that* $|\mu(E)| \ge \pi^{-1}$.

Comment A *measure on* $X$, or what is often called a *Carathéodory outer measure on* $X$, is a function $\mu$ on the subsets of $X$, taking values in $[0,\infty]$, with $\mu(\emptyset) = 0$ , and satisfying (1); that is, it is *increasing and subadditive.*

References [EM, vol. 6, pp. 183], [MPF, pp. 509]; *Saks* [Sa, p. 43].

**Median-Mean Inequality** In statistics this simple inequality, due to Hotelling & Solomons, puts bounds on the measure of skewness $(\mu - m)/\sigma$:

$$|\mu - m| < \sigma.$$

Comment No such bound exists if the median is replaced by the mode.

Reference *Lord* [181].

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**Mediant Inequalities** *(a) If $a, b, c, d \in \mathbb{R}$ and if $b, d > 0$ then*

$$\frac{a}{b} < \frac{c}{d} \implies \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

*(b) If $C$ is a regular continued fraction with $n$-th convergent $C_n = P_n/Q_n$ then the mediant of $C_n$ and $C_{n+1}$ lies between $C$ and $C_n$; hence*

$$|C - C_n| \geq \frac{1}{Q_n(Q_n + Q_{n+1})}. \tag{1}$$

COMMENTS (i) Given two fractions $a/b, c/d$ the quantity $(a+c)/(b+d)$ is called the *mediant of these fractions*.

(ii) For the notation in (b) see **Continued Fraction Inequalities**. Inequality (1) is an inverse of **Continued Fraction Inequalities** (1).

(iii) If all the reduced fractions in $]0, 1[$ with denominators at most $n, n \geq 3$, are written in increasing order then each term, except the first and last, is the mediant of its neighbors. Such a sequence of fractions is called a *Farey sequence*.

REFERENCE [EM, vol. 6, p. 190].

**Metric Inequalities** *If $X$ is any space and $\rho : X \times X \to \mathbb{R}$ a metric on $X$ then for all $x, y, z \in X$*

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z). \tag{1}$$

COMMENTS (i) A function on $X \times X$ with values in $[0, \infty[$ is a metric if it is symmetric, positive except on $\Delta = \{(x, x), x \in X\}$, where it is zero, and satisfies (1); then $X$ is called a *metric space*.

(ii) It follows from (T) that $|\cdot|$ is a metric on $\mathbb{R}^n$, *the Euclidean metric*; and from **Norm Inequalities** (2) it follows that if $||\cdot||$ is a norm on $X$ then $\rho(x, y) = ||x - y||$ gives a metric on $X$.

(iii) If $z, w \in \overline{\mathbb{C}}$ then

$$\rho(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

is a bounded metric on $\overline{\mathbb{C}}$. It is the distance between the stereographic images of $z, w$ on the Riemann sphere in $\mathbb{R}^3$, $|\underline{x}| = 1$.

(iv) Another metric associated with the complex plane is the *hyperbolic metric*, $\rho_h$, defined for $z, w \in D$ by:

$$\rho_h(z, w) = \log \frac{1 + \delta(z, w)}{1 - \delta(z, w)}, \text{ where } \delta(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|;$$

for an application see **Schwarz's Lemma** COMMENTS (ii), EXTENSIONS (d).

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(v) If the requirement that $\rho$ be zero only on $\Delta$ is dropped we get a *pseudo-metric*; the $\mathcal{L}^p([a,b])$ spaces, are examples of pseudo-metric spaces. Equivalently we say they are metric spaces if we identify functions that are equal almost everywhere.

EXTENSIONS *(a) If $x_i \in X, 1 \le i \le n$, and if $\rho$ is a metric on $X$ then*

$$\rho(x_1, x_n) \le \sum_{i=1}^{n-1} \rho(x_i, x_{i+1}).$$

*(b)* [QUADRILATERAL INEQUALITY] *If $x_i \in X, 1 \le i \le 4$, and if $\rho$ is a metric on $X$ then*

$$\left|\rho(x_4, x_1) - \rho(x_4, x_3)\right| \le \rho(x_2, x_1) + \rho(x_2, x_3).$$

COMMENT (vi) (a) follows from (1) by a simple induction and is a generalization of **Triangle Inequality** EXTENSIONS (a) (ii); (b) is a corollary of (a), and is an extension of the **Quadrilateral Inequalities** (1).

REFERENCES [EM, vol. 6, p. 206], [MPF, pp. 481–483]; *Ahlfors* [Ah73, pp. 1–3], [Ah78, pp. 18–20,51].

## Mid-point Inequality

See: **Quadrature Inequalities** (b).

## Mills' Ratio Inequalities

It has been shown that Mills' Ratio is a **completely monotonic function**. See also: **Conte's Inequality** (I), **Error Function Inequalities**, and the reference.

REFERENCE *Gasull & Utzet* [127].

## Milne's Inequality

*If $\underline{a}, \underline{b}$ are positive $n$-tuples then*

$$\sum_{i=1}^{n}(a_i + b_i) \sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i} \le \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i, \tag{1}$$

*with equality if and only if $\underline{a}$ and $\underline{b}$ are proportional.*

COMMENTS (i) Using (C) we easily see that the left-hand side of (1) exceeds $\sum_{i=1}^{n} \sqrt{a_i b_i}$ this result can be considered as a refinement of (C).

(ii) The easiest proof of (1) is that due to Murty who noted that the inequality is equivalent to $\displaystyle\sum_{1 \le i < j \le n} \frac{(a_i b_j - a_j b_i)^2}{(a_i + b_i)(a_j + b_j)} \ge 0.$

INTEGRAL ANALOGUE *If $f, g : [a, b] \mapsto ]0, \infty[$ then*

$$\int_a^b (f + g) \int_a^b \frac{fg}{f + g} \le \int_a^b f \int_a^b g.$$

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Extension and Inverse [Alzer & Kovačec] *If $n > 2$ let $\underline{p}$ be a positive $n$-tuple, with $P_n = 1$, and $\underline{c}$ an $n$-tuple satisfying $0 \le \underline{c} < 1$; then*

$$\Big(\sum_{i=1}^{n} \frac{p_i}{1-c_i^2}\Big)^{\alpha} \le \sum_{i=1}^{n} \frac{p_i}{1-c_i} \sum_{i=1}^{n} \frac{p_i}{1+c_i} \le \Big(\sum_{i=1}^{n} \frac{p_i}{1-c_i^2}\Big)^{\beta} \tag{2}$$

*with the best possible exponents $\alpha = 1$ and $\beta = 2 - \min \underline{p}$.*

Comments (iii) In the case $\alpha = 1, \beta = 2$ this result is due to Rao and the left-hand side of (2) in this case is just (1).

(iv) Integral and matrix analogues of (2) have also been given.

References [HLP, pp. 67–68]; *Alzer & Kovačec* [40], *Murty* [227], *Woeginger* [326].

## Min and Max Inequalities

See: **Inf and Sup Inequalities**.

## Minc-Sathre Inequality

*If $r > 0$ and $n \ge 1$ then*

$$\frac{n}{n+1} < \left(\frac{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} i^r}{\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n+1} i^r}\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1}$$

Comments (i) The outer inequality is the original Minc-Sarthre inequality; the right-hand inequality is due to Martin. The final form above is due to Alzer.

(ii) Considered as bounds on the central ratio the extreme terms are best possible as was pointed out by Alzer.

(iii) Martins has pointed out that in the case of $a_i = i^r$, $i \ge 1$, the right-hand inequality in (1) improves the equal weight case of **Popoviciu's Geometric-Arithmetic Mean Inequality** (1), being just

$$\frac{\mathfrak{A}_{n+1}(\underline{a})}{\mathfrak{G}_{n+1}(\underline{a})} > \frac{\mathfrak{A}_{n}(\underline{a})}{\mathfrak{G}_{n}(\underline{a})}.$$

This lead Alzer to further observe that the following improvement of **Rado's Geometric-Arithmetic Mean Inequality** (1) holds in this special case:

$$\mathfrak{A}_{n+1}(\underline{a}) - \mathfrak{G}_{n+1}(\underline{a}) > \mathfrak{A}_{n}(\underline{a}) - \mathfrak{G}_{n}(\underline{a}).$$

Extensions (a) [Chan T. H., Gao P. & Qi] If $r > 0$ and $n, m \ge 1, k \ge 0$ then

$$\frac{n+k}{n+m+k} < \left(\frac{\dfrac{1}{n}\displaystyle\sum_{i=k+1}^{n+k} i^r}{\dfrac{1}{n+m}\displaystyle\sum_{i=k+1}^{n+m+k} i^r}\right)^{1/r} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}.$$

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(b) [Kuang] If $f$ is strictly increasing and convex, or concave, on $]0,1]$ then

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f.$$

(c) If $n > 1$ and $\phi(n) = (n)!^{1/n}$ then,

$$n\frac{\phi(n+1)}{\phi(n)} - (n-1)\frac{\phi(n)}{\phi(n-1)} > 1.$$

Comments (iv) Taking $f(x) = \log(1+x)$ in the left inequality in (b) gives the outer inequality in (1), the Minc-Sathre inequality.

(v) A right-hand side to Extension (b) has been given:

$$\int_0^1 f > \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n+1}\right) > \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)$$

References *Chan T. H., Gao P. & Qi* [83], *Chen C. P., Qi, Cerone & Dragomir* [85], *Martins* [199], *Minc & Sathre* [212], *Qi* [277].

**Mingarelli's Inequality** *If $\alpha > 0$ and $f \in \mathcal{C}^1(\mathbb{R})$ has compact support then*

$$\int_{\mathbb{R}} |f(x)|^2 e^{\alpha x^2}\,\mathrm{d}x \le \frac{1}{2\alpha}\int_{\mathbb{R}} |f'(x)|^2 e^{\alpha x^2}\,\mathrm{d}x.$$

References *Mingarelli* [213].

**Minimax Theorems** *(a) If $f : [a,b]\times[c,d] \to \mathbb{R}$ is continuous then*

$$\max_{c\le y\le d}\min_{a\le x\le b} f(x,y) \le \min_{a\le x\le b}\max_{c\le y\le d} f(x,y). \tag{1}$$

*(b)* [Fan] *Let $f : X\times X \to \mathbb{R}$, where $X$ is a compact set in $\mathbb{R}^n$, be such that: for all $x \in X, f(x,\cdot)$ is lower semicontinuous, and for all $y \in X, f(\cdot,y)$ is quasi-concave. Then*

$$\min_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

Comments (i) For the definition of quasi-concave see **Quasi-convex Function Inequalities**.

(ii) If there is equality in (1) the *minimax principle* is said to hold.

References [EM, vol. 6, p. 239], [I3, pp. 103–114] ; *Borwein & Lewis* [BL, pp. 205–206], *Pólya & Szegö* [PS, pp. 101–102].

**Minkowski's Inequality** *If $\underline{a}, \underline{b}$ are positive $n$-tuples, and $p > 1$ then*

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}. \tag{M}$$

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*If $p<1, p\neq 0$, then ($\sim$M) holds. There is equality if and only if $\underline{a}\sim\underline{b}$.*

COMMENTS (i) There are several proofs of this famous inequality; the easiest approach is to deduce it from (H).

(ii) One of the most important uses of (M) is to show that $||\underline{a}||_p, 1\le p<\infty$, is a norm, called the *Hölder norm*; see (1) below. See **Norm Inequalities** (1), Comments (iii).

(iii) The case $p=0$ of (M) is the inequality **Power Mean Inequalities** (4).

EXTENSIONS *(a) If $\underline{a},\underline{b}$ are complex $n$-tuples and $p$ is as in (M),*

$$\left(\sum_{i=1}^{n}|a_i+b_i|^p\right)^{1/p}\le\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|b_i|^p\right)^{1/p},$$

*or*

$$||\underline{a}+\underline{b}||_p\le||\underline{a}||_p+||\underline{b}||_p \tag{1}$$

*with equality if and only if $\underline{a}\sim^{+}\underline{b}$. If $p<1, p\neq 0$, then the reverse inequality holds holds.*

*(b)* [GENERALIZED MINKOWSKI'S INEQUALITY] *If $a_{ij}>0$, $1\le i\le m, 1\le j\le n$, and if $p>1$ then*

$$\left\{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}\right)^{p}\right\}^{1/p}\le\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{p}\right)^{1/p}. \tag{2}$$

*There is equality in (2) if and only if the $n$-tuples $(a_{i1},\ldots,a_{in})$, $1\le i\le m$, are linearly dependent.*

*(c)* [FUNCTIONS OF THE INDEX SET] *If $\mathcal{I}$ is an index set let*

$$\mu(\mathcal{I})=\left(\left(\sum_{i\in\mathcal{I}}a_i^p\right)^{1/p}+\left(\sum_{i\in\mathcal{I}}b_i^p\right)^{1/p}\right)^{p}-\sum_{i\in\mathcal{I}}(a_i+b_i)^p,$$

*where $\underline{a},\underline{b},p$ are as in (M) then $\mu\ge 0$, and if $\mathcal{I}\cap\mathcal{J}=\emptyset$*

$$\mu(\mathcal{I})+\mu(\mathcal{J})\le\mu(\mathcal{I}\cup\mathcal{J});$$

*this inequality is strict unless $\left(\sum_{i\in\mathcal{I}}a_i^p,\sum_{i\in\mathcal{I}}b_i^p\right)$ and $\left(\sum_{i\in\mathcal{J}}a_i^p,\sum_{i\in\mathcal{J}}b_i^p\right)$ are proportional.*

*(d)* [MIKOLÁS] *If $a_{ij}>0, 1\le i\le m, 1\le j\le n$, then if $0<r\le 1$, $0<rp\le 1$ then*

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}^{r}\right)^{p}\le n^{1-pr}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}\right)^{n}; \tag{3}$$

*if $r\ge 1$ and $pr\ge 1$ then ($\sim$3) holds.*

COMMENTS (iv) If we take $p=1/r$ then (3) reduces to (M).

(v) (2) remains valid if $p=\infty$; see the right-hand side of **Inf and Sup Inequalities** (1).

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(vi) Further extensions can be found in [AI], [MPF].

Inverse Inequalities [Mond & Shisha] *If $\underline{a}, \underline{b}$ are positive $n$-tuples such that*

$$0 < m \le \frac{a_i}{a_i + b_i}, \ \frac{b_i}{a_i + b_i} \le M, \ M \ne m;$$

*and if $p > 1$ then*

$$\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p} \le \Lambda\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p}, \tag{4}$$

*where*

$$\Lambda = \frac{M^p - m^p}{\big(p(M-m)\big)^{1/p}\big|q(Mm^p - mM^p)\big|^{1/q}},$$

*$q$ being the conjugate index. If $p < 1, p \ne 0$ then ($\sim$4) holds.*

Integral Analogues *(a) If $p$ is as in (M) and if $f_i \in \mathcal{L}^p([a,b]), 1 \le i \le n$, then $\sum_{i=1}^n f_i \in \mathcal{L}_p([a,b])$ and*

$$\left|\left|\sum_{i=1}^n f_i\right|\right|_{p,[a,b]} \le \sum_{i=1}^n ||f_i||_{p,[a,b]}.$$

*There is equality if $A_i f_i = B_i g$ almost everywhere, where not both $A_i, B_i$ are zero, $1 \le i \le n$.*
*If $0 < p < 1$ the reverse inequality holds. (b) If $p$ is as in (M) then*

$$\left(\int_a^b \left(\int_c^d |f(x,y)|\,\mathrm{d}y\right)^p \mathrm{d}x\right)^{1/p} \le \int_c^d \left(\int_a^b |f(x,y)|^p\,\mathrm{d}x\right)^{1/p} \mathrm{d}y,$$

*with equality if and only if $f(x,y) = g(x)h(y)$ almost everywhere.*
*(c) If $f, g \in \mathcal{L}([a,b]), f, g \ge 0$ then*

$$\exp\left(\frac{1}{b-a}\int_a^b \log(f+g)\right) \ge \exp\left(\frac{1}{b-a}\int_a^b \log f\right) + \exp\left(\frac{1}{b-a}\int_a^b \log g\right).$$

Comments (vii) (c) is the case $p = 0, n = 2$ of (a)

(viii) These can be extended to general measure spaces.

(ix) See also: **Jessen's Inequality, Kober & Dianada's Inequality** Comments (ii), **Power Mean Inequalities** (3), (4), **Quasi-arithmetic Mean Inequalities** Comments (iv), **Rahmail's Inequality**.

(x) For other inequalities of the same name see: **Determinant Inequalities**(1), **Minkowski-Mahler Inequality, Mixed-volume Inequalities** (a), (b).

References [AI, pp. 55–57], [BB, pp. 19–29], [EM, vol. 6. pp. 247–248], [H, pp. 189–198], [HLP, pp. 30–32, 85–88, 146–150], [MOA, pp. 660–661], [MPF,

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pp. 107–117, 135–209], [PPT, pp. 114–118, 126–128, 166–167]; *Lieb & Loss* [LL, pp. 41–42], *Pólya & Szegö* [PS, p. 71].

**Minkowski-Mahler Inequality** *Let $F$ be a generalized norm on $\mathbb{R}^n$, define*

$$G(\underline{b}) = \max_{\underline{a}\in\mathbb{R}^n}\left(\frac{\underline{a}.\underline{b}}{F(\underline{a})}\right), \quad \underline{b}\in\mathbb{R}^n.$$

*Then*

$$F(\underline{a}) = \max_{\underline{b}\in\mathbb{R}^n}\left(\frac{\underline{a}.\underline{b}}{G(\underline{b})}\right), \quad \underline{a}\in\mathbb{R}^n,$$

*and for all $\underline{a},\underline{b}\in\mathbb{R}^n$,*

$$\underline{a}.\underline{b} \le F(\underline{a})G(\underline{b}).$$

COMMENTS (i) For the definition of generalized norm see **Norm Inequalities** COMMENTS (i).

(ii) The function $G$ is called the *polar function of* $F$, when, from the above, $F$ is the polar function of $G$.

REFERENCES [BB, pp. 28–29], [EM, vol. 6, p. 248].

**Minkowski's Mixed Volume Inequality** See: **Mixed-Volume Inequalities**.

**Mitrinović & Đoković Inequality** *If $\underline{a}$ is a positive $n$-tuple with $A_n = 1$ and if $p \ge 0$, then*

$$\mathfrak{M}_n^{[p]}\left(\underline{a} + \frac{1}{\underline{a}}\right) \ge n + \frac{1}{n}.$$

COMMENT (i) This inequality can be proved using (J) or **Schur Convex Function Inequalities** (b). The case $p = 0$ is due to Li W. & Chen J.; of course, by (r;s), this is the critical case.

EXTENSION [WANG C. L.] *Let $p, r \ge 0, q > 0$, $\underline{a}, \underline{w}$ positive $n$-tuples with* $\max \underline{a} \le \sqrt{r/q}$ *then writing*

$$f(x) = \left(qx + \frac{r}{x}\right)^p,$$

*we have*

$$\mathfrak{A}_n(f(\underline{a}), \underline{w}) \ge f\left(\mathfrak{G}_n(\underline{a}, \underline{w})\right) \ge f\left(\mathfrak{A}_n(\underline{a}, \underline{w})\right).$$

COMMENT (ii) This follows by an application of (J) and (GA).

REFERENCES [AI, pp. 282–283], [MOA, pp. 103–104], [MPF, pp. 36–37].

**Mitrinović & Pečarić's Inequality** *Let $p, f_i, g_i, 1 \le i \le n$, be continuous functions defined on $[a, b]$ and denote by $(\Pi\Phi\Gamma)_n$ the $n \times n$ matrix*

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*with $i, j$-th entry $\int_a^b pf_ig_j$; let $\phi(\underline{x}), \gamma(\underline{x})$ denote the $n \times n$ matrices with $i, j$-th entries $f_i(x_j), g_i(x_j)$, respectively, where $a \le x_j \le b, 1 \le j \le n$: then,*

$$\det \phi(\underline{x}) \ge 0, \ \det \gamma(\underline{x}) \ge 0 \quad \textit{for all} \quad \underline{x} \quad \Longrightarrow \quad \det(\Pi\Phi\Gamma)_n \ge 0.$$

Comments (i) This inequality is deduced from an identity of Andreev,

$$\det(\Pi\Phi\Gamma)_n = \frac{1}{n!}\int_a^b \cdots \int_a^b \det \phi(\underline{x}) \det \gamma(\underline{x}) \, \mathrm{d}\underline{x}.$$

(ii) Special cases of this result are (C), (Č), **Gram Determinant Inequalities** (1).

References [BB, p. 61], [MPF, pp. 244, 600–601]; *Mitrinović & Pečarić* [MP91b, p. 51].

**Mixed Mean Inequalities** If $\underline{a}$ is a positive $n$-tuple, $1 \le k \le n$, put $\kappa = \binom{n}{k}$, and denote by $\underline{a}_1^{(k)}, \ldots, \underline{a}_\kappa^{(k)}$ the $\kappa$ $k$-tuples formed from the elements of $\underline{a}$. If then $r, s \in \mathbb{R}$ *the mixed mean of order $r$ and $s$ of $\underline{a}$ taken $k$ at a time* is

$$\mathfrak{M}_n(r, s; k; \underline{a}) = \mathfrak{M}_\kappa^{[r]}\big(\mathfrak{M}_k^{[s]}(\underline{a}_i^{(k)}); 1 \le i \le \kappa\big)$$

The following relations with other means are immediate:

$$\mathfrak{M}_n(r, s; 1; \underline{a}) = \mathfrak{M}_n^{[r]}(\underline{a}); \qquad \mathfrak{M}_n(r, s; n; \underline{a}) = \mathfrak{M}_n^{[s]}(\underline{a});$$
$$\mathfrak{M}_n(r, r; k; \underline{a}) = \mathfrak{M}_n^{[r]}(\underline{a}); \qquad \mathfrak{M}_n(k; 0; k; \underline{a}) = \mathfrak{P}_n^{[r]}(\underline{a}).$$

[Carlson, B. C.] *(a) $\mathfrak{M}_n(r, s; k; \underline{a})$ is increasing both as a function of $r$ and $s$.*
*(b) If $-\infty \le r < s \le \infty$ then*

$$\mathfrak{M}_n(r, s; k-1; \underline{a}) \le \mathfrak{M}_n(r, s; k; \underline{a}). \tag{1}$$

*If $r > s$ then (~1) holds. In both cases there is equality if and only if $\underline{a}$ is constant.*
*(c) If $-\infty \le r < s \le \infty$ and $k + \ell > n$ then*

$$\mathfrak{M}_n(s, r; k-1; \underline{a}) \le \mathfrak{M}_n(r, s; \ell; \underline{a}), \tag{2}$$

*with equality if and only if $\underline{a}$ is constant.*

Comments (i) Both results follow by applications of (r;s), and contain (r;s) as a special case.

(ii) Let $r < s$ and consider the following $2 \times n$ matrix, $\mathbb{M}$, in which $\underline{a}$ is not constant;

$$\begin{pmatrix} \mathfrak{M}_n(r, s; 1; \underline{a}) & \mathfrak{M}_n(r, s; 2; \underline{a}) & \ldots & \mathfrak{M}_n(r, s; n-1; \underline{a}) & \mathfrak{M}_n(r, s; n; \underline{a}) \\ \mathfrak{M}_n(r, s; n; \underline{a}) & \mathfrak{M}_n(r, s; n-1; \underline{a}) & \ldots & \mathfrak{M}_n(r, s; 2; \underline{a}) & \mathfrak{M}_n(r, s; 1; \underline{a}) \end{pmatrix}.$$

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The inequalities (1), (2) can be summarized as saying: (i) the rows of $\mathbb{M}$ are strictly increasing to the right; (ii) the columns, except the first and last, strictly increase downwards; of course the entries in the first column both are equal to $\mathfrak{M}_n^{[r]}(\underline{a})$, while both in the last column are equal to $\mathfrak{M}_n^{[s]}(\underline{a})$. Another matrix of the same type is given in **Symmetric Mean Inequalities** Comments (v).

Special Cases *(a) If $a, b, c$ are positive and not all equal then*

$$\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{3} < \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}. \tag{3}$$

*(b) If $\underline{a}$ is a positive $n$-tuple then*

$$\mathfrak{A}_n\big(\mathfrak{G}_{n-1}(\underline{a}'_k), 1 \le k \le n\big) \le \mathfrak{G}_n\big(\mathfrak{A}_{n-1}(\underline{a}'_k), 1 \le k \le n\big).$$

*(c)* [Kuczma]

$$\mathfrak{M}_n(2,0;2;\underline{a}) \le \mathfrak{M}_n(0,1;n-1;\underline{a}), \qquad \mathfrak{M}_n(n-1,0;2;\underline{a}c) \le \mathfrak{M}_n(0,1;n-1;\underline{a}).$$

Comments (iii) Both are obtained from (2), and (3) should be compared with the result in **Symmetric Mean Inequalities** Comments (vi).

(iv) Both inequalities in (c) include (3), which is just $\mathfrak{M}_3(2,0;2;a,b,c) \le \mathfrak{M}_3(0,1;2;a,b,c)$. All are special cases of a conjecture of Carlson, Meany & Nelson:

$$\textit{if } r+m>n \textit{ then } \mathfrak{M}_n(r,0;r;\underline{a}) \le \mathfrak{M}_n(0,1;m;\underline{a}).$$

(v) See also: **Nanjundiah's Mixed Mean Inequalities**.

References [AI, p. 379], [H, pp. 253–256]; *Kuczma* [166].

**Mixed-volume Inequalities** If $K_i$ is a convex body in $\mathbb{R}^p$, $\lambda_i \ge 0, 1 \le i \le n$, then the linear combination $B = \sum_{i=1}^n \lambda_i K_i$ has a volume, $V(B)$, that is a homogeneous polynomial of degree $n$ in the $\lambda_i, 1 \le i \le n$. The coefficients of this polynomial are called the *mixed-volumes of the bodies* $K_i, 1 \le i \le n$. In particular $V(K_{i_1}, \ldots K_{i_n})$ denotes the coefficient of $\lambda_{i_1} \ldots \lambda_{i_n}$.

*With the above notation*

$$V^n(K, L, \ldots, L) \ge V(K)V^{n-1}(L);$$

*and*

$$V^2(K, L, \ldots, L) \ge V(L)V(K, K, L, \ldots, L).$$

Comments (i) The first inequality is called *Minkowski's mixed-volume inequality*, while the second is called *Minkowski's quadratic mixed-volume inequality*.

(ii) Many isoperimetric inequalities are special cases of these results.

Extensions [Aleksandrov, P. S.–Fenchel]

$$V^m(K_1, \ldots, K_m, L_1, \ldots, L_{n-m}) \ge \prod_{j=1}^m V(K_j, \ldots, K_j, L_1, \ldots, L_{n-m});$$

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*in particular*

$$V^n(K_1, \dots, K_n) \ge V(K_1) \dots V(K_n).$$

COMMENT (iii) See also: **Brunn-Minkowski Inequality, Permanent Inequalities**.

REFERENCES [EM, vol. 6, pp. 262–264].

**Modulus of Continuity Inequalities** If $f : I \mapsto \mathbb{R}, I$ some interval in $\mathbb{R}$ then the following definitions are made:

$$(i) \qquad \omega(f;\delta) = \sup_{|x-y|\le\delta} |f(x) - f(y)|,$$

called the *modulus of continuity of* $f$;

$$(ii) \quad \text{if } 0 < \alpha < 1, \quad H(f;\alpha) = \sup_{0\le x,y\le y, x\ne y} \frac{|f(y) - f(x)|}{|y-x|^\alpha},$$

the *coefficient of Hölder continuity of* $f$:

$$(iii) \qquad \omega_1(f, x_0, h) = \sup_{|x-x_0|\le h} |f(x) - f(x_0)|.$$

the *local modulus of continuity of* $f$ *at* $x_0$.

*(a)* [HAYMAN] *If* $f \in \mathcal{C}([0,1])$ *then*

$$\begin{aligned} \omega(f^*;\delta) &\le \quad \omega(f;\delta); \\ H(f^*;\alpha) &\le \quad H(f;\alpha). \end{aligned}$$

*(b)* [ANASTASSIOU] *If* $f \in \mathcal{C}^n(a,b),\ a < x_0 < b$ *and* $0 < h < b - a$*: then*

$$\omega_1(f, x_0, h) \le \sum_{k=1}^n \frac{|f^{(k)}(x_0)|}{k!} h^k + \frac{h^n}{n!}\omega_1(f^{(n)}, x_0, h).$$

COMMENTS (i) See also: **Marchaud's Inequality**.

(ii) A similar result to (b) for the second local modulus of continuity,

$$\omega_2(f, x_0, h) = \sup_{|y|\le h} |f(x_0 + y) + f(x_0 - y) - 2f(x_0)|.$$

is in the first reference.

REFERENCES *Anastassiou* [42], *Yanagihara* [329].

**Moment Inequalities** *(a) If* $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *and if* $k, k_0, k_1 \in \mathbb{R}^+$ *with* $k = (1-t)k_0 + tk_1, 0 \le t \le 1$*, then*

$$\mu_t(f) \le \mu_{k_0}^{1-t}(f)\mu_{k_1}^t(f).$$

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*(b)* [Petschke] *Let* $f:[0,1]\to\mathbb{R}$ *be decreasing,* $p,q\ge 0$, $p<q$, *then if* $0<x<1$

$$x^{q-p}\int_x^1 t^p f(t)\,\mathrm{d}t \le \left(\frac{q-p}{q+1}\right)^{(q-p)/(p+1)}\int_0^1 t^q f(t)\,\mathrm{d}t.$$

*(c)* [Good] *If* $m<n, m,n\in\mathbb{N}$, $X>0$, *and the integrals exist then*

$$\frac{\int_X^\infty f(x)x^m\,\mathrm{d}x}{\int_0^\infty f(x)x^m\,\mathrm{d}x} < \frac{\int_X^\infty f(x)x^n\,\mathrm{d}x}{\int_0^\infty f(x)x^n\,\mathrm{d}x}.$$

Comments (i) The notation in (a) is defined in **Notations** 8.

(ii)This is a variant of **Steffensen's Inequalities** (1).

(iii) Other similar results are given in the references. See also: **Convex Function Integral Inequalities, Gauss's Inequality, Ting's Inequalities**.

References [MPF, pp. 329–330]; *Steele* [S, pp. 149, 262]; *Good* [133]

**Moment of Inertia Inequality** *If $A$ is the area of a plane domain and $I$ the polar moment of inertia about the centre of gravity then*

$$I\ge\frac{A^2}{2\pi}.$$

*Equality occurs only when the domain is a circle.*

Comments (i) The *polar moment of inertia about the center of gravity of a plane domain* $A$ is

$$I=\int_A[(x-\overline{x})^2+(y-\overline{y})^2]\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\overline{x}=\frac{1}{|A|}\int_A x\,\mathrm{d}x\,\mathrm{d}y,$$

and $\overline{y}$ is defined analogously; $(\overline{x},\overline{y})$ is the *center of gravity of* $A$.

(ii) This is an example of **Symmetrization Inequalities**.

References *Pólya & Szegö* [PS51, pp. 2, 8, 153, 195–196].

**Mond & Shisha Inequalities** See: **Geometric-Arithmetic Mean Inequality** Inverse Inequalities, **Minkowski's Inequality** Inverse Inequalities.

**Monotone Matrix Function Inequalities** *(a) if $A, B$ are $n\times n$ Hermitian matrices and if $f$ is a monotone matrix function of order $n$ then*

$$A\ge B \qquad\Longrightarrow\qquad f(A)\ge f(B). \tag{1}$$

*(b) If $f$ is a non-constant monotone matrix function of order $n, n\ge 2$, on $[0,\infty[$ and if $A$ is a positive semi-definite $n\times n$ Hermitian matrix then*

$$f^{-1}(A)_{i_1,\ldots,i_m}\ge f^{-1}(A_{i_1,\ldots,i_m}) \implies f(A)_{i_1,\ldots,i_m}\ge f(A_{i_1,\ldots,i_m}).$$

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*(c) If $A$ is a rank one positive semi-definite Hermitian matrix, and if $\alpha \geq 1$ then*

$$(A_{i_1,\ldots,i_m})^{1/\alpha} \geq A_{i_1,\ldots,i_m}^{1/\alpha}.$$

*(d) If $f$ is a monotone matrix function and $A, B$ are positive semi-definite then for any unitary invariant norm $||\cdot||$.*

$$||f(A)|| + ||f(B)|| \leq 2||f(A+B)||.$$

COMMENTS (i) If $I$ is a interval in $\mathbb{R}$ then a function $f : I \to \mathbb{R}$ is said to be a *monotone matrix function of order* $n$ if (1) holds for all $A, B$ with eigenvalues in $I$; here $A \geq B$ means that $A - B$ is positive semi-definite. To be more precise this is sometimes called a *monotone increasing function*; if on the other hand $A \geq B \Longrightarrow f(A) \leq f(B)$ we say that $f$ is a *monotone decreasing function.*

$x^\alpha, 0 < \alpha \leq 1$, $\log x$, $-x^{-1}$ are all monotone matrix function of all orders on $]0, \infty[$.

(ii) A matrix function that is monotone of all orders on $I$ is called an *operator monotone function on* $I$.

(iii) A matrix norm $||\cdot||$, see **Matrix Norm Inequalities**, is called *unitary invariant* if for all Hermitian $A$, $||UAV|| = ||A||$ for all unitary $U, V$.

(iv) See also: **Convex Matrix Function Inequalities**.

REFERENCES [MOA, pp. 670–675]; *Roberts & Varberg* [RV, pp. 259–261]; *Aujla* [48], *Chollet* [88].

**Monotonic Function Inequalities** See: **Increasing Function Inequalities, Monotone Matrix Function Inequalities**.

**Morrey's Inequality** *If $n < p < \infty$, $B = B_{\underline{x},r} = \{\underline{u}; |\underline{u} - \underline{x}| < r\} \subset \mathbb{R}^n$, and if $f \in \mathcal{W}^{1,p}(\mathbb{R}^n)$ then for almost all $y, z$*

$$|f(y) - f(z)| \leq rC\left(\frac{1}{|B|}\int_B |Df|\right)^{1/p}$$

*where $C$ depends only on $n$ and $p$.*
*In particular $\lim_{r\to 0} \frac{1}{|B|}\int_B f$ is Lipschitz of order $(p-n)/p$.*

COMMENT For the definition of $\mathcal{W}^{1,p}(\mathbb{R}^n)$ see **Sobolev's Inequalities**; and for the definition of Lipschitz of order $\alpha$ see **Lipschitz Function Inequalities**.

REFERENCE *Evans & Gariepy* [EG, pp. 143–144].

**Muirhead's Inequality** See: **Muirhead Symmetric Function and Mean Inequalities** (1).

**Muirhead Symmetric Function and Mean Inequalities** If $\underline{a}$ a positive $n$-tuple and $\underline{\alpha}$ is a non-negative $n$-tuple, put $|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_n$ and define

$$e_n(\underline{a};\underline{\alpha}) = e_n(\underline{a};\alpha_1,\ldots,\alpha_n) = \frac{1}{n!}\sum! \prod_{j=1}^n a_{i_j}^{\alpha_j};$$
$$\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) = \mathfrak{A}_{n,\{\alpha_1,\ldots,\alpha_n\}}(\underline{a}) = \left(e_n(\underline{a};\underline{\alpha})\right)^{1/|\underline{\alpha}|}.$$

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These are the *Muirhead symmetric functions and means* respectively. These means, are called symmetric means in [HLP].

There is no loss in generality in assuming that $\underline{\alpha}$ is decreasing.

It is easily seen that some earlier means are special cases of the Muirhead means:

$$\mathfrak{A}_{n,\{p,0,\ldots,0\}}(\underline{a}) = \mathfrak{M}_n^{[p]}(\underline{a}); \qquad \mathfrak{A}_{n,\{1,1,\ldots,1\}}(\underline{a}) = \mathfrak{G}_n(\underline{a});$$

and if $1 \le r < n$,

$$\mathfrak{A}_{n,\{\underbrace{1,1,\ldots 1}_{r\text{ terms}},0,\ldots,0\}}(\underline{a}) = \mathfrak{P}_n^{[r]}(\underline{a}).$$

*(a) If $\underline{\alpha}, \underline{\beta}$ are non-identical decreasing non-negative $n$-tuples then*

$$e_n(\underline{a};\underline{\alpha}) \le e_n(\underline{a};\underline{\beta}) \tag{1}$$

*if and only if $\underline{\alpha} \prec \underline{\beta}$; further there is equality if and only if $\underline{a}$ is constant.*
*(b) If $\underline{a}$ is a positive $n$-tuple and $\underline{\alpha}$ is a non-negative $n$-tuple then*

$$\min \underline{a} \le \mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) \le \max \underline{a}.$$

*(c) If $\underline{a}$ is a positive $n$-tuple and if $\underline{\alpha}_1, \underline{\alpha}_2$ are non-identical decreasing non-negative $n$-tuples with $\underline{\alpha}_1 \prec \underline{\alpha}_2$ then*

$$\mathfrak{A}_{n,\underline{\alpha}_1}(\underline{a}) \le \mathfrak{A}_{n,\underline{\alpha}_2}(\underline{a}), \tag{2}$$

*with equality if and only if $\underline{a}$ is constant.*
*(d)* [Carlson, B. C., Pittenger] *If $0 < a < b$ and if $\underline{\alpha} = \Big((1+\sqrt{\delta})/2, (1-\sqrt{\delta})/2\Big)$ where $0 \le \delta \le 1/3$ then*

$$\mathfrak{A}_{2,\underline{\alpha}}(a,b) \le \mathfrak{L}(a,b).$$

Comments (i) Inequality (1) is called *Muirhead's Inequality*; (2) follows from (1).

(ii) Inequality (2) implies (GA), and (r;s). In addition if $\underline{\alpha}$ has at least two non-zero entries and $|\underline{\alpha}| = 1$,

$$\mathfrak{G}_n(\underline{a}) \le \mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) \le \mathfrak{A}_n(\underline{a})$$

(iii) Inequality (2) identifies a class of order preserving functions from the pre-ordered positive $n$-tuples to $\mathbb{R}$. See: **Schur Convex Function Inequalities** Comments (ii).

(iv) The inequality in (d) can fail if $1/3 < \delta < 1$.

Extensions *(a) If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and symmetric, and $\underline{\alpha}, \underline{\beta}, \underline{a}$ are as in (a) above with $\underline{\alpha} \prec \underline{\beta}$, then*

$$\sum! f(\alpha_1 a_1, \ldots, \alpha_n a_n) \le \sum! f(\beta_1 a_1, \ldots, \beta_n a_n).$$

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*(b)* [Marcus & Lopes Type] *If* $\underline{a}, \underline{b}, \underline{\alpha}$ *are positive* $n$*-tuples, and if* $0 < |\underline{\alpha}| < 1$ *then*

$$\mathfrak{A}_{n,\underline{\alpha}}(\underline{a}) + \mathfrak{A}_{n,\underline{\alpha}}(\underline{b}) \le \mathfrak{A}_{n,\underline{\alpha}}(\underline{a} + \underline{b}).$$

Integral Analogue [Ryff] *Let* $\alpha, \beta$ *be two bounded measurable functions on* $[0,1]$, $f \in \mathcal{L}^p([0,1])$ *for all* $p \in \mathbb{R}$, $f > 0$*; then*

$$\int_0^1 \log\left(\int_0^1 f(t)^{\alpha(s)}\,\mathrm{d}t\right)\mathrm{d}s \le \int_0^1 \log\left(\int_0^1 f(t)^{\beta(s)}\,\mathrm{d}t\right)\mathrm{d}s,$$

*if and only if* $\alpha \prec \beta$.

Comment (v) A geometric mean analogue of $\mathfrak{A}_{n,\underline{\alpha}}(\underline{a})$ has also been defined, and appropriate inequalities obtained; see [H] for this and further extensions of these means.

References [AI, p. 167], [H, pp. 357–364, 380–381, 390], [HLP, pp. 44–51], [MOA, pp. 125, 159–160], [PPT, pp. 361–364].

## Mulholland's Inequality

*If* $\mu$ *is a probability measure on* $\mathbb{C}$*, and if* $p > 0$ *then*

$$\mathfrak{G}_{\mathbb{C}\times\mathbb{C}}(|z-w|;\mu\otimes\mu) \le \left(\frac{2}{\sqrt{e}}\right)^{1/p}\mathfrak{M}^{[p]}_{\mathbb{C}}(|z|;\mu).$$

*There is equality if* $\mu$ *is defined by*

$$\mu(B) = K\int_B 1_{\{z;|z|<a\}}|z|^{k-2}\,\mathrm{d}\lambda_2,$$

*where* $B$ *a Borel set in* $\mathbb{C}$.

Comments (i) The best value for the constant has been conjectured as $\left(\dfrac{2}{\sqrt{e}}\dfrac{(p/2)!}{p!}\right)^{1/p}$.

(ii) For a discrete analogue of the left-hand side see: **Difference Means of Gini**.

References [EM, vol. 4, p. 277]; *Mulholland* [225].

## Multigamma Function Inequalities

See: **Digamma Function Inequalities**.

## Multilinear Form Inequalities

[Hardy-Littlewood-Pólya] *Hypotheses:* $\underline{a}_i = \{a_{ij}, j \in \mathbb{Z}\}$ *are non-negative sequences with* $||\underline{a}_i||_{p_i} \le A_i$, $p_i \ge 1$, $1 \le i \le n$*;* $\underline{c} = \{c_{j_1,j_2,\ldots,j_n}, j_i \in \mathbb{Z}, 1 \le i \le n\}$ *are non-negative with* $\sum_i{}' \underline{c}^{r_i} \le C_i, r_i > 0$, $1 \le i \le n$*;* $P = \dfrac{1}{n-1}\Big(\displaystyle\sum_{i=1}^n \frac{1}{p_i} - 1\Big)$, $p_i$, $1 \le i \le n$, *satisfying* $0 \le P \le \min_{1\le i\le n}\{p_i^{-1}\}$*;* $\overline{p}_i$ *is defined by* $\dfrac{1}{\overline{p}_i} = \dfrac{1}{p_i} - P$, $1 \le i \le n$,

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*then* $r_i$, $1 \le i \le n$, *satisfies* $\displaystyle\sum_{i=1}^{n} \frac{r_i}{\overline{p}_i} = 1$.

*Conclusion:*

$$\sum_{j_i \in \mathbb{Z}, 1 \le i \le n} c_{j_1, j_2, \ldots, j_n} a_{1j_1} \ldots a_{nj_n} \le \prod_{i=1}^{n} C_i^{1/\overline{p}_i} \prod_{i=1}^{n} A_i. \tag{1}$$

COMMENT (i) The notation $\displaystyle\sum_i{}' \underline{c}$ means that in the multiple summation there is no summing over the $i$-th suffix.

SPECIAL CASES [$n = 2$] *Let* $p, q \ge 1$, $\dfrac{1}{p} + \dfrac{1}{q} \ge 1$, $r, s > 0$, $\dfrac{r}{p'} + \dfrac{s}{q'} = 1$, *where* $p', q'$ *are the conjugate indices of* $p, q$, *respectively. If* $\underline{a} = \{a_i,\, i \in \mathbb{Z}\}$, $\underline{b} = \{a_j,\, j \in \mathbb{Z}\}$, $\underline{c} = \{c_{ij}, i, j \in \mathbb{Z}\}$ *are non-negative with* $||\underline{a}||_p \le A$, $||\underline{b}||_q \le B$, $\sum_{i \in \mathbb{Z}} c_{ij}^r < C$ , $\sum_{j \in \mathbb{Z}} c_{ij}^s < D$ , *then*

$$\sum_{i,j \in \mathbb{Z}} c_{ij} a_i b_j \le C^{1/p'} D^{1/q'} AB.$$

COMMENTS (ii) Another special case is **Young's Convolution Inequality** (b).

(iii) Another important inequality for multilinear forms is the Riesz convexity theorem. See: **Riesz-Thorin Theorem** COMMENTS (ii).

(iv) See also **Bilinear Form Inequalities, Quadratic Form Inequalities**.

REFERENCES [GI3, pp. 205–218], [HLP, pp. 196–225].

## Myers Inequality

*If* $\underline{a}$ *is a real* $n$*-tuple with* $A_n = 0$ *then*

$$\prod_{i=1}^{n} (1 + a_i) \le 1$$

*with equality if and only if* $\underline{a}$ *is null.*

COMMENTS (i) The proof is by induction, or the result follows from **Weierstrass's Inequalities** (b).

(ii) This inequality has been used to prove (GA).

REFERENCE [H, pp. 105–106].

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# 14 Nanjundiah–Number

**Nanjundiah Inequalities** *(a) If* $0 < a < b$ *and*

$$\begin{aligned}
\underline{a} &= \left\{a,\, a+\frac{b-a}{n-1}, a+2\tfrac{b-a}{n-1}, \ldots, a+(n-2)\tfrac{b-a}{n-1}, b\right\},\\
\underline{g} &= \left\{a,\, a\left(\frac{b}{a}\right)^{1/(n-1)}, a\left(\frac{b}{a}\right)^{2/n-1}, \ldots, a\left(\frac{b}{a}\right)^{(n-2)/(n-1)}, b\right\},\\
\underline{h} &= \left\{a,\, \frac{ab}{b-\dfrac{(b-a)}{n-1}}, \frac{ab}{b-\dfrac{2(b-a)}{n-1}}, \ldots, \frac{ab}{b-\dfrac{(n-2)(b-a)}{n-1}}, b\right\};
\end{aligned}$$

*then*

$$\mathfrak{H}_n(\underline{a}) > \mathfrak{A}_n(\underline{g}), \quad \textit{and} \quad \mathfrak{H}_n(\underline{g}) > \mathfrak{A}_n(\underline{h}). \tag{1}$$

*(b) If* $r > 1,\ q > 1,\ r > q,\ ra > (r-1)c > 0,\ qc > (q-1)b > 0$ *then*

$$\frac{(ra-(r-1)c)^r}{(qc-(q-1)b)^{r-1}} \ge r\frac{a^r}{c^{r-1}} - (r-1)\frac{c^q}{b^{q-1}}. \tag{2}$$

Comments (i) The $n$-tuples $\underline{a}, \underline{g}, \underline{h}$ are said to *interpolate* $(n-2)$ *arithmetic, geometric, harmonic means between* $a$ *and* $b$, respectively.

(ii) Inequalities (1) follow from the convexity of $e^x$ and $1/x$ and an application of **Hermite-Hadamard's Inequality** (2).

(iii) Taking limits in (1) leads to special cases of **Logarithmic Mean Inequalities** (1),

$$\mathfrak{L}^{[-2]}(a,b) \le \mathfrak{L}^{[-1]}(a,b) \le \mathfrak{L}^{[0]}(a,b) \le \mathfrak{L}^{[1]}(a,b).$$

(iii) Inequality (2) follows by an application of **Nanjundiah's Inverse Mean Inequalities** (b) and (GA).

Reference [H, pp. 166–167].



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**Nanjundiah's Inverse Mean Inequalities** The *inverse means of Nanjundiah* are defined by

$$(\mathfrak{M}_n^{[p]})^{-1}(\underline{a};\underline{w}) = \begin{cases} \left(\left(\dfrac{W_n}{w_n}\right)a_n^p - \left(\dfrac{W_{n-1}}{w_n}\right)a_{n-1}^p\right)^{1/p}, & \text{if } -\infty < p < \infty,\ p \neq 0; \\ \left(a_n^{W_n/w_n}\right)\Big/\left(a_{n-1}^{W_{n-1}/w_n}\right), & \text{if } p = 0. \end{cases}$$

The cases $p = -1, 0, 1$ are also written $\mathfrak{H}_n^{-1}(\underline{a};\underline{w}), \mathfrak{G}_n^{-1}(\underline{a};\underline{w}), \mathfrak{A}_n^{-1}(\underline{a};\underline{w})$, respectively.

The name is suggested by the fact that applying these means to the corresponding power mean sequence gives the original sequence back.

*Let $\underline{a}$ be a positive $n$-tuple, $n \geq 2$.*
*(a)*

$$\mathfrak{G}_n^{-1}(\underline{a};\underline{w}) \geq \mathfrak{A}_n^{-1}(\underline{a};\underline{w}),$$

*with equality if and only if $a_{n-1} = a_n$.*
*(b)*

$$\mathfrak{G}_n^{-1}(\underline{a};\underline{w}) + \mathfrak{G}_n^{-1}(\underline{b};\underline{w}) \geq \mathfrak{G}_n^{-1}(\underline{a}+\underline{b};\underline{w})),$$

*with equality if and only if $a_{n-1}b_n = a_n b_{n-1}$.*
*(c) If $(a_{n-1}, a_n), (b_{n-1}, b_n)$ are similarly ordered then*

$$\mathfrak{A}_n^{-1}(\underline{a};\underline{w})\mathfrak{A}_n^{-1}(\underline{b};\underline{w}) \geq \mathfrak{A}_n^{-1}(\underline{a}\,\underline{b};\underline{w}),$$

*with equality if and only if $a_n = a_{n-1}$ and $b_n = b_{n-1}$.*
*(d) If $W_1a_1 < W_2a_2 < \cdots$, $W_1/w_1 < W_2/w_2 < \cdots$ then*

$$\mathfrak{G}_n(\underline{\mathfrak{A}}^{-1};\underline{w}) \geq \mathfrak{A}_n(\underline{\mathfrak{G}}^{-1};\underline{w}),$$

*with equality if and only if $a_{n-2} = a_{n-1} = a_n$.*

Comments (i) Inequalities (a), (b), and (c) are analogous to (GA), **Geometric Mean Inequalities** (1) and (Č), respectively.

(ii) (a) is easily seen to be equivalent to (B), and was used by Nanjundiah to give a simultaneous proof of (GA), **Popoviciu's' Geometric-Arithmetic Mean Inequality Extension** (1), and **Rado's Geometric-Arithmetic Mean Inequality Extension** (1).

(iii) The inequality in (a) is also a consequence of **Jensen-Pečarić Inequalities** Reverse Inequality.

(iv) (b) has been used to prove a Popoviciu-type extension of **Power Mean Inequalities** (4).

(v) (c) can be used to give a Rado-type extension of (Č).

(vi) (d) is **Nanjundiah's Inequalities** (2) with a change of notation.

References *Bullen* [77], [79].

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**Nanjundiah's Mixed Mean Inequalities** *If $\underline{\mathfrak{A}}$, $\underline{\mathfrak{G}}$ are, respectively, the sequences of arithmetic, geometric, means of a sequence $\underline{a}$ with weight sequence $\underline{w}$ then*

$$\mathfrak{G}_n(\underline{\mathfrak{A}};\underline{w}) \geq \mathfrak{A}_n(\underline{\mathfrak{G}};\underline{w}),$$

*with equality if and only if $\underline{a}$ is constant.*

COMMENTS (i) This follows from **Nanjundiah's Inverse Mean Inequalities** (d) applied to the sequence $\underline{\mathfrak{A}}$ in place of $\underline{a}$.

(ii) The equal weight case of this result was shown by Nanjundiah to imply **Carleman's Inequality** (1).

EXTENSIONS *If $-\infty < r < s < \infty$ and $\underline{\mathfrak{M}}^{[r]}$, $\underline{\mathfrak{M}}^{[s]}$ are, respectively, the sequences of r-th, s-th, means of a sequence $\underline{a}$ with weight sequence $\underline{w}$, and if $\{W_1a_1, W_2a_2, \ldots\}$ and $\{w_1^{-1}W_1, w_2^{-1}W_2, \ldots\}$ are strictly increasing then*

$$\mathfrak{M}_n^{[r]}(\underline{\mathfrak{M}}^{[s]};\underline{w}) \geq \mathfrak{M}_n^{[s]}(\underline{\mathfrak{M}}^{[r]};\underline{w}).$$

COMMENT (iii) Although these results were proved by Nanjundiah he has not published his proofs. Proofs can be found in the references.

REFERENCES [H, pp. 94–95, 126, 136–141]; *Bullen* [77], [79], *Kedlaya* [152], *Matsuda* [203], *Mond & Pečarić* [221].

**Nanson's Inequality** *If $\underline{a}$ is a convex positive $(2n+1)$-tuple and if*

$$\underline{b} = \{a_2, a_4, \ldots, a_{2n}\}, \qquad \underline{c} = \{a_1, a_3, \ldots, a_{2n+1}\},$$

*then*

$$\mathfrak{A}_n(\underline{b}) \leq \mathfrak{A}_{n+1}(\underline{c}), \tag{1}$$

*with equality if and only if the elements of $\underline{a}$ are in arithmetic progression.*

COMMENTS (i) By taking $a_i = x^{i-1}, 1 \leq i \leq 2n+1, 0 < x < 1$, we get as a particular case *Wilson's inequalities*:

$$\frac{1+x^2+\cdots+x^{2n}}{x+x^3+\cdots+x^{2n-1}} > \frac{n+1}{n}, \text{ equivalently } \frac{1-x^{n+1}}{n+1} > \frac{1-x^n}{n}\sqrt{x}. \tag{2}$$

(ii) Inequality (2) has been improved; see [AI]

EXTENSIONS *(a)* [STEINIG] *Under the above assumptions*

$$\mathfrak{A}_n(\underline{b}) \leq \mathfrak{A}_{2n+1}(\underline{a}) \leq \mathfrak{A}_{n+1}(\underline{c}) \leq \sum_{i=1}^{2n+1}(-1)^{i+1}a_i.$$

*(b)* [ANDRICA, RAŞA & TOADER] *If $m \leq \Delta^2\underline{a} \leq M$ then*

$$\frac{2n+1}{6}m \leq \mathfrak{A}_{n+1}(\underline{c}) - \mathfrak{A}_n(\underline{b}) \leq \frac{2n+1}{6}M.$$

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COMMENT (iii) Extension (b) follows because the hypothesis implies the convexity of the sequences $\alpha_i = a_i - mi^2/2,\ \beta_i = Mi^2/2 - a_i, i = 1, 2, \ldots.$

REFERENCES [AI, pp. 205–206], [H, pp. 170–171], [HLP, p. 99], [PPT, pp. 247–251].

## Napier's Inequality *If $0 < a < b$ then*

$$\frac{1}{b} \leq \frac{\log b - \log a}{b - a} \leq \frac{1}{a}.$$

COMMENTS (i) This inequality has several very simple proofs such as one using the **Mean Value Theorem of Differential Calculus** COMMENT (ii), one using the **Integral Mean Value Theorems** (1) or just considering the area under the graph of the log function.

(ii) This inequality is a very special case of **Logarithmic Mean Inequalities** (1):

$$a < \mathfrak{L}(a, b) < b.$$

## Nash's Inequality *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable then*

$$\left(\int_{\mathbb{R}^n} |f|^2\right)^{(n+2)/n} \leq A_n \int_{\mathbb{R}^n} |\nabla f|^2 \left(\int_{\mathbb{R}^n} |f|\right)^{4/n}.$$

COMMENT A sharp value for the constant $A_n$ has been determined in the reference.

REFERENCE *Lieb & Loss* [LL, pp. 220–222];*Carlen & Loss* [80].

## n-Convex Function Inequalities *(a) If $f$ is $n$-convex on interval $[a, b]$ and if $x_0, \ldots, x_n$ are any $(n + 1)$ distinct points from that interval then*

$$[\underline{x}; f] = [x_0, \ldots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)} \geq 0; \tag{1}$$

*where*

$$w(x) = w_n(x; \underline{x}) = \prod_{i=0}^{n} (x - x_i);$$

*if $f$ is strictly $n$-convex then (1) is strict for all choices of such $\underline{x}$.*
*(b) If $f$ is $n$-convex on $[a, b]$ and if $x_0, \ldots, x_{n-1},\ y_0 \ldots, y_{n-1}$ are two sets of distinct points in $[a, b]$ with $x_i \leq y_i, 0 \leq i \leq n - 1$, then*

$$[x_0, \ldots, x_{n-1}; f] \leq [y_0, \ldots, y_{n-1}; f].$$

*(c)* [PEČARIĆ & ZWICK] *If $f$ is $n + 2$-convex on $[a, b]$, and if $\underline{a} \prec \underline{b}$, where $\underline{a}, \underline{b}$ are $n + 1$-tuples with elements in $[a, b]$ then*

$$[\underline{a}; f] \leq [\underline{b}; f].$$

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COMMENTS (i) (a) is just the definition of $n$-convexity and strict $n$-convexity. In addition the definition of a $n$-concave function, strictly $n$-concave function, is one for which ($\sim$1) holds, strictly.

(ii) The quantity on the left-hand side of (1) is called the *$n$-th divided difference of $f$ at the points* $x_0, \ldots, x_n$. It is sometimes written $[\underline{x}]f = [x_0, \ldots, x_n]f$.

If $n = 2$, $[\underline{x}; f]$ is just the left-hand side of **Convex Function Inequalities** (4), and if $n = 1$ it is the elementary *Newton ratio*, $(f(x_0) - f(x_1))/(x_0 - x_1)$.

(iii) Clearly 2-convex functions are just convex functions; 1-convex functions are increasing functions and 0-convex functions are just non-negative functions.

(iv) If $n \geq 1$ and $f^{(n)} \geq 0$ then $f$ is $n$-convex, and if $f^{(n)} > 0$, except possibly at a finite number of points, then $f$ is strictly $n$-convex. So, for instance, the exponential function is strictly $n$-convex for all $n$, the logarithmic function is strictly $n$-convex if $n$ is odd, but is strictly $n$-concave if $n$ is even, while $x^\alpha$ is strictly $n$-convex if $\alpha > n - 1$, $\alpha < 0$ and $n$ is even, or if $\alpha$ is positive, not an integer an $k - [\alpha]$ is odd.

(v) The inequality in (b) just says that if $f$ is $n$-convex then the $n-1$-th divided difference is an increasing function of each of its variables; in the case $n = 2$ this is just **Convex Function Inequalities** (5).

(vi) The results **Convex Function Inequalities** DERIVATIVE INEQUALITIES extend to $n$-convex functions if the derivative of order $n - 1$ is interpreted in the appropriate way.

(vii) (c) is an example of **Schur Convex Function Inequalities** (b).

(viii) See also: **Čakalov's Inequality** COMMENTS (iv), **Farwig & Zwick's Lemma, Hermite-Hadamard Inequality** COMMENTS (iv), **Levinson's Inequality, Quadrature Inequalities** COMMENTS (iv), **Quasi-Convex Function Inequalities** COMMENTS (v).

REFERENCES [GI3, pp. 379–384], [H, pp. 54–57], [MPF, pp. 4–5], [PPT, pp. 14–18, 76]; *Popoviciu* [PT], *Roberts & Varberg* [RV, pp. 237–240].

## n-Convex Sequence Inequalities

*(a) If $\underline{a}$ is an $n$-convex real sequence, $n \geq 2$, then*

$$\Delta^n \underline{a} \geq 0; \tag{1}$$

*and if $\underline{a}$ is bounded then*

$$\Delta^k \underline{a} \geq 0, \quad 1 \leq k \leq n-1; \tag{2}$$

*in particular $\underline{a} \geq \underline{0}$.*

*(b)* [ŌZEKI] *If $\underline{a}$ in a real $n$-convex sequence so is $\mathfrak{A}_n(\underline{a})$.*

COMMENTS (i) (1) is just the definition of an $n$-convex sequence.

(ii) In particular a sequence that is 1-convex is decreasing, and 2-convex is just convex, see **Convex Sequence Inequalities**. As usual if (1) holds strictly we say that $\underline{a}$ is *strictly $n$-convex*, while if ($\sim$1) holds, strictly, we say that *the sequence is $n$-concave, strictly $n$-concave.*

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(iii) If the function $f$ is such that $(-1)^n f$ is $n$-convex then if $a_i = f(i), i = 1, 2, \ldots$ the sequence $\underline{a}$ is $n$-convex.

(iv) (b) is a generalization of **Convex Sequence Inequalities** (d).

Extension *If $\underline{a}$ in a real $n$-convex sequence then so is the sequence $\mathfrak{A}_n(\underline{a}; \underline{w})$ if and only if $\underline{w}$ is given by $w_n = w_0\binom{\alpha+n+1}{n}, n \geq 1$, for some positive real numbers $w_0, \alpha$.*

Comments (v) The case $n = 2$ of this extension is due to Vasić, Kečkić, Lacković & Ž. Mitrović. The general case is a result of work by Lacković & Simić, and Toader.

(vi) See also **Čebišev's Inequality** Comments (v).

References [H, pp. 11–16], [PPT, pp. 21, 253–257, 277–279].

## Nehari's Inequality

See: **Inverse Hölder's Inequalities** (b).

## Nesbitt's Inequality

*If $a, b, c$ are positive real numbers then:*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Comment This simple inequality, that is a special case of Shapiro's inequality, has many proofs and has been generalized in various ways. See: **Shapiro's Inequality** Comments (iii).

References *Bulajich Manfrino, Ortega & Delgado* [BOD, pp. 16–17]; *Bencze & Pop* [55]; http://en.wikipedia.org/wiki/Nesbitt's_inequality.

## Newman's Conjecture[38]

*If $p$ is a complex polynomial of degree $n$ with coefficients $\pm 1$ then if $n$ is large enough*

$$M_1(p; 1) \leq n$$

Comment In any case by (C) $M_1(p; 1) \leq M_2(p; 1) = n + 1$ and Newman proves that the 1 can be replaced by 0.97. In the reference this is improved to 0.8250041 for $n$ large enough.

Reference *Habsieger* [137].

## Newton's Inequalities

*If $n \geq 2$, $c_0 c_n \neq 0$ and if all the zeros of $\sum_{i=1}^n c_i x^i = \sum_{i=1}^n \binom{n}{i} d_i x^i$ are real then for $1 \leq i \leq n-1$,*

$$d_i^2 \geq d_{i-1}d_{i+1} \quad \textit{and} \quad c_i^2 > c_{i-1}c_{i+1}; \quad (1)$$

$$d_i^{1/i} \geq d_{i+1}^{1/i+1} \quad \textit{and} \quad c_i^{1/i} > c_{i+1}^{1/i+1}. \quad (2)$$

[38] This is D. J. Newman.

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*The inequalities on the left are strict unless the zeros are all equal.*

COMMENTS (i) The right inequalities are weaker than the corresponding left inequalities.

(ii) The first inequality in (2) follows by writing the first inequality in (1) as $d_k^{2k} \geq d_{k-1}^k d_{k+1}^k$ for all $k, 1 \leq k \leq i < n$, and multiplying; the second inequality in (2) follows similarly.

(iii) By writing the $c_i$ in terms of the roots of the polynomial in the above results leads to an important inequality for elementary symmetric functions and means; see **Notations 3** (1), and **Elementary Symmetric Function Inequalities** (1), (2).

(iv) The particular cases of (2) $d_1 \geq d_n^{1/n}$, $c_1 > c_n^{1/n}$ are the equal weight case of (GA).

(v) Various extensions of these results have been given by Mitrinović, see [AI].

REFERENCES [AI, pp. 95–96], [H, pp. 2–3], [HLP, pp. 51–55, 104–105].

**N-function Inequalities** If $p : [0, \infty[ \to [0, \infty[$ is right continuous, increasing, $p(0) = 0, p(t) > 0, t > 0$, with $\lim_{t\to\infty} p(t) = \infty$ we will say that $p \in \mathcal{P}$. Then if $q(s) = \sup_{p(t)\leq s} t, s \geq 0$, $q \in \mathcal{P}$ and is called the *right inverse of* $p$.

A function $M$ is called an *N-function* if it can be written as

$$M(x) = \int_0^{|x|} p, \quad x \in \mathbb{R},$$

for some $p \in \mathcal{P}$; $p$ is the right derivative of $M$. If then $q$ is the right inverse of $p$ the N-function

$$N(x) = \int_0^{|x|} q, x \in \mathbb{R},$$

is called the *complementary N-function to* $M$.

YOUNG'S INEQUALITY[39] *If $M, N$ are complementary N-functions then for all $u, v \in \mathbb{R}$*

$$uv \leq M(u) + N(u),$$

*with equality if and only if either $u = q(v)$ or $v = p(u)$, for $u, v \geq 0$, where $p, q$ are the right derivatives of $M, N$, respectively.*

COMMENTS (i) If $M, N$ are complementary N-functions and $K$ is a compact set in $\mathbb{R}^n$, define

$$||f||_M = \sup \left| \int_K fg \right|,$$

[39]This is W. H. Young.

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where the sup is over all functions $g$ for which

$$\int_K N \circ g \leq 1.$$

Then $||\cdot||_M$ satisfies the **Norm Inequalities** (1).

(ii) Particular cases of N-functions give the standard $||\cdot||_p$.

REFERENCES [EM, vol. 7, pp. 19–20], [MPF, p. 382], [PPT, pp. 241–242]; *Krasnocel'skiĭ & Rutickiĭ* [KR].

**Nikol'skiĭ's Inequality** *If $T_n$ is a trigonometric polynomial of degree at most $n$, and if $1 \leq p < q < \infty$ then*

$$||T_n||_{q,[-\pi,\pi]} \leq An^{(p^{-1}-q^{-1})}||T_n||_{p,[-\pi,\pi]},$$

*where $A$ is a constant.*

COMMENT For a definition of trigonometric polynomial of degree at most $n$ see: **Trigonometric Polynomial Inequalities**.

REFERENCE *Zygmund* [Z, vol. I, p. 154].

**Nirenberg's Inequality** See: **Friederichs's Inequality** COMMENTS (ii), **Sobolev's Inequalities** (b).

**Normal Distribution Function Inequalities** See: **Error Function Inequalities**.

**Norm Inequalities** *If $X$ is any vector space and $||\cdot|| : X \to \mathbb{R}$ is a norm on $X$ then for all $x, y$ in $X$,*

$$||x+y|| \leq ||x|| + ||y||; \tag{1}$$
$$||x-z|| \leq ||x-y|| + ||y-z||. \tag{2}$$

COMMENTS (i) A *norm on a real, (complex), vector space X* is a function $||\cdot|| : X \to [0,\infty[$ that is positive except for $x = 0$, $||0|| = 0$, and satisfies $||\lambda x|| = |\lambda|\,||x||$, $\lambda \in \mathbb{R}, (\mathbb{C})$, and (1). Then $X$ is called a *normed space*. A complete normed space is called a *Banach space*. If only $\lambda \geq 0$ required above then $||\cdot||$ is a *generalized norm*. See also: **Banach Algebra Inequalities**.

(ii) Given that $||0|| = 0$, (1) and (2) are equivalent; (2) is a generalization of (T).

(iii) It follows from **Absolute Value Inequalities**, or (T), that $|\cdot|$ is a norm on $\mathbb{R}^n$. More generally if $1 \leq p \leq \infty$ then by (M), or **Inf and Sup Inequalities** (1), $||\cdot||_p$ is also a norm on $\mathbb{R}^n$; of course $||\underline{a}||_2 = |\underline{a}|$.

(iv) Other examples of a normed spaces are inner product spaces, see: **Inner Product Inequalities** COMMENTS (iv), the *Hardy Spaces*. See: **Analytic Function Inequalities** COMMENTS (i).

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(v) If zero values are allowed for non-zero elements we have what is called a *semi-norm*; so for instance the space $\mathcal{L}^p([a,b]), p \geq 1$ has $||\cdot||_p$ as a semi-norm; equivalently it is a normed space if we identify functions that are equal almost everywhere.

(vi) If the norm satisfies the parallelogram identity, see **Inner Product Inequalities** (3), then the Banach space is a Hilbert space.

EXTENSIONS *(a) If $X$ is a normed space with norm $||\cdot||$ and if $x_i \in X, 1 \leq i \leq n$, and if $\underline{w}$ is a positive $n$-tuple then*

$$\left|\left|\sum_{i=1}^{n} w_i x_i\right|\right| \leq \sum_{i=1}^{n} w_i ||x_i||. \tag{3}$$

*If $\underline{w}$ has $w_1 > 0$ and $w_i < 0, 2 \leq i \leq n$ then ($\sim$3) holds.*
*(b)* [PEČARIĆ & DRAGOMIR] *If $X$ is a normed space with norm $||\cdot||$ and if $x, y \in X$, and if $0 \leq t \leq 1$ then*

$$\left|\left|\frac{x+y}{2}\right|\right| \leq \int_0^1 ||\,\overline{1-t}x + ty||\,\mathrm{d}t \leq \frac{||x|| + ||y||}{2}.$$

*(c) If $X$ is a normed space with norm $||\cdot||$ and if $x, y \in X$, and if $p, q, r \in \mathbb{R}$ with $pq(p+q) > 0, r \geq 1$ then*

$$\frac{||x+y||^r}{p+q} \leq \frac{||x||^r}{p} + \frac{||y|^r}{q}; \tag{4}$$

*if $pq(p+q) < 0$ then ($\sim$4) holds*

COMMENTS (vii) Inequalities (3) and ($\sim$3) should be compared to (J) and ($\sim$J); see **Jensen's Inequality, Jensen-Pečarić Inequalities** REVERSE INEQUALITY. In particular by taking $n = 2, W_2 = 1$ in (3) we see that $||\cdot|| : X \to \mathbb{R}$ is convex. So (b) is a special case of **Hermite-Hadamard Inequality** (1).

(viii) For a particular case of (4) see: **Complex Number Inequalities** (4).

(ix) See also: **Dunkl & Williams Inequality, Hajela's Inequality, Matrix Norm Inequalities, Metric Inequalities, N-functions Inequalities** COMMENTS (i).

REFERENCES [EM, vol. 6, pp. 459–460], [MPF, pp. 483–594].

## Nosarzewska's Inequality See: **Jarníck's Inequality**.

## n-Simplex Inequality *If $R$ is the outer radius, and $r$ the inner radius of an $n$-simplex then*

$$R \geq nr.$$

COMMENT For definitions of the radii see: **Isodiametric Inequalities** COMMENTS (i).

REFERENCES [EM, Supp., p. 469]; *Mitrinović, Pečarić & Volenec* [MPV].

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**Number Theory Inequalities** *(a) If $x > 0$ we define: $\psi(x) = \sum_{n\le x}\Lambda(n)$, where $\Lambda(n) = \log p$ if $n$ is a power of the prime $p$, and otherwise it is $0$; and $\pi(x) =$ the number of primes $p$ with $p \le x$. Then*

$$\frac{\psi(x)}{x} \le \frac{\pi(x)\log x}{x} < \frac{1}{\log x} + \frac{\psi(x)\log x}{x\log(x/\log^2 x)}.$$

*(b) If $x > 10^6$ then*

$$\pi(x) < \frac{x}{\log x - 1.08366};$$

*while if $x \ge 4$,*

$$\pi(x) < \frac{x}{\log x - 1.11}.$$

*(c)* [Dirichlet] *If $d(n) = \sum_{d|n} 1$, the number of positive divisors of $n$, then*

$$\left|\sum_{n\le x} d(n) - \big(x\log x + (2\gamma-1)\big)\right| \le 4\sqrt{x},$$

*where $\gamma$ is Euler's constant.*

Comments (i) As usual $\log^2 x = \log\circ\log x$.

(ii) This inequality in (a) is fundamental in proving the *Prime Number Theorem*,

$$\lim_{x\to\infty}\frac{\pi(x)\log x}{x} = 1.$$

(iii) For a definition of $\gamma$ see: **Euler's Constant Inequalities**.

(iv) See also: **Chi Inequality**.

A different kind of result is the following.

*If $N$ is a positive integer, $N = \sum_{1=0}^{n} a_i 10^i$, $n > 2$, $a_i$ an integer, $0 \le a_i \le 9$, $a_n \ne 0$, $0 \le i \le 9$, then*

$$N \ge \frac{1\overbrace{0\ldots0}^{k+1}\overbrace{9\ldots9}^{n-k}}{1+9(n-k-1)}A_n,$$

*where $A_n = \sum_{i=0}^{n} a_i$, and $k$ is unique integer such that*

$$2 + \sum_{i=0}^{k} 10^i \le n \le 2 + \sum_{i=0}^{k+1} 10^i.$$

References *Rudin* [R91, pp. 212–213], *Mitrinović & Popadić* [MP]; *Cimadevilla Villacorta* [92], *Gu & Liu C.* [136], *Panaitopol* [254].

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# 15 Operator–Ōzeki

**Operator Inequalities** See: **Furuta's Inequality, Halmos's Inequalities, Hansen's Inequality, Heinz-Kato-Furuta Inequality, Hölder-McCarthy Inequality, Löwner-Heinz Inequality, Young Inequality.**

**Opial's Inequalities** *(a) If $f$ is absolutely continuous on $[0, h]$ with $f(0) = 0$ then*

$$\int_0^h |ff'| \le \frac{h}{2}\int_0^h f'^2, \tag{1}$$

*with equality if and only if $f(x) = cx$.*
*The constant is best possible.*
*(b) If $\underline{a}$ is a non-negative $(2n+1)$-tuple satisfying*

$$a_{2k} \le \min\{a_{2k-1}, a_{2k+1}\}, \quad 1 \le k \le n,$$

*and if $a_0 = 0$ then*

$$\left(\sum_{k=0}^{n}(\Delta a_{2k})\right)^2 \ge \sum_{k=0}^{2n+1}(-1)^{k+1}a_k.$$

COMMENT (i) Inequality (1), usually referred to as *Opial's Inequality*, has been the object of much study, and many extensions can be found in the references. In particular there are extensions that allow higher derivatives, and to higher dimensions. Some of the generalizations are known by other names; for instance, *Maroni's Inequality, Godunova-Levin Inequality*.

EXTENSIONS *(a)* [YANG G. S.] *If $f$ is absolutely continuous on $[a, b]$ with $f(a) = 0$ and if $r \ge 0, s \ge 1$ then*

$$\int_a^b |f|^r|f'|^s \le \frac{s}{r+s}(b-a)^r\int_a^b |f'|^{r+s}. \tag{2}$$

*(b)* [PACHPATTE] *Let $f \in \mathcal{C}^1(I)$, where $I = \{\underline{x}; \underline{a} \le \underline{x} \le \underline{b}\}$ is an interval in $\mathbb{R}^n$, with $f$ zero on $\partial I$, then if $r, s \ge 1$,*

$$\int_I |f|^r|\nabla f|^s \le M\int_I |\nabla f|^{r+s},$$



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*where*

$$M = \frac{1}{n2^r}\left(\sum_{i=1}^{n}(b_i - a_i)^{r(r+s)/s}\right)^{s/(r+s)}.$$

*(c)* [Fitzgerald] *If* $f \in \mathcal{C}^2([0,h])$ *with* $f(0) = f'(0) = 0$ *then*

$$\int_0^h |ff'| \le \frac{h^3}{4\ \pi^2}\int_0^h f''^2. \tag{1}$$

Comments (ii) Yang proved (2) for $r, s \ge 1$; the more general result is due to Beesack.

(iii) Extensions of (a) in which different weight functions are allowed in each integral have been given by Beesack & Das.

(iv) (c) can be obtained by combining (1) and **Wirtinger's Inequality**; as a result the constant is not best possible. A sharp result involving higher order derivatives can be found in [GI4, GI7].

Discrete Analogues *(a) If* $\underline{a}$ *is a real n-tuple and if* $a_0 = 0$ *then*

$$\sum_{i=0}^{n-1} a_{i+1}|\Delta a_i| \le \frac{n+1}{2}\sum_{i=0}^{n-1}(\Delta a_i)^2.$$

*(b)* [Lee, C. M.] *If* $\underline{a}$ *is a non-negative, increasing n-tuple with* $a_0 = 0$, $r, s > 0$, $r + s \ge 1$ *or* $r, s < 0$ *then*

$$\sum_{i=1}^{n} a_i^r(\tilde{\Delta} a_i)^s \le K_{n,r,s}\sum_{i=1}^{n}(\tilde{\Delta} a_i)^{r+s},$$

*where*

$$K_{0,r,s} = \frac{s}{r+s}, \quad K_{n,r,s} = \max\{K_{n-1,r,s} + \frac{rn^{r-1}}{r+s}, \frac{s(n+1)^r}{r+s}\}.$$

Comment (iv) Other discrete analogues of (1) have been given by Pachpatte.

References [AI, pp. 154–162, 351], [GI4, pp. 25–36], [GI7, pp. 157–178], [PPT, p. 162]; *Agarwal & Pang* [AP]; *Lee, C. M.* [172],*Pachpatte* [240, 241, 242].

**Oppenheim's Inequality** *Let* $\underline{a}, \underline{b}, \underline{w}$ *be three positive n-tuples with* $\underline{a}, \underline{b}$ *increasing and satisfying*

$$a_i \le b_i,\ 1 \le i \le n - m, \qquad a_i \ge b_i,\ n - m + 2 \le i \le n,$$

*for some* $m, 1 < m < n$. *Then if* $t < s,\ s, t \in \mathbb{R}$,

$$\mathfrak{M}_n^s(\underline{a}, \underline{w}) \le \mathfrak{M}_n^s(\underline{b}, \underline{w}) \Longrightarrow \mathfrak{M}_n^t(\underline{a}, \underline{w}) \le \mathfrak{M}_n^t(\underline{b}, \underline{w}).$$

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COMMENTS (i) The case $n = 3, s = 1, t = 0, w_1 = w_2 = w_3$ is the original result of Oppenheim.

(ii) For other inequalities due to Oppenheim see **Hadamard Product Inequalities** (e), **Oppenheim's Problem**.

REFERENCES [AI, pp. 309-310], [H, pp. 290–294].

**Oppenheim's Problem** *If $p > 0,\ 0 \le x \le \pi/2$ then*

$$\frac{q\sin x}{1+p\cos x} \le x \le \frac{r\sin x}{1+p\cos x} \tag{1}$$

*if any of the following holds:*

$$\begin{array}{llll} (a) & 0 < p < \frac{1}{2} & \textit{and} & q = p+1, r = \frac{\pi}{2}; \\ (b) & \frac{1}{2} \le p < \frac{\pi}{2} - 1 & \textit{and} & q = 4p(1-p^2),\ r = \frac{\pi}{2}; \\ (c) & \frac{\pi}{2} - 1 \le p < \frac{2}{\pi} & \textit{and} & q = 4p(1-p^2),\ r = p+1; \\ (d) & p \ge \frac{2}{\pi} & \textit{and} & q = \frac{\pi}{2},\ r = p+1. \end{array}$$

*In (a) and (d) $q$ and $r$ are the best constants; in (b) and (c) $r$ is the best constant.*

COMMENTS (i) This is a problem posed by Oppenheim. Oppenheim and Carver gave a partial solution and the complete solution is due to Zhu.

(ii) This result has been extended to Bessel functions.

A PARTICULAR CASE *If $0 \le t \le 1$ then*

$$\frac{\pi}{2}\frac{t}{1+\sqrt{1-t^2}} \le \arcsin t \le 2\frac{t}{1+\sqrt{1-t^2}}.$$

*Furthermore $\pi/2$ and $2$ are the best possible constants.*

COMMENT (iii) This follows from (1) by the substitutions: $p = 1, q = \pi/2, r = 2, t = \sin x$. The result is related to the **Shafer-Fink Inequality**.

REFERENCES [AI, p. 238]; *Baricz & Zhu* [50], *Zhu* [337, 338].

**Order Inequalities** *(a)* [RADO] *$\underline{a} \prec \underline{b}$ if and only if there is a doubly stochastic matrix $S$ such that $\underline{a} = \underline{b}S$.*

*(b)* [KARAMATA] *If $\underline{a}, \underline{b} \in I^n$, $I$ an interval in $\mathbb{R}$, then $\underline{a} \prec \underline{b}$ if and only if for all functions $f$, convex on $I$,*

$$\sum_{i=1}^{n} f(a_i) \le \sum_{i=1}^{n} f(b_i). \tag{1}$$

*(c)* [WARD] *If $\underline{a}, \underline{b}$ are $n$-tuples of non-negative integers, with $\underline{a} \prec \underline{b}$ and if $\underline{x}$ is a positive $n$-tuple then*

$$\sum ! x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_n}^{a_n} \le \sum ! x_{i_1}^{b_1} x_{i_2}^{b_2} \dots x_{i_n}^{b_n}.$$

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*(d )* [Tomić] *If* $\underline{a}, \underline{b} \in I^n$*, I an interval in* $\mathbb{R}$*, then* $\underline{a} \prec^w \underline{b}$ *if and only if for all functions f, convex and increasing on I,*

$$\sum_{i=1}^{n} f(a_i) \le \sum_{i=1}^{n} f(b_i).$$

*(e) If* $\underline{a}$ *and* $\underline{b}$ *are decreasing non-negative n-tuples such that*

$$\prod_{i=1}^{k} a_i \le \prod_{i=1}^{k} b_i,\ 1 \le k \le n,$$

*then for all p,* $p > 0$*,* $\underline{a}^p \prec^w \underline{b}^p$*.*

Comments (i) (b) is a fundamental property both of the order and of convex functions.

(ii) If $a_i = \cdots = a_n = \mathfrak{A}_n(\underline{b})$ then (b) reduces to the equal weight case of (J).

(iii) When for some doubly stochastic $S$, $\underline{a} = \underline{b}S$, as in (a), we sometimes say that $\underline{a}$ *is an average of* $\underline{b}$. For this equality it is both necessary and sufficient that $\underline{a}$ lie in the convex hull of the $n!$ points obtained by permuting the elements of $\underline{b}$. This has been improved by Zhan to at most $n$ permutations of the elements of $\underline{b}$; this is best possible as the example $\underline{e} \prec n\underline{e}_1$ shows.

(iv) The expressions on both sides of Ward's inequality are symmetric polynomials that are homogeneous of order $A_n (= B_n)$; (for definitions of these terms see **Segre's Inequalities**). If $\underline{a} = \{1, 1, \ldots, 1\}, \underline{b} = \{n, 0, \ldots, 0\}$ then the inequality is just the equal weight case of (GA).

(v) (e) is an immediate consequence of (d) applied to the function $f(x) = e^{px}$, $p > 0$.

Extension [Fuchs] *If* $\underline{a}, \underline{b} \in I^n$*, I an interval in* $\mathbb{R}$*, and* $\underline{w}$ *is a real n-tuple then*

$$\sum_{i=0}^{n} w_i f(a_i) \le \sum_{i=0}^{n} w_i f(b_i) \tag{2}$$

*for every convex function f if and only if* $\underline{a}, \underline{b}$ *are decreasing and*

$$\sum_{i=0}^{k} w_i a_i \le \sum_{i=0}^{k} w_i b_i, 1 \le k < n, \quad \sum_{i=0}^{n} w_i a_i = \sum_{i=0}^{n} w_i b_i.$$

Comment (vi) Inequality (1) is referred to as *Karamata's inequality* and inequality (2) is called *Fuchs's Inequality*.

Integral Analogue [Fan & Lorentz] $\alpha \prec \beta$ *on* $[a, b]$ *if and only if for all convex functions f*

$$\int_a^b f \circ \alpha \le \int_a^b f \circ \beta.$$

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COMMENT (vii) This concept is used to prove many particular inequalities. See: **Absolutely and Completely Monotonic Function Inequalities** (C), **Barnes's Inequalities** (B), **Chong's Inequalities** (B), **Muirhead Symmetric Function and Mean Inequalities** (1), **Permanent Inequalities** (C), **Schur Convex Function Inequalities** (B), **Shannon's Inequality** (B), **Steffensen's Inequalities** COMMENTS (IV), **Walker's Inequality** COMMENTS, **Weierstrass's Inequality** COMMENTS (II).

REFERENCES [AI, pp. 162–170], [BB, pp. 30–33], [EM, vol. 6, pp. 74–76], [GI4, pp. 41–46], [H, pp. 21–25], [MOA, pp. 29–41, 155–162], [PPT, pp. 319–332]; *König* [Kon, p. 35]; *Ward* [325], *Zhan* [334].

**O'Shea's Inequality** *If* $a_1 \geq \cdots \geq a_n > 0$, $a_1 \neq a_n$, *and if* $a_{n+k} = a_k, 1 \leq k \leq n$, *then*

$$\sum_{i=1}^{n}\left(a_i^k - \prod_{j=1}^{k} a_{i+j}\right) \geq 0; \tag{1}$$

*further (1) is strict if* $k > 1$.

COMMENT With $k = n$ this inequality is just (GA).

REFERENCE [H, pp. 102–103].

**Ostrowski's Inequalities** *(a) If* $\underline{a}, \underline{b}, \underline{c}$ *are real* $n$*-tuples with* $\underline{a} \not\sim \underline{b}$, $\underline{a}.\underline{c} = 0$ *and* $\underline{b}.\underline{c} = 1$, *then*

$$\frac{|\underline{a}|^2}{|\underline{a}|^2|\underline{b}|^2 - |\underline{a}.\underline{b}|^2} \leq |\underline{c}|^2. \tag{1}$$

*with equality if and only if*

$$c_i = \frac{b_i|\underline{a}|^2 - a_i\, \underline{a}.\underline{b}}{|\underline{a}|^2|\underline{b}|^2 - |\underline{a}.\underline{b}|^2},\ 1 \leq i \leq n.$$

*(b) Let* $f$ *be bounded on* $[a,b]$ *with* $\alpha \leq f \leq A$, *and assume that* $g$ *has a bounded derivative on* $[a,b]$ *then*

$$\left|\mathfrak{A}_{[a,b]}(fg) - \mathfrak{A}_{[a,b]}(f)\mathfrak{A}_{[a,b]}(g)\right| \leq \frac{1}{8}(b-a)(A-\alpha)||g'||_{\infty,[a,b]}.$$

*The constant* $1/8$ *is best possible.*
*(c) If* $f \in \mathcal{C}^1([a,b])$ *then*

$$\left|f(x) - \mathfrak{A}_{[a,b]}(f)\right| \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)}||f'||_{\infty,[a,b]}.$$

*The function on the right-hand side cannot be replaced by a smaller function.*

COMMENTS (i) Inequality (1) can be regarded as a special case of **Bessel's Inequality** for non-orthonormal vectors. The result also holds for complex $n$-tuples if throughout the inner product is replaced by $\underline{u}.\overline{\underline{v}}$.

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(ii) (b) is converse of (Č), or of **Power Mean Inequalities** (2), in the case $q = r = s = 1$; see also **Grüsses's Inequalities** (a).

EXTENSIONS *(a)* [FAN & TODD] *If $\underline{a}, \underline{b}$ are real $n$-tuples with $a_ib_j \neq a_jb_i$, $i \neq j$, $1 \leq i, j \leq n$, then*

$$\frac{|\underline{a}|^2}{|\underline{a}|^2|\underline{b}|^2 - |\underline{a}.\underline{b}|^2} \leq \binom{n}{2}^{-2} \sum_{i=1}^{n} \left( \sum_{j \neq i, j=1}^{n} \frac{a_j}{a_jb_i - a_ib_j} \right)^2. \tag{2}$$

*(b)* [ANASTASSIOU] *If $f \in \mathcal{C}^n([a,b]), n \geq 2$ with $f^{(k)}(x) = 0, 1 \leq k \leq n-1$, then*

$$\left| f(x) - \mathfrak{A}_{[a,b]}(f) \right| \leq \frac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)!(b-a)} \sup_{a \leq x \leq b} |f^{(n)}(x)|.$$

*The function on the right-hand side cannot be replaced by a smaller function.*

COMMENTS (iii) Inequality (2) gives **Chassan's Inequality** on putting $a_i = \sin\alpha_i$ and $b_i = \cos\alpha_i$, $0 \leq \alpha_i \leq \pi$, $\alpha_i \neq \alpha_j, , i \neq j$, $1 \leq i, j \leq n$.

(iv) Further extensions can be found in the references. Other inequalities due to Ostrowski are in **Matrix Norm Inequalities, Schur Convex Function Inequalities** (b), **Trigonometric Integral Inequalities**.

REFERENCES [AI, pp. 66–70], [H, p. 198], [MPF, pp. 92–95], [PPT, pp. 209–210]; *Anastassiou* [41], *Fink* [115]

## Ōzeki's Inequalities

*(a) If $\underline{a}$ is a real sequence then*

$$\sum_{i=1}^{n-1} a_ia_{i+1} \leq a_na_1 + \cos\frac{\pi}{n} \sum_{i=1}^{n} a_i^2. \tag{1}$$

*(b) If $\underline{a}$ is a real $n$-tuple and $p > 0$ then*

$$\min_{i \neq j} |a_i - a_j|^p \leq C_{n,p} \min_{x \in \mathbb{R}} \sum_{i=1}^{n} |a_i - x|^p$$

*where*

$$C_{n,p}^{-1} = \begin{cases} 2\sum_{i=1}^{(n-1)/2} i^p, & \text{if } n \text{ is odd,} \\ \min\{1, 2^{1-p}\} \sum_{i=1}^{n/2} (2i-1)^p, & \text{if } n \text{ is even.} \end{cases}$$

COMMENTS (i) (1) can be used to obtain a discrete version of **Wirtinger's Inequality**.

(ii) For other inequalities by Ōzeki see: **Complete Symmetric Function Inequalities** (C), **Convex Sequence Inequalities** (D), **Elementary Symmetric Function Inequalities** EXTENSIONS (F), **n-Convex Sequence Inequalities** (B).

REFERENCES [AI, pp. 202, 340–341], [GI4, pp. 83–86], [H, pp. 261], [MPF, pp. 438–439], [PPT, p. 277].

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# 16 Pachpatte–Ptolemy

**Pachpatte's Series Inequalities** *If $\underline{a}$ is a non-negative sequence, and $p, q, r \geq 1$ then f*

$$\begin{aligned}\sum_{k=1}^{n} a_k A_k &\leq \frac{n+1}{2}\sum_{k=1}^{n} a_k^2;\\ \sum_{k=1}^{n} A_k^{p+q} &\leq \big((p+q)(n+1)\big)^q \sum_{k=1}^{n} a_k^q A_k^p;\\ \sum_{k=1}^{n} a_k^r A_k^{p+q} &\leq \big((p+q+r)(n+1)\big)^q \sum_{k=1}^{n} a_k^{q+r} A_k^p.\end{aligned}$$

COMMENTS (i) These are proved using (H) and **Davies & Petersen Inequality**.

(rn2) For other series results see: **Series Inequalities**.

REFERENCE *Pachpatte* [248].

**Padoa's Inequality** See: **Adamović's Inequality** COMMENTS (ii).

**Paley's Inequalities** *(a) If $1 < p \leq 2$, $f \in \mathcal{L}^p(a,b)$ and if $\phi_n, n \in \mathbb{N}$ is a uniformly bounded orthonormal sequence of complex valued functions defined on an interval $[a,b]$, with $\sup_{a\leq x\leq b}|\phi_n(x)| \leq M, n \in \mathbb{N}$, and if $\underline{c}$ is the sequence of Fourier coefficients of $f$ with respect to $\phi_n, n \in \mathbb{N}$, then*

$$\left(\sum_{n\in\mathbb{N}} |c_n|^p n^{p-2}\right)^{1/p} \leq A_p M^{(2-p)/p}||f||_{p,[a,b]}.$$

*(b) If $\phi_n, n \in \mathbb{N}$, is as in (a), $q \geq 2$ and $\left(\sum_{n\in\mathbb{N}} |c_n|^q n^{q-2}\right)^{1/q} < \infty$, $\underline{c} = \{c_n, n \in \mathbb{N}\}$ a complex sequence, then there is a function $f \in \mathcal{L}^q(a,b)$ having $\underline{c}$ as its sequence of Fourier coefficients with respect to $\phi_n, n \in \mathbb{N}$, and,*

$$||f||_{q,[a,b]} \leq B_q M^{(q-2)/q}\left(\sum_{n\in\mathbb{N}} |c_n|^q n^{q-2}\right)^{1/q}.$$



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COMMENTS (i) The constants $A_p, B_q$ can be taken so that if $p, q$ are conjugate indices then $A_p = B_q$.

(ii) These results can be even further extended by replacing the sequence $\underline{c}$ by the decreasing rearrangement $\underline{c}^*$. That this strengthens the results follows from **Rearrangement Inequalities** (1). The proof consists in rearranging the orthonormal sequence.

(iii) The two parts of all these results are equivalent in that either implies the other.

(iv) The case $p = 2$ of part (a) of the above results is Bessel's Inequality. See: **Bessel's Inequalities** (1).

(v) These results extend the **Hausdorff-Young Inequalities**. Integral analogues are given in **Fourier Transform Inequalities** EXTENSIONS (a).

REFERENCES [MPF, p. 398]; *Zygmund* [Z, vol. II, pp. 120–127].

## Paley-Titchmarsh Inequality

See: **Fourier Transform Inequalities** EXTENSIONS (a).

## Parallelogram Inequality

*If $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ are real $n$-tuples then*

$$|\underline{a} - \underline{b}|^2 + |\underline{b} - \underline{c}|^2 + |\underline{c} - \underline{d}|^2 + |\underline{d} - \underline{a}|^2 \geq |\underline{a} - \underline{c}|^2 + |\underline{d} - \underline{b}|^2, \tag{1}$$

*with equality if and only if the $n$-tuples are co-planar and form a parallelogram in the correct order.*

COMMENTS (i) In the case that the points form a parallelogram, putting $\underline{a} = \underline{0}$, $\underline{b} = \underline{u}$ then $\underline{c} = \underline{u} + \underline{v}$ and (1) becomes $\underline{d} = \underline{v}$,

$$2\big(|\underline{u}|^2 + |\underline{v}|^2\big) = |\underline{u} + \underline{v}|^2 + |\underline{u} - \underline{v}|^2.$$

This is called the *parallelogram identity*. See: **Inner Product Inequalities** COMMENTS (iii).

(ii) The geometrical interpretation of (1) is:

*the sum of the squares of the lengths of the sides of the quadrilateral formed by the $n$-tuples exceed the squares of the lengths of the diagonals.*

(iii) This result can be extended to $2^n$ $n$-tuples.

REFERENCE *Gerber* [130].

## Permanent Inequalities

*(a)* [MARCUS & NEWMAN] *If $A$ is an $m \times n$ complex matrix, $B$ an $n \times m$ complex matrix then*

$$|\mathrm{per}(AB)|^2 \leq \mathrm{per}(AA^*)\mathrm{per}(B^*B).$$

*There is equality if and only if either $A$ has a zero row, $B$ has a zero column or $A = DPB^*$, where $D$ is diagonal and $P$ is a permutation matrix.*
*In particular if $A$ is a complex square matrix*

$$|\mathrm{per}(A)|^2 \leq \mathrm{per}(AA^*).$$

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*(b)* [SCHUR] *If $A$ is a positive semi-definite Hermitian matrix then*

$$\det(A) \le \operatorname{per}(A),$$

*with equality if and only if either $A$ is diagonal or $A$ has a zero row.*
*(c) Let $\underline{w}$ be a positive n-tuple, $\underline{a}, \underline{b}$ be n-tuples of positive integers, and let $A, B$ be $n \times n$ matrices with $(i, j)$ entries $w_i^{a_j}, w_i^{b_j}$, respectively. Then*

$$\operatorname{per}(A) \le \operatorname{per}(B)$$

*if and only if $\underline{a} \prec \underline{b}$. There is equality if and only if either $\underline{a} = \underline{b}$, or $\underline{w}$ is constant.*
*(d)* [ALEKSANDROV-FENCHEL INEQUALITY][40] *If $A$ is an $n \times n$ positive matrix and $A_1$ is obtained from $A$ by replacing the second column by the first, so that the first two columns are the same, and $A_2$ is obtained from $A$ by replacing the first column by the second, so that again the first two columns are the same, then*

$$\operatorname{per}^2(A) \le \operatorname{per}(A_1)\operatorname{per}(A_2),$$

*with equality if and only if the first two columns are dependent.*

COMMENTS (i) (c) is a form of Muirhead's inequality, **Muirhead Symmetric Function and Mean Inequalities** (1).

(ii) The Aleksandrov-Fenchel inequality was used in solving **van der Waerden's Conjecture**.

REFERENCES [MPF, pp. 225–226]; *Marcus & Minc* [MM, p. 118], *Minc* [Mi, pp. 9–10, 25]; *Wang, C. L.* [322]

**Persistence of Inequalities** *(a) If $\underline{a}, \underline{b}$ are real sequences and $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ and if $\underline{a} \le \underline{b}$ then $A \le B$.*
*(b) If $a_1 \le a_2 \le \cdots$ then $\mathfrak{A}_2(a_1, a_2) \le \mathfrak{A}_3(a_1, a_2, a_3) \le \cdots$.*

COMMENTS (i) The result in (a) cannot be improved to strict inequality as $a_n = 0,\ b_n = 1/n,\ n \in \mathbb{N}^*$, shows.

(ii) Clearly (b) can be extended to more general means. More interestingly the hypothesis of increasing can be generalized to $\underline{a}$ being n-convex, with the analogous generalization in the conclusion. See: **n-Convex Sequence Inequalities** (b).

INTEGRAL ANALOGUE *If $f$ is an increasing function on $[a, b]$ then $\dfrac{1}{v-u}\displaystyle\int_u^v f$ is an increasing function of both $u$ and $v$, $a \le u, v \le b$.*

REFERENCES [AI, p. 9], [H, p. 160–161]; *Cloud & Drachman* [CD, p. 6].

[40] This is P. S. Aleksandrov.

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**Petrović's Inequality** *Let $f$ be convex on $[0,a]$ and $a_i \in [0,a], 0 \le i \le n$, with $\sum_{i=1}^n a_i \in [0,a]$, then*

$$\sum_{i=1}^n f(a_i) \le f\left(\sum_{i=1}^n a_i\right) + (n-1)f(0).$$

Extension [Vasić & Pečarić] *Let $f$ be convex on $[0,a]$, $\underline{w}$ be a non-negative $n$-tuple, $a_i \in [0,a[, 1 \le i \le n$, with $\sum_{i-1}^n w_i a_i \in [0,a]$ and $\sum_{i=1}^n w_i a_i \ge \max \underline{a}$; then*

$$\sum_{i=1}^n w_i f(a_i) \le f\big(\sum_{i=1}^n w_i a_i\big) + (W_n - 1)f(0).$$

Comment See also: **Szegő's Inequality**.

References [AI, pp. 22–23], [MPF, pp. 11, 715], [PPT, pp. 151–169].

**Petschke's Inequality** *Let $f, g : [0,1] \to \mathbb{R}$ be concave, increasing and non-negative functions, and let $p, q \ge \lambda \approx 3.939$, where $\lambda$ is a solution of $r + 1 = (3/2)^r$, then*

$$\int_0^1 fg \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{3} ||f||_p ||g||_q. \tag{1}$$

*There is equality if and only if $f(x) = g(x) = x$.*

Comments (i) This is an inverse for (H); see **Inverse Hölder Inequalities**.
(ii) For a similar result related to (M) see: **Rahmail's Inequality**.
(iii) Various other possible relations bewteen $p, q, \lambda$ are considered in the reference.
(iv) For another inequality of Petschke see: **Moment Inequalities** (b).

Reference [MPF, pp. 149–150].

**Phragmén-Lindelöf Inequality** *Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and bounded on $R = \{\alpha \le \Re z \le \beta\}$, and analytic on $\overset{\circ}{R}$, with*

$$|f(\alpha + iy)| \le A, \qquad |f(\beta + iy)| \le B, \qquad y \in \mathbb{R}.$$

*If then $z = x + iy \in \overset{\circ}{R}$*

$$|f(z)| \le A^{\ell(x)} B^{1-\ell(x)},$$

*where*

$$\ell(x) = \frac{\beta - x}{\beta - \alpha}.$$

*There is equality if and only if $f(z) = A^{\ell(z)} B^{1-\ell(z)} e^{i\theta}$.*

Comments (i) Since this result connects the upper bounds of $f$ on the three lines $\Re z = \alpha, x, \beta$ it is often called the *Three Lines Theorem*; it is a statement

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that this upper bound function is log-convex. See also: **Hadamard's Three Circles Theorem** and **Hardy's Analytic Function Inequality**.

(ii) This result was used by Thorin to prove the **Riesz-Thorin Theorem**.

REFERENCES [EM, vol. 7, pp. 152–153]; *Conway* [C, vol. I, pp. 131–133], *Pólya & Szegö* [PS, pp. 166-172], *Rudin* [R87, pp. 243–244], *Titchmarsh* [T75, pp. 176–187], *Zygmund* [Z, vol. II, pp. 93–94].

**Picard-Schottky Theorem** *If $f$ is analytic in $D$ and if $f$ does not take the values $0, 1$ then*

$$\log|f(z)| \le \big(7 + \log^+|f(0)|\big)\frac{1+|z|}{1-|z|}.$$

REFERENCE *Ahlfors* [Ah73, pp. 19–20].

**Pittenger's Inequalities** *If $0 < a \le b$ and $r \in \mathbb{R}$, then*

$$\mathfrak{M}_2^{[r_1]}(a,b) \le \mathfrak{L}^{[r]}(a,b) \le \mathfrak{M}_2^{[r_2]}(a,b),$$

*where*

$$r_1 = \begin{cases} \min\left\{\dfrac{r+2}{3}, \dfrac{r\log 2}{\log(r+1)}\right\}, & \text{if } r > -1,\ r \ne 0,\\ \min\left\{\dfrac{2}{3}, \log 2\right\}, & \text{if } r = 0,\\ \min\left\{\dfrac{r+2}{3}, 0\right\}, & \text{if } r \le -1, \end{cases}$$

*and $r_2$ is defined as $r_1$ but with min replaced by max. There is equality if and only if $a = b$, or $r = 2, 1$ or $1/2$. The exponents $r_1, r_2$ are best possible.*

COMMENTS (i) These inequalities follow from (r;s). Taking $r = -1$ gives an inequality of Lin T. P.; see **Logarithmic Mean Inequalities** (d).

(ii) An interesting variant of the above inequalities has been given by Hästö.

(iii) See also: **Rado's Inequality** COMMENTS (i).

REFERENCES [H, pp. 388-389]; *Hästö* [140].

**Poincaré's Inequalities** *(a) If $1 \le p < n$, $B = B_{\underline{x},r} = \{\underline{u}; |\underline{u} - \underline{x}| < r\} \subset \mathbb{R}^n$, and $f \in \mathcal{W}^{1,p}(B)$, then*

$$\left(\frac{1}{|B|}\int_B |f - \tilde{f}|^{p^*}\right)^{1/p^*} \le C_{n,p}\, r\left(\frac{1}{|B|}\int_B |\nabla f|^p\right)^{1/p},$$

*where $\tilde{f} = \dfrac{1}{|B|}\displaystyle\int_B f$, and $p^*$ is the Sobolev conjugate.*

*(b) If $f \in \mathcal{C}^1(Q), Q = [0,a]^n$, then*

$$\int_Q f^2 \le \frac{1}{a^n}\left(\int_Q f\right)^2 + \frac{na^2}{2}\int_Q |\nabla f|^2.$$

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COMMENTS (i) For a definition of $\mathcal{W}_p^1(B)$ and $p^*$ see: **Sobolev's Inequalities**.
(ii) Both inequalities are related to **Friederichs's Inequality**; and a one-dimensional analogue of (b) is **Wirtinger's Inequality**.

EXTENSION [PACHPATTE] *If* $f, g \in \mathcal{C}^1(Q), Q = [0,a]^n$, *then*

$$\int_Q fg \leq \frac{1}{a^n}\Big(\int_Q f\Big)\Big(\int_Q g\Big) + \frac{na^2}{4}\int_Q \big(|\nabla f|^2 + |\nabla g|^2\big).$$

REFERENCES *Evans & Gariepy* [EG, pp. 141–142], *Mitrović & Žubrinić* [MZ, pp. 184, 242], *Opic & Kufner* [OK, p. 2]; *Pachpatte* [245].

**Poisson Kernel Inequalities** The quantity

$$P(r,x) = \frac{1}{2} + \sum_{i=1}^{\infty} r^i \cos it = \frac{1}{2}\,\frac{1-r^2}{1-2r\cos x + r^2},$$

is called the *Poisson kernel*.

*(a)*

$$P(r,x) > 0; \qquad \frac{1}{2}\,\frac{1-r}{1+r} \leq P(r,x) \leq \frac{1}{2}\,\frac{1+r}{1-r}.$$

*(b) If* $0 \leq t \leq \pi$ *then for some* $C$,

$$P(x,r) \leq C\frac{1-r}{x^2 + (1-r)^2}.$$

*In particular if* $0 \leq r < 1$

$$P(r,x) \leq \frac{1}{1-r}; \qquad P(r,x) \leq \frac{C(1-r)}{x^2},\ 0 < x \leq \pi.$$

COMMENT These should be compared to the similar inequalities in: **Dirichlet Kernel Inequalities, Fejér Kernel Inequalities**.

REFERENCE *Zygmund* [Z, vol. I, pp. 96–97].

**Pólya-Knopp Inequality** See: **Carleman's Inequality** INTEGRAL ANALOGUE.

**Pólya's Inequality** *Let* $f, g, h : [a,b] \to \mathbb{R}$, *with* $f$ *increasing,* $g, h$ *continuously differentiable and* $g(a) = h(a), g(b) = h(b)$ *then*

$$\int_a^b fg' \int_a^b fh' \leq \Big(\int_a^b f\sqrt{(gh)'}\Big)^2. \tag{1}$$

COMMENTS (i) If $a = 0, b = 1$, $g(x) = x^{2p+1}, h(x) = x^{2q+1}$ this becomes:

$$1 - \Big(\frac{b-a}{b+a-1}\Big)^2 \leq \frac{\big(\int_0^1 x^{a+b} f(x)\,\mathrm{d}x\big)^2}{\int_0^1 x^{2a} f(x)\,\mathrm{d}x \int_0^1 x^{2b} f(x)\,\mathrm{d}x}; \tag{2}$$

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and is due to Pólya; the generalization is by Alzer. It is an easy consequence of (C) that the right-hand side of (2) is less than 1.

(ii) Inequality (1) can be written using geometric means which then have been replaced by more general means to obtain other inequalities of the same type.

(iii) See also: **Gauss's Inequality**.

REFERENCES *Pólya & Szegő* [PS, p. 72]; *Alzer* [7], *Pearce, Pečarić & Varošanec* [259].

**Pólya & Szegő Inequality** *If* $0 < a \le \underline{a} \le A$, $0 < b \le \underline{b} \le B$ *then*

$$\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} \le \frac{1}{2}\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)\left(\sum_{i=1}^{n} a_ib_i\right),$$

*with equality if and only if* $\nu = nAb/(Ab+Ba)$ *is an integer and if* $\nu$ *of the* $a_i$ *are equal to* $a$, *and the others equal to* $A$, *with the corresponding* $b_i$ *being* $B$, $b$, *respectively.*

COMMENT (i) This inequality is a converse to (C) and is equivalent to **Kantorović's Inequality**.

EXTENSIONS *(a)* [CASSELS] *If* $\underline{w}$ *is a non-negative* $n$*-tuple with* $W_n \neq 0$ *then*

$$\left(\sum_{i=1}^{n} w_ia_i^2\right)^{1/2}\left(\sum_{i=1}^{n} w_ib_i^2\right)^{1/2} \le \max_{1\le i,j\le n}\frac{1}{2}\left(\sqrt{\frac{a_ib_j}{a_jb_i}}+\sqrt{\frac{a_jb_i}{a_ib_j}}\right)\left(\sum_{i=1}^{n} w_ia_ib_i\right).$$

*(b)* [DÍAZ & METCALF] *If* $\underline{a}, \underline{b}$ *are real* $n$*-tuples with* $a_i \neq 0, 1 \le i \le n$, *and if* $m \le a_i/b_i \le M, 1 \le i \le n$, *then*

$$\sum_{i=1}^{n} b_i^2 + mM\sum_{i=1}^{n} a_i^2 \le (M+m)\sum_{i=1}^{n} a_ib_i,$$

*with equality if and only if for all* $i, 1 \le i \le n$, *either* $b_i = ma_i$ *or* $b_i = Ma_i$.

COMMENTS (ii) Cassels inequality, (a), **Kantorović's Inequality** and **Schweitzer's Inequality** are all special cases of the elementary inequality (b).

(iii) Cassel's inequality is often called the *Greub & Rheinboldt inequality*.

(iv) The result of Díaz & Metcalf is equivalent to **Rennie's Inequality**.

REFERENCES [AI, pp. 59–66], [BB, pp. 44–45], [H, pp. 240–245], [HLP, pp. 62, 166], [PPT, pp. 114–115].

**Pólya-Vinogradov Inequality** *If* $\chi$ *is a primitive character* (mod $p$), $k > 2$, *and if* $s(\chi) = \max_{r\ge 1}\sum_{n=1}^{r}|\chi(n)|$, *then*

$$\frac{s(\chi)}{\sqrt{k}\log k} \le \begin{cases} \dfrac{1}{2\pi}+o(1), & \text{if } \xi(-1)=1,\\[2ex] \dfrac{1}{\pi}+o(1), & \text{if } \xi(-1)=-1.\end{cases}$$

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EXTENSION [SIMALRIDES]

$$s(\chi) \leq \begin{cases} \frac{1}{\pi}\sqrt{k}\log k + \sqrt{k}\big(1 - \frac{\log 2}{\pi}\big), & \textit{if } \xi(-1) = 1, \\ \frac{1}{\pi}\sqrt{k}\log k + \sqrt{k} + \frac{1}{2}, & \textit{if } \xi(-1) = -1. \end{cases}$$

REFERENCE *Simalrides* [298]

**Polygamma Function Inequalities** See: **Multigamma Function Inequalities.**

**Polynomial Inequalities** *(a) If $x \geq 0$, $x \neq 1$ and $n \geq 2$ then*

$$\begin{aligned} x^n - nx + (n-1) &> 0; && (1) \\ (x + n - 1)^n - n^n x &> 0. && (2) \end{aligned}$$

*(b) If $0 \leq x \leq 1$ and $n \in \mathbb{N}$ then*

$$0 \leq x^n(1-x)^n \leq 1. \tag{3}$$

*(c) If $n \in \mathbb{N}^*$ and $x > 1$ then*

$$\frac{x^n - 1}{n} > \left(\frac{x+1}{2}\right)^{n-1}(x-1), \tag{4}$$

*while if $0 \leq x \leq 1$ then ($\sim$4) holds.*
*(d)* [HERZOG] *If $0 < x < 1$, $0 \leq \nu \leq n, n, \nu \in \mathbb{N}$, then*

$$\binom{n}{\nu} x^\nu (1-x)^{n-\nu} < \frac{1}{2enx\sqrt{1-x}}.$$

COMMENTS (i) The proofs of both (1) and (2) follow by noting that $x = 1$ is the only positive root of the polynomials involved. These inequalities imply certain cases of (B).

(ii) The polynomial in (3) and all of its derivative have integer valued derivatives at $0, 1$; it can be used to prove that $e^n$ is irrational for all non-zero integers $n$.

(iii) Inequality (4) follows from **Haber's Inequality**.

EXTENSIONS *(a)* [HADŽIĬVANOV & PRODANOV] *If $n$ is even and $x \neq 1$*

$$x^n - nx + (n-1) > 0,$$

*while if $n$ is odd*

$$\begin{aligned} x^n - nx + (n-1) &> 0, \text{ if } x > x_n,\ x \neq 1, \\ x^n - nx + (n-1) &< 0, \text{ if } x < x_n, \end{aligned}$$

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*where* $x_n, -2 \le x_n < -1 - (1/n)$*, is the unique zero of* $x^n - nx + (n-1)$.
*(b)* [Ostrowski] *If* $0 \le x \le 1$ *and* $n \ge 1$ *then*

$$0 \le x^{n-1}(1-x)^n \le \frac{1}{2^{2(n-1)}}.$$

*(c)* [Ostrowski & Redheffer] *If* $n, \nu \in \mathbb{N}$*,* $n \ge 2$*,* $0 \le \nu \le n$*,* $0 < x < 1$ *and* $q = \nu/n$ *then*

$$\binom{n}{\nu} x^\nu (1-x)^{n-\nu} < \exp\left(-2n(x-q)^2\right). \tag{5}$$

Comments (iv) Inequality (5) is sharp in the sense that the coefficient $-2$ in the exponential and the coefficient in front of the exponential cannot be improved. See, however, **Statistical Inequalities** (b).

(v) See also: **Bernoulli's Inequality, Bernšteĭn Polynomial Inequalities, Binomial Function Inequalities, Brown's Inequalities, Čebišev Polynomial Inequalities, Descartes Rule of Signs, Erdős's Inequality, Erdős & Grünwald Inequality, Integral Mean Value Theorems** Extensions (c), **Kneser's Inequality, Labelle's Inequality, Markov's Inequality, Mason-Stothers Inequality, Newman's Conjecture, Newton's Inequalities, Polynomial Interpolation Inequalities, Pommerenke's Inequality, Schoenberg's Conjecture, Shampine's Inequality, Turán's Inequalities, Zeros of a Polynomial**.

References [AI, pp. 35, 198, 200, 226], [BB, pp. 12–13], [GI1, pp. 125–129, 307], [H, pp. 3–4], [HLP, pp. 40–42, 103], [MPF, pp. 65, 581–582].

**Polynomial Interpolation Inequalities** *(a) Let* $a \le a_i < a_2 < \cdots < a_n \le b$*, and if* $f \in \mathcal{C}^n([a,b])$*, has* $f^{(j)}(a_i) = 0$*,* $1 \le j \le k_i, 1 \le i \le r$*, with* $k_1 + k_2 + \cdots + k_r + r = n$*, then*

$$||f^{(k)}||_{\infty,[a,b]} \le (b-a)^{n-k}\alpha_{n,k}||f^{(n)}||_{\infty,[a,b]}, \quad 0 \le k \le n-1,$$

*where* $\alpha_{n,k} = 1/(n-k)!$*,* $0 \le k \le n-1$*.*
*(b)* [Tumura] *If as above, but* $f \in \mathcal{C}^n([a_1, a_r])$*, then*

$$||f^{(k)}||_{\infty,[a_1,a_r]} \le (a_r - a_1)^{n-k}\beta_{n,k}||f^{(n)}||_{\infty,[a_1,a_r]}, \quad 0 \le k \le n-1,$$

*where* $\beta_{n,0} = \dfrac{(n-1)^{n-1}}{n^n}\alpha_{n,0}$*, and if* $1 \le k \le n-1$*,* $\beta_{n,k} = \dfrac{k}{n}\alpha_{n,k}$*.*

Comments (i) The constants $\beta_{n,k}$ are best possible being exact for the functions, $f(x) = (x-a_1)^{n-1}(a_r - x)$, and $f(x) = (x-a_1)(a_r-x)^{n-1}$. These functions are, up to a constant factor, the only functions for which (b) is exact.

(ii) These results have been generalized by Agarwal.

References [GI3, pp. 371–378].

**Pommerenke's Inequality** *If* $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$ *has some* $a_k \ne 0$ *and if* $E = \{z;\, z \in \mathbb{C}, |f(z)| \le 1\}$ *is connected then*

$$\max\{|f'(z)|;\, z \in E\} \le \frac{en^2}{2}.$$

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COMMENT It was conjectured by Erdős that the right-hand side can be replaced by $n^2/2$.

REFERENCE [GI7, pp. 401–402].

**Popa's Inequality** *If $K$ is a convex set in $\mathbb{R}^3$ put $K_\epsilon = \{\underline{x}; d(\underline{x}, K) < \epsilon\}$, $K^{-\rho} = \{\underline{x}; B(\underline{x}, \rho) \subseteq K)\}$. If then $f : [0, \infty[ \to [0, \infty[$ is a $\mathcal{C}^1$ function,*

$$\int_{K_\epsilon \setminus K^{-\rho}} f(|\underline{x}|)\,\mathrm{d}\underline{x} \le 4\pi(\epsilon + \rho) \int_0^\infty t^2 |f'(t)|\,\mathrm{d}t.$$

COMMENT $d(\underline{x}, K)$ and $B(\underline{x}, \rho)$ are defined by:

$$d(\underline{x}, K) = \inf\{t; t = |\underline{y} - \underline{x}|, \underline{y} \in K\}, \qquad B(\underline{x}, \rho) = \{\underline{y}; |\underline{y} - \underline{x}| < \rho\};$$

the *distance of $\underline{x}$ from $K$* and the *sphere of centre $\underline{x}$, radius $\rho$*, respectively.

REFERENCE *Popa* [273].

**Popoviciu's Convex Function Inequality** *A continuous function on the interval $I$ is convex if and only if for all $x, y, z \in I$*

$$\frac{1}{3}\big(f(x)+f(y)+f(z)\big)+f\Big(\frac{1}{3}(x+y+z)\Big) \ge \frac{2}{3}\Big(f\big(\frac{x+y}{2}\big)+f\big(\frac{y+z}{2}\big)+f\big(\frac{z+x}{2}\big)\Big).$$

*If $f$ is strictly convex this inequality is strict unless $x = y = z$.*

COMMENT In the case $f(x) = |x|$ this reduces to **Hlawka's Inequality**.

REFERENCE *Niculescu & Popovici* [234].

**Popoviciu's Geometric-Arithmetic Mean Inequality Extension** *If $n \ge 2$ then*

$$\left(\frac{\mathfrak{A}_n(\underline{a}; \underline{w})}{\mathfrak{G}_n(\underline{a}; \underline{w}}\right)^{W_n} \ge \left(\frac{\mathfrak{A}_{n-1}(\underline{a}; \underline{w})}{\mathfrak{G}_{n-1}(\underline{a}; \underline{w}}\right)^{W_{n-1}}, \tag{1}$$

*with equality if and only if $a_n = \mathfrak{A}_{n-1}(\underline{a}; \underline{w})$.*

COMMENTS (i) This is a multiplicative analogue of **Rado's Geometric Arithmetic Mean Inequality Extension**, and several proofs have been given.

(ii) Repeated application of (1) leads to

$$\left(\frac{\mathfrak{A}_n(\underline{a}; \underline{w})}{\mathfrak{G}_n(\underline{a}; \underline{w}}\right)^{W_n} \ge \cdots \ge \left(\frac{\mathfrak{A}_1(\underline{a}; \underline{w})}{\mathfrak{G}_1(\underline{a}; \underline{w}}\right)^{W_1} = 1,$$

which exhibits (1) as an extension of (GA), to which it is equivalent. Stopping the above applications one step earlier and remarking that the left-hand side is

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invariant under simultaneous permutations of $\underline{a}$ and $\underline{w}$ leads to an improvement of (GA):

$$\mathfrak{A}_n(\underline{a};\underline{w}) \geq \max_{1\leq i,j\leq n}\left\{\left(\frac{w_i a_i + w_j a_j}{w_i + w_j}\right)^{w_i+w_j} a_i^{w_i} a_j^{w_j}\right\}^{1/W_n} \mathfrak{G}_n(\underline{a};\underline{w}).$$

In the case of equal weights this is,

$$\mathfrak{A}_n(\underline{a}) \geq \sqrt[n]{\max_{1\leq i,j\leq n}\left\{\frac{1}{2}+\frac{1}{4}\left(\frac{\max \underline{a}}{\min \underline{a}}+\frac{\min \underline{a}}{\max \underline{a}}\right)\right\}}\, \mathfrak{G}_n(\underline{a}).$$

A similar discussion can be found in **Rado's Geometric-Arithmetic Mean Inequality Extension** COMMENTS (i), **Jensen's Inequality** COROLLARY.

(iii) Inequalities that extend $P/Q \geq 1$ to $P/Q$ is a decreasing function of index sets are called *Popoviciu-type, or Rado-Popoviciu type, inequalities*; see for instance: **Harmonic Mean Inequalities** (C), **Hölder's Inequality** EXTENSIONS (D), **Induction Inequality, Sierpinski's Inequalities** EXTENSIONS (A), **Symmetric Mean Inequalities** EXTENSIONS.

EXTENSIONS [FUNCTIONS OF INDEX SETS] *Let $\pi$ be the following function defined on the index sets,*

$$\pi(\mathcal{I}) = \left(\frac{\mathfrak{A}_{\mathcal{I}}(\underline{a};\underline{w})}{\mathfrak{G}_{\mathcal{I}}(\underline{a};\underline{w})}\right)^{W_{\mathcal{I}}};$$

*then $\pi \geq 1$, is increasing, and $\log\circ\pi$ is super-additive.*

COMMENT (iv) See also: **Minc-Sathre Inequality**.

REFERENCES [H, pp. 125–129].

## Popoviciu's Inequality
See: **Variance Inequalities** COMMENT (i).

## Popoviciu Type, Rado-Popoviciu Type Inequalities
See: **Popoviciu's Geometric-Arithmetic Mean Inequality Extension** COMMENT (III), **Rado Type, Rado-Popoviciu Type Inequalities**.

## Power Mean Inequalities
*(a) If $-\infty \leq r < s \leq \infty$ then:*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \leq \mathfrak{M}_n^{[s]}(\underline{a};\underline{w}), \qquad (r;s)$$

*with equality if and only if $\underline{a}$ is constant.*
*(b) If $q, r, s \in \overline{\mathbb{R}}$ with $s \leq q$ and $s \leq r$ and if*

$$\frac{1}{q}+\frac{1}{r} \leq \frac{1}{s}; \qquad (1)$$

*then*

$$\mathfrak{M}_n^{[s]}(\underline{a}\,\underline{b};\underline{w}) \leq \mathfrak{M}_n^{[q]}(\underline{a};\underline{w})\mathfrak{M}_n^{[r]}(\underline{b};\underline{w}). \qquad (2)$$

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*If $q \le s$ and $r \le s$ (∼2) holds.*
*Inequality (2) is strict except under the following circumstances:*
*(i) $q, r, s \neq 0, \pm\infty$, (1) strict, $\underline{a}$ and $\underline{b}$ constant; (ii) $q, r, s \neq 0, \pm\infty$, (1) is equality, and, $\underline{a}^q \sim \underline{b}^r$; (iii) $s \neq 0, \pm\infty, q = \pm\infty$ and $\underline{a}$ constant; (iv) $s \neq 0, \pm\infty, r = \pm\infty$ and $\underline{b}$ constant; (v) $s = 0, q = 0$ or $r = 0$ and $\underline{b}$ constant; (vi) $q = r = s = 0$; (vii) $q = r = s = \infty$ and for some $i, 1 \le i \le n$, $\max \underline{a} = a_i$ and $\max \underline{b} = b_i$; (viii) $q = r = \pm\infty, s = \pm\infty$, $\underline{a}$ and $\underline{b}$ constant; (ix) $q = r = s = -\infty$ and for some $i$, $1 \le i \le n$, $\min \underline{a} = a_i$ and $\min \underline{b} = b_i$.*
*(c) If $r \ge 1$ then*

$$\mathfrak{M}_n^{[r]}(\underline{a}+\underline{b};\underline{w}) \le \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) + \mathfrak{M}_n^{[r]}(\underline{b};\underline{w}). \tag{3}$$

*If $r < 1$ inequality (∼3) holds; in particular*

$$\mathfrak{G}_n(\underline{a}+\underline{b};\underline{w}) \ge \mathfrak{G}_n(\underline{a};\underline{w}) + \mathfrak{G}_n(\underline{b};\underline{w}). \tag{4}$$

*If $r \ge 1$ (3) is strict unless: (i) $r = 1$, (ii) $1 < r < \infty$ and $\underline{a} \sim \underline{b}$, or (iii) $r = \infty$ and for some $i$, $1 \le i \le n$, $\max \underline{a} = a_i$ and $\max \underline{b} = b_i$.*
*(d) If $\underline{a}$ is not constant, $\left(\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\right)^r$ is strictly log-convex on $]-\infty, 0[$ and on $]0, \infty[$.*
*(e) $\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})$ is log-convex, and so convex, in $1/r$.*

COMMENTS (i) Simple algebraic arguments show that to prove (r;s) it is sufficient to consider cases (0;1), (r;1), $0 < r < 1$, and (1; s). The case (0;1) is just (GA), and the other cases follow from (H).

Many other proofs have been given. In particular (r;s) follows from (J), and from **Mixed Mean Inequalities** (1), (2).

(ii) While (H) can be used to prove (r;s), the case (r;1), $0 < r < 1$, is, on putting $r = 1/p$ and changing notation, just (H).

(iii) The cases when $q, r$ or $s$ of (b) are infinite are trivial. If $q, r$ and $s$ are finite but not zero the result follows from (H) and **Hölder's Inequality** (5); if one or more of $q, r, s$ is zero then use (r;s). This is really a weighted form of (H).

(iv) If $r \neq 0$ (c) is either a weighted form of (M), or is trivial. The case $r = 0$, (4), which is **Geometric Mean Inequalities** (1), can be proved using (GA); this inequality is sometimes called *Hölder's Inequality*.

(v) The convexity results, (d), (e) follow from **Power Sums Inequalities** COMMENTS (ii).

EXTENSIONS *(a) If $0 < r < s$ and if $W_n \le 1$ then*

$$\left(\sum_{i=1}^n w_i a_i^r\right)^{1/r} \le \left(\sum_{i=1}^n w_i a_i^s\right)^{1/s}, \tag{5}$$

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*and if* $W_n < 1$ *the inequality is strict.*
*(b) If* $0 < r < s < t$ *and if* $\underline{a}$ *is not constant then*

$$1 < \frac{\mathfrak{M}_n^{[t]}(\underline{a}) - \mathfrak{M}_n^{[r]}(\underline{a})}{\mathfrak{M}_n^{[t]}(\underline{a}) - \mathfrak{M}_n^{[s]}(\underline{a})} < \frac{s(t-r)}{r(t-s)}.$$

*(c) If* $0 < r$ *and* $\underline{a}$ *is not constant then*

$$1 < \frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[-r]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[0]}(\underline{a};\underline{w})} < \frac{W_n}{\min \underline{w}}.$$

*(d)* [Popoviciu Type] *If* $-\infty < r \le 0 \le s < \infty$ *then*

$$\left(\frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}\right)^{W_n} \ge \left(\frac{\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})}\right)^{W_{n-1}},$$

*with equality if and only if (i) when* $s = 0$, $a_n = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$*, (ii) when* $r = 0$, $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})$*, (iii) if* $r < 0 < s$ *both conditions in (i) and (ii) hold.*
*(e)* [Rado Type] *If* $-\infty \le r \le 1 \le s \le \infty, r \ne s$ *then*

$$W_n\big(\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w})\big) \ge W_{n-1}\big(\mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})\big)$$

*with equality if and only if one of the following holds: (i)* $s = 1$, $a_n = \mathfrak{M}_{n-1}^{[r]}(\underline{a};\underline{w})$*; (ii)* $r = 1$, $a_n = \mathfrak{M}_{n-1}^{[s]}(\underline{a};\underline{w})$*; (iii)* $r < 1 < s$ *both conditions in (i) and (ii) hold.*
*(f)* [Functions of Index Sets] *If* $\mathcal{I}$ *is an index set define* $\sigma(\mathcal{I}) = W_{\mathcal{I}}\mathfrak{M}_{\mathcal{I}}^{[s]}(\underline{a};\underline{w})$*. If* $s > 1$ *and if* $\mathcal{I} \cap \mathcal{J} = \emptyset$ *then*

$$\sigma(\mathcal{I} \cup \mathcal{J}) \ge \sigma(\mathcal{I}) + \sigma(\mathcal{J}), \tag{6}$$

*with equality in (6) if and only if* $\mathfrak{M}_{\mathcal{I}}(\underline{a};\underline{w}) = \mathfrak{M}_{\mathcal{J}}(\underline{a};\underline{w})$*.*
*If* $s < 1$ *then (∼6) holds, while if* $s = 1$ *(6) becomes an equality.*
*(g) If* $k = 0, 1, 2, \ldots$

$$\big(\mathfrak{A}_n^k(\underline{a}) - \mathfrak{G}_n^k(\underline{a})\big) \ge n^{1-k}\left(\big(\mathfrak{M}_n^{[k]}\big)^k(\underline{a}) - \mathfrak{G}_n^k(\underline{a})\right), \tag{7}$$

*with equality if and only if one of the following holds: (i)* $k = 0, 1$*; (ii)* $n = 1$*; (iii)* $k = n = 2$*; (iv)* $\underline{a}$ *is constant.*
*If* $k < 0$ *(∼7) holds and is strict unless* $\underline{a}$ *is constant.*
*(h)* [Pečarić & Janić] *If* $\underline{a}, \underline{u}, \underline{v}$ *are real* $n$*-tuples with* $\underline{a}$ *increasing, and with*

$$0 \le V_n - V_{k-1} \le U_n - U_{k-1} \le U_n = V_n, 2 \le k \le n,$$

*then*

$$\mathfrak{M}_n^{[r]}(\underline{a};\underline{v}) \le \mathfrak{M}_n^{[r]}(\underline{a};\underline{u}).$$

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*(j)* [Izumi, S., Kobayashi & Takahashi, T.] *If $\underline{a}, \underline{b}, \underline{w}$ are positive $n$-tuples with $\underline{b}$ and $\underline{b}/\underline{a}$ similarly ordered and if $r \le s$, then*

$$\frac{\mathfrak{M}_n^{[r]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{b};\underline{w})} \le \frac{\mathfrak{M}_n^{[s]}(\underline{a};\underline{w})}{\mathfrak{M}_n^{[s]}(\underline{b};\underline{w})}. \tag{8}$$

*If $\underline{b}$ and $\underline{b}/\underline{a}$ are oppositely ordered then (∼8) holds.*

Comments (vi) (5) is a consequence of (r;s).

(vii) If $n > 2$ and $0 < k < 1$ then neither (7) nor (∼7) holds; (7) also holds if $k = 1, 2$ and $n = 2$.

(viii) Inequality (8) follows by a use of (r;s) and (Č).

Inverse Inequalities *(a)* [Cargo & Shisha]

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) - \mathfrak{M}_n^{[r]}(\underline{a};\underline{w}) \le \mathfrak{M}_2^{[s]}(m, M; 1-t_0, t_0) - \mathfrak{M}_2^{[r]}(m, M; 1-t_0, t_0)$$

*where $m, M$ are, respectively, the smallest and the largest of the $a_i$ with associated non-zero $w_i$, and where $\overline{1-t_0}\, m + t_0 M$ is the mean-value point for $f$ on $[m, M]$.*
*There is equality if and only if either all the $a_i$ with associated non-zero $w_i$ are equal or if all the $w_i$ are zero except those associated with $m$ and $M$, and then these have weights $1-t_0, t_0$, respectively.*
*(b)* [Specht] *If $0 < m \le a_i \le M, 1 \le i \le n$, $-\infty < r < s < \infty, rs \ne 0$ then*

$$\frac{\mathfrak{M}_n^{[s]}(\underline{a},\underline{w})}{\mathfrak{M}_n^{[r]}(\underline{a},\underline{w})} \le \left(\frac{r}{\mu^r - 1}\right)^{1/s} \left(\frac{\mu^s - 1}{s}\right)^{1/r} \left(\frac{\mu^s - \mu^r}{s-r}\right)^{1/s-1/r},$$

*where $\mu = M/m$.*

Comments (ix) A mean value point is defined in **Mean Value Theorem of Differential Calculus** Comments (i).

(x) Specht's result, (b), has extensions to allow one or other of the powers to be zero; see [H]. On putting $s = 1, r = -1$ this result reduces to **Kantorović's Inequality**.

(xi) Other inverse inequalities can be found in the references; see also **Ostrowski's Inequalities** (b).

Integral Analogues *(a) If $f$ is defined and not zero almost everywhere on $[a, b]$ and $r < s$,*

$$\mathfrak{M}_{[a,b]}^{[r]}(f, w) \le \mathfrak{M}_{[a,b]}^{[s]}(f, w),$$

*provided the integrals exist.*
*There is equality only if either $r \ge 0$ and $\mathfrak{M}_{[a,b]}^{[r]}(f, w) = \mathfrak{M}_{[a,b]}^{[s]}(f, w) = \infty$, or $s \le 0$ and $\mathfrak{M}_{[a,b]}^{[r]}(f, w) = \mathfrak{M}_{[a,b]}^{[s]}(f, w) = 0$.*

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*(b) If $f, g, w$ are positive functions on $[a,b]$ and if $g, f/g$ are similarly ordered then if $r \le s$*

$$\frac{\mathfrak{M}^{[r]}_{[a,b]}(f;w)}{\mathfrak{M}^{[r]}_{[a,b]}(g;w)} \le \frac{\mathfrak{M}^{[s]}_{[a,b]}(f;w)}{\mathfrak{M}^{[s]}_{[a,b]}(g;w)}.$$

Comments (xii) The power means and their various inequalities have been extended by Kalman to allow the power to be a function; that is $r = r(x) \le s = s(x)$.

(xiii) See also: **Čebišev's Inequality** (2), **Factorial Function Inequalities, Fan Inequality** Extensions (a), **Gauss-Winkler Inequality** Comments (i), **Hamy Mean Inequalities** Comment, **Jessen's Inequality, Levinson's Inequality** Special Cases, **Logarithmic Mean Inequalities** (d), Corollaries (a), Extensions, **Love-Young Inequalities, Lyapunov's Inequality, Minc-Sathre Inequality, Muirhead Symmetric Function and Mean Inequalities** Comments (ii), **Rado's Inequality, Rennie's Inequality** Extension, **Thunsdorff's Inequality**.

References [AI, pp. 76–80, 85–95], [BB, pp. 16–18], [GI3, pp. 43–68], [H, pp. 178, 202–225, 229–240], [HLP, pp. 26–28, 44, 134–145], [MOA, pp. 189–191], [MPF, pp. 14–15, 48, 181], [PPT, pp. 108–112, 117]; *Polyá & Szegö* [PS, p. 69]; *Bullen* [76], *Diananda* [100].

**Power Sum Inequalities** *(a) If $r, s \in \mathbb{R}$, $r < s$ then,*

$$\left(\sum_{i=1}^{n} a_i^s\right)^{1/s} < \left(\sum_{i=1}^{n} a_i^r\right)^{1/r}. \tag{1}$$

*(b) If $r, s \in \mathbb{R}$ and $0 \le \lambda \le 1$ then*

$$\left(\sum_{i=1}^{n} a_i^{(1-\lambda)r+\lambda s}\right) \le \left(\sum_{i=1}^{n} a_i^r\right)^{1-\lambda} \left(\sum_{i=1}^{n} a_i^s\right)^{\lambda}, \tag{2}$$

*and*

$$\left(\sum_{i=1}^{n} a_i^{(1-\lambda)r+\lambda s}\right)^{1/\{(1-\lambda)r+\lambda s\}} \le \left(\sum_{i=1}^{n} a_i^r\right)^{(1-\lambda)/r} \left(\sum_{i=1}^{n} a_i^s\right)^{\lambda/s}. \tag{3}$$

*(c) If $\underline{a}$ is a decreasing non-negative $n$-tuple, $\underline{b}$ a real $n$-tuple then*

$$\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i, 1 \le k \le n \qquad \Longrightarrow \qquad \sum_{i=1}^{n} a_i^2 \le \sum_{i=1}^{n} b_i^2,$$

*with equality if and only if $\underline{a} = \underline{b}$.*

Comments (i) In (1), that is often called *Jensen's Inequality*, we can, without loss in generality, assume that $0 < r < s$ and that the left-hand side of (1) is equal to 1; then $\underline{a} < \underline{e}$ and so $\underline{a}^s < \underline{a}^r$.

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Inequalities (2) and (3) follow by a simple application of (H).

(ii) While (1) says that the power sum is decreasing as a function of the power, (2) and (3) say that certain sums are log-convex functions of their parameters.

Extensions *(a) If* $r > 1$ *then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}^r < \sum_{j=1}^{n}\left(\sum_{i=m}^{n} a_{i,j}\right)^r;$$

*with the opposite inequality if* $0 < r < 1$.

*(b)*

$$\sum_{i=1}^{n} a_i^{\mathfrak{A}_n(\underline{r};\underline{w})} \le \mathfrak{G}_n(\sum_{i=1}^{n} a_i^{r_1}, \ldots, \sum_{i=1}^{n} a_i^{r_n}; \underline{w}).$$

Comments (iii) (b), an extension of (2), is proved by a simple induction. There is a similar extension of (3), but it can be improved by replacing the arithmetic mean on the left-hand side by the harmonic mean; it is an improvement because of (1) and (GA).

(iv) The power sums have been extended by Kalman to allow the power to be a function. See: **Power Mean Inequalities** Comment (xii).

(v) See also: **Bennett's Inequalities** (b)–(e), **Hölder's Inequality, Klamkin & Newman Inequalities, König's Inequality, Love-Young Inequalities** (b), **Minkowski's Inequality, Quasi-arithmetic Mean Inequalities** Comments (iv), **Ōzeki's Inequalities** (b), **Young's Convolution Inequality** Discrete Analogue.

References [AI, pp. 337–338], [BB, pp. 18–19], [H, pp. 185–189], [HLP, pp. 28–30, 32, 72, 196–203], [MPF, p. 181], [PPT, pp. 164–169, 216–217].

**Prékopa-Leindler Inequalities** *(a) If* $p, q \ge 1$ *are conjugate indices then*

$$\int_{\mathbb{R}} \sup_{x+y=t}\{f(x)g(y)\}\,\mathrm{d}t \ge p^{1/p}q^{1/q}||f||_{p,\mathbb{R}}||g||_{q,\mathbb{R}}. \tag{1}$$

*(b) If* $p, q, r \ge 1$ *with* $1/p + 1/q = 1 + 1/r$ *then*

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left(f(x)g(t-x)\right)^r\,\mathrm{d}x\right)^{1/r}\,\mathrm{d}t \ge ||f||_{p,\mathbb{R}}||g||_{q,\mathbb{R}}. \tag{2}$$

*(c) If* $f, g : \mathbb{R}^n \to [0, \infty[$ *and if* $0 < \lambda < 1, \alpha \in \overline{\mathbb{R}}$, *define*

$$h_\alpha(\underline{x}; f, g) = \sup_{\underline{y}\in\mathbb{R}^n}\left\{\mathfrak{M}_2^{[\alpha]}\left(f\big(\frac{1}{\lambda}(\underline{x}-\underline{y})\big), g\big(\frac{1}{1-\lambda}\,\underline{y}\big); \lambda, 1-\lambda\right)\right\}.$$

*Then if* $\alpha \ge -1/n, \nu = \alpha/(1+n\alpha)$,

$$\int_{\mathbb{R}^n} h_\alpha \ge \mathfrak{M}_2^{[\nu]}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda, 1-\lambda\right);$$

*in particular*

$$\int_{\mathbb{R}^n} h_0 \ge \left(\int_{\mathbb{R}^n} f\right)^{\lambda}\left(\int_{\mathbb{R}^n} g\right)^{1-\lambda}. \tag{3}$$

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COMMENTS (i) In general the discrete analogue of (1) is not valid, but there is a discrete analogue of (2).

(ii) A probabilistic interpretation can be given to (1) in the case $p = q = 2$:

*the marginal densities in each direction of a log-concave probability distribution on $\mathbb{R}^2$ are also log-concave.*

(iii) Taking $f = 1_A, g = 1_B$ in (c), then $h_0 = 1_{\lambda A + \overline{1-\lambda}B}$ and (3) reduces to **Brunn-Minkowski Inequalities** (a). These inequalities have been further generalized by Brascamp & Lieb.

REFERENCES [GI1, pp. 303–304], [MPF, pp. 168–173]; *Niculesu & Persson* [NP, pp. 158–165]; *Gardner* [126], *Brascamp & Lieb* [70].

**Probability Inequalities** *If $f$ is an even non-negative Borel function on $\mathbb{R}$, decreasing on $[0, \infty[$ then if $r \geq 0$*

$$\frac{Ef \circ X - f(r)}{||f||_{\infty,P}} \leq P\{|X| \geq r\} \leq \frac{Ef \circ X}{f(r)}.$$

COMMENTS (i) If $f$ is increasing we can replace the central term by $P\{X \geq r\}$.

(ii) Special cases of this result are **Markov's Probability Inequality** and, if $k > 0$,

$$\frac{Ee^{kX} - e^{kr}}{||e^{kX}||_{\infty,P}} \leq P\{X \geq r\} \leq \frac{Ee^{kX}}{e^{kr}}.$$

(iii) See also: **Bernšteĭn's Probability Inequality, Berry-Esseen Inequality, Bonferroni's Inequalities, Čebišev's Probability Inequality, Entropy Inequalities, Gauss-Winkler Inequality, Kolmogorov's Probability Inequality, Lévy's Inequalities, Markov's Probability Inequality, Martingale Inequalities, Prékopa-Leindler Inequalities** COMMENTS (ii), **Statistical Inequalities**.

REFERENCES *Loève* [L, pp. 157–158].

**Pseudo-Arithmetic and Pseudo-Geometric Mean Inequalities** See: **Alzer's Inequalities** (e), COMMENTS (iv).

**Psi Function Inequalities** See: **Digamma Function Inequalities**.

**Ptolemy's Inequality** *If $A, B, C, D$ are four points in $\mathbb{R}^3$ with $AB = a, BC = b, CD = c, AD = d, AC = m, BD = n$ then*

$$|ac - bd| \leq mn \leq ac + bd,$$

*with equality if and only if either the points are co-linear or concyclic.*

COMMENT A set of four points $A, B, C, D$ in $\mathbb{R}^3$ connected by line segments $AB, BC, CD, AC, AD, BD$ is called a *tetron*, with *vertices* the four points, *sides* the four line segments connecting adjacent vertices and *diagonals* the two segments connecting non-adjacent vertices.

REFERENCE *Astapov & Noland* [46].

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# 17 Q-class–Quaternion

**Q-class Function Inequalities** *(a) If $f$ is of class $Q$ on $[a,b]$ then for all $x,y \in [a,b]$ and $\lambda, 0 < \lambda < 1$,*

$$0 \le f\big((1-\lambda)x + \lambda y\big) \le \frac{f(x)}{1-\lambda} + \frac{f(y)}{\lambda}. \tag{1}$$

*(b) If $f$ is of class $Q$ on $[a,b]$ and if $a \le x_i \le b, 1 \le i \le 3$ then*

$$f(x_1)(x_1-x_2)(x_1-x_3)+f(x_2)(x_2-x_3)(x_2-x_1)+f(x_3)(x_3-x_1)(x_3-x_2) \ge 0.$$

*(c) If $f$ is of class $Q$ on $[a,b]$ and if $a \le x_1 < x_2 < x_3 \le b$ then*

$$\begin{aligned} f(x_2) &\le \frac{x_3-x_1}{x_3-x_2}f(x_1) + \frac{x_3-x_1}{x_2-x_1}f(x_3);\\ 0 &\le \frac{f(x_1)}{x_3-x_2} + \frac{f(x_2)}{x_1-x_3} + \frac{f(x_3)}{x_2-x_1}. \end{aligned}$$

*(d) If $f$ is of class $Q$ on $[a,b]$, $a \le a_i \le b, 1 \le i \le n, n \ge 2$, and if $\underline{w}$ is a positive $n$-tuple then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i a_i\right) \le W_n \sum_{i=1}^n \frac{f(a_i)}{w_i}.$$

Comments (i) (1) is just the definition of the $Q$-class, and the inequalities in (b), (c) are equivalent to (1).

(ii) (d) is an analogue of (J) for the $Q$-class. The various inequalities above should be compared to those in **Convex Function Inequalities**.

(iii) It is easy to see that if $f$ is non-negative, and either monotonic or convex, then it is in the $Q$-class. As a result since $f(x) = x^r$ is a $Q$-class function on $]0,\infty[$ we get **Schur's Inequality** (1) as a particular case of (b).

References [MPF, pp. 410–413]; *Mitrinović & Pečarić* [216].

**Quadratic Form Inequalities** *(a) If $a, b, c$ are non-negative real numbers then*

$$2bx \le c + ax^2 \quad \textit{for all} \quad x \ge 0 \qquad \Longrightarrow \qquad b^2 \le ac.$$



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*(b)* [Popoviciu] *Let $\underline{a}, \underline{b}$ be increasing real $n$-tuples and let $w_{ij}, 1 \le i, j \le n$, be real numbers satisfying*

$$\begin{aligned} \sum_{i=r}^{n}\sum_{j=s}^{n} w_{ij} &\ge 0, \quad 2 \le r, s \le n, \\ \sum_{i=r}^{n}\sum_{j=1}^{n} w_{ij} &= 0, \quad 1 \le r \le n, \qquad \sum_{i=1}^{n}\sum_{j=s}^{n} w_{ij} = 0, \quad 1 \le s \le n; \end{aligned} \tag{1}$$

*then*

$$\sum_{i,j=1}^{n} w_{ij} a_i b_j \ge 0. \tag{2}$$

*Conversely if (2) holds for all such $\underline{a}, \underline{b}$ then (1) holds.*

Comment (i) Popoviciu's result has been extended to more general forms. See: [MPF].

Extension *If $a, b, c$ are positive real numbers, and if $0 < u < v$ then*

$$bx^u \le c + ax^v \quad \textit{for all} \quad x > 0 \qquad \Longrightarrow \qquad b^v \le K_{u,v} c^{v-u} a^u.$$

Comments (ii) This follows by taking $x = (bu/av)^{1/(v-u)}$.

(iii) See also: **Aczél & Varga Inequality, Bilinear Form Inequalities, Multilinear Form Inequalities, Schur's Lemma**.

References [BB, p. 177], [MPF, pp. 338, 340–344]; *Herman, Kučera & Šimša* [HKS, pp. 124–125].

**Quadrature Inequalities** These inequalities arise from estimates of the remainders of various quadrature rules, the trapezoidal rule, the mid-point rule, Simpson's rule, etc.

*(a)* [Trapezoidal Inequality] *If $f : [a, b] \mapsto \mathbb{R}$ has a bounded continuous second derivative derivative on $]a, b[$, with bound $M_2$, then*

$$\left| \int_a^b f - \frac{b-a}{2}\left( \frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big) \right) \right| \le \frac{M_2}{48}(b-a)^3.$$

*(b)* [Mid-point Inequality] *If $f : [a, b] \mapsto \mathbb{R}$ has a bounded continuous second derivative derivative on $]a, b[$, with bound $M_2$, then*

$$\left| \int_a^b f - \frac{b-a}{2}\left( f\Big(\frac{3a+b}{4}\Big) + f\Big(\frac{a+3b}{4}\Big) \right) \right| \le \frac{M_2}{96}(b-a)^3.$$

*(c)* [Simpson's Inequality] *If $f : [a, b] \mapsto \mathbb{R}$ has a bounded continuous fourth derivative on $]a, b[$, with bound $M_4$, then*

$$\left| \int_a^b f - \frac{b-a}{3}\left( \frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big) \right) \right| \le \frac{M_4}{2880}(b-a)^5.$$

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COMMENTS (i) Similar inequalities have been obtained with much weaker conditions on the function $f$.

(ii) See also: **Iyengar's Inequality**.

HAMMER'S INEQUALITIES *If $f$ is convex on $[-1,1]$ then*

$$f(-1)+f(1)-\int_{-1}^{1} f \geq \int_{-1}^{1} f - 2f(0) \geq 0.$$

COMMENTS (iii) Clearly the result can be applied to any interval and says: *for convex functions the mid-point rule underestimates the integral and the trapzoidal rule overestimates; further the error for mid-point rule is smaller.*

(iv) This result can be extended to quadrature rules that use more points using functions of higher order convexity.

REFERENCES *Davis & Rabinowitz* [DR]; *Dragomir, Agarwal & Cerone* [105], *Matić, Pečarić & Vukelić* [202].

## Quadrilateral Inequalities

*If $\underline{a}_i, 1 \leq i \leq 4$, are real $n$-tuples then:*

$$\big||\underline{a}_1-\underline{a}_4|-|\underline{a}_3-\underline{a}_4|\big| \leq |\underline{a}_1-\underline{a}_2|+|\underline{a}_3-\underline{a}_2|, \tag{1}$$

*with equality if and only if the four points are in order on the same line;*

$$|\underline{a}_1|+|\underline{a}_2|+|\underline{a}_3|+|\underline{a}_1+\underline{a}_2+\underline{a}_3| \geq |\underline{a}_1+\underline{a}_2|+|\underline{a}_2+\underline{a}_3|+|\underline{a}_3+\underline{a}_1|. \tag{2}$$

COMMENTS (i) Inequality (1) is an easy deduction from **Triangle Inequality** EXTENSIONS (b). See also: **Metric Inequalities** EXTENSIONS (b).

(ii) Inequality (2) is just **Hlawka's Inequality** and spaces in which this result holds are called *quadrilateral spaces*.

REFERENCE *Smiley & Smiley* [302].

## Quasi-arithmetic Mean Inequalities

Let $[a,b] \subseteq \overline{\mathbb{R}}$ and suppose that $M:[a,b] \to \overline{\mathbb{R}}$ is continuous and strictly monotonic. Let $\underline{a}$ be an $n$-tuple with $a \leq a_i \leq b$, $1 \leq i \leq n$, $\underline{w}$ be a non-negative $n$-tuple with $W_n \neq 0$, then *the quasi-arithmetic $\mathfrak{M}$-mean of $\underline{a}$ with weight $\underline{w}$* is

$$\mathfrak{M}_n(\underline{a};\underline{w}) = M^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i M(a_i)\right) = M^{-1}\left(\mathfrak{A}_n\big(M(\underline{a});\underline{w}\big)\right).$$

It is easy to see that if $M(x)=x^p, x>0$, $-\infty<p<\infty$ then the $\mathfrak{M}$-mean is just the $p$-power mean, while if $M(x)=\log x, x>0$ it is the geometric mean; many more special cases can be found in the references.

Different functions can define the same mean, as can easily be seen.

Simple restrictions on the properties of $M$ can imply that the quasi-arithmetic mean reduces to a well-known mean.

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BASIC PROPERTIES *(a)* $\min \underline{a} \le \mathfrak{M}_n(\underline{a};\underline{w}) \le \max \underline{a}$.
*(b) If N is convex with respect to M then*

$$\mathfrak{M}_n(\underline{a};\underline{w}) \le \mathfrak{N}_n(\underline{a};\underline{w}); \tag{1}$$

*further if N is strictly convex with respect to M this inequality is strict unless all the $a_i$ with $w_i > 0$ are equal.*
*(c)*

$$f\left(\mathfrak{M}_n(\underline{a};\underline{w})\right) \le \mathfrak{N}_n\left(f(\underline{a});\underline{w}\right)$$

*if and only if $N \circ f$ is convex with respect to M.*

COMMENTS (i) To say that *a function f is (strictly ) convex with respect to a function g* is to say that both $f$ and $g$ are strictly increasing, and $f \circ g^{-1}$ is (strictly) convex. If both functions have continuous second derivatives and non-zero first derivatives this occurs if $g''/g' \le f''/f'$.

(ii) (1) is just a restatement of (J), the basic property of convex functions.

(iii) (c) is an easy extension of (b). See: **Log-convex Function Inequalities** (b) for a special case.

(iv) By omitting the weights in the definition of $\mathfrak{M}_n(\underline{a};\underline{w})$ we get analogues of the power sums. The results (M) and **Power Sum Inequalities** (1) have been extended to this situation.

(vi) See also: **Čakalov's Inequality** COMMENTS (iv).

REFERENCES [H, pp. 49–50, 266–320], [HLP, pp. 65–101], [MPF, pp. 193–194], [PPT, pp. 107–110, 165–169]; *Páles* [252].

**Quasi-conformal Function Inequalities** *If $\Omega$ is a domain in $\mathbb{R}^n, n \ge 2$, and if $f \in \mathcal{W}^{1,n}(\Omega)$ is quasi-conformal then there is a $k \in \mathbb{R}$ such that*

$$|\nabla f|^n \le k n^{n/2} J(f'),$$

*almost everywhere, where $J(f')$ is the Jacobian of f.*

COMMENT This is one of the definitions of quasi-conformality; the definition of $\mathcal{W}^{1,n}(\Omega)$ is given in **Sobolev's Inequalities**.

REFERENCE [EM, vol. 7, pp. 418–421].

**Quasi-convex Function Inequalities** *(a) If f is quasi-convex on a convex set X then for all $\lambda, 0 \le \lambda \le 1$ and for all $x, y \in X$,*

$$f\big((1-\lambda)x + \lambda y\big) \le \max\big\{f(x), f(y)\big\}.$$

*(b) If f is quasi-convex on a convex set X then for all $y \in X$ and if $P, Q$ are two parallelopipeds, symmetric with respect to y, $P \subseteq Q$,*

$$f(y) + \inf\big\{f(x); x \in P \setminus Q\big\} \le \frac{2}{|P \setminus Q|} \int_{P\setminus Q} f.$$

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*(c) If $f$ is continuous, differentiable and quasi-convex on the open convex set $X$ then*

$$f(x) \leq f(y) \implies f'(y)(x-y) \leq 0.$$

COMMENTS (i) Both (a) and (b) are definitions of quasi-convexity. If $-f$ is quasi-convex then $f$ is said to be *quasi-concave*. Both convex and monotone functions are quasi-convex.

(ii) The converse of (c) is also valid; so within the class functions in (c) this is another definition of quasi-convexity.

(iv) A symmetric quasi-convex function is Schur convex. For definitions see: **Segre's Inequalities, Schur Convex Function Inequalities**.

(v) The concept can be readily extended to higher dimensions; in addition, there is a higher order quasi-convexity.

(vi) See also **Minimax Theorems** (b); another definition of quasi-concavity is given in **Bergh's Inequality** COMMENTS (i).

REFERENCES [MOA, pp. 98–99]; *Roberts & Varberg* [RV, pp. 104, 230]; *Longinetti* [179], *Popoviciu, E.* [274].

## Quaternion Inequalities

*If $\mathfrak{a}_k, \mathfrak{b}_k, 1 \leq k \leq n$, are quaternions then*

$$\left|\left|\sum_{k=1}^{n} \mathfrak{a}_k \overline{\mathfrak{b}}_k\right|\right|^2 \leq \left(\sum_{k=1}^{n} ||\mathfrak{a}_k||^2\right)\left(\sum_{k=1}^{n} ||\mathfrak{b}_k||^2\right),$$

*with equality if and only if either $\mathfrak{a}_k = \mathfrak{o}, 1 \leq k \leq n$, $\mathfrak{b}_k = \mathfrak{o}, 1 \leq k \leq n$, or for some real $\lambda$, $\mathfrak{a}_k = \lambda \mathfrak{b}_k, 1 \leq k \leq n$.*

COMMENTS (i) If $\mathfrak{c}$ is a quaternion then $\mathfrak{c} = c_1 + c_2\mathfrak{i} + c_3\mathfrak{j} + c_4\mathfrak{k}$, $c_m \in \mathbb{R}$, $1 \leq m \leq 4$, and $\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = -1$, $\mathfrak{ij} = -\mathfrak{ji} = \mathfrak{k}$, $\mathfrak{jk} = -\mathfrak{kj} = \mathfrak{i}$, $\mathfrak{ki} = -\mathfrak{ik} = \mathfrak{j}$.

The zero quaternion is $\mathfrak{o} = 0 + 0\mathfrak{i} + 0\mathfrak{j} + 0\mathfrak{k}$.

In addition we define: $\overline{\mathfrak{c}} = c_1 - c_2\mathfrak{i} - c_3\mathfrak{j} - c_4\mathfrak{k}$, and $||\mathfrak{c}||^2 = \mathfrak{c}\overline{\mathfrak{c}} = \sum_{m=1}^{4} c_m^2$.

An important property is that $||\mathfrak{c}\mathfrak{d}|| = ||\mathfrak{c}||\,||\mathfrak{d}||$.

(ii) The inequality above is a generalization of (C).

REFERENCES [AI, p. 43], [EM, vol. 7, pp. 440–442], [MPF, p. 90].

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# 18 Rademacher–Rotation

**Rademacher-Men'šov Inequality** *If $\phi_k : [0,1] \mapsto \mathbb{C}, 1 \le k \le n$, are orthonormal and if $c_k \in \mathbb{C}, 1 \le k \le n$, then:*

$$\int_0^1 \max_{1\le k\le n}\left|\Big(\sum_{j=1}^k c_j\phi_j(x)\Big)^2\right| \mathrm{d}x \le \log_2^2(4n)\sum_{j=1}^n |c_j|^2.$$

Comments (i) This result, found independently and almost at the same time by the two mathematicians named, is one of the most important in the study of the summability of orthonormal series.

(ii) An interesting discussion of this result appears in the paper by Móricz.

References *Kaczmarz & Steinhaus* [KS], *Steele* [S, pp. 217–220]; *Móricz* [223].

**Rado's Geometric-Arithmetic Mean Inequality Extension** *If $n \ge 2$, then*

$$W_n\left\{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})\right\} \ge W_{n-1}\left\{\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})\right\}, \qquad (1)$$

*with equality if and only if $a_n = \mathfrak{G}_{n-1}(\underline{a};\underline{w})$.*

Comments (i) Repeated application of (1) exhibits it as an extension of (GA), to which it is, however, equivalent. If these applications are stopped earlier we get another extension of (GA):

$$\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w}) \ge \frac{1}{W_n}\max_{1\le i,j\le n}\left\{w_ia_i + w_ja_j - (w_i+w_j)\left(a_i^{w_i}a_j^{w_j}\right)^{1/(w_i+w_j)}\right\}.$$

In the case of equal weights this is,

$$\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a}) \ge \frac{1}{n}\left(\sqrt{\max\underline{a}} - \sqrt{\min\underline{a}}\right)^2.$$

A similar discussion can be found in **Jensen's Inequality** Corollary, **Popoviciu's Geometric-Arithmetic Mean Inequality Extension** Comments (ii).

(ii) Inequalities that extend $P - Q \ge 0$ to $P - Q$ is a increasing function of index sets are called *Rado-type or Rado-Popoviciu inequalities*. See: **Induction Inequality, Power Mean Inequalities** Extensions (e).



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(iii) Inequality (1) implies that the ratio

$$\frac{\mathfrak{A}_n(\underline{a};\underline{w}) - \mathfrak{G}_n(\underline{a};\underline{w})}{\mathfrak{A}_{n-1}(\underline{a};\underline{w}) - \mathfrak{G}_{n-1}(\underline{a};\underline{w})}$$

is bounded below as a function of $\underline{a}$ by $W_{n-1}/W_n$. Furthermore this bound is attained only when $a_n = \mathfrak{G}_{n-1}(\underline{a};\underline{w})$ which is not possible for non-constant increasing sequences. A natural question is whether this bound can be improved for such sequences; this is the basis of **Čakalov's Inequality**. No such extension exists for **Popoviciu's Geometric-Arithmetic Mean Inequality Extension** (1).

(iv) Inequality (1) implies that $\lim_{n\to\infty} n\big(\mathfrak{A}_n(\underline{a}) - \mathfrak{G}_n(\underline{a})\big)$ always exists, possibly infinite. The value of this limit has been obtained by Everitt.

(v) Inequality (1) is often just called Rado's inequality. An easy deduction from (1), is

$$\sum_{i=m+1}^{n} a_i \ge n\mathfrak{G}_n(\underline{a}) - m\mathfrak{G}_m(\underline{a}); \tag{2}$$

an inequality that is also called *Rado's Inequality.*

Extension [Functions of Index Set] *Let $\rho$ be the following function defined on the index sets,*

$$\rho(\mathcal{I}) = W_{\mathcal{I}}\left\{\mathfrak{A}_{\mathcal{I}}(\underline{a};\underline{w}) - \mathfrak{G}_{\mathcal{I}}(\underline{a};\underline{w})\right\}.$$

*Then $\rho \ge 0$, is increasing and super-additive.*

Comment (vi) There are many other extensions in the references. See also: **Harmonic Mean Inequalities** (c), **Henrici's Inequality** (c), **Minc-Sathre Inequality** Comment (iii).

References [AI, pp. 90–95], [H, pp. 125–136], [HLP, p. 61].

**Rado's Inequality** *If $a, b, r \in \mathbb{R}$, $a < b$ and if $f \in \mathcal{C}([a,b])$ is positive and convex*

$$\mathfrak{M}_2^{[r_1]}\big(f(a), f(b)\big) \le \mathfrak{M}_{[a,b]}^{[r]}(f) \tag{1}$$

*where*

$$r_1 = \begin{cases} \min\left\{\dfrac{r+2}{3}, \dfrac{r\log 2}{\log(r+1)}\right\}, & \text{if } r > -1,\ r \ne 0,\\ \min\left\{\dfrac{2}{3}, \log 2\right\}, & \text{if } r = 0,\\ \min\left\{\dfrac{r+2}{3}, 0\right\}, & \text{if } r \le -1. \end{cases}$$

*The value $r_1$ is best possible.*
*If $f$ is concave then ($\sim$1) holds if $r_1$ is replaced by $r_2$ where $r_2$ is defined as is $r_1$, but with min replaced by max.*

Comment (i) In particular (1) implies **Pittenger's Inequalities**.

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Extensions [Pearce & Pečarić] *With the above notation*

$$\mathfrak{M}_2^{[r_1]}\big(f(a),f(b)\big) \le \mathfrak{M}_{[a,b]}^{[r]}(f) \le \mathfrak{L}^{[r]}\big(f(a),f(b)\big) \le \mathfrak{M}_2^{[r_2]}\big(f(a),f(b)\big), \qquad (2)$$

*with equality if and only if $f$ is linear. If $f$ is concave then ($\sim$2) holds.*

Comments (ii) The case $r = 2$ of the central inequality is due to Sándor.
(iii) For other inequalities with the same name see: **Rado's Geometric-Arithmetic Mean Inequality Extension** Comments (v).

References *Pearce & Pečarić* [256].

**RadoType, Rado-Popoviciu Type Inequalities** See: **Induction Inequality, Rado's Geometric-Arithmetic Mean Inequality Extension** Comment (ii), **Popoviciu Type, Rado-Popoviciu Type Inequalities**.

**Radon's Inequality** *If $\underline{a}, \underline{b}$ are positive $n$-tuples and $p > 1$ then*

$$\frac{\left(\sum_{i=1}^n a_i\right)^p}{\left(\sum_{i=1}^n b_i\right)^{p-1}} \le \sum_{i=1}^n \left(\frac{a_i^p}{b_i^{p-1}}\right), \qquad (1)$$

*while if $p < 1$ the inequality ($\sim$1) holds. The inequalities are strict unless $\underline{a} \sim \underline{b}$.*

Comment This is a deduction from (H).

References [H, p. 181], [HLP, p. 61]; *Redheffer & Voigt* [280].

**Rahmail's Inequality** *Let $f, g : [a,b] \to \mathbb{R}$ be positive concave functions, then if $k < p \le k+1, k \in \mathbb{N}, k > 0$,*

$$||f+g||_{p,[a,b]} \ge \Lambda\left(||f||_{p,[a,b]} + ||g||_{p,[a,b]}\right),$$

*where*

$$\Lambda = \left(\frac{\left(\frac{k+1}{2}\right)!\left(\frac{k+2}{2}\right)!}{(k+2)!}\right)^{1/k} (p+1)^{1/p} \left(\frac{2p-k}{p-k}\right)^{(p-k)/pk}.$$

Comment This inequality is a converse to (M); for a similar result related to (H) see: **Petschke's Inequality**.

Reference [MPF, p. 150].

**Rank Inequalities** *(a)* [Fröbenius] *If $A, B, C$ are matrices for which $ABC$ is defined, then*

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \le \operatorname{rank}B + \operatorname{rank}(ABC).$$

*(b)* [Sylvester] *If $A$ is a $m \times n$ matrix and $B$ an $n \times p$ matrix then*

$$\operatorname{rank}A + \operatorname{rank}B - n \le \operatorname{rank}(AB) \le \min\{\operatorname{rank}A, \operatorname{rank}B\}.$$

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REFERENCE *Marcus & Minc* [MM, pp. 27–28].

**Rayleigh-Ritz Ratio** *If $A$ is an $n\times n$ Hermitian matrix then for any complex $n$-tuple $\underline{a}$,*

$$\lambda_{(1)} \le \frac{\underline{a}^* A\underline{a}}{\underline{a}^*\underline{a}} \le \lambda_{(n)}.$$

COMMENTS (i) The central term in this inequality is called *the Rayleigh-Ritz ratio.* The maximum of this ratio over all non-zero $\underline{a}$ of this ratio is $\lambda_{(n)}$, whilst the minimum is $\lambda_{(1)}$.

(ii) For a generalization see: [I2].

REFERENCES [BB, pp. 71–72], [GI1, pp. 223–230], [I2, pp. 276–279]; *Horn & Johnson* [HJ, pp. 176–177], *Mitrović & Žubrinić* [MZ, p. 242].

**Real Numbers Inequalities** See: **Arithmetic's Basic Inequalities**.

**Rearrangement Inequalities** *(a)* [HARDY, LITTLEWOOD & PÓLYA] *If $\underline{a}, \underline{b}$ are real $n$-tuples then*

$$\begin{aligned} \sum_{i=1}^{n} a_{(i)}b_{[i]} \quad &\le \quad \sum_{i=1}^{n} a_ib_i \le \sum_{i=1}^{n} a_{(i)}b_{(i)}; \qquad (1)\\ \textit{or} \qquad \underline{a}_*.\underline{b}^* = \underline{a}^*.\underline{b}_* \quad &\le \quad \underline{a}.\underline{b} \le \underline{a}_*.\underline{b}_* = \underline{a}^*.\underline{b}^*.\\ \prod_{i=1}^{n}(a_{(i)}+b_{(i)}) \quad &\le \quad \prod_{i=1}^{n}(a_i+b_i) \le \prod_{i=1}^{n}(a_{[i]}+b_{(i)}). \qquad (2) \end{aligned}$$

*(b) If $f$ is a positive increasing function then*

$$\sum_{n=1}^{\infty} f\big(\mathfrak{A}_n(\underline{a})\big) \le \sum_{n=1}^{\infty} f\big(\mathfrak{A}_n(\underline{a}^*)\big).$$

COMMENT (i) A simple application of (1) gives:

$$A_nB_n \le n\sum_{i=1}^{n} a_ib_i,$$

where $\underline{a}, \underline{b}$ are increasing real $n$-tuples.

(ii) The right-hand side of (1) is a consequence of Fan's inequality on the traces of symmetric matrices. See: **Trace Inequalities** COMMENT (iv).

EXTENSIONS *(a)* [HAUTUS] *If $f$ is strictly convex in the interval $I$, and if $\underline{a}, \underline{b}$ are two $n$-tuples with $a_i, b_i, a_i+b_i \in I, 1\le i\le n$, then*

$$\sum_{i=1}^{n} f\left(a_{(i)}+b_{[i]}\right) \le \sum_{i=1}^{n} f(a_i+b_i) \le \sum_{i=1}^{n} f\left(a_{(i)}+b_{(i)}\right).$$

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*(b)* [Hardy & Littlewood] *If $a_i, b_i, c_i, -n \le i \le n$, are real $(2n+1)$-tuples, with $\underline{c}$ symmetrical, then*

$$\sum_{\substack{i,j,k=-n\\ i+j+k=0}}^{n} a_i b_j c_k \le \sum_{\substack{i,j,k=-n\\ i+j+k=0}}^{n} {}^{+}a_i b_j^{+} c_k^{(*)} = \sum_{\substack{i,j,k=-n\\ i+j+k=0}}^{n} a_i^{+}\,{}^{+}b_j c_k^{(*)}.$$

*(c)* [Geretschläger & Janous] *With the notation in (1), let $T$ be a real function on $(\min \underline{a}, \max \underline{a}) \times (\min \underline{b}, \max \underline{b})$ with $\partial^2 T/\partial x \partial y \ge 0$ and put $S(\underline{a}, \underline{b}) = \sum_{i=1}^{n} T(a_i, b_i)$. Then*

$$S(\underline{a}_*, \underline{b}^*) = S(\underline{a}^*, \underline{b}_*) \le S(\underline{a}, \underline{b}) \le S(\underline{a}_*, \underline{b}_*) = S(\underline{a}^*, \underline{b}^*).$$

Comments (ii) Extension (b) is a generalization of the basic result (1).

(iii) Inequality (1) is the case $T(x, y) = xy$ of (c); and taking $T(x, y) = \log(x+y)$ gives inequality (2). Interesting special cases occur when $T(x, y) = f(x)g(y)$ with $f' \ge 0, g' \ge 0$. Extensions to more than two $n$-tuples are more complicated and more difficult to prove. See: [HLP].

Integral Analogues *(a)* [Hardy-Littlewood] *If $I$ is an interval in $\mathbb{R}$, $fg \in \mathcal{L}(I)$ then*

$$\int_I fg \le \int_I f^* g^*.$$

*(b)* [Brascamp-Lieb-Luttinger] *Let $f_k$, $1 \le k \le m$, be non-negative functions on $\mathbb{R}^n$, vanishing at infinity. Let $k \le m$ and let $B = (b_{ij})_{\substack{1 \le i \le k\\ 1 \le j \le m}}$ be a $k \times m$ matrix, and define*

$$I(f_1, \ldots, f_n) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \left( \sum_{i=1}^{k} b_{ij} x_i \right) \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

*Then $I(f_1, \ldots, f_n) \le I(f_1^{(*)}, \ldots, f_n^{(*)})$.*

Comments (iv) A function $f : \mathbb{R}^n \mapsto \mathbb{C}$ is said to *vanish at infinity* if for all $t$ the set $\{\underline{x}; |f(\underline{x})| > t\}$ has finite measure.

(v) This inequality is used to obtain the exact constants in **Titchmarsh's Theorem** and **Young's Convolution Inequality**.

(vi) See also: **Chong's Inequalities** (a), **Duff's Inequality, Hardy-Littlewood Maximal Inequalities, Hardy-Littlewood-Sobolev Inequality, Modulus of Continuity Inequalities, Riesz's Rearrangement Inequality, Spherical Rearrangement Inequalities, Variation Inequalities**.

References [AI, p. 284], [H, p. 23], [HLP, pp. 248, 261–276], [MPF, p. 17]; *Bulajich Manfrino,Ortega & Delgado* [BOD, pp. 13–19], *Kawohl* [Ka, p. 23], *Lieb & Loss* [LL, p. 5]; *Atanassov* [47], *Geretschläger & Janous* [131].

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**Recurrent Inequalities of Redheffer** *Let $f_k(a_1,\ldots,a_k), g_k(a_1,\ldots,a_k)$ be real-valued functions defined for $a_k$ in the set $D_k$, $1\le k\le n$ and for which there exist real-valued functions $F_k$ such that for all $k$, $1\le k\le n$,*

$$\sup_{a_k\in D_k}\big(\mu f_k(a_1,\ldots,a_k)-g_k(a_1,\ldots,a_k)\big)=F_k(\mu)f_{k-1}(a_1,\ldots,a_{k-1}), \tag{1}$$

*where $f_0=1$. Then*

$$\sum_{k=1}^{n}\mu_k f_k(a_1,\ldots,a_k)\le\sum_{k=1}^{n}g_k(a_1,\ldots,a_k), \tag{2}$$

*provided we can find real numbers $\delta_k$, $1\le k\le n+1$, such that*

$$\begin{aligned}\delta_1\le\ & 0, \qquad \delta_{n+1}=0\\ \mu_k\ =\ & F_k^{-1}(\delta_k)-\delta_{k+1},\end{aligned}$$

*where $F_k^{-1}(y)$ denotes any $x$ such that $F_k(x)=y$.*

Comments (i) The proof is by induction on $n$. Inequalities of this form are called *recurrent inequalities* and the functions $F_k, 1\le k\le n$, are called the *structural functions of the inequality.*

(ii) If we want the inequality ($\sim$2) then the sup in (1) is replaced by inf.

(iii) It is possible to replace the fixed sets $D_1,\ldots,D_n$ by

$$D_k=D_k(a_1,\ldots,a_{k-1}),\ 2\le k\le n.$$

(iv) Many classical inequalities can be put into this form, and usually this not only gives a proof of the inequality but of a significant refinement. See: **Abel's Inequalities** Extensions (b), **Arithmetic Mean Inequalities** (d), **Carleman's Inequality** Extensions (b), **Fan-Taussky-Todd** Extensions (b), **Hardy's Inequality** Extensions (a), **Redheffer's Inequalities**.

References [AI, pp. 129–131], [H, pp. 145–147]; *Redheffer* [279].

**Redheffer's Inequalities** *(a) If $\mathfrak{A}_n(\underline{\mathfrak{G}})=\mathfrak{A}_n\big(\mathfrak{G}_1(\underline{a}),\ldots,\mathfrak{G}_n(\underline{a})\big)$ then*

$$\mathfrak{A}_n(\underline{\mathfrak{G}})\exp\left\{\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{A}_n(\underline{\mathfrak{G}})}\right\}\le e\mathfrak{A}_n(\underline{a}),$$
$$1\le\frac{1}{2}\left\{\frac{\mathfrak{A}_n(\underline{\mathfrak{G}})}{\mathfrak{G}_n(\underline{a})}+\frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{A}_n(\underline{\mathfrak{G}})}\right\}\le\frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})};$$

*there is equality if and only if $\underline{a}$ is constant.*
*(b) If $\underline{w}$ is a positive $n$-tuple then*

$$\sum_{i=1}^{n}\frac{2i+1}{W_i}+\frac{n^2}{W_n}\le 4\sum_{i=1}^{n}\frac{1}{w_i},$$

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*with equality if and only if $w_i = iw_1$, $1 \le i \le n$.*
*(c) If $\underline{a}$ is a positive sequence,*

$$\sum_{i=1}^{n} \mathfrak{G}_i(\underline{a}) + n\mathfrak{G}_n(\underline{a}) < \sum_{i=1}^{n}(1+\frac{1}{i})^i a_i,$$

*with equality if and only if $a_i = 2a_1 i^{i-1}(i+1)^{-i}$, $1 \le i \le n$.*

COMMENTS (i) Inequality (2) follows from (1). The proofs are based on the method of **Recurrent Inequalities of Redheffer**.

(ii) Both (1) and (2) extend (GA). Further inequalities that involve the sequence $\underline{\mathfrak{G}}$ can be found in **Nanjundiah Mixed Mean Inequalities**.

(iii) Since $\exp\left(\mathfrak{G}_n(\underline{a})/\mathfrak{A}_n(\underline{\mathfrak{G}})\right) > 1$, (1) implies the finite form of **Carleman's Inequality** (1).

(iv) See also: **Abel's Inequalities** EXTENSIONS (b), **Carleman's Inequality** EXTENSIONS (b).

REFERENCES [H, pp. 145–147]; *Redheffer* [279].

## Reisner's Inequality See: **Mahler's Inequalities** COMMENTS (ii).

## Rellich's Inequality *If $n \ne 2$, $f \in \mathcal{C}_0^{\infty}(\mathbb{R}^n \setminus \{\underline{0}\})$ then*

$$\int_{\mathbb{R}^n} |\nabla f|^2 \ge \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|f|^2}{|\underline{x}|^4}\,\mathrm{d}\underline{x}.$$

COMMENT This result has been extended, under extra conditions, to the case $n = 2$. In addition the exponent 4 in the integral on the right-hand side can be replaced by $\alpha, \alpha > 4$.

REFERENCES *Pachpatte* [246, 247].

## Rennie's Inequality *If $0 < m \le \underline{a} \le M$, and if $W_n = 1$ then*

$$\sum_{i=1}^{n} w_i a_i + mM \sum_{i=1}^{n} \frac{w_i}{a_i} \le M + m.$$

COMMENT (i) This is equivalent to the **Díaz & Metcalf Inequality**.

EXTENSION [GOLDMAN] *If $0 < m \le \underline{a} \le M$, and if $-\infty < r < s < \infty, rs < 0$ then*

$$(M^s - m^s)\left(\mathfrak{M}_n^{[r]}(\underline{a},\underline{w})\right)^r - (M^r - m^r)\left(\mathfrak{M}_n^{[s]}(\underline{a},\underline{w})\right)^s \le M^s m^r - M^r m^s.$$

*The opposite inequality holds if $rs > 0$.*

COMMENT (ii) Putting $s = 1, r = -1$ Goldman's inequality reduces to Rennie's inequality. Goldman used his result to prove Specht's inequality, see **Power Mean Inequalities** INVERSE INEQUALITIES (b).

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REFERENCES [AI, pp. 62–63], [H, p. 235], [MPF, pp. 123–125], [PPT, pp. 108–109].

**Rényi's Inequalities** Using the notation of **Entropy Inequalities** the *Rényi Entropy* is defined by:

$$H_\alpha(\mathcal{A}) = \begin{cases} \dfrac{1}{1-\alpha}\log_2 \displaystyle\sum_{A\in\mathcal{A}} p^\alpha(A), & \text{if } 0<\alpha<\infty, \alpha\neq 1,\\ -\sum_{A\in\mathcal{A}} p(A)\log_2 p(A), & \text{if } \alpha = 1,\\ \log_2|\mathcal{A}|, & \text{if } \alpha=0,\\ -\log_2 \max_{A\in\mathcal{A}} p(A), & \text{if } \alpha=\infty. \end{cases}$$

*(a) The Renyi entropies are Schur concave.*
*(b) If $0\le\alpha\le\beta\le\infty$ then $H_\alpha(\mathcal{A})\ge H_\beta(\mathcal{A})$.*

COMMENTS (i) The cases $\alpha = 0, 1, \infty$ are limits of the general case.

(ii) The case $\alpha = 1$ is defined in **Shannon's Inequality** COMMENTS (I), **Entropy Inequalities** COMMENTS (I).

REFERENCES [MPF, p. 636]; *Rényi* [Ren].

**Reverse, Inverse, and Converse Inequalities** These terms are used rather loosely in the literature but in this book they are used as explained below

CONVERSE INEQUALITIES If:

*the condition $P$ implies the inequality $A\le B$*: (1)

then

*inequality $A\le B$ implies the condition $P$*

is the converse inequality, or more exactly the converse to the inequality.

This is not very common except in the trivial situation where the inequality is the definition of the property $P$; see for instance **Increasing Function Inequalities** (1). However consider the example:

*if $f\in\mathcal{C}^2([a,b])$ then*

$$f''\ge 0 \Longrightarrow f\big(\frac{x+y}{2}\big)\le \frac{f(x)+f(y)}{2}, \forall x,y, a\le x,y\le b;$$

*and conversely if $f\in\mathcal{C}^2([a,b])$ then*

$$f\big(\frac{x+y}{2}\big)\le \frac{f(x)+f(y)}{2}, \forall x,y, a\le x,y\le b \Longrightarrow f''\ge 0.$$

INVERSE INEQUALITIES This is a more subtle situation. Given inequality (1) and if it can be proved that:

*the condition $Q$ implies the inequality $B\le K_1A$, or $B\le K_2+A$ for some constants $K_1, K_2$* (2)

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then (2) is called an inverse inequality to (1). See: **Inverse Hölder Inequalities**.

The simplest form of such inverse inequalities arise when we note that usually $B/A$ and $B-A$ are continuous functions defined on a compact set and so attain their maximums on that set, $K_1$ and $K_2$ say. Hence have that $A \le B \le K_1 A$ and $A \le B \le K_2 + A$. Of course these constants then depend very much on the particuar circumstances; more desirable, and more subtle, are absolute constants as in the reference given above.

These have also been called *complementary inequalities*. See: [MOA, pp. 102–103].

Reverse Inequalities Given inequality (1) then if we prove:

$$\textit{the condition } Q \textit{ implies the inequality } B \le A: \qquad (\sim 1)$$

then this is called an inequality reverse to the inequality (1), inequality $(\sim 1)$. For instance:

*if* $a, b > 0$ *and* $0 \le t \le 1$ *then* $a^{1-t}b^t \le (1-t)a + tb$;
*if* $a, b > 0$ *and* $t \le 0$ *or* $t \ge 1$ *then* $a^{1-t}b^t \ge (1-t)a + tb$.

More generally the same name is given in circumstances where an analogous quantity gives the the reverse inequality to the standard situation. See: **Alzer's Inequalities** Pseudo Arithmetic and Geometric Means.

## Riesz's Inequalities[41]

See **Conjugate Harmonic Function Inequalities, Hausdorff-Young Inequalities, Riesz Mean Value Theorem, Riesz-Thorin Theorem**.

## Riesz Mean ValueTheorem[42]

*If* $0 < p \le 1$ *and* $y > 0$ *then*

$$\int_0^x (x-t)^{p-1} f(t)\,\mathrm{d}t \ge 0, \quad 0 \le x \le y, \implies \int_0^y (x-t)^{p-1} f(t)\,\mathrm{d}t \ge 0, \quad y \le x < \infty.$$

Comment This result is important as the integral on the left-hand side is, when divided by $(p-1)!$, the fractional integral of order $p$ of $f$; in addition it appears in various summability problems, and in the remainder of Taylor's theorem. The integral on the right-hand side is then its partial integral.

Extensions [Türke & Zeller] *If* $p > 1$ *and* $y > 0$ *then*

$$\int_0^x (x-t)^{p-1} f(t)\,\mathrm{d}t \ge 0, \quad 0 \le x \le y,$$
$$\implies \int_0^y \big((y-t)(x-t)\big)^{p-1} f(t)\,\mathrm{d}t \ge 0, \quad y \le x < \infty.$$

[41] This is M. Riesz.
[42] This is M. Riesz.

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Discrete Analogue [Askey, Gasper & Ismail] *If $p > 1$ and if for some integer $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{m-k+p-1}{m-k} a_k \geq 0, \quad 0 \leq m \leq n,$$

*then*

$$\sum_{k=0}^{n} \binom{n-k+p-1}{n-k} \binom{m-k+p-1}{m-1} a_k \geq 0, \quad n \leq m < \infty.$$

References [GI3, pp. 491–496].

**Riesz's Rearrangement Inequality**[43] *If $f, g, h$ are non-negative measurable functions on $\mathbb{R}^n$ then*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(\underline{v}) g(\underline{u} - \underline{v}) h(\underline{u}) \, \mathrm{d}\underline{u} \mathrm{d}\underline{v} \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} f^{(*)}(\underline{v}) g^{(*)}(\underline{u} - \underline{v}) h^{(*)}(\underline{u}) \, \mathrm{d}\underline{u} \mathrm{d}\underline{v}.$$

Comments (i) The case $n = 1$ is due to Riesz; the general case was given by Sobolev.

(ii) Cases of equality have been given by Burchard; in addition the result has been extended to $m$ functions in $k$ variables. See: [LL].

References [HLP, pp. 279–291]; *Lieb & Loss* [LL, pp. 79–87].

**Riesz-Thorin Theorem**[44] *Let $(X, \mu), (Y, \nu)$ be two measure spaces, $T : X \to Y$ a linear operator with the properties*

$$||Tf||_{1/\beta_1} \leq M_1 ||f||_{1/\alpha_1}, \qquad ||Tf||_{1/\beta_2} \leq M_2 ||f||_{1/\alpha_2},$$

*where $P_i = (\alpha_i, \beta_i), i = 1, 2$ belong to the square $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$. Then for all $(\alpha, \beta)$ on the the segment joining the points $P_1, P_2$, $\alpha = (1-t)\alpha_1 + t\alpha_2, \beta = (1-t)\beta_1 + t\beta_2, , 0 \leq t \leq 1$, say,*

$$||Tf||_{1/\beta} \leq M_1^{1-t} M_2^t ||f||_{1/\alpha}.$$

Comments (i) This is the original example of what has become called an *Interpolation Theorem*. It is also called the *Riesz Convexity Theorem*.

(ii) Originally this theorem was a result about multilinear forms; *if $S$ is the left-hand side of* **Multilinear Form Inequalities** *(1), and if the supremum of $S$ subject to $||\underline{a}_i||_{p_i} \leq 1, 1 \leq i \leq n$, is $M = M(p_1, \ldots, p_n)$, then $M$ is log-convex in each variable in the region $p_i > 0, 1 \leq i \leq n$.*

(iii) See also: **Phragmén-Lindelöf Inequality**.

[43] This is F. Riesz.
[44] This is M. Riesz.

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REFERENCES [EM, vol. 5, pp. 146–148; vol. 8, p. 154], [HLP, pp. 203–204], [MPF, pp. 205–206]; *Hirschman* [Hir, pp. 170–173]; *Zygmund* [Z, vol. II, pp. 93–100].

**Rogers's Inequality** See: **Hölder's Inequality** OTHER FORMS (A), COMMENT (XII).

**Rogosinski-Szegő Inequality** *If* $0 < x < \pi$ *and* $n \geq 1$ *then*

$$\frac{1}{2} + \sum_{i=1}^{n} \frac{\cos ix}{i+1} \geq 0.$$

COMMENT (i) Clearly the constant is best possible being attained when $n = 1$.

EXTENSION *If* $0 < x < \pi$ *and* $n \geq 2$ *then*

$$\frac{41}{96} + \sum_{i=1}^{n} \frac{\cos ix}{i+1} \geq 0.$$

COMMENTS (ii) The constant is best possible being attained when $n = 2$.

(iii) See also: **Fejér-Jackson Inequality, Trigonometric Polynomial Inequalities.**

REFERENCES *Milovanović, Mitrinović & Rassias* [MMR, pp. 323–332]; *Koumandos* [162].

**Root and Ratio Test Inequalities** *If* $a_n, n \in \mathbb{N}$, *is a positive sequence then*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

COMMENTS (i) All the inequalities above can be strict; consider the sequence $a_{2n} = 3^{-n}$, $a_{2n-1} = 2^{-n}, n \geq 1$.

(ii) The name comes from the fact that the sequence of ratios and of roots are used as tests for the convergence of series. Further if $a_n, n \in \mathbb{N}$, is the complex sequence of coefficients of a power series then the radius of convergence of that series is $1\big/ \limsup_{n\to\infty} \sqrt[n]{|a_n|}$; with suitable interpretations if this upper limit is either zero or infinity.

REFERENCE *Rudin* [R76, pp. 57–61].

**Rotation Theorems** *If* $f$ *is univalent in* $\overline{D}$, *with* $f(0) = 0, f'(0) = 1$, *then*

$$|\arg f'(z)| \leq \begin{cases} 4\arcsin|z|, & \text{if} \quad |z| \leq 1/\sqrt{2}, \\ \pi + \log\left(\dfrac{|z|^2}{\big(1-|z|\big)^2}\right), & \text{if} \quad 1/\sqrt{2} \leq |z| < 1. \end{cases}$$

COMMENT Rotation theorems give estimates for the rotation at a point under a conformal map.

REFERENCE [EM, vol. 8, pp. 185–186].

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# 19 Saffari–Székely

**Saffari's Inequality** *If* $f : [0,1] \to \mathbb{R}$ *is bounded and measurable with* $H = \mathfrak{M}_{[0,1]}^{[\infty]}(f) > 0, h = -\mathfrak{M}_{[0,1]}^{[-\infty]}(f) > 0$, *and if* $p = 1$ *or* $p \geq 2$, *then*

$$||f||_p \leq \mathfrak{M}_2^{[p-2]}(H, h).$$

Comment The result is false if $1 < p < 2$.

Reference [GI3, pp. 529–530].

**Samuelson Inequality** See: **Laguerre-Samuelson Inequality**.

**Sándor's Inequality** See: **Rado's Inequality** Comment (ii).

**Schlömilch-Lemonnier Inequality** *If* $n \in \mathbb{N}$, $n > 1$, *then*

$$\log(n+1) < \sum_{i=1}^{n} \frac{1}{i} < 1 + \log n.$$

Comments (i) This is a simple deduction from **Integral Test Inequality**, and many extensions are given by Alzer & Brenner.

(ii) See also: **Euler's Constant Inequalities**, **Logarithmic Function Inequalities**.

References [AI, pp. 187–188]; *Alzer & Brenner* [38].

**Schoenberg's Conjecture** If $a_n \neq 0$ let

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n \prod_{k=1}^{n} (z - z_k).$$

Define the *quadratic mean radius of* $p_n$ by:

$$R(p_n) = \sqrt{\frac{1}{n} \sum_{k=1}^{n} |z_k|^2} = \mathfrak{M}_n^{[2]}(|z_1|, \ldots, |z_n|).$$



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*If $p_n$ is a monic polynomial with the sum of its zeros being zero, equivalently $a_{n-1} = 0$, then*

$$R(p'_n) \le \sqrt{\frac{n-2}{n-1}} R(p_n).$$

COMMENT In the case of real zeros this is easy to verify.

REFERENCE *De Bruin, Ivanov & Sharma* [97]

**Schur Convex Function Inequalities** *(a) If $F: \mathbb{R}^n \to \mathbb{R}$ is Schur convex then for all $i \ne j, 1 \le i, j \le n$,*

$$(x_i - x_j)\left(\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}\right) \ge 0.$$

*(b)* [OSTROWSKI] *If $F$ is a real valued function that is Schur convex on a set $A$ in $\mathbb{R}^n$ and if $\underline{a}, \underline{b} \in A$ then*

$$F(\underline{a}) \le F(\underline{b}) \iff \underline{a} \prec \underline{b}.$$

*(c)* [MARSHALL & OLKIN] *If $f, g$ are both Schur convex on $\mathbb{R}^n$ so is $f \star g$.*

COMMENTS (i) (a) is just the definition of a *Schur convex function on $A$.* If the inequality is always strict then $F$ is said to be *strictly Schur convex on $A$.* It is worth noting that a function that is both convex, (even quasi-convex) and symmetric is Schur convex.

(ii) (b) identifies the class of order preserving functions from the pre-ordered set of real $n$-tuples to $\mathbb{R}$. See also: **Muirhead Symmetric Function and Mean Inequalities** COMMENT (iii).

(iii) Many inequalities can be obtained by using special Schur convex functions; for instance:

*if $\underline{w}$ is a positive $n$-tuple with $W_n = 1$,*

$$\sum_{i=1}^{n} \frac{w_i}{1-w_i} \ge \frac{n}{n-1}; \qquad \sum_{i=1}^{n} \frac{w_i}{\prod_{j=1}^{n} w_j} \ge n^{n-1};$$

$$\sum_{r}! \frac{w_1 + \cdots + w_r}{w_1 \cdots w_r} \ge r\binom{n}{r} n^{r-1}; \qquad \sum_{\substack{i,j=1\\ i\ne j}}^{n} \frac{w_i w_j}{w_i + w_j} \le \frac{n-1}{2}.$$

(iv) See also: **Arithmetic Mean Inequalities** COMMENTS (III), **Complete Symmetric Function Inequalities** COMMENTS (III), **Elementary Symmetric Function Inequalities** COMMENTS (VII), **Geometric-Arithmetic Mean Inequality** COMMENTS (VI), **Geometric Mean Inequalities** COMMENTS (III), **Mitrinović & Đoković Inequality** COMMENTS (I), **n-convex Function Inequalities** COMMENTS (VII).

REFERENCES [AI, pp. 167–168, 209–210], [BB, p. 32], [EM, vol. 6, p. 75], [MOA, pp. 79–131], [PPT, pp. 332–336]; *Steele* [S, pp. 193–194].

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**Schur-Hardy Inequality** See: **Hardy-Littlewood-Pólya-Schur Inequality**.

**Schur's Inequality** *If $a, b, c$ are positive and $r \in \mathbb{R}$ then*

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0, \tag{1}$$

*with equality if and only if $a = b = c$.*

COMMENT (i) Inequality (1) has been the subject of many generalizations. It can be deduced from properties of $Q$-class functions. See: **Q-class Function Inequalities** COMMENTS (iii).

EXTENSION [GUHA] *If $a, b, c, u, v, w > 0, p \in \mathbb{R}$ are such that*

$$a^{1/p} + c^{1/p} \leq b^{1/p}, \tag{2}$$

$$u^{1/p+1} + w^{1/p+1} \geq v^{1/p+1}; \tag{3}$$

*then if $p > 0$*

$$ubc - vca + wab \geq 0. \tag{4}$$

*If $-1 < p < 0$ and we have (2), $(\sim 3)$ then $(\sim 4)$ holds; if $p < -1$ then (4) holds if we have $(\sim 2), (\sim 3)$.*
*In each case there is equality in (4), or $(\sim 4)$ if and only if there is equality in (2), (3), or $(\sim 2), (\sim 3)$, and also*

$$\frac{a^{p+1}}{u^p} = \frac{b^{p+1}}{v^p} = \frac{c^{p+1}}{w^p}.$$

COMMENTS (ii) Guha's result follows from (H). It reduces to Schur's inequality (1) by putting $p = 1, a = y - z, b = x - z, c = x - y, u = x^\lambda, v = y^\lambda, w = z^\lambda$, assuming as we may that $0 \leq z \leq y \leq x$.

(iii) This is a cyclic inequality. See: **Cyclic Inequalities**.

(iv) For other inequalities by Schur see: **Eigenvalue Inequalities** (A), **Hardy-Littlewood-Pólya-Schur Inequalities, Permanent Inequalities** (B), **Erdős's Inequality** COMMENTS (II).

REFERENCES [AI, pp. 119–121], [HLP, p. 64], [MPF, pp. 407–413].

**Schur's Lemma** *If $\underline{x}$ is an $m$-tuple and $\underline{y}$ an $n$-tuple, and $C = (c_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ then*

$$\left|\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_i y_j\right| \leq \sqrt{\Big(\max_i \sum_{j=1}^n |c_{ij}|\Big)\Big(\max_j \sum_{i=1}^m |c_{ij}|\Big)} \Big(\sum_{i=1}^m |x_i|^2\Big)^{1/2} \Big(\sum_{j=1}^n |y_j|^2\Big)^{1/2}$$

COMMENTS (i) If $m = n, c_{ij} = 0, i \neq j, c_{ii} = 1, 1 \leq i \leq m$ this reduces to (C).

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(ii) This result is of great use in bounding quadratic forms.
iii) There are several other better known results with the same name.

REFERENCE *Steele* [S, pp. 15, 229].

**Schwarz's Lemma** *If $f$ is analytic in $D$, $|f| \leq 1, f(0) = 0$, then*

$$|f(z)| \leq |z|,\ z \in D, \quad \textit{and} \quad |f'(0)| \leq 1.$$

*There is equality if and only if $f(z) = e^{i\theta} z$.*

COMMENT (i) Schwarz's Lemma is an easy deduction from the **Maximum-Modulus Principle**.

EXTENSIONS *(a) If $f$ analytic in $D_R = |z| < R$, $|f| \leq M$, and if $z_0 \in D_R$ then*

$$\left| \frac{M(f(z) - f(z_0))}{M^2 - \overline{f(z_0)} f(z)} \right| \leq \left| \frac{R(z - z_0)}{R^2 - \overline{z}_0 z} \right|,\ z \in D_R. \tag{1}$$

*(b) If $f$ is analytic in $D$, $|f| \leq 1$, then*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{2}$$

COMMENT (ii) Inequality (1), with $M = R = 1$, and (2) can be expressed as follows:

*an analytic function on the unit disk with values in the unit disk decreases the hyperbolic distance between two points, the hyperbolic arc length and the hyperbolic area.*

In this form the result is known as the *Schwarz-Pick lemma*.

The definition of the hyperbolic metric, $\rho_h$, is given in **Metric Inequalities** COMMENTS (iv).

EXTENSION [DIEUDONNÉ] *(c) If $f$ is a analytic in $D$ and takes values in $D$, and if $f(0) = 0$, then*

$$|f'(z)| \leq \begin{cases} 1, & \textit{if } |z| \leq \sqrt{2} - 1, \\ \dfrac{(1 + |z|^2)^2}{4|z|(1 - |z|^2)}, & \textit{if } |z| \geq \sqrt{2} - 1. \end{cases}$$

*(d)* [BEARDON] *If $f$ is analytic in $D$ and takes values in $D$ and if $f$ is not a fractional linear transformation, then*

$$\rho_h(f^*(0), f^*(z)) \leq 2\rho_h(0, z),$$

*where*

$$f^*(z) = \frac{1 - |z|^2}{1 - |f(z)|^2} f'(z).$$

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*The inequality is strict unless $f(z) = zh(z)$ where $h$ is a fractional linear transformation of the unit disk onto itself.*

COMMENT (iii) The quantity $f^*(z)$ is called the *hyperbolic derivative of $f$ at $z$*.

(iv) *A fractional linear tranformation* is a function of the form $(az+b)(cz+d)^{-1}$, with $ad-bc \neq 0$.

REFERENCES [EM, vol. 8, pp. 224–225], [I3, pp. 9–21]; *Ahlfors* [Ah78, pp. 135–136, 164], [Ah73, pp. 1–3], *Pólya & Szegö* [PS, pp. 160–163], *Titchmarsh* [T75, p. 168]; *Beardon* [53].

**Schwarz-Pick Lemma** See: **Schwarz's Lemma** COMMENTS (ii).

**Schweitzer's Inequality**[45] See: **Kantorović's Inequality** COMMENTS (I), INTEGRAL ANALOGUES.

**s-Convex Function Inequalities** *(a) If $0 < s \le 1$ and $f$ is an s-convex function of type one, or of the first kind, defined on $[0,\infty[$ then for all $x, y \ge 0$, and $0 \le t \le 1$,*

$$f\left((1-t)^{1/s}x + t^{1/s}y\right) \le (1-t)f(x) + tf(y).$$

*(b) If $0 < s \le 1$ and $f$ is an s-convex function of type two, or of the second kind, defined on $[0,\infty[$ then for all $x, y \ge 0$, and $0 \le t \le 1$,*

$$f\left((1-t)x + ty\right) \le (1-t)^s f(x) + t^s f(y).$$

COMMENTS (i) These are just the definition of s-convexity of types one and two, respectively. Both reduce to the ordinary convexity of $s = 1$. See: **Convex Function Inequalities** (1).

(ii) If $0 < s < 1$ an s-convex function of type one is increasing on $]0,\infty[$, while an s-convex function of type two if non-negative.

(iii) If $f(0) = 0$ and $f$ is an s-convex function of type two then it is also an s-convex function of type one, but not conversely.

REFERENCE *Hudzik & Maligranda* [143].

**Segre's Inequalities** If $f(\underline{x}) = f(x_1, \ldots, x_n)$ is unchanged under all permutations of $x_1, \ldots, x_n$ then $f$ is called a *symmetric function.*

If $f(\underline{x}) = f(\underline{x}; \underline{a}^1, \ldots, \underline{a}^m) = f(x_1, \ldots, x_n, a_1^1, \ldots, a_n^1, \ldots, a_1^m, \ldots, a_n^m)$ is unchanged under any simultaneous permutation of $x_1, \ldots, x_n$, and $a_1^1, \ldots, a_n^1$ and $a_1^m, \ldots, a_n^m$ then $f$ is called an *almost symmetric function.* Here $f$ is regarded as a function of $\underline{x}$ and the $\underline{a}^1, \ldots \underline{a}^n$ are parameters.

For example $\mathfrak{A}_n(\underline{a})$ is symmetric, $\mathfrak{A}_n(\underline{a}, \underline{w})$ is almost symmetric.

If for all real $\lambda$, $f(\lambda\underline{x}) = \lambda^\alpha f(\underline{x})$ then $f$ is called a *homogeneous function (of degree $\alpha$).*

[45] This is P. Schweitzer.

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*Hypotheses: $f(\underline{x})$ is almost symmetric on $\{\underline{x}; a < \underline{x} < b\}$ and satisfies the following conditions:*

$$\begin{aligned} f(x_1, x_1, \ldots, x_1) &= 0, \quad a < x_1 < b, &(1)\\ f_1'(\underline{x}) &\leq \rho_1 f(\underline{x}), \quad a < x_1 \leq x_j < b, \quad 2 \leq j \leq n, &(2)\\ f_j'(x_j, x_2, \ldots, x_n) &= \rho_j f_1'(x_j, x_2, \ldots, x_n), \quad 2 \leq j \leq n, &(3) \end{aligned}$$

*where all the $\rho$ are non-negative, and there is equality in (2), if at all, only when $x_1 = \cdots = x_n$.*
*Conclusions:*

$$f(\underline{x}) \geq 0$$

*with equality if and only if $x_1 = \cdots = x_n$.*

Comments (i) If $a = 0, b = \infty$ and if $f$ is also homogeneous then we need only assume that (1), (2), and (3) hold for $x_1 = 1$.
(ii) Many classical inequalities follow from this result.

Reference [H, pp. 425-427].

**Seitz's Inequality** *If $n \geq 2$ and if $x_i, y_i, z_j, u_j, a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, are real numbers such that $\forall i, j,\ 0 \leq i < j \leq m, 0 \leq r < s \leq n$,*

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \begin{vmatrix} z_r & z_s \\ u_r & u_s \end{vmatrix} \geq 0, \qquad \begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \geq 0$$

*then*

$$\frac{\sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i z_j}{\sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i u_j} \geq \frac{\sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i z_j}{\sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i u_j}$$

Comments (i) If $m = n$ and if $y_i = u_i = 1, 1 \leq i \leq n,\ a_{ij} = 0, i \neq j, a_{ii} = w_i, 1 \leq i, j \leq n$ then (1) reduces to (Č).
(ii) If $m = n$ and if $x_i = z_i, y_i = u_i, 1 \leq i \leq n,\ a_{ij} = 0, a_{ii} = 1, i \neq j, 1 \leq i, j \leq n$ then (1) reduces to (C).
(iii) A Rado-type extension of this result has been given by Wang & Luo.
(iv) In his original paper Seitz gives an integral analogue for this inequality.

References [MPF, pp. 252–253]; *Seitz* [295], *Toader* [311], *Wang, L. C. & Luo* [323].

**Semi-continuous Function Inequalities** *If $f : [a, b] \to \overline{\mathbb{R}}$ is lower semi-continuous at $x$ then*

$$\liminf_{y \to x} f(y) \geq f(x). \tag{1}$$

Comments This is sometimes taken as the definition of lower semi-continuity. As the opposite inequality is trivial the definition can also be taken as equality in (1).

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Reference *Bourbaki* [B60, p. 170].

**Sequentially Convex Function Inequalities** *If $f : \mathbb{R} \mapsto \mathbb{R}$ has a convex non-negative second derivative, $\underline{a}$ is a real $n$-tuple, $1, \leq k \leq n$, and*

$$f_{(k)}(\underline{a}) = \frac{1}{k!\binom{n}{k}} \sum_{k}! f\Big(\frac{a_{i_1} + \cdots + a_{i_k}}{k}\Big),$$

*then*

$$(a) \qquad \Delta^2 f_{(k)} \geq 0, \quad 1 \leq k \leq n-2, \tag{1}$$

*that is $f_{(k)}$, $1 \leq k \leq n-2$, is convex;*

$$(b) \qquad f_{(k)} \geq f_{(k+1)} \quad 1 \leq k \leq n-1. \tag{2}$$

Comments (i) Inequality (1) is the definition of a *sequentially convex function*, due to Gabler. Further the converse holds; that is if (1) holds then $f$ has a convex non-negative second derivative.

(ii) The definition can be made for functions defined on arbitrary intervals but then the converse need not hold.

(iii) Noting that $f_{(1)}(\underline{a}) = \mathfrak{A}_n\big(f(\underline{a})\big)$ and $f_{(n)}(\underline{a}) = f\big(\mathfrak{A}_n(\underline{a})\big)$ we see that (2) provides a series of inequalities that interpolate (J), $f_{(1)}(\underline{a}) \geq f_{(n)}(\underline{a})$.

Examples (i) The functions $e^x$, $x^{2n}$, $n = 1, 2, \ldots$ are all examples of functions defined on $\mathbb{R}$ that are sequentially convex.

(ii) The functions $-\log x$, and $x^\alpha$, if $\frac{\log 3}{\log 3 - \log 2} \leq \alpha < 3$, are functions defined on $]0, \infty[$ that are sequentially convex. The last example shows the validity of the last remark in Comment (ii), as the second derivative is concave.

References *Gabler* [122], *Pečarić* [264].

**Series' Inequalities**

*(a)*
$$\frac{m}{n+m} < \sum_{i=1}^{m} \frac{1}{n+i} < \frac{m}{n}.$$

*(b) If $0 \leq a < 1$ and $k \in \mathbb{N}, k > (3+a)/(1-a)$ then* $\displaystyle\sum_{i=0}^{nk-(n+1)} \frac{1}{n+i} > 1 + a.$

*(c)*

$$\begin{aligned}
\frac{1}{2(2n+1)} &< \left|\sum_{i=0}^{n-1} \frac{(-1)^i}{2i+1} - \frac{\pi}{4}\right| = \sum_{i=n}^{\infty} \frac{(-1)^{n-i}}{2i+1} < \frac{1}{2(2n-1)};\\
\frac{1}{2(n+1)} &< \left|\sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} - \log 2\right| = \sum_{i=n+1}^{\infty} \frac{(-1)^{n+1-i}}{i} < \frac{1}{2(n-1)}.
\end{aligned}$$

*(d) If $n \geq 1$ then*
$$\left(1 + \frac{1}{n}\right)^n < \sum_{k=0}^{n} \frac{1}{k!} < \left(1 + \frac{1}{n}\right)^{n+1}.$$

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*(e)* [Trimble, Wells & F. T. Wright] *If* $p \geq 0, x, y > 0$ *then*

$$\frac{1}{\sum_{k\in\mathbb{N}}(x+k)^{-p}} + \frac{1}{\sum_{k\in\mathbb{N}}(y+k)^{-p}} \leq \frac{1}{\sum_{k\in\mathbb{N}}(x+y+k)^{-p}}. \tag{1}$$

*(f)* [Heywood] *If* $a_1, a_2, \ldots$ *is a real sequence then:*

$$\left(\sum_{n=1}^{m} a_n\right)^2 \leq 3\pi \left(\sum_{n=1}^{m} n^{2/3}a_n^2\right)^{1/2} \left(\sum_{n=1}^{m} n^{4/3}a_n^2\right)^{1/2}. \tag{2}$$

*(g)* [Kupán & Szász] *If* $0 \leq x \leq 1$ *then:*

$$\sum_{n=p}^{\infty} \frac{1}{(n+x)^2} < \frac{1}{p-1}, \qquad \sum_{n=p}^{\infty} \frac{1}{(n+x)^3} < \frac{1}{2p(p-1)}.$$

Comments (i) All the terms in inequality (d) have $e$ as limit when $n \to \infty$.

(ii) Inequality (1) follows from the log-convexity of the function $\big((r+s-1)!/r!s!\big)x^r(1-x)^s$ for $0 < x < 1, r, s > -1$.

(iii) The constant $3\pi$ in (2) is conjectured to be best possible. The inequality with this constant replaced by $\sum_{n=1}^{m} 1/n$ is an easy consequence of (H).

(iv) See also: **Binomial Function Inequalities** (b), **Davies & Petersen Inequality, Enveloping Series Inequalities, Exponential Function Inequalities** Inequalities Involving the Remainder of the Taylor Series, **Pachpatte's Series Inequalities, Sums of Integer Powers Inequalities, Trigonometric Function Inequalities** (o),(p).

References [AI, pp. 187–188]; *Kazarinoff* [K, pp. 40–42, 46–47]; *Heywood* [141], *Kupán & Szász* [168]. *Trimble, Wells & F. T. Wright* [314]

## Shafer-Fink Inequality

*If* $0 \leq x \leq 1$*, then*

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1}$$

*Furthermore,* 3 *and* $\pi$ *are the best constants in (1).*

Comments (i) The left-hand inequality was first proved by Shafer and later Fink gave the right-hand inequality.

(ii) Much work and many generalizations of this result have been given, as well as extensions to the inverse hyperbolic sine function and the generalized trigonometric functions See: **Oppenheim's Problem** A Particular Case, **Shafer-Zhu Inequality**.

References [AI, p. 247]; *Yang C. Y.* [330], *Zhu* [336, 337].

## Shafer-Zhu Inequality

*If* $x \geq 0$*, then*

$$\frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} \leq \text{arctanx} \leq \frac{8x}{3+\sqrt{25+\frac{256}{\pi^2}x^2}}. \tag{1}$$

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*Furthermore,* $80/3$ *and* $256/\pi^2$ *are the best constants in (1).*

COMMENT The left-hand inequality was first proved by Shafer and later Zhu gave the right-hand inequality.

REFERENCES *Shafer* [296, 297], *Zhu* [338].

**Shampine's Inequality** *If* $p$ *is a polynomial of degree* $n$ *with real coefficients, and* $p(x) \geq 0,\ x \geq 0,$ *then*

$$-\left[\frac{n}{2}\right]\int_0^\infty p(x)e^{-x}\,\mathrm{d}x \leq \int_0^\infty p'(x)e^{-x}\,\mathrm{d}x \leq \int_0^\infty p(x)e^{-x}\,\mathrm{d}x.$$

REFERENCE *Skalsky* [300].

**Shannon's Inequality** *(a) If* $\underline{p}, \underline{q}$ *are positive* $n$*-tuples with* $P_n = Q_n = 1$ *then*

$$\sum_{i=1}^n p_i \log p_i \geq \sum_{i=1}^n p_i \log q_i, \tag{1}$$

*with equality if and only if* $p_i = q_i, 1 \leq i \leq n.$
*(b) If* $\underline{p} \prec \underline{q}$ *then*

$$\sum_{i=1}^n p_i \log p_i \leq \sum_{i=1}^n q_i \log q_i.$$

COMMENTS (i) The quantity $H(\underline{p}) = -\sum_{i=1}^n p_i \log_2 p_i$ is called the *entropy of* $\underline{p}$. See: **Entropy Inequalities**.

(ii) The logarithmic function in (1) is essentially the only function for which an inequality of this type holds.

(iii) Inequality (1) can be deduced from **Log-convex Function Inequalities** (2) by taking $f(x) = x^x$; or directly using (J) and the convexity of $x \log x$.

(iv) There are other entropy functions with their own inequalities. See: **Rényi's Inequality**, [T].

REFERENCES [AI, pp. 382–383], [EM, vol. 4, pp. 387–388; Supp., pp. 338–339], [H, p. 278], [MOA, p. 101], [MPF, pp. 635–650]; *Tong, ed.* [T, pp. 68–77].

**Shapiro's Inequality** *If the positive* $n$*-tuple* $\underline{a}$ *is extended to a sequence by defining* $a_{n+r} = a_r, r \in \mathbb{N},$ *then*

$$\sum_{i=1}^n \frac{a_i}{a_{i+1} + a_{i+2}} \geq n\lambda(n);$$

*where,*

$$\lambda(n) = \begin{cases} = \frac{1}{2}, & \text{if } n \leq 12 \text{ or } n = 13, 15, 17, 19, 21, 23; \\ \\ < \frac{1}{2}, & \text{for all other values of } n. \end{cases}$$

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COMMENTS (i) It is known that the limit, $\lim_{n\to\infty}\lambda(n)$ exists, $\lambda$ say. Its exact value $0.494566\ldots$, is given in (1) below.

(ii) This inequality was proposed as a problem by Shapiro, although special cases had been considered earlier. Shapiro conjectured that $\lambda(n) \geq 1/2$. As we see above Shapiro's conjecture is false; it is true if for some $k$, $a_{k-1}, \ldots, a_{k+n}$ is monotonic.

(iii) The case $n = 3$ is **Nesbitt's Inequality**.

(iv) An expository article on this inequality has been given by Mitrinović.

EXTENSIONS *(a)* [DRINFELD] *With the above notation*

$$\sum_{i=1}^{n} \frac{a_i}{a_{i+1} + a_{i+2}} \geq n\frac{0.989133\ldots}{2}. \tag{1}$$

*(b)* [DIANANDA] *If the positive n-tuple $\underline{a}$ is extended to a sequence by defining $a_{n+r} = a_r, r \in \mathbb{N}$ then*

$$\sum_{i=1}^{n} \frac{a_i}{a_{i+1} + \cdots + a_{i+m}} \geq \frac{n}{m},$$

*if one of the following holds:*
*(1).* $\sin(r\pi)/n \geq \sin(2m+1)r\pi/n,\ 1 \leq r \leq \left[\frac{n}{2}\right]$*;*
*(2).* $n|(m+2)$*;*
*(3).* $n|2m+k, k = 0, 1,$ *or* $2$*;*
*(4).* $n = 8, 9,$ *or* $12$ *and* $n|m+3$*;*
*(5).* $n = 12$ *and* $n|m+4$*.*

COMMENTS (v) The constant in the numerator on the right-hand side of (1) is $g(0)$ where $g$ is the lower convex hull of the functions $e^{-x}$ and $2(e^x + e^{x/2})^{-1}$. The proof depends on an ingenious use of the **Rearrangement Inequalities** (1).

(vi) These are examples of **Cyclic Inequalities**.

(vii) For another inequality of the same name see **Leindler's Inequality** COMMENT.

REFERENCES [AI, pp. 132–137], [GI6, pp. 17–31], [MPF, pp. 440–447]; *Mitrinović* [214].

**Shapiro-Kamaly Inequality** *Let $\underline{a} = \{a_n, n \in \mathbb{Z}\}$ be a non-negative sequence such that the following series converge:*

$$\sum_{n\in\mathbb{Z}} a_n, \quad \sum_{n\in\mathbb{Z}} a_n^2; \quad \sum_{n\in\mathbb{Z}}\Big(\sum_{i+j=n} a_i a_j\Big)^2; \quad \sum_{n\in\mathbb{Z}}\Big(\sum_{i-j=n} a_i a_j\Big)^2.$$

*Then if* $0 \leq \delta \leq 1$ *or* $\delta = 2$*,*

$$\sum_{n\in\mathbb{Z}} |n|^\delta \Big(\sum_{i-j=n} a_i a_j\Big) \leq \sum_{n\in\mathbb{Z}} |n|^\delta \Big(\sum_{i+j=n} a_i a_j\Big),$$

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*and*

$$\sum_{n\in\mathbb{Z}} |n|^\delta \big( \sum_{i-j=n} a_i a_j \big)^2 \le \sum_{n\in\mathbb{Z}} |n|^\delta \big( \sum_{i+j=n} a_i a_j \big)^2.$$

REFERENCE [MPF, pp. 552–553].

**Siegel's Inequality** See: **Geometric-Arithmetic Mean Inequality** EXTENSIONS (c).

**Sierpinski's Inequalities** *If $\underline{a}$ is a positive $n$-tuple then*

$$\mathfrak{A}_n(\underline{a}) \ge \mathfrak{G}_n(\underline{a}) \left( \frac{\mathfrak{G}_n(\underline{a})}{\mathfrak{H}_n(\underline{a})} \right)^{1/(n-1)}; \qquad \mathfrak{G}_n(\underline{a}) \ge \mathfrak{H}_n(\underline{a}) \left( \frac{\mathfrak{A}_n(\underline{a})}{\mathfrak{G}_n(\underline{a})} \right)^{1/(n-1)}. \quad (1)$$

COMMENTS (i) The first inequality in (1) extends (GA) and the second, implied by the first, extends (HG).

(ii) On rewriting the left-hand inequality all we have to prove is that

$$\mathfrak{A}_n(\underline{a}) \ge \left( \frac{\mathfrak{G}_n^n(\underline{a})}{\mathfrak{H}_n(\underline{a})} \right)^{1/(n-1)},$$

which is just $S(1; n-1)$.

EXTENSIONS *(a)* [POPOVICIU-TYPE] *If $\underline{a}, \underline{w}$ are positive $n$-tuples, $n \ge 2$, then*

$$\frac{\mathfrak{A}_n^{W_{n-1}}(\underline{a};\underline{w})\mathfrak{H}_n^{w_n}(\underline{a};\underline{w})}{\mathfrak{G}_n^{W_n}(\underline{a};\underline{w})} \ge \frac{\mathfrak{A}_{n-1}^{W_{n-2}}(\underline{a};\underline{w})\mathfrak{H}_{n-1}^{w_{n-1}}(\underline{a};\underline{w})}{G_{n-1}^{W_{n-1}}(\underline{a};\underline{w})},$$

*with equality if and only if $\underline{a}$ is constant.*

*(b)* [ADDITIVE ANALOGUES ] *(i) If $\underline{a}$ is a positive $n$-tuple*

$$n\left(\mathfrak{G}_n(\underline{a}) - \mathfrak{A}_n(\underline{a})\right) \le \mathfrak{H}_n(\underline{a}) - \mathfrak{A}_n(\underline{a}),$$

*with equality if and only if $\underline{a}$ is constant.*

*(ii) If $\underline{a}$ is a positive $n$-tuple with $\mathfrak{A}_n(\underline{a}) > 1$ then*

$$\mathfrak{G}_n^n(\underline{a}) - \mathfrak{A}_n^n(\underline{a}) \le \mathfrak{H}_n(\underline{a}) - \mathfrak{A}_n(\underline{a}),$$

*with equality if and only if $\underline{a}$ is constant.*

COMMENT (iii) Extension (a) has been further extended by Pečaríc.

REFERENCES [H, p. 151], [MPF, pp. 21–25]; *Pečarić* [261].

**Simpson's Inequality** See: **Quadrature Inequalities** (c).

**Sine Integral Inequalities** The *sine integral* is the function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin u}{u}\,\mathrm{d}u.$$

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*If* $0 < x < \pi$ *then*

$$\begin{aligned} 0 &< \sum_{k=1}^{n} \frac{\sin kx}{k} \le \mathrm{Si}(\pi) = 1.8519\ldots < \frac{\pi}{2} + 1, &\quad (1)\\ 1 &< \mathrm{Si}(\pi/2) < \pi/2. &\quad (2)\end{aligned}$$

COMMENTS (i) The left-hand side of (1) is the **Fejér-Jackson inequality**.

(ii) It is well- known that $\mathrm{Si}(\infty) = \pi/2$.

(iii) Various refinements of (a) and (b) can be found in the references.

REFERENCES [MPF, p. 613]; *Qi, Cui & Xu S. L.* [278].

**Skarda's Inequalities** *If* $\underline{a}, \underline{b}$ *are two real absolutely convergent sequences then*

$$\begin{aligned} \underline{a}.\underline{b} &\ge \frac{1}{2}\big(|\underline{a}+\underline{b}| - |\underline{a}| - |\underline{b}|\big)\big(\max\{|\underline{a}|, |\underline{b}|\}\big); &\quad (1)\\ \underline{a}.\underline{b} &\ge \frac{1}{4}\big(|\underline{b}| - |\underline{a}| - |\underline{a}-\underline{b}|\big)\big(|\underline{b}| - |\underline{a}| + |\underline{a}-\underline{b}|\big); &\quad (2)\\ \underline{a}.\underline{b} &\le \frac{1}{4}\big(|\underline{a}+\underline{b}| + |\underline{a}| - |\underline{b}|\big)\big(|\underline{a}+\underline{b}| - |\underline{a}| + |\underline{b}|\big); &\quad (3)\\ \underline{a}.\underline{b} &\le \frac{1}{2}\big(|\underline{b}| + |\underline{a}| - |\underline{a}-\underline{b}|\big)\big(\max\{|\underline{a}|, |\underline{b}|\}\big). &\quad (4)\end{aligned}$$

*There is equality in (1) or (4) if either* $a_n b_n = 0$ *for all* $n$*, or if* $\underline{b}$ *has one non-zero term of the opposite sign to the corresponding entry in* $\underline{a}$ *and* $|\underline{b}| \ge |\underline{a}|$*. There is equality in the (2) or (3), if either of* $\underline{a}, \underline{b}$ *is the null sequence, or if* $\underline{a}, \underline{b}$ *have only one non-zero term that is the same entry and is of the same sign.*

REFERENCE [GI2, pp. 137–142].

**Slater's Inequality** *If* $f$ *is convex and increasing on* $[a, b]$*,* $\underline{a}$ *and* $n$*-tuple with terms in* $[a, b]$*, and if* $\underline{w}$ *is a non-negative* $n$*-tuple with* $W_n > 0$*, then*

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i f(a_i) \le f\left(\frac{\sum_{i=1}^{n} w_i a_i f'_+(a_i)}{\sum_{i=1}^{n} w_i f'_+(a_i)}\right).$$

REFERENCE [PPT, pp. 63-67].

**Sobolev's Inequalities** A *Sobolev inequality* is one that estimates lower order derivatives in terms of higher order derivatives, as distinct from the **Hardy-Littlewood-Landau Derivative Inequalities, Hardy-Littlewood-Pólya Inequalities** and **Kolmogorov Inequalities** that estimate intermediate order derivatives in terms of a higher order and a lower order derivative. Another important difference is the spaces in which the derivatives are considered, and the fact that in general these derivatives are generalized derivatives defined using distributions or something equivalent.

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The *Sobolev Space* $\mathcal{W}^{\kappa,p}(\Omega)$, $1 \leq p \leq \infty, \kappa \in \mathbb{N}$, is the class of functions defined on $\Omega$, a domain in $\mathbb{R}^n$, with finite norm

$$||f||_{p,\kappa,\Omega} = ||f||_{p,\Omega} + \sum_{1 \leq |\underline{k}| \leq \kappa} ||D^{\underline{k}} f||_{p,\Omega};$$

where

$$D^{\underline{k}} f = \frac{\partial^{|\underline{k}|} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}; \quad \underline{k} = (k_1, \ldots, k_n), \quad |\underline{k}| = k_1 + \cdots + k_n.$$

The derivatives, of order $\kappa = |\underline{k}|$, are, as mentioned above, usually interpreted in the generalized or distribution sense; in case $\kappa = 1$ we write $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

*(a) If $f : \mathbb{R}^n \to \mathbb{R}$, $1 \leq m \leq n$, and if $0 \leq k = \ell - n/p + m/q$ then*

$$||f||_{q,[k],\mathbb{R}^m} \leq C||f||_{p,\ell,\mathbb{R}^n},$$

*where $C$ is independent of $f$.*
*(b)* [Gagliardo-Nirenberg-Sobolev] *If $f \in \mathcal{W}^{1,p}(\mathbb{R}^n)$ then*

$$||f||_{p^*} \leq C|| \, |\nabla f| \, ||_p,$$

*where $C$ is independent of $f$.*

Comments (i) $p^*$, the *Sobolev conjugate index*, is given by $1/p^* = 1/p - 1/n$.

(ii) (b) is essentially the case $k = 0, \ell = 1, m = n$ of (a) and is related to Hardy's inequality. See: **Hardy's Inequality** (5), Comments (viii). In the case that $f$ is a $\mathcal{C}^1$ function with compact support it is equivalent to **Isoperimetric Inequalities** (1) for compact domains with $\mathcal{C}^1$ boundaries.

(iii) In some particularly important cases exact values of the constants in these inequalities have been given; see [LL].

(iv) See also: **Logarithmic Sobolev Inequality, Morrey's Inequality, Poincaré's Inequalities, Trudinger's Inequality**. In addition there are many other Sobolev inequalities on the references.

(v) Another inequality by Sobolev is the **Hardy-Littlewood-Sobolev Inequality**.

References [EM, vol. 5, pp. 16–21, vol.8, pp. 379–381], [GI4, pp. 401–408]; *Adams & Fournier* [AF, pp. 95–113], *Bobkov & Houdré* [BH, p. 1], *Evans & Gariepy* [EG, pp. 138–140,189–190], *Lieb & Loss* [LL, pp. 183–199], *Mitrović & Žubrinić* [MZ, pp. 160–213], *Opic & Kufner* [OK, pp. 1–3, 240–314]; *Gardner* [126], *Osserman* [239].

**Spherical Rearrangement Inequalities** *(a) If $f : \mathbb{R}^n \to \mathbb{R}$ and $p \geq 1$ then*

$$||\nabla f^{(*)}||_{p,\mathbb{R}^n} \leq ||\nabla f||_{p,\mathbb{R}^n}.$$

*(b) If $f : \Omega \to \mathbb{R}$, $\Omega$ a domain in $\mathbb{R}^n$, then*

$$\int_{\Omega^*} \sqrt{1 + |\nabla u^{(*)}|^2} \leq \int_{\Omega} \sqrt{1 + |\nabla u|^2},$$

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*where $\Omega^*$ is the n-sphere centered at the origin having the same measure as $\Omega$.*

EXTENSIONS [BROTHERS & ZIEMER] *If $f : [0, \infty[\to [0, \infty[, f(0) = 0$ be in $\mathcal{C}^2$ with $f^{1/p}$ convex, $p \geq 1$, and if $u \in \mathcal{W}^{1,p}(\mathbb{R}^n)$ is non-negative then*

$$\int_{\mathbb{R}^n} f\left(|\nabla u^{(*)}|\right) \leq \int_{\mathbb{R}^n} f\left(|\nabla u|\right).$$

COMMENTS (i) The case of equality has been given by Brothers & Ziemer.

(ii) See also **Hardy-Littlewood Maximal Inequalities, Hardy-Littlewood-Sobolev Inequality, Rearrangement Inequalities, Riesz Rearrangement Inequality**.

REFERENCES *Kawohl* [Ka, pp. 94–95], *Lieb & Loss* [LL, pp. 174–176], *Pólya & Szegö* [PS51, p. 194].

**Starshaped Function Inequalities** *(a) If $f : [0, a] \to \mathbb{R}$ is star-shaped then*

$$f(\lambda x) \leq \lambda f(x), \quad 0 \leq \lambda \leq 1, \qquad 0 \leq x \leq a.$$

*(b) If $\underline{a}$ is a positive n-tuple, and if $f : ]0, \infty[\to \mathbb{R}$ is star-shaped then*

$$\sum_{i=1}^{n} f(a_i) \leq f\left(\sum_{i=1}^{n} a_i\right). \tag{1}$$

*If $f$ is strictly star-shaped and $n > 1$ then (1) is strict.*

COMMENTS (i) (a) is just the definition of a function being *star-shaped (with respect to origin)*; an equivalent formulation is that $f(x)/x$, the slope of the chord from the origin, is increasing. If the slope increases strictly, then the function is said to be *strictly star-shaped*.

(ii) It follows from **Convex Function Inequalities** (5), and COMMENTS (vi), that if $f$ is convex on $[0, a]$ then $f(x) - f(0)$ is star-shaped.

(iii) Since (1) is satisfied by any negative function $f$ such that $\inf f \geq 2 \sup f$, the converse of (b) does not hold.

(iv) (b) is a simple deduction from **Increasing Function Inequalities** (2); just apply that result to $f(x)/x$, taking $w_i = a_i, 1 \leq i \leq n$.

(v) The last inequality in the case $\underline{w} = \underline{e}$ just says that star-shaped functions are super-additive; see **Subadditive Function Inequalities** COMMENTS (ii).

EXTENSION *If $f$ is star-shaped on $[0, a]$ and if $\underline{a}, \underline{v}$ are non-negative n-tuples satisfying*

$$a_j \in [0, a], \quad \sum_{i=1}^{n} v_i a_i \geq a_j, \quad 1 \leq j \leq n; \quad \textit{and} \quad \sum_{i=1}^{n} v_i a_i \in [0, a];$$

*then*

$$\sum_{i=1}^{n} v_i f(a_i) \leq f\left(\sum_{i=1}^{n} v_i a_i\right).$$

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COMMENT (vi) This is an easy deduction from **Increasing Function Inequalities** EXTENSION; apply the result to $f(x)/x$, taking $w_i = v_i a_i, 1 \le i \le n$.

REFERENCES [MOA, pp. 650–651], [PPT, pp. 8, 152–154]; *Trimble, Wells & F. T. Wright* [314].

**Statistical Inequalities** *(a)* [KLAMKIN-MALLOWS] *If $\underline{a}$ is a positive increasing $n$-tuple, and $\overline{\underline{a}} = \mathfrak{A}_n(\underline{a})$, then*

$$\min \underline{a} \le \overline{\underline{a}}\left(1 - \frac{CV}{\sqrt{n-1}}\right) \le \overline{\underline{a}}\left(1 + \frac{CV}{\sqrt{n-1}}\right) \le \max \underline{a},$$

*with equality on the right-hand side if and only if $a_2 = \cdots = a_n$, and on the left-hand side if and only if $a_1 = \cdots = a_{n-1}$.*
*(b)* [REDHEFFER & OSTROWSKI] *If $1 \le \nu \le n-1$, $q = \nu/n$, $n, \nu \in \mathbb{N}$, and $0 < x < 1$ then*

$$\binom{n}{\nu} x^\nu (1-x)^{(n-\nu)} \le \left(1 - \frac{1}{n}\right)^{n-1} e^{-2n(x-q)^2},$$

*with equality if and only if $x = q$ and $\nu = 1$ or $n-1$;*
*and*

$$\binom{n}{\nu} x^\nu (1-x)^{(n-\nu)} \le \sqrt{\frac{1}{2\pi n q(1-q)}} e^{-2n(x-q)^2},$$

*where, for every $q$, $2\pi$ is sharp as $n \to \infty$.*
*(c) If $\underline{a}$ is a real $n$-tuple and if $\mathfrak{A}_n(\underline{a}) = 0$, $\mathfrak{A}_n(\underline{a}^2) = 1$ then:*
*(i)* [WILKINS, CHAKRABARTI]

$$\mathfrak{A}_n(\underline{a}^3) \le \frac{n-2}{\sqrt{n-1}}, \qquad \mathfrak{A}_n(\underline{a}^4) \le n - 2 + \frac{1}{n-1},$$

*with equality if and only if $a_1 = \sqrt{n-1}$, and $a_i = -\frac{1}{\sqrt{n-1}}, 2 \le i \le n$;*
*(ii)* [PEARSON]

$$\mathfrak{A}_n(\underline{a}^4) \le 1 + \mathfrak{A}_n(\underline{a}^3);$$

*(iii)* [LAKSHMANAMURTI]

$$\mathfrak{A}_n^2(\underline{a}^m) + \mathfrak{A}_n^2(\underline{a}^{m-1}) \le \mathfrak{A}_n^2(\underline{a}^{2m}).$$

COMMENT See also: **Bernšteĭn's Probability Inequality, Berry-Esseen Inequality, Bonferroni's Inequalities, Čebišev's Probability Inequality, Copula Inequalities, Correlation Inequalities, Entropy Inequalities, Error Function Inequalities, Gauss-Winkler Inequality, Laguerre-Samuelson Inequality, Lévy's Inequalities, Martingale Inequalities, Median-Mean Inequality, Polynomial Inequalities** (d), **Probability Inequalities, Shannon's Inequality, Totally Positive Function Inequalities, Variance Inequalities, Yao & Iyer Inequality**.

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References [GI1, pp. 125–139, 307], [H, p. 202]; *Klamkin* [155], *Mallows & Richter* [196].

**Steffensen's Inequalities** *(a) If $f, g \in \mathcal{L}([a,b])$, with $f$ decreasing and $0 \le g \le 1$, and if $\lambda = \int_a^b g$ then*

$$\int_{b-\lambda}^{b} f \le \int_a^b fg \le \int_a^{a+\lambda} f. \tag{1}$$

*(b) If $\underline{a}, \underline{b}$ are real $n$-tuples such that*

$$\sum_{i=1}^{k} b_i \le \sum_{i=1}^{k} a_i,\ 1 \le k < n; \quad \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i; \tag{2}$$

*then if $\underline{w}$ is an increasing $n$-tuple,*

$$\sum_{i=1}^{n} w_i a_i \le \sum_{i=1}^{n} w_i b_i. \tag{3}$$

*Conversely if (3) holds for such $n$-tuples $\underline{w}$ then (2) holds.*

Comments (i) Inequality (1) is called *Steffensen's Inequality*. If instead we assume $0 \le g \le A$ and apply the result to $g* = g/A$ we get *Hayashi's Inequality*.

(ii) The right-hand inequality is the basis of Steffensen's proof of **Jensen-Steffensen Inequality**, to which it is equivalent.

(iii) For another deduction from (1) see **Fourier Transform Inequalities** (a), Comments (i).

(iv) The result in (b), is related to **Abel's Inequality** Related Results (b), and follows by two applications of **Abel's Inequality** (1). Condition (2) is similar to the definition of the order relation $\prec$, to which it reduces if the $n$-tuples are decreasing. See **Notations 5** (2).

Extension *(a)* [Godunova & Levin] *If $f \in \mathcal{L}^p[a,b], 0 < p \le 1, g \in \mathcal{L}[a,b], g \ge 0$, and if $\lambda = \left(\int_a^b g\right)^p$, then*

$$\int_a^b fg \le \left(\int_a^{a+\lambda} f^p\right)^{1/p},$$

*holds for all decreasing $f$ if and only if $\left(\int_a^x g\right)^p \le x - a, a \le x \le b$.*

*(b)* [Bergh] *Let $f$ and $g$ be positive functions on $\mathbb{R}^+$, $f$ decreasing and $g$ measurable. Assume that, for some $p > 1, f \in \mathcal{L}^p + \mathcal{L}^\infty$ and $g \in \mathcal{L}^q \cap \mathcal{L}, q$ the conjugate index, with $||f||_p = 1$ and $||g||_q = t$, then:*

$$\int_0^\infty fg \le 2^{1/q} \left(\int_0^{t^p} f^p\right)^{1/p}$$

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*where $2^{1/q}$ cannot be replaced by a smaller constant.*

COMMENTS (v) The result in (a) is a corrected version of an earlier version that contained an error although it was reproduced in both in [AI] and [BB].

(vi) Steffensen's inequality can be regarded as the limit as $p \to 1$ of the inequality in (b).

DISCRETE ANALOGUES *(a) Let $\underline{a}$ be a decreasing $n$-tuple and $\underline{b}$ an $n$-tuple with $0 \le \underline{b} \le 1$. If $1 \le n_2 \le B_n \le n_1 \le n$ then*

$$\sum_{k=n-n_2+1}^{n} a_k \le \sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{n_1} a_k.$$

*(b)* [PEČARIĆ] *Let $p \ge 1$, and $\underline{a}, \underline{b}$ a real $n$-tuples, $\underline{a}$ non-negative and decreasing. If $k, 1 \le k \le n$, is such that such that $k^{1/p} \ge B_n$, and $b_i B_n \le 1, 1 \le i \le k$, then*

$$\sum_{i=1}^{n} a_i b_i \le \Bigl(\sum_{i=1}^{k} b_i^p\Bigr)^{1/p}.$$

INTEGRAL ANALOGUE *If $f, g$ are integrable functions on $[a, b]$ then*

$$\int_a^b fh \le \int_a^b gh$$

*holds for every increasing function $h$ if and only if*

$$\int_a^x g \le \int_a^x f,\ a \le x < b; \quad \int_a^b g = \int_a^b f.$$

COMMENTS (iv) See also: **Gauss's Inequality, Moment Inequalities**.

REFERENCES [AI, pp. 107–119], [BB, p. 49], [MPF, pp. 311–337], [PPT, pp. 181–195]; *Niculescu & Persson* [NP, pp. 190–192]; *Bergh* [58], *Gauchman* [128, 129].

## Stirling's Formula

*If $n \in \mathbb{N}$*

$$\sqrt{2n\pi}\, n^n e^{-n+\frac{1}{12n+1/4}} < n! < \sqrt{2n\pi}\, n^n e^{-n+\frac{1}{12n}}.$$

COMMENT This inequality implies that:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2n\pi}\, n^n e^{-n}} = 1,$$

which is often written:

$$n! \sim \sqrt{2n\pi}\, n^n e^{-n}.$$

This last expression is *Stirling's Formula*.

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Extensions *(a)*

$$\lim_{\Re z\to\infty}\frac{z!}{\sqrt{2z\pi}\,z^n e^{-z}} = 1, \quad \textit{as} \quad \Re z \to \infty.$$

*This is often written*

$$z! \sim \sqrt{2z\pi}\,z^n e^{-z}, \quad \textit{as} \quad \Re z \to \infty.$$

*(b) If* $x \ge x_0 \ge 0$

$$\sqrt{2x\pi}x^x \exp\Big(-x + \frac{1}{12x + \frac{3}{5}(x_0+\frac{1}{2})^{-1}}\Big) < x! < \sqrt{2x\pi}x^x \exp\Big(-x + \frac{1}{12x}\Big).$$

References [AI, pp. 181–185], [EM, vol. 8, p. 540]; *Allasia, Giordano & Pečarić* [3].

**Stirling Number Inequalities** The *Stirling number of the second kind*, written $\mathfrak{S}_n^{(m)}$, $n = 1, 2, \ldots, 1 \le m \le n$, is the number of ways of partitioning a set of $n$ elements into $m$ subsets;

$$\mathfrak{S}_n^{(m)} = \frac{1}{m!}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}i^n; \quad \mathfrak{S}_{n+1}^{(m)} = m\mathfrak{S}_n^{(m)} + \mathfrak{S}_n^{(-1)}.$$

*(a) If* $n \ge 4, \frac{1}{2}(n+1) \le m \le n-1$ *then*

$$(n-m)\mathfrak{S}_n^{(m)} > (m+1)\mathfrak{S}_n^{(m+1)}.$$

*(b) If* $n \ge 2$ *then* $\mathfrak{S}_{2n}^{n)} > \mathfrak{S}_{2n}^{(n+1)}$.

*(c) There is a unique integer* $m_n$, $\sqrt{n+1} \le m_n \le (n+1)/2$, *and*

$$\mathfrak{S}_n^{(1)} < \cdots < \mathfrak{S}_n^{(m_n)} \ge \mathfrak{S}_n^{(m_n+1)} > \cdots > \mathfrak{S}_n^{(n)}.$$

*(d)* $\mathfrak{S}_n^{(m)}$ *is strongly log-concave in both* $m$ *and* $n$*; that is*

$$(\mathfrak{S}_n^{(m)})^2 > \mathfrak{S}_{n-1}^{(m)}\mathfrak{S}_{n+1}^{(m)}; \quad (\mathfrak{S}_n^{(m)})^2 > \mathfrak{S}_n^{(m-1)}\mathfrak{S}_n^{(m+1)}.$$

References *Abramowitz & Stegun* [AS, pp. 824–825]; *Neuman* [230].

**Stolarsky's Inequality** *If* $f : [0,1] \to [0,1]$ *is decreasing and if* $p, q > 0$ *then*

$$\int_0^1 f(x^{1/(p+q)})\,\mathrm{d}x \ge \int_0^1 f(x^{1/p})\,\mathrm{d}x \int_0^1 f(x^{1/q})\,\mathrm{d}x.$$

Comment This has been generalized in the references.

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REFERENCES *Maligranda, Pečarić & Persson* [195], *Pečarić & Varošanec* [267].

**Stolarsky Mean Inequalities** If $r, s \in \mathbb{R}$ and $a, b > 0$, $a \neq b$, then the *extended mean of order $r, s$ of $a$ and $b$*, is given by:

$$\mathfrak{E}^{r,s}(a,b) = \begin{cases} \left(\dfrac{r(b^s - a^s)}{s(b^r - a^r)}\right)^{1/(s-r)}, & \text{if } r \neq s,\ s \neq 0, \\ \left(\dfrac{b^r - a^r}{r(\log b - \log a)}\right)^{1/r}, & \text{if } r \neq 0, s = 0, \\ \dfrac{1}{e^{1/r}}\left(\dfrac{b^{b^r}}{a^{a^r}}\right)^{1/(b^r - a^r)}, & \text{if } r = s \neq 0, \\ \sqrt{ab}, & \text{if } r = s = 0; \end{cases}$$

the definition is completed by defining $\mathfrak{E}^{[p]}(a,a) = a, a > 0$, $p \in \mathbb{R}$. These are also called *the extended means of Leach & Sholander*.

The special cases in the above definition follow from the general case by taking appropriate limits.

*(a)* [LEACH & SHOLANDER] *If $r \neq s, p \neq q, r + s \leq p + q$ then*

$$\mathfrak{E}^{r,s}(a,b) \leq \mathfrak{E}^{p,q}(a,b).$$

*(b)* [ALZER] *If $0 < a < b, r \neq 0$ then*

$$\begin{aligned} \mathfrak{G}_2(a,b) &< \sqrt{\mathfrak{E}^{-r,-r+1}(a,b)\mathfrak{E}^{r,r+1}(a,b)} \\ &< \mathfrak{L}(a,b) < \frac{\mathfrak{E}^{-r,-r+1}(a,b) + \mathfrak{E}^{r,r+1}(a,b)}{2} < \mathfrak{A}_2(a,b). \end{aligned}$$

*(c)* [ALZER] *If $s_1 > s_2 \geq \alpha > 0$ then*

$$\mathfrak{E}^{2\alpha - s_1, s_1}(a,b) \leq \mathfrak{E}^{2\alpha - s_2, s_2}(a,b).$$

*(d)* [ALZER] *If $0 < r_1 \leq \beta \leq s_1, 0 < r_2 \leq \beta \leq s_2$, with $\mathfrak{L}(r_1, s_1) = \mathfrak{L}(r_2, s_2) = \beta$, and if $s_1 \leq s_2$ then*

$$\mathfrak{E}^{2\alpha - s_1, s_1}(a,b) \leq \mathfrak{E}^{2\alpha - s_2, s_2}(a,b).$$

*(e)* [ALZER] *If $0 < a, b \leq 1/2, a \neq b$, and if either $-1 \leq -s < r \leq 2, (r,s) \neq (2,1)$ or $1 \leq r < s \leq 2, (r,s) \neq (1,2)$ then*

$$\frac{\mathfrak{G}_2(a,b)}{\mathfrak{G}_2(1-a,1-b)} \leq \frac{\mathfrak{E}^{r,s}(a,b)}{\mathfrak{E}^{r,s}(1-a,1-b)} \leq \frac{\mathfrak{A}_2(a,b)}{\mathfrak{A}_2(1-a,1-b)}.$$

COMMENT It can be shown that

$$\mathfrak{E}^{r,s}(a,b) = \mathfrak{M}^{[s-r]}_{[0,1]}\big(\mathfrak{M}^{[r]}_2(a,b;t,1-t)\big),$$

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which has led to further generalizations with the power mean being replaced by quasi-arithmetic means, see **Quasi-arithmetic Mean Inequalities**.

Reference [H, pp. 385–393], [MOA, pp. 141–143],[MPF, pp. 45–46]; *Alzer* [10], *Pearce, Pečarić & Šimić* [258],.

**Strongly Convex Function Inequalities** *If $f$ is a strongly convex function defined on the compact interval $[a,b]$ then for some $\alpha > 0$, all $x, y \in [a,b]$ and $0 \le \lambda \le 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \le (1-\lambda)f(x) + \lambda f(y) - \lambda(1-\lambda)\alpha(x-y)^2.$$

Comments (i) This is just the definition of *strong convexity*.

(ii) If $f$ has a second derivative then $f$ is strongly convex if and only if $f'' \ge 2\alpha$.

Reference *Roberts & Varberg* [RV, p. 268].

**Subadditive Function Inequalities** *(a) If $f : \mathbb{R}^n \to \mathbb{R}$ is subadditive then for all $\underline{x}, \underline{y} \in \mathbb{R}^n$.*

$$f(\underline{x} + \underline{y}) \le f(\underline{x}) + f(\underline{y}). \tag{1}$$

*(b)* [Fan] *If $X$ is a lattice and $f : X \to \mathbb{R}$ then for all $x, y \in X$,*

$$f(x \wedge y) + f(x \vee y) \le f(x) + f(y).$$

Comments (i) These are just the definitions of *subadditivity* in the two situations; to avoid confusion the second property is often called *L-subadditivity*; if $-f$ is subadditive then we say that $f$ is *superadditive*. For another usage of this term see: **Measure Inequalities** Comment. For strong sub-additivity see: **Capacity Inequalities** Comments (ii).

(ii) In (a) the domain can be generalized, and if it is a convex cone and if $f$ is also homogeneous then $f$ is convex; for a definition see: **Segre's Inequalities**. A star-shaped function defined on $[0, \infty[$ is superadditive. See: **Star-shaped Function Inequalities**.

References [GI1, pp. 159–160], [MOA, pp. 217–219, 453], [PPT, pp. 175–176].

**Subharmonic Function Inequalities** *If $\Omega$ is a domain in $\mathbb{R}^n$, if $u$ is subharmonic on $\Omega$, if $B_{\underline{x}_0,r} = \{\underline{x}; |\underline{x} - \underline{x}_0| < r\}$ and $\overline{B}_{\underline{x}_0,r} \subset \Omega$ then*

$$u(\underline{x}_0) \le \frac{1}{a_n r^{n-1}} \int_{\partial B_{\underline{x}_0,r}} u; \tag{1}$$

$$u(\underline{x}_0) \le \frac{1}{v_n r^n} \int_{B_{\underline{x}_0,r}} u. \tag{2}$$

Comments (i) (1) says that the value of $u$ at the center of any ball is not greater than the mean value of $u$ on the surface of the ball; while (2) says the same with the mean taken over the ball itself.

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(ii) (1) is just the definition of a *subharmonic function*; such a function being an upper semi-continuous function, with values in $\overline{\mathbb{R}}\setminus\{\infty\}$, not identically equal to $-\infty$, that satisifies (1).

(iii) This class of functions is a natural extension of the idea of **Convex Functions**. The concept can be extended to complex valued functions, and to functions defined on more general spaces.

(iv) If both $u$ and $-u$ are subharmonic then $u$ is harmonic.

(v) A function $u$ on $\Omega$ with continuous second partial derivatives will be subharmonic if and only if $\nabla^2 u \geq 0$.

(vi) It is an immediate consequnce of the integral form of (J) that if $h$ is harmonic and $f$ is convex then $f \circ h$ is subharmonic; in particular if $p \geq 1$ then $|h|^p$ is subharmonic. Similarly if $u$ is subharmonic and $f$ is convex and increasing then $f \circ u$ is subharmonic.

(vii) See also: **Maximum-Modulus Principle**.

References [EM, vol. 9, pp. 59–61]; *Ahlfors* [Ah78, pp. 237–239], *Conway* [C, vol. II, pp. 220–223], *Helms* [Hel, pp. 57–59], *Lieb & Loss* [LL, pp. 202–209], [*Protter & Weinberger* [PW, pp. 54–55], *Rudin* [R87, pp. 328–330].

**Sub-homogeneous Function Inequalities** *If $f : [0, \infty[\mapsto \mathbb{R}$ is sub-homogeneous of degree $t, t > 0$ then*

$$f(\lambda x) \leq \lambda^t f(x), \quad \forall x \geq 0, \lambda > 1;$$

*(b) if in addition $f$ is differentiable then*

$$xf'(x) \leq tf(x), \quad \forall x \geq 0.$$

Comment The inequality in (a) is just the definition of this class of functions.

Reference *Bokharaie, Mason & Wirth* [64].

**Subordination Inequalities** If $g$ is analytic in $D$ then any function $f = g \circ \omega$, for some $\omega$ analytic in $D$, $\omega(0) = 0$ and $|\omega(z)| \leq |z|, z \in D$, is said to be *subordinate to $g$ in $D$*. Clearly $f$ is also analytic in $D$ and $f(0) = g(0)$. We also say that $f$ *is subordinate to $g$ in* $\{z; |z| < R\}$ if $f(Rz)$ is subordinate to $g(Rz)$ in $D$.

*(a) If $f$ is subordinate to $g$ in $D$ then for $0 < r < 1$,*

$$\begin{aligned} M_p(f;r) \leq &\quad M_p(g;r), \quad 0 \leq p \leq \infty; \\ \int_0^{2\pi} \log^+ f(re^{i\theta})\,\mathrm{d}\theta \leq &\quad \int_0^{2\pi} \log^+ g(re^{i\theta})\,\mathrm{d}\theta; \\ M_p(\Re f;r) \leq &\quad M_p(\Re g;r), \quad 1 \leq p \leq \infty. \end{aligned}$$

*(b) If $f, g$ are as in (a) with $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$, $g(z) = \sum_{n\in\mathbb{N}} b_n z^n$, $|z| < 1$ then*

$$a_0 = b_0; \quad |a_1| \leq |b_1|; \quad |a_2| \leq \max\{|b_1|, |b_2|\}; \quad \sum_{n\in\mathbb{N}} |a_n|^2 \leq \sum_{n\in\mathbb{N}} |b_n|^2;$$

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*there is equality in the first inequality if and only if* $f(z) = g(e^{i\theta}z)$, *for some* $\theta \in \mathbb{R}$.

REFERENCES [EM, vol. 9, pp. 64–65], [GI3, pp. 340–348]; *Conway* [C, vol. II, pp. 133–136, 272].

**Sums of Integer Powers Inequalities** *(a) If* $p > 0$ *then*

$$\frac{n^{p+1}-1}{p+1} < \sum_{i=1}^{n} i^p < \frac{n^{p+1}-1}{p+1} + n^p.$$

*(b) If* $p < 0, p \neq -1$ *then*

$$\frac{n^{p+1}-1}{p+1} + n^p < \sum_{i=1}^{n} i^p < \frac{n^{p+1}-1}{p+1} + 1.$$

*(c)*

$$\frac{1}{4n} < \sum_{i=n+1}^{2n} \frac{1}{i^2} < \frac{1}{n}.$$

*(d)*

$$\frac{1}{n} - \frac{1}{2n+1} < \sum_{i=1}^{2n} \frac{1}{i^2} < \frac{1}{n-1} - \frac{1}{2n},\ \textit{if}\ n > 1;$$

$$\sum_{i=n}^{\infty} \frac{1}{i^2} < \frac{2}{2n-1},\ n \geq 1.$$

*(e)*

$$2\left(\sqrt{n+1}-1\right) < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} < 2\sqrt{n} - 1.$$

COMMENTS (i) (a), (b) are easy deductions from the **Integral Test Inequality**.

(ii) See also: **Bennett's Inequalities** (1), (2), **Euler's Constant Inequalities, Logarithmic Function Inequalities** (j), **Mathieu's Inequality, Minc-Sathre Inequality, Schlömilch-Lemonnier Inequality, Zeta Function Inequalities**.

REFERENCES [AI, pp. 190–191], [PPT, p. 39]; *Cloud & Drachman* [CD, p. 11]; *Milisavljević* [210].

**Sundman's Inequalities** The Sundman inequalities are inequalities between the angular momentum, $C$, the moment of inertia, $I$ and the kinetic energy, $K$, and the total energy $h$, of a system of $N$ bodies. The bodies have masses $m_i$, $1 \leq i \leq N$, position at time $t$ given by $(\underline{q}_1, \ldots, \underline{q}_N)$, and are subject to a homogeneous potential

$$U(\underline{q}_1, \ldots, \underline{q}_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{|\underline{q}_i - \underline{q}_j|^\alpha}, \qquad \alpha > 0, \alpha \neq 2.$$

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*Under the above conditins:*

$$(a) \qquad C^2 \le 4IK; \quad (b) \quad 2(\dot{\sqrt{I}})^2 \le \sum_{i=1}^{N} m_i \frac{\underline{q}_i \cdot \dot{\underline{q}}_i}{|\underline{q}_i|^2};$$

$$(c) \qquad \frac{2}{2-\alpha} I + \frac{2\alpha}{2-\alpha}(-h) \ - \ 2(\dot{\sqrt{I}})^2 \ge \frac{C^2}{2I}.$$

COMMENT We have the following relations:

$$h = K - U; \quad C = \sum_{i=1}^{N} m_i \underline{q}_i \times \dot{\underline{q}}_i; \quad K = \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\underline{q}}_i|^2; \quad I = \frac{1}{2}\sum_{i=1}^{N} m_i |\underline{q}_i|^2.$$

REFERENCE *Zhang S.* [335].

**Sup and Inf Inequalities** See: **Inf and Sup Inequalities**.

**Symmetric Elliptic Integral Inequalities** *If $a, b, c$ are positive and not all equal, and if*

$$R = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{(t+a^2)(t+b^2)(t+c^2)}}\,\mathrm{d}t$$

*then*

$$\frac{3}{a+b+c} < R < \frac{2}{\sqrt[3]{(a+b)(b+c)(c+a)}}$$

COMMENTS (i) $R$ is called the *symmetric elliptic integral* and $1/R$ is the value of the electric capacity of an ellipsoid with axes $a, b, c$. If $a = b = c = 1$ then $R = 1$.

(ii) This inequality is due to Greiman and improves a result of Pólya & Szegő.

REFERENCE *Carlson* [81].

**Superadditive Function Inequalities** See: **Subadditive Function Inequalities**.

**Symmetric Function Inequalities** See: **Elementary Symmetric Function Inequalities, Segre's Inequalities.**

**Symmetric Mean Inequalities** *(a) If $1 \le r \le n$ then*

$$\min \underline{a} \le \mathfrak{P}_n^{[r]}(\underline{a}) \le \max \underline{a},$$

*with equality if and only if $\underline{a}$ is constant.*
*(b) If $1 \le r < s \le n$ then*

$$\mathfrak{P}_n^{[s]}(\underline{a}) \le \mathfrak{P}_n^{[r]}(\underline{a}), \qquad S(r;s)$$

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*with equality if and only if $\underline{a}$ is constant.*
*(c)* [Ku, Ku & Zhang X. M.] *If $1 \le r < s \le n$ and $0 < m \le \underline{a} \le M$ then*

$$\mathfrak{A}_n(\underline{a})\left(\frac{\mathfrak{P}_n^{[r]}(\underline{a})^r}{\mathfrak{P}_n^{[s]}(\underline{a})^s}\right)^{1/(s-r)} \le \frac{(M+m)^2}{4Mm},$$

*with equality if and only if $\underline{a}$ is constant.*

Comments (i) The case $s = r+1$ of S(r;s) follows from **Elementary Symmetric Function Inequalities** (2) with $r = t$ and multiplying over $t$, $1 \le t \le r$

(ii) The basic inequality S(r;s) is yet another generalization of (GA) for if $1 \le r \le n$,

$$\mathfrak{P}_n^{[n]}(\underline{a}) = \mathfrak{G}_n(\underline{a}) \le \mathfrak{P}_n^{[r]}(\underline{a}) \le \mathfrak{P}_n^{[1]}(\underline{a}) = \mathfrak{A}_n(\underline{a}).$$

(iii) If $n = 3$ then S(r;s) has a simple geometric interpretation. See: **Geometric Inequalities** (b).

(iv) The inequality in (c) can be considered as a generalization of **Kantorovič's Inequality**.

(v) As was pointed out in **Mixed Means Inequalities**, symmetric means are special cases of mixed means; so consider the matrix $\mathbb{S}$ where

$$\mathbb{S} = \begin{pmatrix} \mathfrak{M}_n(0,1;1;\underline{a}) & \mathfrak{M}_n(0,1;2;\underline{a}) & \dots & \mathfrak{M}_n(0,1;n-1;\underline{a}) & \mathfrak{M}_n(0,1;n;\underline{a}) \\ \mathfrak{M}_n(0,1;n;\underline{a}) & \mathfrak{M}_n(0,1;n-1;\underline{a}) & \dots & \mathfrak{M}_n(0,1;2;\underline{a}) & \mathfrak{M}_n(0,1;1;\underline{a}) \\ \mathfrak{M}_n(n,0;n;\underline{a}) & \mathfrak{M}_n(n-1,0;n-1;\underline{a}) & \dots & \mathfrak{M}_n(2,0;2;\underline{a}) & \mathfrak{M}_n(1,0;1;\underline{a}) \end{pmatrix}.$$

Then by S(r;s) and **Mixed Mean Inequalities** (1) the rows of $\mathbb{S}$ increase strictly to the right, and the entries in the second row are strictly less than those in the first row, and strictly greater than those in the last row, except for the first column all of whose entries are $\mathfrak{G}_n(\underline{a})$, and for the last column all of whose entries are $\mathfrak{A}_n(\underline{a})$. A similar matrix is discussed in **Mixed Mean Inequalities** Comments (ii).

(vi) No general relations are known between the first and last rows of $\mathbb{S}$ except for the following special case: $\mathfrak{M}_3(2,0;2;\underline{a}) \le \mathfrak{M}_3(0,1;2;\underline{a})$ or

$$\sqrt{\frac{ab+bc+ca}{3}} \le \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}. \tag{1}$$

It is proved by using (GA) and S(r;s). This should be compared with **Mixed Means Inequalities** (3).

Extensions [Popoviciu Type] *Let $\underline{a}$ be a positive $(n+m)$-tuple, $r, k$ integers with $1 \le r < k \le n+m$, and put $u = \max\{r-n, 0\}$, $v = \min\{r, m\}$, $w = \max\{k-n, 0\}$, $x = \min\{k, m\}$.*
*(a) If $v \ge w$ and $r - u \le k - x$ then*

$$\left(\frac{\mathfrak{P}_{n+m}^{[r]}(\underline{a})}{\mathfrak{P}_{n+m}^{[k]}(\underline{a})}\right)^k \ge \left(\frac{\mathfrak{P}_n^{[r-u]}(\underline{a})}{\mathfrak{P}_n^{[k-x]}(\underline{a})}\right)^{k-x}\left(\frac{\overline{\mathfrak{P}}_m^{[v]}(\underline{a})}{\overline{\mathfrak{P}}_m^{[w]}(\underline{a})}\right)^w.$$

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*(b) If $v \le w$*

$$\left(\frac{\mathfrak{P}^{[r]}_{n+m}(\underline{a})}{\mathfrak{P}^{[k]}_{n+m}(\underline{a})}\right)^k \le \left(\frac{\overline{\mathfrak{P}}^{[v]}_m(\underline{a})}{\overline{\mathfrak{P}}^{w]}_m(\underline{a})}\right)^w.$$

*Here we use the notation $\overline{\mathfrak{P}}_m(\underline{a}) = \mathfrak{P}_m(a_{n+1}, \ldots, a_{n+m})$.*

Comment (vii) See also: **Complete Symmetric Mean Inequalities** Comments (ii), **Fan's Inequality** Extensions (b), **Hamy Mean Inequalities, Marcus & Lopes Inequality** (b), **Muirhead Symmetric Function, Mean Inequalities**.

References [AI, pp. 95–107], [BB, p. 11], [H, pp. 321–338], [MOA, pp. 114–118], [MPF, pp. 15–16] ; *Ku, Ku & Zhang, X. M.* [163].

**Symmetrical Rearrangement Inequalities** See: **Spherical Rearrangement Inequalities**.

**Symmetrization Inequalities** [Pólya & Szegö; Steiner] *(a) Symmetrization of a domain in $\mathbb{R}^2$ with respect to a line leaves the area unchanged but decreases the perimeter, polar moment of inertia about the center of gravity, capacity, and principal frequency.*
*(b) Symmetrization of a domain in $\mathbb{R}^3$ with respect to a plane leaves the volume unchanged but decreases the surface area and capacity.*

Comments (i) Symmetrization of a domain in $\mathbb{R}^2$ with respect to a line is called *Schwarz symmetrization*, while symmetrization of a domain in $\mathbb{R}^3$ with respect to a plane is called *Steiner symmetrization*. These symmetrizations can be defined in $\mathbb{R}^n, n \ge 1$.

(ii) The importance of symmetrization lies in the fact that the process reduces certain functionals. So by obtaining a sequence of symmetrizations that converge to a ball, on which the minimum occurs and on which the functional can be calculated, inequalities can be proved.

(iii) See also: **Capacity Inequalities, Frequency Inequalities, Isodiametric Inequality, Isoperimetric Inequalities, Moment of Inertia Inequalities**.

References *Evans & Gariepy* [EG, pp. 67–70], *Lieb & Loss* [LL, pp. 79–80], *Pólya & Szegö* [PS51, pp. 6–7, 153–154].

**Szegő's Inequality** *If $\underline{a}$ is a decreasing non-negative n-tuple and if f convex on $[0, a_1]$ with $f(0) \le 0$ then*

$$\sum_{i=1}^n (-1)^{i-1} f(a_i) \ge f\left(\sum_{i=1}^n (-1)^{i-1} a_i\right).$$

Comments (i) The condition $f(0) \le 0$ can be omitted if $n$ is odd, the case given originally by Szegő; the above result is due to Bellman. The case $n$ odd holds under the weaker assumption of Wright convexity. See: **Convex Function Inequalities** Comments (viii).

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(ii) This result is a consequence of **Order Inequalities** (b), and is related to **Steffensen's Inequalities** (1).
(iii) The case of $f(x) = x^r, r > 1$, is *Weinberger's inequality.*

EXTENSIONS *(a)* [BRUNK] *If $f$ is convex on $[0, b]$ with $b \geq a_1 \geq \cdots \geq a_n \geq 0$, and if $1 \geq h_1 \geq \cdots \geq h_n \geq 0$ then*

$$\sum_{i=1}^{n} (-1)^{i-1} h_i f(a_i) \geq f\left(\sum_{i=1}^{n} (-1)^{i-1} h_i a_i\right).$$

*(b)* [OLKIN] *If $\underline{w}$ is a decreasing $n$-tuple with $0 \leq w_i \leq 1, 1 \leq i \leq n$, and $\underline{a}$ is a decreasing non-negative $n$-tuple, and if $f$ is a convex function on $[0, a_1]$, then*

$$\left(1 - \sum_{i=1}^{n} (-1)^{i-1} w_i\right) f(0) + \sum_{i=1}^{n} (-1)^{i-1} w_i f(a_i) \geq f\left(\sum_{i=1}^{n} (-1)^{i-1} w_i a_i\right).$$

COMMENT (iv) For a related inequality, see: **Petrović's Inequality**.

REFERENCES [AI, pp. 112–114], [BB, pp. 47–49], [MOA, p. 140], [MPF, p. 359], [PPT, pp. 156–159].

## Szegő-Kaluza Inequality

See: **Kaluza-Szegö Inequality**.

## Székely, Clark & Entringer Inequality

*(a) If $\underline{a}$ is an increasing real sequence, $p \geq 1, n \geq 2$ then*

$$\sum_{i=1}^{n} \tilde{\Delta} a_{i-1} a_{n+1-i}^{p} \leq \left(\sum_{i=1}^{n} \tilde{\Delta} a_{i-1} a_{n+1-i}^{1/p}\right)^{p},$$

*where $a_0 = 0$.*
*(b)* [INTEGRAL ANALOGUE] *If $f \in \mathcal{L}([0,1]), f \geq 0$ and $p \geq 1$, then*

$$\int_0^1 f(x)\left(\int_0^{1-x} f\right)^{p} \mathrm{d}x \leq \left(\int_0^1 f(x)\left(\int_0^{1-x} f\right)^{1/p} \mathrm{d}x\right)^{p}.$$

*In both inequalities the constant is best possible.*

COMMENTS (i) These are Alzer's extensions of the original results.
(ii) The result (a) is implied by **Alzer's Inequalities** (c).

REFERENCES *Alzer* [24].

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# 20 Talenti–Turán

**Talenti's Inequality** *If $a > 0$ and $f$ is positive and decreasing on $[a, b]$ then*

$$\log\left(1 + \frac{1}{1 + af(a)}\int_a^b f\right) \le \int_a^b \frac{f(t)}{1 + tf(t)}\,\mathrm{d}t.$$

COMMENT This result has been extended and provided with an inverse by Lemmert & Alzer.

REFERENCE [GI6, pp. 441–443].

**Tchakaloff's Inequality** See: **Čakalov's Inequality**.

**Tchebysheff's Inequality** See: **Čebišev's Inequality**.

**Three Chords Lemma** See: **Convex Function Inequalities** (f).

**Three Circles Theorem** See: **Hadamard's Three Circles Theorem**.

**Three Lines Theorem** See: **Phragmén-Lindelöf Inequality**.

**Thunsdorff's Inequality** *If $f$ is non-negative and concave on $[a, b]$ and if*
*$0 < r < s$, then*

$$\left(\frac{1+s}{b-a}\int_a^b f^s \le\right)^{1/s} \le \left(\frac{1+r}{b-a}\int_a^b f^r\right)^{1/r}.$$

COMMENTS (i) This result is due to Thunsdorff but his proof was not published; a proof was given by Berwald. The discrete case below was proved much later.

(ii) Extensions to $n$-convex functions, and to weighted means have been given; see: [MPF].

(iii) This result is an inverse to (r;s), and a particular case is **Favard's Inequalities** (b). The case $r = 1, s = 2$ is the *Frank-Pick Inequality*.

DISCRETE ANALOGUE *If $\underline{a}$ is both increasing and concave and if $0 < r < s$ then*

$$\mathfrak{M}_n^{[s]}(\underline{a};\underline{w}) \le C\mathfrak{M}_n^{[r]}(\underline{a};\underline{w}), \qquad (1)$$



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*where if* $\underline{b} = \{0, 1, \ldots, n-1\}$ *then*

$$C = \frac{\mathfrak{M}_n^{[s]}(\underline{b}; \underline{w})}{\mathfrak{M}_n^{[r]}(\underline{b}; \underline{w})}. \tag{2}$$

*If* $\underline{a}$ *is decreasing then (1) holds with* $\underline{b}$ *in* $C$ *replaced by* $\underline{c} = \{n-1, \ldots, 1, 0\}$. *If* $\underline{a}$ *is convex and* $0 < s < r$ *then (~1) holds with* $C$ *given by (2).*

COMMENT (iv) The method of proof is based on **Power Mean Inequalities** EXTENSIONS (j).

(v) The result has extensions to $n$-convex sequences.

REFERENCES [AI, p. 307], [MPF, pp. 50–56].

**Ting's Inequalities** *If* $f \in \mathcal{C}([0, a])$, *is non-negative and convex, and if* $\alpha > 2$ *then*

$$\frac{a(\alpha-1)}{\alpha+1} \le \frac{\int_0^a x^{\alpha-1} f(x)\,\mathrm{d}x}{\int_0^a x^{\alpha-2} f(x)\,\mathrm{d}x} \le \frac{a\alpha}{\alpha+1},$$

*with equality on the left-hand side if* $f(x) = a - x$, *and on the right-hand side if* $f(x) = x$.

COMMENT Ting gave the case $\alpha$ an integer; the general result is due to Ross, who also gave a simpler proof and other extensions of Ting's result.

REFERENCE [GI4, pp. 119–130].

**Titchmarsh's Theorem** See: **Fourier Transform Inequalities** TITCHMARSH'S THEOREM.

**Totally Positive Function Inequalities** *If* $: \mathbb{R}^2 \to \mathbb{R}$ *is a totally positive of order n function then for all* $m, 1 \le m \le n$, *and* $x_1 < \cdots < x_m, y_1 < \cdots < y_m$,

$$\det\big(f(x_i, y_j)\big)_{1 \le i,j \le m} \ge 0.$$

COMMENT This is just the definition of this class of functions. A related class to the case $n = 2$ can be found in **Copula Inequalities**.

REFERENCES [EM, Supp., pp. 469–470]; *Tong, ed.*[T, p. 54]; *Karlin* [KaS, pp. 11–12].

**Trace Inequalities** [BELLMAN] *If* $A, B$ *are Hermitian matrices then:*

$$(a) \qquad \operatorname{tr}(AB) \le \frac{\operatorname{tr}(A^2) + \operatorname{tr}(B^2)}{2},$$

*with equality if and only if* $A = B$;

$$(b) \qquad \operatorname{tr}(AB) \le \sqrt{\operatorname{tr}(A^2)\operatorname{tr}(B^2)},$$

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*with equality if and only if* $A = kB$, *for some constant* $k$;

$$(c) \qquad \operatorname{tr}(AB) \le \sqrt{\operatorname{tr}(A^2B^2)},$$

*with equality if and only if* $A, B$ *commute.*
*(d)* [Fan] *If* $A, B$ *are* $n \times n$ *real symmetric matrices with eigenvalues, in decreasing order,* $\lambda(A) = \{\lambda_1(A), \dots \lambda_n(A)\}, \lambda(B) = \{\lambda_1(B), \dots \lambda_n(B)\}$ *then,*

$$\operatorname{tr}(AB) \le \lambda(A)^T \lambda(B). \tag{1}$$

*There is equality in (1) if and only if there is an orthogonal matrix* $U$ *such that*

$$A = U^T \operatorname{Diag}\big(\lambda(A)\big)U, \text{ and } B = U^T \operatorname{Diag}\big(\lambda(B)\big)U. \tag{2}$$

Comments (i) The *trace of a square matrix* is the sum of its diagonal elements. If $X$ is a row or column matrix then $\operatorname{Diag}(X)$ is the diagonal matrix with entries along the diagonal those of $X$.

(ii) The condition (2) is called *simultaneous ordered spectral decomposition*; it implies that the matrices commute.

(iii) Inequality (1) can be regarded as a refinement of (C) in the space of real symmetric matrices.

(iv) If in (d) the matrices are diagonal we get as a consequence the right-hand inequality in **Rearrangement Inequalities** (1).

References [GI2, pp. 89–90], [MPF, pp. 224-225]; *Borwein & Lewis* [BL, pp. 10–13].

## Trapezoidal Inequality

See: **Quadrature Inequalities** (a).

## Triangle Inequality

*(a) If* $\underline{a}, \underline{b}$ *are real* $n$*-tuples then*

$$|\underline{a} + \underline{b}| \le |\underline{a}| + |\underline{b}|, \tag{1}$$

*with equality if and only if* $\underline{a} \sim^+ \underline{b}$.
*(b) If* $\underline{a}, \underline{b}$ *are real* $n$*-tuples then*

$$|\underline{a} - \underline{b}| \ge \big||\underline{a}| + |\underline{b}|\big|,$$

*with equality if and only if* $\underline{a} \sim^+ \underline{b}$.
*(c) If* $\underline{a}, \underline{b}, \underline{c}$ *are real* $n$*-tuples then*

$$|\underline{a} - \underline{c}| \le |\underline{a} - \underline{b}| + |\underline{b} - \underline{c}|, \tag{T}$$

*with equality only if* $\underline{b}$ *is between* $\underline{a}$ *and* $\underline{c}$.
*(d)* [Sudbery] *If the triangle, in* $\mathbb{R}^2$, *with vertices* $\underline{a}, \underline{b}, \underline{c}$ *contains the origin,* $\underline{0}$, *then*

$$\begin{aligned} |\underline{a}| + |\underline{b}| &\le |\underline{a} - \underline{c}| + |\underline{b} - \underline{c}|, \\ |\underline{a}| + |\underline{b}| + |\underline{c}| &\le |\underline{a} - \underline{x}| + |\underline{b} - \underline{x}| + |\underline{c} - \underline{x}| + |\underline{x}|, \quad \forall x \in \mathbb{R}^2. \end{aligned}$$

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COMMENTS (i) We say that $\underline{b}$ *is between* $\underline{a}$ *and* $\underline{c}$ if for some $\lambda, 0 \le \lambda \le 1$, we have $\underline{b} = (1-\lambda)\underline{a} + \lambda\underline{c}$.

(ii) $\underline{a}, \underline{b}, \underline{c}$ contain the origin if for some $\lambda, \mu, \nu, \in \mathbb{R}^+$, not all zero, $\lambda\underline{a}+\mu\underline{b}+\nu\underline{c} = \underline{0}$.

(iii) (a) is the $p = 2$ case of (M), and a proof is given in any book on linear algebra.

(iv) (c) is the *Triangle Inequality*, and is a simple deduction from (1), which in turn follows form (T). As a result of this equivalence (1) is also referred to as (T).

(v) The name follows from the geometric interpretation of $|\underline{a} - \underline{b}|$ as the distance between $\underline{a}$ and $\underline{b}$.

EUCLID BOOK 1 PROPOSITION 20 THEOREM *Any two sides of a triangle are together greater than the third side.*

EXTENSIONS (a) If $\underline{a}_j, 1 \le j \le m$ are real $n$-tuples then

$$(i) \qquad \Big|\sum_{j=1}^m \underline{a}_j\Big| \le \sum_{j=1}^m |\underline{a}_j|,$$

with equality if and only if all are on the same ray from the origin;

$$(ii) \qquad |\underline{a}_m - \underline{a}_1| \le \sum_{j=2}^m |\underline{a}_j - \underline{a}_{j-1}|,$$

with equality if and only if $\underline{a}_2, \ldots, \underline{a}_{m-1}$ lie in order between $\underline{a}_1$ and $\underline{a}_m$.
(b) [RYSER] If $\underline{a}, \underline{b}$ are real $n$-tuples and if $\underline{m} = \min\{\underline{a}, \underline{b}\}$, $\underline{M} = \max\{\underline{a}, \underline{b}\}$ then

$$|\underline{a} + \underline{b}| \le |\underline{M}| + |\underline{m}| \le |\underline{a}| + |\underline{b}|.$$

COMMENT (vi) An integral analogue of (a)($i$) is **Integral Inequalities** (b).

INVERSE INEQUALITY [JANOUS] *If* $\underline{a}, \underline{b}$ *are real non-zero* $n$*-tuples then*

$$\underline{a}.\underline{b}\,\frac{|\underline{a}| + |\underline{b}|}{|\underline{a}|\,|\underline{b}|} \le |\underline{a} + \underline{b}|.$$

COMMENTS (vi) For another inverse inequality see **Wilf's Inequality**. In certain geometries the inverse inequality replaces the triangle inequality as the basic inequality.

(vii) The triangle inequality occurs in various guises and in many generalizations. See: **Absolute Value Inequalities** (1)–(3), **Complex Number Inequalities** (1), (2) (3), **Metric Inequalities** (1), **Norm Inequalities** (1), (2); **Quadrilateral Inequalities**.

(viii) For inequalities involving triangles see: **Abi-Khuzam's Inequality, Geometric Inequalities** (D),(E), **Padoa's Inequality**.

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References [AI, pp. 170–171], [EM, vol. 7, p. 363], [MPF, pp. 473–517]; *Sudbery* [305].

**Trigonometric Function Inequalities** *(a) If* $0 < |x| \leq \pi/2$ *then*

$$\cos x < \left(\frac{\sin x}{x}\right)^3 < \frac{\sin x}{x} < 1, \quad \sin x < x < \tan x. \tag{1}$$

*(b) If* $0 \leq x < y \leq \pi/2$ *then* $\displaystyle \frac{x}{y} \leq \frac{\sin x}{\sin y} \leq \left(\frac{\pi}{2}\right)\frac{x}{y}$.

*(c) If* $x \geq \sqrt{3}$ *then:* $\displaystyle (x+1)\cos\frac{\pi}{x+1} - x\cos\frac{\pi}{x} > 1.$

*(d)* [Kubo T.] *If* $0 \leq x \leq \pi$ *then*

$$\sin x(1+\cos x) \leq \left(\sin\frac{x+\pi}{4}\right)\left(1+\cos\frac{x+\pi}{4}\right).$$

*(e)* [Calderón] *If* $1 < p \leq 2$, $-\pi/2 \leq \theta \leq \pi/2$ *then for some positive* $\lambda$,

$$\left(\frac{1+\lambda\cos p\theta}{1+\lambda}\right)^{1/p} \leq \lambda\cos\theta.$$

*(f) If* $x \in \mathbb{R}, n \geq 1$ *then*

$$\left|\left(\frac{\sin x}{x}\right)^{(n)}\right| \leq \frac{1}{n+1}, \quad \text{and} \quad \left|\left(\frac{1-\cos x}{x}\right)^{(n)}\right| \leq \frac{1}{n+1},$$

*with equality in the first inequality only if* $n$ *is even and* $x = 0$, *and in the second only if* $n$ *is odd and* $x = 0$.

*(g) If* $0 < x < \pi/2$ *then:* $\displaystyle \log\sec x < \frac{1}{2}\sin x\tan x.$

*(h) If* $0 \leq x, y \leq 1$ *or* $1 \leq x, y < \pi/2$ *then:* $\tan x\tan y \leq \tan 1\tan xy.$

*(j) If* $0 < x < \pi/2$ *then:* $\displaystyle \frac{x}{\pi-2x} < \frac{\pi}{4}\tan x.$

*(k) If* $\leq x, y \leq 1$ *then:* $\displaystyle \arcsin x\arcsin y \leq \frac{1}{2}\arcsin xy.$

*(ℓ) If* $a > b$ *then for all* $t \in \mathbb{R}, t \neq 0$,

$$\frac{b-a}{\sqrt{(a^2+t)(b^2+t)}} \leq \frac{1}{t}\left(\arctan\frac{b}{t} - \arctan\frac{a}{t}\right).$$

*(m) Let us write*

$$S_{2n-1}(x) = \sum_{i=1}^{n}\frac{(-1)^{i-1}x^{2i-1}}{(2i-1)!}, \quad C_{2n}(x) = \sum_{i=0}^{n}\frac{(-1)^{i-1}x^{2i}}{(2i)!};$$

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*then for all* $x, n$,

$$(-1)^{n+1}\Big(\cos x - C_{2n}(x)\Big) \geq 0; \quad (-1)^{n+1}x\Big(\sin x - S_{2n+1}(x)\Big) \geq 0.$$

*(n) If* $0 < x < y < \sqrt{6}$ *then, with the above notation,*

$$\frac{S_{4n-1}(y)}{S_{4n-1}(x)} < \frac{\sin y}{\sin x} < \frac{S_{4n+1}(y)}{S_{4n+1}(x)}.$$

*(o) If* $n \geq 1, |x| < \pi/2$ *then*

$$\operatorname{cosec}^2 x - \frac{1}{2n+1} < \sum_{i=-n}^{n} \frac{1}{(x-i\pi)^2} < \operatorname{cosec}^2 x,$$

*while if* $0 < x \leq \pi/2$

$$\operatorname{cosec} x - \frac{x}{4n+1} < \sum_{i=-2n} 2n\frac{(-1)^i}{x-i\pi} < \operatorname{cosec} x + \frac{x}{4n+2}.$$

*(p) If* $0 < a_i < \pi, 1 \leq i \leq n$ *then:* $\quad \mathfrak{G}_n(\sin \underline{a}) \leq \sin \mathfrak{A}_n(\underline{a})$.

*(q) If* $z \in \mathbb{C}$ *and* $0 < |z| < 1$ *then:* $\quad |\cos z| < 2 \quad \text{and} \quad |\sin z| < \dfrac{6|z|}{5}$.

*(r)* [Seiffert] *If* $x \neq y$ *then*

$$\mathfrak{G}_2(x,y) < \frac{x-y}{2\arcsin\left(\dfrac{x-y}{x+y}\right)} < \mathfrak{A}_2(x,y) < \frac{x-y}{2\arctan\left(\dfrac{x-y}{x+y}\right)} < \mathfrak{M}_2^{[2]}(x,y).$$

*(s) If* $f \in \mathcal{C}^\infty(0,\pi/2),\ f(0) = 0, |f(\pi/2)| \leq 1$ *and* $\max_{n\geq 1} |f^{(n)}| \leq 1$, *then*

$$|f(x)| \leq \sin x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

*In particular if* $f \in \mathcal{C}^\infty(0,\pi/2),\ f(0) = 0$, *then*

$$\int_0^{\pi/2} |f| \leq \max_{n\geq 0} \max |f^{(n)}|,$$

*(t)* [Sándor] *(i) If* $-1 \leq x \leq 1$ *then:* $\quad \dfrac{x}{\arcsin x} \leq \frac{\sin x}{x}$;

*(ii) If* $-\pi/2 < x < \pi/2$ *then:* $\quad \dfrac{x}{\arctan x} \leq \frac{\tan x}{x}$.

*In both cases there is equality if and only if* $x = 0$.

*(u) If* $r \in \mathbb{R}$ *then*

$$|\cos x|^r + |\sin x|^r \begin{cases} > 1 & \text{if } r < 2, \\ = 1 & \text{if } r = 2, \\ < 1 & \text{if } r > 2. \end{cases}$$

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COMMENTS (i) A geometric proof of parts of (1) can be found in any elementary calculus book where it is needed to show that $\lim_{x\to 0}\sin x/x = 1$; a variant is given in **Jordan's Inequality**. The exponent 3 cannot be increased; this result is due to Adamović & Mitrinović.

(ii) Kubo's result, (d), is a particular case of **Function Inequalities** (b).

(iii) Calderón's inequality, (e), was used in his proof of M. Riesz's theorem; see **Conjugate Harmonic Function Inequalities** (a).

(iv) The inequalities in (f) follow from the identities

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^{(n)} &= \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y+\frac{(n+1)\pi}{2}\right)\,\mathrm{d}y,\\ \left(\frac{1-\cos x}{x}\right)^{(n)} &= \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y+\frac{n\pi}{2}\right)\,\mathrm{d}y.\end{aligned}$$

(v) (h) and (k) are simple deductions from (Č).

(vi) (m) is a special case of **Gerber's Inequality** EXTENSION.

(vii) Inequality (n) follows from ($\sim J$) and the concavity of sin on the interval $[0,\pi]$. Similar inequalities for the cosine and tangent functions can be obtained by using their strict concavity on the intervals $[-\pi/2,\pi/2]$, and $[0,\pi/2[$, respectively.

(viii) (o) follows from the strict concavity of sin on $[0,\pi]$.

(ix) (p) is a consequence of the log-concavity of $1/\sin x$; it should be compared with **Bessel Function Inequalities** COROLLARIES (a).

(x) (q) follows from **Integral Inequalities** (a) applied to the function complex function $(x+ti)^{-2}$.

(xi) Seiffert's inequalities, (r), have been generalized by Toader; see the reference.

(xii) The inequalities of Sándor, (t), are a consquence of **Function Inequalities** (c).

(xiii) The inequality in (u) can be compared to **Geometric Inequalities** (e) ($\beta$).

(xiv) See also: **Enveloping Series Inequalities** COMMENTS (II), **Function Inequalities** (A), **Harker-Kasper Inequality, Huygens's Inequalities, Jordan's Inequality, Lochs's Inequality, Oppenheim's Problem, Ostrowski's Inequalities** COMMENTS (III), **Shafer-Fink Inequality, Shafer-Zhu Inequality, Sine Integral Inequalities, Trigonometric Polynomial Inequalities**.

REFERENCES [AI, pp. 235–240, 243–249], [GI2, pp. 161–162], [GI5, p. 143], [H, p. 165], [MPF, pp. 476–477]; *Abramowicz & Stegun* [AS, p. 75]; *Niven* [NI, pp. 92–110], *Rudin* [R87, pp. 345–346]; *Kubo T.* [165], *Toader* [312], *Wright, E.* [327].

## Trigonometric Integral Inequality

[OSTROWSKI] *If $f \in \mathcal{L}([a,b])$ is monotone then*

$$\left|\int_a^b f(x)\cos x\,\mathrm{d}x\right| \le 2\big(|f(a)-f(b)|+|f(b)|\big).$$

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COMMENTS This follows from **Bounded Variation Function Inequalities**, EXTENSIONS.

(ii) See also: **Fourier Transform Inequalities**.

REFERENCE [AI, p. 301].

**Trigonometric Polynomial Inequalities** A *trigonometric polynomial of degree at most n* is anything of the form

$$\sum_{k=0}^{n} a_k \sin kx + b_i \cos kx,$$

where $n \in \mathbb{N}, a_k, b_k \in \mathbb{R}, 0 \leq k \leq n$; it is a partial sum of the first $n+1$ terms of a *trigonometric series.*

The same name is also used for either $\sum_{k=0}^{n} c_k e^{ikx}, c_k \in \mathbb{C}, 0 \leq k \leq n$, or $\sum_{k=-n}^{n} c_k e^{ikx}$, $c_k \in \mathbb{C}, -n \leq k \leq n$.

*(a) If* $0 \leq x_i \leq \pi,\ 1 \leq k \leq n$, *then*

$$\sum_{k=1}^{n} \sin x_k \leq n \sin\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right).$$

*with equality if and only if* $x_1 = \cdots = x_n$.
*(b) If* $n \geq 1, x \in \mathbb{R}$ *then*

$$0 < \sum_{k=1}^{n} \frac{\sin kx}{k} < \int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x = 1.8519\ldots < 1 + \frac{\pi}{2}.$$

*(c) If* $n \geq 1, x \in \mathbb{R}$ *then:* $$\left|\sum_{k=1}^{n} \frac{(-1)^k}{k} \sin kx\right| < \sqrt{2}|x|.$$

*(d) If* $0 < x < 2\pi$ *then:* $$\left|\sum_{k=m+1}^{m+n} e^{ikx}\right| < \operatorname{cosec}\frac{x}{2}.$$

*(e) If* $0 < \beta < 1, 0 < x \leq \pi$ *then*

$$\left|\sum_{k=1}^{n} \frac{\cos kx}{k^\beta}\right| \leq C_\beta x^{\beta-1}; \qquad \left|\sum_{k=1}^{n} \frac{\sin kx}{k^\beta}\right| \leq C_\beta x^{\beta-1}.$$

*(f) If* $0 < x \leq \pi$ *then:* $$\left|\sum_{k=1}^{n} \frac{\cos kx}{k}\right| \leq C - \log x.$$

*(g)* [BARI] *If* $T_n$ *is a trigonometric polynomial of degree at most n then*
*(i):*

$$||T_n(x)||_{\infty,[-\pi,\pi]} \leq (n+1)||T_n(x)\sin x||_{\infty,[-\pi,\pi]};$$

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*(ii): if* $1 \le p < q < \infty$ *then*

$$||T_n||_{q,[-\pi,\pi]} \le 2^{1+1/p-1/q}||T_n||_{p,[-\pi,\pi]}.$$

*(h)* $$\sum_{k=1}^{n}(n+1-k)\sin ix > 0; \qquad \sum_{k=1}^{n}\sin kx + \frac{1}{2}\sin(n+1)x \ge 0.$$

COMMENTS (i) (a) is an immediate consequence of (J).

(ii) (b), (c) should be compared with the **Fejér-Jackson Inequality**. (1), (2).

(iii) See also: **Bernšteĭn's Inequality** (a), **Bohr-Favard Inequality, Dirichlet Kernel Inequalities, Fejér Kernel Inequalities, Integral Mean Value Theorems** EXTENSIONS (B), **Jackon's Inequality, Littlewood's Problem, Nikol'skiĭ's Inequality, Poisson Kernel Inequalities, Rogosinski-Szegő Inequality, Young's Inequalities (d),** EXTENSIONS.

REFERENCES [MPF, pp. 579–580, 583–584, 613–614]; *Milovanović, Mitrinović & Rassias* [MMR, pp. 299–382], *Zygmund* [Z, vol. I, p. 191]; *Alzer* [17], *Gluchoff & Hartmann* [132].

**Trudinger's Inequality** *If* $\Omega \subseteq \mathbb{R}^n, n \ge 2$ *is an open domain,* $|\Omega| < \infty$ *and* $f \in \mathcal{W}^{1,n}(\Omega)$ *with* $f = 0$ *on* $\partial\Omega$, $\int_\Omega |\nabla f|^n \le 1$, *and if* $\alpha \le \alpha_n = na_n^{1/n-1}$, *then*

$$\int_R e^{\alpha f^{n'}} \le C_n|\Omega|,$$

*where* $n'$ *is the conjugate index, and* $C_n$ *is a constant that depends on* $n$.

COMMENTS (i) The definition of $\mathcal{W}^{1,n}(\Omega)$ is in **Sobolev's Inequalities**. The inequality is a limiting case of **Sobolev's Inequalities** (b).

(ii) If $\alpha > \alpha_n$ the integral is still finite but can be arbitrarily large.

(iii) The result is Trudinger's, but the exact range of $\alpha$, the value of $\alpha_n$, is due to Moser.

(iii) The cases of $f \in \mathcal{W}^{1,p}(\Omega), p \ne n$, are easier than this limiting case. See: *Moser*.

REFERENCES [EM, Supp., pp. 377–378]; *Moser* [224].

**Turán's Inequalities** *(a) If* $a_k, z_k \in \mathbb{C}, 1 \le k \le n$, *with* $\min_{1\le k\le n}|z_k| = 1$ *and if* $p \in \mathbb{R}$ *then*

$$\max_{p=m+1,\ldots,m+n}\left|\sum_{k=1}^{n} a_k z_k^p\right| \ge \left(\frac{n}{2e(m+n)}\right)^n \left|\sum_{k=1}^{n} a_k\right|. \tag{1}$$

*(b) With the notation of (a), but now assuming that* $1 = |z_1| \ge |z_2| \ge \cdots \ge |z_n|$,

$$\max_{p=m+1,\ldots,m+n}\left|\sum_{k=1}^{n} a_k z_k^p\right| \ge 2\left(\frac{n}{4e(m+n)}\right)^n \min_{1\le j\le n}\left|\sum_{k=1}^{j} a_j\right|. \tag{2}$$

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*(c) With the notation of (a), but now assuming that* $z_k \neq 0, 1 \leq k \leq n$, *and* $\max_{1\leq k\leq n} |z_k| = \Lambda$, $\min_{1\leq k,j\leq n, k\neq j} |z_k - z_j| = \lambda > 0$ *then*

$$\max_{p=m+1,\ldots,m+n} \frac{\left|\sum_{k=1}^{n} a_k z_k^p\right|}{\sum_{k=1}^{n} |a_k|\,|z_k|^p} \geq \frac{1}{n}\left(\frac{\lambda}{2\Lambda}\right)^{n-1}.$$

Comments (i) The best value of the constant on the right-hand side of (1), due to Makai and de Bruijn, is

$$\left(\sum_{k=0}^{n-1} \binom{m+k}{k} 2^k\right)^{-1}.$$

(ii) The constant in (2) is best possible and is due to Kolesnik & Straus.

Special Cases *(a) If* $z_k \in \mathbb{C}, 1 \leq k \leq n$, *with* $\min_{1\leq k\leq n} |z_k| = 1$ *and if* $p \in \mathbb{N}$ *then*

$$\max_{1\leq p\leq n} \left|\sum_{k=1}^{n} z_k^p\right| \geq 1,$$

*with equality if and only if the* $z_k$ *are vertices of a regular polygon on the unit circle.*

*(b)* [Turán, Atkinson] *If* $z_k \in \mathbb{C}, |z_k| \leq 1, 1 \leq k \leq n, z_n = 1$ *and if* $p \in \mathbb{N}$ *then*

$$\max_{1\leq p\leq n} \left|\sum_{k=1}^{n} z_k^p\right| > \frac{1}{3}. \tag{3}$$

Comments (iii) The special cases do not follow directly from the main results.

(iv) The best value of the constant on the right-hand side of (3) is not known.

References [AI, pp. 122–125], [MPF, pp. 651–659].

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# 21 Ultraspherical–von Neumann

**Ultraspherical Polynomial Inequalities** *(a)* [Lorch] *If* $C_n^{(\alpha)}$ *is an ultraspherical and if* $0 < \alpha < 1$*, then*

$$\sin^\alpha\theta |C_n^{(\alpha)}(\cos\theta)| \le \frac{1}{(\alpha-1)!}\left(\frac{2}{n}\right)^{1-\alpha}, \quad 0 \le \theta \le \pi.$$

*The constant is best possible.*

*(b) If* $x^2 \le 1 - \left(\dfrac{1-\alpha}{n+\alpha}\right)^2$ *then*

$$\left(C_{n+1}^{(\alpha)}(x)\right)^2 \ge C_n^{(\alpha)}(x) C_{n+2}^{(\alpha)}(x).$$

*(c)* [Nikolov] *If* $x \ge 1, n \in \mathbb{N}$ *then:*

$$\frac{\left(C_n^{(\alpha)}\right)'(x)}{C_n^{(\alpha)}(x)} \begin{cases} \ge \dfrac{n(n+2\alpha)}{(2\alpha+1)x + (n-1)\sqrt{x^2-1}} & \text{if} \quad \alpha > -\frac{1}{2}; \\[2ex] \le \dfrac{n^2(n+\alpha)}{\alpha(n+1)x(n^2-1)\sqrt{x^2-1}} & \text{if} \quad 0 \le \alpha \le 1. \end{cases}$$

Comments (i) Ultraspherical polynomials are also called *Gegenbauer polynomials* and are determined by the generating function:

$$\frac{1}{(1-2xw+w^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x) w^n.$$

An alternative notation is: $P_n(x,\alpha)$ or $P_n^{(\alpha)}(x)$.

In the particular case $\alpha = 1/2$ these polynomials are the Legendre polynomials, [EM].

(ii) The inequality in (b) should be compared with **Bessel Function Inequalities** (c), and **Legendre Polynomial Inequalities** (b).

(iii) In the case $\alpha = \frac{1}{2}$ the second inequality in (c) is slightly stronger than one that is equivalent to Raşa's conjecture that the function $\sum_{k=0}^{n} B_k^2(1,x)$



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is convex on $[0,1]$; for a definition of the terms of this sum see: **Bernšteĭn Polynomial Inequalities**.

EXTENSION [LORCH] *With the above notation,*

$$\sin^{\alpha}\theta|C_n^{(\alpha)}(\cos\theta)| \leq \frac{1}{(\alpha-1)!}\left(\frac{2}{n+\alpha}\right)^{1-\alpha}, \quad 0 \leq \theta \leq \pi.$$

COMMENT (iii) A particular case of this is the first inequality in **Martin's Inequalities**.

REFERENCES [GI1, pp. 35–38]; *Szegő* [Sz, p. 171]; *Nikolov* [237].

**Univalent Function Inequalities** See: **Area Theorems, Bieberbach's Conjecture, Distortion Theorems, Entire Function Inequalities, Grunsky's Inequalities, Rotation Theorems**.

**Upper and Lower Limit Inequalities** *If $\underline{a}, \underline{b}$ are real sequences then*

$$\limsup \underline{a} + \liminf \underline{b} \leq \limsup(\underline{a}+\underline{b}) \leq \limsup \underline{a} + \limsup \underline{b},$$

*provided the terms in the inequalities are defined.*
*(b) If $\underline{a}, \underline{b}$ are non-negative sequences then*

$$\limsup \underline{a} \liminf \underline{b} \leq \limsup \underline{a}\,\underline{b} \leq \limsup \underline{a} \limsup \underline{b},$$

*provided the terms in the inequalities are defined.*

COMMENTS (i) These are simple consequences of **Inf and Sup Inequalities**.

(ii) The operations on these inequalities are understood to be in $\overline{\mathbb{R}}$. In addition we could assume the terms of the sequences to be taking values in $\overline{\mathbb{R}}$.

(iii) There are analogous results for functions taking values in $\overline{\mathbb{R}}$. See also: **L'Hôpital's Rule**.

REFERENCE *Bourbaki* [B60, p. 164].

**Upper Semi-continuous Function Inequalities** See: **Semi-continuous Function Inequalities**.

**van der Corput's Inequality** *If $m, n \in \mathbb{N}^*, 1 \leq m \leq n$, and $z_k \in \mathbb{C}$, $1 \leq k \leq n$, then:*

$$m\left|\sum_{k=1}^{n} z_k\right|^2 \leq (n+m-1)\left[\sum_{k=1}^{n}|z_k|^2 + 2\sum_{j=1}^{m-1}\left((1-j/m)\left|\sum_{k=1}^{n-j} z_k\overline{z}_{k+k}\right|\right)\right].$$

COMMENTS (i) This result has applications in the estimation of exponential sums.

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(ii) For another inequality by van der Corput see: **Beth-van der Corput Inequality**.

REFERENCES *Kuipers & Niederreiter* [KN, p. 25], *Steele* [S, pp. 214–216].

**van der Waerden's Conjecture** *If $S$ is an $n \times n$ doubly stochastic matrix then*

$$\operatorname{per}(S) \geq \frac{n!}{n^n},$$

*with equality if and only if all the entries in $S$ are equal to $1/n$.*

COMMENTS (i) This conjecture was made in 1926 and caused much interest; it was settled by Falikman and Egoričev in 1980–81. The standard reference on permanents, [Mi], was written two years before this solution was published.

(ii) For a definition of a doubly stochastic matrix see: **Order Inequalities** COMMENTS (i).

(iii) See also **Permanent Inequalities**.

REFERENCES [EM, vol. 7, p. 127], [GI3, pp. 23–40]; *Minc* [Mi, pp. 11, 73–102].

## Variance Inequalities

*If $\underline{a}$ is an $n$-tuple with $m \leq \underline{a} \leq M$, then*

$$\frac{1}{n}\sum_{i=1}^{n}\big(a_i - \mathfrak{A}_n(\underline{a})\big)^2 \leq \big(M - \mathfrak{A}_n(\underline{a})\big)\big(\mathfrak{A}_n(\underline{a}) - m\big),$$

*with equality if and only if all the elements of $\underline{a}$ are equal to either $M$ or $m$.*

COMMENTS (i) This implies the following inequality known as *Popoviciu's Inequality*

$$\frac{1}{n}\sum_{i=1}^{n}\big(a_i - \mathfrak{A}_n(\underline{a})\big)^2 \leq \frac{(M-m)^2}{4}.$$

Here there is equality if and only if $n$ is even, and half of the elements of $\underline{a}$ are equal to $m$ and half are equal to $M$.

(ii) This result has been given a simple proof in an abstract setting in the reference where a connection with **Kantorović's Inequality** is made.

REFERENCE *Bhatia & Davis, C.* [61]

## Variation Inequalities

*If $f : [a,b] \to \mathbb{R}$ and $\delta > 0$ define*

$$V(f;\delta) = \sup\left\{\sum |f(y) - f(x)|\right\},$$

*where the sup is taken over all collections of nonoverlapping intervals $[x,y]$ with $\sum |y-x| < \delta$, then*

$$V(f^*;\delta) \leq V(f;\delta).$$

COMMENT See also: **Bounded Variation Function Inequalities**.

REFERENCE *Yanagihara* [329].

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**Vietoris's Inequality** *If*

$$J(x,m,n) = \frac{1}{B(m,n)} \int_x^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t,$$

*then*

$$J\left(\frac{r}{r+s}, r, s+1\right) < \frac{1}{2} < J\left(\frac{r}{r+s}, r+1, s\right).$$

COMMENT J is called the *Incomplete Beta function*; $B$ denotes the Beta function. For a definition see: **Beta Function Inequalities**.

REFERENCE *Lochs* [177].

**Volume of the Unit Ball** The *measures of the volume and of the surface area of* $B_n$ are, respectively,

$$v_n = \frac{\pi^{n/2}}{(n/2)!}, \qquad a_n = nv_n. \tag{1}$$

In particular the unit balls in $\mathbb{R}^n, 1 \le n \le 4$, have volumes and surface areas $v_1 = 2, v_2 = \pi, v_3 = 4\pi/3, v_4, = \pi^2/2; a_1 = 2, a_2 = 2\pi, a_3 = 4\pi, a_4 = 2\pi^2$.

*If $v_n$ is the volume of the unit ball in $\mathbb{R}^n$, then:*

$$\frac{2}{\sqrt{\pi}} v_{n+1}^{n/(n+1)} \le v_n \le \sqrt{e}\, v_{n+1}^{n/(n+1)};$$

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}} \le \frac{v_{n-1}}{v_n} \le \sqrt{\frac{n-1+\pi/2}{2\pi}};$$

$$(1+1/n)^{2-\log\pi/\log 2} \le \frac{v_n^2}{v_{n-1}v_{n+1}} \le \sqrt{(1+1/n)}.$$

*In all inequalities the constants are best possible.*

REFERENCE *Alzer* [31].

**von Neumann & Jordan Inequality** *If $X$ is a Banach space there is a unique constant $C, 1 \le C \le 2$, such that if $x, y \in X$, not both zero, then*

$$\frac{1}{C} \le \frac{||x+y||^2 + ||x-y||^2}{2(||x||^2 + ||y||^2)} \le C.$$

COMMENTS (i) By use of the **Clarkson Inequalities** it can be shown that if $X = \ell_p$ or $\mathcal{L}^p$ then $C = 2^{(2-p)/p}$ if $1 \le p \le 2$, and $C = 2^{(p-2)/p}$ if $p \ge 2$.

(ii) $C = 1$ if and only if $X$ is finite-dimensional or a Hilbert space.

REFERENCES [MPF, p. 550]; *Mitrinović & Vasić* [219].

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# 22 Wagner–Wright

**Wagner's Inequality** See: **Cauchy's Inequality**, EXTENSIONS (c).

**Walker's Inequality** *If $a, b, c, x, y, z > 0$ where*

$$x = b + c - a, \quad y = c + a - b, \quad z = a + b - c,$$

*then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{1}{xyz}.$$

COMMENT This is an example of **Order Inequalities** (a), (b) since $(a, b, c) = (x, y, z)S$, where $S$ is doubly stochastic.

REFERENCE [MOA, p. 103, 280, 281].

**Wallis's Inequality** *If $n \ge 1$ then*

$$\frac{1}{\sqrt{\pi(n + 1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n + 1/4)}}.$$

COMMENTS (i) For the notation see **Factorial Function Inequalities** (f).

(ii) In Wallis's original result the right-hand side was $1/\sqrt{\pi n}$; the improvement is due to Kazarinoff.

Incidentally Wallis introduced the notation $\infty$.

(iii) The important use of this result is to give *Wallis's Formula*:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{(2n)!!^2}{(2n-1)!!^2(2n+1)}.$$

REFERENCES [AI, pp. 192–193, 287], [EM, vol. 9, p. 441]; *Apostol* [A69, vol. II, p. 450–453], *Borwein & Borwein* [Bs, pp. 338, 343], *Kazarinoff* [K, pp. 47–48, 65–67].

**Walsh's Inequality** *If $\underline{a}, \underline{w}$ are positive $n$-tuples*

$$\mathfrak{A}_n(\underline{a}; \underline{w})\mathfrak{A}_n(\underline{a}^{-1}; \underline{w}) \ge 1,$$



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*with equality if and only if $\underline{a}$ is constant.*

COMMENTS (i) The proof is by induction, the case $n = 1$ being trivial.

(ii) This inequality is easily seen to be equivalent to (HA). It can be used to prove the apparently stronger (GA).

(iii) For an inverse inequality see **Kantorović's Inequality**.

REFERENCES [AI, pp. 206–207], [H, pp. 93–94].

**Weak Young Inequality** See **Hardy-Littlewood-Sobolev Inequalities** COMMENTS (iv).

**Weierstrass's Inequalities** *(a) If $0 < a_i < 1,\ w_i \ge 1,\ 1 \le i \le n$, then*

$$\begin{aligned}\prod_{i=1}^n (1-a_i)^{w_i} &> 1 - \sum_{i=1}^n w_i a_i,\\ 1 + \sum_{i=1}^n w_i a_i &< \prod_{i=1}^n (1+a_i)^{w_i} < \frac{1}{\prod_{i=1}^n (1-a_i)^{w_i}}.\end{aligned}$$

*(b) If in addition $\sum_{i=1}^n w_i a_i < 1$ then*

$$\begin{aligned}\prod_{i=1}^n (1+a_i)^{w_i} &< \frac{1}{1 - \sum_{i=1}^n w_i a_i},\\ \prod_{i=1}^n (1-a_i)^{w_i} &< \frac{1}{1 + \sum_{i=1}^n w_i a_i}.\end{aligned}$$

COMMENTS (i) Taking $w_i = 1, 1 \le i \le n$, and combining the above results we get, under the appropriate conditions, the classical *Weierstrass Inequalities*:

$$\begin{aligned}1 + \sum_{i=1}^n a_i &< \prod_{i=1}^n (1+a_i) < \frac{1}{1 - \sum_{i=1}^n a_i}, \qquad (1)\\ 1 - \sum_{i=1}^n a_i &< \prod_{i=1}^n (1-a_i) < \frac{1}{1 + \sum_{i=1}^n a_i}.\end{aligned}$$

(ii) A proof of (1) can be given using **Chong's Inequalities** (2).

RELATED INEQUALITIES *(a) If $0 \le a_i \le 1/2,\ 1 \le i \le n,\ n \ge 2$, then*

$$\frac{1}{2^n} \le \frac{\prod_{i=1}^n (1+a_i)}{\prod_{i=1}^n \big(1 + (1-a_i)\big)} \le \frac{1 + \sum_{i=1}^n a_i}{1 + \sum_{i=1}^n (1-a_i)} \le \frac{1 + \prod_{i=1}^n a_i}{1 + \prod_{i=1}^n (1-a_i)} \le 1,$$

*equalities occur if and only if $a_1 = \cdots = a_n = 1/2$ or $0$.*

*(b)* [FLANDERS] *If $0 < \underline{a} < 1$ then*

$$\frac{1 + \mathfrak{A}_n(\underline{a};\underline{w})}{1 - \mathfrak{A}_n(\underline{a};\underline{w})} \le \mathfrak{G}_n\left(\frac{1+\underline{a}}{1-\underline{a}};\underline{w}\right), \qquad (2)$$

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*with equality if and only if $\underline{a}$ is constant.*

COMMENTS (iii) In the case of equal weights and $A_n = 1$ (2) reduces to **Geometric Mean Inequalities** (2).

(iv) See also: **Geometric-Arithmetic Mean Inequality** EXTENSIONS (B), **Hölder's Inequality** COMMENTS (VII), **Fan's Inequality, Myer's Inequality**.

REFERENCES [AI, pp. 35, 210], [H, p. 24–25], [HLP, p. 6], [MPF, pp. 69–77]; *Bourbaki* [B60, pp. 176–177].

## Weinberger's Inequality

See: **Szegő's Inequality** COMMENTS (iii).

## Weyl's Inequalities

*(a) If $A, B$ are $n \times n$ Hermitian matrices then*

$$\lambda_{(k)}(A) + \lambda_{(1)}(B) \le \lambda_{(k)}(A+B) \le \lambda_{(k)}(A) + \lambda_{(n)}(B),\ 1 \le k \le n. \tag{1}$$

*(b) If $A$ is a square matrix then*

$$\big(\log|\lambda_1(A)|, \ldots, \log|\lambda_n(A)|\big) \prec \big(\log|\sigma_1(A)|, \ldots, \log|\sigma_n(A)|\big),$$

*where $\sigma_s(A), 1 \le s \le n$ denotes the singular values of $A$.*
*(c) With the notation of (b), and $p > 0$,*

$$\sum_{j=1}^{n} |\lambda_j(A)|^p \le \sum_{j=1}^{n} |\sigma_j(A)|^p.$$

COMMENTS (i) The *singular values of a matrix $A$* are the eigenvalues of $(AA^*)^{1/2}$.

(ii) (c) is a corollary of (b) and **Order Inequalities** (b).

EXTENSIONS

(*a*) $$\lambda_{[i+j-1]}(A+B) \le \lambda_{[i]}(A) + \lambda_{[j]}(B),\ 1 \le i+j-1 \le n;$$

*(b)* [FAN]

$$\sum_{j=1}^{k} \lambda_{[i]}(A+B) \le \sum_{j=1}^{k} \lambda_{[i]}(A) + \sum_{j=1}^{k} \lambda_{[i]}(B),\ 1 \le k \le n;$$

*(c)* [LIDSKĬI, WIELANDT] *If $1 \le i_1 < \cdots < i_k \le n$ then*

$$\sum_{j=1}^{k} \lambda_{[i_j]}(A+B) \le \sum_{j=1}^{k} \lambda_{[i_j]}(A) + \sum_{j=1}^{k} \lambda_{[i]}(B),\ 1 \le k \le n;$$

COMMENTS (iii) Putting $j = 1$ in (a) gives the right-hand side of Weyl's inequality, (1).

(iv) For another inequality of Weyl see the **Heisenberg-Weyl Inequality**.

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(v) Other sets of inequalities between the eigenvalues have been given, in particular by Horn, Lidskiĭ, and Wielandt. An excellent discussion of these can be found in the survey article by Bhatia.

References [GI4, pp. 213–219], [MOA, pp. 317–322]; *Horn & Johnson* [HJ, pp. 181–182], *König* [Kon, p. 35], *Marcus & Minc* [MM, pp. 116–117]; *Bhatia* [60].

**Whiteley Mean Inequalities** A natural generalization of the symmetric and complete symmetric functions are the *Whiteley symmetric functions*, $t_n^{[k,s]}, n = 1, 2, \ldots, k \in \mathbb{N}, s \in \mathbb{R}, s \neq 0$, that are generated by:

$$\sum_{k=0}^{\infty} t_n^{[k,s]}(\underline{a})x^k = \begin{cases} \prod_{i=1}^{n}(1 + a_i x)^s, & \text{if } s > 0, \\ \prod_{i=1}^{n}(1 - a_i x)^s, & \text{if } s < 0; \end{cases}$$

where $\underline{a}$ is a positive $n$-tuple.

The *Whiteley means, of order $n$, of the positive $n$-tuple $\underline{a}$* are:

$$\mathfrak{W}_n^{[r,s]}(\underline{a}) = \begin{cases} \left(\dfrac{t_n^{[r,s]}(\underline{a})}{\binom{ns}{r}}\right)^{1/r}, & \text{if } s > 0, \\ \left(\dfrac{t_n^{[r,s]}(\underline{a})}{(-1)^r\binom{ns}{r}}\right)^{1/r}, & \text{f } s < 0. \end{cases}$$

*(a) If $k$ is an integer, $1 \leq k \leq n$, $\neq 0$, then*

$$\min \underline{a} \leq \mathfrak{W}_n^{[k,s]}(\underline{a}) \leq \max \underline{a},$$

*with equality if and only if $\underline{a}$ is constant.*
*(b) If $s > 0$, $k$ an integer, $1 \leq k < s$, when $s$ is not an integer, or $1 \leq k < ns$, when $s$ is an integer,*

$$\left(\mathfrak{W}_n^{[k,s]}(\underline{a})\right)^2 \geq \mathfrak{W}_n^{[k-1,s]}(\underline{a})\mathfrak{W}_n^{[k+1,s]}(\underline{a}). \tag{1}$$

*If $s < 0$ then ($\sim$1) holds. In both cases there is equality if and only if $\underline{a}$ is constant.*
*(c) If $s > 0$, $k, \ell$ are integers, with $1 \leq k < \ell < s + 1$, when $s$ is not an integer, and $1 \leq k < \ell < ns$, when $s$ is an integer, then*

$$\mathfrak{W}_n^{[\ell,s]}(\underline{a}) \leq \mathfrak{W}_n^{[k,s]}(\underline{a}). \tag{2}$$

*If $s < 0$ then ($\sim$2) holds. In both cases there is equality if and only if $\underline{a}$ is constant.*
*(d) If $s > 0, k \in \mathbb{N}$, and if $k < s + 1$ if $s$ is not an integer, then*

$$\mathfrak{W}_n^{[k,s]}(\underline{a} + \underline{b}) \geq \mathfrak{W}_n^{[k,s]}(\underline{a}) + \mathfrak{W}_n^{[k,s]}(\underline{b}).$$

*If $s < 0$ this inequality is reversed. The inequality is strict unless either $k = 1$ or $\underline{a} \sim \underline{b}$.*

Comment (i) These results generalize the analogous results in **Complete Symmetric Mean Inequalities, Elementary Symmetric Function Inequalities,**

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**Symmetric Mean Inequalities**. They have been further generalized by Whiteley and Menon.

REFERENCES [BB, pp. 35–36], [H, pp. 343–356], [MOA, p. 119], [MPF, pp. 166–168].

**Wilf's Inequality** *If* $\alpha \in \mathbb{R}, 0 < \theta < \pi/2$, *and if* $\alpha - \theta \le \arg z_k \le \alpha + \theta$, $1 \le k \le n$ *then*

$$\left|\sum_{k=1}^{n} z_k\right| \ge \cos\theta \sum_{k=1}^{n} |z_k|.$$

COMMENT This is an inverse of **Complex Number Inequalities** EXTENSIONS (a). It seems to have been first proved by Petrović.

EXTENSION [JANIĆ, KEČKIĆ & VASIĆ] *If* $\alpha \in \mathbb{R}, 0 < \theta < \pi/2$, $\alpha \le \arg z_k \le \alpha + \theta$, $1 \le k \le n$, *then*

$$\left|\sum_{k=1}^{n} z_k\right| \ge \max\left\{\cos\theta, \frac{1}{\sqrt{2}}\right\} \sum_{k=1}^{n} |z_k|.$$

INTEGRAL ANALOGUE *If* $f : [a, b] \to \mathbb{C}$ *is integrable and if for some* $\theta$, $0 < \theta < \pi/2$, $-\theta \le \arg f(x) \le \theta$, *then*

$$\left|\int_a^b f\right| \ge \cos\theta \int_a^b |f|.$$

REFERENCES [AI, pp. 310–311], [H, p. 209], [MPF, pp. 492–497], [PPT, pp. 128–132].

**Wilson's Inequalities** See: **Nanson's Inequality** (2).

**Wirtinger's Inequality** *If* $f$ *is a function of period* $2\pi$ *with* $f, f' \in \mathcal{L}^2([0, 2\pi])$ *and* $\int_0^{2\pi} f = 0$, *then*

$$\int_0^{2\pi} f^2 \le \int_0^{2\pi} f'^2,$$

*with equality if and only if* $f(x) = A\cos x + B\sin x$.

COMMENTS (i) A discrete analogue can be found in **Fan-Taussky-Todd Inequalities** and *Pachpatte* [241]; see also **Ōzeki's Inequalities** COMMENTS (i).

(ii) This inequality has given rise to considerable research. A full discussion of its history can be found in [AI]. Higher dimensional analogues are **Friederichs's Inequality** and **Poincaré's Inequalities**.

EXTENSIONS (a) [TANANIKA] If $f' \in \mathcal{L}^2([0, 1]), f \in \mathcal{L}^p([0, 1]), p \ge 1$, and $\int_0^{2\pi} f = 0$, then

$$||f||_{p,[01]} \le \left(\sqrt{\frac{p}{\pi}}\right)\left(\frac{1}{2^{(p-1)/p}(p+2)^{(p+2)/2p}}\right)\left(\frac{((p+2)/2p)!}{(1/p)!}\right)||f'||_{2,[0,1]}.$$

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(b) [Vornicescu] If $f \in \mathcal{C}^1([0, 2\pi])$ with $f(0) = f(2\pi) = 0$ and $\int_0^{2\pi} f = 0$ then

$$\frac{1}{\pi}\sum_{n=2}^{\infty}(n^2-1)(a_n^2+b_n^2) + \int_0^{2\pi} f^2 \le \int_0^{2\pi} f'^2,$$

where $a_n, b_n$ are the Fourier coefficients of $f$.
(c) [Alzer] If $f$ is a real valued continuously differentiable function of period $2\pi$ with $\int_0^{2\pi} f = 0$ then

$$\max_{0\le x\le 2\pi} f^2(x) \le \frac{\pi}{6}\int_0^{2\pi} f'^2.$$

Comments (iv) Vornicescu has given a discrete analogue of his result; his proof depends on applying Wirtinger's inequality to a suitable function. For another inequality involving Fourier coefficients see: **Boas's Inequality**.

(v) See also: **Benson's Inequalities** Comment, **Opial's Inequality**.

References [AI, pp. 141–154], [BB, pp. 177–178], [GI7, pp. 153–155], [HLP, pp. 84–187]; *Zwillinger* [Zw, p. 207]; *Alzer* [16], *Pachpatte* [241], *Vornicescu* [320].

**Wright Convex Function Inequalities**[46] See: **Convex Function Inequalities** Comments (viii), **Szegő's Inequality** Comments (i).

[46] This is E. T. Wright.

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# 23 Yao–Zeta

**Yao & Iyer Inequality** *If $|X|$ and $|Y|$ are random variables with $Y \sim (0,1)$ and $X \sim (\mu, \sigma^2)$, $\mu \in \mathbb{R}, \sigma > 0$ and $(0,1) \neq (\mu, \sigma^2)$ then for all $z > 0$,*

$$\frac{P(|X| < z)}{P(|Y| < z)} > \min\{r(\infty), r(0+)\}.$$

COMMENT $r(\infty) = \lim_{z\to\infty} r(z) = 1$, and $r(0+) = \lim_{z\to 0+} r(z) = \dfrac{\phi(\mu/\sigma)}{\sigma\phi(0)}$, where $\phi$ is the standard normal density.

REFERENCE *Pinelis* [269].

**Young's Convolution Inequality** *If $g \in \mathcal{L}^r(\mathbb{R}^n), h \in \mathcal{L}^s(\mathbb{R}^n)$, $1 \leq r, s, \leq \infty$,*
*and if $\dfrac{1}{r} + \dfrac{1}{s} = 1 + \dfrac{1}{t} \geq 0$ then $g \star h \in \mathcal{L}^t(\mathbb{R}^n)$ and*

$$||g \star h||_t \leq K_1 ||g||_r ||h||_s. \tag{1}$$

*Equivalently: if $f \in \mathcal{L}^t(\mathbb{R}^n), g \in \mathcal{L}^r(\mathbb{R}^n), h \in \mathcal{L}^s(\mathbb{R}^n)$, $1 \leq r, s, t \leq \infty$, where $\dfrac{1}{r} + \dfrac{1}{s} + \dfrac{1}{t} = 2$ then*

$$\left| \int_{\mathbb{R}} g \star h(\underline{x}) f(\underline{x}) \, \mathrm{d}\underline{x} \right| \leq K_2 ||g||_r ||h||_s ||f||_t. \tag{2}$$

*The constants are*

$$K_1 = (C_r C_s C_{t'})^n, \quad K_2 = (C_r C_s C_t)^n \qquad \textit{where} \qquad C_p^2 = \frac{p^{1/p}}{p'^{1/p'}},$$

*the primed numbers being the conjugates of the respective unprimed numbers. The inequalities are strict when $r, s, t > 1$ unless almost everywhere*

$$\begin{aligned} f(x) = &\quad A \exp\big\{ -r' \big\langle \underline{x} - \underline{a}, (\underline{x} - \underline{\alpha})H \big\rangle + i\underline{d}.\underline{x} \big\}, \\ g(x) = &\quad B \exp\big\{ -s' \big\langle \underline{x} - \underline{b}, (\underline{x} - \underline{b})H \big\rangle + i\underline{d}.\underline{x} \big\}, \\ h(x) = &\quad C \exp\big\{ -t' \big\langle \underline{x} - \underline{c}, (\underline{x} - \underline{c})H \big\rangle + i\underline{d}.\underline{x} \big\}, \end{aligned}$$



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*where $A, B, C \in \mathbb{C}$, $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in \mathbb{R}^n$, and $H$ is a real, symmetric, positive-definite matrix.*

COMMENTS (i) Inequalities (1), (2) with $K_1 = K_2 = 1$ were proved by Young. The exact values of the constants, and the cases of equality, are due to Bruscamp & Lieb.

(ii) There are various other forms of this result depending on the function spaces used.

(iii) The case of (1) with $t = \infty$, when $r, s$ are conjugate indices and $g \star h$ is uniformly continuous, has been called a *Backward Hölder's Inequality*:

$$\sup_{a \le t \le b} \int_a^b g(x-t)h(t)\,\mathrm{d}x \le ||g||_r||h||_s.$$

DISCRETE ANALOGUE *If $\underline{a} \in \ell_r$, $\underline{b} \in \ell_s$ and if $t$ is defined as in (1) then $\underline{a} \star \underline{b} \in \ell_t$ and*

$$||\underline{a} \star \underline{b}||_t \le ||\underline{a}||_r||\underline{b}||_s,$$

*with equality if and only if $\underline{a} = \underline{0}$, $\underline{b} = \underline{0}$ or $\underline{a}$ and $\underline{b}$ differ from $\underline{0}$ in one entry only.*

AN ELEMENTARY CASE

$$\int_{\mathbb{R}^2} f(x)g(y)h(x-y)k(x-y)\mathrm{d}x\mathrm{d}y \le \frac{1}{\sqrt{2}}||f||_2||g||_2||h||_2||k||_2.$$

COMMENTS (iv) The constant is best possible as can be seen with the functions: $f(x) = g(x) = e^{-2x^2}, h(x) = k(x) = e^{-x^2}$.

(v) This is an easy consequence of **Cauchy's Inequality** INTEGRAL ANALOGUES and Fubini's theorem.

(vi) For another convolution result see: **Hardy-Littlewood-Sobolev Inequality**.

REFERENCES [EM, vol. 2, pp. 427–428], [HLP, pp. 198–202], [MPF, pp. 178–181]; *Hirschman* [Hir, pp. 168–169]; *Hewitt & Stromberg* [HS, pp. 396–400], *Lieb & Loss* [LL, pp. 90–97], *Zwillinger* [Zw, p. 207], *Zygmund* [Z, vol. I, pp. 37–38].

**Young's Inequalities** *(a) If $f$ is a strictly increasing continuous function on $[0, c]$ with $f(0) = 0$ and $0 \le a, b \le c$ then*

$$ab \le \int_0^a f + \int_0^b f^{-1}. \tag{1}$$

*There is equality if and only if $b = f(a)$.*
*(b) Under the same conditions*

$$ab \le af(a) + bf^{-1}(b).$$

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*(c) If $p, q, r > 0$ with $1/p + 1/q + 1/r = 1$, and if $p', q'$ are, respectively, the conjugate indices of $p, q$ and if $f \in \mathcal{L}^{p'}(\mathbb{R}), g \in \mathcal{L}^{q'}(\mathbb{R})$ then $fg \in \mathcal{L}(\mathbb{R})$ and*

$$\int_{\mathbb{R}} |fg| \le \left( \int_{\mathbb{R}} |f|^{p'} |g|^{q'} \right)^{1/r} \left( \int_{\mathbb{R}} |f|^{p'} \right)^{1/q} \left( \int_{\mathbb{R}} |g|^{q'} \right)^{1/p}. \tag{2}$$

*(d) If $0 < \theta < \pi$ and $-1 < \alpha \le 0$, then*

$$\sum_{k=0}^{n} \frac{\cos k\theta}{k + \alpha} > 0.$$

*(e) If $R$ is the outer radius of a bounded domain in $\mathbb{R}^n, n \ge 2$, of diameter $D$ then*

$$R \le D\sqrt{\frac{n}{n+2}}.$$

Comments (i) Inequality (1), which holds under slightly wider conditions, has a very simple geometric proof.

(ii) Applying (1) to the function $f(x) = x^p, p > 0$ gives a proof of (B) in the form given in **Geometric-Arithmetic Mean Inequality** (2).

(iii) For another application see: **Logarithmic Function Inequalities** (e).

(iv) Inequality (3) is a simple deduction from the integral analogue of **Hölder's Inequality** (2), in the case $n = 3$, and $\rho_3 = 1$.

(v) A definition of outer radius is given in **Isodiametric Inequality** Comments (ii).

Extensions *(a)* [Brown & Koumandos] *If $0 < \theta < \pi$ and $\alpha \ge 1$ then the function*

$$\cos\theta/2 \left( \sum_{k=1}^{n} \frac{\cos k\theta}{k^\alpha} \right)$$

*is strictly decreasing. In particular*

$$\frac{5}{6} + \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0, \qquad 0 \le \theta \le \pi.$$

*(b)* [Hyltén-Cavallius] *If $0 < \theta \le \pi$ then*

$$1 - \log \circ \sin x/2 + \frac{\pi - x}{2} \ge 1 + \sum_{i=1}^{n} \frac{\cos i\theta}{i} > 0.$$

*(c)*[Furuta] *If $A$ is a positive linear operator on a Hilbert space, and if $0 \le t \le 1$ then for all unit vectors $\underline{x}$,*

$$tA - I - t \ge A^t.$$

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COMMENTS (vi) Inequality (c) is equivalent to the **Hölder-McCarthy Inequality**.

(vii) See also: **Conjugate Convex Function Inequalities** (B), **Gale's Inequality, N-function Inequalities**.

REFERENCES [AI, pp. 48–50], [BB, p. 7], [EM, vol. 5, p. 204], [HLP, pp. 111–113], [MPF, pp. 379–389, 615], [PPT, pp. 239–246]; *Titchmarsh* [T86, p. 97]; *Brown & Koumandos* [74], *Furuta* [120].

**Zagier's Inequality** *Let $f, g$ be monotone decreasing non-negative functions on $[0, \infty[$ then for any integrable $F, G : [0, \infty[\to [0, 1]$,*

$$\int_0^\infty fg \ge \frac{\int_0^\infty fF \int_0^\infty Gg}{\max\{\int_0^\infty F, \int_0^\infty G\}}.$$

*In particular if $f, g : [0, \infty[\to [0, 1]$ be monotone decreasing integrable functions*

$$\int_0^\infty fg \ge \frac{\int_0^\infty f^2 \int_0^\infty g^2}{\max\{\int_0^\infty f, \int_0^\infty g\}}.$$

COMMENT (i) This is an inverse of the integral analogue of (C).

DISCRETE ANALOGUE [ALZER, PEČARIĆ] *If $0 < a_i, b_i, c_i, d_i \le 1,\ 1 \le i \le n$, and if both $\underline{a}, \underline{b}$ are decreasing then*

$$\frac{\sum_{i=1}^n a_i c_i \sum_{i=1}^n b_i d_i}{\max\left\{\sum_{i=1}^n c_i, \sum_{i=1}^n d_i\right\}} \le \sum_{i=1}^n a_i b_i.$$

*In particular*

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\max\left\{\sum_{i=1}^n a_i, \sum_{i=1}^n b_i\right\}} \le \sum_{i=1}^n a_i b_i.$$

COMMENT (ii) For another result of Zagier see: **Difference Means of Gini**.

REFERENCES [MPF, pp. 95–96, 155]; *Pečarić* [263], *Zagier* [333].

**Zeros of a Polynomial** *If $p(z) = a_0 + \cdots + a_n z^n$, $a_n \ne 0$, where the coefficients are complex, then all of the zeros $\zeta$ of this polynomial satisfy*

$$|\zeta| \le 1 + \frac{n}{a_n} \max_{0 \le k \le n-1} |a_k|$$

REFERENCE *Cloud & Drachman* [CD, pp. 9–10].

**Zeta Function Inequalities** *(a) If $\sigma > 1, n \ge 1$ then*

$$\frac{1}{\sigma - 1} < n^{\sigma - 1} \sum_{i \ge n} \frac{1}{i^\sigma} \le \zeta(\sigma),$$

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*with equality on the right if and only if* $n = 1$.
*(b) If* $\sigma > 1,\ \gamma \geq 0$ *then*

$$\sum_{i\geq n}\frac{1}{i^\sigma} \leq \frac{n^\gamma(\sigma+\gamma)}{(\sigma-1)\big(n^{\sigma+\gamma}-(n-1)^{\sigma+\gamma}\big)}.$$

*(c) If* $\sigma > 1$ *then*

$$\frac{1}{n[\big(n^{\sigma-1}-(n-1)^{\sigma-1}\big)} < \sum_{i\geq n}\frac{1}{i^\sigma} < \frac{(n+1)^{\sigma-1}}{n^\sigma\big((n+1)^{\sigma-1}-n^{\sigma-1}\big)}.$$

The *Hurwitz zeta function* is defined by

$$\zeta_s(x) = \sum_{i=0}^{\infty}\frac{1}{(x+i)^s}, \quad x > 0,\, \sigma = \Re s > -1.$$

COMMENT (i) Clearly $\zeta_s(0)$ is just $\zeta(s)$.

*(a)* [TRIMBLE, WELLS & WRIGHT] *If* $\sigma \geq 2$ *then*

$$\frac{1}{\zeta_\sigma(x)} + \frac{1}{\zeta_\sigma(y)} < \frac{1}{\zeta_\sigma(x+y)}.$$

*(b)* [ALZER] *If* $1 < \sigma \leq 2$ *then*

$$\zeta_q s(xy) < \zeta_\sigma(x)\zeta_\sigma(y). \tag{1}$$

COMMENT (ii) For no $\sigma > 1$ does ($\sim$1) hold.

EXTENSIONS [ALZER] *The inequality*

$$\big(\zeta_\sigma(x)\big)^\alpha + \big(\zeta_\sigma(y)\big)^\alpha < \big(\zeta_\sigma(x+y)\big)^\alpha, \tag{2}$$

*holds if and only if* $\sigma \geq 1$ *and* $\alpha \leq -1/(\sigma-1)$.
*The inequality (~2) holds if and only if* $\sigma > 1$ *and* $\alpha > -1/\sigma$.

REFERENCES [AI, p. 190]; *Bennett* [Be, p. 14]; *Alzer* [30], *Cochran & Lee* [93].

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