Corner Transfer Matrix Renormalization Group

Efficient Contraction of 2D Classical Lattice Models

Your Name

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January 4, 2026

Outline

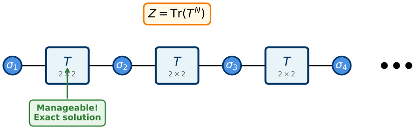
1D Ising Model:

$$H = -J \sum_i \sigma_i \sigma_{i+1}$$

Transfer Matrix:

$$Z = \mathrm{Tr}(T^N), \quad T \in \mathbb{R}^{2\times 2}$$

$$T = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix}$$



Key Point

T is $2 \times 2 \Rightarrow$ **Exact diagonalization is trivial!**

When Even 1D Has an Exponential Wall

Recall: For uniform 1D Ising, $Z = \mathrm{Tr}(T^N)$ with $T \in \mathbb{R}^{2\times 2}$ is **exact**.

But what if...

- Couplings J_i are **site-dependent**?
- Random disorder: $J_i \sim \mathcal{N}(\bar{J}, \sigma)$?
- Open boundary conditions (no translation symmetry)?

Then: Cannot diagonalize T once!

$$Z = \mathrm{Tr}(T_1 \cdot T_2 \cdot T_3 \cdots T_N)$$

Each T_i is different $\Rightarrow$ no simple λ^N formula.

[Figure: 1D chain with non-uniform couplings J_i]

The Problem

Solution: Transfer Matrix Renormalization Group (TMRG)

Key Insight: Not all configurations contribute equally to Z.

$\Rightarrow$ Keep only the **most relevant** ones!

[Figure: $Z \approx Tr(L \cdot T \cdot T \cdot R)$]

TMRG Algorithm:

1. **Grow:** $L' = L \cdot T$
   (add sites $\approx$ decimation)
2. **Truncate:** SVD $\rightarrow$ keep χ largest
   (coarse-grain $\approx$ rescaling)

RG Fixed Point

Iterate until L^*, R^* converge.
$\Rightarrow$ Thermodynamic limit!

2D: The Transfer Matrix Becomes a **Network**

1D: Ordered product of matrices

$$Z = \mathrm{Tr}(T_1 \cdot T_2 \cdots T_N)$$

Contract left-to-right: $O(N)$.

2D: Each site couples to **4 neighbors**!

$$T^{\sigma_i}_{\sigma_{\text{up}},\sigma_{\text{down}},\sigma_{\text{left}},\sigma_{\text{right}}}$$

This is a **rank-4 tensor**, not a matrix!

[Figure: 2D lattice — each site is a rank-4 tensor]

The Fundamental Problem

One Approach: Row-to-Row Transfer (DMRG-style)

Idea: Group one row of L spins $\to$ treat as a "super-spin" with 2^L states.

[Figure: Row $\to$ super-spin, $T_{row} \in \mathbb{R}^{2^L \times 2^L}$]

Then $Z = \mathrm{Tr}(T_{\text{row}}^M)$ looks like 1D!

Apply DMRG/TMRG ideas:

- Row config space $\to$ MPS
- Row-to-row transfer $\to$ MPO
- Truncate via SVD

Pros & Cons

+ Systematic, well-understood
– Breaks 2D symmetry
– Hard to generalize to other lattices

(Details in Appendix)

A More Natural Approach: Think in 2D!

Question: If we only want **local observables** $\langle \sigma_{i,j} \rangle$,
do we really need to contract the *entire* infinite lattice?

[Figure: Local site surrounded by 2D environment]

Key Insight:
Decompose the infinite 2D environment into **geometrically natural** pieces:

- **4 Corners** (quarter-planes)
- **4 Edges** (half-infinite strips)

Baxter's Insight (1968)

The **Corner Transfer Matrix** encodes a quarter of the infinite plane!

2D Analog: Four Corners + Four Edges

Key Idea: Divide infinite 2D lattice into **4 corners** + **4 edges**.

[Figure: 2D environment structure — 4 corners + 4 edges]

The Local Tensor: Building Block

Local Boltzmann weight tensor a:

[Figure: Local tensor a with 4 legs]

For 2D Ising:

$$a_{u,d,l,r} = \sum_{\sigma} W_{\sigma,u} W_{\sigma,d} W_{\sigma,l} W_{\sigma,r}$$

where

$$W_{\sigma,\sigma'} = e^{\frac{\beta J}{2}\sigma\sigma'}$$

Physical Meaning

a encodes the Boltzmann weight at one site, with bond weights split symmetrically.

CTMRG Iteration: Step 1 — Grow (Absorb Row/Column)

Absorption: Add one row **and** one column to expand the environment.

[Figure: Absorption step — corner grows by absorbing a and edges]

CTMRG Iteration: Step 2 — Truncate (The Key Difference!)

Truncation in 2D: Use the **full environment** to determine projectors.

[Figure: Building density matrix from environment for truncation]

Compute projector P:

Apply truncation:

CTMRG: Complete Algorithm

[Figure: CTMRG algorithm flowchart]

Partition function:

$$Z \propto \mathrm{Tr}(C_1 T_1 C_2 T_2 C_3 T_3 C_4 T_4)$$

[Figure: Computing $\langle \sigma \rangle$ by inserting operator]

Numerical Results: 2D Ising Model

[Results to be added: magnetization, critical behavior, etc.]

Summary: The Core of CTMRG

CTMRG in Three Steps

1. **Decompose:** Infinite 2D lattice $\to$ 4 corners $+$ 4 edges
2. **Grow:** Absorb local tensors (add row/column)
3. **Truncate:** SVD-based RG to keep χ most relevant states

[Figure: Visual summary — Decompose $\to$ Grow $\to$ Truncate]

Comparison: iDMRG vs CTMRG

Aspect	**iDMRG (1D)**	**CTMRG (2D)**
Dimension	1D chain	2D square lattice
Environment	Left + Right	4 Corners + 4 Edges
Grow step	Add site pair	Add row + column
Truncation	SVD on center bond	SVD on corner boundaries
Bond dimension	χ (MPS)	χ (environment)
Fixed point	L^*, R^*	C_i^*, T_i^*
Computational cost	$O(\chi^3)$	$O(\chi^6)$ or $O(\chi^5)$

Thank You!

Questions?

Appendix: TMRG Setup in Detail

Goal: Compute $Z = \mathrm{Tr}(T_1 \cdot T_2 \cdots T_N)$ for large N.

[Figure: Left environment L, local site T, right environment R]

Left Environment L:

$$L_\sigma = \sum T_1 \cdot T_2 \cdots T_{k-1}$$

Right Environment R:

$$R_\sigma = \sum T_{k+1} \cdots T_N$$

Absorption: Add one more site to the environment.

[Figure: $L' = L \cdot T_k$, growing the left environment]

Mathematically:

$$L'_{\sigma_k} = \sum_{\sigma_{k-1}} L_{\sigma_{k-1}} \cdot (T_k)_{\sigma_{k-1},\sigma_k}$$

Appendix: TMRG Step 2 — Truncate (SVD)

Idea: Compress L' back to dimension χ using SVD.

Form the "density matrix":

$$\rho = L' \cdot R'^T$$

SVD:

$$\rho = U \Sigma V^\dagger$$

Truncate:

$$P = U_{:,1:\chi}$$

New environment:

$$L_{\text{new}} = P^\dagger L'$$

[Figure: SVD spectrum, keep χ largest]

Eckart-Young Theorem

SVD gives the **optimal** rank-χ approximation

Iterate: Grow $\to$ Truncate $\to$ Grow $\to$ Truncate $\to \cdots$

Convergence criterion:

Fixed point: L^*, R^* such that

$$L^* \xrightarrow{\text{grow+truncate}} L^*$$

Free energy:

$$f = -k_B T \lim_{N\to\infty} \frac{1}{N} \ln Z$$

$$= -k_B T \ln \sigma_1^*$$

Local observables:

$$\langle \sigma_k \rangle = \frac{L^* \cdot \sigma_k \cdot R^*}{L^* \cdot R^*}$$

Correlation functions:

$$\langle \sigma_i \sigma_j \rangle = \frac{L^* \cdot \sigma_i \cdot T^{|i-j|} \cdot \sigma_j \cdot R^*}{Z}$$

Physical Meaning

L^*, R^* encode the **thermodynamic limit**.

Appendix: 2D Row-to-Row — Row as Super-Spin

Idea: Treat one row of L spins as a single "super-spin".

[Figure: Row $\rightarrow$ super-spin with 2^L states]

Row configuration:

$$\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_L)$$

Total states: 2^L.

Row-to-row transfer:

$$(T_{\text{row}})_{\vec{\sigma},\vec{\sigma}'} = \prod_{i=1}^{L} e^{\beta J \sigma_i \sigma_i'} \prod_{i=1}^{L-1} e^{\beta J \sigma_i \sigma_{i+1}}$$

$T_{\text{row}} \in \mathbb{R}^{2^L \times 2^L}$.

Key Point

Now $Z = \text{Tr}(T^M_{\text{row}})$ looks like 1D problem with 2^L dimensional transfer matrix!

Appendix: Row Transfer Matrix = MPO

Key insight: T_{row} has a **tensor network** structure!

Decompose T_{row} into local tensors:

$$T_{\vec{\sigma},\vec{\sigma}'} = \sum_{\alpha_1,\ldots} W^{\sigma_1\sigma_1'}_{\alpha_0\alpha_1} W^{\sigma_2\sigma_2'}_{\alpha_1\alpha_2} \cdots$$

[Figure: MPO structure of T_{row}]

Local tensor W**:**

$$W^{\sigma\sigma'}_{\alpha\beta} = \text{local Boltzmann weight}$$

- σ, σ': spins in rows n, $n+1$
- α, β: auxiliary (horizontal bonds)

No Quantum!

"MPO" is just a factorization of the classical transfer matrix.

Appendix: Boundary = MPS

MPS: Efficient representation of row configuration space.

Full vector:

$$|R\rangle = \sum_{\vec{\sigma}} R_{\vec{\sigma}} |\vec{\sigma}\rangle$$

has 2^L components.

MPS compression:

$$R_{\vec{\sigma}} = A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_L}$$

Only $L \cdot \chi^2 \cdot 2$ parameters!

[Figure: MPS tensor network]

Physical Meaning

MPS compresses 2^L row configs into χ effective states with **limited entanglement**.

Appendix: Row-to-Row DMRG Algorithm

Combine: MPS (boundary) + MPO (transfer) + SVD (truncation).

1. Initialize MPS $|L\rangle$, $|R\rangle$ for boundaries
2. **Grow:** Apply MPO T_{row} to boundaries

$$|L'\rangle = T_{\text{row}}|L\rangle$$

3. **Truncate:** SVD to compress bond dimension back to χ
4. Iterate until convergence to fixed point $|L^*\rangle$, $|R^*\rangle$

Pros

- Systematic, well-understood
- Controlled approximation

Cons

- Breaks 2D rotational symmetry
- Hard to generalize to other lattices

Appendix: No Monte Carlo Sign Problem

Classical models: All Boltzmann weights are **positive**!

$$W = e^{-\beta H} > 0 \quad \text{always}$$

[Figure: Classical (positive) vs Quantum (sign problem)]

Advantage of Tensor Network Methods

Appendix: Why is 3D Difficult?

In 2D: Environment tensors are **1D objects** (edges).
In 3D: Environment would be **2D surfaces** — back to exponential!

[Figure: 2D boundary problem in 3D systems]

Appendix: Extensions and Generalizations

Beyond square lattice Ising:

Different lattices:

- Honeycomb
- Triangular
- Kagome

Different models:

- Potts model
- Clock model
- Vertex models

Quantum systems (via iPEPS):

- 2D Heisenberg model
- Frustrated magnets
- Topological phases

Improvements:

- Directional CTMRG
- Full-update vs simple-update
- Gradient optimization

Appendix: Computational Complexity

CTMRG scaling:

Operation	**Cost**	**Bottleneck?**
Corner absorption	$O(\chi^4 d^2)$	
Edge absorption	$O(\chi^3 d^2)$	
Build density matrix	$O(\chi^4)$	
SVD for projector	$O(\chi^3 d^3)$	$\checkmark$
Apply truncation	$O(\chi^3 d)$	
Total per iteration	$O(\chi^3 d^3)$ to $O(\chi^6)$	

Comparison

- iDMRG (1D): $O(\chi^3)$ per sweep
- CTMRG (2D): $O(\chi^5)$ – $O(\chi^6)$ per iteration
- More expensive, but still **polynomial** in χ!

Appendix: Key References

Original works:

- R. J. Baxter, *J. Math. Phys.* **9**, 650 (1968) — Corner transfer matrices
- R. J. Baxter, *J. Stat. Phys.* **19**, 461 (1978) — CTM method

Modern CTMRG:

- T. Nishino & K. Okunishi, *J. Phys. Soc. Jpn.* **65**, 891 (1996)
- T. Nishino & K. Okunishi, *J. Phys. Soc. Jpn.* **66**, 3040 (1997)

CTMRG for iPEPS:

- R. Orús & G. Vidal, *Phys. Rev. B* **80**, 094403 (2009)
- P. Corboz et al., *Phys. Rev. B* **84**, 041108(R) (2011)

Reviews:

- R. Orús, *Ann. Phys.* **349**, 117 (2014) — Tensor networks review