Corner Transfer Matrix Renormalization Group

Efficient Contraction of 2D Classical Lattice Models

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Outline

Review: 1D Transfer Matrix (Your Pre-knowledge)

1D Ising Model:

$$H = -J \sum_i \sigma_i \sigma_{i+1}$$

Transfer Matrix:

$$Z = \mathrm{Tr}(T^N), \qquad T \in \mathbb{R}^{2\times 2}$$

$$T = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix}$$

[Figure: 1D chain with transfer matrices]

Key Point

T is $2 \times 2 \Rightarrow$ **Exact diagonalization is trivial!**

2D Square Lattice: Row-to-Row Transfer

2D Ising Model:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

Row-to-Row Transfer:

$$Z = \mathrm{Tr}(T_{\mathrm{row}}^M)$$

Row has L spins $\Rightarrow$

$$T_{\mathrm{row}} \in \mathbb{R}^{2^L \times 2^L}$$

[Figure: 2D lattice with row-to-row transfer matrix]

The Exponential Wall: A Dimensional Curse

[Figure: Exponential growth of 2^L with system size]

L	Matrix Size 2^L
10	$\sim 10^3$
20	$\sim 10^6$
50	$\sim 10^{15}$

The Problem

Exponential growth is intrinsic to **higher dimensions**, not specific to row-to-row vs

The Solution: Compress the Correlations!

Key Insight: Not all 2^L configurations are equally important.

[Figure: Eigenvalue spectrum decay — keep only χ largest]

Row Transfer Matrix = Matrix Product Operator (MPO)

Key Point: The row-to-row transfer matrix has a **tensor network** structure!

Decompose T_{row} into local tensors:

$$T_{\{\sigma\},\{\sigma'\}} = \sum_{\text{aux}} W^{\sigma_1\sigma_1'} W^{\sigma_2\sigma_2'} \cdots W^{\sigma_L\sigma_L'}$$

[Figure: MPO = decomposed row transfer matrix]

Physical meaning of W:

$$W_{\alpha\beta}^{\sigma\sigma'} = \text{local Boltzmann weight}$$

- σ, σ': spins in rows n, $n+1$
- α, β: auxiliary indices (encode horizontal bonds)

No Quantum Mechanics!

MPO is just a **factorization** of the classical transfer matrix.

Boundary Vector = Matrix Product State (MPS)

Recall: $Z = \mathrm{Tr}(T^M)$ requires summing over **boundary configurations**.

[Figure: MPS as boundary vector in $Z = \langle L|T^M|R\rangle$]

MPS structure:

$$|R\rangle_{\{\sigma\}} = \sum_{\text{aux}} A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_L}$$

- $|R\rangle$: vector in 2^L-dim config space
- A^{σ_i}: $\chi \times \chi$ matrices
- Total: only $L \cdot \chi^2 \cdot d$ parameters!

Physical Meaning

MPS **compresses** the 2^L boundary configurations into χ effective states.

Transfer Matrix RG: The Setup

Goal: Compute $Z = \mathrm{Tr}(T_{\mathrm{row}}^{M})$ for **infinite** L and M.

[Figure: L and R as compressed half-infinite environments]

Left Environment L**:**

Right Environment R**:**

TMRG Iteration: Step 1 — Grow (Absorb One Row)

Absorption: Multiply environment by one row transfer matrix.

[Figure: Absorb row transfer T into environments L, R]

Mathematically:

TMRG Iteration: Step 2 — Truncate (SVD)

Problem: Dimension grew to $\chi \cdot d$ — must compress back to χ!

Solution: Use **SVD** — a pure matrix factorization (no quantum mechanics!).

[Figure: SVD and truncation to χ largest]

SVD of environment product:

$$L' \cdot R'^T = U \Sigma V^\dagger$$

Truncate to χ largest:

$$P = U_{:,1:\chi}, \quad \tilde{P} = V_{:,1:\chi}$$

New compressed environments:

$$L_{\text{new}} = P^\dagger L', \quad R_{\text{new}} = R' \tilde{P}$$

RG Analogy

Why Does SVD Give the Best Truncation?

Theorem (Eckart-Young): SVD gives the **optimal low-rank approximation**.

[Figure: Singular value spectrum with truncation]

Physical interpretation:

- Large σ_i: configurations with **high Boltzmann weight**
- Small σ_i: configurations with **negligible contribution** to Z

Truncation error:

$$\epsilon = \sum_{i>\chi} \sigma_i^2 \Big/ \sum_i \sigma_i^2$$

Key Point

We keep configs that contribute **most to** Z,

TMRG: Convergence = RG Fixed Point

Iterate: Absorb $\rightarrow$ Truncate $\rightarrow$ Absorb $\rightarrow$ Truncate $\rightarrow \cdots$

[Figure: Convergence of dominant singular value to fixed point]

RG Interpretation

Fixed point: L^*, R^*

The RG transformation

Physical Meaning

L^*, R^* encode the **thermodynamic limit**:

Computing Z and Local Observables

With converged environments L^*, R^*:

Partition function (per site):

$$Z \propto \sigma_1^{(\text{area})}$$

where σ_1 = largest singular value.

Free energy per site:

$$f = -k_B T \ln \sigma_1^*$$

[Figure: Z from contracting L^*, R^*]

Local observable $\langle \sigma_i \rangle$:

Insert operator at site i:

$$\langle \sigma_i \rangle = \frac{L^* \cdot \sigma_i \cdot R^*}{L^* \cdot R^*}$$

[Figure: Insert σ to compute $\langle \sigma \rangle$]

Correlation functions:

2D Analog: Four Corners + Four Edges

Key Idea: Divide infinite 2D lattice into **4 corners** + **4 edges**.

[Figure: 2D environment structure — 4 corners + 4 edges]

The Local Tensor: Building Block

Local Boltzmann weight tensor a:

[Figure: Local tensor a with 4 legs]

For 2D Ising:

$$a_{u,d,l,r} = \sum_{\sigma} W_{\sigma,u} W_{\sigma,d} W_{\sigma,l} W_{\sigma,r}$$

where

$$W_{\sigma,\sigma'} = e^{\frac{\beta J}{2}\sigma\sigma'}$$

Physical Meaning

a encodes the Boltzmann weight at one site, with bond weights split symmetrically.

CTMRG Iteration: Step 1 — Grow (Absorb Row/Column)

Absorption: Add one row **and** one column to expand the environment.

[Figure: Absorption step — corner grows by absorbing a and edges]

CTMRG Iteration: Step 2 — Truncate (The Key Difference!)

Truncation in 2D: Use the **full environment** to determine projectors.

[Figure: Building density matrix from environment for truncation]

Compute projector P:

Apply truncation:

CTMRG: Complete Algorithm

[Figure: CTMRG algorithm flowchart]

Computing Observables with CTMRG

Partition function:

$$Z \propto \mathrm{Tr}(C_1 T_1 C_2 T_2 C_3 T_3 C_4 T_4)$$

[Figure: Computing $\langle \sigma \rangle$ by inserting operator]

Numerical Results: 2D Ising Model

[Results to be added: magnetization, critical behavior, etc.]

Summary: The Core of CTMRG

CTMRG in Three Steps

1. **Decompose:** Infinite 2D lattice $\rightarrow$ 4 corners $+$ 4 edges
2. **Grow:** Absorb local tensors (add row/column)
3. **Truncate:** SVD-based RG to keep χ most relevant states

[Figure: Visual summary — Decompose $\rightarrow$ Grow $\rightarrow$ Truncate]

Comparison: iDMRG vs CTMRG

Aspect	**iDMRG (1D)**	**CTMRG (2D)**
Dimension	1D chain	2D square lattice
Environment	Left + Right	4 Corners + 4 Edges
Grow step	Add site pair	Add row + column
Truncation	SVD on center bond	SVD on corner boundaries
Bond dimension	χ (MPS)	χ (environment)
Fixed point	L^*, R^*	C_i^*, T_i^*
Computational cost	$O(\chi^3)$	$O(\chi^6)$ or $O(\chi^5)$

Thank You!

Questions?

Appendix: No Monte Carlo Sign Problem

Classical models: All Boltzmann weights are **positive**!

$$W = e^{-\beta H} > 0 \quad \text{always}$$

[Figure: Classical (positive) vs Quantum (sign problem)]

Appendix: Why is 3D Difficult?

In 2D: Environment tensors are **1D objects** (edges).
In 3D: Environment would be **2D surfaces** — back to exponential!

[Figure: 2D boundary problem in 3D systems]

Appendix: Extensions and Generalizations

Beyond square lattice Ising:

Different lattices:

- Honeycomb
- Triangular
- Kagome

Different models:

- Potts model
- Clock model
- Vertex models

Quantum systems (via iPEPS):

- 2D Heisenberg model
- Frustrated magnets
- Topological phases

Improvements:

- Directional CTMRG
- Full-update vs simple-update
- Gradient optimization

Appendix: Computational Complexity

CTMRG scaling:

Operation	**Cost**	**Bottleneck?**
Corner absorption	$O(\chi^4 d^2)$	
Edge absorption	$O(\chi^3 d^2)$	
Build density matrix	$O(\chi^4)$	
SVD for projector	$O(\chi^3 d^3)$	$\checkmark$
Apply truncation	$O(\chi^3 d)$	
Total per iteration	$O(\chi^3 d^3)$ to $O(\chi^6)$	

Comparison

- iDMRG (1D): $O(\chi^3)$ per sweep
- CTMRG (2D): $O(\chi^5)$ – $O(\chi^6)$ per iteration
- More expensive, but still **polynomial** in χ!

Appendix: Key References

Original works:

- R. J. Baxter, *J. Math. Phys.* **9**, 650 (1968) — Corner transfer matrices
- R. J. Baxter, *J. Stat. Phys.* **19**, 461 (1978) — CTM method

Modern CTMRG:

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CTMRG for iPEPS:

- R. Orús & G. Vidal, *Phys. Rev. B* **80**, 094403 (2009)
- P. Corboz et al., *Phys. Rev. B* **84**, 041108(R) (2011)

Reviews:

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