Homework #5

Theodor Chakhachiro
UMID: 15801216
ROB501 - Mathematics for Robotics
University of Michigan, Ann Arbor

October 4, 2021

Problem 1

Part a

To calculate the eigenvectors of A_3, we first need to find their correspondent eigenvalues. Since A_3 is an upper triangular matrix, the eigenvalues of A_3 are its diagonal elements:

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = 2 \text{ and } \lambda_3 = 3$$

Solving for the eigenvector v_1 of $\lambda_1 = 1$, we get:

$$A_3 v_1 = \lambda_1 v_1 \implies (A_3 - \lambda_1 I)v_1 = 0$$

$$\begin{bmatrix} 1-\lambda_1 & 4 & 10 \\ 0 & 2-\lambda_1 & 0 \\ 0 & 0 & 3-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{bmatrix} = 0$$

Solving the system of equations, we get:

$$\left.\begin{aligned} 4v_1^2 + 10v_1^3 &= 0 \\ v_1^2 &= 0 \\ 3v_1^3 &= 0 \end{aligned}\right\} \implies v_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \forall t \in \mathbb{R}$$

Repeating the same procedure for v_2 and v_3, we obtain:

$$\begin{bmatrix} 1-\lambda_2 & 4 & 10 \\ 0 & 2-\lambda_2 & 0 \\ 0 & 0 & 3-\lambda_2 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix} = 0 \implies v_2 = \begin{bmatrix} 4s \\ s \\ 0 \end{bmatrix}, \forall s \in \mathbb{R}$$

And:

$$\begin{bmatrix} 1-\lambda_3 & 4 & 10 \\ 0 & 2-\lambda_3 & 0 \\ 0 & 0 & 3-\lambda_3 \end{bmatrix} \begin{bmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{bmatrix} = 0 \implies v_3 = \begin{bmatrix} \frac{10}{2}r \\ 0 \\ r \end{bmatrix} = \begin{bmatrix} 5r \\ 0 \\ r \end{bmatrix}, \forall r \in \mathbb{R}$$

To check whether the eigenvectors of A_3 are linearly independent, we write a linear combination of the eigenvectors and solve the system of equations as such:

$$\alpha_1 \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4s \\ s \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5r \\ 0 \\ r \end{bmatrix} = 0 \tag{1}$$

$$\left.\begin{aligned} \alpha_1 t + 4\alpha_2 s + 5\alpha_3 r = 0 \\ \alpha_2 s = 0 \\ \alpha_3 r = 0 \end{aligned}\right\} \implies \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus we have showed that the eigenvectors of A_3 are linearly independent since the only solution to (1) is the trivial solution. Computing the eigenvectors of A_3 in MATLAB, we notice that they are similar to the vectors computed by hand. In fact, for a specific set of values $\{t, s, r\}$, we check that relations between the elements of each vector are satisfied as such:

$$V = \text{eigv}(A_3) = \begin{bmatrix} V_1, V_2, V_3 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.9701 & 0.9806 \\ 0 & 0.2425 & 0 \\ 0 & 0 & 0.1961 \end{bmatrix}$$

For $S_1 = \{t, s, r\} = \{1, 0.2425, 0.1961\}$, the relations between the elements of $\{v_1, v_2, v_3\}$ are satisfied. Thus, the vectors computed by hand are equal to the ones obtained from MATLAB for the set of values S_1.

Part b

To calculate the eigenvectors of A_4, we first need to find their correspondent eigenvalues. Since A_4 is an upper triangular matrix, the eigenvalues of A_4 are its diagonal elements:

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \lambda_1 = 3 \text{ and } \lambda_2 = 2$$

Solving for the eigenvector v_1 of $\lambda_1 = 3$, we get:

$$A_4 v_1 = \lambda_1 v_1 \implies (A_4 - \lambda_1 I)v_1 = 0$$

$$\begin{bmatrix} 3 - \lambda_1 & 1 & 0 \\ 0 & 3 - \lambda_1 & 0 \\ 0 & 0 & 2 - \lambda_1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{bmatrix} = 0$$

Solving the system of equations, we get:

$$\left.\begin{aligned} v_1^2 = 0 \\ 0 = 0 \\ -v_1^3 = 0 \end{aligned}\right\} \implies v_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \forall t \in \mathbb{R}$$

Repeating the same procedure for v_2, we obtain:

$$\begin{bmatrix} 3-\lambda_2 & 1 & 0 \\ 0 & 3-\lambda_2 & 0 \\ 0 & 0 & 2-\lambda_2 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix} = 0 \implies v_2 = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}, \forall s \in \mathbb{R}$$

Since we only have two eigenvectors, we cannot represent every element $x \in \mathbb{R}^3$ because the basis of $\mathbb{R}^3$ requires 3 vectors, however, we will prove this using the definition of a basis. To check whether the eigenvectors of A_4 constitute a basis B for $\mathbb{R}^3$, we first check if they are linearly independent by writing a linear combination of the eigenvectors and solve the system of equations as such:

$$\alpha_1 \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = 0 \tag{2}$$

$$\left.\begin{aligned} \alpha_1 t = 0 \\ 0 = 0 \\ \alpha_2 s = 0 \end{aligned}\right\} \implies \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus we have showed that the eigenvectors of A_4 are linearly independent since the only solution to (2) is the trivial solution. Now, we also need to check if span(B)=$\mathbb{R}^3$. In order to do this, we need to show that $\forall x \in \mathbb{R}^3$, $\exists \alpha_i$ s.t. $\sum_{i=0}^{n} \alpha_i v_i - x_i = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}$$

Solving the system of equations, we get:

$$\left.\begin{aligned} x_1 &= \alpha_1 t \\ x_2 &= 0 \\ x_3 &= \alpha_2 s \end{aligned}\right\} \implies x = \begin{bmatrix} \alpha_1 t \\ 0 \\ \alpha_2 s \end{bmatrix} = \begin{bmatrix} \tilde{t} \\ 0 \\ \tilde{s} \end{bmatrix} \quad \forall \tilde{t}, \tilde{s} \in \mathbb{R}$$

Thus we can only represent the vectors $x \in \mathbb{R}^3$ whose second element is zero (i.e: $x_2 = 0$). Therefore, we cannot constitute a basis for $\mathbb{R}^3$ using the eigenvectors of A_4.

Problem 2

STATEMENT: If A and B are similar matrices, then they have the same characteristic equation, hence the same eigenvalues.

PROOF: To prove the above statement, we use a direct proof as such:

$$\begin{aligned}
\det(\lambda I - B) &= \det(\lambda I - P^{-1}AP) \\
&= \det(\lambda P^{-1}P - P^{-1}AP) \\
&= \det(P^{-1}(\lambda P - AP)) \\
&= \det(P^{-1}(\lambda I - A)P) \\
&= \det(P^{-1})\det(\lambda I - A)\det(P) \\
&= \frac{1}{\det(P)}\det(P)\det(\lambda I - A) \\
&= \det(\lambda I - A)
\end{aligned}$$

Thus, we have shown that the two similar matrices have the same characteristic polynomial, hence the same characteristic equation and eigenvalues λ.

Problem 3

To show that A_3 is similar to a diagonal matrix Λ, it suffices to show that there exist an invertible matrix M such that $\Lambda = M^{-1}AM$. Let Λ be a diagonal matrix whose elements are the eigenvalues of A_3 denoted as such:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Let M be a matrix in $\mathbb{R}^3$ whose columns are the eigenvectors of A_3 such that $M = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, then M is invertible since the eigenvectors of A_3 are linearly independent. We have:

$$AM = \begin{bmatrix} Av_1 & Av_2 & Av_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \lambda_3 v_3 \end{bmatrix}$$

$$\implies AM = M \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = M\Lambda$$

Thus, we can find a diagonal matrix $\Lambda = M^{-1}AM$ who is similar to A_3 where the elements of the diagonal matrix are the eigenvalues of A_3.

Problem 4

Part a-b

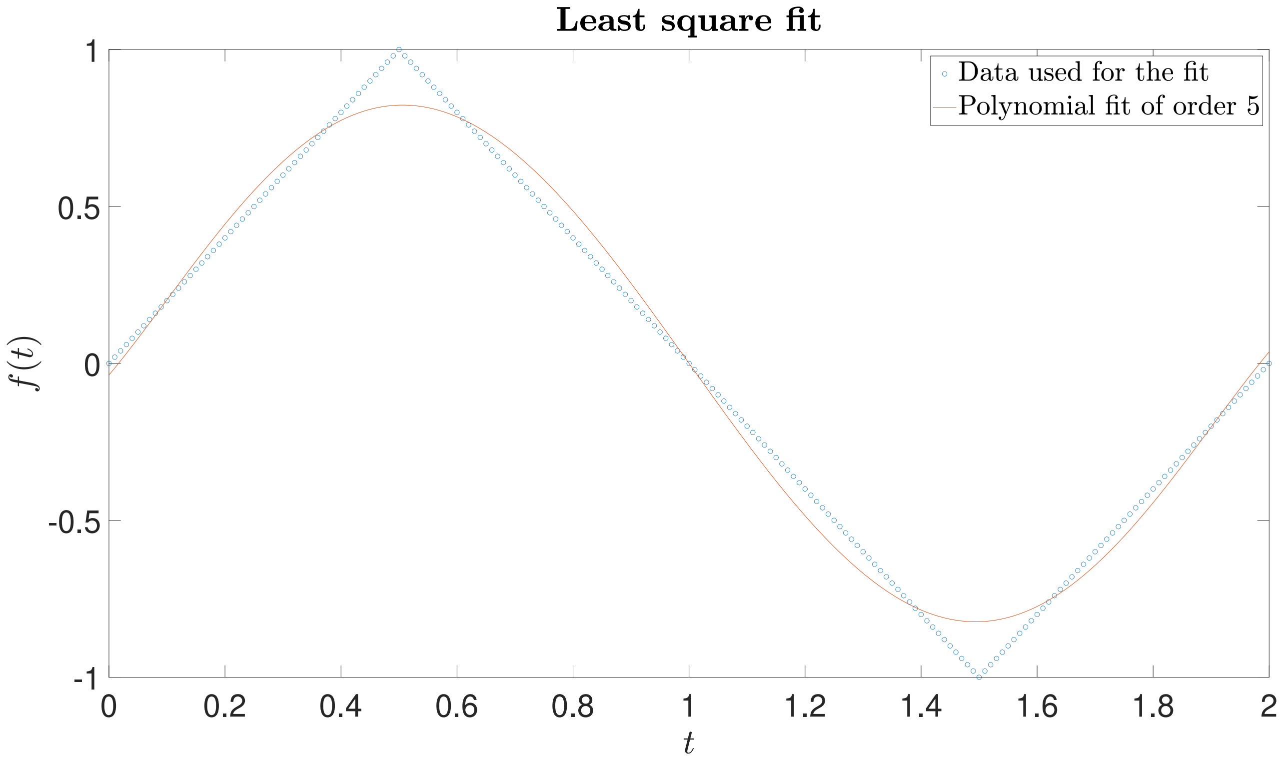

Figure 1: Goodness of Fit

From MATLAB, the polynomial $P(t)$ and the computed coefficients of the fit of order 5 are tabulated below:

Table 1: Coefficients of the Polynomial $P(t)$

Coefficient	**Value**
c_0	-0.0370
c_1	-0.0370
c_2	2.8901
c_3	-11.2271
c_4	7.6978
c_5	-1.5396

$$P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$$

$$\boxed{\implies P(t) = -0.0370 - 0.0370t + 2.8901t^2 - 11.2271t^3 + 7.6978t^4 - 1.5396t^5}$$

Part c

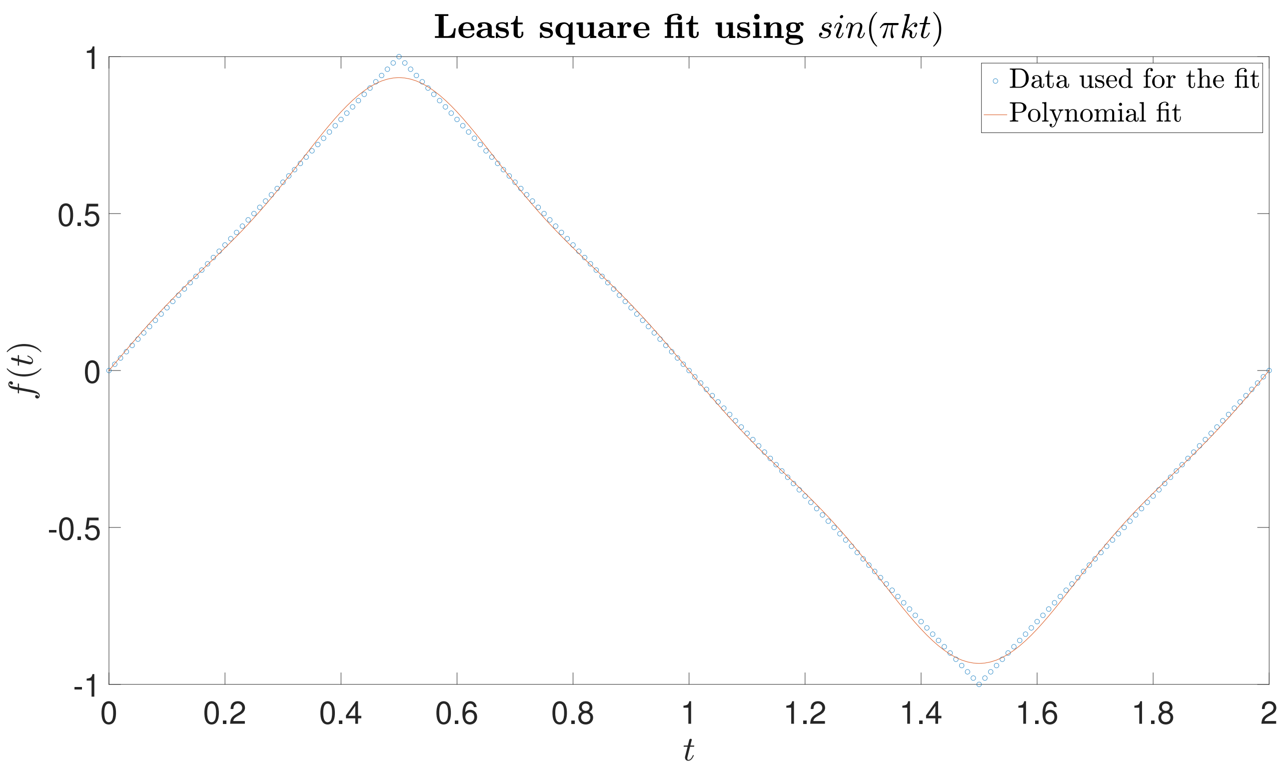

Figure 2: Goodness of Fit using $sin(\pi kt)$

From MATLAB, the polynomial $\tilde{P}(t)$ and the computed coefficients of the fit are tabulated below:

Table 2: Coefficients of the Polynomial $\tilde{P}(t)$

Coefficient	**Value**
c_1	0.8106
c_2	-3.7016e-17 ≈ 0
c_3	-0.0901
c_4	3.6312e-17 ≈ 0
c_5	0.0325

$$\tilde{P}(t) = c_1 sin(\pi t) + c_2 sin(\pi 2t) + c_3 sin(\pi 3t) + c_4 sin(\pi 4t) + c_5 sin(\pi 5t)$$

$$\boxed{\implies \tilde{P}(t) = 0.8106 sin(\pi t) - 0.0901 sin(\pi 3t) + +0.0325 sin(\pi 5t)}$$

Problem 5

From the plot of the data given, we notice the data can fit through an exponential function as well as a tangent function. Thus these two functions will be used and their respective polynomials are denoted below:

$$E(t) = a_0 + a_1 e^t + a_2 e^{t^2}$$
$$T(t) = t_0 + t_1 \tan(t) + t_2 \tan(t^2)$$

The plot of the data as well as the two fits is shown:

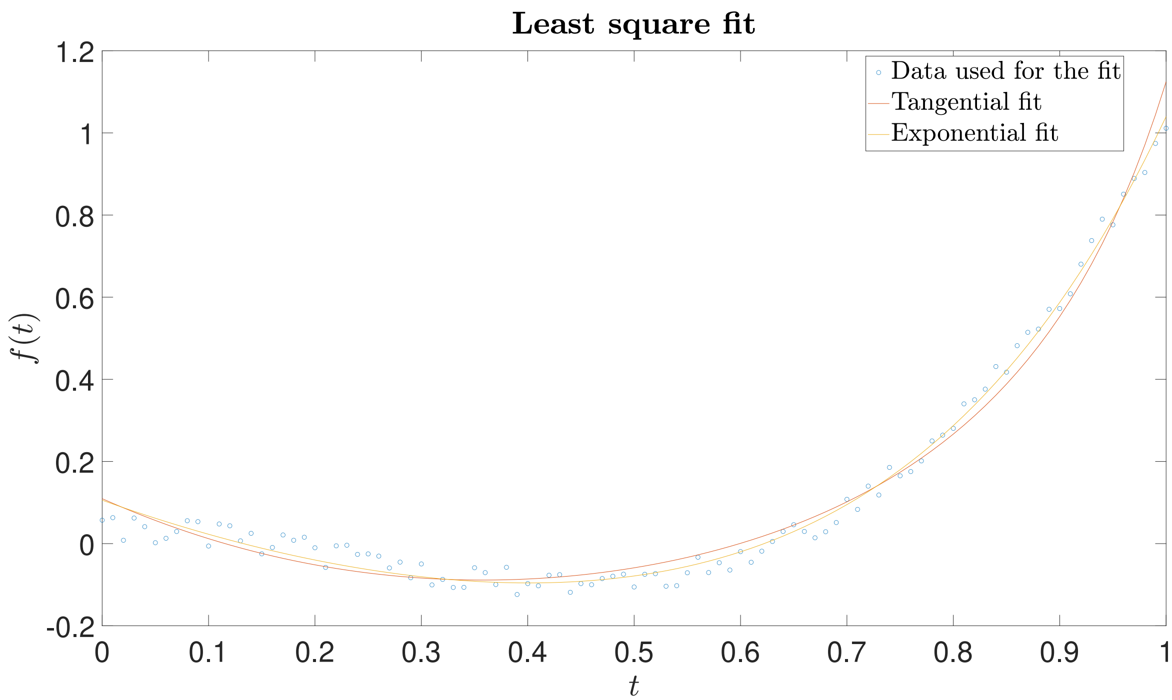

Figure 3: Goodness of Fit using $sin(\pi kt)$

From MATLAB, the polynomials $E(t)$ and $T(t)$ as well as the computed coefficients of each fit are tabulated below:

Table 3: Coefficients of Polynomials $E(t)$ and $T(t)$

Coefficients of $E(t)$	**Value**	**Coefficients of $T(t)$**	**Value**
a_0	-0.4379	t_0	0.1095
a_1	-0.9292	t_1	-1.1508
a_2	1.4729	t_2	1.8026

$$\boxed{E(t) = -0.4379 - 0.9292e^t + 1.4729e^{t^2}}$$

$$\boxed{T(t) = 0.1095 - 1.1508\tan(t) + 1.8026\tan(t^2)}$$

Now computing the derivative at $t = 0.3$, we get:

$$\dot{E}(t) = -0.9292e^t + 2 * 1.4729 * te^{t^2} \implies \dot{E}(0.3) = -0.2873$$
$$\dot{T}(t) = -1.1508\sec^2(t) + 2 * 1.8026 * t\sec^2(t^2) \implies \dot{T}(0.3) = -0.1709$$

Problem 6

DAVID LUENBERGER DEFINITION:

To show that the definition $< x, y >= x^T\bar{y}$ satisfies the definition of inner product proposed by David Luenberger, we show the following three properties:

A. $\forall\, x, y \in \mathbb{C}^n,\ < x, y >= \overline{< y, x >}$

B. $\forall\, x_1, x_2, y \in \mathbb{C}^n, \forall\alpha_1, \alpha_2 \in \mathbb{C},\ < \alpha_1 x_1 + \alpha_2 x_2, y >= \alpha_1 < x_1, y > +\alpha_2 < x_2, y >$

C. $\forall x \in \mathbb{C}^n, < x, x > \geq 0$ and $< x, x >= 0 \Leftrightarrow x = 0$

PROOF:

A. $< x, y >= x^T\bar{y}$, and $\overline{< y, x >} = \overline{y^T\bar{x}} = \bar{y}^T\bar{\bar{x}} = \bar{y}^T x = x^T\bar{y} =< x, y >$

B. $< \alpha_1 x_1 + \alpha_2 x_2, y >= (\alpha_1 x_1 + \alpha_2 x_2)^T\bar{y} = (\alpha_1^T x_1^T + \alpha_2^T x_2^T)\bar{y} = \alpha_1 x_1^T\bar{y} + \alpha_2 x_2^T\bar{y}$
$= \alpha_1 < x_1, y > +\alpha_2 < x_2, y >$

C. $< x, x >= x^T\bar{x} = \begin{bmatrix}\alpha_1 + \beta_1 j & \alpha_2 + \beta_2 j & \ldots & \alpha_n + \beta_n j\end{bmatrix}\begin{bmatrix}\alpha_1 - \beta_1 j \\ \alpha_2 - \beta_2 j \\ \vdots \\ \alpha_n - \beta_n j\end{bmatrix}$
$= (\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + \ldots + (\alpha_n^2 + \beta_n^2) = ||x||^2 \geq 0$ and $(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + \ldots + (\alpha_n^2 + \beta_n^2) = 0 \Leftrightarrow \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n = 0$

NAGY DEFINITION:

To show that the definition $< x, y >= \bar{x}^T y$ satisfies the definition of inner product proposed by nagy, we show the following two properties $\forall x, y, z \in \mathbb{C}^n$ and $\forall a, b \in \mathbb{C}$:

A. $< x, y >= \overline{< y, x >}$

B. $< x, (ay + bz) >= a < x, y > +b < x, z >$

C. $< x, x > \geq 0$ and $< x, x >= 0 \Leftrightarrow x = 0$

PROOF:

A. $< x, y >= \bar{x}^T y$ and $\overline{< y, x >} = \overline{\bar{y}^T x} = y^T\bar{x} = \bar{x}^T y =< x, y >$

B. $< x, (ay + bz) >= \bar{x}^T (ay + bz) = a\bar{x}^T y + b\bar{x}^T z = a < x, y > +b < x, z >$

C. $< x, x >= \bar{x}^T x \begin{bmatrix} \alpha_1 - \beta_1 j & \alpha_2 - \beta_2 j & \dots & \alpha_n - \beta_n j \end{bmatrix} \begin{bmatrix} \alpha_1 + \beta_1 j \\ \alpha_2 + \beta 2j \\ \vdots \\ \alpha_n + \beta_n j \end{bmatrix}$
$= (\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + \ldots + (\alpha_n^2 + \beta_n^2) = ||x||^2 \geq 0$ and $(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + \ldots + (\alpha_n^2 + \beta_n^2) = 0 \Leftrightarrow \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n = 0$

Problem 7

$$< p_0, p_3 >= \int_{-1}^{1} \frac{1}{2}(5x^3 - 3x)dx = \frac{1}{2}[\frac{5}{4}x^4 - \frac{3}{2}x^2]_{-1}^{1} = (\frac{5}{8} - \frac{3}{4}) - (\frac{5}{8} - \frac{3}{4}) = 0$$

$$< p_1, p_2 >= \int_{-1}^{1} \frac{1}{2}(3x^3 - x)dx = \frac{1}{2}[\frac{3}{4}x^4 - \frac{1}{2}x^2]_{-1}^{1} = (\frac{3}{8} - \frac{1}{4}) - (\frac{3}{8} - \frac{1}{4}) = 0$$

Problem 8

Part a

We will prove the statement by multplying the matrix $(A + BCD)$ by its inverse using the matrix inversion lemma and show that their product is the identity matrix:

$$\begin{aligned}
P &= (A + BCD)(A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}) \\
&= (I - B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}) + (BCDA^{-1} - BCDA^{-1}B(C^{-1} + DA^{-1}U)^{-1}DA^{-1}) \\
&= (I + BCDA^{-1}) - (B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} + BCDA^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}) \\
&= I + BCDA^{-1} - (B + BCDA^{-1}U)(C^{-1} + VA^{-1}U)^{-1}DA^{-1} \\
&= I + BCDA^{-1} - BC(C^{-1} + DA^{-1}B)(C^{-1} + DA^{-1}B)DA^{-1} \\
&= I + BCDA^{-1} - BCDA^{-1} \\
&= I
\end{aligned}$$

Part b

We have:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \quad C = 0.2 \quad D = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix}$$

With:

$$C^{-1} = \frac{1}{0.2} = 5 \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Thus we can find the inverse using the matrix inversion lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} -$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \{5 + \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \}^{-1} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 6 \end{bmatrix} \{5 + \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 6 \end{bmatrix} \}^{-1} \begin{bmatrix} 1 & 0 & 4 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 6 \end{bmatrix} \{5 + 27\}^{-1} \begin{bmatrix} 1 & 0 & 4 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \frac{1}{32} \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \frac{1}{32} \begin{bmatrix} 1 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 16 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 24 & 0 & 36 \end{bmatrix} = \boxed{\begin{bmatrix} 31/32 & 0 & -1/8 & 0 & -3/16 \\ 0 & 2 & 0 & 0 & 0 \\ -1/8 & 0 & 3/2 & 0 & -3/4 \\ 0 & 0 & 0 & 1 & 0 \\ -3/16 & 0 & -3/4 & 0 & 7/8 \end{bmatrix} = (A + BCD)^{-1}}$$

Problem 9

Part a

To check whether $f(x) = (x^T Ax)^{\frac{1}{2}}$ is a norm, we check if the properties of the norm are satisfied for $f(x)$ as follows:

A. $\forall x \in \mathbb{R}^2, ||x|| \geq 0$ and $||x|| = 0 \Leftrightarrow x = 0$

B. $\forall x, y \in \mathbb{R}^2,\ ||x + y|| \leq ||x|| + ||y||$

C. $\forall \alpha \in \mathbb{R},\ \forall x \in \mathbb{R}^2,\ ||\alpha x|| = |\alpha|.||x||$

PROOF:

A. Since A is a positive definite matrix, then $\forall x \in \mathbb{R}^2 - \{0\},\ x^T Ax > 0$, thus $\forall x \in \mathbb{R}^2 - \{0\},\ f(x) > 0$ and $f(x) = 0 \Leftrightarrow x = 0$

B. $||x + y||^2 = f(x + y)^2 = ((x + y)^T A(x + y)) = (x^T Ax + y^T Ay + x^T Ay + y^T Ax)$. Since A is symmetric positive definite, then $x^T Ay = y^T Ax$. Thus:

$$||x + y||^2 = (x^T Ax + y^T Ay + 2x^T Ay)$$

Now:

$$(||x|| + ||y||)^2 = x^T Ax + y^T Ay + 2(x^T Ax)^{\frac{1}{2}}(y^T Ay)^{\frac{1}{2}}$$

From the properties of a symmetric positive definite matrix A, we use the Cholesky decomposition on A with S a lower triangular matrix whose diagonal elements are positive:

$$\exists! S \in \mathbb{R}^{n\times n} \quad \text{s.t.} \quad A = SS^T$$

Then we can write:

$$x^T Ay = x^T SS^T y = (S^T x)^T (S^T y)$$

Invoking the Cauchy Schwarz Inequality since every inner product induces a norm, we get:

$$|(S^T x)^T (S^T y)| \leq ||S^T x||.||S^T y||$$

Where:

$$||S^T x|| = (x^T SSx)^{\frac{1}{2}} = (x^T Ax)^{\frac{1}{2}} \quad \text{and} \quad ||S^T y|| = (y^T SSy)^{\frac{1}{2}} = (y^T Ay)^{\frac{1}{2}}$$

Then:

$$|x^T Ay| \leq (x^T Ax)^{\frac{1}{2}}(y^T Ay)^{\frac{1}{2}}$$

Comparing (B) and (B), we can see that the triangular inequality holds for $f(x)$.

C.

$$||\alpha x|| = f(\alpha x) = ((\alpha x)^T A(\alpha x))^{\frac{1}{2}} = (\alpha x^T A\alpha x)^{\frac{1}{2}} = (\alpha^2 x^T Ax)^{\frac{1}{2}} = |\alpha|(x^T Ax)^{\frac{1}{2}} = |\alpha|.||x||$$

Thus, we have shown that $f(x) = (x^T Ax)^{\frac{1}{2}}$ is a norm in $(\mathbb{R}^2, \mathbb{R})$.

Part b

ONE-NORM:

For $A = [A_1|A_2|\ldots|A_n]$ where A_i are the column vectors of A, and $x = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$, we have:

$$f_1(A) = \sup_{x\in\mathbb{R}^n,\ ||x||_1=1} ||Ax||_1$$

Where:

$$||Ax||_1 = || \begin{bmatrix} A_1 & A_2 & \ldots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ||_1 = ||A_1x_1 + A_2x_2 + \ldots + A_nx_n||_1$$

Applying the triangular inequality, we have:

$$\begin{aligned} ||Ax||_1 &\leq ||A_1x_1||_1 + ||A_2x_2||_1 + \ldots + ||A_nx_n||_1 \\ ||Ax||_1 &\leq ||A_1||_1.|x_1| + ||A_2||_1.|x_2| + \ldots + ||A_n||_1.|x_n| \end{aligned}$$

We define:

$$\begin{aligned} A^* &= \text{argmax} \quad A \\ &= \underset{j=1,\ldots,n}{\text{argmax}} \quad [A_1|A_2|\ldots|A_n] \end{aligned}$$

Then:

$$\begin{aligned} \max_{x\in\mathbb{R}^n,\ ||x||_1=1} ||Ax||_1 \quad &\leq \quad ||A^*||_1.|x_1| + ||A*||_1.|x_2| + \ldots + ||A*||_1.|x_n| \\ \max_{x\in\mathbb{R}^n,\ ||x||_1=1} ||Ax||_1 \quad &\leq \quad ||A^*||_1(|x_1| + |x_2| + \ldots + |x_n|) \\ \max_{x\in\mathbb{R}^n,\ ||x||_1=1} ||Ax||_1 \quad &\leq \quad ||A^*||_1.||x||_1 \\ \max_{x\in\mathbb{R}^n,\ ||x||_1=1} ||Ax||_1 \quad &\leq \quad ||A^*||_1 \end{aligned}$$

Where:

$$||A^*||_1 = || \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix} ||_1 = \sum_{i=1}^{n} |a_i^*|$$

This means, that the supremum of the one-norm of Ax is equal to the one-norm of the column vector A^* of A whose one-norm is the greatest, i.e. the sum of the absolute values of its elements $|a_i^*|$. We can thus write:

$$\boxed{f_1(A) = \sup_{x\in\mathbb{R}^n,\ ||x||_1=1} ||Ax||_1 = \max_{A_1\leq A^*\leq A_n} ||A^*||_1 = \max_{A_1\leq A^*\leq A_n} \sum_{i=1}^{n} |a_i^*|} \quad \square$$

∞-NORM:

For $A = [A_1|A_2|\dots|A_n]^T$ where A_i are the row vectors of A, and $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$, we have:

$$f_1(A) = \sup_{x \in \mathbb{R}^n,\ ||x||_\infty = 1} ||Ax||_\infty$$

Where:

$$||Ax||_\infty = || \begin{bmatrix} A_1 \\ A_2 \\ \dots \\ A_n \end{bmatrix} x||_\infty = || \begin{bmatrix} A_1 x \\ A_2 x \\ \dots \\ A_n x \end{bmatrix} ||_\infty$$

Thus:

$$||Ax||_\infty = \max_{1 \le i \le n} |A_i x|$$

$$||Ax||_\infty = \max_{1 \le i \le n} |\sum_{j=1}^{n} a_{i,j} x_j|$$

$$||Ax||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}|.|x_j|$$

$$||Ax||_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}| \ \max_{k} |x_k|$$

$$||Ax||_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}|.||x||_\infty$$

We can then write:

$$\max_{x \in \mathbb{R}^n,\ ||x||_\infty = 1} ||Ax||_\infty \quad \le \quad \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}|.||x||_\infty$$

$$\max_{x \in \mathbb{R}^n,\ ||x||_\infty = 1} ||Ax||_\infty \quad \le \quad \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}|$$

This means, that the supremum of the ∞-norm of Ax is equal to the one-norm of the row vector of A whose one-norm is the greatest, i.e. the sum of the absolute values of its elements $|a_{i,j}|$. We can write:

$$\boxed{f_1(A) = \sup_{x \in \mathbb{R}^n,\ ||x||_\infty = 1} ||Ax||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{i,j}|} \quad \square$$