Homework #6

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Problem 1

We define the set of vectors $v_i | i = 1, \dots, 3$ in $\mathbb{R}^3$ such that:

$$
\begin{aligned}
v_1 &= y_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
v_2 &= y_2 - a_{21} v_1 \ , \quad \text{where} \quad a_{21} = \frac{\langle v_1, y_2 \rangle}{||v_1||^2} \\
v_3 &= y_3 - a_{31} v_1 - a_{32} v_2 \ , \quad \text{where} \quad a_{31} = \frac{\langle v_1, y_3 \rangle}{||v_1||^2} \quad \text{and} \quad a_{32} = \frac{\langle v_2, y_3 \rangle}{||v_2||^2}
\end{aligned}
$$

Thus, we can compute the orthogonal vector v_2 as such:

$$
\begin{aligned}
a_{21} &= \frac{\langle v_1, y_2 \rangle}{||v_1||^2} = \frac{v_1^T y_2}{||v_1||^2} = \frac{1}{1^2 + 2^2 + 1^2} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \frac{4-1}{1+4+1} = \frac{3}{6} = \frac{1}{2} \\
\implies v_2 &= y_2 - a_{21} v_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - \frac{1}{2} \\ 1 \\ -1 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 1 \\ -\frac{3}{2} \end{bmatrix} \\
\implies v_2 &= \frac{1}{2} \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}
\end{aligned}
$$

Computing v_3, we get:

$$a_{31} = \frac{\langle v_1, y_3 \rangle}{||v_1||^2} = \frac{v_1^T y_3}{||v_1||^2} = \frac{1}{1^2+2^2+1^2}\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}-2\\2\\3\end{bmatrix} = \frac{-2-4+3}{1+4+1} = -\frac{3}{6} = -\frac{1}{2}$$

$$a_{32} = \frac{\langle v_2, y_3 \rangle}{||v_2||^2} = \frac{v_2^T y_3}{||v_2||^2} = \frac{1}{(\frac{7}{2})^2+1^2+(-\frac{3}{2})^2}\begin{bmatrix}\frac{7}{2} & 1 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}-2\\2\\3\end{bmatrix} = \frac{-7+2-\frac{9}{2}}{\frac{49}{4}+1+\frac{9}{4}} = \frac{-\frac{19}{2}}{\frac{31}{2}} = -\frac{19}{31}$$

$$\implies v_3 = y_3 - a_{31}v_1 - a_{32}v_2 = \begin{bmatrix}-2\\2\\3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-2\\1\end{bmatrix} + \frac{19}{31}\begin{bmatrix}\frac{7}{2}\\1\\-\frac{3}{2}\end{bmatrix} = \begin{bmatrix}-2+\frac{1}{2}+\frac{19}{31}\cdot\frac{7}{2}\\2-\frac{2}{2}+\frac{19}{31}\\3+\frac{1}{2}-\frac{19}{31}\cdot\frac{3}{2}\end{bmatrix} = \begin{bmatrix}\frac{20}{31}\\\frac{50}{31}\\\frac{80}{31}\end{bmatrix}$$

$$\implies v_3 = \frac{10}{31}\begin{bmatrix}2\\5\\8\end{bmatrix}$$

By using the Gram-Schmidt Process, we obtain the following set of orthogonal vectors:

$$\{v_1, v_2, v_3\} = \left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}7\\2\\-3\end{bmatrix}, \frac{10}{31}\begin{bmatrix}2\\5\\8\end{bmatrix}\right\}$$

Problem 2

In this problem, we use the following regression basis to compute the estimate of the derivative denoted by $\frac{d\hat{y}(t)}{dt}$:

$$\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\} = \left\{1\ ,\ t\ ,\ t^2\right\}$$

Thus, the polynomial function used for the derivative fit is displayed below:

$$P(t) = \sum_{i=1}^{3} c_{i-1}\varphi_i(t) = c_0 + c_1 t + c_2 t^2$$

And:

$$\dot{P}(t) = c_1 + 2c_2 t$$

where the coefficients of the polynomial are evaluated at each time window $T_k = \{t_{k-M+1}, \ldots, t_k\}$.

Part a

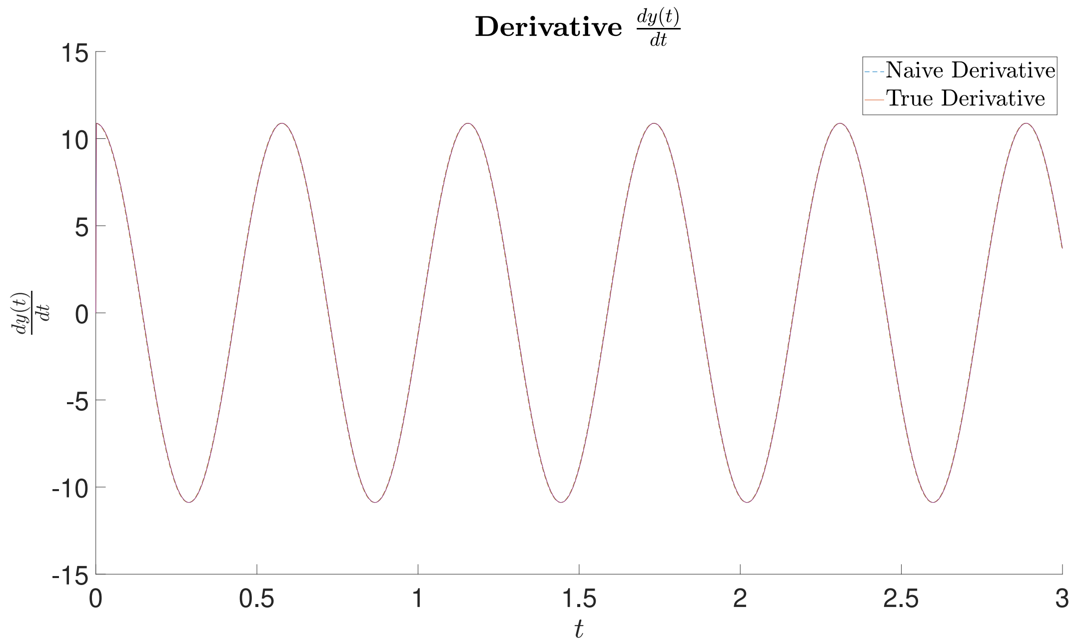

Figure 1: Comparison between Naive and True Derivative

Part b

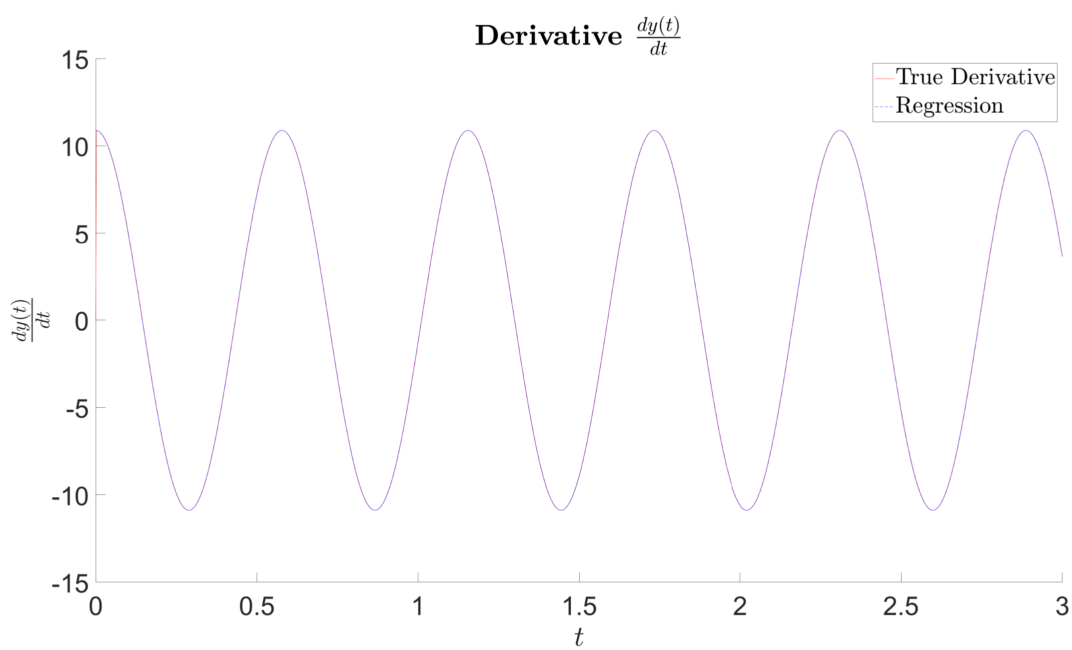

Figure 2: Comparison between Regression and True Derivative

The matlab code used to solve this part is displayed below:

```matlab
%% Initialize
clear all
clc
load DataHW06_Prob2.mat;
dt = t(2) - t(1);
%% Part a
dy_naive = diff(y)/dt;
hold on
plot(t(1:end-1),dy_naive,'--');
plot(t,dy);
legend('Naive Derivative','True Derivative','Interpreter','latex')
title('\textbf{Derivative $\frac{dy(t)}{dt}$}','Interpreter','latex')
xlabel('$t$','Interpreter','latex')
ylabel('$\frac{dy(t)}{dt}$','Interpreter','latex')
set(gca,'fontsize',40)
%% Part b
y_test=y;
dy_test=dy;
M=3;
figure(2)
hold on
for k=M:length(t)
    Y_k=y_test(k-M+1:k);
    T_k=t(k-M+1:k);
    dY_k=dy_test(k-M+1:k);
    N=length(Y_k);
    A=[ones(N,1) T_k T_k.^2];
    alpha_hat = inv(A'*A)*A'*Y_k;
    c0 = alpha_hat(1);
    c1 = alpha_hat(2);
    c2 = alpha_hat(3);
  plot(T_k,dY_k,'-r',T_k,c1+2*c2*T_k,'--b');
end
axis([t(1) t(end) min(dy) max(dy)]);
legend('True Derivative','Regression','Interpreter','latex')
title('\textbf{Derivative $\frac{dy(t)}{dt}$}','Interpreter','latex')
xlabel('$t$','Interpreter','latex')
ylabel('$\frac{dy(t)}{dt}$','Interpreter','latex')
set(gca,'fontsize',40)
%% End
```

Problem 3

In this problem, we use the following regression basis to compute the estimate of the derivative denoted by $\frac{d\hat{y}(t)}{dt}$:

$$\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\} = \left\{1\ ,\ t\ ,\ t^2\right\}$$

Thus, the polynomial function used for the derivative fit is displayed below:

$$P(t) = \sum_{i=1}^{3} c_{i-1}\varphi_i(t) = c_0 + c_1 t + c_2 t^2$$

And:

$$\dot{P}(t) = c_1 + 2c_2 t$$

where the coefficients of the polynomial are evaluated at each time window $T_k = \{t_{k-M+1}, \ldots, t_k\}$.

Part a

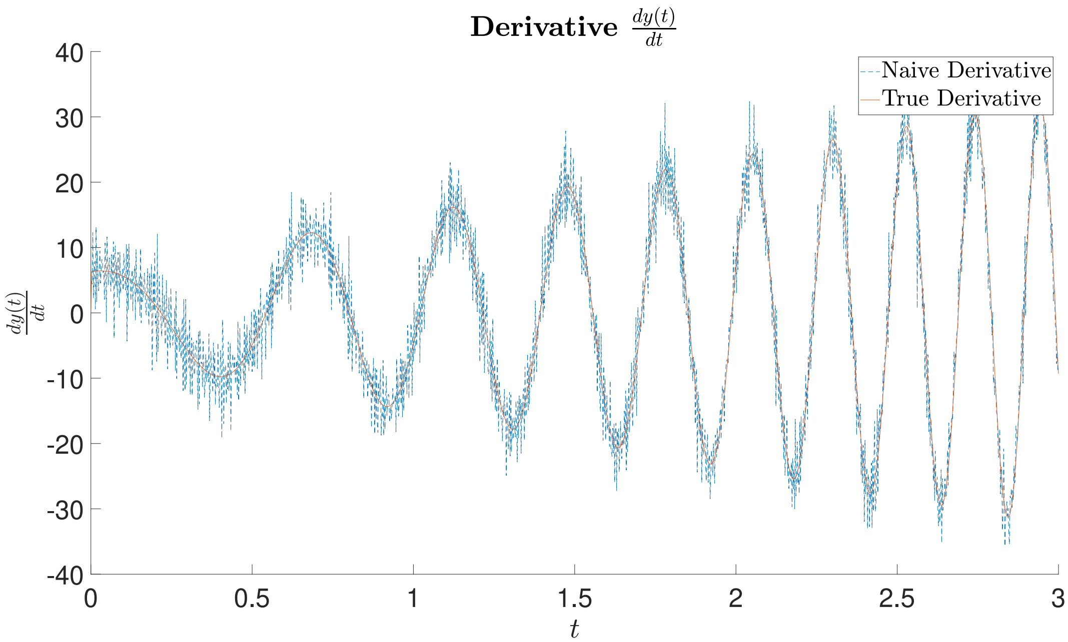

Figure 3: Comparison between Naive and True Derivative

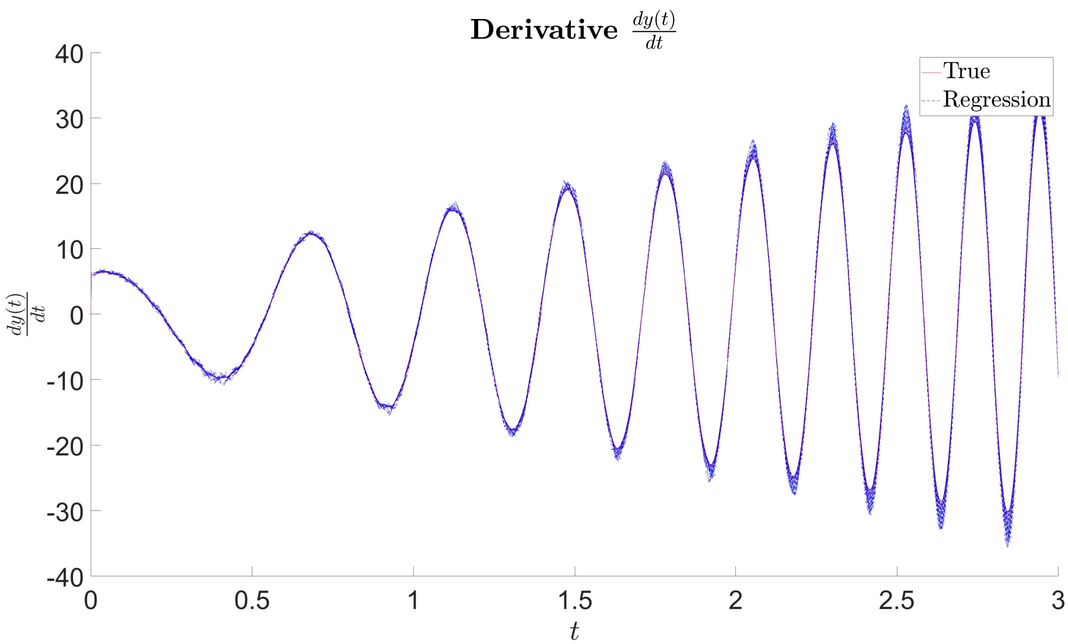

Figure 4: Comparison between Regression and True Derivative

Part b

To compute the root mean square error between the true derivative and its regression estimate, we use two methods. The first method, uses a time window that moves M steps in order to avoid any overlap of data. The second method, overwrites the already computed data from the time window at each time step. The results are tabulated below:

Table 1: RMSE of the estimated derivated

Method	RMSE
1	0.7417
2	1.0878

The matlab code used to solve this part is displayed below:

```
%% Initialize
clear all
clc
load DataHW06_Prob3.mat;
dt = t(2) - t(1);
%% Part a
dy_naive = diff(y)/dt;
hold on
plot(t(1:end-1),dy_naive,'--');
plot(t,dy);
legend('Naive Derivative','True Derivative','Interpreter','latex')
title('\textbf{Derivative $\frac{dy(t)}{dt}$}','Interpreter','latex')
xlabel('$t$','Interpreter','latex')
ylabel('$\frac{dy(t)}{dt}$','Interpreter','latex')
set(gca,'fontsize',40)
%% Part b
Y_derivative = [];
Y_derivative_1 = [];
y_test = y;
dy_test = dy;
M = 10;
figure(2)
hold on
% for k = M:M:length(t) % Method 1
for k = M:length(t) % Method 2
    Y_k = y_test(k-M+1:k);
    T_k = t(k-M+1:k);
    dY_k = dy_test(k-M+1:k);
    N = length(Y_k);
    A = [ones(N,1) T_k T_k.^2];
    alpha_hat = inv(A'*A)*A'*Y_k;
    c0 = alpha_hat(1);
    c1 = alpha_hat(2);
    c2 = alpha_hat(3);
    plot(T_k,dY_k,'r',T_k,c1+2*c2*T_k,'--b');
%     Y_derivative = [Y_derivative;c1+2*c2*T_k];
    Y_derivative_1(k-M+1:k) = c1+2*c2*T_k;
end
legend('True','Regression','Interpreter','latex')
title('\textbf{Derivative $\frac{dy(t)}{dt}$}','Interpreter','latex')
xlabel('$t$','Interpreter','latex')
ylabel('$\frac{dy(t)}{dt}$','Interpreter','latex')
set(gca,'fontsize',40)
%% Calculate RMSE
%Method 1
% error_m1=sqrt(1/length(Y_derivative)*sum((Y_derivative-dy(1:end-1)).^2));
```

```
%Method 2
error_m2=sqrt(1/length(Y_derivative_1)*sum((Y_derivative_1'-dy).^2));
%% End
```

Problem 4

To solve the minimization problem defined below, we first compute the Gram matrix:

$$\hat{x} = \underset{y \in M}{\arg\min} \; ||x - y||$$

The solution is of the form:

$$\hat{x} = \alpha_1 y^1 + \alpha_2 y^2$$

Where the coefficients α_i are computed as follows:

$$\alpha = G^{-T}\beta$$
$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \langle y^1, y^1 \rangle & \langle y^1, y^2 \rangle \\ \langle y^2, y^1 \rangle & \langle y^2, y^2 \rangle \end{bmatrix}^{-T} \begin{bmatrix} \langle x, y^1 \rangle \\ \langle x, y^2 \rangle \end{bmatrix}$$
$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \text{trace}((y^1)^T y^1) & \text{trace}((y^1)^T y^2) \\ \text{trace}((y^2)^T y^1) & \text{trace}((y^2)^T y^2) \end{bmatrix}^{-T} \begin{bmatrix} \text{trace}(x^T y^1) \\ \text{trace}(x^T y^2) \end{bmatrix}$$

Where:

$$G_{1,1} = \text{trace}((y^1)^T y^1) = \text{trace}\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\} = 5$$

$$G_{1,2} = \text{trace}((y^1)^T y^2) = \text{trace}\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} = 3$$

$$G_{2,1} = \text{trace}((y^2)^T y^1) = \text{trace}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\} = 3$$

$$G_{2,2} = \text{trace}((y^1)^T y^1) = \text{trace}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} = 4$$

$$\beta_1 = \text{trace}(x^T y^1) = \text{trace}\left\{ \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\} = 4$$

$$\beta_2 = \text{trace}(x^T y^2) = \text{trace}\left\{ \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} = 1$$

Thus, we obtain:

$$\begin{aligned}
\alpha &= G^{-T}\beta \\
\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} &= \begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix}^{-T}\begin{bmatrix}4\\ 1\end{bmatrix} \\
\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} &= \begin{bmatrix}\frac{4}{11} & -\frac{3}{11}\\ -\frac{3}{11} & \frac{5}{11}\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix} \\
\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} &= \begin{bmatrix}\frac{13}{11}\\ -\frac{7}{11}\end{bmatrix}
\end{aligned}$$

Finally, we can solve for $\hat{x}$:

$$\begin{aligned}
\hat{x} &= \alpha_1 y^1 + \alpha_2 y^2 \\
\hat{x} &= \frac{13}{11}\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix} - \frac{7}{11}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \\
\hat{x} &= \frac{1}{11}\begin{bmatrix}6 & -7\\ 19 & -7\end{bmatrix}
\end{aligned}$$

Thus:

$$\hat{x} = \frac{1}{11}\begin{bmatrix}6 & -7\\ 19 & -7\end{bmatrix}$$

Problem 5

We will solve this problem using a proof by contradiction. Let $m_{1,2} \in M$ such that :

$$\gamma = ||x - m_1|| = d(x, M) \quad \text{and}\gamma = \quad ||x - m_2|| = d(x, M)$$

Now, since M is a vector space, then $\frac{m_1+m_2}{2} \in M$ since it is a linear combination of the elements in M. We can write:

$$||x - m_1|| \leq ||x - \frac{m_1 + m_2}{2}|| \quad \text{and} \quad ||x - m_2|| \leq ||x - \frac{m_1 + m_2}{2}||$$

Then:

$$\begin{aligned}
||x - m_1|| &\leq ||x - \frac{m_1 + m_2}{2}|| \\
||x - m_1|| &\leq ||\frac{x - m_1}{2} + \frac{x - m_2}{2}||
\end{aligned}$$

Also:

$$\begin{aligned}
||x - m_2|| &\leq ||x - \frac{m_1 + m_2}{2}|| \\
||x - m_2|| &\leq ||\frac{x - m_1}{2} + \frac{x - m_2}{2}||
\end{aligned}$$

We consider two cases:

1. $\forall \alpha \in \mathbb{R} \quad \text{s.t.} \quad (x - m_1) \neq \alpha(x - m_2)$
2. $\exists \alpha \in \mathbb{R} \quad \text{s.t.} \quad (x - m_1) = \alpha(x - m_2)$

For the first case, using the triangle inequality and the properties of a strict norm, we can write:

$$||\frac{x-m_1}{2} + \frac{x-m_2}{2}|| < ||\frac{x-m_1}{2}|| + ||\frac{x-m_2}{2}||$$

Thus:

$$\begin{aligned}
\gamma = ||x - m_1|| &< ||\frac{x-m_1}{2}|| + ||\frac{x-m_2}{2}|| \\
\gamma &< |\frac{1}{2}|.||x - m_1|| + |\frac{1}{2}|.||x - m_2|| \\
\implies \gamma &< \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma \quad \Rightarrow\Leftarrow
\end{aligned}$$

For the second case, we write the following for some $\alpha \, in \, \mathbb{R}$:

$$(x - m_1) = \alpha(x - m_2)$$

Taking the norm on both sides, we get:

$$||x - m_1|| = |\alpha|.||x - m_2||$$

Using the properties of the strict norm as well as the triangle inequality, we get:

$$\begin{aligned}
||\frac{x-m_1}{2} + \frac{x-m_2}{2}|| &= ||\frac{x-m_1}{2}|| + ||\frac{x-m_2}{2}|| \\
||\alpha(x - m_2) + (x - m_2)|| &= \gamma + \gamma \\
|\alpha + 1|.||x - m_2|| &= 2\gamma \\
|\alpha + 1|\gamma &= 2\gamma \\
|\alpha + 1| &= 2
\end{aligned}$$

Since α is non-negative, we can write:

$$\begin{aligned}
\alpha + 1 &= 2 \\
\alpha &= 1
\end{aligned}$$

Finally, we show that m^* is unique:

$$\begin{aligned}
x - m_1 &= \alpha(x - m_2) \\
x - m_1 &= x - m_2
\end{aligned}$$

$$\boxed{m_1 = m_2 = m^*}$$

Problem 6

Part a

We prove that the 1-norm is not a strict norm using a counter-example. Let $x = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $y = \begin{bmatrix}1 & 1\end{bmatrix}^T$, we have:

$$\begin{aligned} ||x||_1 &= |1| + |0| = 1 \\ ||y||_1 &= |1| + |1| = 2 \\ ||x+y||_1 &= |1+1| + |1+0| = 3 \\ \implies ||x+y||_1 &= ||x||_1 + ||y||_1 = 3 \end{aligned}$$

Thus:

$$\forall \alpha \in \mathbb{R}^+ \ , \ y \neq \alpha x \ , \ x \neq \alpha y \quad \text{s.t.} \quad ||x+y||_1 = ||x||_1 + ||y||_1$$

Part c

We prove that the ∞-norm is not a strict norm using a counter-example. Let $x = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $y = \begin{bmatrix}1 & 1\end{bmatrix}^T$, we have:

$$\begin{aligned} ||x||_\infty &= \max\{|1|, |0|\} = 1 \\ ||y||_\infty &= \max\{|1|, |1|\} = 1 \\ ||x+y||_\infty &= \max\{|1+1|, |0+1|\} = 2 \\ \implies ||x+y||_\infty &= ||x||_\infty + ||y||_\infty = 2 \end{aligned}$$

Thus:

$$\forall \alpha \in \mathbb{R}^+ \ , \ y \neq \alpha x \ , \ x \neq \alpha y \quad \text{s.t.} \quad ||x+y||_\infty = ||x||_\infty + ||y||_\infty$$

Problem 7

We prove the rank-nullity theorem by using a proof by exhaustion. In fact, we consider the only two possible cases. Let $A \in \mathbb{R}^{m \times n}$, $A: \ \mathbb{R}^n \longrightarrow \mathbb{R}^m$

1. rank(A)$= n$
2. rank(A)$= p < n$

For the first case, since $rank(A) = n$, then A is invertible and the only solution to $Ax = 0$ is the trivial solution $x = 0$. Thus $\mathcal{N}(A) = \{0\}$ and $nullity(A) = \dim(\mathcal{N}(A))= 0$. Thus, the rank-nullity theorem holds.

For the second case, if A is rank deficient, then $\dim(\mathcal{N}(A))= k$ and $\{v_1, v_2, \ldots, v_k\}$ forms a basis for $\mathcal{N}(A)$ where $v_{1,\ldots,k}$ are the column vectors of A that can be written as a linear combination of the vectors $v_{k+1,\ldots,n}$. Define the linear operator $\mathcal{L}(v) = Av$ and denote the following set of vectors $\mathcal{S}$:

$$\mathcal{S} = \{\mathcal{L}(v_{k+1}), \mathcal{L}(v_{k+2}), \ldots, \mathcal{L}(v_n)\}$$

We know that the column vectors of A span the range space of A, thus:

$$\mathcal{R}(A) = \text{span}(\{Av_1, Av_2, \ldots, Av_n\})$$

We also know that $v_{1,\ldots,k}$ can be written as a linear combination of $v_{k+1,\ldots,n}$, thus:

$$\mathcal{R}(A) = \text{span}(\{Av_{k+1}, Av_{k+2}, \ldots, Av_n\})$$

Since $\mathcal{L}(v)$ is a linear operator, we can write:

$$\sum_{i=k+1}^{n} b_i \mathcal{L}(v_i) = \sum_{i=k+1}^{n} b_i A v_i = A \sum_{i=k+1}^{n} b_i v_i$$

We assume the following:

$$\exists b_i \in \mathbb{R} \quad \text{s.t.} \quad A \sum_{i=k+1}^{n} b_i v_i = 0$$

From the previous result, we notice that:

$$\sum_{i=k+1}^{n} b_i v_i \in \mathcal{N}(A)$$

Since $v_{1,\ldots,k}$ forms a basis for $\mathcal{N}(A)$, then $\exists c_i \in \mathbb{R}$ such that:

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$$

Since $\{v_1, v_2, \ldots, v_k\}$ forms a basis for $\mathcal{N}(A)$ and since $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$, then $\{v_1, v_2, \ldots, v_k\}$ forms a basis for $\mathbb{R}^n$ and we can write:

$$\sum_{i=1}^{k} c_i v_i = 0$$

Thus:

$$\sum_{i=k+1}^{n} b_i v_i = 0 \implies \quad b_i = 0 \quad \forall i = k+1, \ldots, n$$

Then the vectors in $\mathcal{S}$ are linearly independent and therefore:

$$\text{rank}(A) = n - (k+1) + 1 = n - k$$

Finally:

$$\boxed{\text{rank}(A) + \text{nullity}(A) = n - k + k = n} \quad \square$$

Problem 8

The matlab code used to solve this part is displayed below:

```
%% Initialize
clear all
clc
format rat
%% Define matrices
A=eye(5);
A(2,2)=0.5;
A(3,3)=0.5;
A(5,5)=0.5;
B=[1;0;2;0;3];
C=0.2;
D=[1 0 2 0 3];
%% Call function
[inverse] = MIL(inv(A),B,C,D);
%% Function definition
function [inverse] = MIL(A,B,C,D)
%function input A is actually the inverse of A
inverse=A-A*B*inv((inv(C)+D*A*B))*D*A;
end
%% End
```

Running the matlab code with the matrices from Problem 8 of HW # 5, we obtain the following result:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \quad C = 0.2 \quad D = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix}$$

$$\implies (A + BCD)^{-1} = \begin{bmatrix} 31/32 & 0 & -1/8 & 0 & -3/16 \\ 0 & 2 & 0 & 0 & 0 \\ -1/8 & 0 & 3/2 & 0 & -3/4 \\ 0 & 0 & 0 & 1 & 0 \\ -3/16 & 0 & -3/4 & 0 & 7/8 \end{bmatrix}$$

Problem 9

Let $\mathcal{X} = \{f : \mathbb{R} \Longleftarrow \mathbb{R}\}, \mathcal{F} = \mathbb{R}$. We define the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$. We define the set $M = \text{span}\{1, t, \frac{1}{2}(3t^2 - 1)\}$ and $x = e^t$. We use the Gram Matrix to solve the following optimization problem:

$$\hat{x} = \underset{y \in M}{\arg\min}\ ||x - y||$$

The solution is of the form:

$$\hat{x} = \alpha_1 y^1 + \alpha_2 y^2 + \alpha_3 y^3$$

Where the coefficients α_i are computed as follows:

$$\alpha = G^{-T}\beta$$
$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}\langle y^1,y^1\rangle & \langle y^1,y^2\rangle & \langle y^1,y^3\rangle\\\langle y^2,y^1\rangle & \langle y^2,y^2\rangle & \langle y^2,y^3\rangle\\\langle y^3,y^1\rangle & \langle y^3,y^2\rangle & \langle y^3,y^3\rangle\end{bmatrix}^{-T}\begin{bmatrix}\langle x,y^1\rangle\\\langle x,y^2\rangle\\\langle x,y^3\rangle\end{bmatrix}$$

Where:

$$G_{1,1} = \langle y^1,y^1\rangle = \int_{-1}^{1} 1dt = [t]_{-1}^{1} = 2$$
$$G_{1,2} = G_{2,1} = \langle y^1,y^2\rangle = \langle y^2,y^1\rangle = \int_{-1}^{1} tdt = [\frac{1}{2}t^2]_{-1}^{1} = 0$$
$$G_{1,3} = G_{3,1} = \langle y^1,y^3\rangle = \langle y^3,y^1\rangle = \int_{-1}^{1} \frac{1}{2}(3t^2-1)dt = \frac{1}{2}[t^3-t]_{-1}^{1} = 0$$
$$G_{2,3} = G_{3,2} = \langle y^2,y^3\rangle = \langle y^3,y^2\rangle = \int_{-1}^{1} \frac{1}{2}(3t^3-t)dt = \frac{1}{2}[\frac{3}{4}t^4-\frac{1}{2}t^2]_{-1}^{1} = 0$$
$$G_{2,2} = \langle y^2,y^2\rangle = \int_{-1}^{1} t^2dt = \frac{1}{3}[t^3]_{-1}^{1} = \frac{2}{3}$$
$$G_{3,3} = \langle y^3,y^3\rangle = \int_{-1}^{1} \frac{1}{4}(3t^2-1)^2dt = \frac{1}{4}[\frac{9}{5}t^5+t-2t^3]_{-1}^{1} = \frac{2}{5}$$
$$\beta_1 = \langle x,y^1\rangle = \int_{-1}^{1} e^tdt = [e^t]_{-1}^{1} = e^1-e^{-1}$$
$$\beta_2 = \langle x,y^2\rangle = \int_{-1}^{1} te^tdt = [(t-1)e^t]_{-1}^{1} = 2e^{-1}$$
$$\beta_3 = \langle x,y^3\rangle = \int_{-1}^{1} \frac{1}{2}(3t^2-1)e^tdt = \frac{1}{2}[(3t^2-6t+5)e^t]_{-1}^{1} = e-7e^{-1}$$

Thus, we obtain:

$$\alpha = G^{-T}\beta$$
$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\0 & \frac{2}{3} & 0\\0 & 0 & \frac{2}{5}\end{bmatrix}^{-T}\begin{bmatrix}e^1-e^{-1}\\2e^{-1}\\e-7e^{-1}\end{bmatrix}$$
$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}1/2 & 0 & 0\\0 & 3/2 & 0\\0 & 0 & 5/2\end{bmatrix}\begin{bmatrix}e^1-e^{-1}\\2e^{-1}\\e-7e^{-1}\end{bmatrix}$$
$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^1-e^{-1}\\6e^{-1}\\5e-35e^{-1}\end{bmatrix}$$

Finally, we can solve for $\hat{x}$:

$$\hat{x} = \alpha_1 y^1 + \alpha_2 y^2 + \alpha_3 y^3$$
$$\hat{x} = \frac{1}{2}(e^1 - e^{-1}) + 3e^{-1}t + \frac{1}{4}(5e - 35e^{-1})(3t^2 - 1)$$
$$\hat{x} = (\frac{33}{4}e^{-1} - \frac{3}{4}e) + 3e^{-1}t - \frac{1}{4}(105e^{-1} - 15e)t^2$$