ROB 501 - Mathematics for Robotics
HW #4

Due 3 PM on Mon, Sept. 27, 2021
To be submitted on Canvas

Reading Assignment: Chapters 6 and 7 of Nagy. Selected chapters of the textbook *Linear Algebra* by Gabriel Nagy are available under `Files` $\rightarrow$ `Handouts (Background Material from the Web)` $\rightarrow$ `02_LinearAlgebraAndGeometry.pdf` on Canvas.

1. Nagy, Page 130, Prob. 4.3.2 (the field is $\mathbb{R}$ and the vector space is $\mathbb{R}^4$)

4.3.2.- Find the dimension of the space spanned by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\}.$$

Figure 1: Q 1

2. Nagy, Page 136, Prob. 4.4.2 (the field is $\mathbb{R}$ and the vector space is $\mathbb{R}^3$) Using the vocabulary from lecture, the problem is to find the representation of the vector

$$x = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

in the basis $\mathcal{U} = \{u_{1s}, u_{2s}, u_{3s}\}$.

4.4.2.- Let $\mathcal{S} = (\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3)$ be the standard basis of $\mathbb{R}^3$. Find the components of the vector

$$\mathsf{v}_s = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

in the ordered basis $\mathcal{U}$ given by

$$\left(\mathsf{u}_{1s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathsf{u}_{2s} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathsf{u}_{3s} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right).$$

Figure 2: Q 2

3. Using the data in Prob. 7 (that is, Nagy, Page 136, Prob. 4.4.2), find the change of basis matrix P from the standard basis $\{e_1, e_2, e_3\}$ to the new basis $\{u_{1s}, u_{2s}, u_{3s}\}$.

4. A sensor is mounted on a robot that can turn in place. Figure 3 shows the robot rotated by an angle θ. Find the change of basis matrix P, ($[x]_R = P[x]_W$) from the world reference frame (X_W, Y_W) to the robot's reference frame (X_R, Y_R).

   Remark: In practice, imagine you have mounted a camera on the robot and observe an object; it will be in the frame of the robot, which moves with the robot. Being able to relate the object's position in a fixed (global or world) reference frame is of obvious interest. In fact, for navigation in driverless cars, one idea being pursued is to provide the car with a 3D enhanced map that will allow the car to tell where it is by what it sees. In our problem, if you know where an object is in the frame of the robot and in the world frame, you can determine the angle θ.

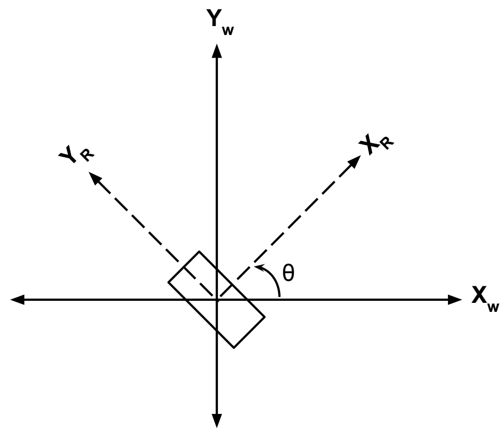

Figure 3: World coordinate system and Robot coordinate system

5. Nagy, Page 136, Prob. 4.4.4

4.4.4.- Let $\mathcal{S}$ be the standard ordered basis of $\mathbb{R}^{2,2}$, that is,

$$\mathcal{S} = (\mathsf{E}_{11}, \mathsf{E}_{12}, \mathsf{E}_{21}, \mathsf{E}_{22}) \subset \mathbb{R}^{2,2},$$

with

$$\mathsf{E}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathsf{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathsf{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{E}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Show that the ordered set $\mathcal{M}$ below is a basis of $\mathbb{R}^{2,2}$, where

$$\mathcal{M} = (\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4) \subset \mathbb{R}^{2,2},$$

with

$$\mathsf{M}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{M}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathsf{M}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathsf{M}_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where the matrices above are written in the standard basis.

(b) Consider the matrix A written in the standard basis $\mathcal{S}$,

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the components of the matrix A in the ordered basis $\mathcal{M}$.

Figure 4: Q 5

6. Nagy, Page 136, Prob. 4.4.3 (Nagy's "ordered bases" are simply called bases in lecture. Finding the components is what we call finding the representation). In addition, find the change of basis matrix P from the basis $\mathcal{S}$ to the basis $\mathcal{Q}$.

4.4.3.- Consider the vector space $V = \mathbb{P}_2$ with the ordered basis $\mathcal{S}$ given by

$$\mathcal{S} = \big(\boldsymbol{p}_0 = 1,\ \boldsymbol{p}_1 = x,\ \boldsymbol{p}_2 = x^2\big).$$

(a) Find the components of the polynomial $\boldsymbol{r}(x) = 2 + 3x - x^2$ in the ordered basis $\mathcal{S}$.

(b) Find the components of the same polynomial $\boldsymbol{r}$ given in part (a) but now in the ordered basis $\mathcal{Q}$ given by

$$\big(\boldsymbol{q}_0 = 1,\ \boldsymbol{q}_1 = 1 - x,\ \boldsymbol{q}_2 = x + x^2,\big).$$

Figure 5: Q 6

7. Let $\mathcal{F} = \mathbb{R}$ and let $\mathcal{X}$ be the set of 2×2 matrices with real coefficients. Define $L : \mathcal{X} \to \mathcal{X}$ by

$$L(M) = \frac{1}{2}(M + M^\top),$$

where $M \in \mathcal{X}$ is a 2×2 real matrix.

   (a) Show that L is a linear operator.

   (b) Compute A, the matrix representation of L, when the basis

$$E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   is used on both copies of $\mathcal{X}$ (i.e., on both the domain and codomain of L).

8. Let A be an $n \times n$ matrix with possibly complex coefficients. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ by $L(x) = Ax$. Note that the field is $\mathcal{F} = \mathbb{C}$.

   (a) Compute the matrix representation of L when the "natural" (also called canonical) basis is used in $\mathbb{C}^n$. Call your representation $\hat{A}$ and find its relation to the original matrix A.

   (b) Suppose that the e-values of A are distinct. Compute the matrix representation L with respect to a basis constructed from the e-vectors of A. Call your representation $\hat{A}$.

Hints

Hints: Prob. 1 Starting from the left and moving to the right, discard a vector if it is linearly dependent on those preceding it. How many vectors remain? The set you obtain is by construction linearly independent and the number of elements is the dimension.

Hints: Prob. 3 Recall that you can compute whichever of P or $\bar{P}$ is easier, and then, if necessary, invert a matrix in MATLAB to get the final answer.

Hints: Prob. 4

- Recall coordinate transformations. Since the origins of both coordinate systems coincide, this is a case of pure rotation.
- Take any generic point (x, y) and use trigonometry to find the relation between (x_W, y_W) and (x_R, y_R).

Hints: Prob. 6 Recall that the change of basis matrix **from** the basis $\{v^1, \ldots, v^n\}$ **to** the basis $\{\tilde{v}^1, \ldots, \tilde{v}^n\}$ is the $n \times n$ matrix P with i-th column given by

$$P_i = [v^i]_{\tilde{v}},$$

the representation of v^i in the basis $\{\tilde{v}^1, \ldots, \tilde{v}^n\}$, and the change of basis matrix **from** the basis $\{\tilde{v}^1, \ldots, \tilde{v}^n\}$ **to** the basis $\{v^1, \ldots, v^n\}$ is the $n \times n$ matrix $\tilde{P}$ with i-th column given by

$$\tilde{P}_i = [\tilde{v}^i]_v.$$

Moreover, $P\tilde{P} = \tilde{P}P = I$, the $n \times n$ identity matrix. Consequently, you should always compute whichever of the two matrices is easier, and then get the other by matrix inversion in MATLAB.

Hints: Prob. 7 A will be 4×4. From lecture, we have a formula for the columns of A.

Hints: Prob. 8 In both parts, you are using the same basis on the domain and the co-domain of L. The point of (a) is that when you view a matrix as linear operator, and write $L(x) = Ax$, you are using the natural basis on $\mathbb{R}^n$. The answer for (b) will show that A is similar to a diagonal matrix, viz Prob. 5, though the perspective is different.