Homework #3

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Problem 1

Part a

Denote the subset $\mathcal{X}$ of $\mathbb{R}^n$ such that $\mathcal{X} = \{x \in \mathbb{R}^n : x_i \geq 0 \quad i = 1; \dots, n\}$. This is not a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$ since is it not closed under multiplication by a constant. In fact:

$$\forall x \in \mathcal{X},\ \exists \alpha \in \mathbb{R},\ \alpha < 0 \quad \text{s.t.} \quad y = \alpha x = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \text{with} \quad y_i \leq 0$$

Part b

Denote the subset $\mathcal{X}$ of $\mathbb{R}^n$ such that $\mathcal{X} = \{x \in \mathbb{R}^n : x_1 = 0\}$. This is a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$, since it contains the zero vector $\vec{0}$ and it satisfies the following property:

$$\forall \alpha_{1,2} \in \mathbb{R} \text{ and } \forall v^{1,2} \in \mathcal{X},\ v = \alpha_1 v^1 + \alpha_2 v^2 \in \mathcal{X} \tag{1}$$

In fact:

$$\alpha_1 v^1 + \alpha_2 v^2 = \alpha_1 \begin{bmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_n^1 \end{bmatrix} + \alpha_2 \begin{bmatrix} v_1^2 \\ v_2^2 \\ \vdots \\ v_n^2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ v_2^1 \\ \vdots \\ v_n^1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ v_2^2 \\ \vdots \\ v_n^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_1 v_2^1 + \alpha_2 v_2^2 \\ \vdots \\ \alpha_1 v_n^1 + \alpha_2 v_n^2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Thus $v = \alpha_1 v^1 + \alpha_2 v^2 \in \mathcal{X} : \{x \in \mathbb{R}^n : x_1 = 0\}$, implying that $\mathcal{X}$ is a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$.

Part c

Denote the subset $\mathcal{X}$ of $\mathbb{R}^n$ such that $\mathcal{X} = \{x \in \mathbb{R}^n : x_1 x_2 = 0 \quad n \geq 2\}$. This is not a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$, since it does not satisfy property (1). In fact, from the attribute of $\mathcal{X}$, we have that for $x \in \mathcal{X}$, either $x_1 = 0$, $x_2 = 0$ or $x_1 = x_2 = 0$. Thus, we have:

$$\exists \alpha_{1,2} \in \mathbb{R},\ \alpha_{1,2} \neq 0 \text{ and } \exists v^{1,2} \in \mathcal{X},\ v = \alpha_1 v^1 + \alpha_2 v^2 \notin \mathcal{X}$$

In fact:

$$\alpha_1 v^1 + \alpha_2 v^2 = \alpha_1 \begin{bmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_n^1 \end{bmatrix} + \alpha_2 \begin{bmatrix} v_1^2 \\ v_2^2 \\ \vdots \\ v_n^2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ v_2^1 \\ \vdots \\ v_n^1 \end{bmatrix} + \alpha_2 \begin{bmatrix} v_1^2 \\ 0 \\ \vdots \\ v_n^2 \end{bmatrix} = \begin{bmatrix} \alpha_2 v_1^2 \\ \alpha_1 v_2^1 \\ \vdots \\ \alpha_1 v_n^1 + \alpha_2 v_n^2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Thus $v = \alpha_1 v^1 + \alpha_2 v^2 \notin \mathcal{X} : \{x \in \mathbb{R}^n : x_1 x_2 = 0 \quad n \geq 2\}$, implying that $\mathcal{X}$ is not a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$.

Part d

Denote the subset $\mathcal{X}$ of $\mathbb{R}^n$ such that $\mathcal{X} = \{x \in \mathbb{R}^n : x_1 + \ldots + x_n = 0\}$. This is a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$, since it satisfies property (1) and it contains the zero vector $\vec{0}$. In fact:

$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \ \sum_{i=0}^{n} \vec{0}_i = \vec{0}_1 + \ldots + \vec{0}_n = 0 \tag{2}$$

We also have:

$$\alpha_1 v^1 + \alpha_2 v^2 = \alpha_1 \begin{bmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_n^1 \end{bmatrix} + \alpha_2 \begin{bmatrix} v_1^2 \\ v_2^2 \\ \vdots \\ v_n^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 v_1^1 + \alpha_2 v_1^2 \\ \alpha_1 v_2^1 + \alpha_2 v_2^2 \\ \vdots \\ \alpha_1 v_n^1 + \alpha_2 v_n^2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Now:

$$\begin{aligned} \sum_{i=0}^{n} v_i = \sum_{i=0}^{n} \alpha_1 v_i^1 + \alpha_2 v_i^2 &= (\alpha_1 v_1^1 + \alpha_2 v_1^2) + \ldots + (\alpha_1 v_n^1 + \alpha_2 v_n^2) \\ &= \alpha_1 (v_1^1 + \ldots + v_n^1) + \alpha_2 (v_1^2 + \ldots + v_n^2) \\ &= \alpha_1 (0) + \alpha_2 (0) \\ &= 0 \end{aligned}$$

Thus $v = \alpha_1 v^1 + \alpha_2 v^2 \in \mathcal{X} : \{x \in \mathbb{R}^n : x_1 + \ldots + x_n = 0\}$, implying that $\mathcal{X}$ is a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$.

Part e

Denote the subset $\mathcal{X}$ of $\mathbb{R}^n$ such that $\mathcal{X} = \{x \in \mathbb{R}^n : x_1 + \ldots + x_n = 1\}$. This is not a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$, since it does not contain the zero vector $\vec{0}$ as seen in equation (2).

Part f

Denote the subset $\mathcal{X}$ of $\mathbb{R}^n$ such that $\mathcal{X} = \{x \in \mathbb{R}^n : Ax = b, A \neq 0, b \neq 0\}$. This is not a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$, since it does not contain the zero vector $\vec{0}$. In fact, let $A_{m,n} \in$

$\{\mathbb{R}^{m,n} - 0^{m,n}\}$ and $b^m \in \mathbb{R}^m$, we have:

$$A^{m,n}\vec{0} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n,1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{m,1}$$

However, since $\mathcal{X}$ imposes that $b \neq 0$, we can then deduce that $\vec{0} \notin \mathcal{X}$ and thus $\mathcal{X}$ is not a subspace of $\{\mathbb{R}^n, \mathbb{R}\}$.

Problem 2

Let $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ be two subsets of the vector space $(\mathcal{X}, \mathcal{F})$. To show that $\text{span}(\mathcal{S}_1 \cup \mathcal{S}_2) = \text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2)$, we must prove the following two inclusions:

$$\text{span}(\mathcal{S}_1 \cup \mathcal{S}_2) \subset \text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2) \tag{3a}$$
$$\text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2) \subset \text{span}(\mathcal{S}_1 \cup \mathcal{S}_2) \tag{3b}$$

We denote the spans of the respective subsets as follows:

$$\begin{aligned} \text{span}(\mathcal{S}_1) &= \{v \in V | \exists k_1 < \infty, v_1^{\mathcal{S}_1}, \ldots, v_{k_1}^{\mathcal{S}_1} \in \mathcal{S}_1, \alpha_1, \ldots, \alpha_{k_1} \in \mathcal{F}, v = \alpha_1 v_1^{\mathcal{S}_1} + \ldots + \alpha_{k_1} v_{k_1}^{\mathcal{S}_1}\} \\ \text{span}(\mathcal{S}_2) &= \{v \in V | \exists k_2 < \infty, v_1^{\mathcal{S}_2}, \ldots, v_{k_2}^{\mathcal{S}_2} \in \mathcal{S}_2, \beta_1, \ldots, \beta_{k_2} \in \mathcal{F}, v = \beta_1 v_1^{\mathcal{S}_2} + \ldots + \beta_{k_2} v_{k_2}^{\mathcal{S}_2}\} \\ \text{span}(\mathcal{S}_1 \cup \mathcal{S}_2) &= \{v \in V | \exists k_{1,2} < \infty, v_1^{\mathcal{S}_1}, \ldots, v_{k_1}^{\mathcal{S}_1} \in \mathcal{S}_1, v_1^{\mathcal{S}_2}, \ldots, v_{k_2}^{\mathcal{S}_2} \in \mathcal{S}_2, \alpha_1, \ldots, \alpha_{k_1} \in \mathcal{F}, \\ &\qquad \beta_1, \ldots, \beta_{k_2} \in \mathcal{F}, v = \alpha_1 v_1^{\mathcal{S}_1} + \ldots + \alpha_{k_1} v_{k_1}^{\mathcal{S}_1} + \beta_1 v_1^{\mathcal{S}_2} + \ldots + \beta_{k_2} v_{k_2}^{\mathcal{S}_2}\} \end{aligned}$$

The union of the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ contains all the elements in either $\mathcal{S}_1$, $\mathcal{S}_2$ or both. Thus $\text{span}(\mathcal{S}_1 \cup \mathcal{S}_2)$ is the subset of all possible linear combinations of the vectors in $\mathcal{S}_1$ and $\mathcal{S}_2$. We first prove the first inclusion (3a). Let $v \in \text{span}(\mathcal{S}_1 \cup \mathcal{S}_2)$, thus we can write:

$$\begin{aligned} v &= \alpha_1 v_1^{\mathcal{S}_1} + \ldots + \alpha_{k_1} v_{k_1}^{\mathcal{S}_1} + \beta_1 v_1^{\mathcal{S}_2} + \ldots + \beta_{k_2} v_{k_2}^{\mathcal{S}_2} \\ &= (\alpha_1 v_1^{\mathcal{S}_1} + \ldots + \alpha_{k_1} v_{k_1}^{\mathcal{S}_1}) + (\beta_1 v_1^{\mathcal{S}_2} + \ldots + \beta_{k_2} v_{k_2}^{\mathcal{S}_2}) \\ &= s_1 + s_2 \end{aligned}$$

where $s_1 \in \text{span}(\mathcal{S}_1)$ and $s_2 \in \text{span}(\mathcal{S}_2)$. Thus we have showed that for $v \in \text{span}(\mathcal{S}_1 \cup \mathcal{S}_2)$, $v \in \text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2)$. This concludes the proof of the first inclusion (3a).
Proving the second inclusion (3b), we denote the vectors s_1 and s_2 mentioned in the first inclusion proof. Let $v \in \text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2)$, then v can be written as a linear combination of vectors in $\mathcal{S}_1$ and $\mathcal{S}_2$. In particular:

$$\begin{aligned} & v = s_1 + s_2 = (\alpha_1 v_1^{\mathcal{S}_1} + \ldots + \alpha_{k_1} v_{k_1}^{\mathcal{S}_1}) + (\beta_1 v_1^{\mathcal{S}_2} + \ldots + \beta_{k_2} v_{k_2}^{\mathcal{S}_2}) \\ \implies & v = \alpha_1 v_1^{\mathcal{S}_1} + \ldots + \alpha_{k_1} v_{k_1}^{\mathcal{S}_1} + \beta_1 v_1^{\mathcal{S}_2} + \ldots + \beta_{k_2} v_{k_2}^{\mathcal{S}_2} \end{aligned}$$

Thus we showed that $v \in \text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2)$ can be written as a linear combination of vectors in $(\mathcal{S}_1 \cup \mathcal{S}_2)$.This concludes the proof of the second inclusion (3b). We conclude the following:

$$\boxed{\text{span}(\mathcal{S}_1 \cup \mathcal{S}_2) = \text{span}(\mathcal{S}_1) + \text{span}(\mathcal{S}_2)} \quad \square$$

Problem 3

To check whether a set of vectors $\mathcal{X}$ is linearly independent or not, we check the following property:

$$\forall v \in \mathcal{X}, \sum_{i=1}^{n} \alpha_i v^i = 0 \implies \alpha_i = 0 \ i = 1, \dots, n < \infty \tag{4}$$

Part a

Checking property (4) for the following set:

$$\mathcal{X} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \}$$

We first write the linear combination of the vectors in $\mathcal{X}$ and equate it to 0:

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 3 & 0 & 9 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

Now we write the generated matrix A in row echelon form and we get:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

We notice $rank(A) = 2 < n = 3$. Thus the vectors generated are linearly dependent. We can also check that one possible set of α_i is $\alpha_1 = 3, \alpha_2 = -1, \alpha_3 = -1$. We can also write a linear combination of the vectors as such:

$$\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Part b

Checking property (4) for the following set:

$$\mathcal{X} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$$

We notice that $\mathcal{X} \subset \mathbb{R}^3$ and we know that the basis of $\mathbb{R}^3$ is made of 3 vectors as such:

$$\text{basis}(\mathbb{R}^3) = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$$

Thus the vectors in $\mathcal{X}$ are linearly dependent since the number of vectors in $\mathcal{X}$ is greater than the number of vectors in the basis of $\mathbb{R}^3$. Following property (4), we can write a linear of combination of the vectors as such:

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Solving the set of equations we get:

$$\left.\begin{aligned} \alpha_1 + \alpha_4 = 0 \\ 2\alpha_1 + 4\alpha_2 + \alpha_4 = 0 \\ 3\alpha_1 + 5\alpha_2 + 6\alpha_3 + \alpha_4 = 0 \end{aligned}\right\} \implies \alpha_1 = -8, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 8$$

We can then write a possible linear combination as such:

$$-8 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Part c

Checking property (4) for the following set:

$$\mathcal{X} = \{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \}$$

We first write the linear combination of the vectors in $\mathcal{X}$ and equate it to 0:

$$\alpha_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0$$

We now write and solve the system of equations:

$$\left.\begin{aligned} 3\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \\ 2\alpha_1 + \alpha_3 = 0 \\ \alpha_1 = 0 \end{aligned}\right\} \implies \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$$

We then conclude that the vectors in $\mathcal{X}$ are linearly independent since our set is finite ($n = 3$) and the only solution for equation (4) is obtained for $\alpha_{1,2,3} = 0$.

Problem 4

Let $\mathcal{X}$ denote a subset of matrices in $\mathbb{R}^{2\times 2}$ as follows:

$$\mathcal{X} = \{\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix}\}$$

We can write a system of linear combination of the matrices in $\mathcal{X}$ as follows:

$$\alpha_1 \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} + \alpha_2 \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} + \alpha_3 \begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\begin{bmatrix}\alpha_1 + 2\alpha_2 + 4\alpha_3 & 2\alpha_1 + \alpha_2 - \alpha_3\\2\alpha_1 + \alpha_2 - \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

We obtain the following set of equations by removing the redundant equality:

$$\left.\begin{aligned}\alpha_1 + 2\alpha_2 + 4\alpha_3 = 0\\2\alpha_1 + \alpha_2 - \alpha_3 = 0\\\alpha_1 + \alpha_2 + \alpha_3 = 0\end{aligned}\right\} \implies \alpha_1 = 2, \alpha_2 = -3, \alpha_3 = 1$$

We can then write a possible linear combination of the matrices as such:

$$2\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} - 3\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} + \begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Thus we can conclude that the matrices in the subset $\mathcal{X} \subset \mathbb{R}^{2\times 2}$ are linearly dependent.

Problem 5

To prove that for a vector space $(\mathcal{X}, \mathcal{F})$ and for a subset $\mathcal{S} \subset \mathcal{X}$, span$(\mathcal{S})$ is the smallest subspace of $\mathcal{X}$ that contains $\mathcal{S}$, we use a proof by contradiction. Let us define a subspace $\mathcal{Y}$ of $\mathcal{X}$ such that $\mathcal{S} \subset \mathcal{Y}$ and $\mathcal{Y} \subset$ span$(\mathcal{S})$, which means that $\mathcal{Y}$ contains all of the elements in $\mathcal{S}$ and that $\mathcal{Y}$ is strictly smaller than span$(\mathcal{S})$, respectively. Since $(\mathcal{Y}, \mathcal{F})$ is a vector space, $\mathcal{Y}$ is closed under vector addition and scalar multiplication as such:

$$\forall \alpha_{1,2} \in \mathcal{F} \text{ and } \forall y_{1,2} \in \mathcal{Y},\ \alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{Y}$$

Since $\mathcal{Y}$ contains all of the elements of $\mathcal{S}$ and from the above definition, $\mathcal{Y}$ is the set of all possible linear combinations of the vectors in $\mathcal{S}$. This contradicts the notion that $\mathcal{Y} \subset$ span$(\mathcal{S})$ since by definition, span$(\mathcal{S})$ is the set of all possible linear combinations of vectors $v \in \mathcal{S}$. Thus by contradiction, $\mathcal{Y} = \mathcal{S}$ and span$(\mathcal{Y})$ = span$(\mathcal{S})$. We have thus proved the following claim:

$$\boxed{\text{If } \mathcal{Y} \text{ is a subspace of } \mathcal{X} \text{ and } \mathcal{S} \subset \mathcal{Y}\text{, then span}(\mathcal{S}) \subset \mathcal{Y}} \quad \square$$

Problem 6

Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\mathcal{V}$ and $\mathcal{W}$ be two subspaces of $\mathcal{X}$, we prove the following claim:

$$\mathcal{V} \cap \mathcal{W} = \{0\} \iff \forall x \in \mathcal{V} + \mathcal{W},\ \exists! v \in \mathcal{V} \text{ and } \exists! w \in \mathcal{W} \quad \text{s.t.} \quad x = v + w$$

We first prove ($\Longleftarrow$) by direct proof:
We have $\mathcal{X} = \mathcal{V} \oplus \mathcal{W} = \mathcal{V} + \mathcal{W}$. Now let us assume $\exists v \in \mathcal{V}$, $\exists w \in \mathcal{W}$ and $\exists x \in \mathcal{V} \cap \mathcal{W}$ such that:

$$\left.\begin{aligned} x = 0 + v \\ x = w + 0 \end{aligned}\right\} \implies v = 0 \text{ and } w = 0, \text{ therefore } \mathcal{V} \cap \mathcal{W} = \{0\}$$

Now we prove ($\Longrightarrow$) by contradiction:
For all $x \in \mathcal{X}$, there exists a $v_{1,2} \in \mathcal{V}$ and $w_{1,2} \in \mathcal{W}$ such that $x = v_1 + w_1$ and $x = v_2 + w_2$. We can write the following:

$$\begin{aligned} 0 &= x - x \\ 0 &= v_1 + w_1 - v_2 - w_2 \\ 0 &= (v_1 - v_2) + (w_1 - w_2) \\ (v_1 - v_2) &= -(w_1 - w_2) \end{aligned}$$

Since $\mathcal{W}$ is a subspace, $\mathcal{W}$ is closed under scalar multiplication and vector addition. Thus, $(v_1 - v_2) \in \mathcal{W}$ since it is a linear combination of the vectors in $\mathcal{W}$. We then can write that $(v_1 - v_2) \in (\mathcal{V} \cap \mathcal{W})$. We also have that $\mathcal{V} \cap \mathcal{W} = \{0\}$, thus:

$$v_1 - v_2 = 0 \implies v_1 = v_2 \tag{5}$$

The same proof is applied for $(w_1 - w_2) \in \mathcal{V}$. Thus $(w_1 - w_2) \in (\mathcal{V} \cap \mathcal{W}) = \{0\}$, we have:

$$w_1 - w_2 = 0 \implies w_1 = w_2 \tag{6}$$

From (5) and (6), we showed that $\exists! v \in \mathcal{V}$ and $\exists! w \in \mathcal{W}$ and $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Finally, we have proved the following:

$$\boxed{\mathcal{V} \cap \mathcal{W} = \{0\} \iff \forall x \in \mathcal{V} + \mathcal{W},\ \exists! v \in \mathcal{V} \text{ and } \exists! w \in \mathcal{W} \quad \text{s.t.} \quad x = v + w} \quad \square$$