Homework #2

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Problem 1

Part a

The statement we want to negate is $(p \wedge q)$, we do by negating each of its parts as follows:

$$\begin{aligned} \textbf{STATEMENT 1} &: (p \wedge q) \\ \textbf{NEGATION} &: \neg(p \wedge q) = \neg p \vee \neg q \end{aligned} \tag{1}$$

Building up the truth table, we notice that the end result aligns with equation (1) as seen below:

Table 1: Statement one truth table

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

Thus, we can state that:

$$\boxed{\neg(p \wedge q) = \neg p \vee \neg q \quad \square}$$

Part b

The statement we want to negate is $(p \vee q)$, we do by negating each of its parts as follows:

$$\begin{aligned} \textbf{STATEMENT 2} &: (p \vee q) \\ \textbf{NEGATION} &: \neg(p \vee q) = \neg p \wedge \neg q \end{aligned} \tag{2}$$

Building up the truth table, we notice that the end result aligns with equation (2) as seen below:

Table 2: Statement one truth table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

Thus, we can state that:

$$\boxed{\neg(p \vee q) = \neg p \wedge \neg q \quad \square}$$

Problem 2

Part a

$$\neg(\forall n \in \mathbb{Z}, 2n+1 \text{ is odd}) = \exists n \in \mathbb{Z}, 2n+1 \text{ is even}$$

Part b

$$\neg(\exists n \in \mathbb{Z}, 2n+1 \text{ is prime}) = \forall n \in \mathbb{Z}, 2n+1 \text{ is composite}$$

Part c

$$\neg(\exists v \in \mathbb{R}^n, v \neq 0 \quad \text{s.t.} \quad Av = \lambda v) = \forall v \in \mathbb{R}^n, v \neq 0 \quad \text{s.t.} \quad Av \neq \lambda v$$

Part d

$$\neg(\forall \eta > 0, \exists \delta > 0 \quad \text{s.t.} \quad |x| \leq \delta \implies |f(x)| \leq \eta |x|) = \exists \eta > 0, \forall \delta > 0 \quad \text{s.t.} \quad |x| \leq \delta \quad \wedge \quad |f(x)| > \eta |x|$$

Problem 3

To prove that $\sqrt{7}$ is irrational, we resort to a proof by contradiction, that is, our statement is denoted by p and we show that $\neg p$ is false.

$$\textbf{STATEMENT} : \sqrt{7} \text{ is irrational}$$
$$\textbf{NEGATION} : \sqrt{7} \text{ is rational}$$

The negation of this statement implies by definition that $\sqrt{7}$ can be written as a ratio of two integers m and n as follows:

$$\sqrt{7} \text{ is rational } : \{\sqrt{7} \in \mathbb{Q} \mid m, n \in \mathbb{Z}\ ,\ n \neq 0 \text{ with no common factors s.t. } \sqrt{7} = \frac{m}{n}\}$$

Then we can write:

$$\begin{aligned} \sqrt{7} &= \frac{m}{n} \\ \implies n^2 &= \frac{m^2}{7} \end{aligned} \tag{3}$$

With $n \in \mathbb{Z}$, we can say that 7 divides m^2 and thus 7 divides m. We can then write that $m = 7k_1$ with $k_1 \in \mathbb{Z}$. Plugging in this result in equation (3) we obtain:

$$\begin{aligned} n^2 &= \frac{m^2}{7} = \frac{(7k_1)^2}{7} = 7k_1^2 \\ \implies k_1^2 &= \frac{n^2}{7} \end{aligned}$$

With $k \in \mathbb{Z}$, we can say that 7 divides n^2 and thus 7 divides n. We can then write that $n = 7k_2$ with $k_2 \in \mathbb{Z}$. Thus, there is a contradiction in the negation of our statement since m and n have 7 as a common factor showing that the negation of our statement if false, and that our statement that $\sqrt{7}$ is rational is true since $\neg p$ is false $\implies p$ is true.

Problem 4

To prove that if $det(A) = 0$ then A is not invertible, we resort to a proof by contradiction as such:

$$\begin{gathered} \text{if } det(A) = 0 \text{ then } A \text{ is not invertible} \\ p : det(A) = 0 \\ q : A \text{ is not invertible} \\ p \iff q \\ (p \implies q) \wedge (q \implies p) \\ (p \implies q) \iff \neg(p \wedge (\neg q)) \\ (q \implies p) \iff \neg(q \wedge (\neg p)) \end{gathered}$$

It suffices to show either of the two statements below since they are equivalent:

$$\begin{gathered} p \wedge (\neg q) \text{ is false} \\ q \wedge (\neg p) \text{ if false} \end{gathered}$$

We will show the former:

$$p \wedge (\neg q) \ : \ (det(A) = 0) \ \wedge (A \text{ is invertible})$$

We have:

$$\begin{gathered} A \text{ is invertible} \implies AA^{-1} = I \\ det(AA^{-1}) = det(I) \\ det(A)det(A^{-1}) = det(I) = 1, \text{ since } A \in \mathbb{R}^{n \times n} \\ 0 * det(A^{-1}) = 1 \\ 0 = 1 \end{gathered}$$

Thus we reached a contradiction, implying that $p \wedge (\neg q)$ is false, therefore $(p \implies q)$ is true.

Problem 5

To show that $\forall n \geq 1,\ P(n):\ \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$, we resort to a proof by induction:

$$\textbf{Base Case}:\ P(1):\sum_{k=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2} = \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}, \quad \textbf{True}$$

$$\textbf{Induction Step}:\ P(j):\ \sum_{k=1}^{j} \frac{1}{k(k+1)} = \frac{j}{j+1}$$

$$\textbf{Show}:\ P(j+1):\ \sum_{k=1}^{j+1} \frac{1}{k(k+1)} = \frac{j+1}{j+2}$$

Proof:

$$\begin{aligned}
\sum_{k=1}^{j+1} \frac{1}{k(k+1)} &= \sum_{k=1}^{j} \frac{1}{k(k+1)} + \frac{1}{(j+1)(j+2)} \\
&= \frac{j}{j+1} + \frac{1}{(j+1)(j+2)} \\
&= \frac{j(j+2)+1}{(j+1)(j+2)} \\
&= \frac{j^2+2j+1}{(j+1)(j+2)} \\
&= \frac{(j+1)^2}{(j+1)(j+2)} \\
&= \frac{j+1}{j+2}
\end{aligned}$$

Therefore:

$$\boxed{\forall n \geq 1,\ P(n):\ \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \quad \square}$$

Problem 6

Part a

To show that $\forall n \geq 12,\ n \in \mathbb{Z},\ \exists k_{1,2} \in \mathbb{Z}^+$ s.t. $n = 4k_1 + 5k_2$, we resort to a proof by strong induction:

$$\textbf{Base Case}:\ P(12):\ 12 = 4*3 + 5*0$$

$$\textbf{Induction Step}: 12 \leq j \leq k,\ P(j):\ j = 4k_1 + 5k_2$$

$$\textbf{Show}: \forall k+1 \geq 13,\ P(k+1):\ k+1 = 4k_1 + 5k_2$$

For $n = 13, .., 15$:

$$13 = 4 * 2 + 5 * 1$$
$$14 = 4 * 1 + 5 * 2$$
$$15 = 4 * 0 + 5 * 3$$

Now we show that $\forall k + 1 \geq 16,\ P(k+1):\ k + 1 = 4k_1 + 5k_2$ as follows:

$$\begin{aligned} k + 1 &\geq 16 \\ k + 1 - 4 &\geq 12 \\ k + 1 - 4 &= 4k_1 + 5k_2 \\ k + 1 &= 4(k_1 + 1) + 5k_2 \\ k + 1 &= 4k_3 + 5k_2,\ \exists k_{2,3} \in \mathbb{Z}^+ \end{aligned}$$

Thus we have:

$$\boxed{\forall n \geq 12,\ n \in \mathbb{Z},\ \exists k_{1,2} \in \mathbb{Z}^+ \text{ s.t. } n = 4k_1 + 5k_2 \quad \square}$$

To show that $\forall n \geq 8,\ n \in \mathbb{Z},\ \exists k_{1,2} \in \mathbb{Z}^+$ s.t. $n = 4k_1 + 5k_2$, we only prove the cases where $n = 8, .., 11$ since we have proved the statement $\forall n \geq 12$:

$$\begin{aligned} 8 &= 4 * 2 + 5 * 0 \\ 9 &= 4 * 1 + 5 * 1 \\ 10 &= 4 * 0 + 5 * 2 \\ 11 &\neq 4k_1 + 5k_2,\ \forall k_{1,2} \in \mathbb{Z}^+ \end{aligned}$$

Thus, the statement is not true $\forall n \geq 8$.

Part b

To show that $\forall n \geq 6,\ n = 2k,\ k \in \mathbb{Z},\ \exists k_{1,2} \in \mathbb{Z}^+$ s.t. $n = 3k_1 + 5k_2$, we resort to a proof by strong induction:

$$\begin{aligned} \textbf{Base Case}&: P(6):\ 6 = 3 * 2 + 5 * 0 \\ \textbf{Induction Step}&: 6 \leq j \leq k,\ P(j):\ j = 3k_1 + 5k_2 \\ \textbf{Show}&: \forall k + 2 \geq 8,\ P(k+1):\ k + 2 = 3k_1 + 5k_2 \end{aligned}$$

For $n = 8, .., 12$:

$$\begin{aligned} 8 &= 3 * 1 + 5 * 1 \\ 10 &= 3 * 0 + 5 * 2 \\ 12 &= 3 * 4 + 5 * 0 \end{aligned}$$

Now we show that $\forall n \geq 14,\ n = 2k,\ k \in \mathbb{Z},\ \exists k_{1,2} \in \mathbb{Z}^+$ s.t. $n = 3k_1 + 5k_2$ as follows:

$$\begin{aligned} k + 2 &\geq 14 \\ k + 2 - 8 &\geq 6 \\ k + 2 - 8 &= 3k_1 + 5k_2 \\ k + 2 &= 3(k_1 + 1) + 5(k_2 + 1) \\ k + 2 &= 3k_3 + 5k_4,\ \exists k_{3,4} \in \mathbb{Z}^+ \end{aligned}$$

Thus we have:

$$\boxed{\forall n \geq 6,\ n = 2k,\ k \in \mathbb{Z},\ \exists k_{1,2} \in \mathbb{Z}^+ \text{ s.t. } n = 3k_1 + 5k_2 \quad \square}$$

Problem 7

Part a,b

We want to solve the following optimization problem using the Lagrange multipliers method for $x \in \mathbb{R}^n$ and $M \in \mathbb{S}^n$:

$$\begin{aligned} \max_x \text{ or } \min_x \quad & x^T M x \\ \text{s.t.} \quad & x^T x = 1 \end{aligned}$$

The Lagrangian function $\mathcal{L}(x_1, x_2, \lambda)$ is defined as follows:

$$\mathcal{L}(x, \lambda) = F(x) - G(x)$$

where $F(x)$ is our optimization function and $G(x)$ is the constraint function, both defined as follows:

$$\begin{aligned} F(x) &= x^T M x \\ G(x) &= x^T x - 1 \end{aligned}$$

Thus, we have:

$$\mathcal{L}(x, \lambda) = x^T M x - \lambda(x^T x - 1) = x^T M x + \lambda(1 - x^T x)$$

Now let us solve this minimization problem by first getting the partial derivatives of $\mathcal{L}(x, \lambda)$, we have:

$$\begin{aligned} \mathcal{L}_x(x, \lambda) &= 2Mx - 2\lambda x \\ \mathcal{L}_\lambda(x, \lambda) &= 1 - x^T x \end{aligned}$$

Now equating each of these partial derivatives to 0, we get:

$$\begin{aligned} \mathcal{L}_x(x, \lambda) = 0 &\implies Mx = \lambda x \\ \mathcal{L}_\lambda(x, \lambda) = 0 &\implies G(x) = 0 \end{aligned}$$

From the latter, we can see that $\mathcal{L}_x(x, \lambda) = 0$ leads to $Mx = \lambda x$, which implies that λ and x are the eigenvalues and their corresponding eigenvectors respectively. Since $M \in \mathbb{S}^n$, the eigenvalues and eigenvectors of M are real. Substituting the results obtained in our optimization function, we get:

$$F(x) = x^T M x = x^T \lambda x = \lambda x^T x = \lambda$$

Thus to maximize(minimize) our optimization function, we choose the largest(smallest) eigenvalue of M denoted by $\lambda_{max}(\lambda_{min})$. For the choice of x, we realize from the result of $\mathcal{L}_x(x, \lambda) = 0$ that x is the eigenvector of M corresponding to the eigenvalue λ. Thus a correct choice of x in this optimization problem is the eigenvector corresponding to $\lambda_{max}(\lambda_{min})$.