Homework #4

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Problem 1

To find the dimension of a space V in the vector space $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^4, \mathbb{R})$ spanned by the vectors in (1), we can find $\exists \alpha_i \neq 0 \in \mathcal{F},\ \forall v_i \in \text{span(V)}$ s.t. $\sum_{i=0}^{n} \alpha_i v_i = 0$ and iterate until we reach $\sum_{i=0}^{n} \alpha_i v_i = 0 \implies \forall \alpha_i = 0$. A second approach is to construct a matrix A whose columns are the vectors in span(V) and write it in row echelon form. We will do the later:

$$\text{V} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\} \tag{1}$$

We now compute the row echelon form of A :

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 2 & 0 & 8 & 1 & 3 \\ -1 & 0 & -4 & 1 & 0 \\ 3 & 2 & 8 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 4 & -1 & -3 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & -1 & 2 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & 1/2 & 3/2 \\ 0 & 1 & -2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 4 & 1/2 & 3/2 \\ 0 & 1 & -2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 3/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 1/2 & 3/2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking closely at A, we can see that the highlighted column vectors in (2) form a basis in $\mathbb{R}^3$, and since the last row vector of A is the zero row vector, the dimension of A, dim(A)= 3.

$$A = \begin{bmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Problem 2

Given the two basis $\mathcal{S} = \{e_1, e_2, e_3\}$ and $\mathcal{U} = \{u_{1s}, u_{2s}, u_{3s}\}$ as well as the vector x, we can find the components of the vector x in the basis $\mathcal{U}$ as follows:

$$x = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}, \quad \mathcal{S} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \quad \text{and} \quad \mathcal{U} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \}$$

We can write x as a linear combination of the vectors in basis $\mathcal{U}$ as follows:

$$x = [x]_{\mathcal{S}} = \alpha_1 u_{1s} + \alpha_2 u_{2s} + \alpha_3 u_{3s}$$
$$\begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}_{\mathcal{S}} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We now solve the system of equations:

$$\left.\begin{aligned} 8 &= \alpha_1 \quad +1\alpha_2 + 1\alpha_3 \\ 7 &= \alpha_1 \quad +2\alpha_2 + 2\alpha_3 \\ 4 &= \alpha_1 \quad +2\alpha_2 + 3\alpha_3 \end{aligned}\right\} \implies \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Thus, we find:

$$[x]_{\mathcal{U}} = \begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix}$$

Problem 3

To find the change of basis matrix $P : \mathcal{S} \to \mathcal{U}$, we first represent the elements of the basis $\mathcal{S}$ in $\mathcal{U}$ by writing each element in the basis of $\mathcal{S}$ as linear combination of the elements in $\mathcal{U}$ and then solving the system of equations:

$$e_1 = \alpha_1 u_{1s} + \alpha_2 u_{2s} + \alpha_3 u_{3s}$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We get:

$$[e_1]_{\mathcal{U}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

We do the same for e_2 and e_3 and obtain:

$$[e_2]_{\mathcal{U}} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad [e_3]_{\mathcal{U}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Now finally, we can assemble the transformation matrix P as such:

$$P = \begin{bmatrix} P_1 | P_2 | P_3 \end{bmatrix} \quad \text{where} \quad P_i = [e_i]_{\mathcal{U}} \quad \forall i = 1, \ldots, 3$$

We write:

$$P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Problem 4

Given a robot pose $(x_{\mathcal{R}}, y_{\mathcal{R}})$ in the robot coordinate system $\mathcal{R}$, as well as the world coordinate system $\mathcal{W}$, we can find the transformation matrix from the world frame to the robot frame denoted as $P : \mathcal{W} \to \mathcal{R}$

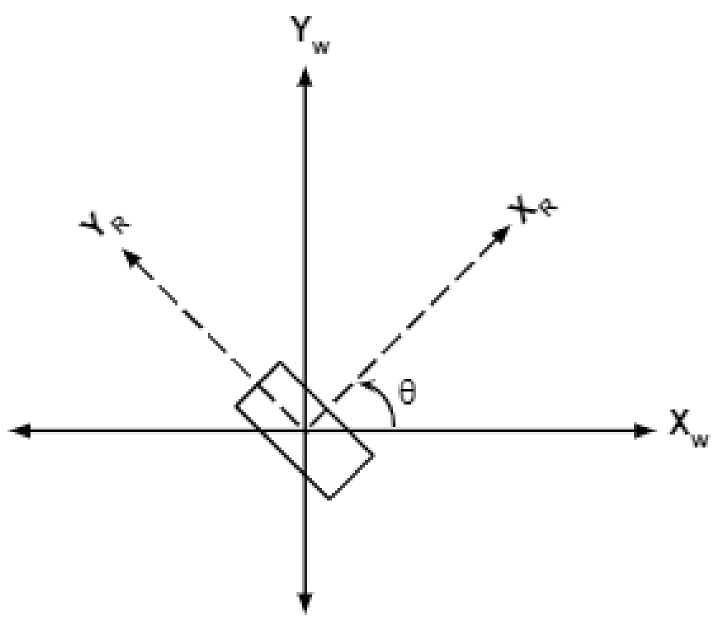

Figure 1: World coordinate system $\mathcal{W}$ and Robot coordinate system $\mathcal{R}$

From figure 1, we can write the following relation between $(x_{\mathcal{R}}, y_{\mathcal{R}})$ and $(x_{\mathcal{W}}, y_{\mathcal{W}})$:

$$x_{\mathcal{W}} = x_{\mathcal{R}} \cos\theta - y_{\mathcal{R}} \sin\theta$$
$$y_{\mathcal{W}} = x_{\mathcal{R}} \sin\theta + y_{\mathcal{R}} \cos\theta$$

Now we can find the transformation matrix from the robot frame to the world frame as such:

$$P^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \implies P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Problem 5

Part a

To show that the ordered matrix $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4) \in \mathbb{R}^{2\times 2}$ is a basis of $\mathbb{R}^{2\times 2}$, we first show that all elements in $\mathcal{M}$ are linearly independent i.e. $\forall \alpha \in \mathbb{R},\ \forall \mathcal{M}_i \in \mathcal{M},\ \sum_{i=0}^{n} \alpha_i \mathcal{M}_i = 0 \implies \alpha_i = 0 \quad i = 1, \ldots, n$ and we show that span($\mathcal{M}$)=$\mathbb{R}^{2\times 2}$:

$$\alpha_1 \mathcal{M}_1 + \alpha_2 \mathcal{M}_2 + \alpha_3 \mathcal{M}_3 + \alpha_4 \mathcal{M}_4 = 0$$
$$\alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We now solve the system of equations:

$$\left.\begin{aligned} 0 &= \alpha_1 - \alpha_2 \\ 0 &= \alpha_3 + \alpha_4 \\ 0 &= \alpha_3 + \alpha_2 \\ 0 &= \alpha_3 - \alpha_4 \end{aligned}\right\} \implies \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the elements that span $\mathcal{M}$ are linearly independent. We now prove that span($\mathcal{M}$)=$\mathbb{R}^{2\times 2}$, i.e every matrix A in $\mathbb{R}^{2\times 2}$ can be written as a linear combination of the elements in $\mathcal{M}$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving the system of we equation, we can find that every component of A can be written as follows:

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} \alpha_3 + \alpha_4 \\ \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 \\ \alpha_3 - \alpha_4 \end{bmatrix}$$

Thus we proved that $\mathcal{M}$ is a basis of $\mathbb{R}^{2\times 2}$ by showing that span($\mathcal{M}$)=$\mathbb{R}^{2\times 2}$ and that the matrices in $\mathcal{M}$ are linearly independent.

Part b

To find the components of the matrix A in the ordered basis $\mathcal{M}$, we write the matrix A as a linear combination of the elements in $\mathcal{M}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving the system of equation, we obtain:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \\ 5/2 \\ -3/2 \end{bmatrix}$$

No we can represent A in basis of $\mathcal{M}$:

$$[A]_{\mathcal{M}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 5/2 & 1/2 \\ 5/2 & -3/2 \end{bmatrix}_{\mathcal{M}}$$

Problem 6

Part a

To find the components of the polynomial $r(x)$ in the ordered basis $\mathcal{S}$, we write the polynomial $r(x)$ as a linear combination of the elements in $\mathcal{S}$:

$$\begin{aligned} r(x) &= \alpha_1 p_0 + \alpha_2 p_1 + \alpha_3 p_2 \\ 2 + 3x - x^2 &= \alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) \end{aligned}$$

Solving the system of equations, we get:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \implies [r(x)]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Part b

To find the transformation matrix $P : \mathcal{S} \to \mathcal{Q}$, we write each element in the basis $\mathcal{S}$ as a linear combination of the elements in $\mathcal{Q}$ as follows:

$$\begin{aligned} p_0 &= \alpha_1 q_0 + \alpha_2 q_1 + \alpha_3 q_2 \\ 1 &= 1(1) + 0(1-x) + 0(x+x^2) \end{aligned}$$

Thus, we get:

$$[p_0]_{\mathcal{Q}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly for $p_{1,2}$, we obtain:

$$[p_1]_{\mathcal{Q}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad [p_2]_{\mathcal{Q}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

We construct the transformation matrix P as follows where $P_i = [p_i]_{\mathcal{Q}},\ \forall i = 0, \ldots, 2$:

$$P = [P_0|P_1|P_2] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the representation $[r(x)]_{\mathcal{Q}}$, we use the following equation:

$$[r(x)]_{\mathcal{Q}} = P[r(x)]_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$$

Problem 7

Part a

To show that $L(M)$ is a linear operator, we show that:

$$\forall \alpha_{1,2} \in \mathbb{R}, \forall M_{1,2} \in \mathbb{R}^{2\times 2} \text{ s.t. } L(\alpha_1 M_1 + \alpha_2 M_2) = \alpha_1 M_1 + \alpha_2 M_2$$

We have:

$$\begin{aligned}
L(\alpha_1 M_1 + \alpha_2 M_2) &= \frac{1}{2}((\alpha_1 M_1 + \alpha_2 M_2) + (\alpha_1 M_1 + \alpha_2 M_2)^T) \\
&= \frac{1}{2}(\alpha_1 M_1 + \alpha_2 M_2 + \alpha_1^T M_1^T + \alpha_2^T M_2^T) \\
&= \frac{1}{2}(\alpha_1 M_1 + \alpha_2 M_2 + \alpha_1 M_1^T + \alpha_2 M_2^T) \\
&= \frac{1}{2}(\alpha_1 M_1 + \alpha_1 M_1^T) + \frac{1}{2}(\alpha_2 M_2 + \alpha_2 M_2^T) \\
&= \frac{\alpha_1}{2}(M_1 + M_1^T) + \frac{\alpha_2}{2}(M_2 + M_2^T) \\
&= \alpha_1 L(M_1) + \alpha_2 L(M_2)
\end{aligned}$$

Part b

To compute A, the matrix representation of the linear operator $L(.)$, we compute the mapping $L : \mathcal{X} \to \mathcal{X}$ of the elements of the basis E:

$$L(E_{11}) = L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Similarly for $E_{12,21,22}$, we get:

$$L(E_{12}) = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad L(E_{21}) = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad L(E_{22}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we construct the matrix $A = [A_1|\dots|A_4]$, where A_i is a column vector representation of the elements of the basis E as such:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 8

Part a

Let E denote the basis of the vector space $(\mathcal{X}, \mathcal{F}) = (\mathbb{C}^n, \mathbb{C})$, we can write the basis as follows:

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

To compute $\hat{A} = [\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n]$, the matrix representation of the linear operator $L(.)$, we compute the mapping $L : \mathbb{C}^n \to \mathbb{C}^n$ of the elements of the basis E:

$$\hat{A}_i = L(E_i)$$

Thus, for $\hat{A}_1$, we have:

$$\hat{A}_1 = L(E_1) = L \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Similarly for $\hat{A}_{1,\ldots,n}$, we construct the matrix $\hat{A}$ as follows:

$$\hat{A} = \left[A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, A \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right] = A \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = AI = A$$

Part b

We will prove that if A has distinct eigenvalues, then A can be written as a diagonal matrix whose elements are the eigenvalues of A. Thus we can represent A as such:

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

First, introducing the basis B whose elements are the eigenvectors v_i of A:

$$B = \{v_1, v_2, \ldots, v_n\}$$

To compute $\hat{A} = [\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n]$, the matrix representation of the linear operator $L(.)$, we compute the mapping $L : \mathbb{C}^n \to \mathbb{C}^n$ of the elements of the basis B: Thus, for $\hat{A}_1$, we have:

$$\hat{A}_1 = L(v_1) = Av_1 = \lambda_1 v_1$$

We construct the matrix $\hat{A}$ as follows:

$$\begin{aligned}\hat{A} &= [\hat{A}_1|\hat{A}_2|\ldots|\hat{A}_n] \\ &= [L(v_1), L(v_2), \ldots, L(v_n)] \\ &= [Av_1, Av_2, \ldots, Av_n] \\ &= [\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n]\end{aligned}$$

Now representing the components of matrix $\hat{A}$ in terms of the elements of the basis B, we obtain the following for $\hat{A}_1$:

$$[\hat{A}_1]_B = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Similarly for $\hat{A}_{1,\ldots,n}$, $\hat{A}$ can be written as:

$$\hat{A} = [\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n] = \left[\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Thus we notice that for a specific basis B, we obtain a unique representation matrix $\hat{A}$ of a linear operator $L(.)$, and when the basis' elements are the eigenvectors of A, we obtain:

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$