Exam Number: I

ROB 501 Exam-I

From Thursday, October 28, 2021 NOON to Friday, October 29, 2021 11:59pm

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RULES:

1. The exam is open book, open lecture handouts and slides, open recitation notes, open HW solutions, open internet (under the communication and usage restrictions mentioned below).
2. If you use MATLAB or any other scientific software to complete some parts of the exam. You are required to submit your script along with your solution in such case.
3. You are not allowed to communicate with anyone other than the Course instructor and the GSIs related to the exam during the entire period. If you have questions, you can post a private Piazza post for the instructors or email `necmiye@umich.edu` with GSIs on cc.
4. You are not allowed to use any online "course helper" sites like Chegg, Course Hero, and Slader, in any part of the exam. You are not allowed to post exam questions on the internet or discuss them online.
5. Please do not wait until the last minute to upload your solution to Gradescope and double-check to make sure you uploaded the correct pdf. If you run into problems with Gradescope, email your .pdf file as an attachment to Prof. Ozay as soon as practicable at `necmiye@umich.edu`.

SUBMISSION AND GRADING INSTRUCTIONS:

1. The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly.
2. You must submit your solutions in a single pdf. You will be asked to mark where each solution is.
3. **Honor Code:** The first page of your submitted pdf should include a hand-written and signed honor code (see the first page of this pdf). Without this, your exam will not be graded.
4. **For problems 1-5** Use this page to record your answers. We will NOT grade other pages and we do not care if you make a mistake when copying your answers to this page. Please be careful. If you are submitting handwritten (or word-processed) documents, make sure to make a similar table where you record all your True/False answers. There is no partial credit on these questions. You are welcome to leave some justification but we will not look at them.
5. **For problems 6-7** Record your final answer in the box provided. If you are submitting handwritten (or word-processed) documents, make sure to box or highlight the final result. However, you MUST show your work to get credit. In other words, a correct result with no reasoning or wrong reasoning could lead to no points.
6. **For problems 8a, 8b** These are proof questions. You should show all the steps of your proof carefully.

Answers for the True/False Part

	(a)	(b)	(c)	(d)
Problem 1	True	True	False	False
Problem 2	True	False	False	True
Problem 3	Fasle	True	True	True
Problem 4	False	True	False	False
Problem 5	False	False	False	False

Problem 6

Part a

To find the matrix representation A of the linear operator $L(.)$ with respect to the basis $\{v_1, v_2, v_3, v_4\}$, we first apply the linear operator to every element in the basis as such:

$$\begin{aligned}
L(v_1) &= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} v_1 \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - v_1^T \\
&= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0.9 & 0 \\ -0.1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.81 & 0.09 \\ -0.09 & -0.01 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -0.19 & 0.09 \\ -0.09 & -0.01 \end{bmatrix}
\end{aligned}$$

Similarly for $v_{2,3,4}$:

$$\begin{aligned}
L(v_2) &= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} v_2 \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - v_2^T \\
&= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.04 & 0.2 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.04 & 0.2 \\ 0.2 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
L(v_3) &= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} v_3 \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - v_3^T \\
&= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 0.9 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.18 & 0.9 \\ -0.02 & -0.1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.18 & 0.9 \\ -1.02 & -0.1 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
L(v_4) &= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} v_4 \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - v_4^T \\
&= \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0.2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.18 & 0.02 \\ 0.9 & 0.1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.18 & -0.98 \\ 0.9 & 0.1 \end{bmatrix}
\end{aligned}
$$

Now, we construct the matrix $A = [A_1| \dots |A_4]$, where A_i is a column vector representation of the linear operator $L(.)$ applied on the elements of the basis $\{v_1, v_2, v_3, v_4\}$ as such:

$$
\boxed{A = \begin{bmatrix} -0.19 & 0.04 & 0.18 & 0.18 \\ -0.01 & 0 & -0.1 & 0.1 \\ 0.09 & 0.2 & 0.9 & -0.98 \\ -0.09 & 0.2 & -1.02 & 0.9 \end{bmatrix}}
$$

Part b

To find the change or basis P from $\{v_1, v_2, v_3, v_4\}$ to $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$, we first find the change or basis $\bar{P}$ from $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ to $\{v_1, v_2, v_3, v_4\}$ and compute its inverse, as such:

$$
[\bar{v}_1]_V = [1 * v_1 + 0 * v_2 + 0 * v_3 + 0 * v_4]_V = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
[\bar{v}_2]_V = [1 * v_1 + 1 * v_2 + 0 * v_3 + 0 * v_4]_V = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
[\bar{v}_3]_V = [3 * v_1 + 1 * v_2 + 1 * v_3 + 0 * v_4]_V = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
[\bar{v}_3]_V = [1 * v_1 - 1 * v_2 + 1 * v_3 - 1 * v_4]_V = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}
$$

Constructing the matrix $\bar{P} = \begin{bmatrix}[\bar{v}_1]_V & [\bar{v}_2]_V & [\bar{v}_3]_V & [\bar{v}_4]_V\end{bmatrix}$ as such:

$$\bar{P} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Finally, we can compute the change or basis P from $\{v_1, v_2, v_3, v_4\}$ to $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$:

$$P = \bar{P}^{-1} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Thus:

$$\boxed{P = \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$$

Problem 7

Part a

To find a basis u for S, we need to check first if the matrices that span S are linearly dependent:

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, we can form a system of 4 equations:

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 - \alpha_2 + \alpha_3 + 3\alpha_4 &= 0 \\ \alpha_3 - \alpha_4 &= 0 \\ \alpha_1 + 2\alpha_4 &= 0 \end{aligned}$$

Thus, setting $\alpha_4 = 1$, we get:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the matrices that span S are linearly dependent. We check for the first 3 matrices:

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, we can form a system of 4 equations:

$$\begin{aligned}\alpha_1 + \alpha_2 &= 0\\ \alpha_1 - \alpha_2 + \alpha_3 &= 0\\ \alpha_3 &= 0\\ \alpha_1 &= 0\end{aligned}$$

Thus, the only solution is the trivial solution and the matrices are linearly independent:

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Thus, a basis for S is:

$$\boxed{u = \left\{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\}}$$

Part b

To find the representation of W with to the basis u denoted by $[W]_u$, we write W as a linear combination of the elements in the basis u:

$$\begin{bmatrix}2 & 5\\ -1 & 4\end{bmatrix} = \alpha_1 \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} + \alpha_2 \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} + \alpha_3 \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Thus, we can form a system of 4 equations:

$$\begin{aligned}\alpha_1 + \alpha_2 &= 2\\ \alpha_1 - \alpha_2 + \alpha_3 &= 5\\ \alpha_3 &= -1\\ \alpha_1 &= 4\end{aligned}$$

Thus, we obtain:

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}4\\-2\\-1\end{bmatrix}$$

Finally:

$$\boxed{[W]_u = \begin{bmatrix}4\\-2\\-1\end{bmatrix}}$$

Part c

To find a basis for $S^\perp$, the orthogonal complement of S, we first need to define $S^\perp$:

$$S^\perp := \{X \mid \forall Y \in S,\ \langle X, Y\rangle = 0\}$$

Since the $\dim(\mathbb{R}^{2\times 2}) = 4$, from the rank nullity theorem, we conclude that there is only one element in the basis of $S^{\perp}$. Denote this element as:

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

Writing the inner of product of X with each element in the basis of S, we get:

$$\langle X, u_1 \rangle = \mathrm{tr}(X^T u_1) = \mathrm{tr}\left(\begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = tr\left(\begin{bmatrix} x_{11} & x_{11}+x_{21} \\ x_{12} & x_{12}+x_{22} \end{bmatrix}\right) = x_{11} + x_{12} + x_{22} = 0$$
$$\langle X, u_2 \rangle = \mathrm{tr}(X^T u_2) = \mathrm{tr}\left(\begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = tr\left(\begin{bmatrix} x_{11} & -x_{11} \\ x_{12} & -x_{12} \end{bmatrix}\right) = x_{11} - x_{12} = 0$$
$$\langle X, u_3 \rangle = \mathrm{tr}(X^T u_3) = \mathrm{tr}\left(\begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = tr\left(\begin{bmatrix} x_{21} & x_{11} \\ x_{22} & x_{12} \end{bmatrix}\right) = x_{21} + x_{12} = 0$$

Thus, we can form a system of 3 equations:

$$\begin{aligned} x_{11} + x_{12} + x_{22} &= 0 \\ x_{11} - x_{12} &= 0 \\ x_{21} + x_{12} &= 0 \end{aligned}$$

Setting $x_{22} = 2$, we get:

$$X = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

Thus, a basis for $S^{\perp}$ is:

$$\boxed{\mathrm{basis}(S^{\perp}) = \left\{\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\right\}}$$

Part d

$$\hat{X} = \underset{X \in S}{\arg\min}\ d(X, Y)$$

Where:

$$d(X, Y) = ||X - Y|| = \langle X, Y \rangle^{\frac{1}{2}} = \sqrt{\mathrm{tr}((X-Y)^T(X-Y))}$$

And:

$$Y = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

The solution is of the form:

$$\hat{X} = \sum_{i=1}^{3} \alpha_i u_i = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$$

Solving for the coefficients $\alpha_{1,3}$:

$$\alpha = (G^{-1})^T \beta$$

Thus:

$$\langle u_1, u_1\rangle = \mathrm{tr}(u_1^T u_1) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\right) = 3$$

$$\langle u_2, u_2\rangle = \mathrm{tr}(u_2^T u_2) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right) = 2$$

$$\langle u_3, u_3\rangle = \mathrm{tr}(u_3^T u_3) = \mathrm{tr}\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 2$$

$$\langle u_1, u_2\rangle = \langle u_2, u_1\rangle = \mathrm{tr}(u_1^T u_2) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\right) = 0$$

$$\langle u_1, u_3\rangle = \langle u_3, u_1\rangle = \mathrm{tr}(u_1^T u_3) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\right) = 1$$

$$\langle u_3, u_2\rangle = \langle u_2, u_3\rangle = \mathrm{tr}(u_2^T u_3) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\right) = -1$$

$$\langle u_1, Y\rangle = \langle Y, u_1\rangle = \mathrm{tr}(u_1^T Y) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}0 & 1\\1 & 3\end{bmatrix}\right) = 3$$

$$\langle u_2, Y\rangle = \langle Y, u_2\rangle = \mathrm{tr}(u_2^T Y) = \mathrm{tr}\left(\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\right) = -1$$

$$\langle u_3, Y\rangle = \langle Y, u_3\rangle = \mathrm{tr}(u_3^T Y) = \mathrm{tr}\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}\right) = \mathrm{tr}\left(\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\right) = 2$$

Constructing the Gram matrix, we obain:

$$G = \begin{bmatrix}\langle u_1, u_1\rangle & \langle u_1, u_2\rangle & \langle u_1, u_3\rangle\\ \langle u_2, u_1\rangle & \langle u_2, u_2\rangle & \langle u_2, u_3\rangle\\ \langle u_3, u_1\rangle & \langle u_3, u_2\rangle & \langle u_3, u_3\rangle\end{bmatrix} = \begin{bmatrix}3 & 0 & 1\\0 & 2 & -1\\1 & -1 & 2\end{bmatrix}$$

$$\beta = \begin{bmatrix}\langle u_1, Y\rangle\\ \langle u_2, Y\rangle\\ \langle u_3, Y\rangle\end{bmatrix} = \begin{bmatrix}3\\-1\\2\end{bmatrix}$$

Using the following MATLAB code, we compute the coefficients as follows:

```
%% Exam-I Problem 7d
format rat
G = [3 0 1;0 2 -1;1 -1 2];
b = [3;-1;2];
u_1 = [1 1;0 1];
u_2 = [1 -1;0 0];
u_3 = [0 1;1 0];
alphas=inv(G)'*b;
x_hat=alphas(1)*u_1+alphas(2)*u_2+alphas(3)*u_3;
```

We get:

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \frac{1}{7}\begin{bmatrix}6\\-2\\3\end{bmatrix}$$

Finally:

$$\boxed{\hat{X} = \frac{1}{7}\begin{bmatrix} 4 & 11 \\ 3 & 6 \end{bmatrix}}$$

Problem 8

Part a

To prove that $\langle f, g\rangle_\eta = \int_{-2}^{2} f(t)\eta(t)g(t)dt$ is an inner product on $(\mathcal{X}, \mathbb{R})$ where $\mathcal{X} = \{f : [-2, 2] \longrightarrow \mathbb{R},\ f \text{ continuous}\}$, we prove that it satisfies the three following properties:

- $$\forall f, g \in \mathcal{X},\ \langle f, g\rangle_\eta = \langle g, f\rangle_\eta$$
- $$\forall \alpha_{1,2} \in \mathbb{R},\ \forall f_{1,2} \in \mathcal{X},\ \forall g \in \mathcal{X},\ \langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle_\eta = \alpha_1\langle f_1, g\rangle_\eta + \alpha_2\langle f_2, g\rangle_\eta$$
- $$\forall f \in \mathcal{X},\ \langle f, f\rangle_\eta \geq 0 \text{ and } \langle f, f\rangle_\eta = 0 \Longleftrightarrow f(t) = 0$$

We check these properties as follows:

$$\langle f, g\rangle_\eta = \int_{-2}^{2} f(t)\eta(t)g(t)dt = \int_{-2}^{2} g(t)\eta(t)f(t)dt = \langle g, f\rangle_\eta$$

$$\Longrightarrow \langle f, g\rangle_\eta = \langle g, f\rangle_\eta$$

For the second property, noting that the integral is definite, we can write:

$$\begin{aligned}
\langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle_\eta &= \int_{-2}^{2} (\alpha_1 f_1(t) + \alpha_2 f_2(t))\eta(t)g(t)dt \\
&= \int_{-2}^{2} (\alpha_1 f_1(t)\eta(t)g(t) + \alpha_2 f_2(t)\eta(t)g(t))dt \\
&= \int_{-2}^{2} \alpha_1 f_1(t)\eta(t)g(t)dt + \int_{-2}^{2} \alpha_2 f_2(t)\eta(t)g(t)dt \\
&= \alpha_1 \int_{-2}^{2} f_1(t)\eta(t)g(t)dt + \alpha_2 \int_{-2}^{2} f_2(t)\eta(t)g(t)dt \\
&= \alpha_1\langle f_1, g\rangle + \alpha_2\langle f_2, g\rangle
\end{aligned}$$

$$\Longrightarrow \langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle_\eta = \alpha_1\langle f_1, g\rangle_\eta + \alpha_2\langle f_2, g\rangle_\eta$$

For the third and final property:

$$\langle f, f\rangle_\eta = \int_{-2}^{2} f(t)\eta(t)f(t)dt = \int_{-2}^{2} f^2(t)\eta(t)dt$$

Since $f(t)$ is continuous, so is $f^2(t)$, and we can split the integral into three distinct regions over [-2,2]:

$$\langle f, f\rangle_\eta = \int_{-2}^{-1} f^2(t)\eta(t)dt + \int_{-1}^{1} f^2(t)\eta(t)dt + \int_{1}^{2} f^2(t)\eta(t)dt$$

The function $\eta(t)$ is defined as follows:

$$\eta(t) = \begin{cases} |t| & \forall t \in [-2,-1] \cup [1,2] \\ 1 & \forall t \in [-1,1] \end{cases}$$

Thus, we obtain:

$$\begin{aligned} \langle f, f\rangle_\eta &= \int_{-2}^{-1} f^2(t)\eta(t)dt + \int_{-1}^{1} f^2(t)\eta(t)dt + \int_{1}^{2} f^2(t)\eta(t)dt \\ &= \int_{-2}^{-1} |t| f^2(t)dt + \int_{-1}^{1} f^2(t)dt + \int_{1}^{2} |t| f^2(t)dt \end{aligned}$$

Since the integral of a nonegative function is nonegative and since $|t|f^2(t) \geq 0$ and $f^2(t) \geq 0$ for all $t \in [-2, 2]$, then we can write that:

$$\forall f \in \mathcal{X},\ \langle f, f\rangle_\eta \geq 0$$

Finally, solving for the following:

$$\langle f, f\rangle_\eta = \int_{-2}^{-1} |t| f^2(t)dt + \int_{-1}^{1} f^2(t)dt + \int_{1}^{2} |t| f^2(t)dt = 0$$

This implies that every integral is equal to zero, thus:

$$\begin{aligned} \int_{-2}^{-1} |t| f^2(t)dt = 0 &\implies f(t) = 0 \text{ since } t \neq 0\ \forall t \in [-2,-1] \\ \int_{-1}^{1} f^2(t)dt = 0 &\implies f(t) = 0\ \forall t \in [-2,-1] \\ \int_{1}^{2} |t| f^2(t)dt = 0 &\implies f(t) = 0 \text{ since } t \neq 0\ \forall t \in [1,2] \\ \implies \forall f \in \mathcal{X},\ \langle f, f\rangle_\eta \geq 0 &\text{ and } \langle f, f\rangle_\eta = 0 \iff f(t) = 0 \end{aligned}$$

Part b

We denote S_1 and S_2 as follows:

$$S_1 := \{x \mid \exists y \in Y,\ \exists z \in Z,\ x = f(y) + 5 + z\}$$

$$S_2 := \{x \in \mathbb{R} \mid \exists z \in Z,\ f^{-1}(x-5-z) \in Y\}$$

To show that $S_1 = S_2$, we show that $S_1 \subset S_2$ and that $S_2 \subset S_1$. We start with proving the former:

$$\begin{aligned} S_1 &:= \{x \mid \exists y \in Y,\ \exists z \in Z,\ x = f(y) + 5 + z\} \\ &:= \{x \mid \exists y \in Y,\ \exists z \in Z,\ x - 5 - z = f(y)\} \\ &:= \{x \mid \exists y \in Y,\ \exists z \in Z,\ f(y) = x - 5 - z\} \end{aligned}$$

Since $x \in \mathbb{R}$, $y \in Y \subseteq \mathbb{R}$ and $z \in Z \subseteq \mathbb{R}$, then $x - 5 - z \in \mathbb{R}$ and we can use the properties of an invertible function f as such:

$$S_1 := \{x \mid \exists y \in Y,\ \exists z \in Z,\ y = f^{-1}(x-5-z)\}$$

Thus:

$$\forall x \in S_1, \exists y \in Y, \quad \text{s.t.} \quad y = f^{-1}(x-5-z) \in Y$$

Since S_2 is a subset that includes $x \in \mathbb{R}$ such that $f^{-1}(x-5-z) \in Y$:

$$\implies S_1 \subset S_2$$

Now we prove the latter:

$$S_2 := \{x \in \mathbb{R} \mid \exists z \in Z,\ f^{-1}(x-5-z) \in Y\}$$

This means that there exist an element $y \in Y$ such that $y = f^{-1}(x-5-z)$ for some $x \in \mathbb{R}$ and some $z \in Z$. Thus:

$$\begin{aligned} S_2 &:= \{x \in \mathbb{R} \mid \exists z \in Z,\ \exists y \in Y,\ y = f^{-1}(x-5-z)\} \\ &:= \{x \in \mathbb{R} \mid \exists z \in Z,\ \exists y \in Y,\ f(y) = x - 5 - z\} \\ &:= \{x \in \mathbb{R} \mid \exists z \in Z,\ \exists y \in Y,\ x = f(y) + 5 + z\} \end{aligned}$$

Since S_1 is a subset that includes $x \in \mathbb{R}$ such that $x = f(y) + 5 + z$ for some $y \in Y$ and some $x \in Z$:

$$\implies S_2 \subset S_1$$

Finally:

$$\boxed{S_1 = S_2} \quad \square$$