Homework #11

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December 29, 2021

Problem 1

In this problem, for each proof, the following norms are used for $x \in \mathbb{R}^2$:

$$||x||_2 = \sqrt{x_1^2 + x_2^2}$$
$$||x||_1 = |x_1| + |x_2|$$
$$||x||_\infty = \max(x_1, x_2)$$

STATEMENT

$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$$

PROOF
Starting with the left hand side inequality:

$$||x||_2 \leq ||x||_1$$

Since a norm is always positive:

$$\begin{aligned} ||x||_2^2 &\leq ||x||_1^2 \\ x_1^2 + x_2^2 &\leq (|x_1| + |x_2|)^2 \\ x_1^2 + x_2^2 &\leq x_1^2 + x_2^2 + 2|x_1 x_2| \end{aligned}$$

The above inequality is true since $2|x_1 x_2| \geq 0$. Now for the right hand side:

$$||x||_1 \leq \sqrt{n}||x||_2$$

Again. since a norm is always positive:

$$\begin{aligned} ||x||_1^2 &\leq 2||x||_2^2 \\ (|x_1| + |x_2|)^2 &\leq 2(x_1^2 + x_2^2) \\ x_1^2 + x_2^2 + 2|x_1 x_2| &\leq 2x_1^2 + 2x_2^2 \\ 0 &\leq x_1^2 + x_2^2 - 2|x_1 x_2| \\ 0 &\leq (|x_1| - |x_2|)^2 \end{aligned}$$

The above inequality is true and thus we have proved the statement.

STATEMENT

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

PROOF

Starting with the left hand side inequality:

$$||x||_\infty \leq ||x||_2$$

Considering the case where $x_1 \geq x_2$ and since a norm is always positive:

$$\begin{aligned} ||x||_\infty^2 &\leq ||x||_2^2 \\ x_1^2 &\leq x_1^2 + x_2^2 \\ 0 &\leq x_2^2 \end{aligned}$$

Now considering the case where $x_2 \geq x_1$:

$$\begin{aligned} ||x||_\infty^2 &\leq ||x||_2^2 \\ x_2^2 &\leq x_1^2 + x_2^2 \\ 0 &\leq x_1^2 \end{aligned}$$

The above inequality is true. Now for the right hand side:

$$||x||_2 \leq \sqrt{n}||x||_\infty$$

Again. considering the case where $x_1 \geq x_2$ and since a norm is always positive:

$$\begin{aligned} ||x||_2^2 &\leq 2||x||_\infty^2 \\ x_1^2 + x_2^2 &\leq 2x_1^2 \\ x_2^2 &\leq x_1^2 \end{aligned}$$

The above inequality is true since $x \in \mathcal{X} \longrightarrow [0, \infty)$. Now for the case where $x_2 \geq x_1$:

$$\begin{aligned} ||x||_2^2 &\leq 2||x||_\infty^2 \\ x_1^2 + x_2^2 &\leq 2x_2^2 \\ x_1^2 &\leq x_2^2 \end{aligned}$$

The above inequality is true since $x \in \mathcal{X} \longrightarrow [0, \infty)$ and the statement has been proved.

STATEMENT

$$||x||_\infty \leq ||x||_1 \leq n||x||_\infty$$

PROOF

Starting with the left hand side inequality:

$$||x||_\infty \leq ||x||_1$$

Considering the case where $x_1 \geq x_2$ and since a norm is always positive:

$$\begin{aligned} ||x||_\infty^2 &\leq ||x||_1^2 \\ x_1^2 &\leq (|x_1| + |x_2|)^2 \\ x_1^2 &\leq x_1^2 + x_2^2 + 2|x_1x_2| \\ 0 &\leq x_2^2 + 2|x_1x_2| \end{aligned}$$

Now considering the case where $x_2 \geq x_1$:

$$\begin{aligned} ||x||_\infty^2 &\leq ||x||_1^2 \\ x_2^2 &\leq (|x_1| + |x_2|)^2 \\ x_2^2 &\leq x_1^2 + x_2^2 + 2|x_1x_2| \\ 0 &\leq x_1^2 + 2|x_1x_2| \end{aligned}$$

The above inequality is true. Now for the right hand side:

$$||x||_1 \leq n||x||_\infty$$

Again. considering the case where $x_1 \geq x_2$ and since a norm is always positive:

$$\begin{aligned} ||x||_1^2 &\leq 4||x||_\infty^2 \\ (|x_1| + |x_2|)^2 &\leq 4x_1^2 \\ x_1^2 + x_2^2 + 2|x_1x_2| &\leq x_1^2 + x_1^2 + 2x_1^2 \\ x_2^2 + 2|x_1x_2| &\leq x_1^2 + 2x_1^2 \end{aligned}$$

The above inequality is true since $x \in \mathcal{X} \longrightarrow [0, \infty)$, thus $x_1^2 \geq x_2^2$ and $2x_1^2 \geq 2|x_1x_2|$. Now for the case where $x_2 \geq x_1$:

$$\begin{aligned} ||x||_1^2 &\leq 4||x||_\infty^2 \\ (|x_1| + |x_2|)^2 &\leq 4x_2^2 \\ x_1^2 + x_2^2 + 2|x_1x_2| &\leq x_2^2 + x_2^2 + 2x_2^2 \\ x_1^2 + 2|x_1x_2| &\leq x_2^2 + 2x_2^2 \end{aligned}$$

The above inequality is true since $x \in \mathcal{X} \longrightarrow [0, \infty)$, thus $x_2^2 \geq x_1^2$ and $2x_2^2 \geq 2|x_1x_2|$. The statement is proved true.

Problem 2

Part a

STATEMENT

$$\tilde{B}_{\frac{a}{K_2}}(x_0) \subset B_a(x_0) \subset \tilde{B}_{\frac{a}{K_1}}(x_0)$$

PROOF
We first prove the left hand side inclusion $\tilde{B}_{\frac{a}{K_2}}(x_0) \subset B_a(x_0)$ by showing that $x \in \tilde{B}_{\frac{a}{K_2}}(x_0) \implies x \in B_a(x_0)$. Let us first define the definition of each open ball in the above equation:

$$B_a(x_0) := \{x \in \mathcal{X} \mid ||x - x_0|| < a\}$$
$$\tilde{B}_{\frac{a}{K_2}}(x_0) := \{x \in \mathcal{X} \mid |||x - x_0||| < \frac{a}{K_2}\}$$

Since $||.||$ and $|||.|||$ are equivalent norms and for $x \in \tilde{B}_{\frac{a}{K_2}}(x_0)$, we have:

$$\frac{1}{K_2}||x - x_0|| \leq |||x - x_0|||$$
$$||x - x_0|| \leq K_2|||x - x_0|||$$

Since $x \in \tilde{B}_{\frac{a}{K_2}}(x_0)$, then :

$$|||x - x_0||| < \frac{a}{K_2}$$
$$K_2|||x - x_0||| < a$$

Thus we have:

$$||x - x_0|| \leq K_2|||x - x_0||| < a \quad \implies \quad ||x - x_0|| < a \quad \implies \quad x \in B_a(x_0)$$

Now proving the right hand side inclusion by showing that $x \in B_a(x_0) \implies x \in \tilde{B}_{\frac{a}{K_1}}(x_0)$, we have:

$$B_a(x_0) := \{x \in \mathcal{X} \mid ||x - x_0|| < a\}$$
$$\tilde{B}_{\frac{a}{K_1}}(x_0) := \{x \in \mathcal{X} \mid |||x - x_0||| < \frac{a}{K_1}\}$$

Since $||.||$ and $|||.|||$ are equivalent norms and for $x \in \tilde{B}_{\frac{a}{K_1}}(x_0)$, we have:

$$K_1|||x - x_0||| \leq ||x - x_0||$$
$$|||x - x_0||| \leq \frac{1}{K_1}||x - x_0||$$

Since $x \in B_a(x_0)$, then :

$$||x - x_0|| < a$$
$$\frac{1}{K_1}||x - x_0|| < \frac{a}{K_1}$$

Thus we have:

$$|||x - x_0||| \leq \frac{1}{K_1}||x - x_0|| < \frac{a}{K_1} \quad \implies \quad |||x - x_0||| < \frac{a}{K_1} \quad \implies \quad x \in \tilde{B}_{\frac{a}{K_1}}(x_0)$$

Now since we proved both inclusions, we can deduce that $x \in \tilde{B}_{\frac{a}{K_2}}(x_0) \implies x \in \tilde{B}_{\frac{a}{K_1}}(x_0)$ and we have proved the statement.

Part b

STATEMENT

$$P \text{ is open in } (\mathcal{X}, \mathbb{R}, ||.||) \iff P \text{ is open in } (\mathcal{X}, \mathbb{R}, |||.|||)$$

PROOF

Starting by proving $\implies$:

$$P \text{ is open} \implies \forall x \in P,\ \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subset P$$

Define the open balls below:

$$B_{\epsilon_1}(x) := \{y \in X \mid ||x-y|| < \epsilon_1\}$$

$$\tilde{B}_{\epsilon_2}(x) := \{y \in X \mid |||x-y||| < \epsilon_2\}$$

Since $||.||$ and $|||.|||$ are equivalent norms, we have:

$$K_1|||x-y||| \leq ||x-y|| \leq K_2|||x-y|||$$

Thus:

$$K_1|||x-y||| \leq ||x-y|| < \epsilon_1 \quad \implies \quad K_1|||x-y||| < \epsilon_1 \quad \implies \quad |||x-y||| < \frac{\epsilon_1}{K_1}$$

Now defining $\epsilon_2 = \frac{\epsilon_1}{K_1}$, we get:

$$|||x-y||| < \epsilon_2$$

Thus:

$$\forall x \in P,\ y \in B_{\epsilon_1}(x) \implies y \in \tilde{B}_{\epsilon_2}(x)$$

Finally, we can write:

$$\forall x \in P,\ \exists \epsilon_2 > 0 \text{ s.t. } \tilde{B}_{\epsilon_2}(x) \subset P$$

Now proving $\Longleftarrow$:
Since $||.||$ and $|||.|||$ are equivalent norms, we have:

$$\frac{1}{K_2}||x-y|| \leq |||x-y||| \leq \frac{1}{K_1}||x-y||$$

Thus:

$$\frac{1}{K_2}||x-y|| \leq |||x-y||| < \epsilon_2 \quad \implies \quad \frac{1}{K_2}||x-y|| < \epsilon_2 \quad \implies \quad ||x-y|| < K_2\epsilon_2$$

Now defining $\epsilon_1 = K_2\epsilon_2$, we get:

$$||x-y|| < \epsilon_1$$

Thus:

$$\forall x \in P,\ y \in \tilde{B}_{\epsilon_2}(x) \implies y \in B_{\epsilon_1}(x)$$

Finally, we can write:

$$\forall x \in P,\ \exists \epsilon_1 > 0 \text{ s.t. } B_{\epsilon_1}(x) \subset P$$

Part c

STATEMENT

$$(x_n) \text{ is Cauchy in } (\mathcal{X}, \mathbb{R}, ||.||) \Longleftrightarrow (x_n) \text{ is Cauchy in } (\mathcal{X}, \mathbb{R}, |||.|||)$$

PROOF

Starting by proving $\implies$:

$$(x_n) \text{ is Cauchy in } (\mathcal{X}, \mathbb{R}, ||.||) \implies \forall\epsilon > 0\ \exists N(\epsilon) < \infty \text{ s.t. } \forall n, m \geq N,\ ||x_n - x_m|| < \epsilon$$

Since $||.||$ and $|||.|||$ are equivalent norms, we have:

$$K_1|||x_n - x_m||| \leq ||x_n - x_m|| \leq K_2||x_n - x_m|||$$

Thus:

$$K_1|||x_n - x_m||| \leq ||x_n - x_m|| < \epsilon \quad \implies \quad K_1|||x_n - x_m||| < \epsilon \quad \implies \quad |||x_n - x_m||| < \frac{\epsilon}{K_1}$$

Now defining $\tilde{\epsilon} = \frac{\epsilon}{K_1}$, we get:

$$|||x_n - x_m||| < \tilde{\epsilon}$$

Thus:

$$\forall\tilde{\epsilon} > 0\ \exists N(\tilde{\epsilon}) < \infty \text{ s.t. } \forall n, m \geq N,\ |||x_n - x_m||| < \tilde{\epsilon}$$

Finally, we can write:

$$\forall\epsilon > 0\ \exists N(\epsilon) < \infty \text{ s.t. } \forall n, m \geq N,\ ||x_n - x_m|| < \epsilon$$
$$\implies \forall\tilde{\epsilon} > 0\ \exists N(\tilde{\epsilon}) < \infty \text{ s.t. } \forall n, m \geq N,\ |||x_n - x_m||| < \tilde{\epsilon}$$

Now proving $\Longleftarrow$:

$$(x_n) \text{ is Cauchy in } (\mathcal{X}, \mathbb{R}, |||.|||) \implies \forall\epsilon > 0\ \exists N(\epsilon) < \infty \text{ s.t. } \forall n, m \geq N,\ |||x_n - x_m||| < \epsilon$$

Since $||.||$ and $|||.|||$ are equivalent norms, we have:

$$\frac{1}{K_2}||x_n - x_m|| \leq |||x_n - x_m||| \leq \frac{1}{K_1}||x_n - x_m||$$

Thus:

$$\frac{1}{K_2}||x_n - x_m|| \leq |||x_n - x_m||| < \epsilon \quad \implies \quad \frac{1}{K_2}||x_n - x_m|| < \epsilon \quad \implies \quad ||x_n - x_m|| < K_2\epsilon$$

Now defining $\tilde{\epsilon} = K_2\epsilon$, we get:

$$||x_n - x_m|| < \tilde{\epsilon}$$

Thus:

$$\forall\tilde{\epsilon} > 0\ \exists N(\tilde{\epsilon}) < \infty \text{ s.t. } \forall n, m \geq N,\ ||x_n - x_m|| < \tilde{\epsilon}$$

Finally, we can write:

$$\forall\epsilon > 0\ \exists N(\epsilon) < \infty \text{ s.t. } \forall n, m \geq N,\ |||x_n - x_m||| < \epsilon$$
$$\implies \forall\tilde{\epsilon} > 0\ \exists N(\tilde{\epsilon}) < \infty \text{ s.t. } \forall n, m \geq N,\ ||x_n - x_m|| < \tilde{\epsilon}$$

Thus, we have proved the statement.

Problem 3

STATEMENT

$$f:\mathbb{R}\longrightarrow\mathbb{R} \text{ is continuous at } x_0 \text{ and } \lim_{n\longrightarrow\infty} x_n = x_0 \quad\Longleftrightarrow\quad \lim_{n\longrightarrow\infty} f(x_n) = f(x_0)$$

PROOF

First, we prove $\Longrightarrow$:

From the definition of a continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ at x_0, we have:

$$\forall\epsilon>0,\ \exists\delta(\epsilon,x_0)>0 \text{ s.t. } ||x-x_0||<\delta \implies ||f(x)-f(x_0)||<\epsilon$$

From the definition of convergence, we have:

$$\lim_{n\longrightarrow\infty} x_n = x_0 \implies \forall\delta>0,\ \exists N(\delta)<\infty \text{ s.t. } \forall n\geq N\ ,\ ||x_n-x_0||<\delta$$

Thus, for $\delta>0$ and $x=x_n$, $\exists\epsilon>0$ and $N\geq 0$ s.t. $n\geq N \implies ||x_n-x_0||<\delta \implies ||f(x_n)-f(x_0)||<\epsilon$ and $\lim_{n\longrightarrow\infty} f(x_n)=f(x_0)$.

Now using a proof by contrapositive for $\Longleftarrow$:

From the definition of a discontinuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ at x_0, we have:

$$\exists\epsilon>0,\ \forall\delta(\epsilon,x_0)>0 \text{ s.t. } ||x-x_0||<\delta \implies ||f(x)-f(x_0)||\geq\epsilon$$

From the definition of divergence, we have:

$$\lim_{n\longrightarrow\infty} x_n \neq x_0 \implies \exists\delta>0,\ \forall N(\delta)<\infty \text{ s.t. } \exists n\geq N\ ,\ ||x_n-x_0||\geq\delta$$

For $\delta=\frac{1}{n}$, we have from the discontinuity property for $x=x_n$:

$$||x_n-x_0||<\frac{1}{n}$$

Since a norm is always positive, we have:

$$0\leq||x_n-x_0||<\frac{1}{n}$$

And:

$$\lim_{n\longrightarrow\infty}||x_n-x_0||=0$$

However, from the discontinuity property for $x-x_n$:

$$||f(x_n)-f(x_0)||\geq\epsilon$$

Thus, $f(x_n)$ does not converge to $f(x_0)$ but x_n converges to x_0 and the statement has been proved.

Problem 4

Given:

$$F(x_1, x_2) = F(x) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x - xx^T \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We can apply the Newton-Raphson algorithm to find x_0 such that $F(x_0) = 0$. the governing equation for this algorithm is:

$$x_{k+1} = x_k - (\frac{\partial F}{\partial x}(x_k))^{-1}(F(x_k) - y)$$

Where:

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} 3 + x_1 + 2x_2 - x_1^2 - 2x_1x_2 \\ 4 + 3x_1 + 4x_2 - x_2x_1 - 2x_2^2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, we can write $\frac{\partial F(x)}{\partial x}$ as:

$$\frac{\partial F(x)}{\partial x} = \begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \frac{\partial F_1(x)}{\partial x_2} \\ \frac{\partial F_2(x)}{\partial x_1} & \frac{\partial F_2(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - 2x_1 - 2x_2 & 2 - 2x_1 \\ 3 - x_2 & 4 - x_1 - 4x_2 \end{bmatrix}$$

We obtain the following solution using MATLAB with an initial condition of $x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$:

$$x_0 = \begin{bmatrix} -1.4295 \\ -2.7774 \end{bmatrix}$$

We notice that when the initial condition $x \preceq 0$, the Newton-Raphson alogirhtm always converges to the minima. However, for some initial conditions such as $x = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ or $x = \begin{bmatrix} 7 & 9 \end{bmatrix}^T$ or for large positive initial conditions, the Newton-Raphson algorithm diverges.

The MATLAB code used for this part is displayed below:

```
%% HW11 P4
clear all
close all
clc
%% Initialize
x_0 = [-1;0];
n = 100;
x = zeros(2,n);
x(:,1) = x_0;
%% Newton Raphson
for i=1:n
    [h,del_h] = hw11p4(x(:,i));
    x(:,i+1) = x(:,i) - inv(del_h)*h;
end
%% Solution
% x = [-1.4295;-2.7774] with initial condition [0;0]
% x = [NaN;NaN] with initial conditions [1;-1] [7;9]
% We notice that for negative initial conditions, it does not matter how
% far off it is, it will always converge to the minima
%% Functions
function  [h,del_h] = hw11p4(x)
h = [3;4] - [1 2;3 4]*x-x*x'*[1;2];
del_h = [1-2*x(1)-2*x(2) 2-2*x(1); 3-x(2) 4-x(1)-4*x(2)];
end
```

Problem 5

Part a: TRUE

$$P(X \leq 365) = 1 - e^{-0.001*365} = 0.3058 < 0.31$$

Part b: FALSE

$$\begin{aligned}
E(\hat{x}) &= E(Ky) \\
&= E(KCx + K\epsilon) \\
&= E(KCx) + E(K\epsilon) \quad \text{(since } x \text{ and } \epsilon \text{ are independent)} \\
&= E(x) + KE(\epsilon) \\
&= x + K\mu \neq x
\end{aligned}$$

Thus, $\hat{x} = Ky$ is an unbiased estimator if and only if $\mu = 0$.

Part c: TRUE

$$\text{cov}(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}) = \begin{bmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) \end{bmatrix} = \begin{bmatrix} \sigma^2(X_1) & \sigma(X_1, X_2) \\ \sigma(X_2, X_1) & \sigma^2(X_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Thus, $\sigma(X_1, X_2) = \sigma(X_2, X_1) = 0 \implies X_1$ and X_2 are independent.

Problem 6

Part a

To solve the following under determined problem with the ℓ-2 norm, we use the MATLAB command `quadprog`:

$$\hat{x} = \underset{A_{eq}x = b_{eq}, A_{in}x \leq b_{in}}{\arg\min} \quad ||x||_2$$

Where:

$$A_{eq} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b_{eq} = \begin{bmatrix} 2 \end{bmatrix}$$

$$A_{in} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \quad \text{and} \quad b_{in} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

Rewriting the cost function, we get:

$$||x||_2 = (x^T x)^{\frac{1}{2}} = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$$

Minimizing the square root of a sum of squares in the the same as minimizing the SOS. Thus, the new minimization problem can be written as:

$$\hat{x} = \underset{A_{in}x \leq b_{in}}{\arg\min} \quad \frac{1}{2} x^T H x + f^T x$$

Where:

$$H = 2I \quad \text{and} \quad f = 0$$

We get using MATLAB:

$$\hat{x} = \begin{bmatrix} 1.4 & 0.8 & 0.2 & -0.4 \end{bmatrix}^T$$

Part b

In this part, we use the MATLAB command `quadprog` to solve the following minimization problem:

$$\hat{x} = \underset{A_{in}x \leq b_{in}}{\arg\min} \quad (x - x_0)^T Q (x - x_0)$$

Where:

$$Q = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 8 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$$

$$A_{in} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \quad \text{and} \quad b_{in} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

Rewriting the cost function, we get:

$$(x - x_0)^T Q (x - x_0) = x^T Q x - 2x_0^T Q x + x_0^T Q x_0^T$$

The last element in the right hand side is considered a constant offset and thus can be ignored when trying to minimize the cost function. Thus, the new minimization problem can be written as:

$$\hat{x} = \underset{A_{in}x \leq b_{in}}{\arg\min} \quad \frac{1}{2} x^T H x + f^T x$$

Where:

$$H = 2Q \quad \text{and} \quad f = -2Q^T x_0$$

We get using MATLAB:

$$\hat{x} = \begin{bmatrix} -3.2957 & 0.4005 & 1.0860 & 2.0591 \end{bmatrix}^T$$

The MATLAB code used for this problem is displayed below:

```
%% HW11 P6
clear all
close all
clc
%% Initialize
A_eq = ones(1,4);
b_eq = 2;
A_in = [1 2 3 4;5 6 7 8];
b_in = [9;10];
Q = [2 1 0 0;1 4 1 0;0 1 6 1;0 0 1 8];
x_0 = [1;2;3;4];
%% Part a
X_a = quadprog(2*eye(4),zeros(4,1),A_in,b_in,A_eq,b_eq);
%% Part b
X_b = quadprog(2*Q,(-2*x_0'*Q)',A_in,b_in,[],[]);
%% Solution
% X_a_sol = [1.4;0.8;0.2;-0.4];
% X_b_sol = [-3.2957;0.4005;1.0860;2.0591];
```

Problem 7

Part a

To solve the following over determined problem with the ℓ-1 norm, we use the MATLAB command `linprog`:

$$\hat{x} = \arg\min \quad ||Ax - b||_1$$

Where:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 7 \\ 4 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ 12 \end{bmatrix}$$

This is equivalent to solving the following minimization problem:

$$\begin{aligned} \min \quad & f^T X \\ \text{s.t.} \quad & A_{in} X \preceq b_{in} \end{aligned}$$

Where:

$$X = \begin{bmatrix} x \\ s \end{bmatrix}, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad x \in \mathbb{R}^n, \quad s \in \mathbb{R}^m$$

$$A_{in} := \begin{bmatrix} A & -I_{m \times m} \\ -A & -I_{m \times m} \end{bmatrix}, \quad b_{in} := \begin{bmatrix} b \\ -b \end{bmatrix} \quad \text{and} \quad f := \begin{bmatrix} 0_{1 \times n} & 1_{1 \times m} \end{bmatrix}$$

Solving this minimization problem in MATLAB, we get:‘

$$\hat{x} = \begin{bmatrix} 1.7209 \\ 1.0233 \end{bmatrix}$$

Part b

Repeating the same procedure but now using the ℓ-∞ norm, the minimization problem is equivalent to:

$$\begin{aligned} \min \quad & f^T X \\ \text{s.t.} \quad & A_{in} X \preceq b_{in} \end{aligned}$$

Where:

$$X = \begin{bmatrix} x \\ s \end{bmatrix}, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad x \in \mathbb{R}^n, \quad s \in \mathbb{R}$$

$$A_{in} := \begin{bmatrix} A & -1_{m \times 1} \\ -A & -1_{m \times 1} \end{bmatrix}, \quad b_{in} := \begin{bmatrix} b \\ -b \end{bmatrix} \quad \text{and} \quad f := \begin{bmatrix} 0_{1 \times n} & 1 \end{bmatrix}$$

Solving this minimization problem in MATLAB, we get:‘

$$\hat{x} = \begin{bmatrix} 1.6949 \\ 0.9322 \end{bmatrix}$$

The MATLAB code used for this problem is displayed below:

```matlab
%% HW11 P7
clear all
close all
clc
%% Initialize
A = [1 2;-3 7;4 5];
b = [3;2;12];
[m,n] = size(A);
%% One Norm
X_one_norm = linprog([zeros(1,n) ones(1,m)],[A -eye(m);-A -eye(m)],[b;-b]);
%% Inifinity Norm
X_inf_norm = linprog([zeros(1,n) 1],[A -ones(m,1); -A -ones(m,1)],[b;-b]);
%% Solution
% X_one_norm_sol = [1.7209;1.0233];
% X_inf_norm_sol = [1.6949;0.9322];
```