ROB 501 - Mathematics for Robotics
HW #5

Due Oct. 4, 2021
3pm on Canvas

1. (a) Calculate by hand a set of e-vectors of the matrix A_3 below (It is noted that the e-values are obvious since the matrix is upper triangular.)

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Verify that the e-vectors you computed are linearly independent.

Remark: Check you answers in MATLAB. We noted in lecture that e-vectors are not unique. Do your e-vectors agree with those computed by MATLAB using the `eig` command?

(b) For the matrix A_4 below, compute its e-values and e-vectors.

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Can you find a basis for $\mathbb{R}^3$ consisting of e-vectors?

2. Two square matrices A and B are said to be **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$. Show that if A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues. **Note:** The characteristic polynomial of a square matrix A is $\det(\lambda I - A)$; the characteristic equation is $\det(\lambda I - A) = 0$.

3. For the matrix A_3 in Prob. 1a above, show that A is similar to a diagonal matrix. You can use results from lecture.

4. Read the handout on 'Linear Regression' and then work this problem. We will be developing the theoretical basis for the handout in lecture. In the meantime, it is instructive to "play" with the ideas a bit.

(a) From the MATLAB folder on CANVAS, download the file `HW05_Prob4_Data.mat`. Load the data into the MATLAB workspace with the following command

```
>> load HW05_Prob4_Data
>> % This makes the plot
>> plot(t,f,'r','linewidth',3)
>> grid on
>> xlabel('t','FontSize',18)
```

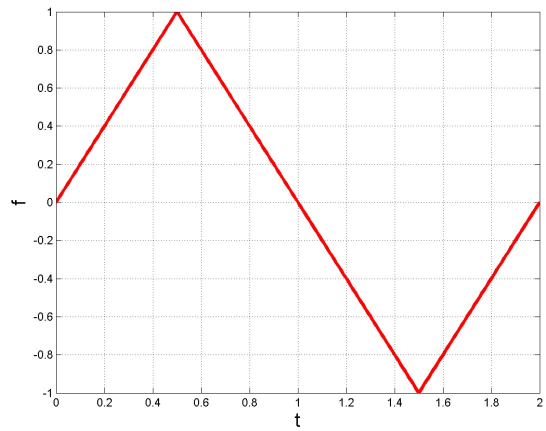

Figure 1: Triangle Wave for Problem 4

```
>> ylabel('f','FontSize',18)
>> print Prob4TriWave -dpng
```

There is nothing to turn in for this part of the problem.

(b) Compute a least squares fit of the data to $\{1, t, t^2, t^3, t^4, t^5\}$. As in the handout, build your basis vectors by evaluating the functions t^k at the provided time points. Turn in the coefficients you obtain and a plot of your fit versus the function. Do not turn in your code.

(c) Repeat the fitting process, but this time use the functions $\{\sin(\pi t), \sin(2\pi t), \cdots, \sin(5\pi t)\}$. Turn in the coefficients you obtain and a plot of your fit versus the function. To be clear, you are seeking α_k to minimize the error of

$$||f - \sum_{k=1}^{5} \alpha_k \sin(k\pi t)||^2$$

5. From the MATLAB folder on CANVAS, download the file `HW05_Prob5_Data.mat`. Fit a set of functions to the data; to be clear, you get to chose your favorite functions for the regression. Report the functions you used, the coefficients you obtained, and a plot of your fit versus the function. In addition, from your fit, estimate the derivative of the function $f(t)$ at $t = 0.3$. (Differentiate the function you fit to the data) *Do not obsess over this. It is meant to be fun, not a burden! It should be very easy to modify your MATLAB code from Prob. 4 for a solution to this problem.*

6. For $z \in \mathbb{C}$ or $\mathbb{C}^n$, let $\bar{z}$ denote the complex conjugate. Show that on $(\mathbb{C}^n, \mathbb{C})$, the definition $< x, y >= x^\top \bar{y}$ satisfies the definition of an inner product used in lecture (which comes from David Luenberger, Optimization by Vector Spaces) while $< x, y >= \bar{x}^\top y$ satisfies Definition 6.2.1, page 185, of Nagy. *You have now learned that in the case of inner product spaces with the field as $\mathbb{C}$, there is no consistency on putting the linearity on the right versus the left!* So, when you read a paper, you have to carefully check which definition is being used, unless the vector space is real, in which case, linearity on the left and right both hold. Mamma mia!

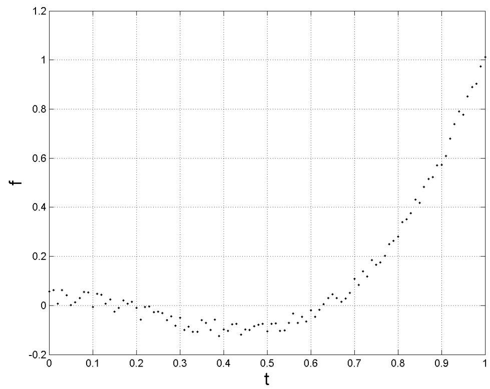

Figure 2: Noisy Data for Problem 5

Definition 6.2.1. *Let V be a vector space over the scalar field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A function $\langle\,,\rangle : V \times V \to \mathbb{F}$ is called an* ***inner product*** *iff for every $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in V$ and every a, $b \in \mathbb{F}$ the function $\langle\,,\rangle$ satisfies:*

(a1) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$, *(Symmetry, for $\mathbb{F} = \mathbb{R}$);*

(a2) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$, *(Conjugate symmetry, for $\mathbb{F} = \mathbb{C}$);*

(b) $\langle \boldsymbol{x}, (a\boldsymbol{y} + b\boldsymbol{z}) \rangle = a\langle \boldsymbol{x}, \boldsymbol{y} \rangle + b\langle \boldsymbol{x}, \boldsymbol{z} \rangle$, *(Linearity on the second argument);*

(c) $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geqslant 0$, *and* $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ *iff* $\boldsymbol{x} = 0$, *(Positive definiteness).*

An ***inner product space*** *is a pair $(V, \langle\,,\rangle)$ of a vector space with an inner product.*

Figure 3: Q6

7. Nagy, Page 198, Prob. 6.3.4 Note that the problem does NOT require the basis to be *orthonormal*. Because it is really boring to check all of them, only verify that $< p_0, p_3 >= 0$ and $< p_1, p_2 >= 0$.

6.3.4.- Let $\mathbb{P}_3([-1,1])$ be the space of polynomials up to degree three defined on the interval $[-1,1] \subset \mathbb{R}$ with the inner product

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle = \int_{-1}^{1} \boldsymbol{p}(x)\boldsymbol{q}(x)\, dx.$$

Show that the set $(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$ is an orthogonal basis of $\mathbb{P}_3$, where

$$\boldsymbol{p}_0(x) = 1,$$
$$\boldsymbol{p}_1(x) = x,$$
$$\boldsymbol{p}_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$\boldsymbol{p}_3(x) = \frac{1}{2}(5x^3 - 3x).$$

(These polynomials are the first four of the Legendre polynomials.)

Figure 4: Q7

8. This problem is dealing with the **Matrix Inversion Lemma**. It is a very useful result for reducing the complexity of matrix inversions. We will use it when doing recursive least squares and in the Kalman filter.

   (a) Either look up a proof of the following fact and copy it down, or develop your own proof: Suppose that A, B, C and D are compatible[1] matrices. If A, C, and $(C^{-1} + DA^{-1}B)$ are each square and invertible, then $A + BCD$ is invertible and

   $$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

   (b) In many important applications, the inverse of A may be already known or easy to compute. Here is a made up example, but it gets the point across: By hand, evaluate $(A + BCD)^{-1}$ when

   $$A = \text{diag}([1,\ 0.5,\ 0.5,\ 1,\ 0.5]),\ B = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, C = 0.2, D = B^\top$$

9. (a) In $(\mathbb{R}^2, \mathbb{R})$, given a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$, define $f(x) = (x^\top Ax)^{1/2}$. Is it a norm? Prove or disprove.

   (b) Suppose $(\mathbb{R}^n, \mathbb{R}, ||\cdot||_V)$ is a normed space with some kind of norm $||\cdot||_V$ defined. In $(\mathbb{R}^{n\times n}, \mathbb{R})$, define $f_V(A) = \sup\limits_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \dfrac{||Ax||_V}{||x||_V}$. In the recitation, we showed that this is a norm. By using homogeneity of norms, we can also show that $f_V(A) = \sup\limits_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \dfrac{||Ax||_V}{||x||_V} = \sup\limits_{\substack{x\in\mathbb{R}^n \\ ||x||=1}} ||Ax||_V$. Try to calculate $f_1(A)$ and $f_\infty(A)$.

[1]The sizes are such the matrix products and sum in $A + BCD$ make sense.

Hints

Hints: Prob. 1 The answer is NO, you cannot find such a basis The matrix is "defective" (check definition of defective matrix on the web), meaning it does not have a full set of e-vectors. To treat such matrices, one has to learn about the Jordan canonical form, a subject we will not cover in ROB 501. The point of the problem is simply that when e-values are not distinct, you may not have enough e-vectors to build a basis. Jordan canonical forms are treated in EECS 560 = ME 564 = AERO 550.

Hints: Prob. 2 Recall that for compatible square matrices A and B, $\det(AB) = \det(A)\det(B)$.

Hints: Prob. 3 Translate the equation $Av^i = \lambda_i v^i$ into an equation involving matrices

$$P = [v^1 \mid \cdots \mid v^n] \text{ and } \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

where in this problem, $n = 3$. Questions to ask yourself: Is P invertible? And which is correct, $AP = \Lambda P$ or $AP = P\Lambda$? Perhaps neither?

Hints: Prob. 4 and 5 For your convenience t and f are given as column vectors in the data file. An easy way in MATLAB to obtain a column of 1's the same length as t is $1 + 0 * t$. There are many other ways too! If you're using python instead of MATLAB, you can load the .mat files with the package scipy.io.loadmat (`https://docs.scipy.org/doc/scipy-0.19.0/reference/generated/scipy.io.loadmat.html`)

Hints: Prob. 6 Nothing exciting here. Just straightforward manipulations of complex numbers.

Hints: Prob. 8 You can multiply $(A + BCD)$ by the proposed inverse and check that you obtain the identity [it is enough to multiply on the left or the right, you do not have to check both, and you will find one of the two is easier to do than the other]. Alternatively, there are dozens of proofs on the web. You can choose a proof that seems to satisfy you and report that. You will not be faulted for your choice. Most of the results on the web are not stated quite as precisely as our HW problem, but that's life!

Hints: Prob. 9b Here $f_1(A)$ and $f_\infty(A)$ are essentially defining matrix norms $||A||_1$ and $||A||_\infty$, respectively, using the corresponding vector norms $||x||_1$ and $||x||_\infty$. These type of matrix norms are called induced matrix norms. You can look their definitions up from the internet and type as your answer but try to convince yourself using the definition of f.