Homework #9

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Problem 1

Part a

For $x_0 \in V \cap \text{span}\{y_1, \dots, y_p\}$, we can write the inner product $\langle x_0, y_i \rangle$ $\forall i = 1, \dots, p$ as follows:

$$\begin{aligned}
\langle x_0, y_i \rangle &= \langle \sum_{k=1}^{p} \alpha_k y_k, y_i \rangle \\
&= \langle \alpha_1 y_1, y_i \rangle + \langle \alpha_2 y_2, y_i \rangle + \ldots + \langle \alpha_p y_p, y_i \rangle = c_i \\
&= \alpha_1 \langle y_1, y_i \rangle + \alpha_2 \langle y_2, y_i \rangle + \ldots + \alpha_p \langle y_p, y_i \rangle = c_i
\end{aligned}$$

Thus, $\forall i = 1, \dots, p$, we have:

$$\begin{aligned}
\langle x_0, y_1 \rangle &= \alpha_1 \langle y_1, y_1 \rangle + \alpha_2 \langle y_2, y_1 \rangle + \ldots + \alpha_p \langle y_p, y_1 \rangle = c_1 \\
\langle x_0, y_2 \rangle &= \alpha_1 \langle y_1, y_2 \rangle + \alpha_2 \langle y_2, y_2 \rangle + \ldots + \alpha_p \langle y_p, y_2 \rangle = c_2 \\
&\vdots \\
\langle x_0, y_p \rangle &= \alpha_1 \langle y_1, y_p \rangle + \alpha_2 \langle y_2, y_p \rangle + \ldots + \alpha_p \langle y_p, y_p \rangle = c_p
\end{aligned}$$

From the above system of equations we can construct a Gram matrix of the form:

$$G = \begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle & \cdots & \langle y_1, y_p \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle & \cdots & \langle y_2, y_p \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_1, y_p \rangle & \langle y_1, y_2 \rangle & \cdots & \langle y_1, y_p \rangle \end{bmatrix} \quad \beta = \begin{bmatrix} \langle x_0, y_1 \rangle \\ \langle x_0, y_2 \rangle \\ \vdots \\ \langle x_0, y_p \rangle \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

Since $y_1, \dots, y_p$ are linearly independent, then G is invertible and we obtain a unique solution for the coefficient vector α. Thus, $x_0 = \sum_{i=1}^{p} \alpha_i y_i$ is unique.

Part b

Define the set $M = (\text{span}\{y_1, \ldots, y_p\})^\perp$ and its orthogonal complement $M^\perp = \text{span}\{y_1, \ldots, y_p\}$. We need to prove the following statement:

$$x \in V \iff (x - x_0) \perp \text{span}\{y_1, \ldots, y_p\}$$

We first prove ($\Rightarrow$):

$$\begin{aligned} x \in V &\implies \langle x, y_i \rangle = c_i, \quad \forall i = 1, \ldots, p \\ x_0 \in V &\implies \langle x_0, y_i \rangle = c_i, \quad \forall i = 1, \ldots, p \end{aligned}$$

Now, we have:

$$\langle x - x_0, y_i \rangle = \langle x, y_i \rangle - \langle x_0, y_i \rangle = c_i - c_i = 0, \quad \forall i = 1, \ldots, p$$

$$\implies \forall i = 1, \ldots, p, \quad (x - x_0) \perp y_i$$

$$\implies (x - x_0) \in M$$

Now we prove ($\Leftarrow$):

$$(x - x_0) \perp \text{span}\{y_1, \ldots, y_p\} \implies (x - x_0) \in M$$

Thus:

$$\langle x - x_0, y_i \rangle = 0, \quad \forall i = 1, \ldots, p$$

$$\implies \langle x, y_i \rangle - \langle x_0, y_i \rangle = 0$$

$$\implies \langle x, y_i \rangle = \langle x_0, y_i \rangle = c_i$$

$$\implies x \in V$$

Thus:

$$\boxed{x \in V \iff (x - x_0) \perp \text{span}\{y_1, \ldots, y_p\}} \quad \square$$

Part c

We need to prove the following statement:

$$\exists! v^* \in V, \ ||v^*|| = \inf_{v \in V} ||v||, \ v^* \perp M$$

Let $v \in V$ and $m \in M$, then from Lemma 2 we have $v = x_0 + m$. Since M is a subspace, then $v = x_0 - m \in V$. Now, from the minimum norm, we can write:

$$\inf_{v \in V} ||v|| = \inf_{m \in M} ||x_0 - m|| = d(x_0, M)$$

From the projection theorem, we can write:

$$m^* = \arg\min_{m \in M} ||x_0 - m||$$

Thus, we obtain:

$$v^* = \arg\min_{v \in V} ||v|| \implies v^* = x_0 - m^*$$

Since $||v^*|| = d(x_0, M)$, then $v^* = x_0 - m^* \in M^\perp$, we have:

$$x_0 - m^* = \sum_{i=1}^{p} \alpha_i y_i$$

Now, since $x_0 \in M^\perp$ from Lemma 1, then:

$$x_0 = \sum_{i=1}^{p} \beta_i y_i$$

Finally:

$$\begin{aligned} x_0 - m^* &= \sum_{i=1}^{p} \alpha_i y_i \\ \sum_{i=1}^{p} \beta_i y_i - m^* &= \sum_{i=1}^{p} \alpha_i y_i \\ \implies m^* &= \sum_{i=1}^{p} \beta_i y_i - \sum_{i=1}^{p} \alpha_i y_i = \sum_{i=1}^{p} \gamma_i y_i \in M^\perp \end{aligned}$$

Thus, $m^* \in M \cap M^\perp \implies m^* = 0$, and we have:

$$\boxed{v^* = x_0}$$

Since x_0 is unique from Lemma 1, then v^* is also unique and we have:

$$\boxed{\exists! v^* \in V,\ ||v^*|| = \inf_{v \in V} ||v||,\ v^* \perp M}$$

Problem 2

From Lemma 2, we have:

$$v = x_0 - m$$

Now from Lemma 3, we have proved that $m^* = \arg\min_{m \in M} ||x_0 - m|| = 0$ and thus:

$$\exists! v^* \in V \quad \text{s.t.} \quad v^* = \arg\min_{v \in V} ||v|| = x_0$$

From Lemma 1, we have that $x_0 = \sum_{i=1}^{9} \beta_i y_i$ is unique. Thus:

$$v^* = \sum_{i=1}^{9} \beta_i y_i$$

Also from Lemma 1, we have that $\langle x_0, y_i\rangle = \langle v^*, y_i\rangle \ \forall i = 1, \ldots, p$. Similarly to problem 1 part a, we can construct the Gram matrix as follows:

$$\begin{aligned}
\langle v^*, y_1\rangle &= \alpha_1\langle y_1, y_1\rangle + \alpha_2\langle y_2, y_1\rangle + \ldots + \alpha_p\langle y_p, y_1\rangle = c_1\\
\langle v^*, y_2\rangle &= \alpha_1\langle y_1, y_2\rangle + \alpha_2\langle y_2, y_2\rangle + \ldots + \alpha_p\langle y_p, y_2\rangle = c_2\\
&\vdots\\
\langle v^*, y_p\rangle &= \alpha_1\langle y_1, y_p\rangle + \alpha_2\langle y_2, y_p\rangle + \ldots + \alpha_p\langle y_p, y_p\rangle = c_p
\end{aligned}$$

From the above system of equations we can construct a Gram matrix of the form:

$$G = \begin{bmatrix} \langle y_1, y_1\rangle & \langle y_1, y_2\rangle & \cdots & \langle y_1, y_p\rangle \\ \langle y_1, y_2\rangle & \langle y_2, y_2\rangle & \cdots & \langle y_2, y_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_1, y_p\rangle & \langle y_1, y_2\rangle & \cdots & \langle y_1, y_p\rangle \end{bmatrix} \qquad \beta = \begin{bmatrix} \langle v^*, y_1\rangle \\ \langle v^*, y_2\rangle \\ \vdots \\ \langle v^*, y_p\rangle \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

Since $y_1, \ldots, y_p$ are linearly independent, then G is invertible and we obtain a unique solution for the coefficient vector α. Thus, $v^* = \sum_{i=1}^{p} \beta_i y_i$ is unique and we can solve for the coefficients β_i as follows:

$$\begin{bmatrix} \langle y_1, y_1\rangle & \langle y_1, y_2\rangle & \cdots & \langle y_1, y_p\rangle \\ \langle y_1, y_2\rangle & \langle y_2, y_2\rangle & \cdots & \langle y_2, y_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_1, y_p\rangle & \langle y_1, y_2\rangle & \cdots & \langle y_1, y_p\rangle \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

Problem 3

Part a

The mean and covariance of X conditioned on $Y = y$ are given by:

$$\begin{aligned}
\mu_{X|Y=y} &= \mu_X + \Sigma_{12}\Sigma_{22}^{-1}(y - \mu_Y)\\
\mu_{X|Y=y} &= \bar{x} + PC^T(CPC^T + Q)^{-1}(y - \bar{y})
\end{aligned}$$

Now for the conditioned covariance, we have:

$$\begin{aligned}
\Sigma_{X|Y=y} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\\
\Sigma_{X|Y=y} &= P - PC^T(CPC^T + Q)^{-1}CP
\end{aligned}$$

Looking back at Problem 6 of HW8, we notice that the MVE estimate $\hat{x}$ and its covariance are obtained using the mean and covariance of X conditioned on $Y = y$:

$$\hat{x} = \bar{x} + PC^T(CPC^T + Q)^{-1}(y - \bar{y})$$

$$E\{(x - \hat{x})(x - \hat{x})^T\} = P - PC^T(CPC^T + Q)^{-1}CP$$

Part b

Given the following covariance matrix:

$$\Sigma = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} P & PC^T \\ CP & CPC^T + Q \end{bmatrix}$$

We can compute the Schur complement of $(CPC^T + Q)$ in Σ as follows:

$$Schur(\Sigma) = A - BC^{-1}B^T = P - PC^T(CPC^T + Q)^{-1}CP$$

Comparing with the conditioned covariance $\Sigma_{X|Y=y}$, we notice that it is similar to the Schur complement of $(CPC^T + Q)$ in Σ.

Problem 4

Part a

From Lemma 1 through 3, we know that the vector of the minimum norm is given by:

$$v^* = \sum_{i=1}^{p} \alpha_i y_i$$

We define the set $V = \{x \in \mathcal{X} \mid \langle x, t \rangle = 2\}$. We can find the coefficients α_i as follows:

$$G = \langle t, t \rangle = \int_0^2 t^2 dt = \frac{1}{3}[t^3]_0^2 = \frac{8}{3}$$

$$\beta = \langle v^*, t \rangle = 2$$

Thus:

$$\alpha = G^{-T}\beta = \frac{3}{8} * 2 = \frac{3}{4}$$

$$\implies v^* = \frac{3}{4}t$$

Part b

Following the same procedure as in part a, we have:

$$G = \begin{bmatrix} \langle t, t \rangle & \langle t, \sin(\pi t) \rangle \\ \langle t, \sin(\pi t) \rangle & \langle \sin(\pi t), \sin(\pi t) \rangle \end{bmatrix}$$

$$\beta = \begin{bmatrix} \langle v^*, t \rangle \\ \langle v^*, \sin(\pi t) \rangle \end{bmatrix} = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$$

Where:

$$\langle t, t \rangle = \int_0^2 t^2 dt = \frac{1}{3}[t^3]_0^2 = \frac{8}{3}$$

$$\langle t, \sin(\pi t)\rangle = \int_0^2 t\sin(\pi t)dt = [\frac{\sin(\pi t) - \pi t\cos(\pi t)}{\pi^2}]_0^2 = -\frac{2}{\pi}$$

$$\langle \sin(\pi t), \sin(\pi t)\rangle = \int_0^2 \sin^2(\pi t)dt = [\frac{t}{2} - \frac{\sin(2\pi t)}{4\pi}]_0^2 = 1$$

Thus:

$$\alpha = G^{-T}\beta = \begin{bmatrix} \frac{8}{3} & -\frac{2}{\pi} \\ -\frac{2}{\pi} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ \pi \end{bmatrix} = \begin{bmatrix} \frac{3\pi^2}{2\pi^2-3} \\ \frac{2\pi^3+3\pi}{2\pi^2-3} \end{bmatrix}$$

Finally:

$$v^* = \frac{3\pi^2}{2\pi^2 - 3} * t + \frac{2\pi^3 + 3\pi}{2\pi^2 - 3} * \sin(\pi t)$$

Problem 5

Part a

Considering $Ax = b$, we can decompose A as follows:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} \implies \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}$$

We can construct a system of equations as follows:

$$\begin{aligned} a_1 x &= b_1 \\ a_2 x &= b_2 \\ &\vdots \\ a_p x &= b_p \end{aligned}$$

Using the standard inner on product on $\mathbb{R}$:

$$\langle x, a_i^T\rangle = b_i \quad \forall i = 1, \ldots, p$$

From problem 2, we can write:

$$\hat{x} = \underset{x\in Ax=b}{\arg\min}\, ||x|| = \sum_{i=1}^{p} \alpha_i a_i^T$$

Constructing the Gram matrix, we have:

$$G = \begin{bmatrix} \langle a_1^T, a_1^T\rangle & \langle a_1^T, a_2^T\rangle & \cdots & \langle a_1^T, a_p^T\rangle \\ \langle a_1^T, a_2^T\rangle & \langle a_2^T, a_2^T\rangle & \cdots & \langle a_2^T, a_p^T\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_1^T, a_p^T\rangle & \langle a_1^T, a_2^T\rangle & \cdots & \langle a_1^T, a_p^T\rangle \end{bmatrix} = AA^T \quad \beta = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}$$

If G is invertible, which means that the rows of A are linearly independent, then:

$$\implies \alpha = G^{-T}\beta = (AA^T)^{-1}b$$

The estimate $\hat{x}$ is given by:

$$\hat{x} = \sum_{i=1}^{p} \alpha_i a_i^T = \begin{bmatrix} a_1^T & a_2^T & \dots & a_p^T \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = A^T\alpha$$

Thus:

$$\boxed{\hat{x} = A^T(AA^T)^{-1}b} \quad \square$$

Part b

We first define $\tilde{A}$ as follows, given that $Q \succ 0 \implies Q^{-1}$ exists:

$$\tilde{A} = AQ^{-1}$$

We now decompose A as follows:

$$A = \begin{bmatrix} a_i \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

Defining $v_i \in \mathbb{R}^n$ as follows:

$$v_i = (a_iQ^{-1})^T = Q^{-1}a_i^T \quad \forall i = 1, \dots, p$$

Thus, we can write:

$$a_i = v_i^T Q$$

Now following the same procedure as part a:

$$Ax = b$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}$$

$$\implies a_i x = b_i \quad \forall i = 1, \dots, p$$

$$\implies v_i^T Q x = \langle v_i, x \rangle = b_i \quad \forall i = 1, \dots, p$$

And we have:

$$\langle v_i, v_i \rangle = v_i^T Q v_i = a_i Q^{-1} Q Q^{-1} a_i^T = a_i Q^{-1} a_i^T$$

From problem 2, we can write:

$$\hat{x} = \underset{x \in Ax=b}{\arg\min} ||x|| = \sum_{i=1}^{p} \alpha_i v_i$$

Constructing the Gram matrix, we have:

$$G = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_p \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_p \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, v_p \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_p \rangle \end{bmatrix} \quad \beta = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}$$

Thus:

$$G = \begin{bmatrix} a_1 Q^{-1} a_1^T & a_1 Q^{-1} a_2^T & \cdots & a_1 Q^{-1} a_p^T \\ a_2 Q^{-1} a_1^T & a_2 Q^{-1} a_2^T & \cdots & a_2 Q^{-1} a_p^T \\ \vdots & \vdots & \ddots & \vdots \\ a_p Q^{-1} a_1^T & a_p Q^{-1} a_2^T & \cdots & a_p Q^{-1} a_p^T \end{bmatrix} = AQ^{-1}A^T$$

If G is invertible, which means that the rows of A are linearly independent, then:

$$\implies \alpha = G^{-T}\beta = (AQ^{-1}A^T)^{-1}b$$

The estimate $\hat{x}$ is given by:

$$\hat{x} = \sum_{i=1}^{p} v_i \alpha_i = \begin{bmatrix} Q^{-1}a_1^T & Q^{-1}a_2^T & \ldots & Q^{-1}a_p^T \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = Q^{-1}A^T\alpha$$

Thus:

$$\boxed{\hat{x} = Q^{-1}A^T(AQ^{-1}A^T)^{-1}b} \quad \square$$

Problem 6

Decompose the following A matrix:

$$A = [A_1|A_2] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Applying the Gram Schmidt process, we get:

$$v_1 = \frac{A_1}{||A_1||} = \frac{1}{\sqrt{1+3^2+5^2}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{35}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.1690 \\ 0.5071 \\ 0.8452 \end{bmatrix}$$

$$
\begin{aligned}
v_2 &= A_2 - \langle A_2, v_1 \rangle v_1 \\
&= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0.1690 \\ 0.5071 \\ 0.8452 \end{bmatrix} * \begin{bmatrix} 0.1690 \\ 0.5071 \\ 0.8452 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - 7.4374 \begin{bmatrix} 0.1690 \\ 0.5071 \\ 0.8452 \end{bmatrix} \\
&= \begin{bmatrix} 0.7429 \\ 0.2286 \\ -0.2857 \end{bmatrix}
\end{aligned}
$$

Normalizing:

$$
v_2 = \frac{v_2}{||v_2||} = \frac{1}{\sqrt{0.7429^2 + 0.2286^2 + 0.2857^2}} \begin{bmatrix} 0.7429 \\ 0.2286 \\ -0.2857 \end{bmatrix}
$$

$$
v_2 = \begin{bmatrix} 0.8971 \\ 0.2760 \\ -0.3450 \end{bmatrix}
$$

Thus:

$$
Q = [v_1 | v_2] = \begin{bmatrix} 0.1690 & 0.8971 \\ 0.5071 & 0.2760 \\ 0.8452 & -0.3450 \end{bmatrix}
$$

Now for R, we have:

$$
\begin{aligned}
QR_1 &= A_1 \\
\begin{bmatrix} 0.1690 & 0.8971 \\ 0.5071 & 0.2760 \\ 0.8452 & -0.3450 \end{bmatrix} \begin{bmatrix} r_{11} \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}
\end{aligned}
$$

Since this is an underdetermined system of equations, the solution is of the form:

$$
R_1 = (Q^T Q)^{-1} Q^T A_1
$$

From MATLAB, we obtain:

$$
R_1 = \begin{bmatrix} 5.9161 \\ 0 \end{bmatrix}
$$

Similarly for R_2:

$$
\begin{aligned}
QR_2 &= A_2 \\
\begin{bmatrix} 0.1690 & 0.8971 \\ 0.5071 & 0.2760 \\ 0.8452 & -0.3450 \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \end{bmatrix} &= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}
\end{aligned}
$$

Solving this underdetermined system of equations, we obtain using MATLAB:

$$R_2 = \begin{bmatrix} 7.4374 \\ 0.8281 \end{bmatrix}$$

Finally:

$$R = [R_1|R_2] = \begin{bmatrix} 5.9161 & 7.4374 \\ 0 & 0.8281 \end{bmatrix}$$

Using MATLAB's qr command we obtain the following:

$$[Q_1, R_1] = \text{qr}(A)$$

$$Q_1 = \begin{bmatrix} -0.1690 & 0.8971 & 0.4082 \\ -0.5071 & 0.2760 & -0.8165 \\ -0.8452 & -0.3450 & 0.4082 \end{bmatrix} \quad R_1 = \begin{bmatrix} -5.9161 & -7.4374 \\ 0 & 0.8281 \\ 0 & 0 \end{bmatrix}$$

Now using the following arguments:

$$[Q_2, R_2] = \text{qr}(A, 0)$$

$$Q_2 = \begin{bmatrix} -0.1690 & 0.8971 \\ -0.5071 & 0.2760 \\ -0.8452 & -0.3450 \end{bmatrix} \quad R_2 = \begin{bmatrix} -5.9161 & -7.4374 \\ 0 & 0.8281 \end{bmatrix}$$

We notice that the first column of $Q_1 \in \mathbb{R}^{3\times3}$ is similar to the first column of Q multiplied by -1 and the second column of Q_1 is exactly the same as the second column of Q, however there is a third column in Q_1 unlike Q. Now for $R_1 \in \mathbb{R}^{3\times2}$, the first row of R_1 is the first row of R but multiplied by -1 and the second row is the same. However, R_1 has an extra row which is not present in R. Comparing the economic-size decomposition solution, we notice that $Q_2 \in \mathbb{R}^{3\times2}$ and $R_2 \in \mathbb{R}^{2\times2}$ have the same size as Q and R respectively. However, we notice that the first column of Q_2 is similar to the first column of Q multiplied by -1 and the second column is the same as Q. Now, for R_2, we notice that the first row is similar to the first row of R multiplied by -1 and the second row is the same. The multiplication by -1 does not affect the result since Q is orthonormal and R is obtained by solving $QR_i = A_i$. Thus, the solution by hand using Gram Schmidt process and the closed form solution of an undetermined system is similar to the economy-size decomposition solution using MATLAB's qr function.
The MALTAB code used to solve this part is displayed below:

```
%% HW9 P6
A = [1 2; 3 4;5 6];
%% Gram Schmidt
A_1 = A(:,1);
A_2 = A(:,2);
v_1 = A_1/norm(A_1);
v_2 = A_2 - A_2'*v_1*v_1;
v_2 = v_2/norm(v_2);
Q = [v_1 v_2];
R_1= inv(Q'*Q)*Q'*A_1;
R_2= inv(Q'*Q)*Q'*A_2;
R = [R_1 R_2];
% Sanity Check
Q'*Q - eye(2)
[Q_1,R_1] = qr(A);
[Q_2,R_2] = qr(A,0);
```