Graph Algorithms (Chapter 23)
Minimum Spanning Trees

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Goals

Weighted graph definition and examples

The minimum spanning tree problem

A generic solution to the problem

Algorithm of Kruskal

Algorithm of Prim

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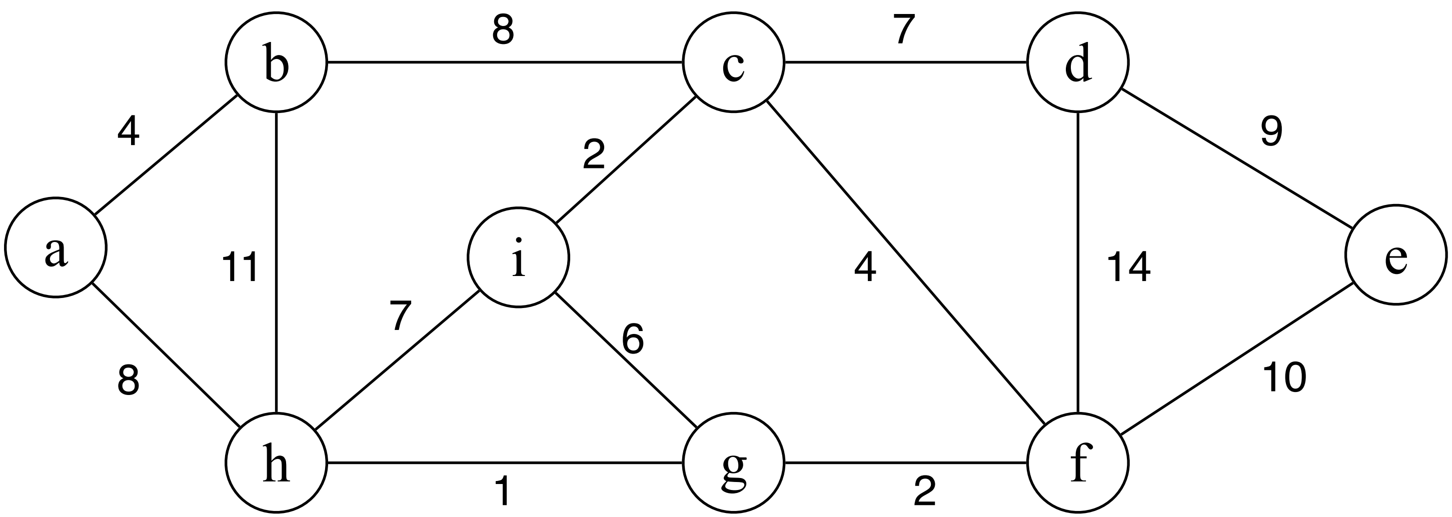
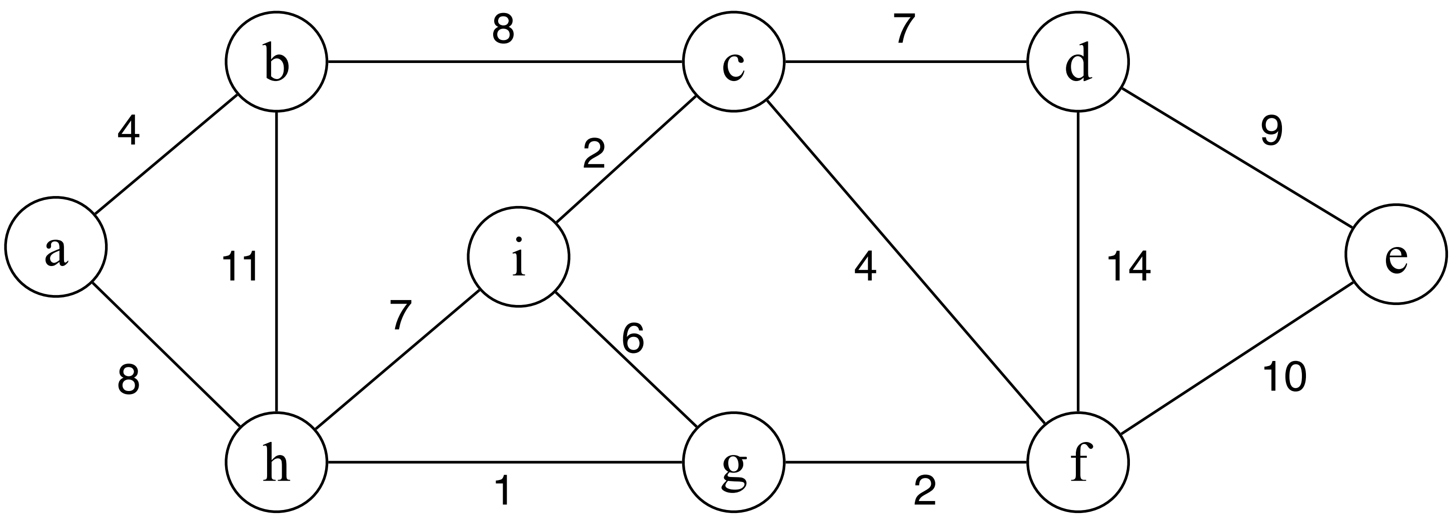
$G = (V, E)$ is an undirected graph where every edge is associated with a weight



Graph Algorithms (Chapter 23) Minimum Spanning Trees

- A tree is a connected acyclic graph
- A spanning tree of a graph is a tree containing all vertices of G
- A minimum spanning tree (MST) is a minimum total weight spanning tree



Find the MST of this graph !

Generic greedy strategy:

1. Prior to each iteration, A is a subset of some minimum spanning tree.
2. At each step determine and edge $e = (u, v)$ such that $A \cup \{e\}$ does not violate condition 1. Meaning it is safe to add e (safe edge).

```
GENERIC-MST(G, w)
A = ∅
while A does not form a spanning tree
    find an edge (u, v) that is safe for A
    A = A ∪ {(u, v)}
return A
```

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Minimum Spanning Trees

Generic greedy strategy:

We reason in a way that there exists a spanning Tree T and $A \subseteq T$. This means that an edges $(u, v) \in T$ and $(u, v) \notin A$ is a safe edge to add to A

How to recognise that safe edge (u, v) when you don't know T yet ?!

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GENERIC-MST(G, w)
  A = Ø
  while A does not form a spanning tree
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```

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Cut definition

- Cut: cut $(S, S - V)$ of $G = (V, E)$ is a partition of V
- An edge $(u, v) \in E$ crosses the cut $(S, V - S)$ if one of its endpoints is in S and the other in $V - S$
- A cut respects A if no edge in A crosses the the cut
- An edge is light edge crossing a cut if its weight is minimum of any edge crossing the cut.

Theorem 23.1

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A.

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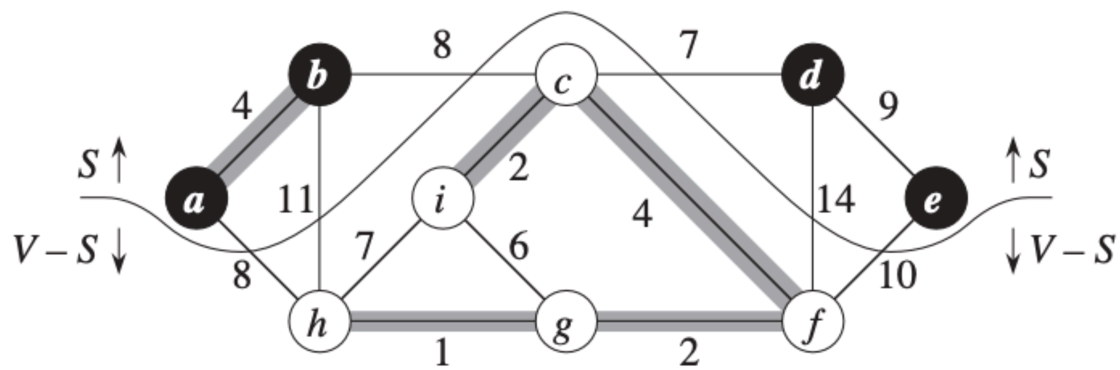
How to recognise a safe edge (u, v)? Option 1



(a)

Theorem 23.1
Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A.

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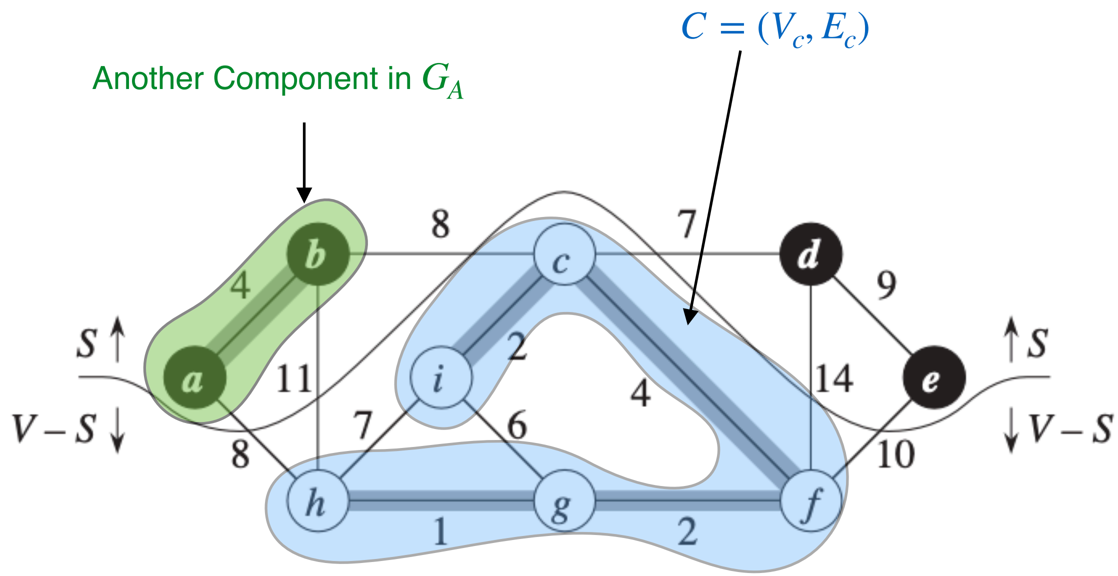
How to recognise a safe edge (u, v)? Option 2



Corollary 23.2
Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A, then (u, v) is safe for A.

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Minimum Spanning Trees

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```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
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    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

Two algorithms following the same generic approach

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Start with disjoint sets and merge them to form one single set. Merge starting from the smallest weight edge (greedy choice)

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? We will come back to this

Disjoint-set operations(from chapter 21.3)

- MAKE-SET(x): Makes a set of element x where x is the set representative
- UNION(x,y): merge the set containing x with the set containing y
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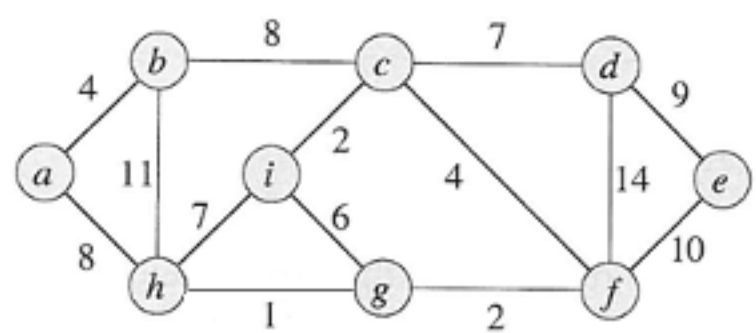
$A = \{\}$



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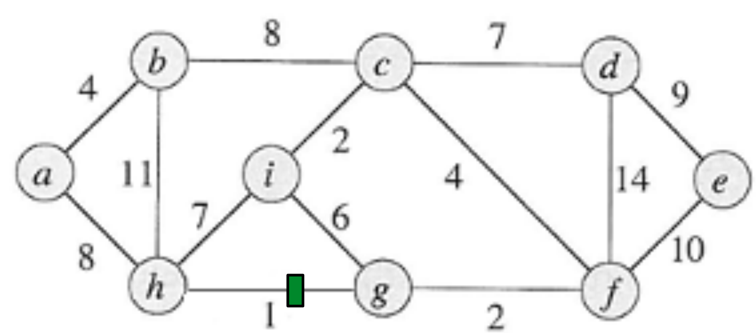
$A = \{(h, g)\}$



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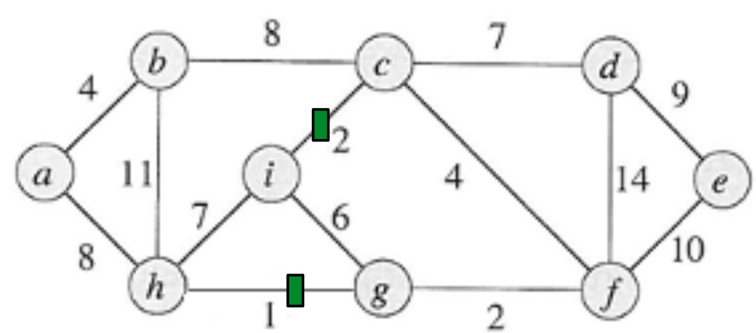
$A = \{(h, g), (i, c)\}$



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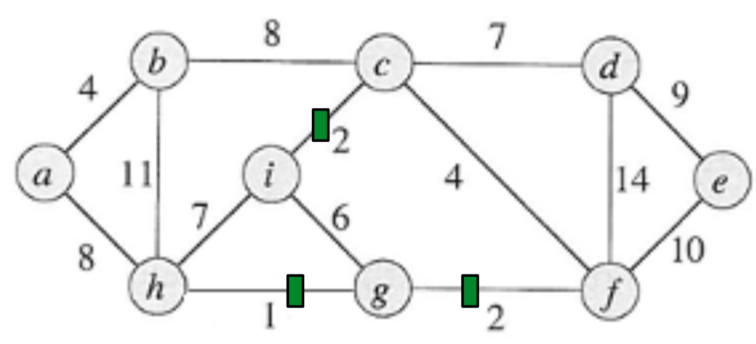

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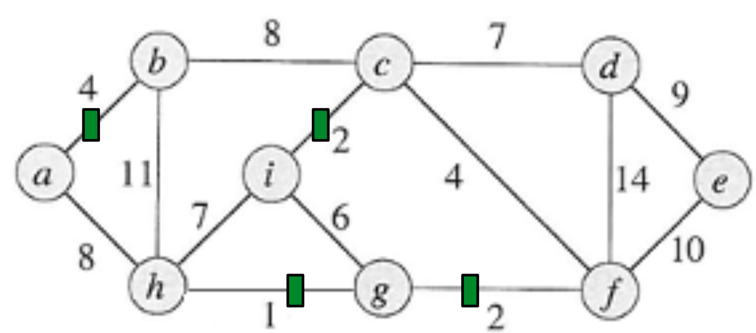
$A = \{(h,g),(i,c),(g,f),(a,b)\}$



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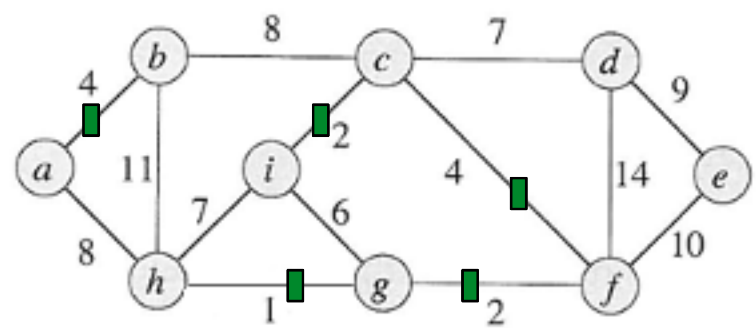

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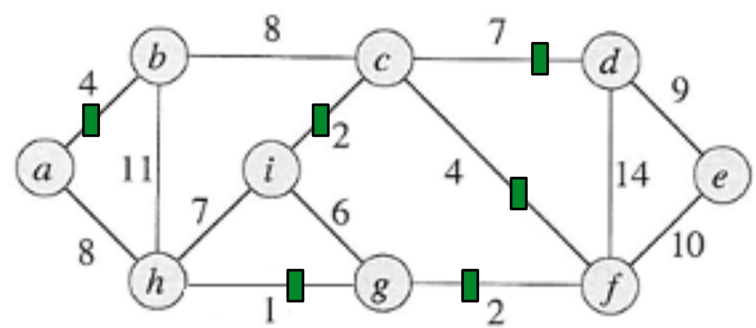

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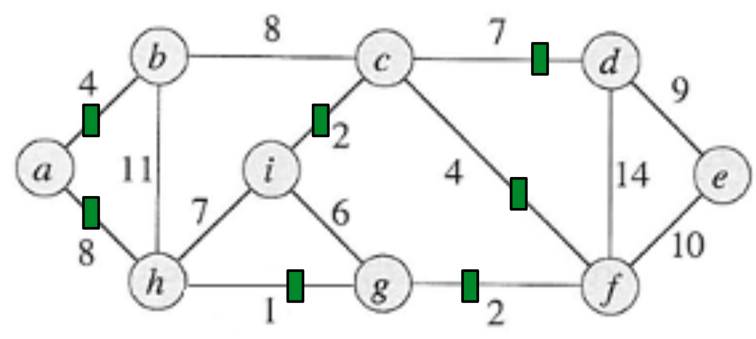

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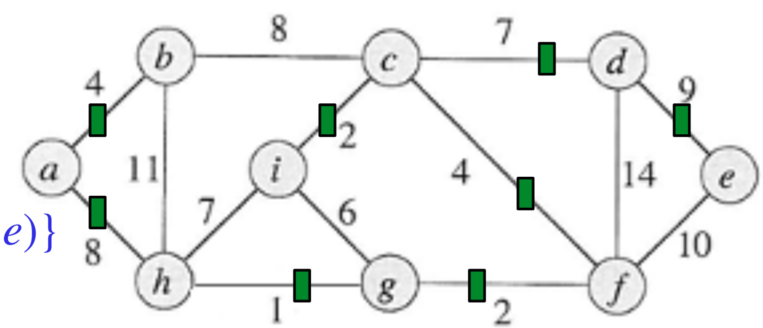

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Disjoint-set operations(from chapter 21.3)

- MAKE-SET(x): Makes a set of element x where x is the set representative
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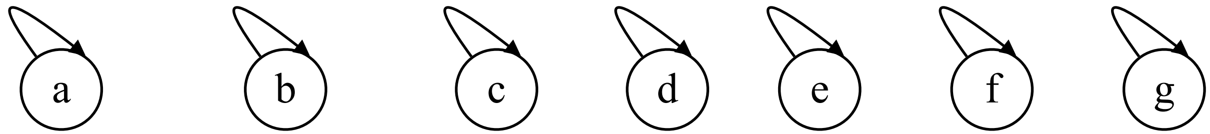
Explanation

Graph Algorithms (Chapter 23) Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

MAKE-SET(x) for {a,b,c,d,e,f,g}



MAKE-SET(x)

1 $x.p = x$

2 $x.rank = 0$

What is the runtime complexity of MAKE-SET(x)?

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Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

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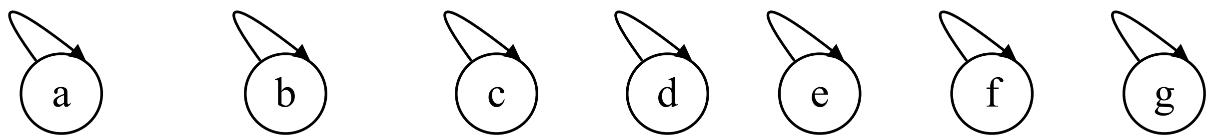

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$\Theta(1)$

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Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

```
MAKE-SET(x)
    x.p = x
    x.rank = 0

UNION(x, y)
    LINK(FIND-SET(x), FIND-SET(y))

LINK(x, y)
    if x.rank > y.rank
        y.p = x
    else x.p = y
        if x.rank == y.rank
            y.rank = y.rank + 1

FIND-SET(x)
    if x ≠ x.p
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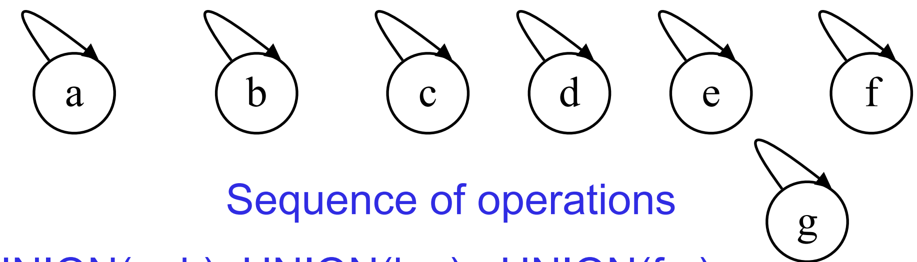
```



Sequence of operations

UNION(a, b), UNION(b,c), ,UNION(f,g), UNION(c, d), UNION(d,f)?

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Disjoint-set operations(from chapter 21.3)

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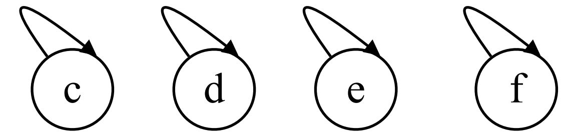

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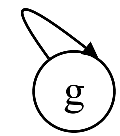
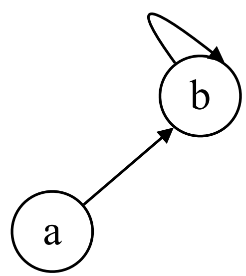
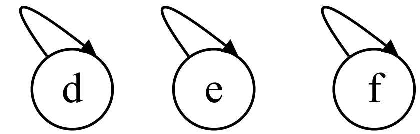


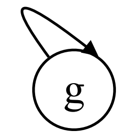
~~UNION(a, b)~~, UNION(b,c), ,UNION(f,g), UNION(c, d), UNION(d,f)?



Disjoint-set operations(from chapter 21.3)



Sequence of operations



~~UNION(a, b)~~, ~~UNION(b,c)~~, ,UNION(f,g), UNION(c, d), UNION(d,f)?

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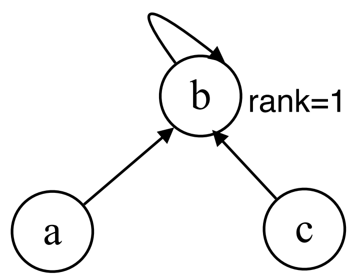

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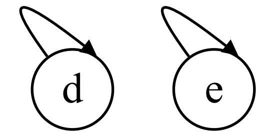

Sequence of operations

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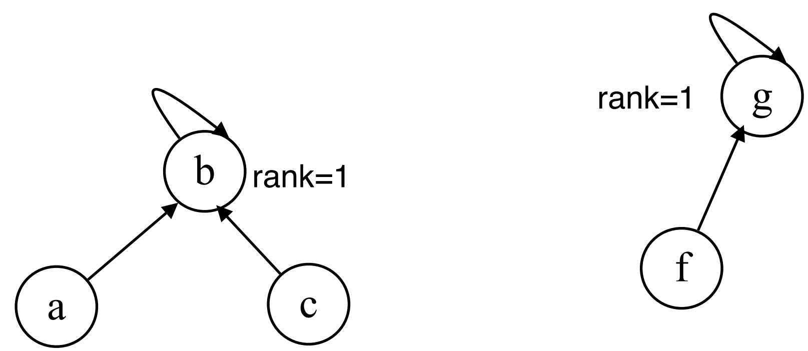
UNION(c, d), UNION(d,f)?



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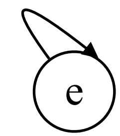

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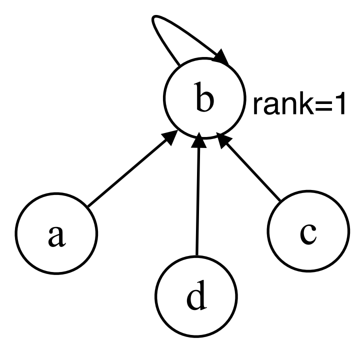

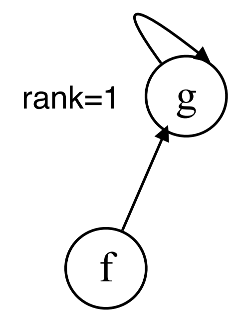

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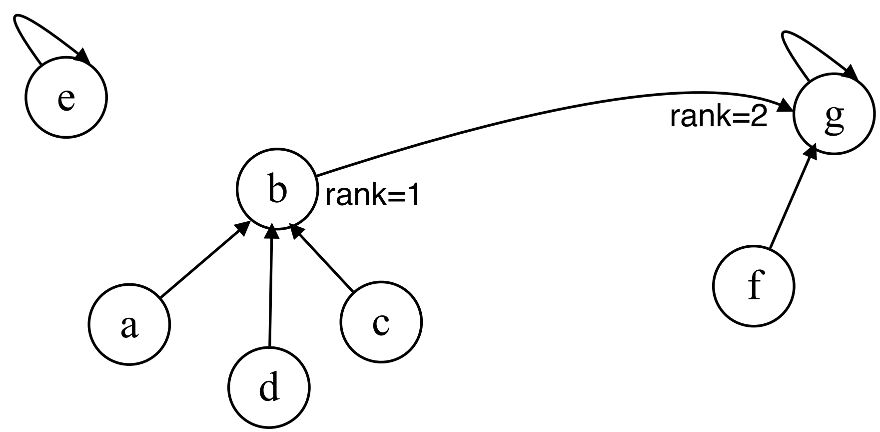

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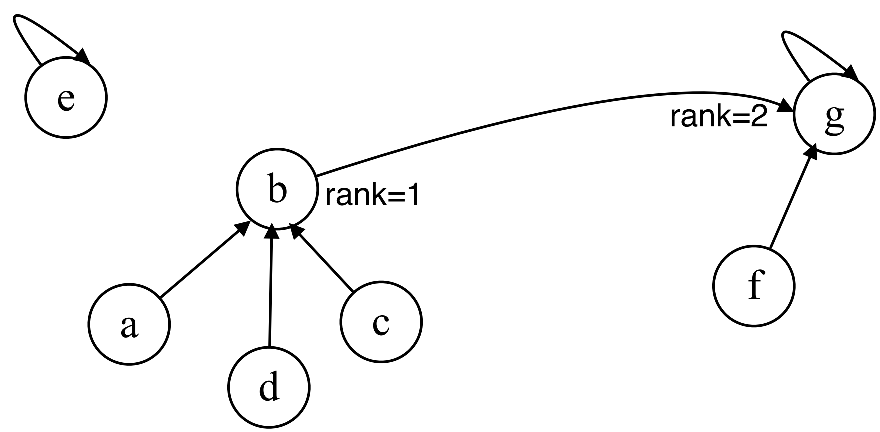
What is the time complexity of

UNION(x,y)?



Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

```
MAKE-SET(x)
  x.p = x
  x.rank = 0

UNION(x, y)
  LINK(FIND-SET(x), FIND-SET(y))

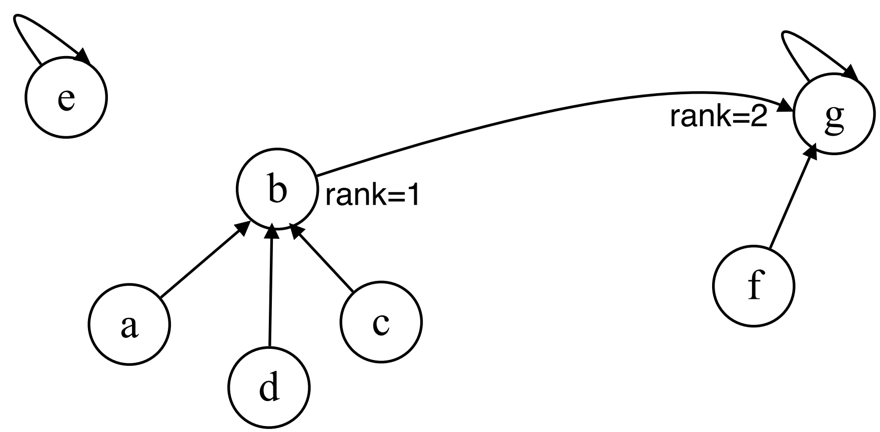
LINK(x, y)
  if x.rank > y.rank
      y.p = x
  else x.p = y
      if x.rank == y.rank
          y.rank = y.rank + 1

FIND-SET(x)
  if x ≠ x.p
      x.p = FIND-SET(x.p)
  return x.p
```

What is the time complexity of

UNION(x,y)?

Depends on FIND-SET



Graph Algorithms (Chapter 23) Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

MAKE-SET(x)

$x.p = x$
$x.rank = 0$

UNION(x, y)

LINK(FIND-SET(x), FIND-SET(y))

LINK(x, y)

if $x.rank > y.rank$
    $y.p = x$
else $x.p = y$
    if $x.rank == y.rank$
        $y.rank = y.rank + 1$

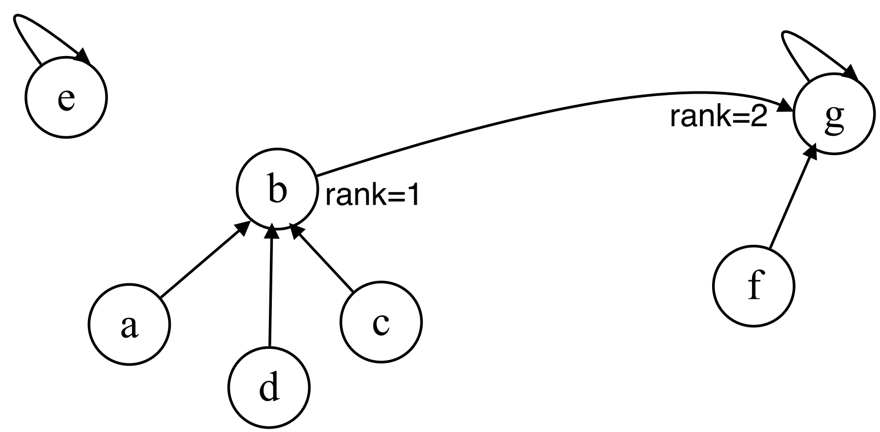
FIND-SET(x)

if $x \neq x.p$
    $x.p =$ FIND-SET$(x.p)$
return $x.p$

What is the time complexity of

UNION(x,y)?

FIND-SET(x)= $\Theta(log(n))$



Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

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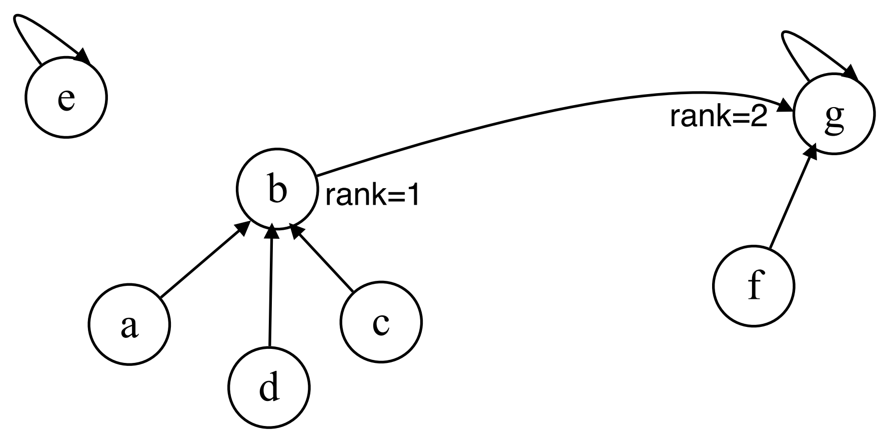
FIND-SET(x)
  if x ≠ x.p
      x.p = FIND-SET(x.p)
  return x.p
```

Can we do better?

FIND-SET(x)= $\Theta(log(n))$



Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Disjoint-set operations(from chapter 21.3)

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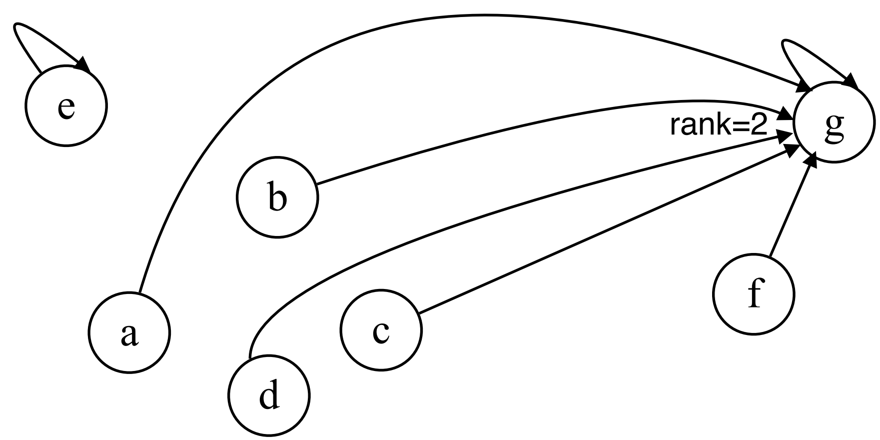
    if x ≠ x.p
        x.p = FIND-SET(x.p)
    return x.p
```

Can we do better?

Path compression

On FIND-SET each element on the path gets as parent the set representative



University of Stavanger

What is the runtime complexity of Kruskal Algorithm?

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

What is the runtime complexity of Kruskal Algorithm?

$O(E\,.\,lgE)$ and since $|V^2| \geq |E| \geq |V| - 1$ we have $lg|E| = O(lgV)$,

we can also say that the runtime time is $O(E\,.\,lgV)$

MST-KRUSKAL(G, w)

$A = \emptyset$
for each vertex $v \in G.V$ — $O(V)$
  MAKE-SET(v) — $O(1)$
sort the edges of $G.E$ into nondecreasing order by weight w — $O(E\,.\,lgE)$
for each edge $(u, v) \in G.E$, taken in nondecreasing order by weight — $O(E)$
  if FIND-SET$(u) \neq$ FIND-SET(v) — $O(lgV)$
    $A = A \cup \{(u, v)\}$ — $O(1)$
    UNION(u, v) — $O(lgV)$
return A

or less by compressed path

Graph Algorithms (Chapter 23)
Minimum Spanning Trees

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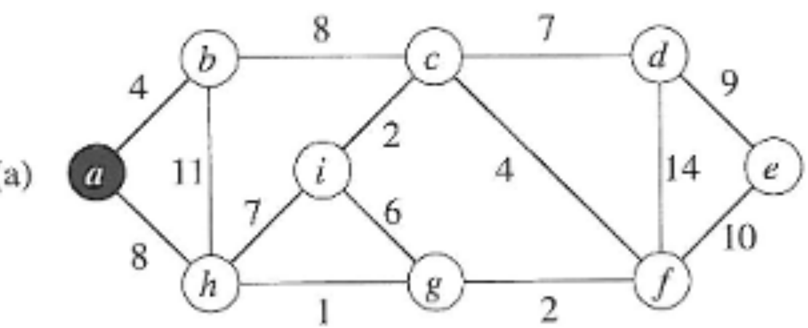
Use a priority queue holding the vertices to be extracted by minimum edge weight. Update the parents of the vertices to construct a tree

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```



Start at r, and follow the minimum weighted edge, update the keys in Q.

After a, b will have b.key=4 and h.key=8.

The next to extract will be b because b.key is the minimum

Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Use a priority queue holding the vertices to be extracted by minimum edge weight. Update the parents of the vertices to construct a tree

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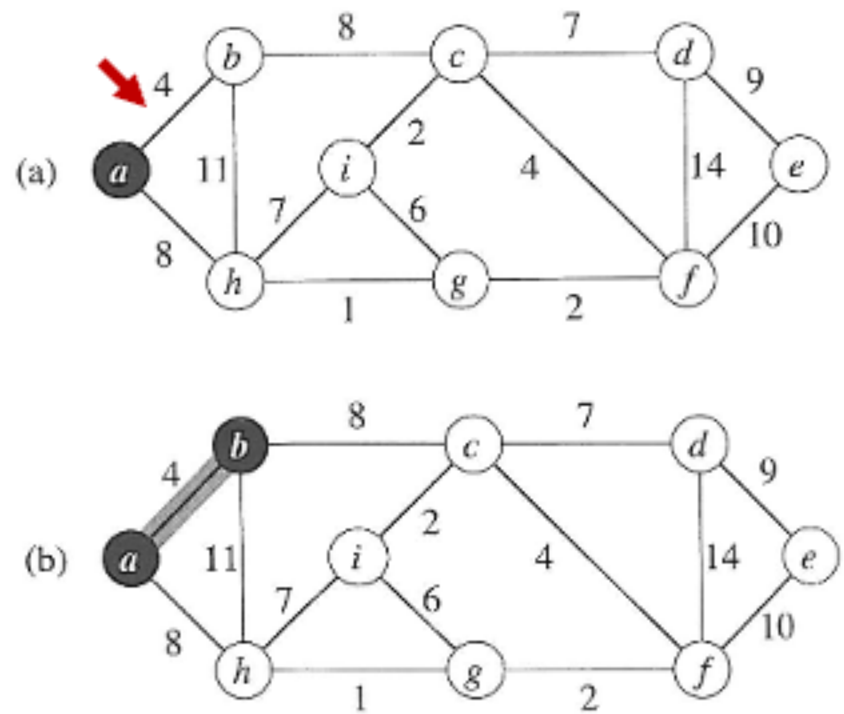
```



Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Use a priority queue holding the vertices to be extracted by minimum edge weight. Update the parents of the vertices to construct a tree

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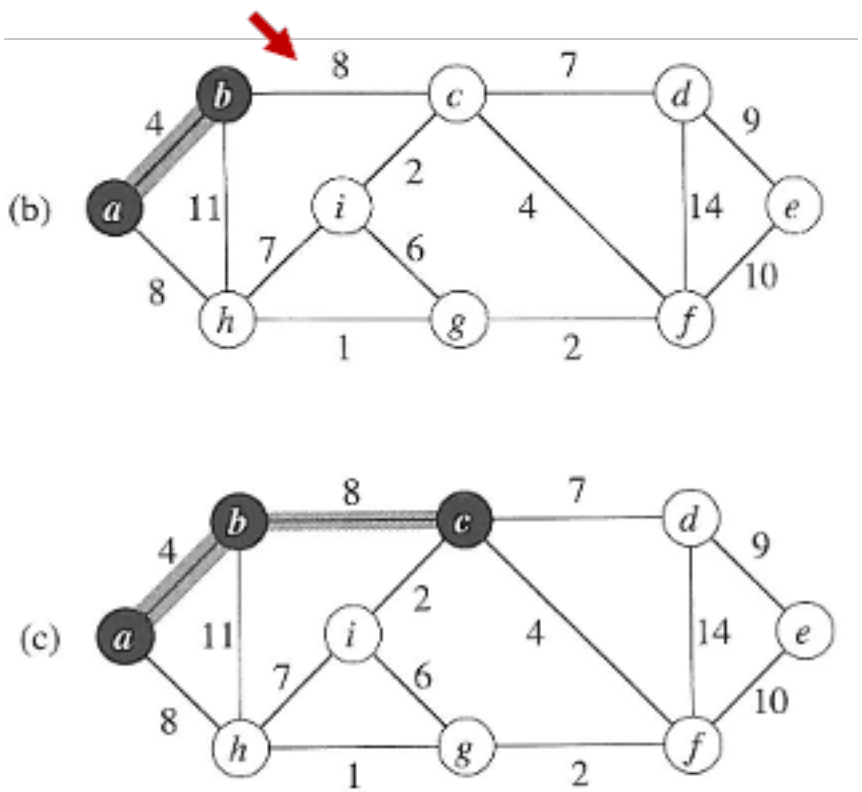
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```



Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Use a priority queue holding the vertices to be extracted by minimum edge weight. Update the parents of the vertices to construct a tree

What the runtime of MST-PRIM?

```
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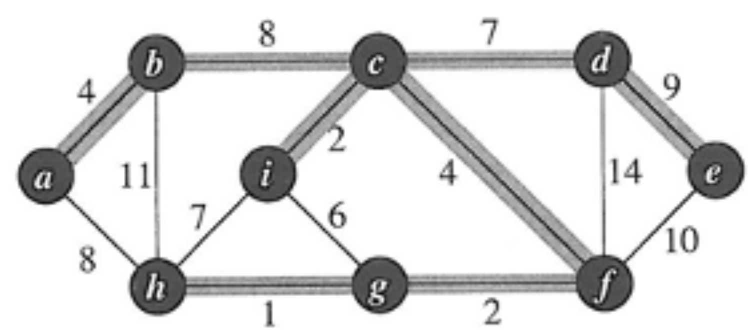
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

(i)



Graph Algorithms (Chapter 23)
Minimum Spanning Trees

Use a priority queue holding the vertices to be extracted by minimum edge weight. Update the parents of the vertices to construct a tree

What the runtime of MST-PRIM?

$$O(V\,.\,lgV + E\,.\,lgV) = O(E\,.\,lgV)$$

because $|E| \geq |V| - 1$

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
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r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

$O(V)$

$O(VlgV)$ total times we extract min

$O(E)$ total times we enter the loop

$O(lgV)$ from chapter 6.5

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

Summary Chapter 23

- Minimum spanning tree applies to undirected graphs
- A generic solution focuses on always picking a safe edge to add
- A safe edge to add to a minimum spanning tree is the greedy choice that can be taken
- Kruskal starts with a forest of trees and builds a minimum spanning tree by linking the trees with safe edges
- Prims grows a minimum spanning tree by maintaining a min-priority queue
- Both algorithms run in $O(E \,.\, lgV)$ depending on how the priority queue and disjoint set operations are implemented.

Exercices

Will come