MOD550
Applied Data Analytics and Statistics for Spatial and Temporal Modeling

University of Stavanger

05 – Bayes' Theorem

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Bayes' Theorem

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Terminology

- $P(A)$ is the marginal probability of an event or a proposition A
- $P(A \text{ and } B) = P(A,B) = P(B,A)$ is the joint probability of A and B, the probability that both are true
- $P(A|B)$ is the conditional probability of A given that B is true. The vertical line between A and B is pronounced "given"

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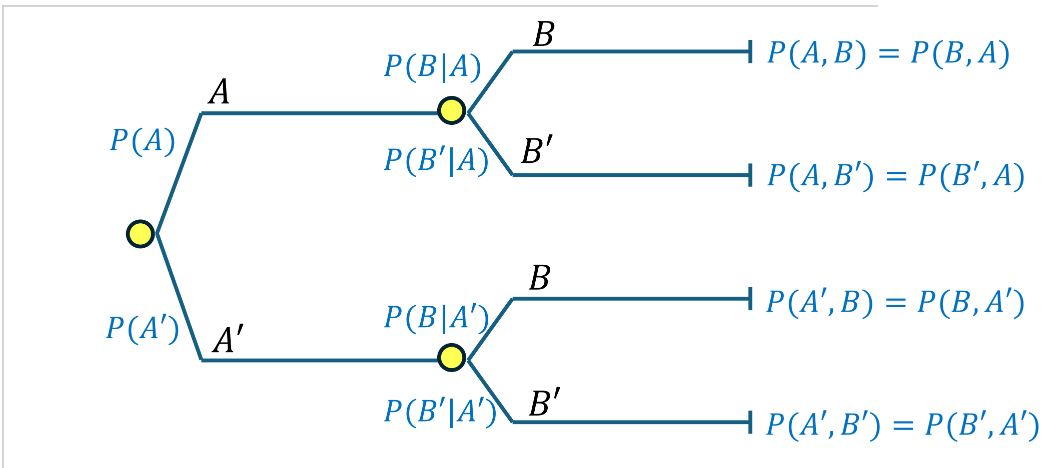
Terminology in a probability tree



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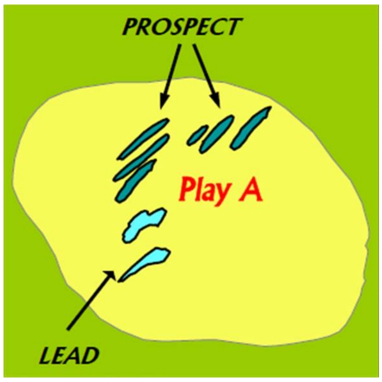
Probabilistic dependence in exploration drilling

- 30 previous 3D seismic studies have been carried out in an oil play. In the 10 times oil was present the seismic correctly predicted it 8 times. In the 20 times oil was not present the seismic correctly predicted the non-presence only 14 times.
- An initial evaluation of a prospect indicates a 10% chance of commercial oil.
- A 3D seismic survey is run and indicates presence of commercial hydrocarbons.
- What is the revised probability that hydrocarbons are indeed present?



$$P(Oil \mid "Oil") = ?$$

Revision of Probabilities:
Example – Seismic Study

Seismic Reliability (Number)		Prospect is Oil	Prospect is Dry
Seismic Says	Oil	8	6
Seismic Says	Dry	2	14
		10	20

Seismic Reliability (Conditional Probability)		Prospect is Oil	Prospect is Dry
Seismic Says	Oil	0.80	0.30
Seismic Says	Dry	0.20	0.70

OMV Norway

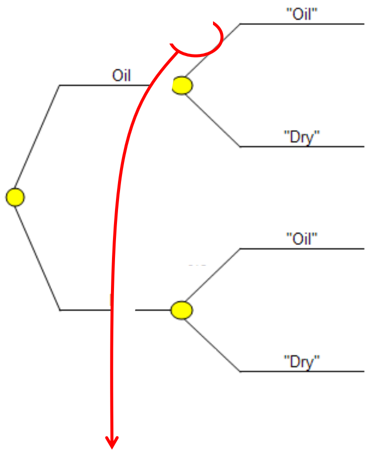
We can summarize the information we have in a probability tree

Let

- Oil – Prospect has Oil
- Dry – Prospect is Dry
- “Oil“ – Seismic says Oil
- “Dry“ – Seismic says Dry

We need some means to reverse the assessment from

$P(''Oil''|Oil)$ to $P(Oil|''Oil'')$



$$P(''Oil''|Oil) = 0.8$$

Bayes’ Theorem Provides the Means for Updating Probabilities Given New Information

- Named after Thomas Bayes (1702 – 1761), who was an English minister and mathematician.
- In his work *Essay Toward Solving a Problem in the Doctrine of Chance*, published posthumously, an early attempt is made at establishing what we now refer to as Bayes’ theorem.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

The probability revision process

Prior probabilities →

Likelihood probabilities usually based on observations →

Bayes' Theorem calculation.

→ Posterior probabilities

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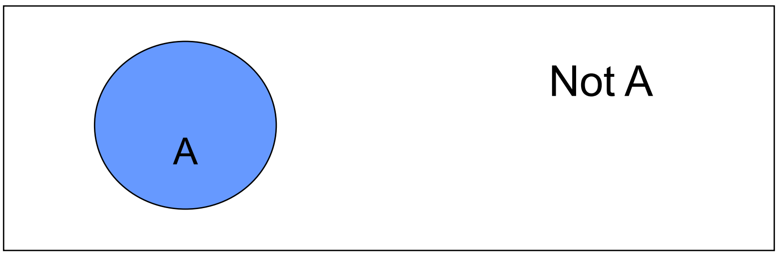
Probability Concepts: Venn Diagrams & Complement



- Area of rectangle = 1
- Area of blue circle is probability of A

Probability A occurs = 1 – Probability "not A" occurs

$$P(A) = 1 - P(A')$$

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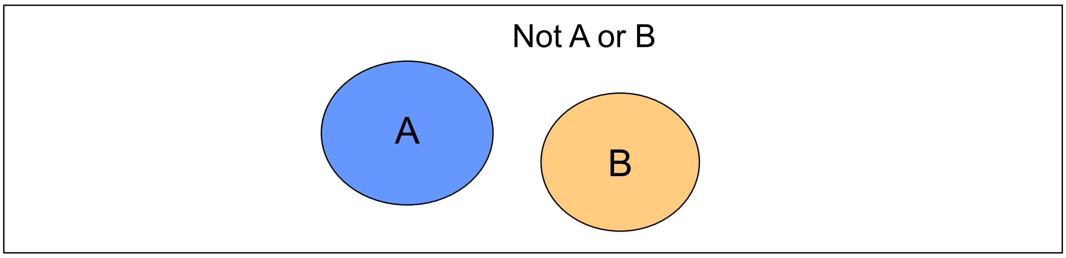
Probability Concepts:
Mutual Exclusivity and Addition Rule

Event A and Event B cannot occur at the same time
(Probability(A and B) = 0)



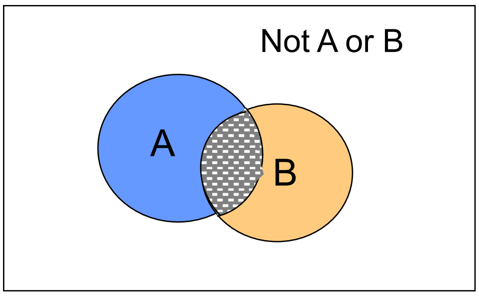
- Addition Rule for Mutually Exclusive events

P(A or B) = P(A) + P (B)

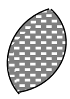
P(♣ or ♥) = .25 + .25 = 0.5

Probability Concepts:
Addition Rule for Non-Mutually Exclusive Events

Event A and Event B can occur at the same time



Joint probability =
P (A and B)

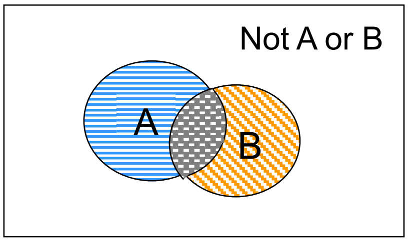


- Addition Rule for non-mutually exclusive events

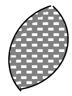
P(A or B) = P(A) + P (B) – P (A and B)

P(Q or ♣) = 1/13 + 1/4 – 1/52

Probability Concepts: Conditional Probability in Venn Diagram

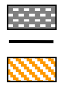


Joint probability = P (A <u>and</u> B)



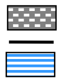
The probability of A happening, given that B has occurred

$$P(A|B) = \frac{P\,(A \text{ and } B)}{P(B)} =$$



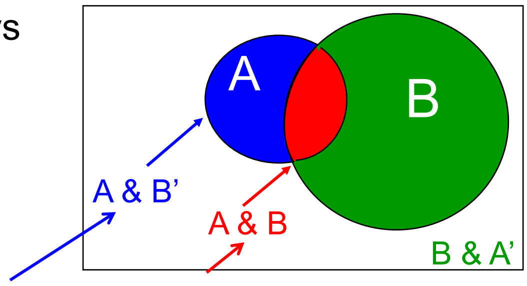
The probability of B happening, given that A has occurred

$$P(B|A) = \frac{P\,(A \text{ and } B)}{P(A)} =$$



The Law of Total Probability - Decomposing Total (or Marginal) Probability

Event A can occur two ways
- with B
- without B



$$P(A) = P(A \text{ and } B') + P(A \text{ and } B)$$
$$= P(A \mid B')P(B') + P(A \mid B)P(B)$$

Similarly, the total probability of B is

$$P(B) = P(B \text{ and } A') + P(A \text{ and } B)$$
$$= P(B \mid A')P(A') + P(A \mid B)P(B)$$

Conditional Probabilities & Bayes' Theorem

From $P(A,B) = P(B,A) = P(B \mid A)P(A) = P(A \mid B)P(B)$

$$P(B \mid A)P(A) = P(A \mid B)P(B)$$

or, rearranged

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A)}}$$

Usual form of Bayes' Theorem

and substituting for total probability of A

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B')P(B')}}$$

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Generalization of Bayes' Rule and Prior, Posterior, Likelihood terminology

Given a collection of n mutually exclusive and collectively exhaustive events B_1, B_2, …. B_n and another event A

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_i P(A \mid B_i)P(B_i)}$$

The denominator (summation) is the total probability of A, that is, all the ways that A can occur and $P(A|B_i)$ is called the likelihood function

$$\text{Posterior probability} = \frac{likelihood \cdot prior\ probability}{\sum_i likelihood \cdot prior\ probability}$$

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Example – Seismic Indicates "Oil"

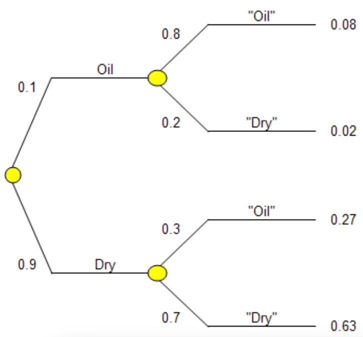
Bayes'

$$P(Oil\,|\,"Oil")$$

$$= \frac{P("Oil"\,|\,Oil)P(Oil)}{P("Oil")}$$

$$= \frac{P("Oil"\,|\,Oil)P(Oil)}{P("Oil"\,|\,Oil)P(Oil) + P("Oil"\,|\,Dry)P(Dry)}$$

$$= \frac{0.8 \cdot 0.1}{0.8 \cdot 0.1 + 0.3 \cdot 0.9} = 0.23$$



- The revised (updated) probability of there being Oil given that the seismic indicates "Oil" is 23%.
- This is up from the initial assessment of 10%.

Summary

- We can compute a conditional probability using a joint probability

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

- We can compute a joint probability using a conditional probability

$$P(A \text{ and } B) = P(A,B) = P(A|B)P(B)$$

- Bayes' Rule give us a way to get from $P(A|B)$ to $P(B|A)$, or the other way around

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- The Law of Total Probability provides a way to compute probabilities by adding up the pieces

$$P(A) = \sum_i P(B_i)P(A|B_i)$$

So what? What's the relevance?

In decision tree modeling, and decision analysis in general, we make extensive use of conditional probabilities, and their formulation as Bayes' Theorem, to revise prior probabilities in the light of new information?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

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Quiz - Newspaper Problem

- It is Saturday morning at 0800, and I must decide whether to walk down to the bottom of my driveway to get the newspaper.
- On the basis of past experience, I judge that there is an 80% chance that the paper has been delivered by now.
- Looking out of the kitchen window, I can see exactly half of the bottom of the driveway, and the paper is not in the half that I see.
  - If the paper has been delivered there's an equal chance that it will fall in each half of the driveway
- What is the probability that the paper has been delivered?

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Newspaper Problem

Let

- D be the event “paper has been delivered”,
- S’ the event “I don’t see the paper from my window”

Want to calculate P(D|S’)

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Newspaper Problem

$$P(D \mid S') = \frac{P(S' \mid D) \cdot P(D)}{P(S')}$$

$$= \frac{P(S' \mid D) \cdot P(D)}{P(S' \mid D) \cdot P(D) + P(S' \mid D') \cdot P(D')}$$

$$= \frac{0.5 \cdot 0.8}{0.5 \cdot 0.8 + 1 \cdot 0.2} = 2/3$$

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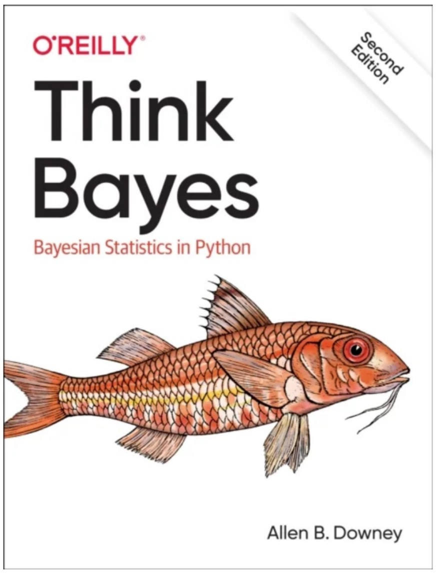
Jupyter Notebooks Draw on Alex Downey's book Think Bayes



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