Pythagorean Quintuples and Quaternions

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Abstract

This paper, which is in number theory, explores the connections between quaternions and primitive Pythagorean quintuples. It is known that the square of a Gaussian integer (a complex number) is a Pythagorean triple $a^2 + b^2 = c^2$. Less is known about the relationship between quaternions, an extension of complex numbers, and Pythagorean quintuples $a^2 + b^2 + c^2 + d^2 = e^2$. We show that squaring a quaternion produces a subfamily of Pythagorean quintuples. Motivated by Conway and Smith's unique factorization theorem for the Hurwitz integers, we present a more general version of squaring a quaternion which generates a larger subfamily of Pythagorean quintuples. Using a counting argument and Jacobi's Four Square Theorem, we show that unlike the characterization for Pythagorean triples, the preceding characterization for Pythagorean quintuples is sparse. Finally, we use a geometric approach to characterize all Pythagorean quintuples. We notice a similarity between the geometric approach and the quaternion squaring approach in that they differ by a geometrically defined constant.

Todo list

x	y	$x^2 - y^2$	$2xy$	$x^2 - y^2$	Check
1	0	1	0	1	$1^2 + 0^2 = 1^2$
2	1	3	4	5	$3^2 + 4^2 = 5^2$
3	2	5	12	13	$5^2 + 12^2 = 13^2$
3	1	8	6	10	$8^2 + 6^2 = 10^2$
4	1	15	8	17	$15^2 + 8^2 = 17^2$
4	2	12	16	20	$12^2 + 16^2 = 20^2$
4	3	7	24	25	$7^2 + 24^2 = 25^2$

Figure 1: Generating pairs for Pythagorean triples where $0 \leq y < x \leq 4$.

1 Pythagorean Triples

Recall nonnegative integers a, b and c are called a *Pythagorean triple* if

$$a^2 + b^2 = c^2.$$

Furthermore, a *primitive Pythagorean triple* is one where $\gcd(a, b, c) = 1$, that is, a, b, and c have no common factor greater than 1. Pythagorean triples have the geometric interpretation that they are the side lengths of a right triangle having integer side lengths.

The following theorem characterizes primitive Pythagorean triples. It was known by Diophantus [12, Page 93] and probably discovered by Euclid [14].

A proof can be found in Hardy and Wright's book; see [10, XIII, 13.2].

Theorem 1.1. *Given a primitive Pythagorean triple* (a, b, c)*, there are integers* x *and* y *such that* $0 < y < x$*, the integers* x *and* y *are of opposite parity, and* $\gcd(x, y) = 1$ *so that*

$$a = x^2 - y^2, \quad b = 2xy, \ \textit{and} \ \ c = x^2 + y^2. \tag{1.1}$$

Example 1.2. For instance, for $a = 5$, $b = 12$, and $c = 13$, we have $x = 3$ and $y = 2$. Geometrically this gives the primitive Pythagorean triple $(5, 12, 13)$ corresponding to the 5-12-13 right triangle.

There is a well-known connection between primitive Pythagorean triples and complex numbers. Consider any *Gaussian integer* $x + yi$, where $x, y \in \mathbb{Z}$. If $x + yi$ is squared, the real and imaginary parts of the resulting Gaussian integer make up two legs of a Pythagorean triangle as follows.

Theorem 1.3. *Any primitive Pythagorean triple* (a, b, c) *with* $a^2 + b^2 = c^2$*, can be represented by the square of a Gaussian integer,* $x + yi$*, that is,*

$$a = \Re(x + yi)^2, \quad b = \Im(x + yi)^2, \ \textit{and}\ c = |(x + yi)^2|,$$

where $\Re$ *and* $\Im$ *denote the real and imaginary parts of a complex number and* $|\cdot|$ *denotes the modulus of a complex number.*

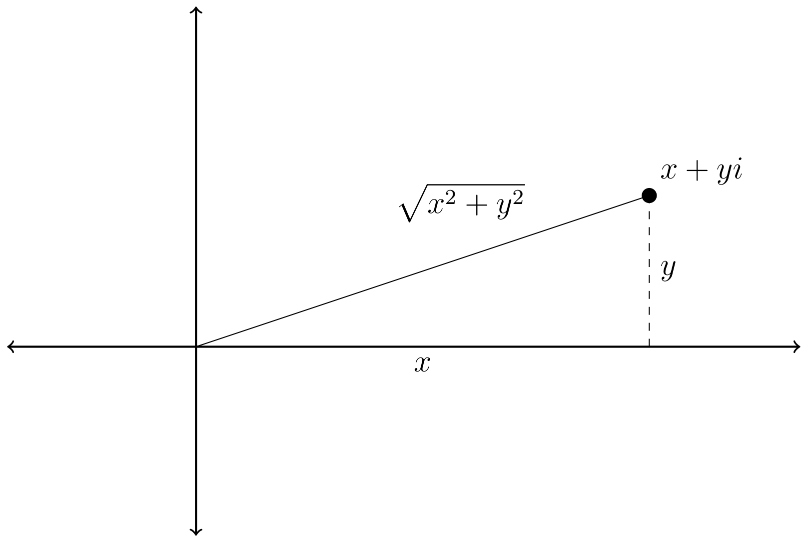

Figure 2: Geometric representation of a Gaussian integer.

Of course, this fact can be proved with algebra, but a more intuitive idea of a proof relies on the definition of multiplication. That is, the number $x + yi$ has a modulus, or distance from the origin, of $\sqrt{x^2 + y^2}$. The modulus of the product of two complex numbers is the same as the product of the moduli of the numbers. So $(x + yi)^2$ has a modulus of $x^2 + y^2$. Thus, the square of a Gaussian integer has integer coordinates and an integer distance from the origin, giving a right triangle with integer sides. See Figures 2 and 3 for diagrams of this concept.

2 Pythagorean Quintuples via Squaring Quaternions

Consider the system of quaternions, discovered by Sir William Hamilton in 1843 [9]. The quaternions can be thought of as complex numbers extended to four dimensions. The general form of a quaternion is $x+yi+zj+wk$, where $x, y, z, w \in \mathbb{R}$. Up to a sign they have 4 units: 1, i, j, and k. Their most striking property is that multiplication is not commutative. For example, $i \cdot j = k \neq j \cdot i = -k$. See Figure 4 for the multiplication table of the quaternions.

Definition 2.1. The *modulus* of a quaternion $q = x + yi + zj + wk$ is given by

$$|q| = \sqrt{x^2 + y^2 + z^2 + w^2}.$$

The *norm* of a quaternion is the modulus squared, that is,

$$N = N(q) = x^2 + y^2 + z^2 + w^2.$$

It is straightforward to check the following property.

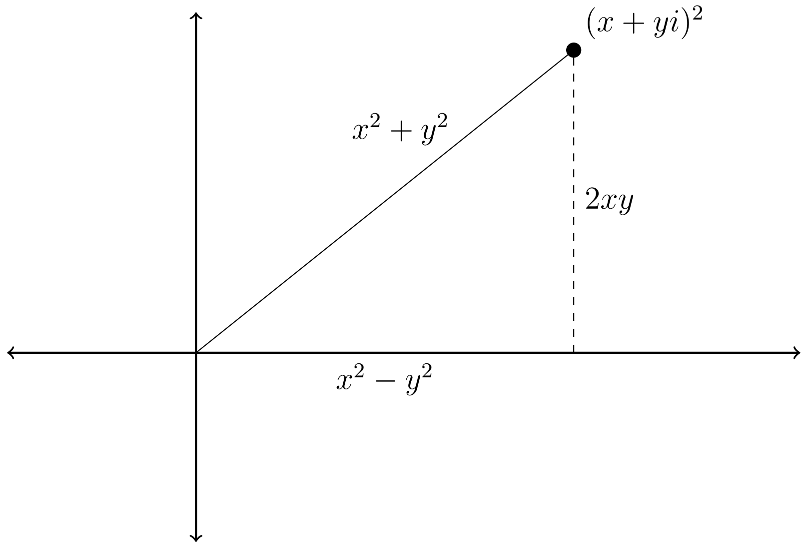

Figure 3: A Gaussian integer $x + yi$ after squaring.

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Figure 4: Multiplication table for the quaternions.

Lemma 2.2. *For two quaternions q_1 and q_2*

$$N(q_1 \cdot q_2) = N(q_1) \cdot N(q_2).$$

Theorem 2.3. *There are Pythagorean quintuples $a^2 + b^2 + c^2 + d^2 = e^2$ which can be represented by the square of a quaternion $x + yi + zj + wk$ where $x, y, z, w \in \mathbb{Z}$. The form of these Pythagorean quintuples is*

$$(a, b, c, d, e) = (2xy, 2xz, 2xw, x^2 - y^2 - z^2 - w^2, x^2 + y^2 + z^2 + w^2).$$

Proof. The general form of a quaternion is $q = x + yi + zj + wk$. Squaring this gives

$$q^2 = (x + yi + zj + wk)^2 \quad = \quad (x^2 - y^2 - z^2 - w^2) + (2xy)i + (2xz)j + (2xw)k$$

Since the modulus of q is $|q| = \sqrt{x^2 + y^2 + z^2 + w^2}$, the modulus of q squared is

$$|q^2| = |q|^2 = (x^2 + y^2 + z^2 + w^2)^2 \quad = \quad (x^2 - y^2 - z^2 - w^2)^2 + (2xy)^2 + (2xz)^2 + (2xw)^2.$$

This means that $(2xy, 2xz, 2xw, x^2 - y^2 - z^2 - w^2, x^2 + y^2 + z^2 + w^2)$ is a Pythagorean quintuple. $\square$

Observe that $1^2 + 1^2 + 3^2 + 5^2 = 6^2$ is a primitive Pythagorean quintuple, but none of the legs 1, 1, 3 and 5 are even. Thus Theorem 2.3 is a partial characterization of Pythagorean quintuples.

3 Pythagorean Quintuples via Multiplying two Hurwitz Integers

John Conway and Derek Smith [5] have investigated the relationship of quaternions to symmetry groups. See the review by Baez [1] for a synopsis of their work. Conway and Smith's unique factorization theorem (see Theorem 3.3) suggested to the author a way to extend the relationship between Pythagorean quintuples and quaternions.

In order to do this, we need to define Hurwitz integers.

Definition 3.1. A quaternion $q = x + yi + zj + wk$ is a *Hurwitz integer* if either $x, y, z, w \in \mathbb{Z}$ or $x, y, z, w \in \mathbb{Z} + \frac{1}{2}$, where $\mathbb{Z} + \frac{1}{2}$ denotes the half integers. In the case that $x, y, z, w \in \mathbb{Z}$ we say q is a *Lipschitz integer.*

Let $\mathbb{H}$ denote the set of Hurwitz integers. Because the Hurwitz integers form a well-packed lattice, they are suitable for error-correcting codes. See the recent paper of Güzeltepe [7]. Conway and Smith found that the Hurwitz integers satisfy the following unique factorization property [5, Ch. 5, Theorem 2]. To do this we need one more definition.

Definition 3.2. A Hurwitz integer $q \in \mathbb{H}$ is *primitive* if there is no natural number $n > 1$ that divides it.

Theorem 3.3 (Conway and Smith)**.** *Let $q \in \mathbb{H}$ be a primitive Hurwitz integer with norm N. Suppose $N = p_1 p_2 \cdots p_k$ where $p_1, p_2, \ldots, p_k$ are prime numbers. Then q can be factored as a product of Hurwitz integers*

$$q = P_1 \cdot P_2 \cdots P_k,$$

where the norm of P_i is given by $N(P_i) = p_i$ for $i = 1, \ldots, k$. This factorization is unique up to unit migration, that is, all other factorizations corresponding to the product $N = p_1 p_2 \cdots p_k$ are of the form

$$q = (P_1 U_1) \cdot (U_1^{-1} P_2 U_2) \cdots (U_{k-1}^{-1} P_k),$$

where the U_i are Hurwitz units, that is, Hurwitz integers with norm 1.

The 24 Hurwitz units are:

$$\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k).$$

Just as the square of a Gaussian integer must have an integer modulus, the square of a Hurwitz integer has an integer modulus and therefore a norm that is a perfect square. As a result of Theorem 3.3, a quaternion built by using the coefficients of a Pythagorean quintuple can always be factored into two quaternions with the same norm.

I can also deduce the last sentence from Sarnak et al.'s book

Thank Professor Sarnak

What does this mean in concrete terms? We give an example.

Example 3.4. The quintuple

$$1^2 + 1^2 + 3^2 + 5^2 = 6^2$$

cannot be represented by the squaring method of Theorem 2.3, but it can be translated into a quaternion

$$1 + i + 3j + 5k.$$

We can factor out a 2, leaving us with a primitive Hurwitz integer

$$h = \frac{1}{2} + \frac{1}{2}i + \frac{3}{2}j + \frac{5}{2}k.$$

Note the norm of h is $N(h) = 9 = 3 \cdot 3$. Since

$$\frac{1}{2} + \frac{1}{2}i + \frac{3}{2}j + \frac{5}{2}k = \left(\frac{1}{2} + \frac{1}{2}i + \frac{3}{2}j - \frac{1}{2}k\right)\left(\frac{1}{2} - \frac{3}{2}i + \frac{1}{2}j + \frac{1}{2}k\right)$$

then the four half integers $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{3}{2}$ generate $1^2 + 1^2 + 3^2 + 5^2 = 6^2$.

What is nice in this example is that the two Hurwitz integers P_1 and P_2 whose product gives the primitive Hurwitz integer h have the property that their coefficients, in absolute value and reordered, coincide.

Theorem 3.5. *There are Pythagorean quintuples $a^2 + b^2 + c^2 + d^2 = e^2$ which can be represented by the product of two Hurwitz integers $x + yi + zj + wk$ and $x' + y'i + z'j + w'k$ where the elements $|x|, |y|, |z|, |w|$ correspond to some ordering of the elements $|x'|, |y'|, |z'|, |w'|$.*

Proof. Consider the product of the two Hurwitz integers $P_1 = x + yi + zj + wk$ and $P_2 = x' + y'i + z'j + w'k$. Since the second Hurwitz integer P_2 is just a reordering of the coefficients x, y, z and w of P_1, with possible sign changes, the norms of these two Hurwitz integers are the same, that is, $N(P_1) = N(P_2) = \varepsilon$.

We claim $\varepsilon \in \mathbb{Z}$. If P_1 (and hence P_2) is a Lipschitz integer then the norm ε is obviously an integer. Otherwise, write

$$P_1 = \frac{\alpha}{2} + \frac{\beta}{2}i + \frac{\gamma}{2}j + \frac{\delta}{2}k,$$

where α, β, γ and δ are all odd integers. The norm of P_1 is then

$$N(P_1) = \frac{1}{4} \cdot (\alpha^2 + \beta^2 + \gamma^2 + \delta^2).$$

Since

$$\alpha^2 \equiv \beta^2 \equiv \gamma^2 \equiv \delta^2 \equiv 1 \mod 4,$$

their sum is divisible by 4. Hence $N(P_1) = \varepsilon \in \mathbb{Z}$. By Lemma 2.2, $N(P_1 \cdot P_2) = \varepsilon^2$ and $|P_1 \cdot P_2| = \varepsilon$. Since $|P_1 \cdot P_2|$ is an integer and $P_1 \cdot P_2 = a + bi + cj + dk$ has coefficients which are either integers or half-integers then either (a, b, c, d, e) is a Pythagorean quintuple or both (a, b, c, d, e) and $(2a, 2b, 2c, 2d, 2e)$ are Pythagorean quintuples, where $e = \varepsilon$. $\square$

One can check that $1^2 + 2^2 + 8^2 + 10^2 = 13^2$ is a primitive Pythagorean quintuple which cannot be written as the product of two quaternions using the same coefficients. For example

$$-2 + 10i + j + 8k = (2 + 3i) \cdot (2 + 2i + 2j + k).$$

See Appendix A for more details. Thus Theorem 3.5 is a partial characterization of Pythagorean quintuples which arise from products of two related Hurwitz integers.

4 An Asymptotic Analysis of Theorem 3.5

In this section we would like to analyze the general question of finding primitive Pythagorean quintuples of magnitude ϵ using a polynomial parametrization in four parameters.

Let us review some definitions. We say a *generating quadruple* is a list of either four integers or four half-integers. The *magnitude* of a Pythagorean quintuple is the value of its largest element, that is, the magnitude of $a^2 + b^2 + c^2 + d^2 = e^2$ is e. For instance, the magnitude of $0^2 + 0^2 + 8^2 + 15^2 = 17^2$ is 17.

This section will require the Jacobi Four Square Theorem, proved by Jacobi in 1834 [13]. This is a counting version of Lagrange's theorem that every integer n can be written as a sum of four squares. A proof using Hurwitz integers appears in Hardy and Wright [10, Chapter XX, Theorem 386]. [15] expressed this theorem differently.

Theorem 4.1 (Jacobi)**.** *The number of solutions to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$ is*

$$8 \sum_{d|n, 4\nmid d} d$$

This is eight times the sum of all positive integers that divide n but are not multiples of 4. Note that the order and sign of a solution (x_1, x_2, x_3, x_4) matter.

Theorem 4.2. *For any prime $\epsilon >> 0$, the number of primitive Pythagorean quintuples of magnitude ϵ is greater than the number of Pythagorean quintuples of magnitude ϵ that can be generated using any parametrization in four variables x_1, x_2, x_3, x_4 that obeys the following conditions:*

(i) The parametrization always generates Pythagorean quintuples

(ii) Their squares give the sum

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \epsilon$$

(iii) x_1, x_2, x_3, x_4 *are either all integers or all half-integers.*

Observe that the 57 representations found as a consequence of Theorem 3.5 are such parametrizations.

We now give a proof of Theorem 4.2.

Proof. Let ϵ be a large prime. By Lagrange's Theorem every integer can be written as the sum of four squares, so there must be at least one solution to $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \epsilon^2$, where α, β, γ, and δ are nonnegative integers. This is a Pythagorean quintuple of magnitude ϵ.

We are interested in the number of Pythagorean quintuples of magnitude ϵ where the order of α, β, γ, and δ does not matter. Since ϵ is a prime, the integers α, β, γ, and δ share no common factor other than 1 or ϵ. Thus, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \epsilon^2$ is a primitive Pythagorean quintuple.

Let ψ be the number of primitive Pythagorean quintuples of magnitude ϵ. Note that ψ does not count Pythagorean quintuples whose addends are squares of half-integers. Let λ be the number of solutions to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \epsilon^2$ where order and sign do matter. The only divisors of ϵ^2 are ϵ^2, ϵ, and 1. Using Jacobi's Four Square Theorem 4.1,

$$\lambda = 8(1 + \epsilon + \epsilon^2).$$

However, we are interested in the number of Pythagorean quintuples, where the order and sign do not matter. Since λ counts at least one solution where one of x_1, x_2, x_3, x_4 is negative, $\lambda > \psi$.

We next look for a lower bound for ψ. In the case that a Pythagorean quintuple counted by ψ has distinct and positive $\alpha, \beta, \gamma, \delta$, Jacobi's theorem vastly overestimates the number of Pythagorean quintuples. Such a Pythagorean quintuple can have its terms permuted in $4! = 24$ ways and can have $2^4 = 16$ different arrangements of sign.

For example, the Pythagorean quintuple $1^2 + 2^2 + 10^2 + 16^2 = 19^2$ is overcounted by Jacobi's theorem as $(1, 2, 10, 16)$, $(2, 1, -16, 10)$, $(-16, 1, -2, 10)$, etc. All are valid solutions to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 19^2$. Thus, such a Pythagorean quintuple is overcounted by $24 \cdot 16 = 384$ times in λ. This is the maximum amount that such a Pythagorean quintuple could be overcounted. Thus,

$$\psi \geq \frac{\lambda}{384}.$$

Since we know the value of λ when ϵ is prime,

$$\psi \geq \frac{8(1 + \epsilon + \epsilon^2)}{384}.$$

In the case of finding parametrizations for the primitive Pythagorean quintuples whose addends are all squares of half-integers, let κ be the total number of primitive Pythagorean quintuples of magnitude ϵ. Since κ is clearly at least ψ, we know that

$$\kappa \geq \frac{8(1+\epsilon+\epsilon^2)}{384} \tag{4.1}$$

Let us shift our focus to the number of generating quadruples that could generate a Pythagorean quintuple of magnitude ϵ. The quadruple has the form (a, b, c, d), where $a^2 + b^2 + c^2 + d^2 = \epsilon$. Here a, b, c, and d are all integers or all half-integers.

Since $a^2 + b^2 + c^2 + d^2 = \epsilon$, we multiply by 4 to obtain:

$$\begin{aligned} &4a^2 + 4b^2 + 4c^2 + 4d^2 = 4\epsilon \\ &(2a)^2 + (2b)^2 + (2c)^2 + (2d)^2 = 4\epsilon \end{aligned}$$

Let $m_1 = 2a$, $m_2 = 2b$, $m_3 = 2c$, and $m_4 = 2d$. Note $m_i \in \mathbb{Z}$ for $i = 1, \dots, 4$ and

$$m_1^2 + m_2^2 + m_3^2 + m_4^2 = 4\epsilon \tag{4.2}$$

Note that this gives a simple bijection between a generating quadruple (a, b, c, d) and a quadruple of four integers (m_1, m_2, m_3, m_4) whose norm is 4ϵ.

We again refer to Jacobi's Theorem to determine how many solutions (4.2) has. Since ϵ is a large prime, 4ϵ is divisible by 1, 2, 4, ϵ, 2ϵ, and 4ϵ. Leaving out the divisors that are divisible by 4, the number of solutions to (4.2) is

$$8(1 + 2 + \epsilon + 2\epsilon) = 8(3 + 3\epsilon) = 24(1 + \epsilon).$$

Assume that there are τ parametrizations that meet the three conditions in the theorem. Let ω be the number of Pythagorean quintuples produced by applying the τ parametrizations to the $24(1 + \epsilon)$ generating quadruples. We have an upper bound for ω.

$$\omega \leq 24\tau(1 + \epsilon) \tag{4.3}$$

Observe that in (4.1) κ grows with the square of ϵ, but in (4.3) ω grows linearly in ϵ.

For sufficiently large prime ϵ,

$$\begin{aligned} &\frac{8(1+\epsilon+\epsilon^2)}{384} > 24\tau(1+\epsilon) \\ &\kappa \geq \frac{8(1+\epsilon+\epsilon^2)}{384} > 24\tau(1+\epsilon) \geq \omega \\ &\kappa > \omega \end{aligned}$$

where in the second inequality we used (4.1) and (4.3). Thus, for any prime ϵ above a certain bound, the number of primitive Pythagorean quintuples of magnitude ϵ is greater than the number of Pythagorean quintuples of magnitude ϵ that can be generated using parametrizations of this type. Since there are an infinite number of primes larger than that bound, there is an infinite number of primitive Pythagorean quintuples that cannot be generated by any finite list of such parametrizations. $\square$

Note that this result also excludes the possibility of using the Hurwitz units as multipliers to generate all primitive Pythagorean quintuples. An infinite number would still evade representation.

5 Characterization of Pythagorean Quintuples: A Geometric Approach

This was done algebraically in Mordell.

In this section we derive an expression which characterizes *all* Pythagorean quintuples.

Theorem 5.1. *All primitive Pythagorean quintuples can be written in the form*

$$(2ad, 2bd, 2cd, a^2 + b^2 + c^2 - d^2, a^2 + b^2 + c^2 + d^2)/k,$$

where $k = \gcd(2ad, 2bd, 2cd, a^2 + b^2 + c^2 - d^2)$ *and* $a, b, c, d \in \mathbb{Z}$.

Proof. Consider the 3-dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ in four dimensions, denoted by $\mathbb{S}^3$. Let $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \epsilon^2$ be a primitive Pythagorean quintuple.

Since

$$\left(\frac{\alpha}{\epsilon}\right)^2 + \left(\frac{\beta}{\epsilon}\right)^2 + \left(\frac{\gamma}{\epsilon}\right)^2 + \left(\frac{\delta}{\epsilon}\right)^2 = 1,$$

the point $(\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}, \frac{\gamma}{\epsilon}, \frac{\delta}{\epsilon})$ is on the sphere $\mathbb{S}^3$. Note that there exists a bijection between rational points on $\mathbb{S}^3$ and primitive Pythagorean quintuples. If a line is drawn between the points $(0, 0, 0, 1)$ and $(\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}, \frac{\gamma}{\epsilon}, \frac{\delta}{\epsilon})$, it has rational slope. Furthermore, this line passes through the three-dimensional subspace $w = 0$ at the point $(m, n, p, 0)$, where m, n, and p are positive rational numbers. Notice that the line through $(0, 0, 0, 1)$ gives a bijection between rational points on $\mathbb{S}^3$ and rational points on $w = 0$.

The line satisfies the following conditions:

$$\frac{\partial y}{\partial x} = \frac{n}{m}, \quad \frac{\partial z}{\partial x} = \frac{p}{m}, \quad \frac{\partial w}{\partial x} = -\frac{1}{m}.$$

Therefore, the line satisfies these conditions as well:

$$y = \frac{n}{m}x, \quad z = \frac{p}{m}x, \quad w = -\frac{1}{m}x + 1.$$

We use these conditions to solve for the intersection of the line and the sphere $\mathbb{S}^3$, which we already know are the two points $(0,0,0,1)$ and $(\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}, \frac{\gamma}{\epsilon}, \frac{\delta}{\epsilon})$.

We have

$$\begin{aligned}
&x^2 + y^2 + z^2 + w^2 = 1 \\
&x^2 + \left(\frac{n}{m}x\right)^2 + \left(\frac{p}{m}x\right)^2 + \left(-\frac{1}{m}x + 1\right)^2 = 1 \\
&x^2 + \frac{n^2}{m^2}x^2 + \frac{p^2}{m^2}x^2 + \frac{1}{m^2}x^2 - \frac{2}{m}x + 1 = 1 \\
&x^2\left(1 + \frac{n^2}{m^2} + \frac{p^2}{m^2} + \frac{1}{m^2}\right) - \frac{2}{m}x = 0 \\
&x\left(x\left(1 + \frac{n^2}{m^2} + \frac{p^2}{m^2} + \frac{1}{m^2}\right) - \frac{2}{m}\right) = 0
\end{aligned}$$

The solution $x = 0$ refers to the point $(0,0,0,1)$. We continue solving for the other intersection point.

$$\begin{aligned}
&x\left(1 + \frac{n^2}{m^2} + \frac{p^2}{m^2} + \frac{1}{m^2}\right) - \frac{2}{m} = 0 \\
&x\left(\frac{m^2 + n^2 + p^2 + 1}{m^2}\right) - \frac{2}{m} = 0 \\
&x\left(\frac{m^2 + n^2 + p^2 + 1}{m^2}\right) = \frac{2}{m} \\
&x = \frac{2m}{m^2 + n^2 + p^2 + 1}
\end{aligned}$$

This solution is the x-coordinate of the point $\left(\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}, \frac{\gamma}{\epsilon}, \frac{\delta}{\epsilon}\right)$. We solve for the other coordinates of this point.

$$y = \frac{n}{m}x = \frac{n}{m} \cdot \frac{2m}{m^2 + n^2 + p^2 + 1} = \frac{2n}{m^2 + n^2 + p^2 + 1}$$

$$z = \frac{p}{m}x = \frac{p}{m} \cdot \frac{2m}{m^2 + n^2 + p^2 + 1} = \frac{2p}{m^2 + n^2 + p^2 + 1}$$

$$\begin{aligned}
w &= -\frac{1}{m}x + 1 = -\frac{1}{m} \cdot \frac{2m}{m^2 + n^2 + p^2 + 1} + 1 \\
&= \frac{-2}{m^2 + n^2 + p^2 + 1} + \frac{m^2 + n^2 + p^2 + 1}{m^2 + n^2 + p^2 + 1} \\
&= \frac{m^2 + n^2 + p^2 - 1}{m^2 + n^2 + p^2 + 1}.
\end{aligned}$$

Since m, n, and p are rational, we can write the following:

$$m = \frac{a}{d}, \;\; n = \frac{b}{d}, \;\; p = \frac{c}{d},$$

where a, b, and c are non-negative integers and d is a positive integer.

Rewriting the coordinates gives:

$$x = \frac{2m}{m^2+n^2+p^2+1} = \frac{2 \cdot \frac{a}{d}}{\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 + 1} = \frac{2ad}{a^2+b^2+c^2+d^2}$$

$$y = \frac{2n}{m^2+n^2+p^2+1} = \frac{2 \cdot \frac{b}{d}}{\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 + 1} = \frac{2bd}{a^2+b^2+c^2+d^2}$$

$$z = \frac{2p}{m^2+n^2+p^2+1} = \frac{2 \cdot \frac{c}{d}}{\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 + 1} = \frac{2cd}{a^2+b^2+c^2+d^2}$$

$$w = \frac{m^2+n^2+p^2-1}{m^2+n^2+p^2+1} = \frac{\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 - 1}{\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 + 1} = \frac{a^2+b^2+c^2-d^2}{a^2+b^2+c^2+d^2}$$

This particular (x, y, z, w) is the same point as $\left(\frac{\alpha}{\epsilon}, \frac{\beta}{\epsilon}, \frac{\gamma}{\epsilon}, \frac{\delta}{\epsilon}\right)$. Setting these equal to each other gives:

$$x = \frac{2ad}{a^2+b^2+c^2+d^2} = \frac{\alpha}{\epsilon}$$

$$y = \frac{2bd}{a^2+b^2+c^2+d^2} = \frac{\beta}{\epsilon}$$

$$z = \frac{2cd}{a^2+b^2+c^2+d^2} = \frac{\gamma}{\epsilon}$$

$$w = \frac{a^2+b^2+c^2-d^2}{a^2+b^2+c^2+d^2} = \frac{\delta}{\epsilon}.$$

Therefore, we can write

$$\begin{aligned}
k\alpha &= 2ad \\
k\beta &= 2bd \\
k\gamma &= 2cd \\
k\delta &= a^2+b^2+c^2-d^2 \\
k\epsilon &= a^2+b^2+c^2+d^2
\end{aligned}$$

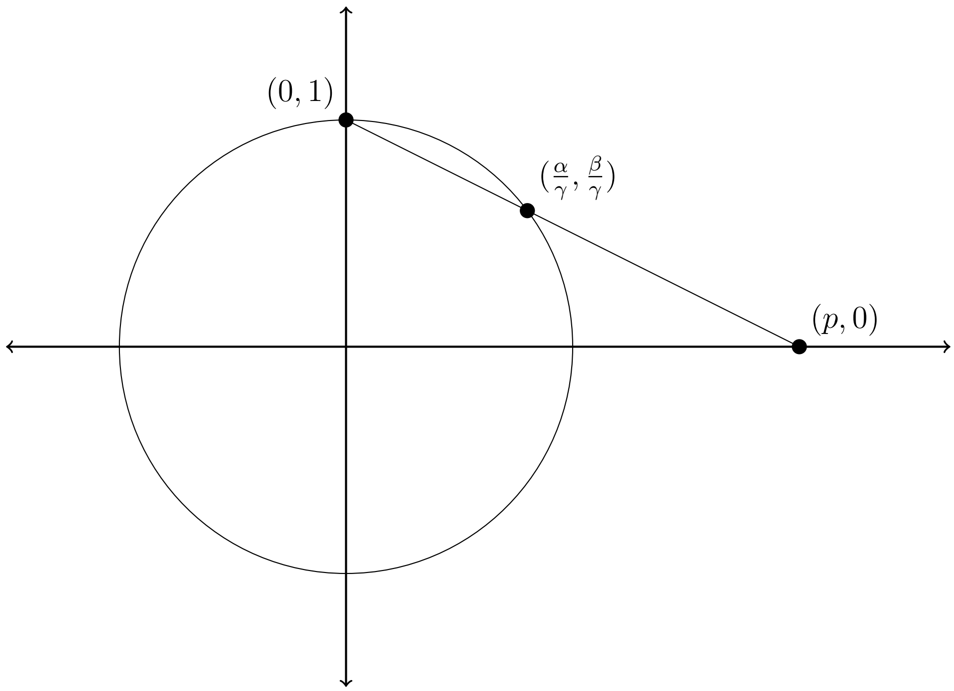

Figure 5: A two-dimensional version of the geometric argument to characterize Pythagorean triples $\alpha^2 + \beta^2 = \gamma^2$

where k is a positive rational number. Write k as $\frac{t}{u}$, where t and u are positive integers with no common factor greater than 1. Thus $\frac{\alpha}{u}$, $\frac{\beta}{u}$, $\frac{\gamma}{u}$, $\frac{\delta}{u}$, and $\frac{\epsilon}{u}$ are all positive integers. Therefore, u is a common factor of α, β, γ, δ, and ϵ. Since this Pythagorean quintuple is primitive, u must be 1. Therefore, k is a positive integer. $\square$

Remarks 5.2. We end with two comments. First, the approach we give to prove Theorem 5.1 is similar to the geometric proof given in Hatcher's notes [11] for Pythagorean triples and quadruples. However, it does not appear in the literature. Secondly, the expression in Theorem 5.1 looks similar to the square of a quaternion from the proof of Theorem 2.3, namely

$$q^2 = (x + yi + zj + wk)^2 = (x^2 - y^2 - z^2 - w^2) + (2xy)i + (2xz)j + (2xw)k.$$

The difference is in Theorem 5.1 we divide by the greatest common divisor of the four terms. This projects Pythagorean quintuples onto primitive Pythagorean quintuples.

6 Conclusion

The original goal of this project was to find a parametrization, preferably a polynomial one, of Pythagorean quintuples in four parameters. Such a parametrization of quintuples has been done using 12 parameters [6], but the method involved a specific case of sextuples, not a direct relationship between quintuples and quaternions. This project has shown that a direct relationship is not trivial.

There are several paths for future research:

1. Conway and Smith have developed connections between the quaternions and orthogonal groups (rotations of space). Once I understand more group theory, their ideas may provide an important geometrical interpretation.

2. Perhaps the obvious geometrical analogue of Pythagorean quintuples — the volumes of the facets of what we will call a tetrarectangular 4-dimensional simplex — has a deeper meaning.

3. In the 1930s, B. Berggren [3] discovered a set of transformations that iteratively generate all primitive Pythagorean triples. F. J. M. Barning [2] independently found this method and reformulated it into a set of matrices, and Conrad [4] provided a geometric interpretation. Notably, these transformations can be shown as the Barning-Hall tree, which contains all primitive Pythagorean triples. This approach is completely different from the methods in this paper, but I plan to explore this path.

Appendix A Detailed Analysis of a Pythagorean Quintuple That Does Not Arise from Theorem 3.5

Consider the Pythagorean quintuple $1^2 + 2^2 + 8^2 + 10^2 = 13^2$. It cannot be represented as in Theorem 3.5. If it could, it would use one of the following generating lists. The coefficients are the only Hurwitz integers to have a modulus of 13:

Get rid of the words generating list, generating quadruple, etc.

1. $\frac{7}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
   How could this list create the 1? The terms that could add to 1 are

   - $\pm\frac{49}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}$
   - or $\pm\frac{7}{4}, \pm\frac{7}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}$.

   Neither of these possibilities work.

2. $3, 2, 0, 0$
   How could this list create the 10? The terms that could add to 10 are

   - $\pm 9, \pm 4, 0, 0,$

- $\pm 6, \pm 6, 0, 0$,
- $\pm 6, 0, 0, 0$,
- $\pm 4, 0, 0, 0$,
- $0, 0, 0, 0$,
- or $\pm 9, 0, 0, 0$.

None of these possibilities work.

3. $\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}$
How could this list create the 1? The terms that could add to 1 are

- $\pm\frac{25}{4}, \pm\frac{9}{4}, \pm\frac{9}{4}, \pm\frac{9}{4}$
- or $\pm\frac{15}{4}, \pm\frac{15}{4}, \pm\frac{9}{4}, \pm\frac{9}{4}$.

Neither of these possibilities work.

4. $\frac{5}{2}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2}$
How could this list create the 1? The terms that could add to 1 are

- $\pm\frac{25}{4}, \pm\frac{25}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}$,
- $\pm\frac{25}{4}, \pm\frac{5}{4}, \pm\frac{5}{4}, \pm\frac{1}{4}$,
- or $\pm\frac{5}{4}, \pm\frac{5}{4}, \pm\frac{5}{4}, \pm\frac{5}{4}$.

None of these possibilities work.

5. $2, 2, 2, 1$
How could this list create the 10? The terms that could add to 10 are

- $\pm 4, \pm 4, \pm 4, \pm 1$
- or $\pm 4, \pm 4, \pm 2, \pm 2$.

Neither of these possibilities work.

Appendix B An Elementary Way to Generate Pythagorean Quintuples

This method was inspired by the method used in [8], which dealt with Pythagorean Quadruples. I strongly suspect that this concept is a consequence of [16].

Choose 3 nonnegative integers a, b, and c, excluding the case where two of them are odd and one is even.

Choose integers p and q that obey the following conditions:

(i) $p|(a^2+b^2+c^2)$

(ii) $pq = a^2+b^2+c^2$

(iii) $p \equiv q \pmod 2$

Let $d = \frac{p-q}{2}$ and $e = \frac{p+q}{2}$. Note that if $p \not\equiv q \pmod 2$, then d and e would not be integers. Then $a^2+b^2+c^2+d^2 = e^2$. All Pythagorean quintuples can be generated in this way.

I (as of June 27, 2019) believe the Heinz 57 result resulted from an error in my computer code. I am also suspicious of the other table in this section, and I don't see how it is directly relevant.

Appendix C Testing the formulas from Theorem 3.5

The author wrote a Python program to determine the number of Pythagorean quintuples that Theorem 3.5 represents. This program found all primitive Pythagorean quintuples with all legs less than or equal to 20. The program then takes a form, applies it to a set of four integers or half integers, and records the resulting Pythagorean quintuple.

Example C.1. Example of a generating list:

$$4, 4, 0, 1$$

Example of a form:

$$(x + yi + zj + wk)(y + zi + wj - xk)$$

The program applies the form:

$$(4 + 4i + 0j + 1k)(4 + 0i + 1j - 4k) = 20 + 15i + 20j - 8k$$

The result:

$$8, 15, 20, 20$$

The result is always a Pythagorean quintuple:

$$8^2 + 15^2 + 20^2 + 20^2 = 33^2$$

The author then analyzed the forms to see if they capture all of the test set of primitive Pythagorean quintuples. To select the forms in Figure 6 the author began with the quaternion $(x + yi + zj + wk)^2$ with x, y, z and w all positive. Theorem 3.5 required permutations and sign changes within the factors. Without loss of generality, the first factor is of the form $(x \pm yi \pm zj \pm wk)$, while the second factor has all possible permutations and sign changes of x, y, z and w to produce $(x' + y'i + z'j + w'k)$. This gives $2^7(4!)$ possible expressions.

1. $(x+yi+zj+wk)(x+yi+zj+wk)$	283	30. $(x+yi+zj+wk)(y+zi+xj-wk)$	353
2. $(x+yi+zj+wk)(x+yi+wj+zk)$	349	31. $(x+yi+zj+wk)(y+zi+wj-xk)$	405
3. $(x+yi+zj+wk)(x+zi+yj+wk)$	349	32. $(x+yi+zj+wk)(y+wi+xj-zk)$	291
4. $(x+yi+zj+wk)(x+zi+wj+yk)$	361	33. $(x+yi+zj+wk)(z+xi+yj-wk)$	353
5. $(x+yi+zj+wk)(x+wi+yj+zk)$	352	34. $(x+yi+zj+wk)(z+xi+wj-yk)$	291
6. $(x+yi+zj+wk)(x+wi+zj+yk)$	349	35. $(x+yi+zj+wk)(z+yi+xj-wk)$	291
7. $(x+yi+zj+wk)(y+xi+zj+wk)$	379	36. $(x+yi+zj+wk)(z+wi+xj-yk)$	5
8. $(x+yi+zj+wk)(y+xi+wj+zk)$	283	37. $(x+yi+zj+wk)(z+wi+yj-xk)$	291
9. $(x+yi+zj+wk)(y+zi+xj+wk)$	352	38. $(x+yi+zj+wk)(w+xi+yj-zk)$	405
10. $(x+yi+zj+wk)(y+zi+wj+xk)$	349	39. $(x+yi+zj+wk)(w+yi+zj-xk)$	405
11. $(x+yi+zj+wk)(y+wi+xj+zk)$	379	40. $(x+yi+zj+wk)(w+zi+xj-yk)$	291
12. $(x+yi+zj+wk)(y+wi+zj+xk)$	393	41. $(x+yi+zj+wk)(w+zi+yj-xk)$	321
13. $(x+yi+zj+wk)(z+xi+yj+wk)$	393	42. $(x+yi+zj-wk)(x+yi+zj+wk)$	321
14. $(x+yi+zj+wk)(z+xi+wj+yk)$	349	43. $(x+yi+zj-wk)(x+zi+wj+yk)$	353
15. $(x+yi+zj+wk)(z+yi+xj+wk)$	379	44. $(x+yi+zj-wk)(x+wi+yj+zk)$	353
16. $(x+yi+zj+wk)(z+yi+wj+xk)$	352	45. $(x+yi+zj-wk)(y+zi+xj+wk)$	353
17. $(x+yi+zj+wk)(z+wi+xj+yk)$	283	46. $(x+yi+zj-wk)(z+xi+yj+wk)$	353
18. $(x+yi+zj+wk)(w+xi+zj+yk)$	352	47. $(x+yi+zj-wk)(z+wi+xj+yk)$	321
19. $(x+yi+zj+wk)(w+yi+xj+zk)$	393	48. $(x+yi+zj-wk)(w+zi+yj+xk)$	321
20. $(x+yi+zj+wk)(w+zi+xj+yk)$	349	49. $(x+yi+zj-wk)(x+yi+wj-zk)$	349
21. $(x+yi+zj+wk)(w+zi+yj+xk)$	283	50. $(x+yi+zj-wk)(x+zi+yj-wk)$	349
22. $(x+yi+zj+wk)(x+yi+zj-wk)$	321	51. $(x+yi+zj-wk)(x+wi+zj-yk)$	349
23. $(x+yi+zj+wk)(x+yi+wj-zk)$	405	52. $(x+yi+zj-wk)(y+xi+zj-wk)$	379
24. $(x+yi+zj+wk)(x+zi+yj-wk)$	405	53. $(x+yi+zj-wk)(y+zi+wj-xk)$	379
25. $(x+yi+zj+wk)(x+zi+wj-yk)$	353	54. $(x+yi+zj-wk)(y+wi+xj-zk)$	349
26. $(x+yi+zj+wk)(x+wi+yj-zk)$	353	55. $(x+yi+zj-wk)(z+xi+wj-yk)$	379
27. $(x+yi+zj+wk)(x+wi+zj-yk)$	291	56. $(x+yi+zj-wk)(z+wi+yj-xk)$	349
28. $(x+yi+zj+wk)(y+xi+zj-wk)$	405	57. $(x+yi+zj-wk)(w+xi+yj-zk)$	349
29. $(x+yi+zj+wk)(y+xi+wj-zk)$	321		

Figure 6: The 57 possible forms of a Pythagorean quintuple in its primitive Hurwitz form as seen in Theorem 3.5. The number next to a form represents the number of times the program used the form to generate a small Pythagorean quintuple.

The author let $x = \pi$, $y = e$, $z = \sqrt{2}$ and $w = \sqrt{5}$ and evaluated each of the $2^7(4!)$ expressions. Of these, there were 57 different resulting values. The author chose one representative expression for each of the 57 values. Notice that there exists a great deal of redundancy among these 57 expressions; any 2 expressions characterize roughly the same set of primitive Pythagorean quintuples, at least for small examples. Intuitively, this redundancy arises because each expression describes a way in which a quaternion can be factored. By Theorem 3.3, a quaternion that represents a primitive Pythagorean quintuple can be factored in many ways, so many of these 57 expressions characterize it.

There are 337 Pythagorean quintuples with all legs less than or equal to 20. See Figure 7 for 50 of those quintuples and the number of ways to generate them:

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Pythagorean Quintuple	Times found	Pythagorean Quintuple	Times found
$0^2+0^2+0^2+1^2=1^2$	255	$2.5^2+3.5^2+6.5^2+19.5^2=21^2$	0
$0^2+0^2+3^2+4^2=5^2$	180	$2.5^2+12.5^2+12.5^2+14.5^2=23^2$	36
$0^2+0^2+5^2+12^2=13^2$	180	$2.5^2+13.5^2+16.5^2+19.5^2=29^2$	0
$0^2+0^2+8^2+15^2=17^2$	180	$3.5^2+5.5^2+8.5^2+10.5^2=15^2$	0
$0^2+3^2+4^2+12^2=13^2$	252	$3.5^2+5.5^2+9.5^2+12.5^2=17^2$	0
$1^2+2^2+8^2+10^2=13^2$	0	$3.5^2+5.5^2+15.5^2+18.5^2=25^2$	0
$1^2+2^2+10^2+16^2=19^2$	0	$3.5^2+5.5^2+17.5^2+19.5^2=27^2$	0
$1^2+4^2+4^2+4^2=7^2$	36	$3.5^2+6.5^2+6.5^2+8.5^2=13^2$	60
$4^2+12^2+13^2+20^2=27^2$	0	$3.5^2+6.5^2+11.5^2+18.5^2=23^2$	0
$4^2+13^2+16^2+20^2=29^2$	144	$3.5^2+7.5^2+10.5^2+10.5^2=17^2$	36
$4^2+16^2+17^2+20^2=31^2$	0	$3.5^2+7.5^2+10.5^2+13.5^2=19^2$	0
$8^2+13^2+14^2+14^2=25^2$	36	$3.5^2+7.5^2+11.5^2+15.5^2=21^2$	24
$10^2+10^2+19^2+20^2=31^2$	36	$3.5^2+8.5^2+8.5^2+11.5^2=17^2$	60
$12^2+16^2+17^2+20^2=33^2$	288	$4.5^2+4.5^2+7.5^2+8.5^2=13^2$	84
$13^2+16^2+20^2+20^2=35^2$	36	$4.5^2+4.5^2+10.5^2+14.5^2=19^2$	84
$13^2+20^2+20^2+20^2=37^2$	36	$4.5^2+4.5^2+13.5^2+17.5^2=23^2$	36
$0.5^2+0.5^2+0.5^2+0.5^2=1^2$	30	$8.5^2+15.5^2+17.5^2+18.5^2=31^2$	0
$0.5^2+0.5^2+1.5^2+2.5^2=3^2$	84	$9.5^2+10.5^2+19.5^2+19.5^2=31^2$	36
$0.5^2+0.5^2+2.5^2+6.5^2=7^2$	84	$9.5^2+12.5^2+14.5^2+16.5^2=27^2$	48
$0.5^2+0.5^2+3.5^2+3.5^2=5^2$	36	$10.5^2+10.5^2+11.5^2+16.5^2=25^2$	36
$0.5^2+0.5^2+3.5^2+12.5^2=13^2$	84	$10.5^2+12.5^2+12.5^2+17.5^2=27^2$	36
$0.5^2+0.5^2+5.5^2+9.5^2=11^2$	36	$10.5^2+16.5^2+16.5^2+17.5^2=31^2$	36
$1.5^2+8.5^2+13.5^2+16.5^2=23^2$	0	$11.5^2+16.5^2+18.5^2+18.5^2=33^2$	36
$1.5^2+10.5^2+10.5^2+17.5^2=23^2$	36	$12.5^2+12.5^2+17.5^2+18.5^2=31^2$	84
$1.5^2+10.5^2+15.5^2+16.5^2=25^2$	0	$12.5^2+14.5^2+18.5^2+19.5^2=33^2$	48

Figure 7: Generating program results for some small Pythagorean quintuples,

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