Applied Numerical Methods
(MATH 151B-Lecture 1, Spring 2016)
Assignment 2

Note:

- **Due day**: 9:00 a.m., 25th April (Monday). Assignments handed after the due date will not be counted.

1. Consider the following multi-step method:

$$w_i - w_{i-1} = \frac{1}{12}h[23f(t_{i-1}, w_{i-1}) - af(t_{i-2}, w_{i-2}) + 5f(t_{i-3}, w_{i-3})]$$

where $a \in R$ is a constant and $h > 0$ is the step size.

   (a) Find a such that the above scheme is consistent.

   (b) Evaluate the order of the above scheme if a is such that the above scheme is consistent (by either directly expanding the local truncation error $\tau_n(h)$ up to a suitable order of h, or by checking relation).

   (c) We can see that the above scheme is consistent iff it converges (which is not the case for general multi-step method.) Please explain.
   Hint: check stability and then use Dahlquist Equivalence Theorem.

2. Consider the following initial value problem:

$$\begin{cases} x'(t) = f(t, x(t)) \text{ on } [0,1] \\ x(0) = x_0 \end{cases}$$

Write it in an integral form:

$$\begin{cases} x(t+2h) - x(t) = \int_t^{t+2h} f(s, x(s))ds \\ x(0) = x_0 \end{cases}$$

where h is a step-size. Let the function $G(t) = f(t, x(t))$ be smooth and let $t_i = ih$, $x_i = x(t_i)$.

   (a) Let $P(t)$ be the Lagrange interpolation polynomial of $G(t)$ at the nodes $t = t_i, t_{i+1}, t_{i+2}$ with its values $P(t_{i+k}) = G(t_{i+k}) = f(t_{i+k}, x_{i+k})$ for $k = 0, 1, 2$. Find an formula for the approximation of the following integral

$$\int_t^{t+2h} f(s, x(s))ds \approx \int_t^{t+2h} P(s)ds$$

by the integral of the polynomial $P(t)$. (Please notice that we are now assuming that we know the exact values of $x(t)$ at $t = t_i, t_{i+1}, t_{i+2}$, i.e. x_{i+k} for $k = 0, 1, 2$, for some unknown reasons.)

(b) Suggest an implicit multistep scheme from the above quadrature approximation.

(c) Show that this scheme is not stable.

3. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Octave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.)
Consider the following multi-step method:

$$w_i - w_{i-1} = \frac{h}{4} f(t_i, w_i) + \frac{3h}{4} f(t_{i-1}, w_{i-1})$$

where $a \in R$ is a constant and $h > 0$ is the step size.

   (a) Find the region of absolute stability of this scheme. Plot the region on the complex plane (with the complex argument $h\lambda$).

   (b) Is the scheme A-stable? Is the scheme L-stable?

4. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Octave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.)
Consider

$$\begin{cases} y'(t) = (t+2)^{-2} \sin(y) \text{ on } [0, 1] \\ y(0) = 0 \end{cases}$$

Recall the 2-step Adam-Bashforth method as

$$w_i - w_{i-1} = \frac{h}{2}[3f(t_{i-1}, w_{i-1}) - f(t_{i-2}, w_{i-2})]$$

and the 2-step Adam Moulton

$$w_i - w_{i-1} = \frac{1}{12} h[5f(t_i, w_i) + 8f(t_{i-1}, w_{i-1}) - f(t_{i-2}, w_{i-2})]$$

   (a) Use the 2-step Adam-Bashforth method method to numerically approximate the solution $y(t)$ with a mesh size 0.1. Tabulate your result and plot the graph.
   (Notice that in order to use the 2-step Adam-Bashforth method, you need to have initial values w_0 and w_1. Please compute w_1 from w_0 by a 4-rd Taylor series method. Then with w_0 and w_1, use Adam-Bashforth)

   (b) Combine the 2-step Adam-Bashforth and the 2-step Adam-Moulton method to provide a predictor-corrector scheme. Use this scheme to numerically approximate the solution $y(t)$ with a mesh size 0.1. Tabulate your result and plot the graph.
   (Notice that in order to use the predictor-corrector method, again, you need to have initial values w_0 and w_1. Please compute w_1 from w_0 by a 4-rd Taylor series method. Then with w_0 and w_1, use predictor corrector.)

5. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Octave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.)

   Consider

$$\begin{cases} y'(t) = -15y \text{ on } [0,5] \\ y(0) = 1 \end{cases}$$

   Remark: In order to finish this question efficiently, please look through the MATLAB command "hold on" for comparison of the graphs.

   (a) Use the Euler's method method to numerically approximate the solution $y(t)$ with a mesh size $1/4$. Plot the graph only.

   (b) Use the Euler's method method to numerically approximate the solution $y(t)$ with a mesh size $2/15$. Plot the graph only.

   (c) Use the Euler's method method to numerically approximate the solution $y(t)$ with a mesh size $1/8$. Plot the graph only.

   (d) Use the 2-step Adam-Moulton method to numerically approximate the solution $y(t)$ with a mesh size $1/8$. Plot the graph only.
   (Notice that in order to use the 2-step Adam-Moulton method, you need to have initial values w_0 and w_1. Please compute w_1 from w_0 by a 4-rd Taylor series method. Then with w_0 and w_1, use Adam-Moulton.)