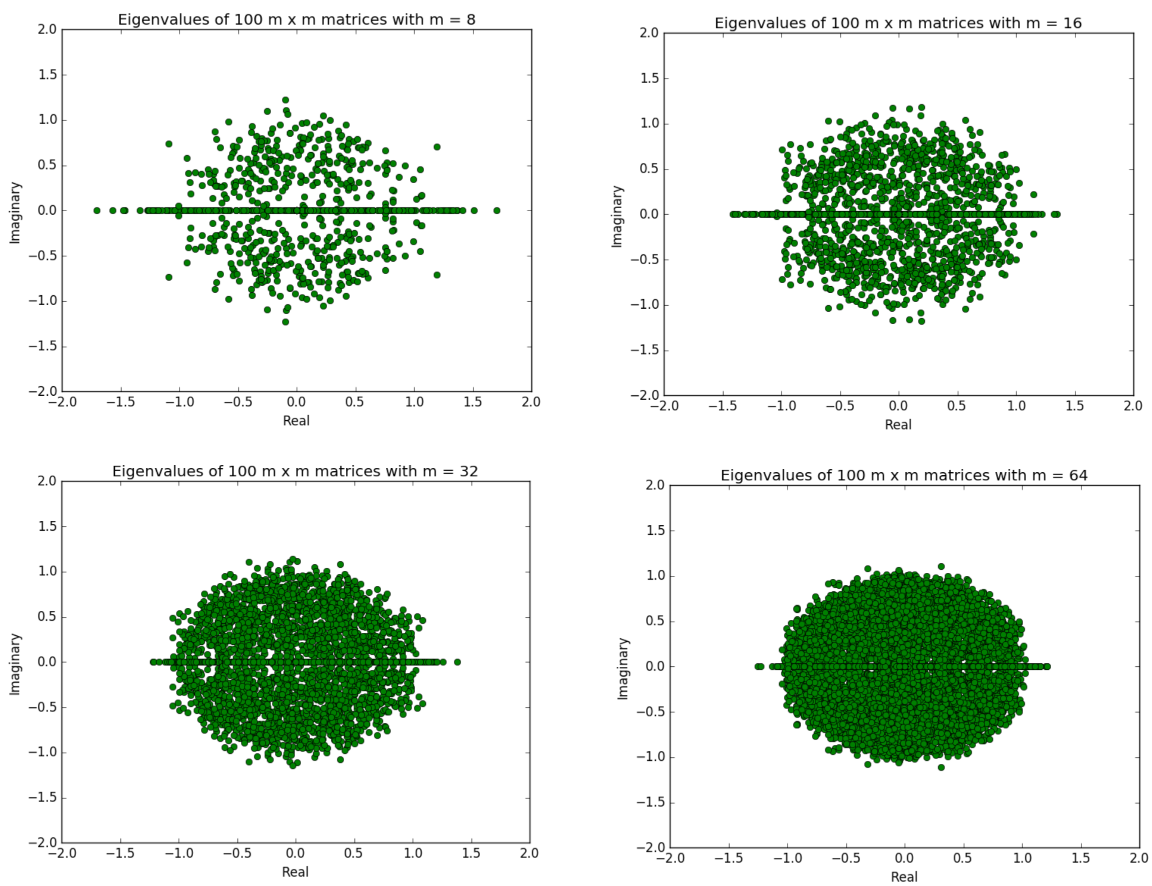
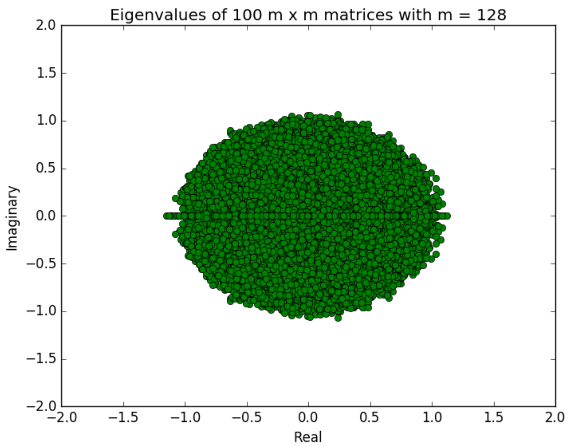
12.3. The goal of this problem is to explore some properties of random matrices. Your job is to be a laboratory scientist, performing experiments that lead to conjectures and more refined experiments. Do not try to prove anything. Do produce well-designed plots, which are worth a thousand numbers.

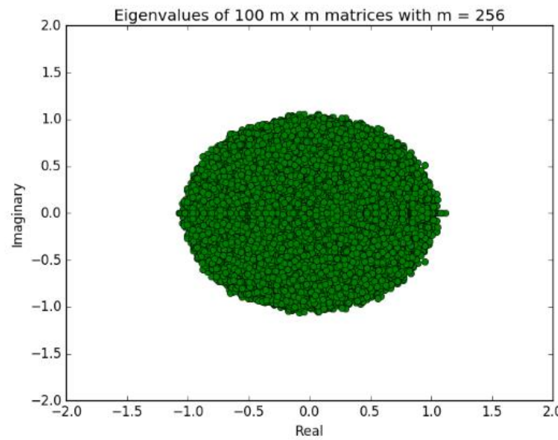
Define a *random matrix* to be an $m \times m$ matrix whose entries are independent samples from the real normal distribution with mean zero and standard deviation $m^{-1/2}$. (In MATLAB, `A = randn(m,m)/sqrt(m)`.) The factor $\sqrt{m}$ is introduced to make the limiting behavior clean as $m \to \infty$.

(a) What do the eigenvalues of a random matrix look like? What happens, say, if you take 100 random matrices and superimpose all their eigenvalues in a single plot? If you do this for $m = 8, 16, 32, 64, \ldots$, what pattern is suggested? How does the spectral radius $\rho(A)$ (Exercise 3.2) behave as $m \to \infty$?

Plots of superimposed eigenvalues for 100 square matrices, with dimensions m = 8, 16, 32, 64, 128, 256

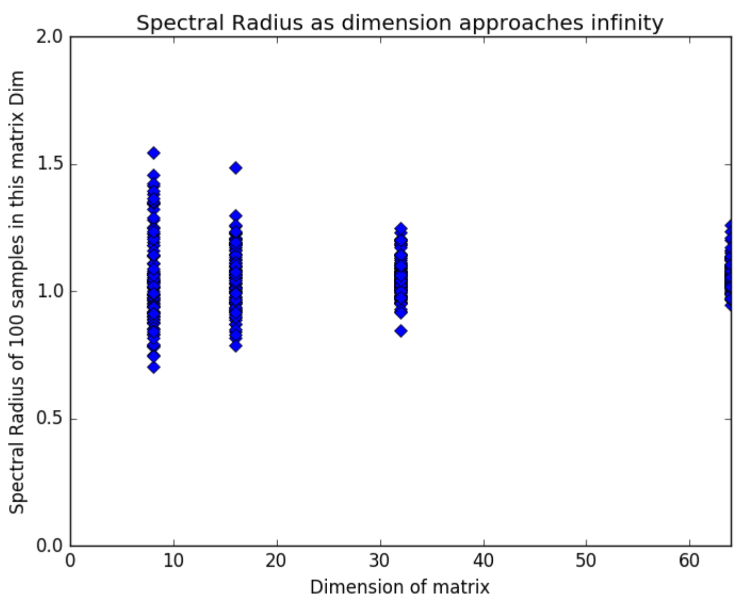






The figures show the scatterplots of the imaginary against the real eigenvalues of 100 matrices of the specified dimensions normalized as by the directions. The eigenvalues of all the matrices appear to lie within a circular formation – notably the unit circle – and this becomes even more evident as the dimension increases. Therefore, we can hypothesize that as the dimension of the matrix increases, the eigenvalues of the matrix follow a distribution that appears to be uniform on the unit circle.

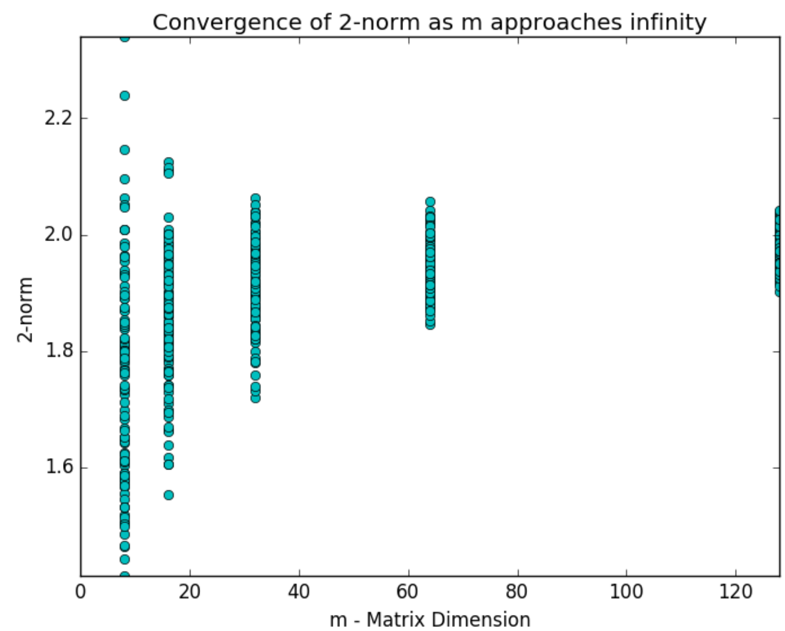
The following plot shows the effect of increasing dimension on the spectral radius of a matrix:



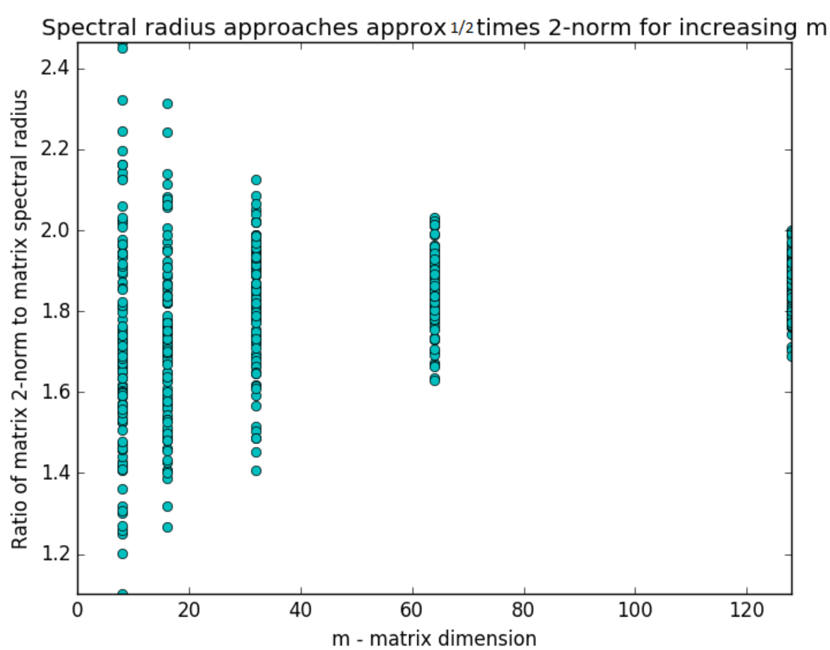
Each individual dot represents the spectral radius of a particular sampled matrix; there are 100 of these for each dimension value (here we have m = 8, 16, 32, and 64). The spectral radius of a particular matrix is defined as the eigenvalue of said matrix with the largest magnitude. As m increases, the spectral radius becomes stable and has less deviation, approaching a value just slightly greater than 1.

(b) What about norms? How does the 2-norm of a random matrix behave as $m \to \infty$? Of course, we must have $\rho(A) \le \|A\|$ (Exercise 3.2). Does this inequality appear to approach an equality as $m \to \infty$?

The following is a plot of the induced 2-norm of 100 random matrices as the matrix dimension approaches infinity:



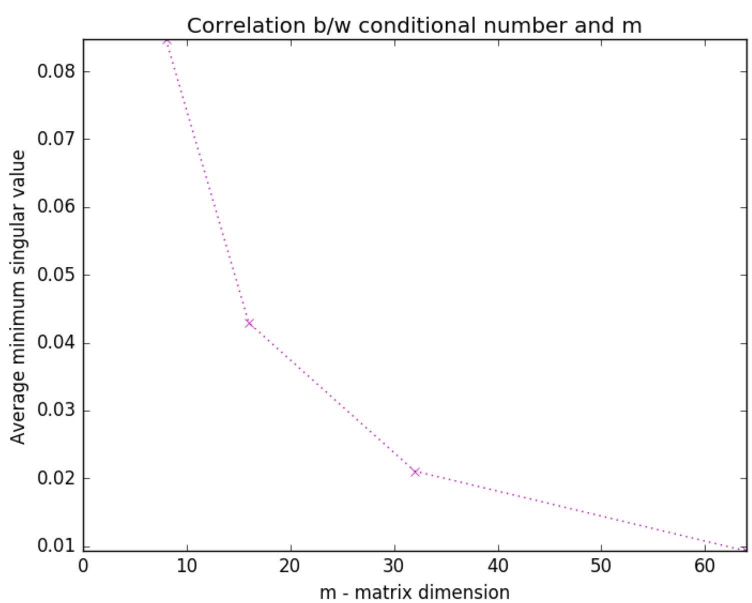
Below is the plot of the **ratio** of *the induced 2-norm of the matrix* to *the spectral radius of the matrix*



We see that the induced 2-norm of a matrix becomes more stable and converges to a number approximately less than 2 as the dimension grows. We computed the induced 2-norm as the maximum singular value of the matrix. On the other hand, the ratio of the induced 2-norm to the spectral radius also seems to converge to a number slightly less than 2 as well; it does not appear that the inequality is approaching an equality, as the latter converging value does not seem to be anywhere near a 1:1 ratio. If we consider that converging to a 2:1 ratio is grounds for equality (assuming that the 2-norm is scaled by 2), then yes, it may be considered as approaching an equality.

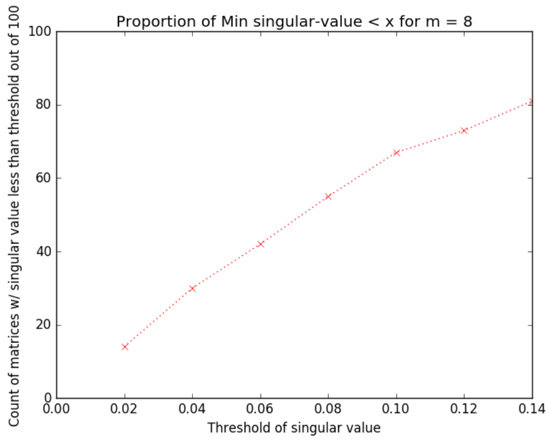
(c) What about condition numbers—or more simply, the smallest singular value $\sigma_{\min}$? Even for fixed m this question is interesting. What proportions of random matrices in $\mathbb{R}^{m\times m}$ seem to have $\sigma_{\min} \leq 2^{-1}, 4^{-1}, 8^{-1}, \ldots$? In other words, what does the tail of the probability distribution of smallest singular values look like? How does the scale of all this change with m?

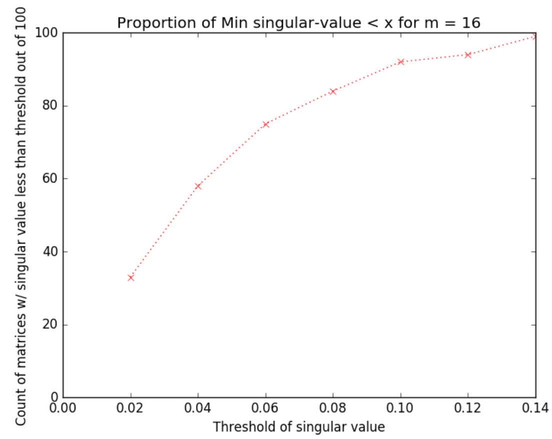
The following plot shows the correlation between the condition number of a matrix (computed as the smallest singular value of the matrix) and the matrix dimension. The condition numbers for each dimension m = 8, 16, 32, 64, were averaged before performing the plot.

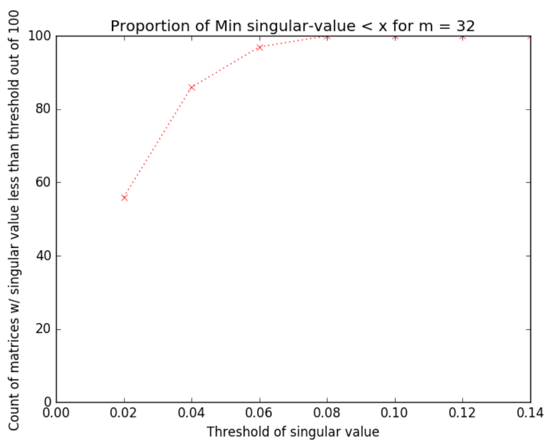


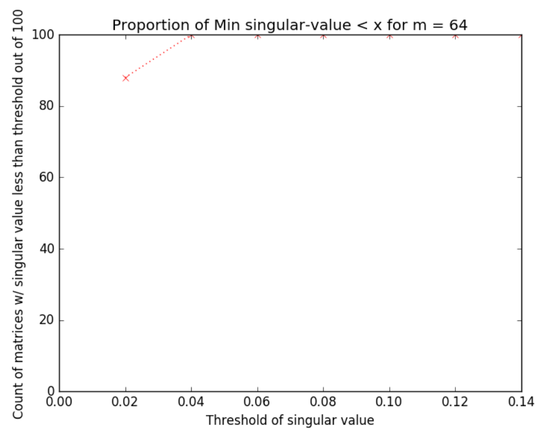
We see that there is a negative correlation: The condition number, or the minimum singular value, of a matrix decreases as the matrix dimension increases.

The following are plots of the count of square matrices with side dimensions m=8, 16, 32, 64 that have condition numbers which are less than or equal to the defined threshold on the x-axis.









As we can see, as the matrix size increases, there are significantly more matrices that have condition numbers less than (any of) the thresholds. This can be rationalized in the sense that as the matrix size increases, there is less variety and deviation for the singular values. As we saw in the plot on the last page, as the matrix dimension increases, the average minimum singular value – that is, the average condition number of a matrix of this dimension – decreases. Therefore, we will have more condition numbers of smaller value, which are more likely to be beneath the threshold, as shown by these four plots.

These plots were generated with Python, numpy, and matplotlib. The associated python code is uploaded on CCLE.