Applied Numerical Methods
(MATH 151B-Lecture 1, Spring 2016)
Assignment 5

Note:

- **Due day**: 9:00 a.m.,3rd Jun (Friday). Assignments handed after the due date will not be counted.

1. For a given vector $w \in \mathbb{R}^n$. Let us define
$$P^w = I - \frac{2}{||w||_2^2} ww^T .$$
   (a) Show that $(P^w)^T = P^w$ and $(P^w)^T(P^w) = (P^w)(P^w)^T = I$.

   (b) Given a vector v. Let $w = v + ||v||_2 e_1$ where $e_1 = (1, 0, 0.., 0)$. Show that the matrix P^w satisfy the property
$$P^w v = -||v||_2 e_1 .$$
   (c) Consider a matrix
$$A^{(1)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 9 & 7 \\ 3 & 9 & 1 & 1 \\ 4 & 7 & 1 & 10 \end{pmatrix}$$
   Find a $w_1 \in \mathbb{R}^3$ such that P^{w_1} is a 3×3 matrix such that
$$\begin{pmatrix} 1 & 0 \\ 0 & P^{w_1} \end{pmatrix} A^{(1)} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$
   (Hint: consider the vector formed by the first column A without the first entry, i.e.: $[a_{21}a_{31}...]^T$.)

   (d) Check that
$$A^{(2)} := \begin{pmatrix} 1 & 0 \\ 0 & P^{w_1} \end{pmatrix} A^{(1)} \begin{pmatrix} 1 & 0 \\ 0 & P^{w_1} \end{pmatrix}$$

is of the form

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

(e) Now consider the lower 3×3 submatrix of the matrix $A^{(2)}$. Describe how to find $w_2 \in \mathbb{R}^2$ such that P^{w_2} is a 2×2 matrix such that

$$\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & P^{w_2} \end{pmatrix} A^{(2)} \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & P^{w_2} \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

2. Consider a family of N points $\{(x_i, y_i)\}_{i=1,..,N}$. Let $P(x)$ be a polynomial of degree $M-1$ of the following form

$$P(x) = \sum_{j=0}^{M-1} a_j x^j = \begin{pmatrix} 1 & x & x^2 & ... & x^{M-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{M-1} \end{pmatrix}.$$

Now let

$$\begin{aligned} J : \mathbb{R}^M &\to \mathbb{R} \\ J(a) &:= \frac{1}{2} \sum_{i=1}^{N} |P(x_i) - y_i|^2 . \end{aligned}$$

In this question, we consider the minimization problem

$$\min_{a \in \mathbb{R}^M} J(a) .$$

(Heuristically speaking, we are minimizing the distance between the graph of a degree $M-1$ polynomial and the given data (x_i, y_i)'s over all possible coefficients a_j's)

(a) Show that

$$J(a) = \frac{1}{2} ||Ba - Y||_2^2$$

where B is the following N-by-M matrix

$$B = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{M-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^{M-1} \end{pmatrix}, \text{ i.e. } B_{ij} = x_i^j .$$

and $Y = (y_1, y_2, ..., y_N)^T$.

(b) Check (as in what we have learnt in steepest descent) that for $h \in \mathbb{R}^n$, we have

$$J(a+h) = J(a) + \left\langle B^T Ba - B^T Y, h \right\rangle + \frac{1}{2}||Bh||_2^2 \,.$$

Hence, show that if a satisfies

$$B^T Ba = B^T Y \,,$$

then

$$J(a+h) \geq J(a)\,.$$

(Notice the $\geq$ sign instead of $>$ sign, because now B can be singular.)

(c) Simplify

$$B^T Ba = B^T Y \,,$$

to get $\{a_j\}_{i=0}^{M-1}$ such that:

$$\sum_{j=0}^{M-1} \left(\sum_{k=1}^{N} x_k^{i+j} \right) a_j = \sum_{k=1}^{N} x_k^i y_k \text{ for } i = 0, \cdots, M-1\,.$$

3. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Ovtave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.

Given a matrix A, consider the following QR algorithm:
Initialize $T^{(0)} = A$, $U^{(0)} = I$. For $k = 0, 1, ...,$ we do

- Find $Q^{(k)}, R^{(k)}$ such that $Q^{(k)}R^{(k)} = T^{(k)}$.
- Compute $T^{(k+1)} = R^{(k)}Q^{(k)}$.
- Compute $U^{(k+1)} = U^{(k)}Q^{(k)}$.

Now let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0.5 \end{pmatrix}\,.$$

Compute 100 steps of QR algorithm. Write down the matrix $T^{(100)}$.

(Hint: You may directly use the MATLAB code to compute QR factorization by `[Q,R] = qr(A).`)