Applied Numerical Methods
(MATH 151B-Lecture 1, Spring 2016)
Assignment 3

Note:

- **Due day**: 9:00 a.m., 9th May (Monday). Assignments handed after the due date will not be counted.

1. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Ovtave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.) Consider the following system boundary value problem:

$$\begin{cases} y'' = \sin(y) + 2y \text{ on } (0, 2.5) \\ y(0) = 1\,; \quad y(2.5) = 15\,. \end{cases}$$

   (a) Show that the BVP has a unique solution.

   (b) Write the BVP in a system of first order ODE by letting $u_1 = y, u_2 = y'$.

   (c) Let $x(t, \alpha)$ be the unique solution to

$$\begin{cases} x'' = \sin(x) + 2x \text{ on } (0, 2.5) \\ x(0) = 1\,; \quad x'(0) = \alpha\,. \end{cases}$$

   Let $F(\alpha) := x(2.5; \alpha) - 15$.
   Write down the secant method for the function F.
   Try to solve BVP by using the secant method with initial values $\alpha_0 = 1$ and $\alpha_1 = -0.7$. You only need to calculate 3 (new) secent-steps, i.e. $\alpha_2, \alpha_3, \alpha_4$. Plot the graphs of $\{x(t; \alpha_i)\}$.
   (For a given α, use the 2-step Adam-Bashforth method method to numerically approximate the solution $u_1(t) = x(t; \alpha), u_2 = x'(t; \alpha)$ with a mesh size 0.01.
   Please use the initial values w_0 and w_1 by computing w_1 from w_0 with a 2-nd Taylor series method. Then with w_0 and w_1, use Adam-Bashforth.)

2. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Ovtave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.) Consider the following system boundary value problem:

$$\begin{cases} y'' - \sin(t)y' + \cos(t)y + \sin(t) = 0 \text{ on } (\pi/2, 3\pi/2) \\ y(\pi/2) = 1\,; \quad y(3\pi/2) = 1\,. \end{cases}$$

(a) Show that the BVP has a unique solution. (You may directly apply a theorem you learnt in class.)

(b) Write down the finite difference method for solving the above BVP (using mid point formulae for first and second derivatives). You may treat $w = [w_1, ...w_N]$ as one vector and set $w_0 = 1, w_{N+1} = 1$.

(c) Show that the linear system coming from the finte difference method also have a unique solution provided that $h < 2$. (You may directly apply a theorem you learnt in class.)

(d) Use the finite difference method to numerically approximate the solution $y(t)$ with a mesh size 0.01 by solving the system of linear euqation that we get above. Plot the graph only.

3. (It is required to use a program to finish this question. Only programs written in C / C++ / Matlab / Ovtave is acceptable. Hand in the code via CCLE. Print your tabulated results and graphs.) Consider the following system of nonlinear equations:

$$\begin{cases} x + y + z + t = 1 \\ x^2 + y^2 + z^2 + t^2 = 1 \\ x^3 + y^3 + z^3 + t^3 = 1 \\ x^4 + y^4 + z^4 + t^4 = 1 \end{cases}$$

Now define $F(x, y, z, t) = \begin{bmatrix} x + y + z + t - 1 \\ x^2 + y^2 + z^2 + t^2 - 1 \\ x^3 + y^3 + z^3 + t^3 - 1 \\ x^4 + y^4 + z^4 + t^4 - 1 \end{bmatrix}$,

(a) Find the differential F, DF, at the point $(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$.

(b) Compute the first iterate of the Newton's Iteration which solves $F(x, y, z, t) = 0$ with the starting point $(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$.

4. Consider the following non-linear equation for $x \in \mathbb{R}^d$

$$F(x) = 0$$

where F is smooth.

(a) Write down the Broyden's method for solving a general non-linear system (with A_k approximating the $J(x_k)$ as the Jacobian in the Newton's step.)

(b) Prove the following Sherman-Morrison Lemma:
Lemma 1 Suppose A is non-singular $d \times d$ matrix, and $x, y \in \mathbb{R}^d$ be such that $y^T A^{-1} x \neq 1$, then $A + xy^T$ is non-singular and

$$(A + xy^T)^{-1} = A^{-1} - \frac{A^{-1} x y^T A^{-1}}{1 + y^T A^{-1} x}. \tag{1}$$

(c) Suggest an improved Broyden's method by replacing the update of A_k by the update of $B_k := A_k^{-1}$ using Sherman-Morrison Lemma.