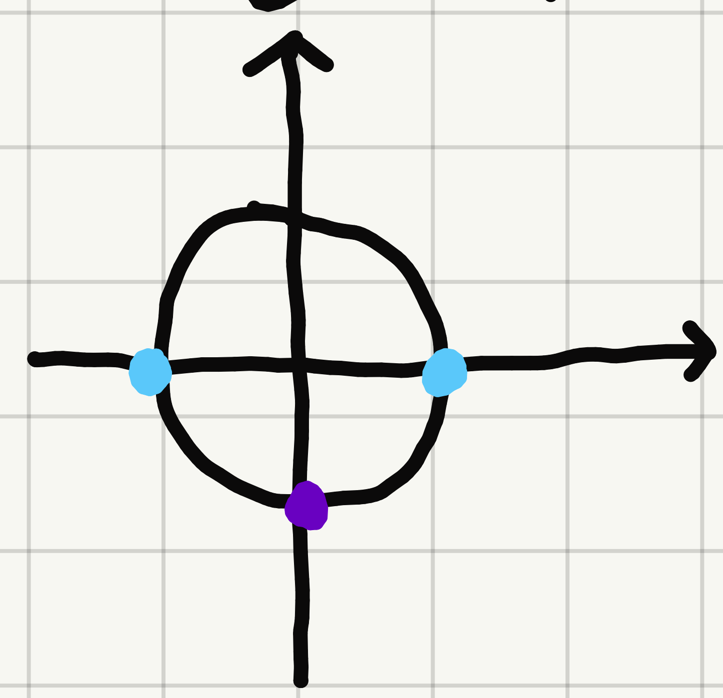
# 8 - EQUAZIONI E DISEQUAZIONI GONIOMETRICHE

1a) $\cos^2 x - \sin x - 1 = 0$

$1 - \sin^2 x - \sin x - 1 = 0$

$\sin^2 x + \sin x = 0 \qquad \sin x(1 + \sin x) = 0$

$$\begin{cases} \sin x = 0 \\ \sin x = -1 \end{cases}$$



$x = k\pi$

$x = \frac{3}{2}\pi + 2k\pi$

$x = k\pi \vee \frac{3}{2}\pi + 2k\pi$

1b) $2\cos x - \sin x - 2 = 0$

FUNZIONI PARAMETRICHE $\cos x = \frac{1-t^2}{1+t^2}$ $\sin x = \frac{2t}{1+t^2}$ $t = \tan\frac{x}{2}$

$2 \cdot \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} - 2 = 0$

$\cancel{2} - 2t^2 - 2t - \cancel{2} - 2t^2 = 0 \rightarrow 4t^2 + 2t = 0$

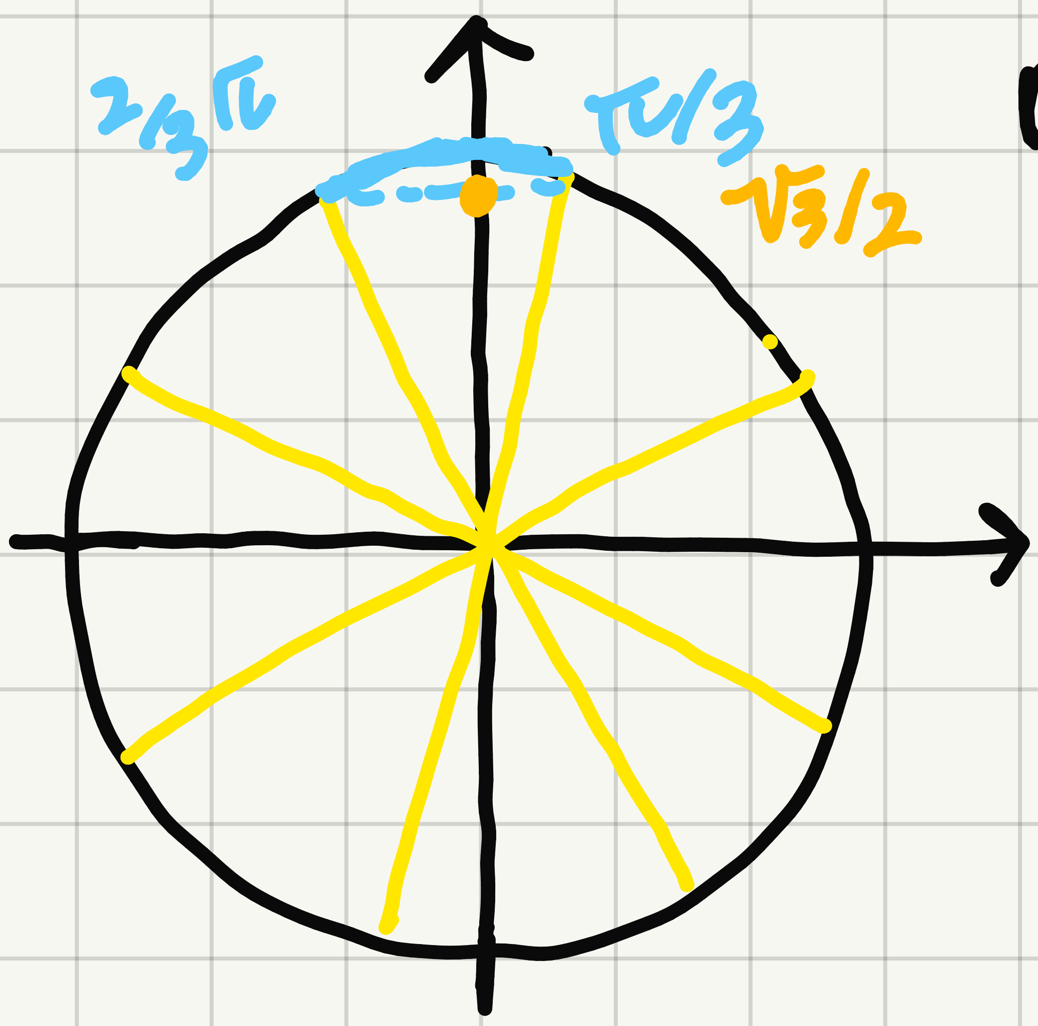
$t(2t + 1) = 0$

- $t = 0 \quad \rightarrow \quad \tan\left(\frac{x}{2}\right) = 0 \Rightarrow x = 2k\pi$
- $t = -\frac{1}{2} \quad \rightarrow \quad \tan\left(\frac{x}{2}\right) = -\frac{1}{2} \Rightarrow \frac{x}{2} = -\arctan\left(\frac{1}{2}\right) + k\pi$

$\Rightarrow x = -2\arctan\left(\frac{1}{2}\right) + 2k\pi$

$x = 2k\pi \vee -2\arctan\left(\frac{1}{2}\right) + 2k\pi$

2a) $\sin x > \frac{\sqrt{3}}{2}$




$\arcsin\left(\frac{\sqrt{3}}{2}\right) + 2k\pi < x < \pi - \arcsin\left(\frac{\sqrt{3}}{2}\right)$

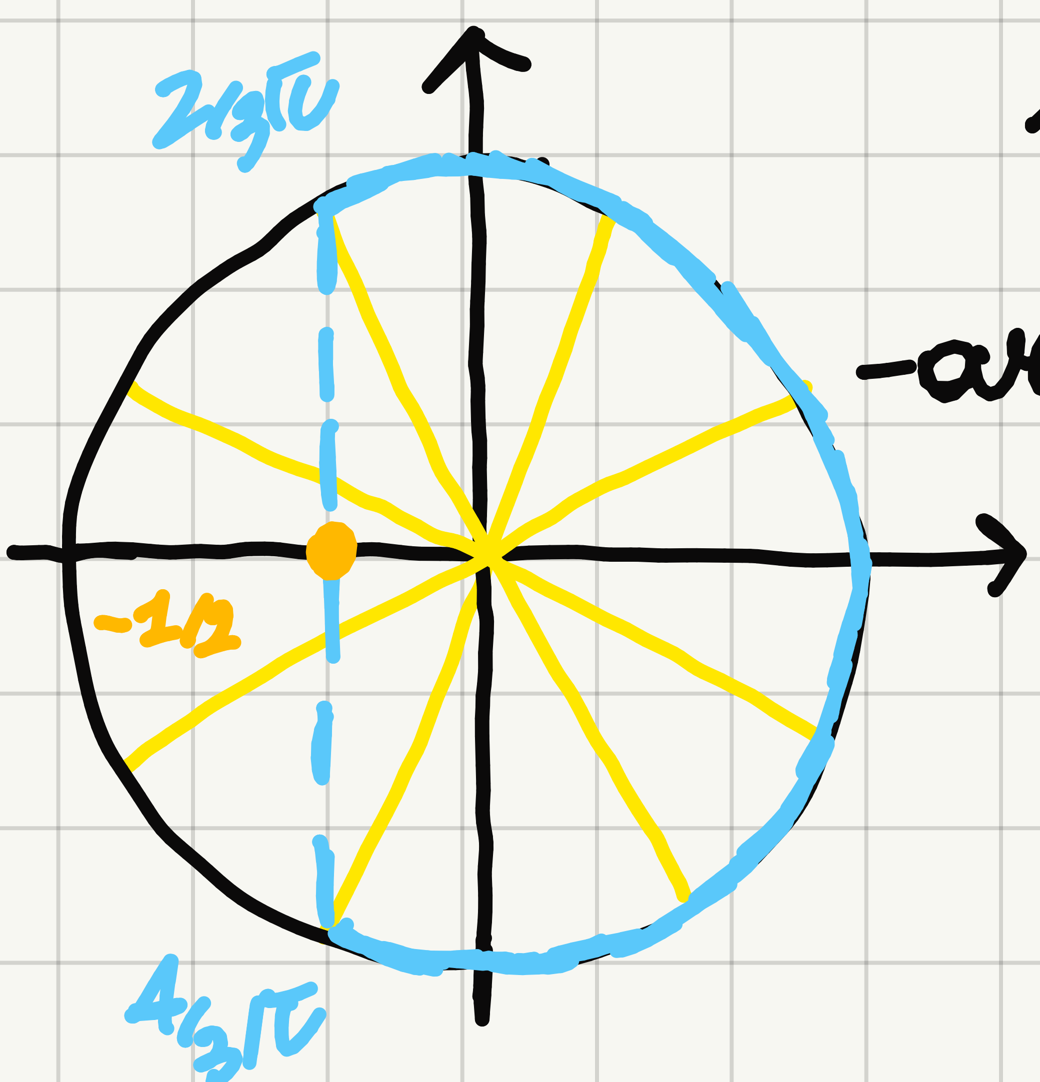
$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\pi - \arcsin\left(-\frac{\sqrt{3}}{2}\right) = \frac{2}{3}\pi$

$\frac{\pi}{3} + 2k\pi < x < \frac{2}{3}\pi + 2k\pi$

2b) $\cos\left(3x - \frac{\pi}{3}\right) > -\frac{1}{2}$

$3x - \frac{\pi}{3} = t$




$\cos(t) > -\frac{1}{2}$

$-\arccos\left(-\frac{1}{2}\right) + 2k\pi < t < \arccos\left(-\frac{1}{2}\right) + 2k\pi$

$\arccos\left(-\frac{1}{2}\right) = \frac{2}{3}\pi$

$-\frac{2}{3}\pi + 2k\pi < t < \frac{2}{3}\pi + 2k\pi$

$-\frac{2}{3}\pi + 2k\pi < 3x - \frac{\pi}{3} < \frac{2}{3}\pi + 2k\pi$

$-\frac{\pi}{3} + 2k\pi < 3x < \pi + 2k\pi$

$-\frac{\pi}{9} + \frac{2}{3}k\pi < x < \frac{\pi}{3} + \frac{2}{3}k\pi$

2c) $\sin x(\sin x+\cos x)\le 0$

N1 $\sin x\ge 0$

N2 $\sin x+\cos x\ge 0$

FUNZIONI PARAMETRICHE $\cos x=\frac{1-t^2}{1+t^2}$ $\sin x=\frac{2t}{1+t^2}$ $t=\tan\frac{x}{2}$

N1 $2k\pi\le x\le\pi+2k\pi$

N2 $\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\ge 0$

N2 $2t+1-t^2\ge 0$

$$t=\frac{-2\pm\sqrt{8}}{-2}=\frac{-2\pm2\sqrt{2}}{-2}=\begin{cases}\frac{-2+2\sqrt{2}}{-2}=1-\sqrt{2}\\ \frac{-2-2\sqrt{2}}{-2}=1+\sqrt{2}\end{cases}$$

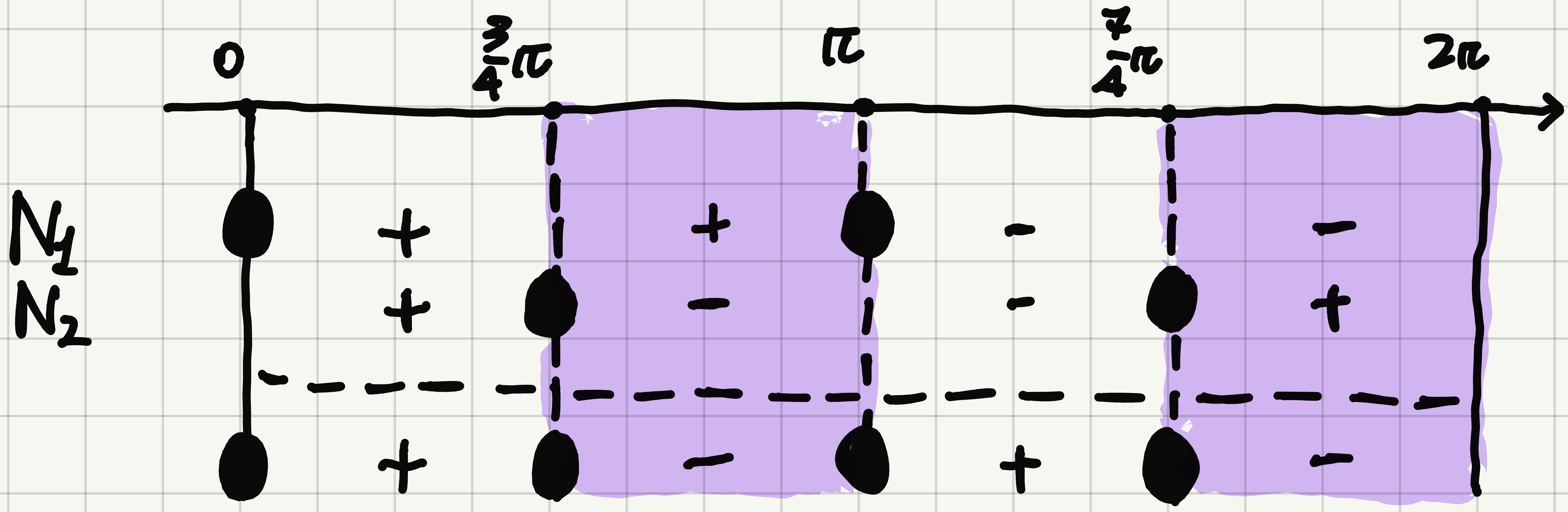
N2 $1-\sqrt{2}\le t\le 1+\sqrt{2}\Rightarrow \arctan(1-\sqrt{2})\le\frac{x}{2}\le\arctan(1+\sqrt{2})$

$\Rightarrow -\frac{\pi}{8}+k\pi\le\frac{x}{2}\le\frac{3}{8}\pi+k\pi$

N1 $2k\pi\le x\le\pi+2k\pi$

N2 $-\frac{\pi}{4}+2k\pi\le x\le\frac{3}{4}\pi+2k\pi$

ANALISI DEL PERIODO $[0,2\pi)$

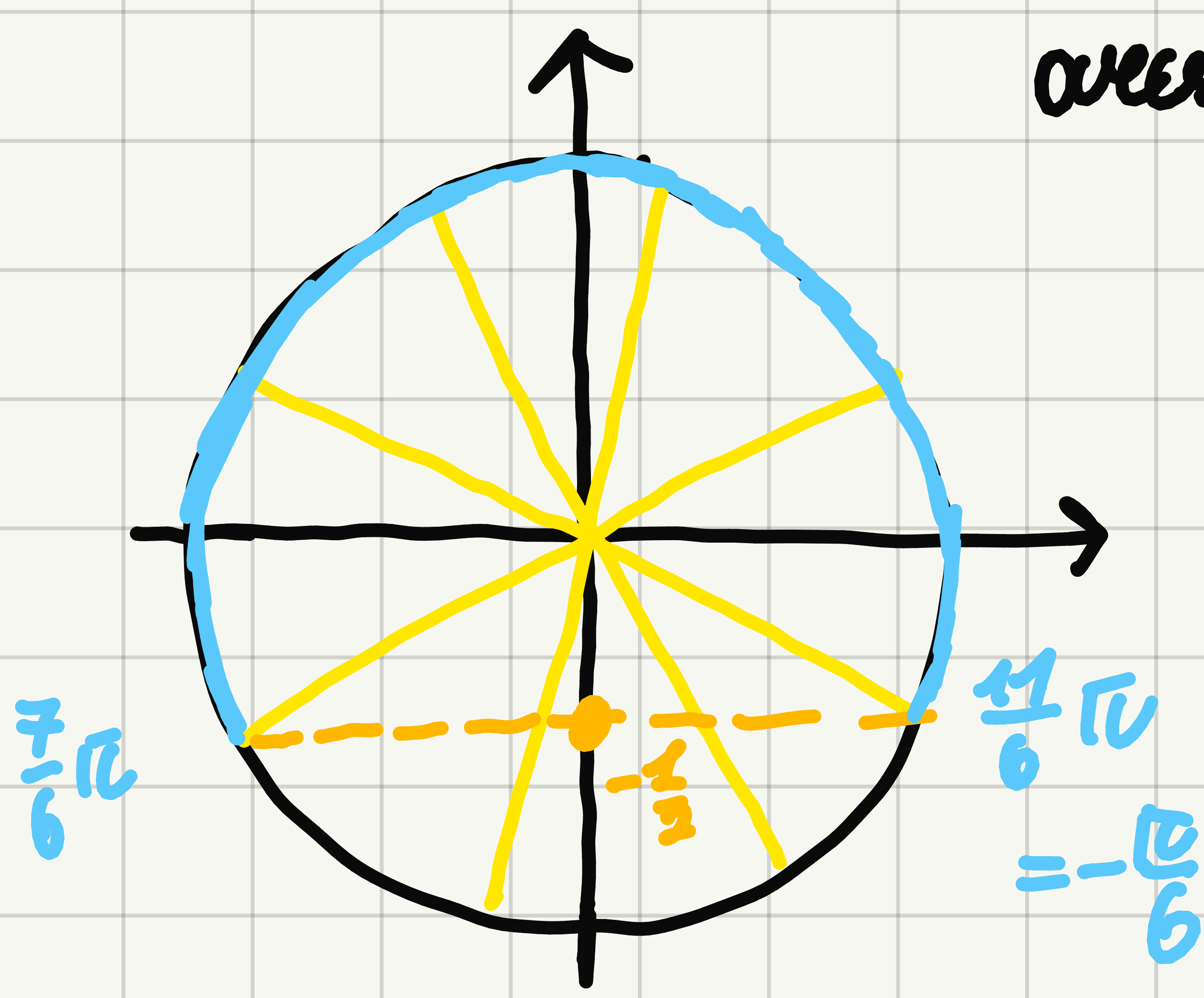



$\frac{3}{4}\pi+k\pi\le x\le\pi+k\pi$

2d) $\dfrac{2\sin x+1}{(2+\sin x)^2} > 0$

$2\sin x+1>0$ (DENOMINATORE > 0 $\forall x \in \mathbb{R}$)

$\sin x > -\frac{1}{2}$




$\arcsin(-\frac{1}{2})+2k\pi < x < \pi-\arcsin(-\frac{1}{2})+2k\pi$

$-\frac{\pi}{6}+2k\pi < x < \pi-(-\frac{\pi}{6})+2k\pi$

$-\frac{\pi}{6}+2k\pi < x < \frac{7}{6}\pi+2k\pi$

# 9 - LIMITI E SUCCESSIONI

1a) $\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{2n}=$ $\lim_{n\to+\infty}\left(1+\frac{a}{n}\right)\simeq e^{a}$

$=\lim_{n\to+\infty}\left[\left(1+\frac{1/3}{n}\right)^{n}\right]^{2}=e^{\frac{1}{3}\cdot 2}=e^{2/3}$

1b) $\lim_{n\to+\infty}\frac{\sqrt{n}-n+n^{2}}{2n^{2}-n^{3/2}+1}=\lim_{n\to+\infty}\frac{n^{2}\left(1-\frac{1}{n}+\left(\frac{1}{n}\right)^{2/3}\right)}{n^{2}\left(2-\left(\frac{1}{n}\right)^{1/2}+\frac{1}{n^{2}}\right)}=\frac{1}{2}$

1c) $\lim_{n\to\infty}\frac{2^{n}-3^{n}}{1+3^{n}}=\lim_{n\to\infty}\frac{\cancel{3^{n}}\left(-1+\overset{\to 0}{\frac{2^{n}}{3^{n}}}\right)}{\cancel{3^{n}}\left(1+\underset{\to 0}{\frac{1}{3^{n}}}\right)}=-1$

1d) $\lim_{n\to\infty}\frac{2^{n}+n^{2}}{3^{n}+n^{3}}=\lim_{n\to\infty}\overset{\to 0}{\left(\frac{2}{3}\right)^{n}}\cdot\frac{1+\frac{n^{2}}{2^{n}}}{1+\frac{n^{3}}{3^{n}}}=0$ $\to 0$ PER GERARCHIA INFINITI

1e) $\lim_{n\to\infty}\frac{n\log n}{(n+1)(n+2)}=\lim_{n\to\infty}\frac{n\log n}{n^{2}+3n+2}=\lim_{n\to\infty}\frac{\cancel{n}\log n}{n^{\cancel{2}}\left(1+\underset{0\leftarrow}{\frac{3}{n}+\frac{2}{n^{2}}}\right)}=\lim_{n\to\infty}\frac{\log n}{n}=0$

1f) $\lim_{n\to\infty}\frac{1+\log n}{\sqrt{n}-\log n}=\lim_{n\to\infty}\frac{\log n\left(1+\overset{0\leftarrow}{\frac{1}{\log n}}\right)}{\sqrt{n}\left(1-\frac{\log n}{\sqrt{n}}\right)}=\lim_{n\to\infty}\frac{\log n}{\sqrt{n}}=0$

$\to 0$ PER GERARCHIA INFINITI

1g) $\lim_{n\to\infty}\underbrace{(-1)^{n}}_{\text{LIM}}\cdot\overset{\to 0}{\frac{n}{n^{2}+1}}=0$

1h) $\lim_{n\to\infty}\underbrace{(-1)^{n}}_{\text{LIM}}\cdot\frac{n^{2}+1}{n+1}=\pm\infty$ NON RISPETTA L'UNICITÀ DEL LIMITE

$\Rightarrow\nexists\lim_{n\to\infty}(-1)^{n}\cdot\frac{n^{2}+1}{n+1}$

1i) $\lim_{n\to\infty}\sqrt[n]{2^{n}+3^{n}}=\lim_{n\to\infty}3\sqrt[n]{\overset{\to 0}{\left(\frac{2}{3}\right)^{n}}+1}=3$

1l) $\lim_{n\to\infty}\sqrt[n]{\frac{2n}{3n^{2}+1}}=\lim_{n\to\infty}\frac{\sqrt[n]{n}\cdot\sqrt[n]{2}}{\sqrt[n]{n^{2}}\cdot\sqrt[n]{3+\frac{1}{n^{2}}}}=\lim_{n\to\infty}\frac{2^{1/n}}{\sqrt[n]{n}\cdot 3^{1/n}}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^{\overset{\to 0}{1/n}}\cdot\frac{1}{e^{\frac{\log n}{n}}}=$

$=\lim_{n\to\infty}1\cdot\frac{1}{e^{0}}=1$

1m) $\lim_{n\to\infty}\frac{n^{2}(3^{n}-3^{-n})}{4^{n}+n^{2}}=\lim_{n\to\infty}\frac{n^{2}\cdot 3^{n}\left(1-\overset{\to 0}{3^{-2n}}\right)}{4^{n}\left(1+\frac{n^{2}}{4^{n}}\right)}=\lim_{n\to\infty}\frac{n^{2}}{(4/3)^{n}}=0$ PER G.I.

$\to 0$ PER G.I.

1n) $$\lim_{n\to\infty}\frac{n^6+\log n+3^n}{2^n+n^4+\log^3 n}=\lim_{n\to\infty}\left(\frac{3}{2}\right)^n\frac{1+\frac{n^6}{3^n}+\frac{\log n}{3^n}}{1+\frac{n^4}{2^n}+\frac{\log^3 n}{2^n}}=+\infty$$

0 PER G.I.

1o) $$\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^n=\lim_{n\to\infty}\left(\frac{\cancel{n}(1+3/n)}{\cancel{n}(1+1/n)}\right)^n=\frac{e^3}{e}=e^2$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$$

1p) $$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n^2}=\lim_{n\to\infty}\left[\left(1+\frac{-1}{n}\right)^n\right]^n=\left(e^{-1}\right)^{\infty}=\left(\frac{1}{e}\right)^{\infty}=0$$

1q) $$\lim_{n\to\infty}\frac{(n^2+1)^n}{n^{2n}}=\lim_{n\to\infty}\left(\frac{n^2+1}{n^2}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n=\lim_{n\to\infty}\left(1+\frac{1/n}{n}\right)^n$$

$$=\lim_{n\to\infty}e^{1/n}=1$$

1r) $$\lim_{n\to\infty}\frac{\log(n+1)}{\log n}=\lim_{n\to\infty}\frac{\log(n(1+\frac{1}{n}))}{\log n}=\lim_{n\to\infty}\frac{\log n+\log(1+\frac{1}{n})}{\log n}=$$

$$=1+\lim_{n\to\infty}\frac{\log(1+\frac{1}{n})}{\log n}=1+\frac{e}{\infty}=1$$

1s) $$\lim_{n\to\infty}\left(\sqrt[n]{3}-1\right)^n=\lim_{n\to\infty}\left(3^{\frac{1}{n}}\left(1-\frac{1}{\sqrt[n]{3}}\right)\right)^n=0^{\infty}=0$$

1t) $$\lim_{n\to\infty}\sqrt[n]{n\log n}=$$ G.I. $1\le\log n\le n\to n\le n\log n\le n^2$

$$=\lim_{n\to\infty}\sqrt[n]{n\cdot n}=\lim_{n\to\infty}\sqrt[n]{n}\cdot\sqrt[n]{n}=1\cdot1=1$$

1u) $$\lim_{n\to\infty}n^2\cdot 2^{-\sqrt{n}}=\lim_{n\to\infty}e^{\ln n^2}\cdot e^{\ln 2^{-\sqrt{n}}}=\lim_{n\to\infty}e^{2\ln n-\sqrt{n}\ln 2}=$$

$$=\lim_{n\to\infty}e^{-\sqrt{n}}\left(e^{\frac{2}{\sqrt{n}}\ln n-\ln 2}\right)=e^{-\infty}\cdot e^{0}=0$$

1v) $$\lim_{n\to\infty}\left(n^{\sqrt{n}}-2^n\right)=\lim_{n\to\infty}\left(e^{\sqrt{n}\ln n}-e^{n\ln 2}\right)=$$

$$=\lim_{n\to\infty}e^{n}\left(e^{1/n\,\ln n}-e^{\ln 2}\right)=+\infty\cdot\left(-e^{\ln 2}\right)=-\infty$$

# 10 – SERIE NUMERICHE

2a) $\sum_{n=2}^{+\infty} \frac{1}{\log(n+1)} =$ SERIE ARMONICA $\frac{1}{n^{\alpha}}$ $\alpha \leq 1$ DIVERGENTE, $\alpha > 1$ CONVERGENTE

$\log(n+1) = O(n+1) \Rightarrow$ PER $n \to +\infty$, $\frac{1}{n+1} = O\left(\frac{1}{\log(n+1)}\right)$

$\simeq \sum_{n=2}^{+\infty} \frac{1}{n+1} \to$ DIVERGE POSITIVAMENTE PER CONFRONTO ASINTOTICO

2b) $\sum_{n=1}^{+\infty} \frac{\log n}{n^4} \simeq \sum_{n=1}^{+\infty} \frac{1}{n^3} \to$ CONVERGE

2c) $\sum_{n=1}^{+\infty} \frac{\log n}{n^{3/2}} =$ SICURAMENTE $> 0$ E $\log(n) = O(n^{3/2})$

$\Rightarrow \frac{\log n}{n^{3/2}} = O\left(\frac{1}{n^{7/6}}\right)$

$\simeq \sum_{n=1}^{+\infty} \frac{1}{n^{7/6}} \to$ CONVERGE

2d) $\sum_{n=1}^{+\infty} \log\left(\frac{n+1}{n^2}\right)$ $\frac{n+1}{n^2} \to 0 \Rightarrow \log\left(\frac{n+1}{n^2}\right) \to -\infty$

$\to$ DIVERGE NEGATIVAMENTE

2e) $\sum_{n=1}^{+\infty} \operatorname{arccotan}\left(\frac{1}{\sqrt{n}}\right) = \sum_{n=1}^{+\infty}\left(\frac{1}{\sqrt{n}} + O\left(\frac{1}{\sqrt{n}}\right)\right) \simeq \sum_{n=1}^{+\infty} \frac{1}{n^{1/2}} \to$ DIVERGE POSITIVAMENTE

2f) $\sum_{n=2}^{\infty} \frac{\sqrt{n+2} - \sqrt{n-2}}{n} = \sum_{n=2}^{\infty} \frac{4}{n(\sqrt{n+2} + \sqrt{n-2})} \simeq \sum_{n=2}^{\infty} \frac{2}{n^{3/2}} \to$ CONVERGE

2g) $\sum_{n=1}^{\infty} \log\left(\frac{1}{\sqrt{n}}\right)$ SICURAMENTE $< 0$

$\simeq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \to$ DIVERGE NEGATIVAMENTE

2h) $\sum_{n=2}^{+\infty} \log\left(1/\sqrt{n^3}\right) = \sum_{n=2}^{+\infty} -\log(\sqrt{n^3}) \to$ DIVERGE NEGATIVAMENTE

2i) $\sum_{n=2}^{+\infty} \frac{1}{\sqrt{n}\log n^3} = \quad \log n^3 = o(n)$

$= \sum_{n=2}^{+\infty} \frac{1}{n^{1/2}} \to$ DIVERGE POSITIVAMENTE

2j) $\sum_{n=1}^{+\infty} \frac{1}{2^{\ln(n!)}} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{\ln(n^2)}} = \sum_{n=1}^{+\infty} \frac{1}{2^{2\ln(n)}} =$

$a^{\log b} = b^{\log a} \quad = \sum_{n=1}^{+\infty} \frac{1}{4^{\ln(n)}} = \sum_{n=1}^{+\infty} \frac{1}{n^{\ln 4}} \quad \ln 4 > 1 \to$ CONVERGE

2k) $\sum_{n=1}^{+\infty} \frac{1}{2^{\ln n}} = \sum_{n=1}^{+\infty} \frac{1}{n^{\ln 2}} \quad \ln 2 < 1 \to$ DIVERGE POSITIVAMENTE

2l) $\sum_{n=1}^{+\infty} 3^{2n} \cos^n(n\pi) = \sum_{n=1}^{+\infty} (-1)^n \cdot 9^n = \sum_{n=1}^{+\infty} (-9)^n$

$-9 < 0 \Rightarrow$ SEGNO ALTERNO $\Rightarrow$ INDETERMINATA

2m) $\sum_{n=1}^{+\infty} \frac{3^{n^2}}{(n!)^n}$ CRITERIO DELLA RADICE

$\lim_{n\to+\infty} \sqrt[n]{\frac{3^{n^2}}{(n!)^n}} = \lim_{n\to+\infty} \frac{3^n}{n!} = 0 < 1 \to$ CONVERGE

2n) $\sum_{n=1}^{+\infty} \frac{n^{43}}{6^n}$ CRITERIO DELLA RADICE

$\lim_{n\to+\infty} \sqrt[n]{\frac{n^{43}}{6^n}} = \lim_{n\to+\infty} \frac{n^{43/n}}{6} = \frac{1}{6} < 1 \to$ CONVERGE

$n^{43/n} \longrightarrow 0$ PER G.I

2o) $\sum_{n=1}^{+\infty} \frac{1}{\binom{4n}{3n}} = \sum_{n=1}^{+\infty} \frac{3n!\cdot n!}{4n!}$ CRITERIO DEL RAPPORTO

$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{[3(n+1)]!\,(n+1)!}{[4(n+1)]!} \cdot \frac{(4n)!}{(3n)!\,n!} =$

$\lim_{n\to+\infty} \frac{(3n+3)(3n+2)\cancel{(3n)!}(n+1)\cancel{n!}\cdot\cancel{(4n)!}}{(4n+4)(4n+3)(4n+2)(4n+1)\cancel{(4n)!}\cdot\cancel{(3n)!}\cancel{n!}} = \frac{9}{256} < 1 \to$ CONVERGE

2p) $\sum_{n=1}^{\infty} \frac{2}{\binom{3n+2}{3n}} = \sum_{n=1}^{+\infty} \frac{4\cdot(3n)!}{(3n+2)!} = \sum_{n=1}^{+\infty} \frac{4\cdot\cancel{(3n)!}}{(3n+2)(3n+1)\cancel{(3n)!}} =$

$= \sum_{n=1}^{+\infty} \frac{4}{9n^2} \to 0 \Rightarrow$ CONVERGE

2q) $\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\log n}}$ $\quad \frac{1}{\sqrt[n]{\log n}} \geq \frac{1}{\log n} \to$ DIVERGE POSITIVAMENTE

2r) $\sum_{n=0}^{\infty} \left(\frac{1}{n+2}\right)^n$ CRITERIO DELLA RADICE $\lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{1}{n+2}\right)^{\cancel{n}}} = \lim_{n\to\infty} \frac{1}{n+2} < 1$

$\to$ CONVERGE

2s) $\sum_{n=1}^{\infty} \frac{\sin(4n^3)}{n(n+1)} \simeq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \simeq \sum_{n=1}^{\infty} \frac{1}{n^2} \to 0 \to$ CONVERGE

2t) $\sum_{n=1}^{\infty} \frac{1}{5^n}\left(\frac{n+2}{n}\right)^{n^2}$ CRITERIO DELLA RADICE

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{5^n}\left(\frac{n+2}{n}\right)^{n^2}} = \lim_{n\to\infty} \frac{1}{5}\left(\frac{n+2}{n}\right)^n = +\infty \to$ DIVERGE POSITIVAMENTE

2u) $\sum_{n=2}^{\infty} 3^n\left(\frac{n-2}{n}\right)^{n^2}$ CRITERIO DELLA RADICE

$\lim_{n\to\infty} \sqrt[n]{3^n\left(\frac{n-2}{n}\right)^{n^2}} = \lim_{n\to\infty} 3\left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} 3\left(1-\frac{2}{n}\right)^n = \frac{3}{e^2}$

$< 1 \to$ CONVERGE

2v) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$

$n^a = \theta(\log n)^{\log n}$

$\frac{n^a = e^{a\log n}}{(\log n)^{\log n}}$ $\log n = t$ $= \frac{e^{at}}{t^t} = e^{t\log\frac{e^a}{t}} = e^{t(a-\log t)} = 0$

$a > 1 \to$ CONVERGE

2w) $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ CRITERIO DEL RAPPORTO

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{(n+1)^{(n+1)}}{[2(n+1)!]} \cdot \frac{(2n)!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n (n+1) \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} =$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^n}_{e} \cdot \frac{n+1}{(2n+2)(2n+1)} = 0 < 1 \to$$ CONVERGE

2x) $$\sum_{n=1}^{\infty} n\sqrt{1+\frac{4}{n^3}} = \lim_{n\to+\infty} \sqrt{n^2+\frac{4}{n}} = \lim_{n\to+\infty} \sqrt{\frac{n^3+4}{n}} = +\infty \to$$ DIVERGE POSITIVAMENTE

2y) $$\sum_{n=1}^{+\infty} \frac{n^{(n+1/n)}}{(n+\frac{1}{n})^n} = \lim_{n\to+\infty} \frac{n^{n+1/n}}{n^n(1+1/n^2)} = \lim_{n\to+\infty} \frac{n^{1/n}}{(1+\frac{1}{n^2})} = 1 \to$$ DIVERGE POSITIVAMENTE

2z) $$\sum_{n=1}^{+\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right) = \lim_{n\to+\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right) =$$

$$= \lim_{n\to+\infty} \left(\frac{1}{n} - \left(\frac{1}{n} + \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right)\right) \simeq \lim_{n\to+\infty} \frac{1}{6n^3} = 0 < 1$$

$\to$ CONVERGE