c) $\exists x \forall y (Rxy \land \quad x \to (Rxx \leftrightarrow Ryy))$

- SATISFIABILITY

SKOLEMIZATION: φ IS SATISFIABLE $\Leftrightarrow$ φ_s IS SATISFIABLE

$\varphi_s = \forall y (Ray \land \neg Rya \to (Raa \leftrightarrow R_{yy}))$

$$\forall y (Ray \land \neg Rya \to (Raa \leftrightarrow R_{yy}))$$

$$|$$

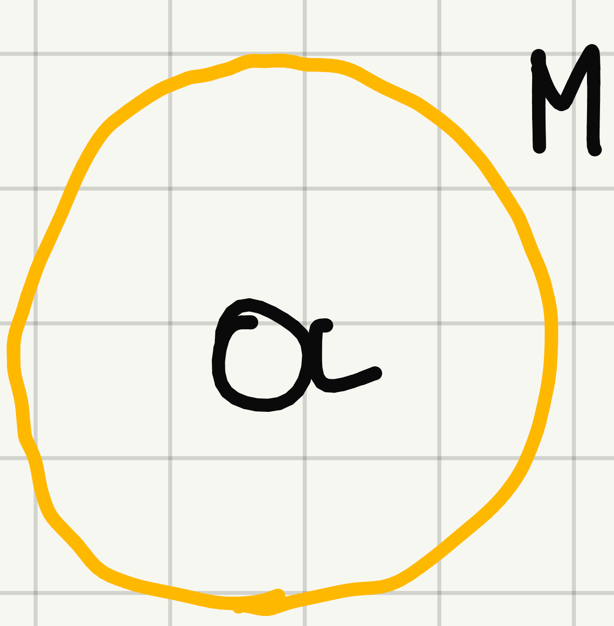
$$Raa \land \neg Raa \to (Raa \leftrightarrow Raa),\ \forall y(\dots)$$

$$\swarrow \qquad \searrow$$

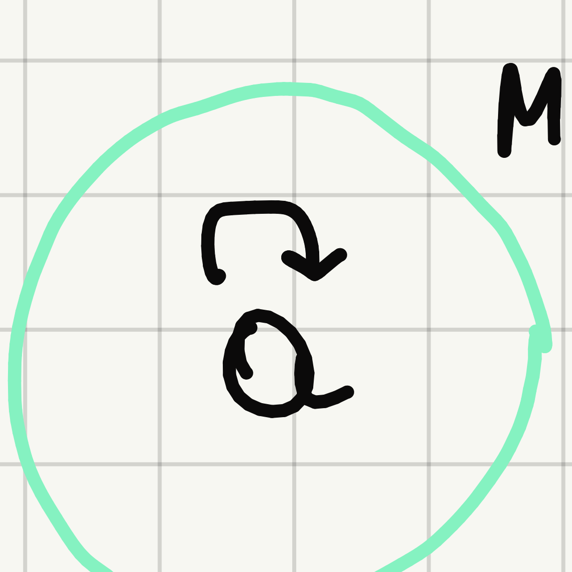
$\neg(Raa \land \neg Raa), \forall y(\dots)$ $\qquad$ $Raa \leftrightarrow Raa, \forall y(\dots)$

Left branch: $\neg(Raa \land \neg Raa), \forall y(\dots)$
- ① $\neg Raa$
- $\neg\neg Raa$ → ② Raa

Right branch: $Raa \leftrightarrow Raa, \forall y(\dots)$
- $Raa \to Raa$
  - Raa ②
  - $\neg Raa$ ①



$M = \{a\}$ $[\![R]\!] = \emptyset$



$M = \{a\}$ $[\![R]\!] = \{(a, a)\}$

THOSE SOLUTIONS ARE BOTH SATISFIABLE FOR φ_s

VALIDITY

$\neg\varphi = \neg(\exists x \forall y (Rxy \land \neg Rxx \to (Rxx \leftrightarrow Ryy)))$

$\equiv \forall x \exists y \neg (Rxy \land \neg Ry \to (Rxx \leftrightarrow Ryy))$

$\equiv \forall x \exists y (Rxy \land \neg Ryx \land \neg (Rxx \leftrightarrow Ryy))$ PRENEX FORM

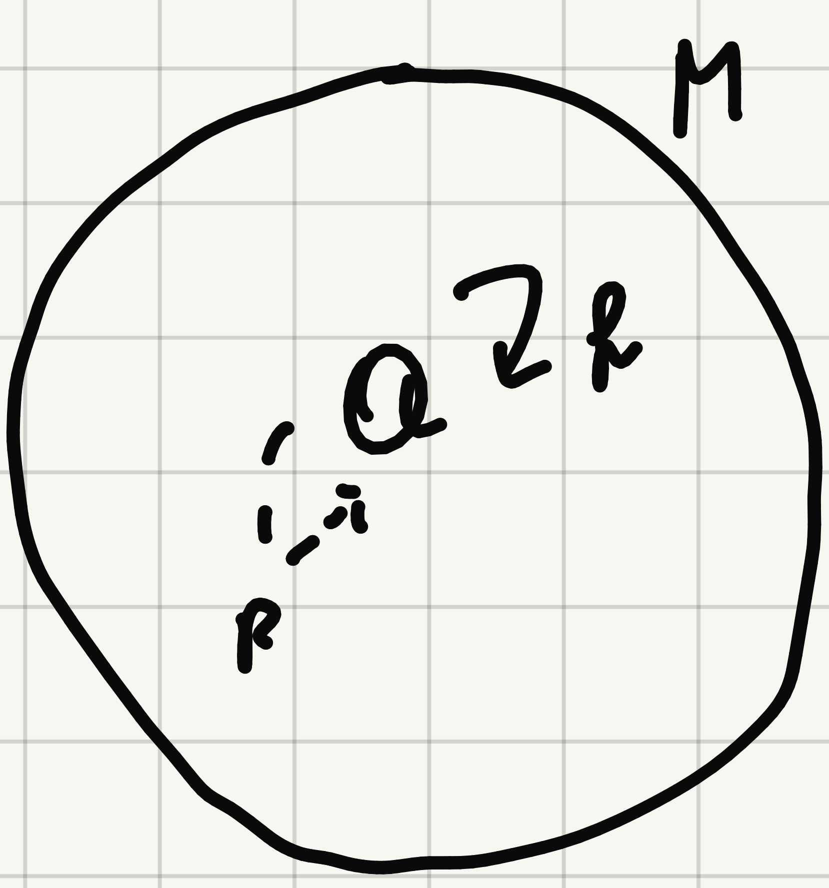
$\mapsto \forall x (R(x, fx) \land \neg R(fx, x) \land \neg (Rxx \leftrightarrow R(fx, fx)))$

$\equiv \underline{\forall x (R(x, fx))} \land \forall x (\underline{\neg R(fx, x)}) \land \forall x (\underline{\neg Rxx \leftrightarrow R(fx, fx)})$

WE NEED M SUCH THAT: $\qquad a \not R b$ a NOT IN RELATION WITH b

- $x R f(x)$
- $\neg(f(x) R x)$, so $f(x) \not R x$
- $x R x \leftrightarrow f(x) \not R f(x)$

1^{st} TRY: $M = \{a\}$

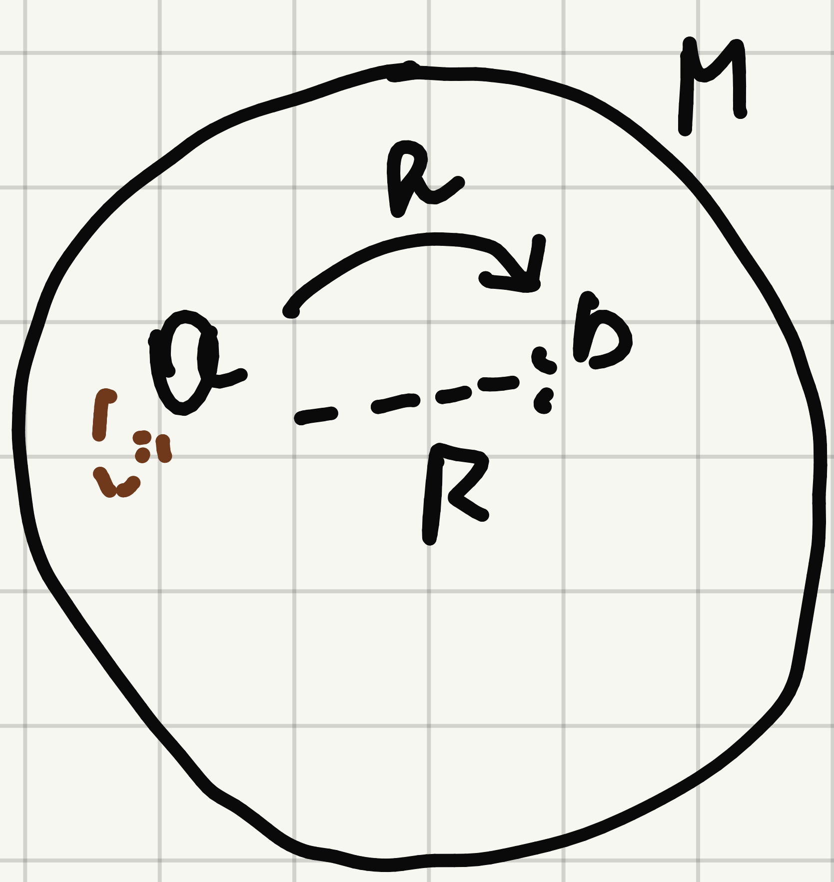


$x = a \quad f(x) = a$

✓

✗ ($a \not R a$, BUT FROM THE PREVIOUS CONDITION $a R a$)

2^{nd} TRY: $M = \{a, b\}$



$x = a \quad f(x) = f(a) = b$

✓

✓

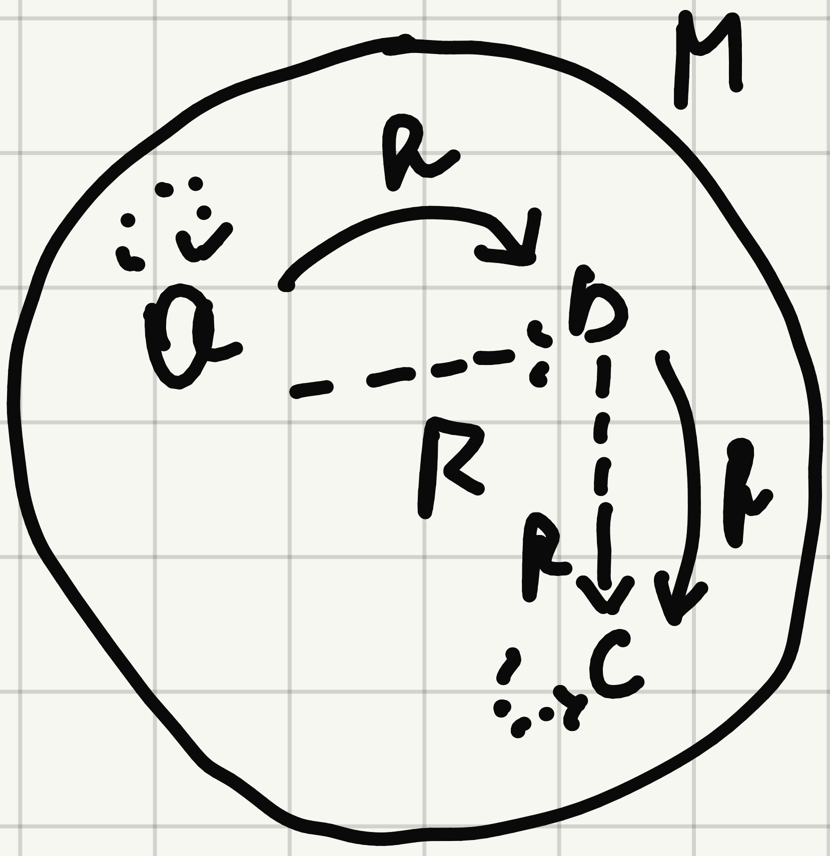
✓

$x = b \quad f(b) = ?$

- a ? NO
- b ? NO

- $b \not R a$
- $a R a \leftrightarrow b \not R b$

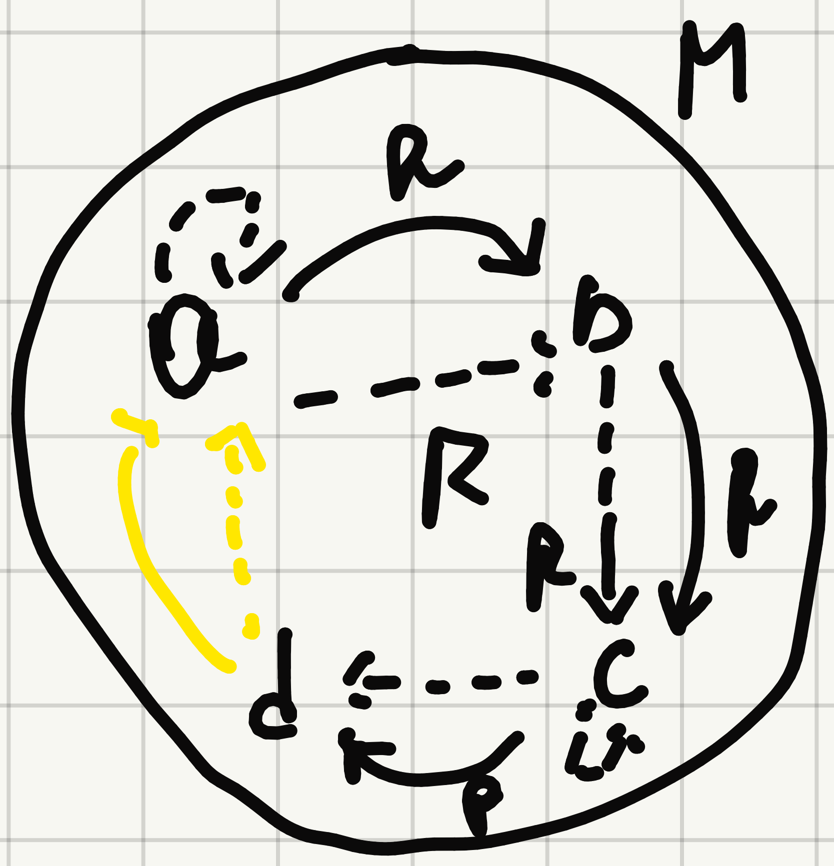
THUS, WE NEED A 3rd ELEMENT c AND $f(b)=c$



$x=a$ ✓ (CONDITIONS PREVIOUSLY VERIFIED)

$x=b$
bRc
cRb
$bRb \leftrightarrow c \not R c$

$x=c$ $f(c)$?
• c? NO $cRc \leftrightarrow c \not R c$
• b? NO cRb $b \not R c$
• a? NO, OTHERWISE: $c f c \leftrightarrow a \not f a$ BUT WE HAVE $cRc \leftrightarrow aRa$



$f(d)$?

• a ✓
dRa
$a \not R d$
$aRa \leftrightarrow d \not R d$

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$M = \{a,b,c,d\}$

$[[R]] = \{(a,b),(b,c),(c,d),(d,a),(a,a),(c,c)\}$

$[[f]] = \{(a,b),(b,c),(c,d),(d,a)\}$

$\neg\varphi$ IS SATISFIABLE $\rightarrow$ φ IS NOT VALID