PROPOSITIONAL LOGIC

TRUTH FUNCTIONS

- ARITY 0 $\quad \bot_\Omega = 0 \quad \top_\Omega = 1$
- ARITY 1 $\quad \neg\bot_\Omega = 1 \quad \neg\top_\Omega = 0$
- ARITY 2 $\quad x \wedge_\Omega y = \begin{cases} 1 & x=y=1 \\ 0 & \text{OTHERWISE} \end{cases} \qquad x \vee_\Omega y = \begin{cases} 1 & \text{OTHERWISE} \\ 0 & x=y=0 \end{cases}$

$$x \rightarrow_\Omega y = \begin{cases} 1 & x \le y \\ 0 & \text{OTHERWISE} \end{cases} \qquad x \leftrightarrow_\Omega y = \begin{cases} 1 & x=y \\ 0 & \text{OTHERWISE} \end{cases}$$

TRUTH AND SATISFIABILITY

GIVEN A VALUATION v, A FORMULA φ IS:

- VALID IF $\forall v \in \Omega^V \mid [[\varphi]](v) = 1$ TAUTOLOGY
- SATISFIABLE IF $\exists v \in \Omega^V \mid [[\varphi]](v) = 1$
- UNSATISFIABLE IF $\forall v \in \Omega^V \mid [[\varphi]](v) = 0$ CONTRADICTION

SET OF FORMULAS:

- SATISFIABLE IF $\exists v \in \Omega^V \mid [[\varphi_1]](v) = \ldots = [[\varphi_n]](v) = 1$
- UNSATISFIABLE IF $\exists v \in \Omega^V \mid [[\varphi_k]](v) = 0$ FOR AT LEAST A $k \in [0, n]$

EQUIVALENCE: $\forall v \in \Omega^V, [[\varphi]](v) = [[\psi]](v) \quad \varphi \equiv \psi$

BOOLEAN ALGEBRA RULES

	AND	OR
ASSOCIATIVITY	$(\varphi\land\chi)\land\psi\equiv\varphi\land(\chi\land\psi)$	$(\varphi\lor\chi)\lor\psi\equiv\varphi\lor(\chi\lor\psi)$
COMMUTATIVITY	$\varphi\land\chi\equiv\chi\land\varphi$	$\varphi\lor\chi\equiv\chi\lor\varphi$
DISTRIBUTIVITY	$\varphi\land(\chi\lor\psi)\equiv(\varphi\land\chi)\lor(\varphi\land\psi)$	$\varphi\lor(\chi\land\psi)\equiv(\varphi\lor\chi)\land(\varphi\lor\psi)$
ABSORPTION	$\varphi\land(\chi\lor\varphi)\equiv\varphi$	$\varphi\lor(\chi\land\varphi)\equiv\varphi$
NEUTRAL ELEMENT	$\varphi\land\top\equiv\varphi$	$\varphi\lor\bot\equiv\varphi$
COMPLEMENT	$\varphi\land\neg\varphi\equiv\bot$	$\varphi\lor\neg\varphi\equiv\top$

α-FORMULAS AND β-FORMULAS

α	α_1	α_2	β	β_1	β_2
$\varphi\land\chi$	φ	χ	$\neg(\varphi\land\chi)$	$\neg\varphi$	$\neg\chi$
$\neg(\varphi\lor\chi)$	$\neg\varphi$	$\neg\chi$	$\varphi\lor\chi$	φ	χ
$\neg(\varphi\to\chi)$	φ	$\neg\chi$	$\varphi\to\chi$	$\neg\varphi$	χ

TREE STRUCTURE

$$\begin{array}{c} \ldots, \alpha_1\,\alpha_2, \ldots \\ | \\ \ldots\ \alpha_1, \alpha_2, \ldots \end{array} \qquad\qquad \begin{array}{ccc} & \ldots, \beta_1\,\beta_2, \ldots & \\ / & & \backslash \\ \ldots, \beta_1, \ldots & & \ldots, \beta_2, \ldots \end{array}$$

CONSEQUENCE

$F\vDash\varphi$ PREMISES $\vDash$ CONCLUSION

PROPOSITION $F\vDash\varphi \Leftrightarrow F\cup\{\neg\varphi\}\vDash\bot$

COMPLETE CONNECTIVES $\{\bot,\to\}$; $\{\top,\neg,\land\}$; $\{\bot,\neg,\lor\}$

CLAUSE: DISJUNCTION OF LITERALS (EXAMPLE: $\neg x \vee y \vee z$)

REDUCED WHEN:

- NO LITERAL OCCURS TWICE
- NO LITERAL OCCURS WITH OPPOSITE SIGNS
- $\bot$ IS OMITTED IF AT LEAST ONE OTHER LITERAL IS PRESENT

DUAL CLAUSE: CONJUNCTION OF LITERALS (EXAMPLE: $\bot \wedge x \wedge \neg y \wedge z$)

(ST. 1 EP. 7)

PANINO SIMMONS
& FITS IN MISS. CONWARD
- PROSCIUTTO
- MOZZARELLA
- PESTO
- SALSA PREZZEMOLO (CON AGLIO)

REDUCED WHEN:

- NO LITERAL OCCURS TWICE
- NO LITERAL OCCURS WITH OPPOSITE SIGNS
- $\top$ IS OMITTED IF AT LEAST ONE OTHER LITERAL IS PRESENT

CNF: CONJUNCTION OF CLAUSES **DNF:** DISJUNCTION OF DUAL CLAUSES

RESOLUTION CALCULUS **RULES:**

- **FALSEHOOD:** $\bot \subset \varphi, \varphi[\emptyset/\bot] \Rightarrow \neg\top \subset \varphi, \varphi[\emptyset/\neg\top]$
- **DOUBLE NEGATION:** $\neg\neg\psi \subset \varphi; \varphi \Rightarrow \varphi[\psi/\neg\neg\psi]$
- **α-EXPANSION:** α-FORMULA $\subset \varphi \Rightarrow \varphi[\alpha_1/\alpha] \vee \varphi[\alpha_2/\alpha]$
- **β-EXPANSION:** β-FORMULA $\subset \varphi \Rightarrow \varphi[\beta_1 \beta_2/\beta]$
- **RESOLUTION:** $\psi \subset \varphi_i \wedge \neg\psi \subset \varphi_j \Rightarrow \varphi_i[\emptyset/\psi] \cup \varphi_j[\emptyset/\neg\psi]$
- **CLOSED EXPANSION:** INCONSISTENCY $\rightarrow$ ONE OF ITS CLAUSES IS $\bot$

RELATIONS AND FUNCTIONS

GENERAL DEFINITIONS

UNION $A \cup B = \{x \in U \mid x \in A \vee x \in B\}$

INTERSECTION $A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$

COMPLEMENT $A' = \{x \in U \mid x \notin A\}$

BOOLEAN ALGEBRA RULES

- ASSOCIATIVITY $(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
- COMMUTATIVITY $A \cap B = B \cap A$ $A \cup B = B \cup A$
- ABSORPTION $A \cap (A \cup B) = A$ $A \cup (A \cap B) = A$
- DISTRIBUTIVITY $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- NEUTRAL ELEMENT $A \cap U = A$ $A \cup \emptyset = A$
- COMPLEMENTARY $A \cap A' = \emptyset$ $A \cup A' = U$
- IDEMPOTENCY $A \cap A = A$ $A \cup A = A$
- DUALITY $(A \cap B)' = A' \cup B'$ $(A \cup B)' = A' \cap B'$
- ABSORPTIVE ELEMENT $A \cap \emptyset = \emptyset$ $A \cup \emptyset = \emptyset$
- INVOLUTION $(A')' = A$

RELATION $R: A \to B$

$xRy \Rightarrow x \in A, y \in B$

PRODUCT $R: A \to B, S: B \to C$ $RS \Rightarrow aRSc \Leftrightarrow \exists b (aRb \wedge bSc)$

PRODUCT AND BOOLEAN RULES

- $\bot R = \bot$ $\qquad R\bot = \bot$
- $TR \subseteq T$ $\qquad RT \subseteq T$
- $(\bigcup_{i \in I} R_i)S = \bigcup_{i \in I} R_i S$ $\qquad R(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} RS_i$
- $(\bigcap_{i \in I} R_i)S \subseteq \bigcap_{i \in I} R_i S$ $\qquad R(\bigcap_{i \in I} S_i) \subseteq \bigcap_{i \in I} RS_i$

PROPERTIES OF RELATIONS

POWERS $R^0 = 1 \quad R^{n+1} = R^n R \quad R^m R^n = R^{m+n} \quad (R^m)^n = R^{mn}$

OPPOSITE $R : A \to B \Rightarrow R^{op} : B \to A \quad b R^{op} a \Leftrightarrow a R b$

$(R^{op})^{op} = R \quad R \subseteq S \Rightarrow R^{op} \subseteq S^{op} \quad \bot^{op} = T \quad T^{op} = \bot$

$(\bigcup_{i \in I} R_i)^{op} = \bigcup_{i \in I} R_i^{op} \quad (\bigcap_{i \in I} R_i)^{op} = \bigcap_{i \in I} R_i^{op} \quad (R')^{op} = (R^{op})'$

$1^{op} = 1 \quad (RS)^{op} = S^{op} R^{op}$

DIRECT AND INVERSE IMAGE $R : A \to B$

- DIRECT IMAGE OF $S \subseteq A$ $\qquad R_*(S) := SR \subseteq B$
- INVERSE IMAGE OF $T \subseteq B$ $\qquad R^*(S) := TR^{op} \subseteq A$

FUNCTIONS $f : A \to B$

DEFINITION $\forall a \in A (|f_*(a)| = 1)$

- $f_*(a)$ HAS AT LEAST ONE ELEMENT $\qquad \forall a \in A \Leftrightarrow A \subseteq f f^{op}$
- $f^*(a)$ HAS AT MOST ONE ELEMENT $\qquad \forall a \in A \Leftrightarrow f^{op} f \subseteq B$
- PRODUCT OF FUNCTIONS $\qquad fg = g(f(x)) \quad g : B \to C$

INVERSE

- LEFT INVERSE $\Leftrightarrow gf = B$ $\quad f(g(b)) = b \ \forall b$
- RIGHT INVERSE $\Leftrightarrow fg = A$ $\quad g(f(a)) = a \ \forall a$

$\}$ BOTH LEFT AND RIGHT INVERSE

BIJECTIVITY

- f EPIMORPHISM/SURJECTIVE $\leftrightarrow$ IT HAS LEFT INVERSE $f^{op}f = B$

IN INCIDENT MATRIX, EVERY COLUMN HAS AT LEAST ONE 1

- f MONOMORPHISM/INJECTIVE $\leftrightarrow$ IT HAS RIGHT INVERSE $A = ff^{op}$

IN INCIDENT MATRIX, EVERY COLUMN HAS AT MOST ONE 1

- f ISOMORPHISM/BIJECTIVE $\leftrightarrow$ BOTH INJECTIVE AND SURJECTIVE

IN INCIDENT MATRIX, EVERY COLUMN HAS EXACTLY ONE 1

DIRECT AND INVERSE IMAGE

- DIRECT IMAGE OF R ALONG f $\quad S = fRf^{op}$
- INVERSE IMAGE OF R ALONG f $\quad S = f^{op}Rf$

EQUIVALENCE RELATION $E: A \to A$

PROPERTIES

- REFLEXIVE $\quad \forall x.(xEx)$ $\quad I \subseteq M_E$
- SYMMETRIC $\quad \forall xy.(xEy \to yEx)$ $\quad M_{E^{op}} \leq M_E$
- TRANSITIVE $\quad \forall xyz.((xEy) \wedge (yEz) \to (xEz))$ $\quad M_{E^2} \leq M_E$

- REFLEXIVE & SYMMETRIC CLOSURE $\quad M_S = M_R + I + M_{R^{op}} \quad R \cup I \cup R^{op}$
- EQUIVALENCE CLOSURE $\quad M_{Re} = \bigcup_{n=1}^{\infty} M^n$
- QUOTIENT $\quad [x] = \{y \in A \mid yEx\}$
- DIRECT IMAGE OF R ALONG f $\quad f_*(R) = f^{op}Rf$ ON B
- INVERSE IMAGE OF R ALONG f $\quad f^*(R) = fRf^{op}$ ON A

PARTIAL ORDER $R: A \to A$

PROPERTIES

- REFLEXIVE $\forall x.(xEx)$ $\quad I \subseteq M_E$
- ANTISYMMETRIC $\forall x y.(xRy \wedge yRx \to x=y)$ "ONLY A DIRECTION FOR EACH ARROW"
- TRANSITIVE $\forall x y z.((xEy) \wedge (yEz) \to (xEz))$ $\quad M_{E^2} \leq M_E$

- IDEMPOTENCY $R^2 = R$
- STRICT ANTISYMMETRISM $R \cap R^{op} = I$

TOTAL ORDER RELATION IF:

- PARTIAL ORDER
- $\forall x y.(xRy \vee yRx)$

HASSE DIAGRAM ALGORITHM:

1. DRAW GRAPH
2. REMOVE LOOPS
3. OMIT ARROWS $x_1 \to x_n$ IF THERE IS A PATH LIKE $x_1 \to \dots \to x_{n-1} \to x_n$
4. REARRANGE THE ARROWS SO THAT $x \to y$ ("y SITS ABOVE x")

MORPHISM OF A RELATION FROM R TO S: FUNCTION $f \mid fS = SR$

INDUCED ORDER: TRANSITIVE CLOSURE $S = R^{rt} \to$ EQUIVALENCE RELATION

$E = S \cap S^{op} \to$ PROJECTION $p: A \to A/E \to$ INDUCED ORDER $\bar{S} = p^{op} S p$

PROJECTION $A \xrightarrow{p} A/E$

$x \longmapsto [x]$

$pp^{op} = E; \quad p^{op} p = I$

MAX/MIN

GIVEN $B \subseteq A$ AND $x \in A$, x IS:

- UPPER BOUND IF $y \leq x \ \forall y \in B$
- LOWER BOUND IF $y \geq x \ \forall y \in B$

- MINIMUM FOR B ($x = \max(B)$) IF $x \geq y \ \forall y \in B$
- MINIMUM FOR B ($x = \min(B)$) IF $x \leq y \ \forall y \in B$

- SUPREMUM FOR B: MINIMUM IN THE SET OF UPPER BOUNDS
- INFINUM FOR B: MAXIMUM IN THE SET OF LOWER BOUNDS

FACTORIZATION OF A FUNCTION $f: A \to B$ $\quad f = em$

- $e: A \to C$ SURJECTIVE
- $m: C \to B$ INJECTIVE

$\}$ (e, m) IS THE KERNEL PAIR OF THE EPI-MONO FACTORIZATION

$h = fg$

- f, g SURJECTIVE $\Rightarrow$ h SURJECTIVE
- h SURJECTIVE $\Rightarrow$ g SURJECTIVE
- f, g INJECTIVE $\Rightarrow$ h INJECTIVE
- h INJECTIVE $\Rightarrow$ f INJECTIVE

FIRST ORDER LOGIC

EVALUATING FORMULAS: GIVEN VALUATION v

- $v \models \top$ — ALWAYS
- $v \models \bot$ — NEVER
- $v \models R(t_1, \dots, t_n)$ — IF $(v(t_1), \dots, v(t_n)) \in [\![R]\!]$
- $v \models t_1 = t_2$ — IF $v(t_1) = v(t_2)$
- $v \models \neg\varphi$ — IF $v \not\models \varphi$
- $v \models \varphi \wedge \psi$ — IF $v \models \varphi \wedge v \models \psi$
- $v \models \varphi \vee \psi$ — IF $v \models \varphi \vee v \models \psi$
- $v \models \varphi \rightarrow \psi$ — IF $v \models \psi$ WHENEVER $v \models \varphi$
- $v \models \varphi \leftrightarrow v \models \psi$ — IF $v \models \varphi$ IF AND ONLY IF $v \models \psi$
- $v \models \forall x.\varphi$ — IF $v \models \varphi$ FOR EVERY x-VARIANT u OF v
- $v \models \exists x.\varphi$ — IF $v \models \varphi$ FOR AT LEAST AN x-VARIANT u OF v

γ-FORMULAS AND δ-FORMULAS

γ	δ
$\forall xy.\varphi \equiv \forall yx.\varphi$	$\exists xy.\varphi \equiv \exists yx.\varphi$
$\forall x.(\varphi \wedge \psi) \equiv \forall x.\varphi \wedge \forall x.\psi$	$\exists x.(\varphi \vee \psi) \equiv \exists x.\varphi \vee \exists x.\psi$
$\forall x.\varphi \equiv \varphi$	$\exists x.\varphi \equiv \varphi$
$\neg\forall x.Rxx \rightarrow \exists x.\neg Rxx$	$\neg\exists x.Rxx \rightarrow \forall x.\neg Rxx$

RULES FOR EXPANSION

$$\forall xy\,(Rxy)$$
$$|$$
$$Rab,\ \forall xy\,(Rxy)$$
$$\vdots$$
$$|$$
$$Rab,\ Rcd$$

$$\exists xy\,(Rxy)$$
$$|$$
$$Rab$$

RULES FOR UNIFICATION

- $\forall xy Rxy \mapsto \{Rxy\}$
- $\exists xy Rxy \mapsto \{Rab\}$
- $\forall x \exists y Rxy \mapsto \{Rxfx\}$
- $\exists x \forall y Rxy \mapsto \{Ray\}$

ALGEBRAIC STRUCTURES

GROUPS

- ASSOCIATIVITY $(xy)z = x(yz)$
- NEUTRAL $1 \cdot x = x = x \cdot 1$
- INVERSE $x^{-1} \cdot x = 1 = x \cdot x^{-1}$

SUBGROUP $H \leq G$ (G IS A GROUP) IF:

- $H \neq \emptyset$
- $\forall x, y \in H, xy^{-1} \in H$

MORPHISM OF GROUP $h: G \to H \Leftrightarrow h(xy) = h(x)h(y)$

NORMAL SUBGROUP $H \trianglelefteq G$ IF $\forall h \in H, g \in G \Rightarrow g^{-1} h g \in H$

KERNEL NORMAL SUBGROUPS OF G ARE KERNELS OF GROUP MORPHISM

$h: G \to G'$ $\quad \mathrm{Ker}(h) = \{g \in G \mid h(g) = \underbrace{1}_{\text{NEUTRAL ELEMENT}} \in G'\}$

RINGS

- $(x+y)+z = x+(y+z)$ ASSOCIATIVITY
- $x+y = y+x$ COMMUTATIVITY
- 0 NEUTRAL ELEMENT
- $x+(-x) = 0$ OPPOSITE
- $(xy)z = x(yz)$ ASSOCIATIVITY
- $1x = x$ LEFT NEUTRAL ELEMENT
- $x1 = x$ RIGHT NEUTRAL ELEMENT

! $xy \neq yx$

- $x(y+z) = xy+xz$ DISTRIBUTIVITY
- $(x+y)z = xz+yz$ DISTRIBUTIVITY

SUBRING $S \subseteq R$ (S IS A RING) IF:

- $1 \in S$
- $\forall x,y \in S \to x-y \in S$
- $\forall x,y \in S \to xy \in S$

MORPHISM OF RING $h: R \to S$ IF

- $h(x+y) = h(x)+h(y)$
- $h(xy) = h(x)h(y)$
- $h(1) = 1$

IDEAL $a \subseteq R$ IF

- $0 \in a$
- $x \in a \Rightarrow -x \in a$
- $x,y \in a \Rightarrow x+y \in a$
- $x \in a, y \in R \Rightarrow xy \in a \wedge yx \in a$

KERNEL $h: R \to S$

$\mathrm{Ker}(h) = h^*(\{0\}) = \{x \in G \mid h(x) = 0\}$