Assume R is a binary relation and f, g are unary functions. Prove that $\forall x R(x,x), \forall xy (R(fx,y) \leftrightarrow R(x,gy)) \vdash \forall x (R(x,gfx) \wedge R(fgx,x))$

Herbrand Model

	Step	Formula	Rule
	1	$\{\forall x R(x,x)\}$	Assumption
	2	$\{\forall xy (R(fx,y) \leftrightarrow R(x,gy))\}$	Assumption
	3	$\{\neg \forall x (R(x,gfx) \wedge R(fgx,x))\}$	Assumption
	4	$\{\neg (R(a,gfa) \wedge R(fga,a))\}$	3, δ expansion
	5	$\{\neg R(a,gfa), \neg R(fga.a)\}$	4, β expansion
×	6	$\{\forall xy (R(fx.y)) \to R(x,gy)\}$	2, α expansion
O	7	$\{\forall xy (R(fa,y) \to R(a,gy))\}$	2, α expansion
×	8	$\{\forall y (R(fa,y) \to R(a,gy))\}$	6, γ expansion
	9	$\{R(fa,fa) \to R(a,gfa))\}$	8, γ expansion
	10	$\{\neg R(fa,fa), R(a.gfa)\}$	9, β expansion
O	11	$\{\forall y (R(a,g(y)) \to R(fa,y))\}$	7, γ expansion
	12	$\{R(a,g(fa)) \to R(fa,fa))\}$	11, γ expansion
	13	$\{\neg R(a,gfa), R(fa,fa)\}$	12, β expansion
	14	$\{R(fa,fa)\}$	1, γ expansion

15	$\{R(a, gfa)\}$	14, 10 RESOLUTION
16	$\{\neg R(a, gfa)\}$	13, 14 RESOLUTION
17	$\emptyset$	15, 16 RESOLUTION

UNIFICATION

- $\{\forall x R(x,x)\} \mapsto \{R(x,x)\}$
- $\{\forall xy (R(fx,y) \leftrightarrow R(x, gy))\} \mapsto \{(R(fx,y) \leftrightarrow R(x, gy))\}$

  $\mapsto \{R(fx,y) \to R(x,gy)\}, \{R(x,gy) \to R(fx,y)\}$

  $\mapsto \{\neg R(fx,y), R(x,gy)\}, \{\neg R(x,gy), R(fx,y)\}$
- $\{\neg \forall x (R(x, gfx) \wedge R(fgx, x))\} \mapsto \{\exists x. \neg (R(x.gfx) \wedge R(fgx,x))\}$

  $\mapsto \{\neg R(a.gfa), \neg R(fga.a)\}$

$$\frac{\{R(x,x)\} \quad \{\neg R(x,gy), R(fx,y)\}}{\{R(f.gy.y)\}} \left[\frac{gy}{x}\right]$$

$$\frac{\{R(f.gy.y)\} \quad \{\neg R(a,gfa), \neg R(fga,a)\}}{\{\neg R(a,gfa)\}} \left[\frac{a}{y}\right]$$

$$\left[\frac{x}{a}, \frac{y}{fa}\right] \quad \frac{\{\neg R(a,gfa)\} \quad \{\neg R(fx,y), R(x,gy)\}}{\{\neg R(fa,fa)\}}$$

$$\left[\frac{fa}{x}\right] \quad \frac{\{\neg R(fa,fa)\} \quad \{R(x,x)\}}{\emptyset}$$

LET L BE THE F.O.L. GENERATED BY A BINARY RELATION SYMBOL R INTENDED TO GENERALIZE EQUALITY AND BY THE UNARY FUNCTION SYMBOLS f AND g.

CONSIDERING THE FOLLOWING TERMINOLOGY:

- f IS R-INJECTIVE IF $\forall xy(R(fx,fy) \to R(x,y))$
- f IS R-SURJECTIVE IF $\forall x \exists y(R(fy,x))$
- f IS R-COMPATIBLE IF $\forall xy(R(x,y) \to R(fx,fy))$

PROVE THE FOLLOWING ASSERTIONS:

a) IF fg IS R-COMPATIBLE AND f IS R-INJECTIVE, THEN g IS R-COMPATIBLE

$$\forall xy(R(x,y) \to R(fgx,fgy)) \land \forall xy(R(fx,fy) \to R(x,y)) \vdash \forall xy(R(x,y) \to R(gx,gy))$$

$$\Gamma = \{\forall xy(R(x,y) \to R(fgx,fgy)), \forall xy(R(fx,fy) \to R(x,y)), \neg\forall xy(R(x,y) \to R(gx,gy))\}$$

UNIFICATION

- $\{\forall xy(R(x,y) \to R(fgx,fgy))\} \mapsto \{\neg R(x,y), R(fgx,fgy)\}$
- $\{\forall xy(R(fx,fy) \to R(gx,gy))\} \mapsto \{\neg R(fx,fy), R(x,y)\}$
- $\{\neg\forall xy(R(x,y) \to R(gx,gy))\} \mapsto \{\exists xy.\neg(R(x,y) \to R(gx,gy))\}$
  $\mapsto \{\exists xy(R(x,y) \land \neg R(gx,gy))\} \mapsto \{R(a,b)\}, \{\neg R(ga,gb)\}$

$$C(F): \{\{\neg R(x,y), R(fgx,fgy)\}, \{\neg R(fx,fy), R(x,y)\}, \{R(a,b)\}, \{\neg R(ga,gb)\}\}$$

$$\frac{\{\neg R(x,y), R(fgx,fgy)\} \qquad \{R(a,b)\}}{\{R(fga,fgb)\}} \left[\frac{a}{x}, \frac{b}{y}\right]$$

$$\frac{\{R(fga,fgb)\} \qquad \{\neg R(fx,fy), R(x,y)\}}{\{R(ga,gb)\}} \left[\frac{ga}{x}, \frac{gb}{y}\right]$$

$$\frac{\{R(ga,gb)\} \qquad \{\neg R(ga,gb)\}}{\varnothing}$$