a) $E = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

- REFLEXIVITY $\Leftrightarrow I \subseteq E \Leftrightarrow M(I) \leq M$

IT SUFFICES TO NOTICE THAT THE DIAGONAL CONTAINS ONLY "1"

$\Rightarrow$ IT IS REFLEXIVE ✓

- SYMMETRY $\Leftrightarrow E^{op} \subseteq E \Leftrightarrow M^T_{ij} \leq M_{ij} \quad \forall i,j$

$M^T = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = M \Rightarrow$ IT IS SYMMETRIC ✓

- TRANSITIVITY $\Leftrightarrow E^2 \subseteq E \Leftrightarrow M^2_{ij} \leq M_{ij} \quad \forall i,j$

$M^2 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = M \Rightarrow$ IT IS TRANSITIVE ✓

$\Rightarrow E$ IS AN EQUIVALENTE RELATION $\quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

$E^* = PEP^{op} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \cdot$

$$\bullet \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} \bullet$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{vmatrix}$$

- REFLEXIVITY $\Leftrightarrow I \subseteq E \Leftrightarrow M(I) \leq M(E) \Leftrightarrow \forall i,j,\ M(I)_{ij} \leq M(E)$

IT SUFFICES TO NOTICE THAT THE DIAGONAL CONTAINS ONLY "1"

$\Rightarrow$ IT IS REFLEXIVE ✓

- SYMMETRY $\Leftrightarrow E^{*op} \subseteq E^* \Leftrightarrow M(E^*)^T_{ij} \leq M(E^*)_{ij}\ \forall i,j$

$$M(E^*)^{op} = \begin{vmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{vmatrix} = M(E^*) \Rightarrow$$ IT IS SYMMETRIC ✓

- TRANSITIVITY $\Leftrightarrow (E^*)^2 \subseteq E^* \Leftrightarrow M((E^*)^2)_{ij} \leq M(E^*)_{ij}\ \forall i,j$

$$M((E^*)^2) = \begin{vmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{vmatrix} \bullet$$

$$\begin{vmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{vmatrix}$$

$= M(E^*) \Rightarrow$ IT IS TRANSITIVE

$\Rightarrow$ E^* IS AN EQUIVALENCE RELATION

b) E EQUIVALENCE RELATION, P SURJECTIVE $\quad E^* = PEP^{op}$

$a\,PEP^{op}\,b \Leftrightarrow \exists c,d\,(a\,p\,c \wedge c\,E\,d \wedge d\,p^{op}\,b) \Leftrightarrow \exists c,d\,(a\,p\,c \wedge c\,E\,d \wedge b\,p\,d)$

$\Leftrightarrow p(a) = c,\ p(b) = d \Leftrightarrow p(a)\,E\,p(b)$

R) $p(a)\,E\,p(a)$ ✓ (R IS REFLEXIVE) AND $p(a) = c = d$

$\Rightarrow \exists c,d\,(a\,p\,c \wedge c\,E\,c \wedge a\,p\,c) \Leftrightarrow \exists c,d\,(a\,p\,c \wedge c\,E\,c \wedge c\,p^{op}\,a)$

$\Rightarrow a\,PEP^{op}\,a$ ✓

S) $p(a)\,E\,p(b)$ E IS SYMMETRIC $\Rightarrow p(b)\,E\,p(a)$. HENCE,

$a\,PEP^{op}\,b \Rightarrow b\,PEP^{op}\,a$ ✓

T) $p(a)\,E\,p(b) \wedge p(b)\,E\,p(c)$ E TRANSITIVE $\Rightarrow p(a)\,E\,p(c)$.

HENCE, $a\,PEP^{op}\,b \wedge b\,PEP^{op}\,c \Rightarrow a\,PEP^{op}\,b$ ✓

C) PROVE $\exists\, q$ SURJECTIVE $\mid PP_E = P_{E^*}q$

WE NEED TO PROVE THAT E^* IS THE KERNEL PAIR OF PP_E

$\mathrm{Ker}(PP_E)$: $(PP_E)(PP_E)^{op} = PP_EP_E^{op}P = PEP^{op} = E^*$ ✓

BY THE 1[st] ISOMORPHISM THEOREM, $\exists$ UNIQUE $PP_E = P_{E^*}q$

WITH AN INJECTIVE q. P, P_E ARE BOTH SURJECTIVE $\Rightarrow$ BY

EPIMORPHISM, q IS SURJECTIVE $\Rightarrow q$ IS BIJECTIVE