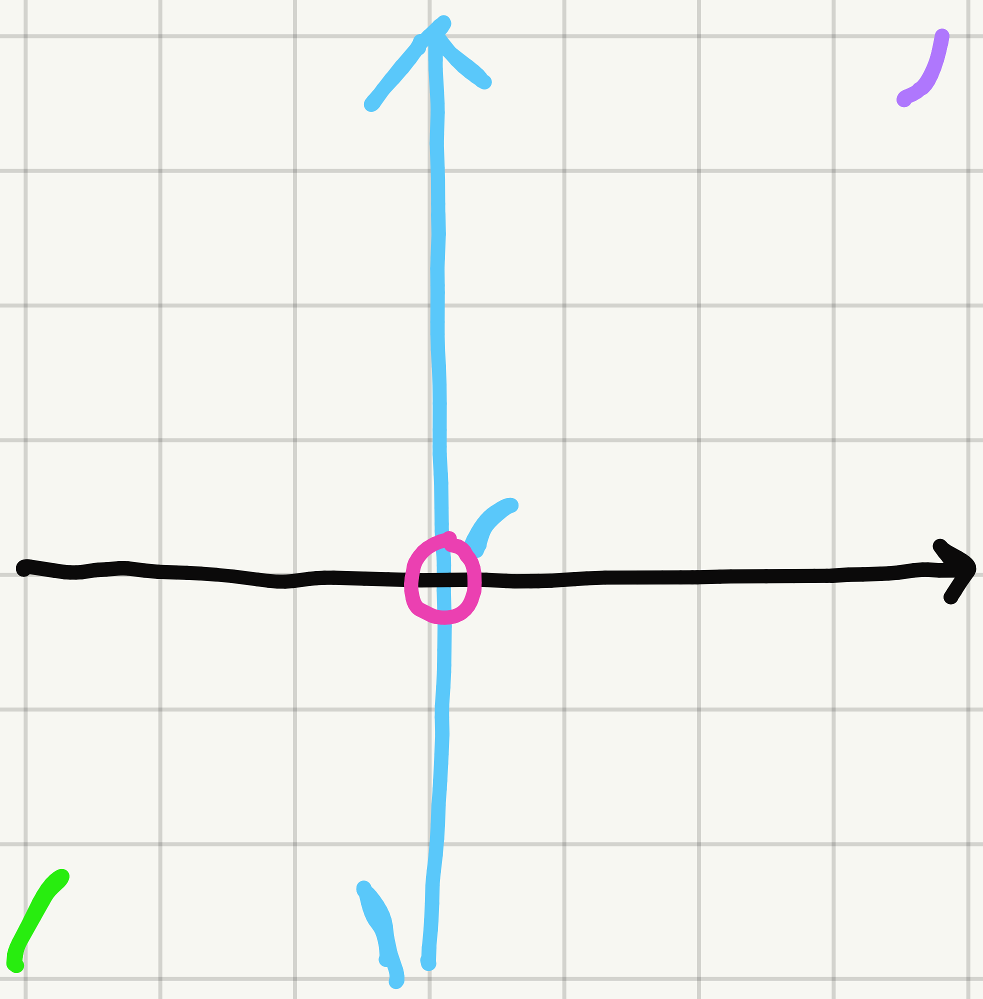
14 – STUDIO DI FUNZIONE

$f(x) = xe^{-\frac{1}{x}}$

$D = x \neq 0 = (-\infty, 0) \cup (0, +\infty)$



PARITÀ O DISPARITÀ

$$f(-x) = -xe^{1/x} \begin{cases} \neq f(x) & \text{NO PARI} \\ \neq -f(x) & \text{NO DISPARI} \end{cases}$$

COMPORTAMENTO AGLI ESTREMI DEL DOMINIO

$$\lim_{x\to-\infty} xe^{-\frac{1}{x}} = -\infty \cdot e^{-\frac{1}{-\infty}} = -\infty$$

$$\lim_{x\to0^-} xe^{-1/x} \overset{y=-1/x}{=} \lim_{y\to+\infty} -\frac{1}{y}e^{y} = -\infty$$ PER GERARCHIA

$\Rightarrow$ $x=0$ È ASINTOTO VERTICALE

$$\lim_{x\to0^+} xe^{-1/x} = 0 \cdot e^{-\infty} = 0 \qquad \lim_{x\to+\infty} xe^{-\frac{1}{x}} = +\infty$$

IN GENERALE:

A.V.) $\lim_{x\to x_0} f(x) = \pm\infty$ A.O.) $\lim_{x\to\pm\infty} f(x) = l \in \mathbb{R}$

A.Q.) $y = mx+q$ ASINTOTO $\Leftrightarrow$ $m = \lim_{x\to\pm\infty} \frac{f(x)}{x} + \frac{q}{x}$, $\frac{q}{x} \to 0$ ($m=0 \to$ A.O.)

$\Rightarrow \lim_{x\to\pm\infty} f(x) - mx = q \in \mathbb{R}$

RICERCA A.Q. $m = \lim_{x\to\pm\infty} \frac{xe^{-1/x}}{x} = 1$

$$\lim_{x\to\pm\infty} x(e^{-1/x} - 1) = x\left(-\frac{1}{x}\right) = 1$$

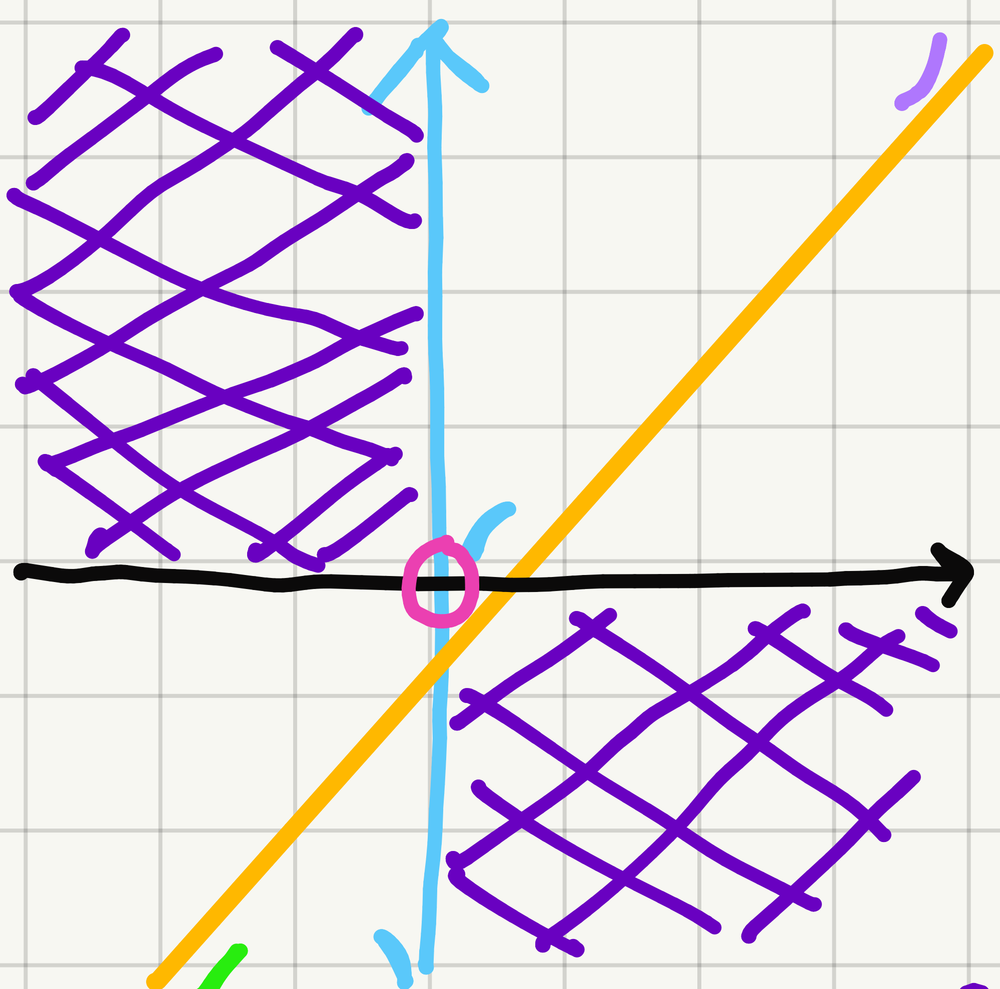
$\Rightarrow$ $y = x-1$

! IN GENERALE, $\lim_{x\to+\infty} \frac{f(x)}{x} \neq \lim_{x\to-\infty} \frac{f(x)}{x}$ E, A $-\infty$ E $+\infty$ SI POSSONO AVERE $\neq$ A.Q.

INTERSEZIONE CON GLI ASSI E STUDIO DEL SEGNO

! NON SEMPRE IMMEDIATO

- ABBIAMO GIÀ DETTO CHE $x \neq 0$ => NO INTERSEZIONI CON LE ORDINATE
- $f(x) = 0 \quad \cancel{x} e^{-1/x} = 0$ NO SOLUZIONI (PROPRIETÀ ESPONENZIALE)
- $f(x) > 0 \qquad x e^{-1/x} = 0$



		0	
x	$-$		$+$
$e^{-1/x}$	$+$		$+$
$f(x)$	$-$		$+$

$\Rightarrow f(x) < 0 \ \forall x < 0 \wedge f(x) > 0 \ \forall x > 0$

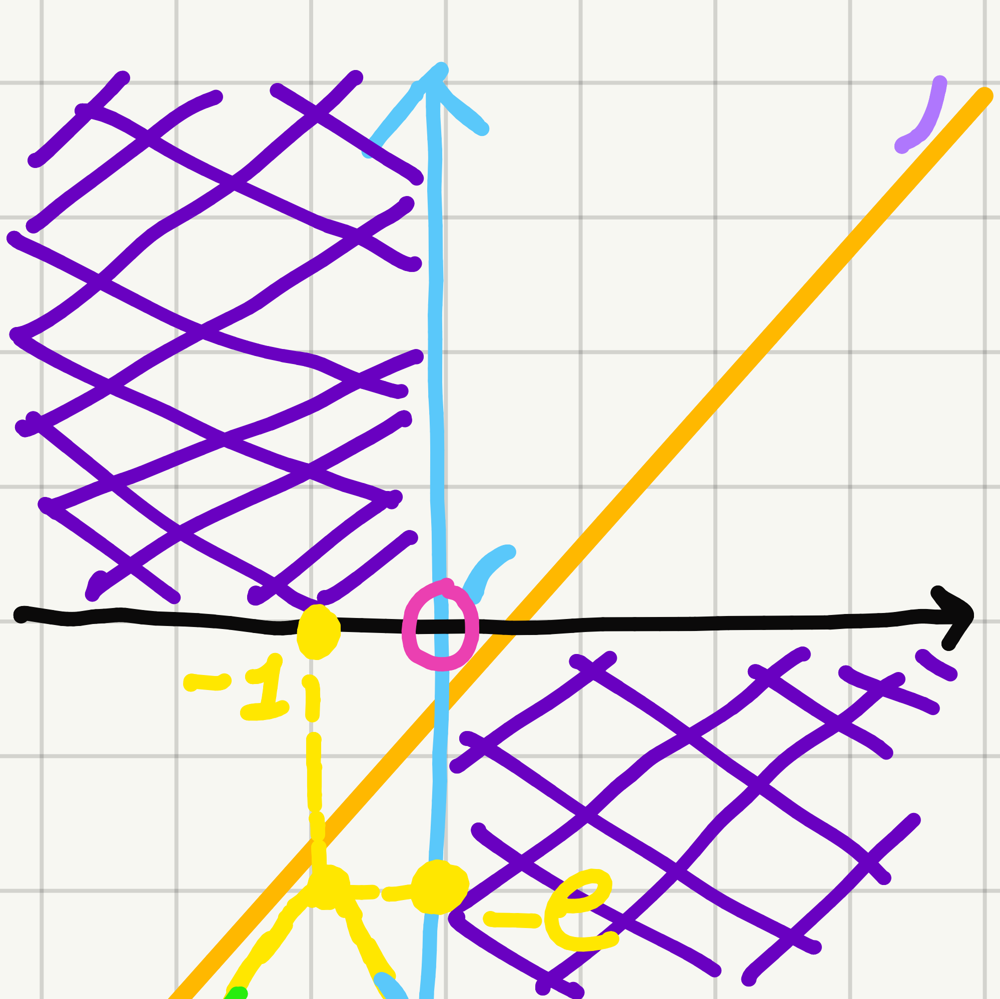
EVENTUALI PUNTI DI NON CONTINUITÀ

ABBIAMO GIÀ DETTO CHE $x = 0 \notin D \Rightarrow f(x)$ NON CONTINUA IN $x = 0 \Rightarrow f(x)$ NON DERIVABILE IN $x = 0$

MASSIMI/MINIMI/FLESSI

$f'(x) > 0 \qquad f'(x) = e^{-1/x} + (x \cdot e^{-\frac{1}{x}} \cdot (-\frac{1}{x^2})) =$

$= e^{-1/x} + \frac{e^{-1/x}}{x} = \cancel{e^{-1/x}}\left(\frac{x+1}{x}\right) > 0$



		-1		0	
N	$-$		$-$		$+$
D	$-$		$+$		$+$
f'	$+$	MAX	$-$		$+$
f	↗		↘		↗

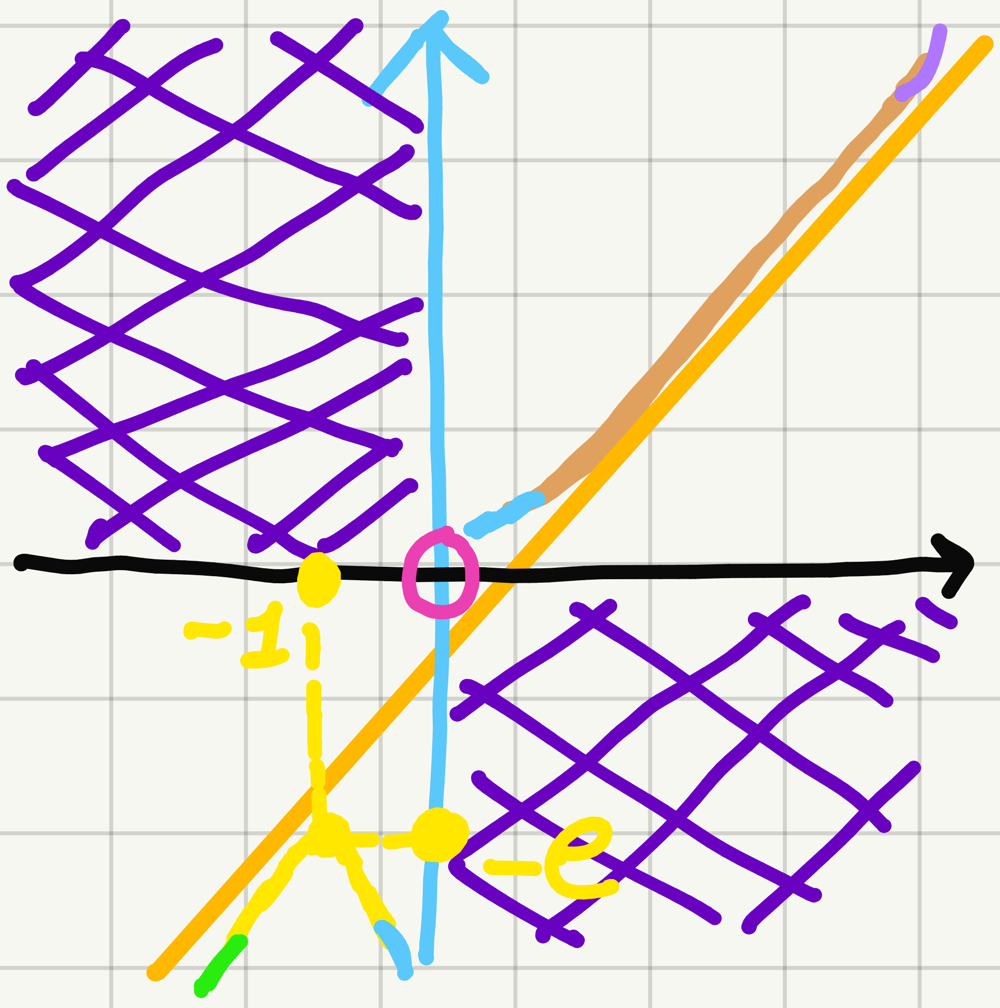
$f(-1) = -e$ MASSIMO RELATIVO

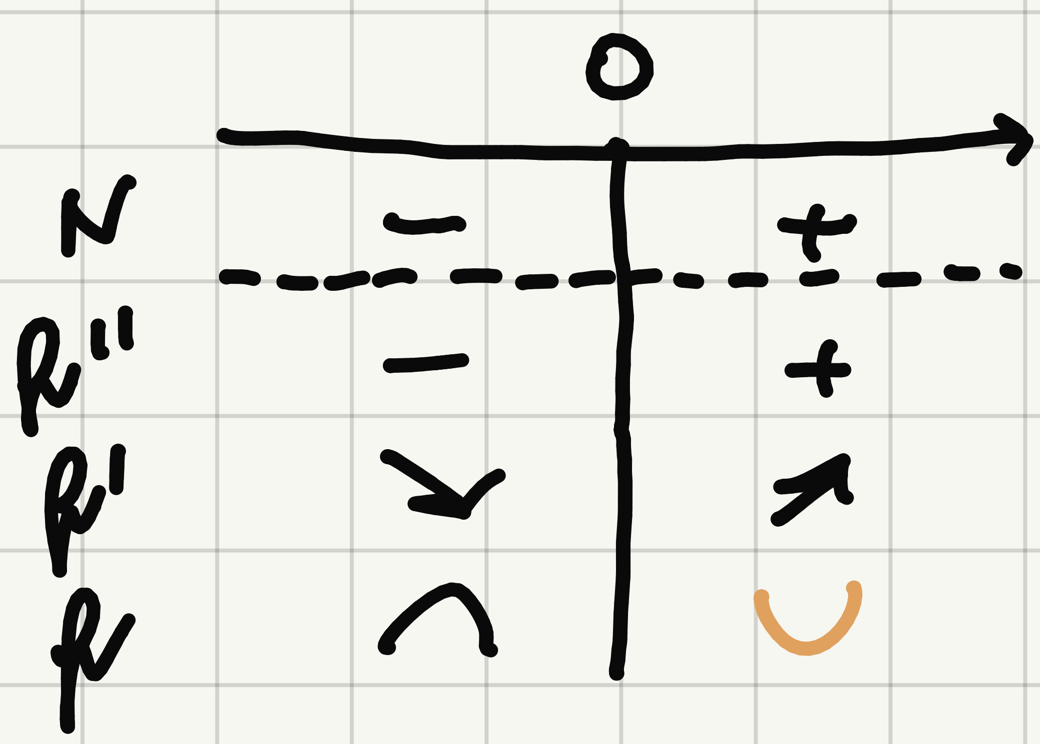
$f(x)$ È STRETTAMENTE CRESCENTE IN $(-\infty;-1]$, STRETTAMENTE DECRESCENTE IN $(-1,0)$ E STRETTAMENTE CRESCENTE IN $(0,+\infty)$

MASSIMI/MINIMI/FLESSI

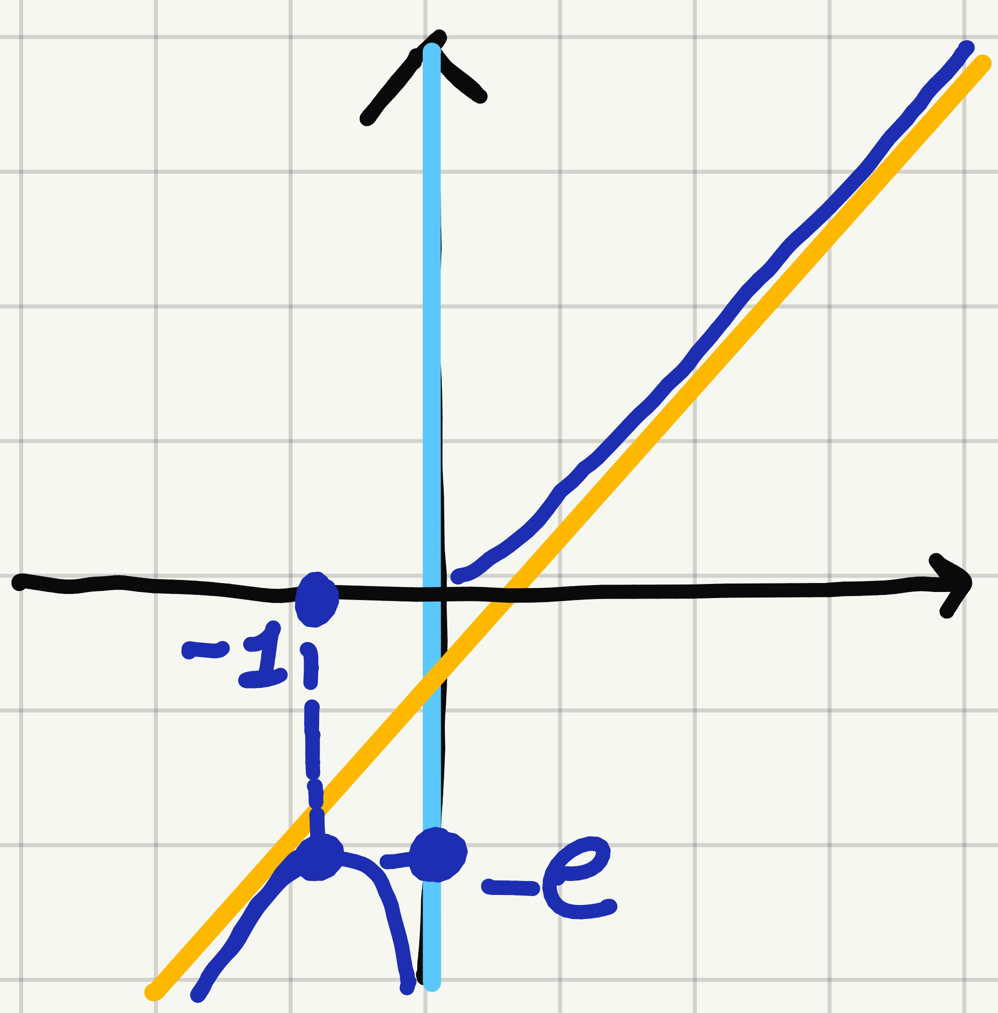
$$f'(x)=e^{-1/x}\left(\frac{x+1}{x}\right) \qquad f''(x)=e^{-1/x}\left(\frac{1}{x^2}\right)\left(\frac{x+1}{x}\right)+e^{-1/x}\left(\frac{x-x-1}{x^2}\right)$$

$$f''(x)>0 \qquad e^{-1/x}\cdot\frac{1}{x^2}\left(\frac{\cancel{x}+1}{x}-\cancel{1}\right)=e^{-1/x}\cdot\frac{1}{x^3}>0 \qquad \frac{1}{x^3}>0$$





FUNZIONE CONCAVA IN $(-\infty,0)$, CONVESSA IN $(0,+\infty)$



$f(x)=xe^{-\frac{1}{x}}$

15 - INTEGRALI

a) $\int \sqrt{2x+5}\,dx = \frac{1}{2}\int 2\,(2x+5)^{1/2}\,dx = \frac{1}{2}\,\frac{(2x+5)^{3/2}}{3/2} + c = \frac{1}{3}\sqrt{2x+5} + c$

b) $\int \frac{x}{\sqrt{(x^2+5)^3}}\,dx = \frac{1}{2}\int 2x\,(x^2+5)^{-3/2}\,dx = \frac{1}{2}\,\frac{(x^2+5)^{-1/2}}{-1/2} + c = -\frac{1}{\sqrt{x^2+5}} + c$

c) $\int x^3(8+x^4)^{-5/3}\,dx = \frac{1}{4}\int 4x^3(8+x^4)^{-5/3}\,dx = \frac{1}{4}\,\frac{(8+x^4)^{-2/3}}{-2/3} + c = -\frac{3}{8}\,\frac{1}{\sqrt[3]{(8+x^4)^2}} + c$

d) $\int \frac{3e^x}{1+e^{2x}}\,dx = 3\int \frac{e^x}{1+(e^x)^2}\,dx = 3\arctan(e^x) + c$

e) $\int \frac{1}{x\sqrt{1-\ln^2 x}}\,dx = \int \frac{1/x}{\sqrt{1-\ln^2 x}}\,dx = \arcsin(\ln x) + c$

f) $\int x \ln^2(5x)\,dx =$

$f'(x) = x \rightarrow f(x) = \frac{x^2}{2}$

$g(x) = \ln^2(5x) \rightarrow g'(x) = 2\ln(5x)\frac{5}{5x}$

$= \frac{x^2}{2}\ln^2(5x) - \int \frac{x^{\cancel{2}}}{\cancel{2}}\left(\cancel{2}\ln(5x)\frac{\cancel{5}}{\cancel{5}\cancel{x}}\right)dx + c =$

$= \frac{x^2}{2}\ln^2(5x) - \int x\ln(5x)\,dx + c =$

$= \frac{x^2}{2}\ln^2(5x) - \frac{x^2}{2}\ln(5x) + \int \frac{x^{\cancel{2}}}{2}\,\frac{1}{\cancel{5}\cancel{x}}\,\cancel{5}\,dx + c =$

$= \frac{x^2}{2}\ln^2(5x) - \frac{x^2}{2}\ln(5x) + \frac{x^2}{4} + c$

g) $\int (x+1)^2\cos x\,dx =$ $f'(x) = \cos x \rightarrow f(x) = -\sin x$

$g(x) = (x+1)^2 \rightarrow g'(x) = 2(x+1)$

$$=(x+1)^2\sin x-\int 2(x+1)\sin x\,dx+c=(x+1)^2\sin x-$$

$$-2\left[-(x+1)\cos x-\int \cos x\,dx\right]=$$

$$=(x+1)^2\sin x+2(x+1)\cos x-2\sin x+c$$

h) $\int 2x\arctan x\,dx=$

$f'(x)=2x\rightarrow f(x)=x^2 \quad g(x)=\arctan x\rightarrow g'(x)=\frac{1}{1+x^2}$

$$=x^2\arctan x-\int x^2\cdot\frac{1}{1+x^2}dx+c=$$

$$=x^2\arctan x-\int\frac{x^2+1-1}{1+x^2}dx+c=$$

$$=x^2\arctan x-\int\frac{x^2+1}{1+x^2}dx+\int\frac{1}{1+x^2}dx+c=$$

$$=x^2\arctan x-x+\arctan x+c$$

i) $\int x^3e^{-x^2}dx=\int x^2(xe^{-x^2})dx=-\frac{1}{4}\int x^2(4xe^{-x^2})dx=$

$f'(x)=-4xe^{-2x^2}\rightarrow f(x)=e^{-2x^2} \quad g(x)=x^2\rightarrow g'(x)=2x$

$$=-\frac{1}{4}(x^2e^{-2x^2}-\int 2xe^{-2x^2}dx)+c=$$

$\int e^{f(x)}f'(x)=e^{f(x)}+c \quad f'(-2x^2)=-4x\checkmark$

$$=-\frac{1}{4}(x^2e^{-2x^2}+\frac{1}{2}\int -4xe^{-2x^2}dx)+c=$$

$$=-\frac{1}{4}(x^2e^{-2x^2}+\frac{1}{2}e^{-2x^2})+c=-\frac{1}{4}x^2e^{-2x^2}-\frac{1}{8}e^{-2x^2}+c$$

j) $\int\frac{3x-4}{x^2-6x+8}dx=$

$\Delta=36-32=4>0$

$$\begin{array}{c|cc|c} & 1 & -6 & 8 \\ & & 2 & -8 \\ \hline 2 & 1 & -4 & 0 \end{array}$$

$$=\int \frac{3x-4}{(x-2)(x-4)}\,dx =$$

$$\frac{3x-4}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} = \frac{(A+B)x-2A-4B}{(x-2)(x-4)}$$

$$\begin{cases} A+B=3 \\ -2A-4B=-4 \end{cases} \Rightarrow \begin{cases} A=4 \\ B=-1 \end{cases}$$

$$= \int \frac{4}{x-4}\,dx + \int \frac{-1}{x-2}\,dx = 4\ln|x-4| - \ln|x-2| + C$$

K) $\int \frac{3x}{x^3-1}\,dx =$

$$\frac{3x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} = \frac{(A+B)x^2+(A-B+C)x+A}{(x-1)(x^2+x+1)}$$

$$\begin{cases} A+B=0 \\ A-B+C=3 \\ A-C=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=1 \end{cases}$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x-1}{x^2+x+1}\,dx = \ln|x-1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1} +$$

$$+ \underbrace{\frac{3}{2}\int \frac{1}{x^2+x+1}\,dx}_{I} + C = \ln|x-1| - \frac{1}{2}\ln|x^2+x+1| + I + C$$

- $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2\right) =$

$$= \frac{3}{4}\left(1+\left(\frac{2x+1}{\sqrt{3}}\right)^2\right)$$

$$I = \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{1}{3/4\left(1+\left(\frac{2x+1}{\sqrt{3}}\right)^2\right)}\,dx =$$

$$=\sqrt{3}\int \frac{2/\sqrt{3}}{1+\left(\frac{2x+1}{\sqrt{2}}\right)}dx=\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\int\frac{3x}{x^3-1}dx=\log|x-1|-\frac{1}{2}\ln|x^2+x+1|+\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

l) $\displaystyle\int\frac{9x+8}{x^3+2x^2+x+2}dx=$

$$\frac{9x+8}{x^3+2x^2+x+2}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}=$$

$$=\frac{A(x^2+1)+(2B+C)x+A+2C}{(x+2)(x^2+1)}$$

$$\begin{cases}A+B=0\\2B+C=9\\A+2C=8\end{cases}\Rightarrow\begin{cases}A=-2\\B=2\\C=5\end{cases}$$

$$=\int\frac{-2}{x+2}dx+\int\frac{2x+5}{x^2+1}dx=\int\frac{-2}{x+2}dx+\int\frac{2x}{x^2+1}dx+$$

$$+5\int\frac{1}{x^2+1}dx=-2\ln|x+2|+\ln|x^2+1|+5\arctan x+c$$

m) $\displaystyle\int\frac{x^5-x+1}{x^4+x^2}dx=$

$$\begin{array}{rrr|l} x^5 & & -x+1 & x^4+x^2 \\ \hline - \; x^5 & +x^3 & & x \\ \hline & -x^3 & -x+1 & \end{array}$$

$$= \int \left(x - \frac{x^3+x-1}{x^4+x^2} \right) dx =$$

$$\frac{x^3+x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{(A+C)x^3+(B+D)x^2+Ax+B}{x^2(x+1)}$$

$$\begin{cases} A+C=1 \\ B+D=0 \\ A=1 \\ B=-1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \\ D=1 \end{cases}$$

$$= \int x - \left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+1} \right) dx = \frac{x^2}{2} - \ln|x| - \frac{1}{x} - \arctan x + C$$

n) $\displaystyle\int \frac{e^x}{e^{2x}-3e^x+2}\,dx$

$e^x = t \Rightarrow x = \ln t \rightarrow dx = \frac{1}{t}\,dt$

$$\int \frac{t}{t^2-3t+2} \cdot \frac{1}{t}\,dt =$$

	1	-3	2
1		1	-2
	1	-2	0

$$= \int \frac{1}{(t-2)(t-1)}\,dt =$$

$$\frac{1}{(t-2)(t-1)} = \frac{A}{t-2} + \frac{B}{t-1} = \frac{(A+B)t-A-2B}{(t-2)(t-1)}$$

$$\begin{cases} A+B=0 \\ -A-2B=1 \end{cases} \rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$= \int \frac{1}{t-2} - \frac{1}{t-1}\,dt = \ln|t-2| - \ln|t-1| + C =$$

$$= \ln|e^x-2| - \ln|e^x-1| + C$$

o) $\int \frac{1}{\sqrt{2x}(\sqrt[3]{2x}+1)}dx \quad t=\sqrt[6]{2x} \to 2x=t^6 \to dx=3t^5dt$

$$\int \frac{3t^5}{t^3(t^2+1)}dt=\int \frac{3t^2}{t^2+1}dt=3\int \frac{t^2+1-1}{t^2+1}dt=$$

$$=3\int\left(1-\frac{1}{t^2+1}\right)dt=3t-3\arctan t+C=$$

$$=3\sqrt[6]{2x}-3\arctan(\sqrt[6]{2x})+C$$

p) $\int \frac{2}{(1+\tan x)^2}dx \quad \tan x=t \to x=\arctan t \to dx=\frac{1}{1+t^2}dt$

$$\int \frac{2}{(1+t)^2(1+t^2)}dt=$$

$$\frac{2}{(1+t)^2(1+t^2)}=\frac{A}{1+t}+\frac{B}{(1+t)^2}+\frac{Ct+D}{1+t^2}=$$

$$=\frac{(A+C)t^3+(A+B+2C+D)t^2+(A+C+2D)t+(A+B+D)}{(1+t)^2(1+t^2)}$$

$$\begin{cases} A+C=0 \\ A+B+2C+D=0 \\ A+C+2D=0 \\ A+B+D=2 \end{cases} \longrightarrow \begin{cases} A=1 \\ B=1 \\ C=-1 \\ D=0 \end{cases}$$

$$=\int \frac{1}{1+t}dt+\int \frac{1}{(1+t)^2}dt-\frac{1}{2}\int \frac{2t}{1+t^2}dt=$$

$$=\ln|1+t|-\frac{1}{1+t}-\frac{1}{2}\ln|1+t^2|+C=$$

$$=\ln|1+\tan x|-\frac{1}{1+\tan x}-\frac{1}{2}\ln|1+\tan^2 x|+C$$

9) $\int \frac{\cos x - 3}{\sin^2 x - \cos^3 x + 1} \sin x \, dx$ $\cos x = t$ $\sin x \, dx = dt$

$$\int \frac{t-3}{1-t^2-t^3+1} dt = \int \frac{3-t}{t^3+t^2-2} dt$$

	1	1	0	-2
1		1	2	2
	1	2	2	0

$$\int \frac{3-t}{(t-1)(t^2+2t+2)} dt =$$

$$\frac{3-t}{(t-1)(t^2+2t+2)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+2t+2} = \frac{(A+B)t^2+(2A-B+C)t+2A-C}{(t-1)(t^2+2t+2)}$$

$$\begin{cases} A+B=0 \\ 2A-B+C=-1 \\ 2A-C=3 \end{cases} \quad \begin{cases} B=-A \\ 2A+A+2A-3=-1 \\ C=2A-3 \end{cases} \quad \begin{cases} A=2/5 \\ B=-2/5 \\ C=-11/5 \end{cases}$$

$$= \int \left(\frac{2/5}{t-1} + \frac{-\frac{2}{5}t - \frac{11}{5}}{t^2+2t+2} \right) dt = \frac{1}{5}\Big(2 \ln|t-1| -$$

$$- \int \frac{2t+2}{t^2+2t+2} dt - \int \frac{9}{1+(1+t)^2} dt \Big) = \frac{2}{5} \ln|t-1| - \frac{1}{5} \ln|t^2+2t+2|$$

$$-\frac{9}{5} \arctan(t+1) + c = \frac{2}{5} \ln|\cos x - 1| - \frac{1}{5} \ln|\cos^2 x + 2\cos x + 2|$$

$$-\frac{9}{5} \arctan(\cos x + 1) + c$$