ESERCIZIO SIMIL-ESAME: CALCOLARE LA RISPOSTA TOTALE DEL SISTEMA $y''(t)+2y'(t)+5y(t)=3 \quad \forall t \geq t_0=0$ SAPENDO CHE $y(0)=y'(0)=0$.

SAPPIAMO CHE $y(t)=y_L(t)+y_F(t)$ E $u(t)=\delta_{-1}(t)$

È POSSIBILE RISOLVERE L'ESERCIZIO SIA LAVORANDO DIRETTAMENTE NEL DOMINIO t SIA PASSANDO PER LAPLACE

SOLUZIONE y_L NEL DOMINIO t

$$P(\lambda)=\lambda^2+2\lambda+5=0 \qquad \lambda=\frac{-2\pm\sqrt{-16}}{2}$$

$$\Rightarrow \begin{matrix} \lambda=-1+2i, \ \nu=1 \\ \lambda^*=-1-2i, \ \nu^*=1 \end{matrix} \Rightarrow \begin{matrix} c_\lambda(t)=e^{(-1+2i)t} \\ c^*_\lambda(t)=e^{(-1-2i)t} \end{matrix}$$

$$c(t)=Ae^{(-1+2i)t}+A^*e^{(-1-2i)t} \qquad A,A^*\in\mathbb{C}$$

$$y_L(t)=c(t)\Big|_{y(0)=0,\ y'(0)=0} \qquad A=\alpha+i\omega,\ A^*=\alpha-i\omega$$

$$c(t)=(\alpha+i\omega)e^{(-1+2i)t}+(\alpha-i\omega)e^{(-1-2i)t}$$

$$c'(t)=(\alpha+i\omega)(-1+2i)e^{(-1+2i)}+(\alpha-i\omega)(-1-2i)e^{(-1-2i)t}$$

$$y_L(t)=\begin{cases} \alpha+\cancel{i\omega}+\alpha-\cancel{i\omega}=0 \\ -\alpha+\cancel{2\alpha i}-\cancel{i\omega}-2\omega-\cancel{2\alpha i}+\cancel{i\omega}-2\omega=0 \end{cases}$$

$$\begin{cases} 2\alpha = 0 \\ -2\alpha - 4w = 0 \end{cases} \Rightarrow A = A^* = 0 \Rightarrow y_L(t) = 0$$

$$y_F(t) = \int_{t_0}^{t} U(t-\tau) \cdot y_\delta(\tau)\, d\tau \qquad y_\delta(t) = C(t)\delta_{-1}(t) + A_0\delta(t)$$

$$A_0 = \begin{cases} 0 & n > m \\ \dfrac{a_n}{b_n} & n = m \end{cases}$$

$$\begin{vmatrix} b_0 \\ b_1 \\ b_2 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & 0 \\ a_2 & 0 & 0 \end{vmatrix} \bullet \begin{vmatrix} A_0 \\ C(0) \\ C'(0) \end{vmatrix}$$

$$\begin{vmatrix} 3 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \bullet \begin{vmatrix} A_0 \\ 2\alpha \\ -2\alpha-4w \end{vmatrix}$$

$$\begin{vmatrix} 3 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & -4 \end{pmatrix} \bullet \begin{vmatrix} A_0 \\ \alpha \\ w \end{vmatrix}$$

$$\Rightarrow \begin{cases} A_0 = 0 \\ \alpha = 0 \\ w = -\frac{3}{4} \end{cases} \Rightarrow y_\delta(t) = \left(-\frac{3}{4} i e^{(-1+2i)t} + \frac{3}{4} i e^{(-1-2i)t}\right)\delta_{-1}(t)$$

$$y_f(t) = \int_0^t \delta_{-1}(t) \cdot \left[\left(-\frac{3}{4} i e^{(-1+2i)\tau} + \frac{3}{4} i e^{(-1-2i)\tau} \right) \delta_{-1}(\tau) \right] d\tau$$

$$= \int_0^t \left(-\frac{3}{4} i e^{(-1+2i)\tau} + \frac{3}{4} i e^{(-1-2i)\tau} \right) \delta\tau \cdot \underbrace{\int_0^t \delta_{-1}(\tau) d\tau}_{1} \cdot \delta_{-1}(t)$$

$$= \int_0^t \frac{3}{4} \left(-i e^{(-1+2i)\tau} + i e^{(-1-2i)\tau} \right) d\tau \cdot \delta_{-1}(t)$$

$$= \int_0^t \frac{3}{2} e^{-\tau} \sin(2\tau) d\tau \cdot \delta_{-1}(t) =$$

$$= \int_0^t \frac{3}{2} e^{-\tau} \cdot \cos\left(2\tau + \frac{\pi}{2}\right) d\tau \cdot \delta_{-1}(t) = \frac{3}{5} \left(1 - e^{-t} \left(\cos(2t) + \frac{1}{2} \sin(2t) \right) \delta_{-1}(t) \right.$$

$$y_{tot}(t) = \frac{3}{5} \left(1 - e^{-t} \left(\cos(2t) + \frac{1}{2} \sin(2t) \right) \delta_{-1}(t) \right.$$

SOLUZIONE 2 NEL DOMINIO s

$$\mathcal{L}[y''(t) + 2y'(t) + 5y(t)] = \mathcal{L}[3\delta_{-1}(t)]$$

$$\mathcal{L}[y''(t)] + 2\mathcal{L}[y'(t)] + 5\mathcal{L}[y(t)] = 3\mathcal{L}[u(t)]$$

$$\left(s^2 Y(s) - sY(0) + Y'(0)\right) + 2\left(sY(s) - Y(0)\right) + 5Y(s) = 3U(s)$$

$$(s^2+2s+5)Y(s) = 3U(s)$$

$$Y(s) = \frac{3}{s^2+2s+5}\cdot U(s) = \underbrace{0}_{Y_L(s)} + \underbrace{\frac{3}{s^2+2s+5}\cdot U(s)}_{Y_F(s)}$$

$$Y(s) = \frac{1}{s}\cdot\frac{3}{s^2+2s+5}$$

$$\frac{3}{s(s^2+2s+5)} = \frac{R_1}{s} + \frac{R_2}{s-(-1+2i)} + \frac{R_2^*}{s+(-1-2i)}$$

$$R_1 = \lim_{s\to 0} \cancel{s}\cdot\frac{3}{\cancel{s}(s^2+2s+5)} = \frac{3}{5}$$

$$R_2 = \lim_{s\to -1+2i} \cancel{(s-(-1+2i))}\frac{3}{s(\cancel{s^2+2s+5})} = \lim_{s\to -1+2i}\frac{3}{s^2+s(-1-2i)} =$$

$$= -\frac{3}{10} + \frac{3}{20}i \qquad R_2^* = -\frac{3}{10} - \frac{3}{20}i$$

$$y(t) = \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \left(-\frac{3}{10} + \frac{3}{20}i\right)\mathcal{L}^{-1}\left[\frac{1}{s-(-1+2i)}\right] +$$

$$+\left(-\frac{3}{10} - \frac{3}{20}i\right)\mathcal{L}^{-1}\left[\frac{1}{s+(-1-2i)}\right]$$

$$y(t) = \left[\frac{3}{5} + \left(-\frac{3}{10} + \frac{3}{20}i\right)e^{(-1+2i)t} + \left(-\frac{3}{10} - \frac{3}{20}i\right)e^{(-1-2i)t}\right]\delta_{-1}(t) =$$

1ª FORMA

$$\overset{2^a}{=}\left[\frac{3}{5}-\frac{3}{5}e^{-t}\cos(2t)-\frac{3}{10}e^{-t}\sin(2t)\right]\delta_{-1}(t)=$$

$$=\left[\frac{3}{5}-\frac{3}{5}e^{-t}\left[\cos(2t)+\frac{1}{2}\sin(2t)\right]\right]\delta_{-1}(t)$$

RISULTATI OTTENUTI CON I DUE PROCEDIMENTI COINCIDONO ✓

PASSAGGIO MODELLO IU → VS

RICORDIAMO CHE $a_n y^n(t)+\ldots+a_1 y'(t)+a_0 y(t)=b_0 u(t)+\ldots+b_m u^m(t)$.

VOGLIAMO RISCRIVERLO IN FORMA $\begin{cases} \dot{x}(t)=A\cdot x(t)+B\cdot u(t) \\ y(t)=C\cdot x(t)+D\cdot u(t) \end{cases}$, DOVE

$A\in\mathcal{M}_{n\times m}$, $B\in\mathcal{M}_{n\times r}$, $C\in\mathcal{M}_{p\times n}$, $D\in\mathcal{M}_{p\times r}$.

CASO 1: $n=m$] $a_0 y(t)=b_0 u(t) \rightarrow y(t)=\frac{b_0}{a_0}u(t)$ $\begin{cases} A=0 \quad B=0 \\ C=0 \quad D=\frac{b_0}{a_0} \end{cases}$

CASO 2: $n>0, m=0$] $a_n y^n(t)+\ldots+a_1 y'(t)+a_0 y(t)=b_0 u(t)$

$y^n(t)=-\frac{a_0}{a_n}y(t)-\frac{a_1}{a_0}y'(t)-\ldots-\frac{a_{n-1}}{a_n}y^{n-1}(t)+\frac{b_0}{a_0}u(t)$. DEFINIAMO

LE SEGUENTI VARIABILI DI STATO:

$$\begin{cases} x_1(t)=y(t) \\ x_2(t)=y'(t)=\dot{x}_1(t) \\ \vdots \\ x_n(t)=y^{n-1}(t)=\dot{x}_{n-1}(t) \\ \dot{x}_n(t)=y^n(t)=-\frac{a_0}{a_n}x_1(t)-\frac{a_1}{a_0}x_2(t)-\ldots-\frac{a_{n-1}}{a_n}x_n(t)+\frac{b_0}{a_0}u(t) \end{cases}$$

Riscrivendo il sistema in forma compatta,

$$\begin{cases} |\dot{x}(t)| = \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & & -\frac{a_{n-1}}{a_n} \end{vmatrix} \cdot |x(t)| + \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{b_0}{a_n} \end{vmatrix} \cdot |u(t)| \\ |y(t)| = | 1 \quad 0 \quad \dots \quad 0 | \cdot |x(t)| + | 0 | \cdot |u(t)| \end{cases}$$

Può essere utile definire la **matrice di realizzazione**

$$M = \left|\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right|,$$ che nel caso in esame è definita

$$M = \left|\begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & & 1 & 0 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & & -\frac{a_{n-1}}{a_n} & \frac{b_0}{a_n} \\ \hline 1 & 0 & \dots & & 0 & 0 \end{array}\right|$$

Esercizio: Scrivere la matrice di realizzazione del sistema assegnato, in modello ingresso-uscita, definito come

$$2y''(t) + 6y'(t) + y(t) = 5u(t)$$

DALLA **DEFINIZIONE DELLE VARIABILI DI STATO** VISTA POCO FA,

$$|\dot{x}(t)| = \begin{vmatrix} 0 & 1 \\ -\frac{1}{2} & -3 \end{vmatrix} \cdot |x(t)| + \begin{vmatrix} 0 \\ \frac{5}{2} \end{vmatrix} \cdot |u(t)|$$

$$|y(t)| = |\, 1 \quad 0 \,| \cdot |x(t)| + |\, 0 \,| \cdot |u(t)|$$

$$\Rightarrow M = \left|\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{2} & -3 & \frac{5}{2} \\ \hline 1 & 0 & 0 \end{array}\right|$$

CASO 3: $n \geq m > 0$] $a_n y^n(t) + \ldots + a_1 y'(t) + a_0 y(t) = b_0 u(t) + \ldots + b_m u^m(t)$

LA SCRITTURA IN VARIABILI DI STATO DI UN SISTEMA DI QUESTO TIPO È POSSIBILE GRAZIE ALLA SEGUENTE **PROPOSIZIONE: SE DUE SEGNALI $u(t)$ E $y(t)$ SONO LEGATI DA UNA RELAZIONE DEL MODELLO INGRESSO-USCITA, LINEARE, STAZIONARIO, ALLORA ESISTE UN SEGNALE $s(t)$ CHE SODDISFA LE ESPRESSIONI:**

$$\begin{cases} u(t) = a_n s^n(t) + \ldots + a_1 s'(t) + a_0 s(t) \\ y(t) = b_m s^m(t) + \ldots + b_1 s'(t) + b_0 s(t) \end{cases}$$

3A: $n = m$]
$$\begin{cases} x_1(t) = s(t) \\ x_2(t) = s'(t) = \dot{x}_1(t) \\ \vdots \\ x_n(t) = s^{n-1}(t) = \dot{x}_{n-1}(t) \\ \dot{x}_n(t) = s^n(t) \end{cases}$$

$$\rightarrow U(t) = a_n \dot{x}_n(t) + \ldots + a_1 x_2(t) + a_0 x_1(t)$$

$$\Rightarrow \dot{x}_n(t) = -\frac{a_0}{a_n} x_1(t) - \frac{a_1}{a_n} x_2(t) - \ldots - \frac{a_{n-1}}{a_n} x_n(t) + \frac{1}{a_n} U(t)$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \vdots \\ \dot{x}_n(t) = -\frac{a_0}{a_n} x_1(t) - \frac{a_1}{a_n} x_2(t) - \ldots - \frac{a_{n-1}}{a_n} x_n(t) + \frac{1}{a_n} U(t) \\ y(t)? \end{cases}$$

$n=m$

$$y(t) = b_m s^m(t) + \ldots + b_1 s'(t) + b_0 s(t) = b_n s^n(t) + \ldots + b_1 s'(t) + b_0 s(t)$$

$$y(t) = b_n \dot{x}_n(t) + \ldots + b_1 \dot{x}_1(t) + b_0 x_1(t) = b_0 x_1(t) + b_1 x_2(t) + \ldots +$$

$$+ b_n \left[-\frac{a_0}{a_n} x_1(t) - \ldots - \frac{a_{n-1}}{a_n} x_n(t) + \frac{1}{a_n} U(t) \right] = \left(b_0 - \frac{a_0}{a_n} b_n \right) x_1(t) +$$

$$+ \left(b_1 - \frac{a_1}{a_n} b_n \right) x_2(t) + \ldots + \left(b_{n-1} - \frac{a_{n-1}}{a_n} b_n \right) x_n(t) + \frac{b_n}{a_n} U(t)$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \vdots \\ \dot{x}_n(t) = -\frac{a_0}{a_n} x_1(t) - \frac{a_1}{a_n} x_2(t) - \ldots - \frac{a_{n-1}}{a_n} x_n(t) + \frac{1}{a_n} U(t) \\ y(t) = \left(b_0 - \frac{a_0}{a_n} b_n \right) x_1(t) + \ldots + \left(b_{n-1} - \frac{a_{n-1}}{a_n} b_n \right) x_n(t) + \frac{b_n}{a_n} U(t) \end{cases}$$

In matriciale, $\mathcal{M} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & & 1 & 0 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & & -\frac{a_{n-1}}{a_n} & 1/a_n \\ \hline (b_0 - \frac{b_m}{a_n} a_0) & \dots & & & (b_{n-1} - \frac{b_m}{a_n} a_{n-1}) & b_m/a_n \end{array}\right]$

3B: $n > m > 0$] Con procedimenti analoghi, otterremo

$$\mathcal{M} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & & 1 & 0 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & & -\frac{a_{n-1}}{a_n} & 1/a_n \\ \hline b_0 & b_1 & \dots\ b_m & 0 \dots & 0 & 0 \end{array}\right]$$

Esercizio: scrivere in variabili di stato il sistema

$$4y'''(t) + y''(t) + 4y'(t) + 2y(t) = 2u(t) + 3u'(t) + 2u''(t) + 3u'''(t)$$

Ci troviamo nel caso 3A, poiché $n = m$. In questo caso, $n = m = 3$. Applicando la $\mathcal{M}$ generale e scomponendola, otterremo

$$
\begin{cases}
\begin{vmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -1 & -\frac{1}{4} \end{vmatrix} \cdot \begin{vmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ \frac{1}{4} \end{vmatrix} \cdot u(t) \\
\begin{vmatrix} y(t) \end{vmatrix} = \begin{vmatrix} \overset{2 \cdot 2 \cdot \frac{4}{2}}{-2} & -5 & -4 \end{vmatrix} \cdot \begin{vmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{vmatrix} + \begin{vmatrix} 2 \end{vmatrix} \cdot u(t)
\end{cases}
$$

MODELLO VS: ANALISI NEL DOMINIO DEL TEMPO

DATO IL SISTEMA $\Sigma: \begin{cases} x'(t) = A \cdot x(t) + B \cdot u(t) \\ y(t) = C \cdot x(t) + D \cdot u(t) \end{cases}$, VOGLIAMO

DETERMINARE LE USCITE NOTE, $\forall t \geq t_0$, L'INGRESSO $u(t)$ E LO

STATO INIZIALE $x(t_0) = \begin{vmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{vmatrix}$.

IL SISTEMA È SCRITTO IN FORMA IMPLICITA. PER PASSARE ALLA FORMA

ESPLICITA, PONIAMO CHE $\begin{cases} x(t) = f(t, t_0, x(t_0), u(\cdot)) \\ y(t) = g(t, t_0, x(t_0), u(\cdot)) \end{cases}$. IN ANALOGIA

A QUANTO VISTO IN IU, $\begin{cases} x(t) = x_L(t) + x_F(t) \\ y(t) = y_L(t) + x_F(t) \end{cases}$

$x_L(t) = f(t, t_0, x(t_0), 0)$ $\quad y_L(t) = g(t, t_0, x(t_0), 0)$

$x_F(t) = f(t, t_0, 0, u(\cdot))$ $\quad y_F(t) = g(t, t_0, 0, u(\cdot))$

$$\begin{cases} x^{\bullet}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$x^{\bullet}(t) = x_L^{\bullet}(t) + x_F^{\bullet}(t) = A(x_L(t) + x_F(t)) + Bu(t)$$

$$\Longrightarrow \begin{cases} x_L^{\bullet}(t) = A \cdot x_L(t) \\ x_F^{\bullet}(t) = A \cdot x_F(t) + B \cdot u(t) \end{cases}$$

$$y_L(t) = y_L(t) + y_F(t) = C \cdot (x_L(t) + x_F(t)) + D \cdot u(t)$$

$$\Longrightarrow \begin{cases} y_L(t) = C \cdot x_L(t) \\ y_F(t) = C \cdot x_F(t) + D \cdot u(t) \end{cases}.$$

QUINDI, DATO

$$\Sigma : \begin{cases} x^{\bullet}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

, LO SI PUÒ SCOMPORRE IN

$$\Sigma_L + \Sigma_F \qquad \Sigma_L : \begin{cases} x_L^{\bullet}(t) = A \cdot x_L(t) \\ y(t) = C \cdot x_L(t) \end{cases}$$

$$\Sigma_F : \begin{cases} x_F^{\bullet}(t) = A \cdot x_F(t) + B \cdot u(t) \\ y_F(t) = C \cdot x_F(t) + D \cdot u(t) \end{cases}$$

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A PARTIRE DAL SISTEMA $\Sigma_L : \begin{cases} x_L^{\bullet}(t) = A \cdot x_L(t) \\ y(t) = C \cdot x_L(t) \end{cases}$, SE $A, C \in \mathcal{M}_{1\times1}$ POSSIAMO, INDICANDO $a = A$ E $c = C$, SCRIVERE LA LEGGE

$$\begin{cases} x_L(t) = e^{a(t-t_0)} \cdot x(t_0) \\ y_L(t) = c \cdot e^{a(t-t_0)} \cdot x(t_0) \end{cases}$$ Per deduzione, si verifica che

questa legge vale anche per casi generali e abbiamo

$$\begin{cases} x_L(t) = e^{A(t-t_0)} \cdot x(t_0) \\ y_L(t) = C \cdot e^{a(t-t_0)} \cdot x(t_0) \end{cases}$$, dove $e^{A(t-t_0)} = \Phi(t, t_0)$ è

detta matrice di transizione dello stato, e $C \cdot e^{A(t-t_0)} = \Psi(t, t_0)$.

In forma compatta, $\begin{cases} x_L(t) = \Phi(t, t_0) \cdot x(t_0) \\ y_L(t) = \Psi(t, t_0) \cdot x(t_0) \end{cases}$

Dettagli sulla matrice di transizione

Data una matrice quadrata $A \in \mathcal{M}_{n \times n}$, $e^{A} = \sum_{k=0}^{+\infty} \frac{A^k}{k!} = \frac{A^0}{0!} + \frac{A^1}{1!} + \ldots$ $= I_n + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \ldots$ Moltiplicando A per un parametro t, otteniamo che $e^{At} = \sum_{k=0}^{+\infty} \frac{(At)^k}{k!} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \cdot A^k$ e, ponendo $t - t_0$ al posto di t, otteniamo la matrice $e^{A(t-t_0)} = \sum_{k=0}^{+\infty} \frac{[A(t-t_0)]^k}{k!} =$ $= \sum_{k=0}^{+\infty} \frac{(t-t_0)^k}{k!} \cdot A^k$.

Proprietà principali:

- $\frac{d}{dt}[e^{At}] = \frac{d}{dt}\left[\sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k\right] = \frac{d}{dt}\left[I_n + A + A^2 \cdot \frac{t^2}{2!} + A^3 \cdot \frac{t^3}{3!} + \ldots\right]$

$$= 0_n + A + \frac{2}{2!} A^2 t + \frac{3}{3!} A^3 \cdot t^2 + \ldots = A\left[I_n + \frac{2}{2!} A \cdot t + \frac{3}{3!} A^2 t^2 + \ldots\right]$$

$$= A\left[I_n + A \cdot t + \frac{1}{2} A^2 t^2 + \ldots\right] = A \cdot \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k = A \cdot e^{At}$$

- SE A È UNA MATRICE DIAGONALE, ALLORA LO È ANCHE e^{At}.

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{a_{11}t} & 0 & \cdots & 0 \\ 0 & e^{a_{22}t} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & e^{a_{nn}t} \end{bmatrix}$$

- DATI $A, B \in \mathcal{M}_{n \times n}$, IN GENERALE $e^{At} \cdot e^{Bt} \neq e^{Bt} \cdot e^{At}$. QUESTO VALE SOLO SE A E B COMMUTANO E, IN PARTICOLARE, AVREMO CHE

$$e^{At} \cdot e^{Bt} = e^{(A+B)t}$$

- INVERSA $\rightarrow$ $(e^{At})^{-1} = e^{-At}$ $\quad e^{At} \cdot (e^{At})^{-1} = e^{At} \cdot e^{-At} = e^{(A-A)t} = e^{0 \cdot t} = I_n$

$$\begin{cases} x_L(t) = e^{A(t-t_0)} \cdot x(t_0) \\ y_L(t) = C \cdot e^{A(t-t_0)} \cdot x(t_0) \end{cases} \rightarrow \begin{cases} x_L(t) = \sum_{k=0}^{+\infty} \frac{(A \cdot t)^k}{k!} \cdot x(t_0) \\ y_L(t) = C \cdot \sum_{k=0}^{+\infty} \frac{(A \cdot t)^k}{k!} \cdot x(t_0) \end{cases}$$

SE LA MATRICE A È DIAGONALE, LA MATRICE È FACILMENTE CALCOLABILE, ALTRIMENTI OCCORRE PASSARE PER LO SVILUPPO DI SYLVESTER

$$e^{At} = \sum_{k=0}^{+\infty} \frac{(A \cdot t)^k}{k!} = \sum_{i=0}^{n-1} \underbrace{\beta_i(t)}_{\text{COEFFICIENTE SCALARE}} \cdot A^i = \beta_0(t) \cdot A^0 + \beta_1(t) \cdot A^1 + \ldots + \beta_{n-1}(t) A^{n-1} =$$

$$\beta_0(t) \cdot I_n + \beta_1(t) \cdot A + \ldots + \beta_{n-1}(t) \cdot A^{n-1}.$$

OCCORRE QUINDI CALCOLARE I COEFFICIENTI $\beta_i(t)$, LA CUI PROCEDURA VARIA CASO PER CASO:

1. AUTOVALORI DI A $\lambda \mid \lambda \in \mathbb{R}, \gamma = 1 \ \forall \lambda$
2. AUTOVALORI DI A $\lambda \mid \lambda \in \mathbb{R}, \gamma > 1$ PER ALMENO UN λ
3. AUTOVALORI DI A $\lambda \mid \lambda, \lambda^* \in \mathbb{C}$ PER ALMENO UN λ

RICORDANDO CHE $P(\lambda) = \det(\lambda I_n - A) = \det \begin{vmatrix} \lambda - a_{1,1} & \dots & -a_{1,n} \\ \vdots & & \vdots \\ -a_{n,1} & \dots & \lambda - a_{n,n} \end{vmatrix}$

1. $A \in \mathbb{R}_{n\times n} \rightarrow \lambda_i \in \mathbb{R}, \gamma_i = 1, i = 1,\dots,n \Rightarrow \lambda_i \neq \lambda_j \ \forall i,j \mid i \neq j$

$$\begin{cases} \lambda_1 \ (\gamma_1 = 1) \rightarrow \lambda_1^0 \beta_0(t) + \lambda_1^1 \beta_1(t) + \dots + \lambda_1^{n-1} \beta_{n-1}(t) = e^{\lambda_1 t} \\ \lambda_2 \ (\gamma_2 = 1) \rightarrow \lambda_2^0 \beta_0(t) + \lambda_2^1 \beta_1(t) + \dots + \lambda_2^{n-1} \beta_{n-1}(t) = e^{\lambda_2 t} \\ \vdots \\ \lambda_n \ (\gamma_n = 1) \rightarrow \lambda_n^0 \beta_0(t) + \lambda_n^1 \beta_1(t) + \dots + \lambda_n^{n-1} \beta_{n-1}(t) = e^{\lambda_n t} \end{cases}$$

$$\begin{cases} \beta_0(t) + \lambda_1 \beta_1(t) + \dots + \lambda_1^{n-1} \beta_{n-1}(t) = e^{\lambda_1 t} \\ \beta_0(t) + \lambda_2 \beta_1(t) + \dots + \lambda_2^{n-1} \beta_{n-1}(t) = e^{\lambda_2 t} \\ \vdots \\ \beta_0(t) + \lambda_n \beta_1(t) + \dots + \lambda_n^{n-1} \beta_{n-1}(t) = e^{\lambda_n t} \end{cases}$$

IN FORMA MATRICIALE, $\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \\ 1 & \lambda_n & \lambda_n^2 & \dots \lambda_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} \beta_0(t) \\ \beta_1(t) \\ \vdots \\ \beta_{n-1}(t) \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{vmatrix}$

ESERCIZIO: DATA $A = \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix}$, CALCOLARE e^{At} USANDO SYLVESTER

$P(\lambda) = \det(\lambda I_2 - A) = \det \begin{vmatrix} \lambda+1 & -1 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda+1)(\lambda+2) = 0$

$$\begin{cases} \lambda_1 = -1, & \gamma_1 = 1 \\ \lambda_2 = -2, & \gamma_2 = 1 \end{cases} \rightarrow e^{At} = \sum_{i=0}^{2-1} \beta_i(t) \cdot A^i = \beta_0(t) \cdot I_2 + \beta_1(t) \cdot A$$

$$\begin{cases} \beta_0(t) + \lambda_1 \beta_1(t) = e^{\lambda_1 t} \\ \beta_0(t) + \lambda_2 \beta_1(t) = e^{\lambda_2 t} \end{cases} \Rightarrow \begin{cases} \beta_0(t) - \beta_1(t) = e^{-t} \\ \beta_0(t) - 2\beta_1(t) = e^{-2t} \end{cases}$$

$$\Rightarrow \begin{cases} \beta_0(t) = 2e^{-t} - e^{-2t} \\ \beta_1(t) = e^{-t} - e^{-2t} \end{cases} \quad e^{At} = (2e^{-t} - e^{-2t}) \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} +$$

$$+ (e^{-t} - e^{-2t}) \cdot \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{vmatrix}$$

2. $A \in \mathbb{R}_{n \times n} \rightarrow \lambda_i \in \mathbb{R}, \gamma_i \geqslant 1, i = 1, \dots, r$

$$\{(\lambda_1, \gamma_1 = 1); (\lambda_2, \gamma_2 = 1); \dots; (\lambda_i, \gamma_i > 1); \dots; (\lambda_r, \gamma_r = 1)\} \in \mathbb{R}$$

$$\begin{cases} \lambda_1 (\gamma_1 = 1) \rightarrow \lambda_1^0 \beta_0(t) + \lambda_1^1 \beta_1(t) + \dots + \lambda_1^{n-1} \beta_{n-1}(t) = e^{\lambda_1 t} \\ \lambda_2 (\gamma_2 = 1) \rightarrow \lambda_2^0 \beta_0(t) + \lambda_2^1 \beta_1(t) + \dots + \lambda_2^{n-1} \beta_{n-1}(t) = e^{\lambda_2 t} \\ \vdots \\ \lambda_i (\gamma_i > 1) \rightarrow \lambda_i^0 \beta_0(t) + \lambda_i^1 \beta_1(t) + \dots + \lambda_i^{n-1} \beta_{n-1}(t) = e^{\lambda_i t} \\ \frac{d}{d\lambda_i} [\lambda_i^0 \beta_0(t) + \lambda_i^1 \beta_1(t) + \dots + \lambda_i^{n-1} \beta_{n-1}(t)] = \frac{d}{d\lambda_i} [e^{\lambda_i t}] \\ \vdots \\ \frac{d^{\gamma_i - 1}}{d\lambda_i^{\gamma_i - 1}} [\lambda_i^0 \beta_0(t) + \lambda_i^1 \beta_1(t) + \dots + \lambda_i^{n-1} \beta_{n-1}(t)] = \frac{d^{\gamma_i - 1}}{d\lambda_i^{\gamma_i - 1}} [e^{\lambda_i t}] \\ \vdots \\ \lambda_r (\gamma_1 = 1) \rightarrow \lambda_r^0 \beta_0(t) + \lambda_r^1 \beta_1(t) + \dots + \lambda_r^{n-1} \beta_{n-1}(t) = e^{\lambda_r t} \end{cases}$$

ESERCIZIO: DATA $A = \begin{vmatrix} 3 & 0 & 1 \\ 2 & -1 & 3/2 \\ 0 & 0 & 3 \end{vmatrix}$, CALCOLARE e^{At} USANDO SYLVESTER

$$P(\lambda) = \det(\lambda I_3 - A) = \det \begin{vmatrix} \lambda-3 & 0 & -1 \\ -2 & \lambda+1 & -3/2 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda+1)(\lambda-3)^2 = 0$$

$$\begin{cases} \lambda_1 = -1, \; r_1 = 1 \\ \lambda_2 = 3, \; r_2 = 2 \end{cases} \longrightarrow e^{At} = \sum_{i=0}^{3-1} \beta_i(t) \cdot A^i = \beta_0(t) \cdot I_3 + \beta_1(t) \cdot A + \beta_2(t) \cdot A^2$$

$$\begin{array}{l} \lambda_1 \{ \\ \\ \lambda_2 \{ \end{array} \begin{cases} \beta_0(t) + \lambda_1 \beta_1(t) + \lambda_1^2 \beta_2(t) = e^{\lambda_1 t} \\ \beta_0(t) + \lambda_2 \beta_1(t) + \lambda_2^2 \beta_2(t) = e^{\lambda_2 t} \\ 0 + \beta_1(t) + 2\lambda_2 \beta_2(t) = t e^{\lambda_2 t} \end{cases}$$

$$\begin{cases} \beta_0(t) - \beta_1(t) + \beta_2(t) = e^{-t} \\ \beta_0(t) + 3\beta_1(t) + 9\beta_2(t) = e^{3t} \\ \beta_1(t) + 6\beta_2(t) = t e^{3t} \end{cases} \Longrightarrow$$

$$\begin{cases} \beta_0(t) = \frac{1}{16}(7e^{3t} - 12te^{3t} - 9e^{-t}) \\ \beta_1(t) = \frac{1}{8}(3e^{3t} - 4te^{3t} - 3e^{-t}) \\ \beta_2(t) = \frac{1}{10}(-3e^{3t} + 4te^{3t} + e^{-t}) \end{cases}$$

$$e^{At} = \begin{vmatrix} e^{3t} & 0 & te^{3t} \\ \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} & e^{-t} & \frac{1}{4}e^{3t} + \frac{1}{2}te^{3t} - \frac{1}{4}e^{-t} \\ 0 & 0 & e^{3t} \end{vmatrix}$$

3. $A \in \mathbb{R}_{n\times n} \to \lambda_i \in \mathbb{R};\ \lambda_i, \lambda_i^* \in \mathbb{C}\ (\gamma_i = \gamma_i^*)$

$$\{(\lambda_1, \gamma_1 = 1); \dots; (\lambda_{i-1}, \gamma_{i-1} = 1)\} \in \mathbb{R}$$

$$\{(\lambda_i, \gamma_i = 1); (\lambda_i^*, \gamma_i^* = \gamma_i = 1)\} \in \mathbb{C}$$

$$\{(\lambda_{i+1}, \gamma_{i+1} = 1); \dots; (\lambda_r, \gamma_r = 1)\} \in \mathbb{R}$$

$$\begin{cases}
\lambda_1\ (\gamma_1 = 1) \to \lambda_1^0 \beta_0(t) + \lambda_1^1 \beta_1(t) + \dots + \lambda_1^{n-1} \beta_{n-1}(t) = e^{\lambda_1 t} \\
\vdots \\
\lambda_{i-1}\ (\gamma_{i-1} = 1) \to \lambda_{i-1}^0 \beta_0(t) + \lambda_{i-1}^1 \beta_1(t) + \dots + \lambda_{i-1}^{n-1} \beta_{n-1}(t) = e^{\lambda_{i-1} t} \\
\lambda_i\ (\gamma_i = 1) \to \lambda_i^0 \beta_0(t) + \lambda_i^1 \beta_1(t) + \dots + \lambda_i^{n-1} \beta_{n-1}(t) = e^{\lambda_i t} \\
\lambda_i^*\ (\gamma_i^* = 1) \to \lambda_i^{*0} \beta_0(t) + \lambda_i^{*1} \beta_1(t) + \dots + \lambda_i^{*\,n-1} \beta_{n-1}(t) = e^{\lambda_i t} \\
\vdots \\
\lambda_r\ (\gamma_r = 1) \to \lambda_r^0 \beta_0(t) + \lambda_r^1 \beta_1(t) + \dots + \lambda_r^{n-1} \beta_{n-1}(t) = e^{\lambda_r t}
\end{cases}$$

$$+\ -\ \begin{cases}
\lambda_i^0 \beta_0(t) + \lambda_i^1 \beta_1(t) + \dots + \lambda_i^{n-1} \beta_{n-1}(t) = e^{\lambda_i t} \\
\lambda_i^{*0} \beta_0(t) + \lambda_i^{*1} \beta_1(t) + \dots + \lambda_i^{*\,n-1} \beta_{n-1}(t) = e^{\lambda_i t}
\end{cases}$$

$+$

$$2\beta_0(t) + (\lambda_i + \lambda_i^*)\beta_1(t) + \dots + (\lambda_i^{n-1} + \lambda_i^{*\,n-1})\beta_{n-1}(t) = e^{\lambda_i t} + e^{\lambda_i^* t}$$

$$2\beta_0(t) + 2\cdot \mathrm{Re}[\lambda_i]\beta_1(t) + \dots + 2\mathrm{Re}[\lambda_i^{n-1}]\beta_{n-1}(t) = e^{\lambda_i t} + e^{\lambda_i^* t}$$

$$\beta_0(t) + Re[\lambda_i]\beta_1(t) + \ldots + Re[\lambda_i^{n-1}]\beta_{n-1}(t) = \frac{e^{\lambda_i t} + e^{\lambda_i^* t}}{2}$$

$$\frac{e^{\lambda_i t} + e^{\lambda_i^* t}}{2} = \frac{e^{(Re[\lambda_i]t + i Im[\lambda_i]t)} + e^{(Re[\lambda_i]t + i Im[-\lambda_i]t)}}{2}$$

$$= e^{Re[\lambda_i]t} \cdot \frac{e^{i Im[\lambda_i]t} + e^{-i Im[\lambda_i]t}}{2} = e^{Re[\lambda_i]t} \cdot$$

$$\cdot \frac{\cos(Im[\lambda_i]t) + \cancel{i \sin(Im[\lambda_i]t)} + \cos(Im[\lambda_i]t) - \cancel{i \sin(Im[\lambda_i]t)}}{2}$$

$$= e^{Re[\lambda_i]t} \cos(Im[\lambda_i]t)$$

$$\Rightarrow \beta_0(t) + \lambda_i \beta_1(t) + \ldots + \lambda_i^{n-1}\beta_{n-1}(t) = e^{Re[\lambda_i]t} \cos(Im[\lambda_i]t)$$

$$(\lambda_i - \lambda_i^*)\beta_1(t) + \ldots + (\lambda_i^{n-1} - \lambda_i^{*n-1})\beta_{n-1}(t) = e^{\lambda_i t} - e^{\lambda_i^* t}$$

$$2i Im[\lambda_i]\beta_1(t) + \ldots + 2i Im[\lambda_i^{n-1}]\beta_{n-1}(t) = e^{\lambda_i t} - e^{\lambda_i^* t}$$

$$i Im[\lambda_i]\beta_1(t) + \ldots + i Im[\lambda_i^{n-1}]\beta_{n-1}(t) = \frac{e^{\lambda_i t} - e^{\lambda_i^* t}}{2}$$

$$\frac{e^{\lambda_i t} - e^{\lambda_i^* t}}{2} = \frac{e^{(Re[\lambda_i]t + i Im[\lambda_i]t)} - e^{(Re[\lambda_i]t + i Im[\lambda_i]t)}}{2} =$$

$$= e^{Re[\lambda_i]t} \cdot \frac{e^{i Im[\lambda_i]t} - e^{-i Im[\lambda_i]t}}{2i} = e^{Re[\lambda_i]t} \cdot$$

$$\cdot\ \frac{\cancel{\cos(Im[\lambda_i]t)} + i\sin(Im[\lambda_i]t) - \cancel{\cos(Im[\lambda_i]t)} + i\sin(Im[\lambda_i]t)}{2i}$$

$$= e^{Re[\lambda_i]t} \cdot \sin(Im[\lambda_i]t)$$

$$\Rightarrow \beta_0(t) + \lambda_i^* \beta_1(t) + \ldots + \lambda_i^{*\,n-1} \beta_{n-1}(t) = e^{Re[\lambda_i]t} \sin(Im[\lambda_i]t)$$

ESERCIZIO: DATO UN SISTEMA LINEARE E STAZIONARIO

$$\begin{cases} x^{\circ}(t) = A \cdot x(t) + B \cdot u(t) \\ y(t) = C \cdot x(t) + D \cdot u(t) \end{cases}, \quad A = \begin{vmatrix} \alpha & \omega \\ \omega & -\alpha \end{vmatrix}, \quad C = |1 \quad \alpha|, \quad x(t_0) = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

CALCOLARE $x_L(t)$ E $y_L(t)$ SAPENDO CHE $t_0 = 0$

$$\begin{cases} x_L(t) = e^{A(t-t_0)} \cdot x(t_0) = e^{A \cdot t} \cdot x(0) \\ y_L(t) = C \cdot e^{A(t-t_0)} \cdot x(t_0) = C \cdot x_L(t) \end{cases}$$

$$e^{At} = \sum_{i=0}^{1} \beta_i(t) \cdot A^i = \beta_0(t) \cdot I_2 + \beta_1(t) \cdot A$$

$$P(\lambda) = \det(\lambda I_2 - A) = \lambda^2 - \alpha^2 - \omega^2 = 0 \quad \begin{matrix} \lambda = \alpha + i\omega & \nu = 1 \\ \lambda^* = \alpha - i\omega & \nu^* = 1 \end{matrix}$$

$$\begin{cases} \lambda^0 \beta_0(t) + \lambda_1^1 \beta_1(t) = e^{\lambda t} \\ \lambda^{*0} \beta_0(t) + \lambda^{*1} \beta_1(t) = e^{\lambda^* t} \end{cases}$$

$$\begin{cases} \beta_0(t) + Re[\lambda]\beta_1(t) = e^{Re[\lambda]t} \cos(Im[\lambda]t) \\ \qquad Im[\lambda]\beta_1(t) = e^{Re[\lambda]t} \sin(Im[\lambda]t) \end{cases}$$

$$\longrightarrow \begin{cases} \beta_0(t) = e^{Re[\lambda]t} \cos(Im[\lambda]t) - \frac{Re[\lambda]}{Im[\lambda]} e^{Re[\lambda]t} \sin(Im[\lambda]t) \\ \beta_1(t) = \frac{1}{Im[\lambda]} e^{Re[\lambda]t} \sin(Im[\lambda]t) \end{cases}$$

$$e^{At} = \beta_0(t)\cdot I_2 + \beta_1(t)\cdot A =$$

$$= \begin{vmatrix} \beta_0(t)+Re[\lambda_2]\beta_1(t) & Im[\lambda_2]\beta_1(t) \\ -Im[\lambda_2]\beta_1(t) & \beta_0(t)+Re[\lambda_2]\beta_1(t) \end{vmatrix}$$

$$x_L(t) = e^{At}\cdot x(0) = \begin{vmatrix} e^{Re[\lambda_2]t}\sin(Im[\lambda_2]t) \\ e^{Re[\lambda_2]t}\cos(Im[\lambda_2]t) \end{vmatrix}$$

$$y_L(t) = C\cdot x_L(t) = e^{Re[\lambda_2]t}\sin(Im[\lambda_2]t)$$

RISPOSTA FORZATA

$$\Sigma_F : \begin{cases} \dot{x}_F(t) = A\cdot x_F(t) + B\cdot u(t) \\ y_F(t) = C\cdot x_F(t) + D\cdot u(t) \end{cases}$$

. LA RISOLUZIONE PASSA ATTRAVERSO LA SEGUENTE PROPOSIZIONE: LA SOLUZIONE DELL'EQUAZIONE DIFFERENZIALE DI STATO $\dot{x}_F(t) = A\cdot x_F(t) + B\cdot u(t)$ È DATA DALLA FORMULA $x_F(t) = \int_{t_0}^{t} e^{A(t-\tau)}\cdot B\cdot u(\tau)\,d\tau \quad \forall t \geq t_0$ DIMOSTRAZIONE

$$\dot{x}_F(t) = \frac{d}{dt}[x_F(t)] = \frac{d}{dt}\left[\int_{t_0}^{t} e^{A(t-\tau)}\cdot B\cdot u(\tau)\,d\tau\right]$$

$$\frac{d}{dt}\left[\int_{t_0}^{t} f(t,\tau)\,d\tau\right] = \int_{t_0}^{t}\frac{d}{dt}\left[\int_{t_0}^{t} f(t,\tau)\right]d\tau = f(t,\tau)$$