IN THE FIRST DIAGRAM BELOW, ASSUME f IS A FUNCTION AND R IS AN EQUIVALENCE RELATION ON A. PROVE THAT THE RELATION $f^*(R) := fRf^{op}$ IS AN EQUIVALENCE RELATION ON B. CONVERSELEY, ASSUME, AS IN THE

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, dashed, "f^*(R)"'] & A \arrow[d, "R"] \\ B \arrow[r, "f"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] \arrow[d, "S"'] & A \arrow[d, dashed, "f_*(S)"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

SECOND DIAGRAM, THAT WE ARE GIVEN AN EQUIVALENCE RELATION S ON B CONTAINING THE KERNEL PAIR OF f. PROVE THAT THE RELATION $f_*(S) = f^{op}Sf$ IS AN EQUIVALENCE RELATION ON A IF AND ONLY IF AND ONLY IF f IS SURJECTIVE

A RELATION IS EQUIVALENT IF IT IS:

- REFLEXIVE
- SYMMETRIC
- TRANSITIVE

$a f R f^{op} b \Leftrightarrow \exists c,d\,(a f c \wedge c R d \wedge d f^{op} b) \Leftrightarrow \exists c,d\,(a f c \wedge c R d \wedge b f d)$

$\Leftrightarrow f(a) = c,\ f(b) = d \Leftrightarrow f(a) R f(b)$

R) $f(a) R f(a)$ ✓ (R IS REFLEXIVE) AND $f(a) = c = d$

$\Rightarrow \exists c,d\,(a f c \wedge c R c \wedge a f c) \Leftrightarrow \exists c,d\,(a f c \wedge c R c \wedge c f^{op} a)$

$\Rightarrow a f R f^{op} a$ ✓

S) $f(a) R f(b)$ R IS SYMMETRIC $\Rightarrow f(b) R f(a)$. HENCE,

$a f R f^{op} b \Rightarrow b f R f^{op} a$ ✓

T) $f(a) R f(b) \wedge f(b) R f(c)$ R TRANSITIVE $\Rightarrow f(a) R f(c)$.

HENCE, $a f R f^{op} b \wedge b f R f^{op} c \Rightarrow a f R f^{op} b$ ✓

$a f^{op} S f b \Leftrightarrow \exists c,d\,(a f^{op} c \wedge c S d \wedge d f b) \Leftrightarrow \exists c,d\,(c f a \wedge c S d \wedge d f b)$

$\Leftrightarrow \exists c,d\,(f(c)=a \wedge c S d \wedge f(d)=b)$.

S IS AN EQUIVALENCE RELATION, f IS SURJECTIVE

R) S IS REFLEXIVE $\Rightarrow cSc \Rightarrow \exists c\,(f(c)=a \wedge c S d \wedge f(d)=a)$

f IS SURJECTIVE $\Rightarrow |f^{-1}(a)| \geq 1 \Rightarrow \exists c\,(f(c)=a \wedge c S d \wedge f(d)=a)$

$\Leftrightarrow \exists c\,(c f a \wedge c S d \wedge d f a) \Leftrightarrow \exists c\,(a f^{op} c \wedge c S d \wedge d f a)$

$\Leftrightarrow a f^{op} S f a$ ✓

S) $a f^{op} S f b \wedge b f^{op} S f a \Rightarrow a=b$?

$c=h \quad d=k$

$\exists c,d,h,k\,(\underline{f(c)=a} \wedge c S d \wedge \underline{f(d)=b} \wedge \underline{f(h)=a} \wedge hSk \wedge$

$c=d$

$\wedge\, \underline{f(k)=b}) \Leftrightarrow \exists c,d\,(f(c)=a \wedge cSd \wedge f(d)=b \wedge dSc)$

$\Leftrightarrow (f(c)=a \wedge cSc \wedge f(c)=b) \Rightarrow a=b$ ✓

T) $a f^{op} S f b \wedge b f^{op} S f c \Rightarrow a f^{op} S f c$?

$n=h$

$\exists m,n,h,k\,(f(m)=a \wedge mSn \wedge \underline{f(n)=b} \wedge \underline{f(h)=b} \wedge \underline{h}Sk \wedge f(k)=c)$

mSk

$\Rightarrow a f^{op} S f c$ ✓

$\Leftrightarrow \exists m,h,k\,(f(m)=a \wedge \underline{mSn} \wedge \underline{nSk} \wedge f(k)=c) \Leftrightarrow (a f^{op} m \wedge mSk \wedge k f c)$

GIVEN $A=\{1,2,3\}$ AND $B=\{1,2,3,4,5\}$, CONSIDER THE RELATION $E: A \to A$ AND THE FUNCTION $p: B \to A$ DEFINED BY THE MATRICES

$$M=\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \qquad N=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}.$$

PROVE THAT E IS AN EQUIVALENCE RELATION ON A AND $E^* = pEp^{op}$ IS AN EQUIVALENCE RELATION ON B. MORE GENERALLY, PROVE THAT FOR ANY A AND B, IF E IS AN EQUIVALENCE RELATION AND p A SURJECTIVE FUNCTION AS IN THE DIAGRAM BELOW, THEN $E^* = pEp^{op}$ IS ALSO AN EQUIVALENCE RELATION. PROVE THAT, IF p_E AND p_{E^*} ARE THE PROJECTION ON THE QUOTIENT SETS OF E AND E^* RESPECTIVELY, THEN p INDUCES A BIJECTION q BETWEEN THE QUOTIENT, SO THAT THERE EXIST A UNIQUE FUNCTION q SUCH THAT $pp_E = p_{E^*}q$ AND q IS BIJECTIVE

$$\begin{tikzcd}
B \arrow[r, "p"] \arrow[d, "E^*"'] & A \arrow[d, "E"] \\
B \arrow[r, "p"] \arrow[d, "p_{E^*}"'] & A \arrow[d, "p_E"] \\
B/E^* \arrow[r, dashed, "q"] & A/E
\end{tikzcd}$$