8) (LEFT) DISTRIBUTIVITY

$r(f+g) = rf + rg$?

$(r(f+g))(x) = r(f+g)(x) =$ // PROPERTY OF THE GIVEN SET

$= r(f(x)+g(x)) =$

$= rf(x) + rg(x) =$ // $f, g \in K$, WHERE DISTRIBUTIVITY IS VALID

$= (rf)(x) + (rg)(x)$

OBSERVE THAT $\mathbb{R}^n = \{f : [n] \to \mathbb{R}\}$, WHERE $[n] = \{1, \dots, n\}$

SUPPOSE T IS AN ALGEBRAIC THEORY AND M, N ARE T-MODELS. PROVE THAT $M \times N$

IS A T-MODEL WITH:

$[[f]]_{M \times N}((a_1, b_1), \ldots, (a_n, b_n)) = ([[f]]_M(a_1, \ldots, a_n), [[f]]_N(b_1, \ldots, b_n))$

SPECIAL CASE SUPPOSE T IS THE THEORY OF VECTOR SPACES AND M, N ARE

T-MODELS $M = \mathbb{R}^3, N = \mathbb{R}^2$

$M \times N = \{(a, b) \mid a \in \mathbb{R}^3, b \in \mathbb{R}^2\} = \{(a_1, a_2, a_3), (b_1, b_2)\} = \{(a_1, a_2, a_3, b_1, b_2)\}$

SUPPOSE $f = +$

$(a, b) \underset{M \times N}{+} (a', b') = (a \underset{M}{+} a', b \underset{N}{+} b') = ((a_1 + a_1', a_2 + a_2', a_3 + a_3'), ((b_1 + b_1'), (b_2 + b_2')))$

HOW TO PROVE $M \times N$ IS A T-MODEL? IF $S = t$ IS AN AXIOM. THEN $M \times N \models S = t$

SUPPOSE $v : X \to M \times N$

SUPPOSE $v(x) = (x_1, x_2) \in M \times N$. DEFINE $\begin{array}{c} X \xrightarrow{v_1} M \\ x \mapsto x_1 \end{array}$ AND $\begin{array}{c} X \xrightarrow{v_2} N \\ x \mapsto x_2 \end{array}$. IN

OTHER WORDS, $v_1 = P_M \circ v$ AND $v_2 = P_N \circ v$.

THEN, $v(S) = (v_1(S), v_2(S)) = (v_1(t), v_2(t))$. THUS, $v \models S = t \Rightarrow M \times N \models T$