EXTRA: LET L BE THE PROPOSITIONAL LANGUAGE GENERATED BY VARIABLES x AND y AND LET $\psi = x \leftrightarrow y \in L$. DETERMINE THE ORDER INDUCED BY THE RELATION DEFINED ON L BY THE FORMULA

$\varphi_1 \prec \varphi_2 \iff \models \psi \wedge \varphi_1 \to \varphi_2$

OBSERVE: $\varphi_1 \prec \varphi_2 \iff \models \psi \wedge \varphi_1 \to \varphi_2$

// HILBERT CALCULUS $\iff \models \psi \to (\varphi_1 \to \varphi_2)$

$$\iff \psi \models \varphi_1 \to \varphi_2$$

VALUATION

$$\iff \forall v, (v(\psi) = 1 \Rightarrow v(\varphi_1 \to \varphi_2) = 1)$$

$\mathcal{V} : \{ v \in \Omega^V \mid v(\psi) = 1 \}$

$$\iff \{\forall v, v(\psi) = 1 \Rightarrow v(\varphi_1) \leq v(\varphi_2)\}$$

$$\iff \varphi_1 \models_{\mathcal{V}} \varphi_2$$

$\models_{\mathcal{V}}$ IS REFLEXIVE AND TRANSITIVE.

IT IS NOT ANTISYMMETRIC. IN FACT, $x \equiv \neg\neg x \Rightarrow x \models_{\mathcal{V}} \neg\neg x$ AND

$\neg\neg x \models_{\mathcal{V}} x \Rightarrow (x \prec \neg\neg x) \wedge (\neg\neg x \prec x)$

WE CAN FIND THE INDUCED ORDER

$$\begin{tikzcd} L \arrow[r, "L"] \arrow[d, "p"'] & L \arrow[d, "p"] \\ L/E \arrow[r, dashed, "\leq"] & L/E \end{tikzcd} \qquad [[\varphi_1]] \leq [[\varphi_2]] \iff \varphi_1 \prec \varphi_2$$

$\varphi_1 E \varphi_2 \iff (\varphi_1 \prec \varphi_2) \wedge (\varphi_2 \prec \varphi_1) \iff \varphi_1 \models_{\mathcal{V}} \varphi_2 \wedge \varphi_2 \models_{\mathcal{V}} \varphi_1 \iff \varphi_1 \equiv_{\mathcal{V}} \varphi_2$

$V = \{ v \in \Omega^X \mid v(\psi) = 1 \}$ $\psi = X \leftrightarrow Y$

	X	Y	$[\bot]$	$[X]$	$[\neg X]$	$[\top]$
v_1	0	0	0	0	1	1
v_2	1	1	0	1	0	1

POSSIBLE TRUTH TABLE FOR L/E

$\Longrightarrow L/E = \{[\bot], [X], [\neg X], [\top]\}$

$[\varphi] \leq [\psi] \iff \varphi \preceq \psi \iff \varphi \models_V \psi$

$$\begin{tikzcd}
 & {[\top]} & \\
{[X]} \arrow[ur, dash] & & {[\neg X]} \arrow[ul, dash] \\
 & {[\bot]} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{vmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{vmatrix}$$

FIRST ORDER LOGIC

THE LANGUAGE OF THE NATURAL LANGUAGE IS THE FIRST ORDER LANGUAGE WITH EQUALITY L GENERATED BY THE SIGNATURE $\{0, 1, +, \cdot, \leq\}$. TRANSLATE THE FOLLOWING ASSERTIONS INTO FORMULAS OF L

1) ANY TWO NONZERO NATURAL NUMBERS HAVE A UNIQUE GREATEST COMMON DIVISOR

- $x \neq 0 := \neg(x = 0)$
- x DIVIDES y $\quad \frac{x}{y} := \exists z(xz = y)$! DIVISION IS NOT IN THE SET
- z IS A COMMON DIVISOR OF x AND y: $\frac{z}{x} \wedge \frac{z}{y}$
- z IS A GREATEST COMMON DIVISOR OF x AND y:

$$G(x,y,z) = \underbrace{\frac{z}{x} \wedge \frac{z}{y}}_{\text{COMMON DIVISOR}} \wedge \forall w\left(\frac{w}{x} \wedge \frac{w}{y} \to \frac{w}{z}\right)$$

- $\forall xy((x \neq 0) \wedge (y \neq 0) \to \exists z(G(x,y,z) \wedge \forall w(G(x,y,w) \to$
$\to w = z))$

2) THE DIVISION THEOREM WHICH STATES THAT GIVEN TWO NATURAL NUMBERS, THE DIVISOR, IS NONZERO, WE CAN OBTAIN A QUOTIENT AND A REMINDER OF THE DIVISION AND THAT THESE TWO NUMBERS ARE UNIQUE IF WE REQUEST THAT THE REMAINDER BE STRICTLY LESS THAN THE DIVISOR