LET L BE A PROPOSITIONAL LANGUAGE AND $\psi \in L$ A FIXED FORMULA

1) PROVE THAT THE RELATION DEFINED ON L BY THE FORMULA

$\varphi_1 \prec \varphi_2 \Leftrightarrow \models \psi \wedge \varphi_1 \to \varphi_2$ IS REFLEXIVE AND TRANSITIVE

2) PROVE THAT $\varphi_1 \sim \varphi_2 \Leftrightarrow \models \varphi_1 \leftrightarrow \psi \wedge \varphi_2$ IS AN EQUIVALENCE RELATION

3) PROVE THAT $\prec$ INDUCES AN ORDER RELATION ON $L/\sim$ BY THE FORMULA $[\varphi_1] \leq [\varphi_2] \Leftrightarrow \varphi_1 \prec \varphi_2$

4) DETERMINE $L/\sim$ AND THE INDUCED ORDER WHEN L IS THE PROPOSITIONAL LANGUAGE GENERATED BY TWO VARIABLES x, y AND $\psi = x \leftrightarrow y$

$$v(\psi \wedge \varphi_1) = 1 \Rightarrow v(\varphi_2)$$

$v(\psi) \wedge v(\varphi_1) \leq v(\varphi_2)$ $\quad v(\psi) = 0$ IS NOT ACCEPTABLE

$$v(\psi) = 1 \Rightarrow \begin{cases} v(\varphi_1) = 0 \Rightarrow v(\varphi_2) = 0 \\ v(\varphi_1) = 1 \Rightarrow v(\varphi_2) = 1 \end{cases} \Rightarrow \varphi_1 \models \varphi_2$$

1) • REFLEXIVE $\quad \varphi_1 \models \varphi_1$ ✓

• TRANSITIVE $\quad \varphi_1 \models \varphi_2 \wedge \varphi_2 \models \varphi_3 \Rightarrow \varphi_1 \models \varphi_3$ ✓

$$v(\varphi_1) \leq v(\varphi_2) \wedge v(\varphi_2) \leq v(\varphi_3) \Rightarrow v(\varphi_1) \leq v(\varphi_3)$$

2) $$\psi \wedge \varphi_1 \leftrightarrow \psi \wedge \varphi_2 \equiv (\psi \wedge \varphi_1 \to \psi \wedge \varphi_2) \wedge (\psi \wedge \varphi_2 \to \psi \wedge \varphi_1)$$

$$\equiv (\psi \wedge \varphi_1 \to \varphi_2) \wedge (\psi \wedge \varphi_2 \to \varphi_1)$$

$$\equiv (\varphi_1 \prec \varphi_2) \wedge (\varphi_2 \prec \varphi_1)$$

• SIMMETRY $\quad \varphi_1 \models \varphi_2 \wedge \varphi_2 \models \varphi_1 \Rightarrow \varphi_1 = \varphi_2$ ✓

3) $\sim$ IS THE EQUIVALENCE RELATION INDUCED BY $\preceq$. HENCE, $\leq$ IS THE ORDER INDUCED BY $\preceq$ ON THE QUOTIENT SET

4)

x	y	$[\varphi \wedge \bot]$	$[\varphi \wedge (\neg x \wedge \neg y)]$	$[\varphi \wedge (x \wedge y)]$	$[\varphi \wedge \top]$
0	0	0	1	0	1
0	1	0	0	0	0
1	0	0	0	0	0
1	1	0	0	1	1

! $x \leftrightarrow y \Rightarrow v = 0$ WHEN $x \neq y$

$L/\sim\ = \{[\bot], [\neg x \wedge \neg y], [x \wedge y], [\top]\}$

LET $f : B \to A$ BE A FUNCTION AND R AN ORDER RELATION ON A. PROVE THAT IF $f = pm$ IS THE CANONICAL FACTORIZATION OF f, THEN THE ORDER RELATION S ON B INDUCED BY INVERSE IMAGE $f^*(R)$ OF R ALONG f IS $m^*(R)$. USE THIS TO COMPUTE S WHEN

$$M_f = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad M_R = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$S = f^*(R)$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "S"] & A \arrow[d, "R"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

$f^*(R) = f R f^{op} = pm R m^{op} p^{op}$

$= p^*(m^*(R)) \Rightarrow S = m R m^{op}$

$m = p^{op} f$

$$k = f f^{op} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix} \quad P = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$p^{op} = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$