$\varphi(x) = 3 \rightarrow \quad 3 = y^2 + z^2$, IMPOSSIBLE $\mathbb{N} \Rightarrow$ NON VALID

FORMALLY,

$$v \models \exists y z\,(x = y^2 + z^2) \Leftrightarrow \exists a \in \mathbb{N} \mid v[a/y] \models \exists z\,(x = y^2 + z^2)$$

$$\Leftrightarrow \exists a, b \in \mathbb{N} \mid v[a/y, b/z] \models x = y^2 + z^2$$

$$\Leftrightarrow \exists a, b \in \mathbb{N} \mid v[a/y, b/z](x) = v[a/y, b/z](y^2 + z^2)$$

$$\Leftrightarrow \exists a, b \in \mathbb{N} \mid v[a/y, b/z](x) = v[a/y, b/z](y^2) + v[a/y, b/z](z^2)$$

$$\Leftrightarrow \exists a, b \in \mathbb{N} \mid v(x) = a^2 + b^2$$

FOR $v(x) = 3 \in \mathbb{N}$, $\nexists\, a, b \mid v(x) = a^2 + b^2$

3) $\forall y\,(\frac{y}{x} \rightarrow (y = 1) \vee (y = x))$ (PRIME NUMBER)

$v(x)$ $\qquad v \models \varphi$?

$$v \models \forall y\,(\tfrac{y}{x} \rightarrow (y = 1) \vee (y = x)) \Leftrightarrow \forall a \in \mathbb{N},\ v[a/y] \models \tfrac{y}{x} \rightarrow (y = 1) \vee (y = x)$$

$$\Leftrightarrow \forall a \in \mathbb{N},\ v[a/y](y) = v[\tfrac{y}{x} \rightarrow (y = 1) \vee (y = x)]$$

$$\Leftrightarrow \forall a \in \mathbb{N},\ v[a/y](y) = v[\tfrac{y}{x} \rightarrow y = 1] \vee v[\tfrac{y}{x} \rightarrow y = x]$$

$$\Leftrightarrow \forall a \in \mathbb{N},\ \frac{a}{v(x)} \rightarrow a = 1 \vee a = v(x)$$

$v(x) = 3$ ✓ $\Rightarrow$ SATISFIABLE

$v(x) = 4$ ✗ IN FACT $a = 1, 2, 4 \Rightarrow$ NON VALID

PROVE THAT THE FORMULA $\forall x(Rx \vee Sx \to \forall x.Rx \vee \forall x.Sx)$ IS SATISFIABLE BUT NOT VALID

$v: X \to M$ R, S UNARY RELATION SYMBOLS

$(M, [[R]], [[S]])$ $[[R]], [[S]] \subseteq M$

β-FORMULA

$v \models \forall x(Rx \vee Sx \to \forall x.Rx \vee \forall x.Sx)$ $\varphi \to \psi$ $\neg\varphi, \psi$

$\Leftrightarrow v \models \neg \forall x(Rx \vee Sx)$ OR $v \models \forall x.Rx \vee x.Sx$

$\Leftrightarrow v \models \neg\forall(Rx \vee Sx)$ OR $v \models x.Rx$ OR $v \models x.Sx$

WE HAVE THE CHOICE OF SATISFYING ANY OF THE 3 CONDITIONS

- $v \models \neg\forall x(Rx \vee Sx) \Leftrightarrow v \not\models \forall x(Rx \vee Sx)$

$\Leftrightarrow \exists a \in M \mid v[a/x] \not\models Rx \vee Sx$

$\Leftrightarrow \exists a \in M \mid v[a/x] \not\models Rx$ AND $v[a/x] = Sx$

$\Leftrightarrow \exists a \in M \mid v[a/x](x) \notin [[R]]$ AND $v[a/x](x) \in [[S]]$

$\Leftrightarrow \exists a \in M \mid a \notin [[R]]$ AND $a \notin [[S]]$

EXAMPLE: $M = \{a, b\}$ $[[R]] = \{b\}, [[S]] = \emptyset \to$ IT IS SATISFIABLE

- $v \models x.Rx \Leftrightarrow \forall a \in M, v[\frac{a}{x}] \models Rx$

$\Leftrightarrow \forall a \in M, v[\frac{a}{x}](x) \in [[R]]$

$\Leftrightarrow \forall a \in M, a \in [[R]]$ ✓

IF $M = \{a\}. [[R]] = \{a\}$. S HAS NO RESTRICTION $\to [[S]] = \emptyset \to$ IT IS SATISFIABLE

- $v \models X.Rx \iff \forall \alpha \in M, v[\frac{\alpha}{x}] \models Sx$

$\iff \forall \alpha \in M, v[\frac{\alpha}{x}](x) \in [[S]]$

$\iff \forall \alpha \in M, \alpha \in [[S]]$ ✓

IF $M = \{a, b\}$, $[[S]] = \{a, b\}$. R HAS NO RESTRICTION $\rightarrow$ $[[R]] = \emptyset$ $\rightarrow$ IT IS SATISFIABLE

φ IS SATISFIABLE

USING SEMANTIC TREE

$\forall x(Rx \vee Sx) \rightarrow \forall x Rx \vee \forall x Sx$

β-FORMULA $\swarrow$ $\searrow$

$\neg \forall x(Rx \vee Sx)$ $\quad$ $\forall x Rx \vee \forall x Sx$

δ-FORMULA

$\exists x. \neg(Rx \vee Sx)$ $\quad$ $\forall x Rx$ $\quad$ $\forall x Sx$

$\neg(Rx \vee Sx)$ $\quad$ $Ra, \forall x Rx$ $\quad$ $Sa, \forall x Sx$

α-FORMULA

$\neg Ra, \neg Sa$ $\quad$ $Ra, R_b, \forall x Rx$

- IS SATISFIABLE IF $[[M]] = \{a\}$, $[[R]] = [[S]] = \emptyset$
- $[[M]] = \{a\}$, $[[R]] = \{a\}$, $[[S]] = \emptyset$
- $[[M]] = \{a\}$, $[[R]] = \emptyset$, $[[S]] = \{a\}$