COMBINAZIONE LINEARE DEI MODI NATURALI PER RADICI COMPLESSE

$$P(\lambda)=0 \rightarrow a_n\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0=0$$

$$\lambda_\ell \in \mathbb{C}\ (r_\ell) \rightarrow e^{\lambda_\ell t},\ te^{\lambda_\ell t},\dots,\ t^{r_\ell-1}e^{\lambda_\ell t} \rightarrow C_{r_\ell}(t)$$

$$\lambda_\ell^* \in \mathbb{C}\ (r_\ell^*=r_\ell) \rightarrow e^{\lambda_\ell^* t},\ te^{\lambda_\ell^* t},\dots,\ t^{r_\ell^*-1}e^{\lambda_\ell^* t} \rightarrow C_{r_\ell^*}(t)$$

$$\rightarrow C_{r_\ell}(t)+C_{r_\ell^*}(t)=\sum_{k=0}^{r_\ell-1} t^k\left[A_{\ell k}e^{\lambda_\ell t}+A_{\ell k}^* e^{\lambda_\ell^* t}\right]$$

SUPPONENDO CHE CI SIANO 2S RADICI COMPLESSE CONIUGATE, SI HA:

$$P(\lambda=0) \rightarrow C_{r_1}(t)=e^{\lambda_1 t},\ te^{\lambda_1 t},\dots,\ t^{r_1-1}e^{\lambda_1 t}$$

$$C_{r_1^*}(t)=e^{\lambda_1^* t},\ te^{\lambda_1^* t},\dots,\ t^{r_1^*-1}e^{\lambda_1^* t}$$

$$\vdots$$

$$C_{r_s}=e^{\lambda_s t},\ te^{\lambda_s t},\dots,\ t^{r_s-1}e^{\lambda_s t}$$

$$C_{r_s^*}=e^{\lambda_s^* t},\ te^{\lambda_s^* t},\dots,\ t^{r_s^*-1}e^{\lambda_s^* t}$$

$$\Rightarrow C_S(t)=\sum_{\ell=1}^{s}\left[C_{r_\ell}(t)+C_{r_\ell^*}(t)\right]=\sum_{\ell=1}^{s}\sum_{k=0}^{r_\ell-1} t^k\left[A_{\ell k}e^{\lambda_\ell t}+A_{\ell k}^* e^{\lambda_\ell^* t}\right]$$

COMBINAZIONE LINEARE DEI MODI NATURALI PER RADICI REALI E COMPLESSE

$$P(\lambda)=0 \rightarrow a_n\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0=0$$

AVREMO UN TOTALE DI R RADICI REALI E 2S RADICI COMPLESSE.

DUNQUE $C(t)=C_R(t)+C_S(t)=\left[\sum_{i=1}^{R}\left(\sum_{k=0}^{r_i-1} A_{ik}t^k e^{\lambda_i t}\right)\right]+$

$$+\sum_{\ell=1}^{s}\sum_{k=0}^{r_\ell-1} t^k\left[A_{\ell k}e^{\lambda_\ell t}+A_{\ell k}^* e^{\lambda_\ell^* t}\right]=$$

$$= \left[\sum_{i=1}^{R} \left(\sum_{k=0}^{r_i-1} A_{i,k} t^k e^{\lambda_i t}\right)\right] + \left[\sum_{i=R+1}^{R+S} \left(\sum_{k=0}^{r_i-1} t^k \left(A_{i,m} t^k \left[A_{i,k} e^{\lambda_i t} + A_{i,k} e^{\bar{\lambda}_i t}\right]\right)\right)\right]$$

1^a FORMA

TEOREMA: UNA FUNZIONE REALE $\overline{y(t)}$ È SOLUZIONE DELL'EQUAZIONE OMOGENEA ASSOCIATA $\Longleftrightarrow$ $\overline{y(t)}$ È UNA COMBINAZIONE LINEARE DEI MODI NATURALI ASSOCIATI ALLE RADICI DEL POLINOMIO CARATTERISTICO DEL SISTEMA CONSIDERATO

$\Rightarrow$ $p(\lambda)=0 \rightarrow$ "r" RADICI $\lambda_1, \dots, \lambda_r$ REALI E/O COMPLESSE

$$c(t) = \sum_{i=1}^{r} c_{\lambda_i}(t) = c_{\lambda_1}(t) + \dots c_{\lambda_r}(t) = \left(\sum_{k=0}^{r_1-1} A_{1,k} e^{\lambda_1 t}\right) + \dots + \left(\sum_{k=0}^{r_r-1} A_{1,k} e^{\lambda_r t}\right)$$

$$= \sum_{i=1}^{r} \sum_{k=0}^{r_i-1} A_{i,k} t^k e^{\lambda_i t} = \overline{y(t)}$$

QUESTA RESTITUISCE INFINITE SOLUZIONI. OCCORRE INTRODURRE UNA

PROPOSIZIONE: LA RISPOSTA LIBERA DEL SISTEMA CONSIDERATO È UNA PARTICOLARE COMBINAZIONE LINEARE DEI MODI NATURALI ASSOCIATI ALLE RADICI DEL POLINOMIO CARATTERISTICO $p(\lambda)$

$\Rightarrow$ $a_n y_e^n(t) + \dots + a_1 y'(t) + a_0 y(t) = 0$

$$y_e(t) = c(t)\Big|_{y(t_0), y'(t_0), \dots, y^{n-1}(t_0)} \Rightarrow \begin{cases} y_e(t_0) = c(t_0) \\ y_e'(t_0) = c'(t_0) \\ y^{n-1}(t_0) = c^{n-1}(t_0) \end{cases}$$

ESERCIZIO: DETERMINARE LA RISPOSTA LIBERA DEL SISTEMA

$$y'''(t) + 8y''(t) + 21y'(t) + 18y(t) = 3u(t) + 5u'(t) - 7u''(t)$$

SAPENDO CHE $y(0)=2,\ y'(0)=1,\ y''(0)=-20$

$$P(\lambda)=\lambda^3+8\lambda^2+21\lambda+18=0$$

$$\begin{array}{c|ccc|c} & 1 & 8 & 21 & 18 \\ -2 & & -2 & -12 & -18 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$$(\lambda+2)(\lambda^2+6\lambda+9)=0$$

$$(\lambda+2)(\lambda+3)^2=0$$

- $\lambda_1=-2 \quad r_1=1 \rightarrow e^{-2t}$
- $\lambda_2=-3 \quad r_2=2 \rightarrow e^{-3t},\ te^{-3t}$

$$c(t)=A_1e^{-2t}+A_{2,1}e^{-3t}+A_{2,2}te^{-3t}$$

$$c'(t)=-2A_1e^{-2t}-3A_{2,1}e^{-3t}+A_{2,2}e^{-3t}-3A_{2,2}te^{-3t}$$

$$c''(t)=4A_1e^{-2t}+9A_{2,1}e^{-3t}-6A_{2,2}e^{-3t}+9A_{2,2}te^{-3t}$$

$$\begin{array}{l} c(0) \\ c'(0) \\ c''(0) \end{array}\begin{cases} A_1+A_{2,1}=2 \\ -2A_1-3A_{2,1}+A_{2,2}=1 \\ 4A_1+9A_{2,1}-6A_{2,2}=-20 \end{cases} \qquad \begin{cases} A_1=2-A_{2,1} \\ -4-A_{2,1}+A_{2,2}=1 \\ 8+5A_{2,1}-6A_{2,2}=-20 \end{cases}$$

$$\begin{cases} A_1=2-A_{2,1} \\ 8+5A_{2,1}-6-6A_{2,1}-24=-20 \\ A_{2,2}=1+A_{2,1}+4 \end{cases} \qquad \begin{cases} A_1=4 \\ A_{2,1}=-2 \\ A_{2,2}=3 \end{cases}$$

$$\Rightarrow y_e(t)=4e^{-2t}-2e^{-3t}+3te^{-3t}$$

ESERCIZIO: TROVARE LA $y_e(t)$ DEL SEGUENTE SISTEMA DATE LE CONDIZIONI INIZIALI

$$\begin{cases} y'''(t)+2y''(t)+5y'(t)=u(t)-3u'(t) \\ y(0)=3 \\ y'(0)=2 \\ y''(0)=1 \end{cases}$$

$$A_{1,0}=\mathrm{Re}[A_{1,0}]+i\,\mathrm{Im}[A_{1,0}]$$

$$A_{1,0}^{*}=\mathrm{Re}[A_{1,0}]-i\,\mathrm{Im}[A_{1,0}]$$

$$P(\lambda)=\lambda^3+2\lambda^2+5\lambda$$

$$P(\lambda)=0 \qquad \lambda(\underbrace{\lambda^2+2\lambda+5}_{})=0 \quad \Delta=-16 \quad \begin{cases} \lambda_1=0 \\ \lambda_2=-1+2i \quad \lambda_2^{*}=-1-2i \end{cases}$$

$$\rightarrow c(t)=A_0+A_{1,0}e^{(-1+2i)t}+A_{1,0}^{*}e^{(-1-2i)t}$$

$$c'(t)=(-1+2i)A_{1,0}e^{(-1+2i)t}+(-1-2i)A_{1,0}^{*}e^{(-1-2i)t}$$

$$c''(t)=(-1+2i)^2A_{1,0}e^{(-1+2i)t}+(-1-2i)^2A_{1,0}^{*}e^{(-1-2i)t}$$

$$=(-3-4i)A_{1,0}e^{(-1+2i)t}+(-3+4i)A_{1,0}^{*}e^{(-1-2i)t}$$

$$\begin{cases} A_0+A_{1,0}+A_{1,0}^{*}=3 \\ (-1+2i)A_{1,0}+(-1-2i)A_{1,0}^{*}=2 \\ (-3-4i)A_{1,0}+(-3+4i)A_{1,0}^{*}=1 \end{cases}$$

$A_{1,0} = v_{1,0} + i\, u_{1,0}$ $\qquad$ DA CUI $\qquad v = -\frac{1}{2}$

$A_{1,0}^{*} = v_{1,0} - i\, u_{1,0}$ $\qquad\qquad\qquad u = \frac{1}{4}$

$$\begin{cases} A_0 = 4 \\ A_{1,0} = \frac{1}{2} + \frac{1}{4} i \\ A_{1,0}^{*} = \frac{1}{2} - \frac{1}{4} i \end{cases}$$

$$\Rightarrow y_e(t) = 4 + \left(\frac{1}{2} + \frac{1}{4} i\right) e^{(-1+2i)t} + \left(\frac{1}{2} - \frac{1}{4} i\right) e^{(-1-2i)t}$$

$$= 4 + \left[-e^{-t}\cos(2t) - \frac{1}{2} e^{-t}\sin(2t)\right] \qquad B_{e,k} = 2 \cdot \frac{-1}{2} = -1$$

$$C_{e,k} = -2 \cdot \frac{1}{4} = -\frac{1}{2}$$

$$= 4 + \left[\frac{\sqrt{5}}{2} e^{-t}\cos\left(2t + \arctan\left(\frac{1}{2}\right)\right)\right] \qquad M_{e,k} = 2\sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{\sqrt{5}}{2}$$

$$\Phi_{e,k} = \arctan\left(\frac{-1/2}{-1/4}\right) = \arctan\left(\frac{-1}{-\frac{1}{2}}\right)$$

I MODI NATURALI

SI FACCIAMO DELLE IMPORTANTI CONSIDERAZIONI SUI MODI NATURALI. NE DISTINGUIAMO DUE DIVERSE FORME:

MODI APERIODICI

$\lambda_i \in \mathbb{R}\ (\mu_i)$

$t^i e^{\lambda_i t} \quad i = 0, 1, \dots, \mu_i - 1$

MODI PSEUDOPERIODICI

$\lambda_i \in \mathbb{C}\ (\mu_i) \quad i = 0, 1, \dots, \mu_i - 1$

2^a FORMA $\quad t^i e^{Re[\lambda_i]t} \cos(Im[\lambda_i]t)$

$t^i e^{Re[\lambda_i]t} \sin(Im[\lambda_i]t)$

CONSIDERAZIONI:

① $\cos(Im[\lambda_i]t) = \sin(Im[\lambda_i]t + \pi/2)$ ② $t^i e^{Re[\lambda_i]t}$ FORNISCE COMPORTAMENTO APERIODICO ③ sin, cos FORNISCONO COMPORTAMENTO PERIODICO

I MODI APERIODICI

COME VISTO POCO FA, RIGUARDIAMO IL CASO DI RADICI REALI E QUINDI UN MODO DEL TIPO $m(t)=t^k e^{\lambda t}$, $k=1,\dots,\mu_i-1$

DEFINIAMO LA COSTANTE DI TEMPO DEL MODO NATURALE m COME $\tau_i=-\frac{1}{\lambda_i} \Rightarrow m_i(t)=t^k e^{-\frac{1}{\tau_i}t}$

STUDIAMO, AD ESEMPIO, IL COMPORTAMENTO DI m NEL CASO DI RADICI NEGATIVE $\lambda_i<0 \Rightarrow \tau_i>0$

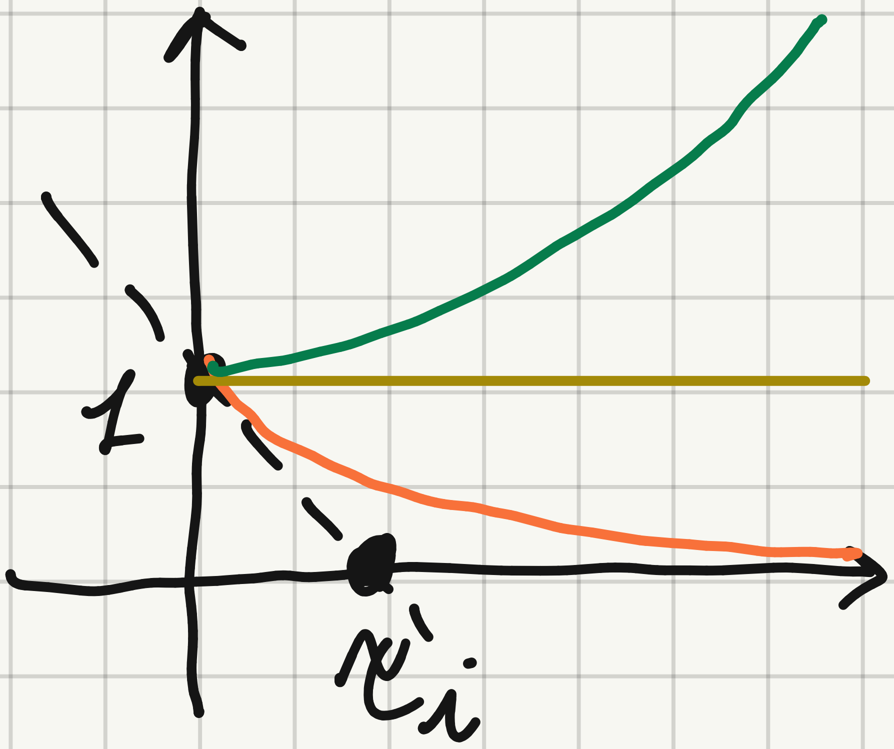
t	$=0$	$=\tau_i$	$=3\tau_i$	...
$e^{-\frac{t}{\tau_i}}$	$=1$	$=0{,}37$	$0{,}05$	...

SI OSSERVA, DUNQUE, CHE COL PASSARE DEL TEMPO IL MODO È SEMPRE MENO INFLUENTE, CIOÈ $m_i(t)$ È UN MODO APERIODICO CHE SI ESAURISCE

MODI APERIODICI ASSOCIATI A RADICI REALI DI MOLTEPLICITÀ $=1$

$m_i(t)=t^{\mu_i-1}e^{\lambda_i t}$ $\quad \mu_i=1 \Rightarrow m_i(t)=e^{\lambda_i t}$

- $\lambda_i<0 \rightarrow$ MODO STABILE (O ASINTOTICAMENTE STABILE)
  - $\rightarrow$ RISPOSTA LIBERA CONVERGENTE
- $\lambda_i=0 \rightarrow$ MODO AL LIMITE DI STABILITÀ (O SEMPLICEMENTE STABILE)
- $\lambda_i>0 \rightarrow$ MODO INSTABILE RISPOSTA LIBERA DIVERGENTE



$$\begin{cases} m_i(t) = e^{\lambda_i t} \Rightarrow m_i'(t) = \lambda_i e^{\lambda_i t} \quad m_i'(0) = \lambda_i \\ h_i(t) = at + b = \lambda_i t + b \Rightarrow h_i(0) = 1 = b \Rightarrow b = 1 \end{cases}$$

$$\rightarrow \lambda_i t + 1 = 0 \Rightarrow t = -\frac{1}{\lambda_i} = \tau_i$$

MODI APERIODICI ASSOCIATI A RADICI REALI DI MOLTEPLICITÀ >1

$$m_i(t) = t^k e^{-\frac{1}{\tau_i} t} \qquad k = 0, 1, \dots, \nu_i - 1$$

STUDIAMO IL COMPORTAMENTO DEL MODO ALL'AUMENTARE DI K

PER $K=0$, ABBIAMO IL CASO VISTO IN PRECEDENZA

PER $K>0$, ABBIAMO $m_i(t) = t^k e^{\lambda_i t}$

- $\lambda_i < 0 \Rightarrow m$ DECRESCE $\Rightarrow$ MODO STABILE
- $\lambda_i > 0 \Rightarrow m$ DIVERGE $\Rightarrow$ MODO INSTABILE

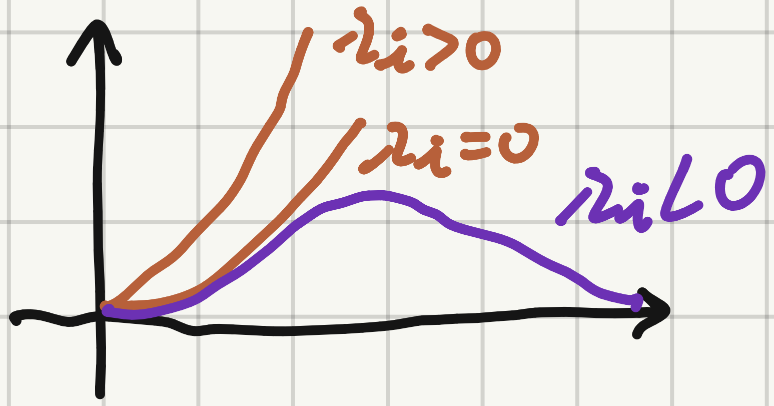


GRAFICO PER UN GENERICO $K \geq 1$

I MODI PSEUDOPERIODICI

RIGUARDA MODI DEL TIPO $m_\ell(t) = \cos(Im[\lambda_\ell]t) \cdot t^k \cdot e^{Re[\lambda_\ell]t}$

E QUINDI RADICI COMPLESSE λ_ℓ E IL RISPETTIVO CONIUGATO λ_ℓ^*

$$\left.\begin{aligned} \lambda_\ell &= Re[\lambda_\ell] + i\, Im[\lambda_\ell] \\ \lambda_\ell^* &= Re[\lambda_\ell] - i\, Im[\lambda_\ell] \end{aligned}\right\} \rightarrow m_\ell(t) = \cos(Im[\lambda_\ell]t) \cdot t^k \cdot e^{Re[\lambda_\ell]t}$$

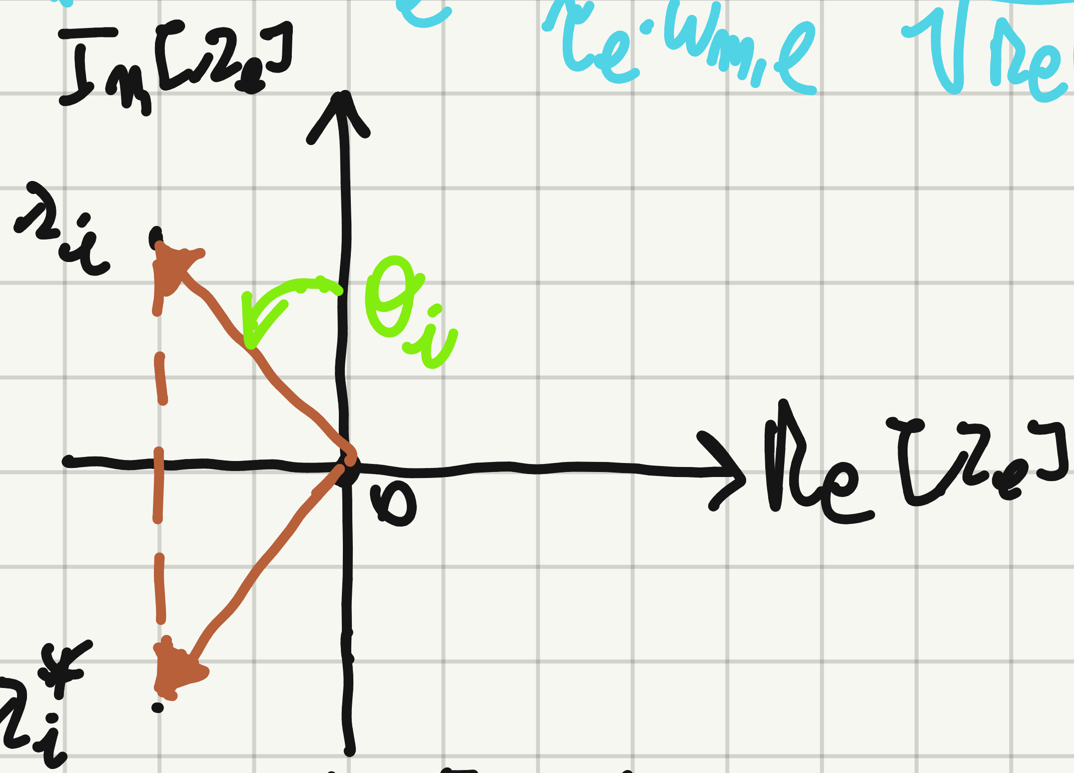
$k = 0, 1, \dots, \tau_\ell - 1$

DEFINIAMO LA COSTANTE DI TEMPO DEL MODO NATURALE m COME $\tau_\ell = -\frac{1}{\lambda_\ell}$, LA PULSAZIONE NATURALE

$$\omega_{n,\ell} = \sqrt{Re[\lambda_\ell]^2 + Im[\lambda_\ell]^2}$$

E IL COEFFICIENTE DI SMORZAMENTO

$$\xi_\ell = \frac{1}{\tau_\ell \cdot \omega_{n,\ell}} = -\frac{Re[\lambda_\ell]}{\sqrt{Re[\lambda_\ell]^2 + Im[\lambda_\ell]^2}}$$



IMMAGINARIO PURO

REALE PURO

$$\sin(\theta_i) = -\frac{Re[\lambda_\ell]}{\sqrt{Re[\lambda_\ell]^2 + Im[\lambda_\ell]^2}} = \xi_\ell \Rightarrow 0 \le \xi_\ell \le 1$$

$\theta_i = \arcsin(\xi_\ell)$ $\quad \omega_{n,\ell} = \sqrt{Re[\lambda_\ell]^2 + Im[\lambda_\ell]^2}$

$\Rightarrow \omega_{n,\ell}^2 = Re[\lambda_\ell]^2 + Im[\lambda_\ell]^2 \Rightarrow Re[\lambda_\ell]^2 = \omega_{n,\ell}^2 - Im[\lambda_\ell]^2$

DA CUI $Re[\lambda_\ell] = \xi_\ell^2 \cdot \omega_{n,\ell}^2$

$$\rightarrow \xi_e^2 \cdot \omega_{n,e}^2 = Im[\lambda_e]^2 - \omega_{n,e}^2 \qquad Im[\lambda_e]^2 = \omega_{n,e}^2(1-\xi_e^2)$$

DA CUI $Im[\lambda_e] = \omega_{n,e} \cdot \sqrt{1-\xi_e^2}$

MODI PSEUDOPERIODICI ASSOCIATI A RADICI COMPLESSE CON MOLTEPLICITÀ =1

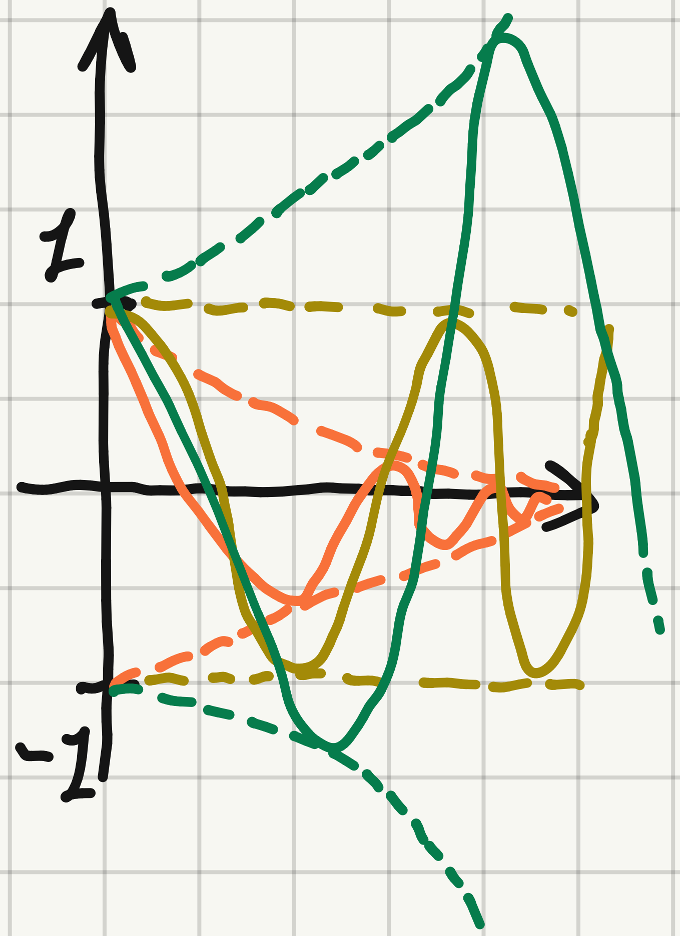
$$\lambda_e \in \mathbb{C} \; (\gamma_e = 1) \rightarrow \lambda_e = Re[\lambda_e] + i\, Im[\lambda_e]$$

$$\lambda_e^* \in \mathbb{C} \; (\gamma_e^* = \gamma_e = 1) \rightarrow \lambda_e^* = Re[\lambda_e] - i\, Im[\lambda_e]$$

$$\begin{cases} m_e(t) = e^{Re[\lambda_i]t} \cos(Im[\lambda_i] \cdot t) \\ m_e^*(t) = e^{Re[\lambda_i]t} \sin(Im[\lambda_i] \cdot t) \end{cases}$$

$$\rightarrow m_e(t) = e^{Re[\lambda_i]t} \cos(Im[\lambda_i]t) = \begin{cases} -e^{Re[\lambda_i]t}, & t = (2k+1) \cdot \frac{\pi}{Im[\lambda_i]} \quad k \in \mathbb{N} \\ e^{Re[\lambda_i]t}, & t = 2k \cdot \frac{\pi}{Im[\lambda_i]} \end{cases}$$

- $Re[\lambda_i] < 0 \Rightarrow$ MODO STABILE
- $Re[\lambda_i] = 0 \Rightarrow$ MODO AL LIMITE DI STABILITÀ
- $Re[\lambda_i] > 0 \Rightarrow$ MODO INSTABILE



MODI PSEUDOPERIODICI ASSOCIATI A RADICI COMPLESSE con MOLTEPLICITÀ = 1

$\lambda_e \in \mathbb{C}\ (\mu_e = 1) \to \lambda_e = Re[\lambda_e] + i\,Im[\lambda_e]$

$\lambda_e^* \in \mathbb{C}\ (\mu_e^* = \mu_e = 1) \to \lambda_e^* = Re[\lambda_e] - i\,Im[\lambda_e]$

$$\begin{cases} m_e(t) = t^k e^{Re[\lambda_i]t} \cos(Im[\lambda_i]\cdot t) \\ m_e^*(t) = t^k e^{Re[\lambda_i]t} \sin(Im[\lambda_i]\cdot t) \end{cases}$$

$$\to m_e(t) = t^k e^{Re[\lambda_i]t} \cos(Im[\lambda_i]t) = \begin{cases} -t^k e^{Re[\lambda_i]t}, & t = (2k+1)\cdot\frac{\pi}{Im[\lambda_i]} \quad k \in \mathbb{N} \\ t^k e^{Re[\lambda_i]t}, & t = 2k\cdot\frac{\pi}{Im[\lambda_i]} \end{cases}$$

- $Re[\lambda_i] < 0 \Rightarrow m$ DECRESCE $\Rightarrow$ MODO STABILE
- $Re[\lambda_i] > 0 \Rightarrow m$ DIVERGE $\Rightarrow$ MODO INSTABILE

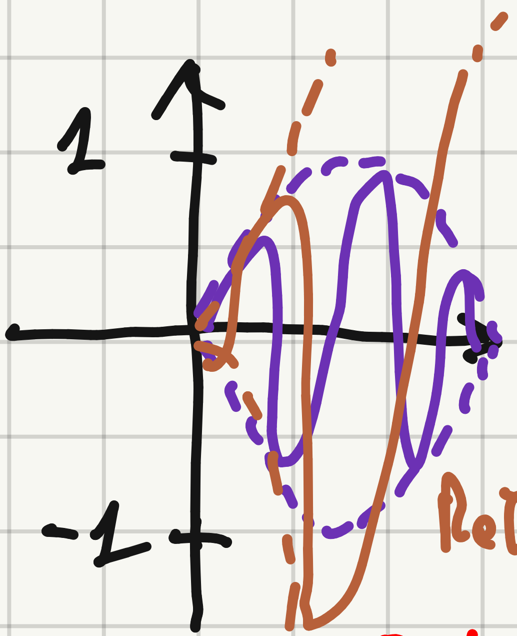


GRAFICO PER UN GENERICO $K \geq 1$

PER $Re[\lambda_i] > \Rightarrow$ GRAFICO // PIÙ ALLARGATO

RISPOSTA FORZATA

$u(t) \neq 0$, NULLE CONDIZIONI INIZIALI

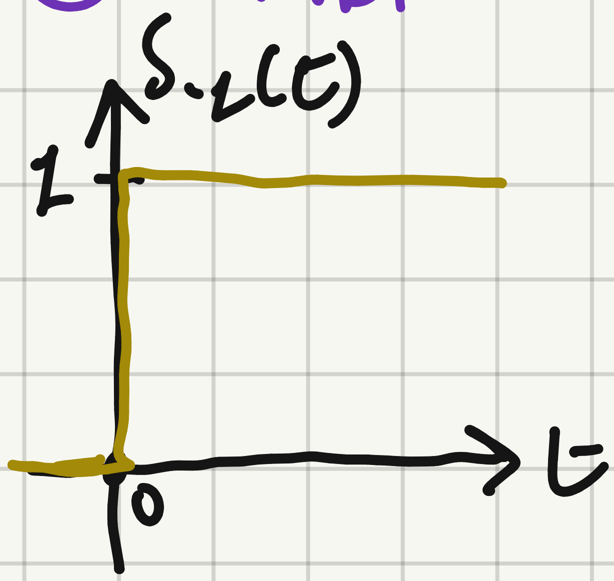
$a_n y^n(t) + \ldots + a_1 y'(t) + a_0 y(t) = 0$

$y(t_0) = y'(t_0) = \ldots = y^n(t_0) = 0$

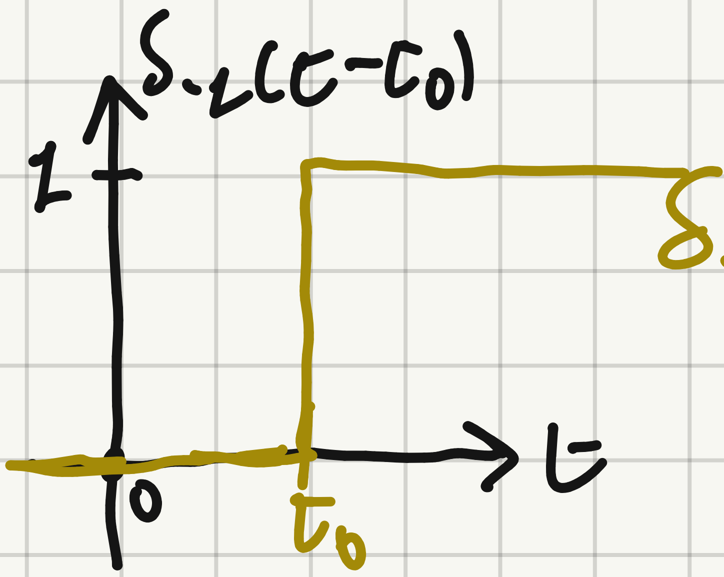
$$U(t) = \sum_{i=1}^{p} U_i(t) = U_1(t) + U_2(t) + \dots + U_p(t) = \frac{s}{R_1}(t) + \frac{s}{R_2} + \dots + \frac{s}{R_n}(t)$$

GLI INGRESSI CANONICI

① GRADINO UNITARIO $\delta_{-1}(t)$

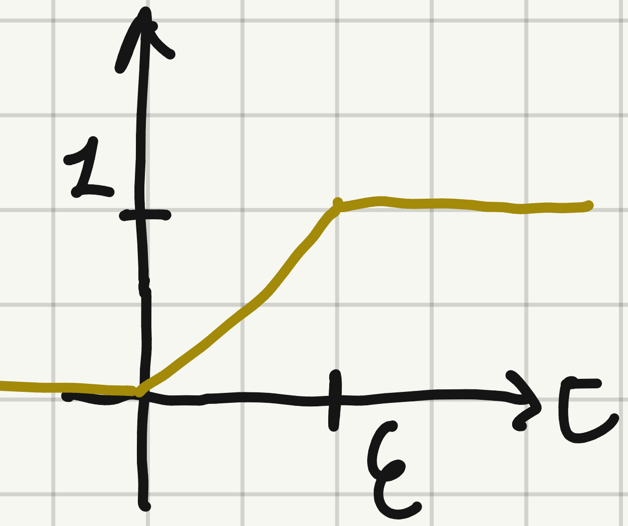


$$\delta_{-1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



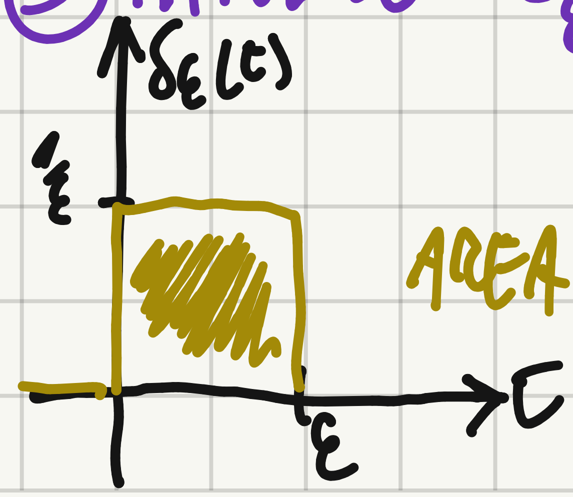
$$\delta_{-1}(t) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

② GRADINO + RAMPA $\delta_{-1,\varepsilon}(t)$



$$\delta_{-1,\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ t/\varepsilon & 0 \leq t \leq \varepsilon \\ 1 & t > \varepsilon \end{cases}$$

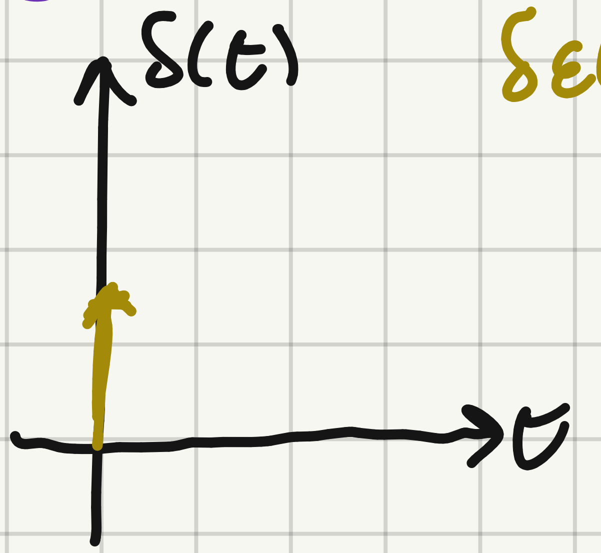
③ IMPULSO $\delta_\varepsilon(t)$



AREA = 1

$$\delta_\varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1/\varepsilon & 0 \leq t \leq \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

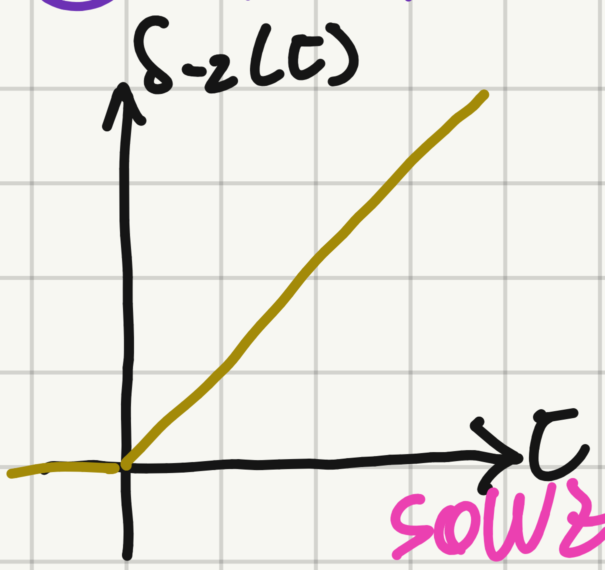
④ IMPULSO UNITARIO $\delta(t)$



$\delta_\varepsilon(t)$ PER $\varepsilon \to 0$

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

⑤ RAMPA $\delta_{-2}(t)$



$$\delta_{-2}(t) = \begin{cases} 0 & t<0 \\ t & t\geq 0 \end{cases}$$

SOLUZIONE RISPOSTA FORZATA: RISPOSTA IMPULSIVA

- $a_n y^n(t) + \ldots + a_1 y'(t) + a_0 y(t) = b_0 u(t) + b_1 u'(t) + \ldots + b_m u^m(t)$
- $y(0) = y'(0) = \ldots = y^{n-1}(0) = 0$
- $u(t) = \delta(t)$

PROPOSIZIONE: LA RISPOSTA FORZATA DEL SISTEMA AD UN INGRESSO $u(t) = \delta(t)$ È UGUALE A $y_f(t) = y_\delta(t) = \begin{cases} 0 & t<0 \\ c(t)\cdot\delta_{-1}(t) + A_0\,\delta(t) & t\geq 0 \end{cases}$

$$A_0 = \begin{cases} 0 & n>m \\ \frac{b_m}{a_m} = \frac{b_n}{a_n} & n=m \end{cases}$$

COME CALCOLARE LA RISPOSTA IMPULSIVA $y_\delta(t)$?

$$y_p(t) = y_\delta(t) = C(t)\cdot\delta_{-1}(t) + A_0\cdot\delta(t)$$

$$y_p'(t) = C'(t)\delta_{-1}(t) + C(t)\delta(t) + A_0\cdot\delta_1(t) = C'(t)\delta_{-1}(t) + C(0)\delta(t) + A_0\delta_1(t)$$

$$y_p''(t) = C''(t)\delta_{-1}(t) + C'(t)\delta(t) + C(0)\delta_1(t) + A_0\delta_2(t) = C''(t)\delta_{-1}(t) + C'(0)\delta(t) + C(0)\delta_1(t) + A_0\delta_2(t)$$

$\vdots$

$$y_p^n(t) = C^n(t)\delta_{-1}(t) + C^{n-1}(0)\cdot\delta(t) + C^{n-2}(0)\cdot\delta_1(t) + \dots + A_0\cdot\delta_n(t)$$

$$a_n y^n(t) + \dots + a_1 y'(t) + a_0 y(t) = b_0 u(t) + b_1 u'(t) + \dots + b_m u^m(t)$$

$$u(t) = \delta(t)$$

$$a_n y^n(t) + \dots + a_1 y'(t) + a_0 y(t) = b_0\delta(t) + b_1\delta_1(t) + \dots + b_m\delta_m(t)$$

$$a_n y^n(t) + \dots + a_1 y'(t) + a_0 y(t) = b_0\delta(t) + \dots + b_m\delta_m(t) + b_{m+1}\delta_{m+1}(t) + \dots + b_n\delta_n(t)$$

$$y(t) = y_p(t) \rightarrow a_n[C^n(t)\delta_{-1}(t) + C^{n-1}(t)\delta(t) + C^{n-2}(t)\delta_1(t) + \dots + A_0\delta_n(t)]$$

$$+ \dots + a_1[C'(t)\delta_{-1}(t) + C(0)\delta(t) + A_0\delta_1(t)] + a_0[C(t)\delta_{-1}(t) + A_0\delta(t)] =$$

$$= b_0\delta(t) + \dots + b_m\delta_m(t) + b_{m+1}\delta_{m+1}(t) + \dots + b_n\delta_n(t)$$

– CASO $n = m$

EGUAGLIAMO I $\delta_i,\ i = 1, \dots, n$

$$\delta \Rightarrow b_0 = a_n C^{n-1}(0) + \dots + a_1 C(0) + a_0 A_0$$

$$\delta_1 \Rightarrow b_1 = a_n C^{n-1}(0) + \dots + a_1 A_0$$

$\vdots$

$$\delta_{n-1} \Rightarrow b_{n-1} = a_n C(0) + a_{n-1} A_0$$

$$\delta_n \Rightarrow b_n = a_n A_0 \longrightarrow A_0 = \frac{b_n}{a_n}$$

$$\begin{vmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & \dots & 0 \end{vmatrix} \cdot \begin{vmatrix} A_0 \\ C(0) \\ C'(0) \\ \vdots \\ C^{n-1}(0) \end{vmatrix}$$

— CASO $n > m$

CON RAGIONAMENTI ANALOGHI,

$$\begin{vmatrix} b_0 \\ b_m \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & \dots & 0 \end{vmatrix} \cdot \begin{vmatrix} A_0 \\ C(0) \\ \vdots \\ C^{n-1}(0) \end{vmatrix}$$

$0 = A_0 \cdot a_n$, COME GIÀ VISTO PER SISTEMI STRETTAMENTE PROPRI

ESEMPIO: TROVARE LA RISPOSTA IMPULSIVA DEL SISTEMA

$$2y''(t) + 6y'(t) + 4y(t) = 3u(t) + u'(t)$$

$$y_\delta(t) = C(t) \cdot \delta_{-1}(t) + A_0 \cdot \delta(t)$$

$$C(t) = ?$$

$$P(\lambda) = 0 \to 2\lambda^2 + 6\lambda + 4 = 0 \quad \lambda^2 + 3\lambda + 2 = 0 \quad (\lambda+1)(\lambda+2) = 0$$

$$\cdot \lambda_1 = -1 \quad \mu_1 = 1 \Rightarrow e^{-t}$$

$$\cdot \lambda_2 = -2 \quad \mu_2 = 1 \Rightarrow e^{-2t} \qquad \Rightarrow C(t) = A_1 e^{-t} + A_2 e^{-2t}$$

$$C'(t) = -A_1 e^{-t} - 2A_2 e^{-2t}$$

$$C(0) = A_1 + A_2 \quad C'(0) = -A_1 - 2A_2$$

$$\begin{vmatrix} 3 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 2 & 6 & 4 \\ 6 & 4 & 0 \\ 4 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} A_0 \\ A_1 + A_2 \\ -A_1 - 2A_2 \end{vmatrix}$$

$$\begin{vmatrix}3\\1\\0\end{vmatrix} = \begin{vmatrix}2 & 6 & 4\\ 6 & 4 & 0\\ 4 & 0 & 0\end{vmatrix} \cdot \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -1 & -2\end{vmatrix} \cdot \begin{vmatrix}A_0\\A_1\\A_2\end{vmatrix}$$

DA CUI $A_0=0$ (COME DA ASPETTATIVE), $A_1=1$, $A_2=-\frac{1}{2}$

$\to y_\delta(t) = \left(e^{-t} - \frac{1}{2}e^{-2t}\right)\delta_{-1}(t)$ (1^a FORMA)

ESEMPIO: TROVARE LA RISPOSTA IMPULSIVA DI $y'''(t)+2y''(t)+5y'(t)=u(t)+4u'(t)$ SAPENDO CHE L'IMPULSO È UNITARIO

$P(\lambda)=0 \quad \lambda^3+2\lambda^2+5\lambda=0 \quad \lambda(\lambda^2+2\lambda+5)=0$

$\lambda_1=0 \quad \mu_1=1 \qquad \dfrac{-2\pm\sqrt{-16}}{2}=\dfrac{-2\pm 4i}{2}$

$\lambda_2=-1+2i \quad \mu_2=1$

$\lambda_2^*=-1-2i \quad \mu_2^*=\mu_2=1$

$\to c(t) = A_1 + A_2 e^{(-1+2i)t} + A_2^* e^{(-1-2i)t}$

$c'(t) = (-1+2i)A_2 e^{(-1+2i)t} + (-1-2i)A_2^* e^{(-1-2i)t}$

$c''(t) = (-1+2i)^2 A_2 e^{(-1+2i)t} + (-1-2i)^2 A_2^* e^{(-1-2i)t}$

$\quad = (-3-4i)A_2 e^{(-1+2i)t} + (-3+4i)A_2^* e^{(-1-2i)t}$

$c(0) = A_1 + A_2 + A_2^* = A_1 + 2\mathrm{Re}[A_2]$

$c'(0) = (-1+2i)A_2 + (-1-2i)A_2^* = -(A_2+A_2^*) + 2i(A_2-A_2^*)$

$\quad = -2\mathrm{Re}[A_2] - 4\mathrm{Im}[A_2]$

$$C''(0) = (-3-4i)A_2 + (-3+4i)A_2^* = -3(A_2+A_2^*) - 4i(A_2 - A_2^*)$$

$$= -6\,Re[A_2] + 8\,Im[A_2]$$

$$\begin{vmatrix} 1 \\ 4 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 2 & 1 \\ 5 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} A_0 \\ A_1 + 2Re[A_2] \\ -2Re[A_2] - 4Im[A_2] \\ -6Re[A_2] + 8Im[A_2] \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 4 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 2 & 1 \\ 5 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -6 & 8 \end{vmatrix} \cdot \begin{vmatrix} A_0 \\ A_1 \\ Re[A_2] \\ Im[A_2] \end{vmatrix}$$

Da cui

$$\begin{vmatrix} A_0 \\ A_1 \\ Re[A_2] \\ Im[A_2] \end{vmatrix} = \begin{vmatrix} 0 \\ 1/5 \\ -1/10 \\ -19/20 \end{vmatrix}$$

$$\Rightarrow y_s(t) = \left(\frac{1}{5} + \left(-\frac{1}{10} - \frac{19}{20}i\right)e^{(-1+2i)t} + \left(-\frac{1}{10} + \frac{19}{20}i\right)e^{(-1-2i)t}\right)\delta_{-1}(t)$$