TRANSLATE THE FOLLOWING ASSERTION INTO FORMULAS OF THE FIRST ORDER LANGUAGE OF THE ORDERED FIELD OF REAL NUMBERS $\langle \mathbb{R}, 0, 1, -, +, \times, \leq \rangle$:

a) THE SQUARE OF A POSITIVE NUMBER IS POSITIVE. SIMILARLY, THE SQUARE OF A NEGATIVE NUMBER IS POSITIVE. THEREFORE, THE FACT THAT 0 IS NOT NEGATIVE IMPLIES THAT NO NEGATIVE NUMBER IS A SQUARE

SQUARE $s(n) := n \cdot n$

POSITIVE $a > 0 := \neg(a \leq 0)$

NEGATIVE $a < 0 := a \leq 0 \land \neg(a = 0)$

NON NEGATIVE $a \geq 0 := a > 0 \lor a = 0$

$$\forall n(n>0 \to s(n)>0, n<0 \to s(n)>0 \models 0 \geq 0 \to \neg\exists n(s(n)<0))$$

b) THE SET OF POSITIVE REAL NUMBERS WHOSE SQUARE DOES NOT EXCEED 2 IS NONEMPTY AND HAS AN UPPER BOUND. THEREFORE, SINCE THIS SET HAS A SUPREMUM, IT FOLLOWS THAT THERE EXISTS A POSITIVE SQUARE ROOT OF 2

POSITIVE $a > 0 := \neg(a \leq 0)$

SQUARE $s(n) := n \cdot n$

UPPER BOUND $U(n) := \exists k(\forall n(n < k))$

SUPREMUM $M(n) := \exists k(U(n) \lor n = k)$

$$\forall n(n>0 \land s(n) < 2 \to \exists U(n) \models \exists M(n) \to \exists n(s(n)=2)$$

THE LANGUAGE OF BINARY RELATIONS IS THE FIRST ORDER LANGUAGE WITH EQUALITY L GENERATED BY A SINGLE RELATION SYMBOL R OF ARITY TWO. TRANSLATE THE FOLLOWING ASSERTIONS INTO FORMULAS OF L:

a) SUPPOSE R IS A PARTIAL ORDER. THEN R IS SYMMETRIC IF AND ONLY IF IT IS DISCRETE (A DISCRETE RELATION IS A RELATION CONSISTING OF THE PAIRS (x,x))

- $R: A \to A$ PARTIAL ORDER
- SYMMETRIC $\forall xy\,(xRy \to yRx)$
- DISCRETE $\exists x\,(xRx)$

$\forall xy((xRy \wedge yRx) \leftrightarrow \exists x(xRx))$

b) SUPPOSE R IS SYMMETRIC AND TRANSITIVE. THEN R IS SERIAL IF AND ONLY IF IT IS REFLEXIVE

- SYMMETRIC $\forall xy\,(xRy \to yRx)$
- TRANSITIVE $\forall xyz(xRy \wedge yRz \to xRz)$
- SERIAL $\forall x \exists y\,(xRy)$
- REFLEXIVE $\forall x\,(xRx)$

$\forall xy(xRy \to yRx), \forall xyz(xRy \wedge yRz \to xRz) \models$

$\forall x \exists y(xRy) \leftrightarrow \forall x(xRx)$

DETERMINE WHETHER THE FOLLOWING FORMULAS ARE TRUE, SATISFIABLE OR UNSATISFIABLE IN $\mathbb{N}$ FOR THE STANDARD INTERPRETATION OF THE SYMBOL

1) $x \neq 0 \to \exists y (xy = 1)$

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X IS FREE VARIABLE $\to \varphi(x)$ $\quad v \models \varphi$

$v(x) = a \to$ CHECK WHETHER $\varphi(a)$ IS TRUE

$\varphi(0) = 0 \neq 0 \to \exists y (0y = 1)$

PREMISE FALSE, IMPLICATION TRUE (?) $\to$ φ SATISFIABLE

VALID

EXAMIN $\varphi(a)$ FOR SMALL VALUE ON a

$\varphi(1) = 1 \neq 0 \to \exists y (1y = 1)$

PREMISE FALSE, SO $v \models \varphi \Leftrightarrow \exists y (1y = 1) \Leftrightarrow \exists a \in \mathbb{N} \mid v[b/y] \models 1y \overset{}{=} 1$

$\to 1b = 1 \to b = 1 \checkmark$

BUT WHAT ABOUT $v(2) = 2$?

$\varphi(2) = 2 \neq 0 \Leftrightarrow \exists y (2y = 1) \Leftrightarrow \exists b \in \mathbb{N} \mid 2b = 1$. IT IS NOT POSSIBLE $\Rightarrow$ φ IS NOT VALID

2) $\exists y z (x = y^2 + z^2)$

X FREE VARIABLE $\to \varphi(x)$ $\quad v \models \varphi$

$\varphi(x) = 0 \to \quad 0 = y^2 + z^2 \quad \checkmark \quad \to \varphi$ SATISFIABLE