LET $f: B \to A$ BE ANY FUNCTION. GIVEN A BINARY RELATION R ON A, THE INVERSE IMAGE OF R ALONG f IS THE BINARY RELATION S ON B DEFINED BY THE FORMULA $(x,y) \in S \Leftrightarrow (fx, fy) \in R$.

PROVE THAT $S = fRf^{op}$. PROVE ALSO THAT IF R IS AN ORDER RELATION AND f IS INJECTIVE, THEN S IS ALSO AN ORDER RELATION. SHOW THAT THE ASSUMPTION THAT f IS INJECTIVE IS NECESSARY BY SHOWING THAT IF

$$R = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad f = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

THEN S IS NOT AN ORDER ON B. DETERMINE THE ORDER INDUCED BY S.

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "S"'] & A \arrow[d, "R"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

a) $xSy \Leftrightarrow x fRf^{op} y \Leftrightarrow$

$\exists c,d\,(x f c \wedge c R d \wedge d f^{op} y) \Leftrightarrow$

$\exists c,d\,(x f c \wedge c R d \wedge y f d) \Leftrightarrow$

$\exists c,d\,(f(x) = c \wedge c R d \wedge f(y) = d) \Leftrightarrow f(x) R f(y)$

b) R IS AN ORDER RELATION, f INJECTIVE $\Rightarrow$ S ORDER RELATION?

- REFLEXIVITY $\quad x fRf^{op} x$

$\exists c,d\,(x f c \wedge c R d \wedge x f d) \quad f(x) = c \wedge f(x) = d$

f IS A FUNCTION $\Rightarrow c = d \quad \exists c\,(x f c \wedge \overbrace{cRc}^{\checkmark} \wedge x f c)$

$\Leftrightarrow \exists c\,(x f c \wedge c R c \wedge c f^{op} x) \Leftrightarrow x fRf^{op} x$ ✓

- TRANSITIVITY $\quad x f R f^{op} y \wedge y f R f^{op} z \Rightarrow x f R f^{op} z$?

$\exists c,d,h,k \,(x f c \wedge c R d \wedge d f^{op} y \wedge y f h \wedge h R k \wedge k f^{op} z)$

$\Leftrightarrow \exists c,d,h,k \,(x f c \wedge c R d \wedge \underline{y f d} \wedge \underline{y f h} \wedge h R k \wedge z f k)$

$f(y)=d;\ f(y)=h \Rightarrow d=h$

$\Rightarrow c R k$

$\Leftrightarrow \exists c,d,k \,(x f c \wedge \underline{c R d} \wedge \cancel{y f d} \wedge \underline{d R k} \wedge z f k)$

$\Leftrightarrow \exists c,k \,(x f c \wedge c R k \wedge k f^{op} z) \Leftrightarrow x f R f^{op} z$ ✓

- ANTISYMMETRY $\quad x f R f^{op} y \wedge y f R f^{op} x \Rightarrow x = y$

$\exists c,d,h,k \,(x f c \wedge c R d \wedge d f^{op} y \wedge y f h \wedge h R k \wedge k f^{op} x)$

$d=h \qquad c=k$

$\Leftrightarrow \exists c,d,h,k \,(\underline{x f c} \wedge c R d \wedge \underline{y f d} \wedge \underline{y f h} \wedge h R k \wedge \underline{x f k})$

$\Leftrightarrow \exists c,d \,(x f c \wedge \underline{c R d} \wedge y f d \wedge \underline{d R c}) \quad c=d$

$\Leftrightarrow \exists c \,(x f c \wedge c R c \wedge y f c)$

$f(x)=c,\ f(y)=c$. BUT f INJECTIVE $\Rightarrow |f^*(c)| \leq 1 \Rightarrow x=y$

$\Leftrightarrow \exists c \,(x f c \wedge c R c \wedge c f^{op} x) \Leftrightarrow x f R f^{op} x$

c)

$$R = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad f = \begin{array}{c} \\ \end{array}\begin{array}{cccc} & \text{NON INJECTIVE} & & \\ \end{array}$$

$$f = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$S := f R f^{op} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

PROOF OF NON ANTISYMMETRISM   SURELY, $S = R^{rt}$

$$d)\ E = S \cap S^{op} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cap \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad P = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$P^{op} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\bar{S} = P^{op} S P = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$