THEOREM 15: FIRST ISOMORPHISM THEOREM

IF $f: A \to B$ IS A FUNCTION, THEN f INDUCES AN ISOMORPHISM

$h: A/\mathrm{Ker}(f) \to \mathrm{im}(f)$ DEFINED BY THE FORMULA $h([x]) = f(x)$

EQUIVALENCE CLASS OF x

[EPI-MONO FACTORIZATION LEMMA: $pg = f$

$\mathrm{Ker}(f) = f f^{op}$

$\mathrm{im}(g) = \mathrm{im}(f)$ $\quad g = hi$]

PROOF

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, shift left, "i^{op}"] \\
A/\mathrm{Ker}(f) \arrow[ur, dashed, "g"] \arrow[r, dashed, "h"] & \mathrm{im}(f) \arrow[u, shift left, "i"]
\end{tikzcd}
$$

PROJECTION ON THE QUOTIENT SET P: $(x) = [x]$

INCLUSION OF THE IMAGE OF f IN B: $i(x) = x$

- OBSERVE THAT p IS SURJECTIVE $\Rightarrow \mathrm{Ker}(p) = \mathrm{Ker}(f)$
- BY EPI-MONO FACTORIZATION, $f = pg$ AND g MONO (INJECTIVE)

  $\Rightarrow \mathrm{im}(g) = \mathrm{im}(f)$. g INJECTIVE $\Rightarrow \mathrm{Ker}(g) = 1$
- i IS INJECTIVE AND $\mathrm{im}(i) = \mathrm{im}(f) = \mathrm{im}(g) \Rightarrow g = hi$ UNIQUELY
- $\mathrm{Ker}(h) = \mathrm{Ker}(g) = 1 \Rightarrow h$ IS MONOMORPHISM
- $\mathrm{im}(h) = i^*(\mathrm{im}(f)) = \mathrm{im}(f) \Rightarrow h$ IS EPIMORPHISM

$\Rightarrow h$ IS ISOMORPHISM

$h: g i^{op} = p^{op} f i^{op} \Rightarrow h([x]) = f(x)$

THEOREM 16: EQUISATISFIABILITY OF SKOLEMIZATION

φ IS SATISFIABLE IF AND ONLY IF φ_s (EXPANSION) IS SATISFIABLE

⌜COROLLARY OF PRENEX FORM: EVERY FORMULA OF A FIRST ORDER LANGUAGE
IS EQUIVALENT TO A FORMULA OF L IN PRENEX FORM⌟

PROOF

$\Leftarrow$

BY THE RECALLED COROLLARY, WE CAN ASSUME THAT φ IS IN PRENEX FORM ($\Rightarrow$ IT IS QUANTIFIER-FREE). TAKING $y = f(x_1, \dots, x_n)$, WE HAVE THAT

$$\forall x_1, \dots, x_n \, \psi(x_1, \dots, x_n, f(x_1, \dots, x_n)) \models \forall x_1, \dots, x_n \exists y \, \psi(x_1, \dots, x_n, y)$$

THUS, $M \models \varphi_s \Rightarrow M \models \varphi$

$\Rightarrow$

1. SUPPOSE $M \models \forall x_1, \dots, x_n \exists y \, \psi(x_1, \dots, x_n, y)$
2. GIVEN $(a_1, \dots, a_n) \in M^n$, $\exists b \in M \mid M \models \psi\left[\frac{a_1}{x_1}, \dots, \frac{a_n}{x_n}, \frac{b}{y}\right]$
3. LET M_f BE THE EXPANSION OF M BY f, WHERE $[\![f]\!](a_1, \dots, a_n) = b$
4. GIVEN $(a_1, \dots, a_n) \in M_f^n$, $M_f \models \psi\left[\frac{a_1}{x_1}, \dots, \frac{a_n}{x_n}, \frac{f(a_1, \dots, a_n)}{y}\right]$
5. HENCE, $M_f \models \forall x_1, \dots, x_n \, \psi(x_1, \dots, x_n, f(x_1, \dots, x_n))$

THUS, $M \models \varphi \Rightarrow M_f \models \varphi_s$

$\rightarrow M \models \varphi \Leftrightarrow M \models \varphi_s$

THEOREM 17: HINTIKKA LEMMA (FOR F.O.L.)

ASSUME L HAS CLOSED TERMS. IF H IS HINTIKKA FOR L, THEN H HAS A HERBRAND MODEL

A HINTIKKA SET OF CLOSED FORMULAS $H \subseteq L$ SATISFIES THE FOLLOWING CONDITIONS:

- SAME AS FOR P.L. (SEE THEOREM 2, SUBSTITUTE X WITH φ
- $\gamma \in H \Rightarrow \gamma(t) \in H$ FOR EVERY CLOSED TERM t
- $\delta \in H \Rightarrow \delta(t) \in H$ FOR SOME CLOSED TERM t

A HERBRAND MODEL M FOR L IS A STRUCTURE FOR WHICH:

- M IS THE SET OF CLOSED TERMS OF L
- $[[f]](t_1, \dots, t_n) = f(t_1, \dots, t_n)$

PROOF

WE PROVE BY STRUCTURAL INDUCTION THAT $\varphi \in H \Rightarrow M \models \varphi$

1. ATOMIC FORMULAS IF $R(t_1, \dots, t_n) \in H$, THEN THE t_i'S MUST BE CLOSED BECAUSE H CONSISTS OF CLOSED FORMULA. THUS $(t_1, \dots, t_n) \in [[R]]$ BY DEFINITION AND $M \models R(t_1, \dots, t_n)$

2. CONNECTIVES AS IN THE PROPOSITIONAL LOGIC

3. γ-FORMULAS SUPPOSE $\gamma(t) \in H \Rightarrow M \models \gamma(t)$ FOR EVERY CLOSED TERM t AND SUPPOSE $\gamma \in H$. THEN $\gamma(t) \in H$ FOR EVERY CLOSED TERM t BECAUSE H IS HINTIKKA. BY THE INDUCTIVE HYPOTHESIS, $M \models \gamma(t)$ FOR EVERY CLOSED t. SINCE M IS HERBRAND AND CONTAINS ONLY CLOSED TERM, $M \models \gamma$

4. δ-FORMULAS SUPPOSE $\delta(t) \in H \Rightarrow M \models \delta(t)$ FOR EVERY CLOSED TERM t AND SUPPOSE $\delta \in H$. THEN $\delta(t) \in H$ FOR SOME CLOSED TERM t BECAUSE H IS HINTIKKA. BY THE INDUCTIVE HYPOTHESIS, $M \models \delta(t)$. SINCE M IS HERBRAND, $t \in M$ AND THEREFORE $M \models \delta$

THEOREM 18: SOUNDNESS OF RESOLUTION (FOR F.O.L.)

THE RESOLUTION CALCULUS IS SOUND

PROOF

ASSUMPTION, FALSEHOOD, DOUBLE NEGATION, α AND β FORMULAS HAVE THE SAME PROOF AS IN P.L. THEN, WE SHOW THAT γ AND δ RULES PRESERVE SATISFIABILITY OF CLOSED FORMULAS

1. SUPPOSE $M \models \gamma$. SINCE t IS CLOSED, $\exists c \in M \mid v(t) = c \ \forall \underset{\text{VALUATION}}{v}: X \to M$. OBSERVE THAT $v \models \gamma(t) \Leftrightarrow v \models \gamma[t/x] \Leftrightarrow v[c/x] \models \gamma(x)$. THEN

   $M \models \gamma \Rightarrow M \models \forall x \gamma(x) \Rightarrow \forall v: X \to M, v \models \forall x \gamma(x)$

   $\Rightarrow \forall v: X \to M, a \in M, v[a/x] \models \gamma(x) \Rightarrow \forall v: X \to M, v[c/x] \models \gamma(x)$

   $\Rightarrow \forall v: X \to M, v \models \gamma(t) \Rightarrow M \models \gamma(t)$

2. SINCE p DOES NOT APPEAR IN δ, WE DO NOT NEED TO INTERPRET IT YET

   $M \models \delta \Rightarrow M \models \exists x \delta(x) \Rightarrow \exists v: X \to M, v \models \exists x \delta(x) \Rightarrow \exists u: X \to M, u \models \delta(x)$

   WHERE u IS AN x-VARIANT OF v. DEFINE THE EXPANSION M_p OF M SETTING $[\![p]\!] := u(x)$

   $\Rightarrow \exists u: X \to M_p, u \models \delta(x) \Rightarrow \exists u: X \to M_p, u \models \delta(p) \Rightarrow M \models \delta(p)$

NOW, SINCE ALL RULES ARE SOUND, $F \vdash \varphi \Rightarrow \bot \in Res(F \cup \{\neg\varphi\}) \Rightarrow \forall v \in M^X, v \not\models Res(F \cup \{\neg\varphi\}) \Rightarrow \forall v \in M^X, v \not\models F \cup \{\neg\varphi\} \Rightarrow F \cup \{\neg\varphi\} \models \bot \Rightarrow F \models \varphi$

THEOREM 19: COMPLETENESS OF RESOLUTION

THE RESOLUTION CALCULUS IS COMPLETE. $F \models \varphi \Rightarrow F \vdash \varphi$

LEMMA : THE CLASS C CONSISTING OF FINITE SETS $S \subseteq L_P$ OF SENTENCES WHICH HAVE NO CLOSED RESOLUTION EXPANSION IS A CONSISTENCY CLASS

PROOF

BY THE PREVIOUS LEMMA AND P.L. THEORY, CONSISTENT SETS FORMS A CONSISTENCY CLASS. THEREFORE, EVERY CONSISTENT SET IS SATISFIABLE, THAT IS THE DEFINITION OF COMPLETENESS. THEREFORE, INFERENCE IS COMPLETE

THEOREM 20: COMPLETENESS OF HILBERT CALCULUS

HILBERT CALCULUS IS COMPLETE: GIVEN $\Gamma \subseteq L$ AND $\varphi \in L$, $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$

PROOF

WE FIRST PROVE THAT $\Gamma \in C_\varphi \Rightarrow \Gamma \cup \{\neg\varphi\} \in C_\varphi$, WHERE C_φ IS THE CLASS OF SETS IN WHICH φ IS NOT DERIVABLE IN THE HILBERT CALCULUS $C_\varphi := \{\Gamma \subseteq L \mid \Gamma \nvdash \varphi\}$. FOR SUPPOSE $\Gamma \cup \{\neg\varphi\} \notin C_\varphi$; WE THEN OBTAIN $\Gamma \notin C_\varphi$ AS FOLLOWS

STEP	STATEMENT	REASON
1	$\Gamma, \neg\varphi \vdash \varphi$	HYPOTHESIS
2	$\Gamma \vdash \neg\varphi \to \varphi$	1, DEDUCTION THEOREM
3	$\Gamma \vdash \neg\varphi \to \varphi \to \varphi$	2, LEMMA $(\neg\varphi \to \varphi) \to \varphi$
4	$\Gamma \vdash \varphi$	2,3 MODUS PONENS

SUPPOSE, NOW, THAT $\Gamma \nvdash \varphi$. THEN, $\Gamma \in C_\varphi$ AND, HENCE, $\Gamma \cup \{\neg\varphi\} \in C_\varphi$ $\Rightarrow \Gamma \cup \{\neg\varphi\}$ SATISFIABLE $\Rightarrow \Gamma \nvDash \varphi$

THEOREM 21: SUBMODELS CHARACTERIZATIONS

LET T BE AN ALGEBRAIC THEORY AND LET M BE A T-MODEL. A SUBSET $N \subseteq M$ CARRIES A T-SUBMODEL STRUCTURE IF AND ONLY IF FOR EVERY FUNCTION SYMBOL $f \in L$ OF ARITY n AND, FOR EVERY TUPLE $(a_1, \dots, a_n) \in N^n$, WE HAVE $[[f]]_M (a_1, \dots, a_n) \in N$.

IN THIS CASE, THE T-MODEL STRUCTURE OF N IS UNIQUE AND $[[f]]_N (a_1, \dots, a_n) = [[f]]_M (a_1, \dots, a_n)$ FOR $a \in N^n$

PROOF

$\Rightarrow$ SUPPOSE N IS A SUBMODEL OF M. THEN $[[f]]_N : N^n \to N$ AND, HENCE,

$[[f]]_M (a_1, \dots, a_n) := [[f]]_N (a_1, \dots, a_n)$

$\Leftarrow$ IF N CARRIES A SUBMODEL STRUCTURE, WE MUST HAVE $[[f]]_N (a_1, \dots, a_n) = [[f]]_M (a_1, \dots, a_n)$, HENCE $[[f]]_N$ IS UNIQUELY DEFINED AND THE T-STRUCTURE IS UNIQUE.

NOW, WE DEFINE $[[f]]_N (a_1, \dots, a_n) := [[f]]_M (a_1, \dots, a_n)$

WE CLAIM THAT N IS A T-MODEL

SUPPOSE $s = t$ IS AN AXIOM OF T. IF $v : X \to N \subseteq M$ IS A VALUATION, THEN $v(s) = v(t)$ BECAUSE $M \models s = t$. HENCE, $N \models s = t$

NOTICE THAT $v(s), v(t) \in N$

THEOREM 22: QUOTIENT MODELS STRUCTURE

LET T BE AN ALGEBRAIC THEORY AND LET M BE A T-MODEL. IF E IS A CONGRUENCE OF M, THEN M/E IS A T-MODEL FOR THE STRUCTURE:

$$[[f]]_{M/E}([a_1],\dots,[a_n]) = [[[f]]_M(a_1,\dots,a_n)]$$

PROOF

$[a_i]=[b_i] \Rightarrow a_i \sim b_i$

$\Rightarrow [[f]]_M(a_1,\dots,a_n) \sim [[f]]_M(b_1,\dots,b_n)$ } DEFINITION OF CONGRUENCE

$\Rightarrow [[[f]]_M(a_1,\dots,a_n)] = [[[f]]_M(b_1,\dots,b_n)]$

$\Rightarrow [[f]]_{M/E}([a_1],\dots,[a_n]) = [[f]]_{M/E}([b_1],\dots,[b_n])$

$M/E \vDash T$. IF $s=t$ IS AN AXIOM, v IS A VALUATION IN M/E AND u LIFTS v TO M ($[u(x_i)] = v(x_i)$), THEN $M \vDash s=t \Rightarrow u(s)=u(t)$

$\Rightarrow v(s)=v(t)$

$\Rightarrow M/E \vDash s=t$

THEOREM 23: CHARACTERIZATION OF SUBMODELS AND QUOTIENT VIA MORPHISM

ASSUME T IS AN ALGEBRAIC THEORY AND M A T-MODEL

1) SUBMODELS OF M ARE PRECISELY THE IMAGES OF MORPHISM WITH CODOMAIN M

× 1 IS NOT PROVED

2) CONGRUENCES ON M ARE PRECISELY KERNEL PAIRS OF MORPHISM WITH DOMAIN M. MOREOVER, THE STRUCTURE ON $M/_E$ IS THE ONLY ONE THAT MAKES THE PROJECTION $p: M \to M/_E$ A MORPHISM

⌜GIVEN MODELS M AND N, A MORPHISM IS A FUNCTION $h: M \to N$ SUCH THAT FOR EVERY FUNCTION f OF A LANGUAGE L,

$h([[f]]_M(x_1,\dots,x_n)) = [[f]]_N(h(x_1),\dots,h(x_n))$⌟

PROOF

IF E IS THE KERNEL PAIR OF A T-MORPHISM $h: M \to N$, IT IS AN EQUIVALENCE RELATION.

IF f IS A FUNCTION SYMBOL AND $a_i E b_i$, THEN $h(a_i) = h(b_i)$ AND, HENCE,

$h([[f]]_M(a_1,\dots,a_n)) = [[f]]_N(h(a_1),\dots,h(a_n)) = [[f]]_N(h(b_1),\dots,h(b_n)) = h([[f]]_M(b_1,\dots,b_n))$

$\Rightarrow [[f]]_M(a_1,\dots,a_n) \, E \, [[f]]_M(b_1,\dots,b_n)$. THUS, E IS A CONGRUENCE

CONVERSELY, IF $E \subseteq M^2$ IS A CONGRUENCE, THEN $p([[f]]_M(a_1,\dots,a_n)) =$

$= [[[f]]_M(a_1,\dots,a_n)] = [[f]]_{M/_E}([a_1],\dots,[a_n]) = [[f]]_{M/_E}(p(a_1),\dots,p(a_n))$

HENCE, p IS A T-MORPHISM AND $E = \ker(p)$

THEOREM 24: FIRST ISOMORPHISM THEOREM

ASSUME h IN THE DIAGRAM BELOW IS A T-MORPHISM. THEN h FACTORS UNIQUELY THROUGH THE PROJECTION p OVER ITS KERNEL PAIR THROUGH A T-MORPHISM g WHICH IS INJECTIVE AND $\mathrm{im}(g) = \mathrm{im}(h)$

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "p"'] & N \\ M/E \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

PROOF

BY THE FIRST ISOMORPHISM THEOREM FOR FUNCTIONS, IT SUFFICES TO PROVE THAT

$g([a]) = h(a)$ IS A T-MORPHISM.

$$\begin{aligned}
g([\![f]\!]_{M/E}([a_1],\dots,[a_n])) &= g([\![f]\!]_{M/E}(p(a_1),\dots,p(a_n))) \\
&= g(p([\![f]\!]_M(a_1,\dots,a_n))) \\
&= h([\![f]\!]_M(a_1,\dots,a_n)) \\
&= [\![f]\!]_N(h(a_1),\dots,h(a_n)) \\
&= [\![f]\!]_N(g(p(a_1)),\dots,g(p(a_n))) \\
&= [\![f]\!]_N(g([a_1]),\dots,g([a_n]))
\end{aligned}$$

THEOREM 25: DIRECT AND INVERSE IMAGES OF SUBMODELS

SUPPOSE $h: M \to N$ IS A T-MORPHISM

1) IF $M' \subseteq M$ IS A SUBMODEL, ITS DIRECT IMAGE $h_*(M') := \{h(x) : x \in M'\}$ IS A SUBMODEL OF N

2) IF $N' \subseteq N$ IS A SUBMODEL, ITS INVERSE IMAGE $h^*(N') := \{x \in M : h(x) \in N'\}$ IS A SUBMODEL OF M

PROOF

1) ASSUME f IS A FUNCTION SYMBOL OF ARITY n AND $y_i \in h_*(M')$ FOR $i = 1, \dots, n$

THEN $y_i = h(x_i)$ WITH $x_i \in M'$. IF $x = [[f]]_M(x_1, \dots, x_n)$, THEN

$$[[f]]_N(y_1, \dots, y_n) = [[f]]_N(h(x_1), \dots, h(x_n)) = h([[f]]_M(x_1, \dots, x_n)) = h(x) \in h_*(M').$$

$h_*(M')$ IS CLOSED UNDER THE OPERATIONS AND, THEREFORE, A SUBMODEL OF N

2) IF $x_i \in h^*(N')$ FOR $i = 1, \dots, n$, THEN $y_i := h(x_i) \in N'$ AND $y := [[f]]_N(y_1, \dots, y_n) \in N'$.

HENCE, $h([[f]]_M(x_1, \dots, x_n)) = [[f]]_N(h(x_1), \dots, h(x_n)) = [[f]]_N(y_1, \dots, y_n) =$

$= y \in N'$. HENCE, $[[f]]_M(x_1, \dots, x_n) \in h^*(N')$