$$\begin{vmatrix}1&0&2\\0&5&0\\2&0&4\end{vmatrix} = \begin{vmatrix}1&0&-2\\0&1&0\\2&0&1\end{vmatrix} \quad \text{ORTONORMALE} \Longrightarrow \begin{vmatrix}1/\sqrt5&0&-2/\sqrt5\\0&1/\sqrt5&0\\2/\sqrt5&0&1/\sqrt5\end{vmatrix}$$

INFATTI $\left\langle \begin{vmatrix}1\\0\\2\end{vmatrix}, \begin{vmatrix}0\\1\\0\end{vmatrix} \right\rangle = \left\langle \begin{vmatrix}1\\0\\2\end{vmatrix}, \begin{vmatrix}-2\\0\\1\end{vmatrix} \right\rangle = \left\langle \begin{vmatrix}0\\1\\0\end{vmatrix}, \begin{vmatrix}-2\\0\\1\end{vmatrix} \right\rangle = 0 \longrightarrow \overset{0}{\overset{\shortparallel}{0}} = -2+2 = 0$ ✓

$$\frac{1}{\sqrt5}\begin{vmatrix}1&0&-2\\0&1&0\\2&0&1\end{vmatrix} \Longrightarrow D = \begin{vmatrix}0&0&0\\0&5&0\\0&0&5\end{vmatrix}$$

2) a) $F\begin{vmatrix}1\\0\\0\\0\end{vmatrix} = \begin{vmatrix}1\\2\\1\end{vmatrix}$ $\qquad F\begin{vmatrix}0\\1\\0\\0\end{vmatrix} = F\begin{vmatrix}0\\1\\1\\0\end{vmatrix} - F\begin{vmatrix}0\\0\\1\\0\end{vmatrix} = \begin{vmatrix}1\\3\\-1\end{vmatrix} - F\begin{vmatrix}0\\0\\1\\0\end{vmatrix}^{*} = \begin{vmatrix}0\\0\\0\end{vmatrix}$

$F\begin{vmatrix}0\\0\\1\\0\end{vmatrix} = F\begin{vmatrix}0\\0\\1\\1\end{vmatrix} - F\begin{vmatrix}0\\0\\0\\1\end{vmatrix} = \begin{vmatrix}3\\8\\-1\end{vmatrix} - \begin{vmatrix}2\\5\\0\end{vmatrix}^{*} = \begin{vmatrix}1\\3\\-1\end{vmatrix}$ $\qquad F\begin{vmatrix}0\\0\\0\\1\end{vmatrix} = \begin{vmatrix}2\\5\\0\end{vmatrix}$

$$F\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} = \begin{vmatrix}1&0&1&2\\2&0&3&5\\1&0&-1&0\end{vmatrix}$$

b) $\mathrm{Im}\,F = \begin{vmatrix}1&0&1&2\\2&0&3&5\\1&0&-1&0\end{vmatrix} \cdot \begin{vmatrix}x\\y\\z\\w\end{vmatrix} = \begin{vmatrix}x+z+2w\\2x+3z+5w\\x-z\end{vmatrix}$

$$\mathrm{rk}\begin{vmatrix}1&0&1&2\\2&0&3&5\\1&0&-1&0\end{vmatrix} = \mathrm{rk}\begin{vmatrix}1&0&2&2\\0&0&5&1\\0&0&0&-2\end{vmatrix}$$

$$= \mathrm{rk}\begin{vmatrix}1&0&0&2\\0&0&0&1\\0&0&0&-2\end{vmatrix} = 2 \Longrightarrow \dim B_F = 2$$

$B_F = \left\{ \begin{vmatrix}1\\2\\1\end{vmatrix}, \begin{vmatrix}1\\3\\-1\end{vmatrix} \right\}$

c) $\det\begin{vmatrix}x&1&1\\y&2&3\\z&1&-1\end{vmatrix} = 0$

$-5x+2y+z=0$

d) $\left\{ \overset{v_1}{\begin{vmatrix}1\\2\\1\end{vmatrix}}, \overset{v_2}{\begin{vmatrix}1\\3\\-1\end{vmatrix}}, \overset{v_3}{\begin{vmatrix}2\\5\\0\end{vmatrix}}, \overset{v_4}{\begin{vmatrix}1\\0\\5\end{vmatrix}}, \overset{v_5}{\begin{vmatrix}0\\1\\-2\end{vmatrix}} \right\}$

e) $\left\{ \overset{v_1}{\begin{vmatrix}2\\5\\0\end{vmatrix}}, \overset{v_2}{\begin{vmatrix}4\\10\\0\end{vmatrix}} \right\}$ INFATTI $v_2 = 2v_1$

③ a) $C \quad \cancel{x}^2+y^2+ax+by+c=0$

$$C\begin{cases}P\\Q\\R\end{cases}$$

POI VERIFICA CHE $S \in C$

$$\begin{cases}4+2a+c=0\\1+a+c=0\\1+b+c=0\end{cases}\quad\begin{cases}4+2a-a-1=0\\c=-a-1\\b=-c-1\end{cases}\quad\begin{array}{l}a=-3\\ \Downarrow\\ c=2\\ b=-3\end{array}$$

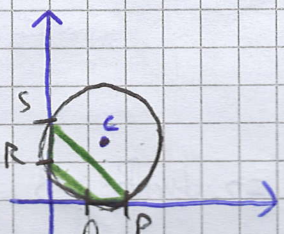
$$\Rightarrow x^2+y^2-3x-3y+2=0$$

$S \rightarrow 0+4-0-6+2=0 \qquad 0=0\checkmark$

b) $C=\begin{vmatrix}-\frac{a}{2}\\-\frac{b}{2}\end{vmatrix}=\begin{vmatrix}3/2\\3/2\end{vmatrix}$

$$r=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}-c}=\sqrt{\frac{9}{4}+\frac{9}{4}-2}=\sqrt{\frac{5}{2}}$$

c)



$\cdot PS // RQ$? $\quad\begin{vmatrix}2-0\\0-2\end{vmatrix}=2\begin{vmatrix}0-1\\1-0\end{vmatrix}\quad -2=-2\ \checkmark$

$\cdot PQ // RS$? $\quad\begin{vmatrix}2-1\\0-0\end{vmatrix}=2\begin{vmatrix}0-0\\1-2\end{vmatrix}\quad$ IMP!

$\Rightarrow$ LA FIGURA È UN TRAPEZIO

$$P=PQ+QR+RS+SP=\sqrt{(2-1)^2+(0-0)^2}+\sqrt{(0-1)^2+(0-1)^2}+\sqrt{(0-0)^2+(1-2)^2}$$

$$+\sqrt{(2-0)^2+(0-2)^2}=1+\sqrt{2}+\sqrt{2}+2\sqrt{2}=1+4\sqrt{2}$$

$$A=\frac{(PS+RQ)\cdot d_{R\,r_{PS}}}{2}$$

r_{PS}: $\det\begin{vmatrix}x-0 & 2-0\\y-2 & 0-2\end{vmatrix}=0 \rightarrow -2x-2y+4=0 \quad x+y-2=0$

$$d_{R\,r_n} = \frac{|a\cdot x_R + b\cdot y_R + c|}{\sqrt{a^2+b^2}} = \frac{|0+1-2|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$A = \frac{(2\sqrt{2}+\sqrt{2})\cdot \frac{1}{\sqrt{2}}}{2} = \frac{3}{2}$$

4 a) $rkA = 3 \wedge rkA|b = 4$

$$A = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow rkA = 3 \checkmark$$

$$A|b = \begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \Rightarrow rkA = 4 \checkmark$$

LE DUE RETTE SONO SGHEMBE

b) $v_1 = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix}$ $v_2 = \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix}$ $v_r = v_1 \times v_2 = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = +\hat{i} + \hat{j} + \hat{k} \Rightarrow v_r = \begin{vmatrix} +1 \\ 1 \\ 1 \end{vmatrix}$

ANALOGAMENTE $v_s = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & +1 \\ 0 & 1 & -3 \end{vmatrix} = -\hat{i} + 3\hat{j} + \hat{k} \Rightarrow v_s = \begin{vmatrix} -1 \\ 3 \\ 1 \end{vmatrix}$

c) $r: \begin{cases} x = t-1 \\ y = t \\ z = t \end{cases}$ $s: \begin{cases} x = -k+2 \\ y = 3k \\ z = k \end{cases}$

d) $v = v_r \times v_s = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 3 & 1 \end{vmatrix} = -2\hat{i} - 2\hat{j} + 4\hat{k} \Rightarrow v = \begin{vmatrix} -2 \\ -2 \\ 4 \end{vmatrix}$

e) $d(r, s) = d(\pi_r, \pi_s)$

$\pi_r: \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} t + \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix} s$ $P \in s = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$

$\det\begin{vmatrix} x+1 & y & z \\ 1 & 1 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \longrightarrow -2x-2-2y+4z=0$ $x+y-2z+2=0$

$$d(\pi_r, \pi_s) = d_{\pi_r P} = \frac{|2+0+0+2|}{\sqrt{1^2+1^2+(-2)^2}} = \frac{4}{\sqrt{6}}$$

⑤

a) $A = \begin{vmatrix} 1 & 1 & 3k \\ k & 1 & k \\ k & 0 & k \end{vmatrix}$ MATRICE DEI COEFFICIENTI $\quad A|b = \left|\begin{array}{ccc|c} 1 & 1 & 3k & 1 \\ k & 1 & k & 2 \\ k & 0 & k & 2 \end{array}\right|$ MATRICE COMPLETA

b) $\det A = \det \begin{vmatrix} 1 & 1 & 3k \\ k & 1 & k \\ k & 0 & k \end{vmatrix} = k(k-3+k) + k(1-k) = 2k^2 - 3k + k - k^2 =$

$\det A = k^2 - 2k = k(k-2)$

c) TEOREMA DI ROUCHÉ-CAPELLI:

- $rkA \neq rkA|b \Rightarrow 0$ SOL
- $rkA = rkA|b$
  - $= n \Rightarrow 1$ SOL $\quad n =$ NUMERO DI COLONNE
  - $< n \Rightarrow \infty^{n-rkA}$ SOL

$\begin{vmatrix} 1 & 1 & 3k \\ k & 1 & k \\ k & 0 & k \end{vmatrix}$

$rkA = 3 \Leftrightarrow \det A \neq 0 \Rightarrow k \neq 0 \wedge k \neq 2$

$\left|\begin{array}{cccc} 1 & 1 & 3k & 1 \\ k & 1 & k & 2 \\ k & 0 & k & 2 \end{array}\right|$ M $\quad rkA|b = 3 \Leftrightarrow \det M \neq 0$

$\det A|b \; 0 - (6 - 2k - k) = 3k - 6 \neq 0 \Rightarrow k \neq 2$

$k = 0$

$rk(A|b) \left|\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right| = 3$

$rkA \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 2$

$k = 2$

$rk(A|b) \left|\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 2 & 0 & 2 & 2 \end{array}\right| = \left|\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right|$

$= \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} \; rk(A|b) = 2$

$rkA = rk \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 0 & 2 \end{vmatrix} = rk \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2$

IN CONCLUSIONE:

$$\begin{cases} k = 0 & 0 \text{ SOLUZIONI} \\ k = 2 & \infty^1 \text{ SOLUZIONI} \\ k \in \mathbb{R} \setminus \{0, 2\} & 1 \text{ SOLUZIONE} \end{cases}$$

TEOREMA DI CRAMER

$$x_i = \frac{\det B_i}{\det A}$$, con B_i la matrice A in cui la colonna i-esima è sostituita dalla colonna b dei termini noti

$$\det B_x = \det \begin{vmatrix} 1 & 1 & 3-k \\ 2 & 1 & k \\ 2 & 0 & k \end{vmatrix} = 2(k-3+k)+k(1-2) = 4k-6-k = 3k-6$$

$$\det B_y = \det \begin{vmatrix} 1 & 1 & 3k \\ k & 1 & k \\ k & 2 & k \end{vmatrix} = \det \begin{vmatrix} 1 & 1 & 3k \\ k & 1 & k \\ 0 & 1 & 0 \end{vmatrix} = -(k-3k+k^2) = -k^2+2k = k(-k+2)$$

$$\det B_z = \det \begin{vmatrix} 1 & 1 & 1 \\ k & 1 & 2 \\ k & 0 & 2 \end{vmatrix} = \det \begin{vmatrix} 1 & 1 & 1 \\ k & 1 & 2 \\ 0 & -1 & 0 \end{vmatrix} = 2-k$$

$$x = \frac{3(k-2)}{k(k-2)} = \frac{3}{k} \qquad y = \frac{-k(k-2)}{k(k-2)} = -1 \qquad z = \frac{2-k}{k(k-2)} = \frac{1}{k}$$

$\forall k \neq 0,2$