11 - CONTINUITÀ E DERIVABILITÀ

STUDIARE LA CONTINUITÀ DELLE FUNZIONI NEL PUNTO INDICATO

$! \lim_{x\to x_p^-} f(x) = \lim_{x\to x_p^+} f(x)$

a) $f(x) = \begin{cases} \dfrac{\sin(x^2)}{x(\sqrt{1+x}-1)} & x < 0 \\ a\cdot 2^x + 3 & x \geq 0 \end{cases}$ $\quad x_p = 0$

- $\lim_{x\to 0^+} a\cdot 2^x + 3 = a+3$
- $\lim_{x\to 0^-} \dfrac{\sin(x^2)}{x(\sqrt{1+x}-1)} = \lim_{x\to 0^-} \dfrac{x^2+o(x^2)}{x(\frac{1}{2}x)+o(x)} = \lim_{x\to 0^-} \dfrac{2x^2}{x^2} = 2$

$\Rightarrow 2 = a+3 \rightarrow a=-1$

STUDIARE LA DERIVABILITÀ DELLE SEGUENTI FUNZIONI

a) $f(x) = |x^2-1|$

PER PROPRIETÀ DEL MODULO, $f(x)$ È NON DERIVABILE IN $f(x)=0$, OSSIA IN $x=\pm 1$. SI VERIFICHI CON LA DEFINIZIONE DI DERIVATA

$$\lim_{x\to -1} \frac{f(x)-f(-1)}{x+1} = \lim_{x\to -1} \frac{|x^2-1|-0}{x+1} = \lim_{x\to -1} \frac{|x-1|\cdot|x+1|}{x+1} = \pm 2$$

$\Rightarrow$ NON È RISPETTATO IL CRITERIO DI UNICITÀ

b) $f(x) = e^{-|x|}$ IN $x=0$ $\qquad \lim_{x\to 0} e^x - 1 = x$

$$\left.\begin{aligned} f'_+ &= \lim_{x\to 0^+} \frac{e^{-x}-e^{-0}}{x} = \lim_{x\to 0^+} \frac{e^{-x}-1}{x} = \lim_{x\to 0^+} \frac{-x}{x} = -1 \\ f'_- &= \lim_{x\to 0^-} \frac{e^{x}-1}{x} = \lim_{x\to 0^-} \frac{x}{x} = 1 \end{aligned}\right\} \Rightarrow f'_+ \neq f'_-$$

$\Downarrow$

NON DERIVABILE

c) $x^2 \lt \frac{1}{x^2}$

SI OSSERVI CHE $f(x)=\begin{cases} x^2 & |x|\leq 1 \\ \frac{1}{x^2} & |x|>1 \end{cases}$ $\quad x\neq 0$

CONTINUITÀ IN $x=1$ $\quad \lim_{x\to 1^-} x^2 = \lim_{x\to 1^+} \frac{1}{x^2} \quad 1=1 \quad f(1)=1$

$$f'_- = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} (x+1) = 1$$

$$f'_+ = \lim_{x\to 1^+} \frac{1/x^2-1}{x-1} = \lim_{x\to 1^+} \frac{1-1}{x^2(x-1)} \quad !$$

$$= \lim_{x\to 1^+} \frac{\cancel{\frac{1}{x^2}}(1-x^2)}{\cancel{\frac{1}{x^2}}(x^3-x^2)} = \lim_{x\to 1^+} \frac{(1-x)(1+x)}{-x^2(1-x)} = -2$$

$\Longrightarrow$ NON DERIVABILE IN $x=1$

DETERMINARE a E/O b IN MODO CHE LA FUNZIONE ASSEGNATA SIA SEMPRE CONTINUA E DERIVABILE

a) $f(x)=\begin{cases} (x-b)^2-2 & x\geq 0 \\ a\sin x & x\lt 0 \end{cases}$

CONTINUITÀ: $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) \qquad x_0=0$

$$\lim_{x\to 0^-} (x-b)^2-2 = \lim_{x\to 0^+} a\sin(x) \to b^2-2=0$$

DERIVABILITÀ: $\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$

$$\lim_{x\to 0^-} \frac{(x-b)^2-2-b^2+2}{x} = \lim_{x\to 0^+} \frac{a\sin x - 0}{x} \; 1$$

$$\lim_{x\to 0^-} \frac{x^{\cancel{2}}-2\cancel{x}b+\cancel{b^2}-\cancel{2}-\cancel{b^2}+\cancel{2}}{\cancel{x}} = a \quad \to -2b=a$$

$\begin{cases} b^2-2=0 \\ a=-2b \end{cases} \qquad \begin{cases} b=\pm 2 \\ a=\mp 2\sqrt{2} \end{cases}$

b) $f(x) = \begin{cases} 2x & x \leq 0 \\ x+a & x > 0 \end{cases}$

CONTINUITÀ: $\lim_{x \to 0^-} 2x = \lim_{x \to 0^+} x+a$ CONTINUA PER $a=0$ $f(0)=0$

DERIVABILITÀ: $\lim_{x \to 0^-} \frac{2x}{x} = \lim_{x \to 0^+} \frac{x}{x}$ $\quad 2=1 \Rightarrow$ NON DERIVABILE IN $x=0$

c) $f(x) = \begin{cases} x^2 & x \leq 1 \\ ax+b & x > 1 \end{cases}$

CONTINUITÀ: $\lim_{x \to 1^-} x^2 = \lim_{x \to 1^+} ax+b$

DERIVABILITÀ: $\lim_{x \to 1^-} \frac{x^2-1}{x-1} = \lim_{x \to 1^+} \frac{ax+b-a-b}{x-1}$

$$\lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1^+} \frac{a\cancel{(x-1)}}{\cancel{x-1}}$$

$$\begin{cases} 1 = a+b \\ 2 = a \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$$

d) $f(x) = \begin{cases} ax^2+3x+b & x \leq 1 \\ x+a & x > 1 \end{cases}$

$$\begin{cases} \lim_{x \to 1^-} (ax^2+3x+b) = \lim_{x \to 1^+} (x+a) \\ \lim_{x \to 1^-} \frac{ax^2+3x+\cancel{b}-a-3-\cancel{b}}{x-1} = \lim_{x \to 1^+} \frac{\cancel{x+a-1-a}}{\cancel{x-1}} \quad 1 \end{cases}$$

$$\begin{array}{c|cc|c} & a & 3 & -a-3 \\ 1 & & a & a+3 \\ \hline & a & 3+a & 0 \end{array}$$

$(x-1)(ax+3+a)$

$$\begin{cases} a+3+b = a+1 \\ \lim_{x \to 1^-} \frac{\cancel{(x-1)}(ax+3+a)}{\cancel{x-1}} = 3+2a = 1 \end{cases} \qquad \begin{cases} a = -1 \\ b = -2 \end{cases}$$

e) $$f(x)=\begin{cases}3x+a & x\le 2\\ x^2+bx+2 & x>2\end{cases}$$

$$\begin{cases}\lim_{x\to 2^-} 3x+a=\lim_{x\to 2^+}(x^2+bx+2)\\ \lim_{x\to 2^-}\dfrac{3x+a-6-a}{x-2}=\lim_{x\to 2^+}\dfrac{x^2+bx+2-4-2b-2}{x-2}\end{cases}$$

	1	b	$-4-2b$
		2	$4+2b$
2	1	$2+b$	0

$$\lim_{x\to 2^-}\frac{3(x-2)}{x-2}=3 \qquad \lim_{x\to 2^+}\frac{x^2+bx-4-2b}{x-2}=\lim_{x\to 2^+}\frac{(x-2)(x+2+b)}{x-2}=4+b$$

$$\begin{cases}a+6=6+2b\\ 3=4+b\end{cases}\qquad\begin{cases}a=-2\\ b=-1\end{cases}$$

f) $$f(x)=\begin{cases}2e^{ax^2}-b & x\ge 0\\ -ax^2+b & x<0\end{cases}$$

$$\begin{cases}\lim_{x\to 0^-}2e^{ax^2}-b=\lim_{x\to 0^+}(-ax^2+b)\\ \lim_{x\to 0^-}\dfrac{2e^{ax^2}-b-2+b}{x}=\lim_{x\to 0^+}\dfrac{-ax^2+b-b}{x}\end{cases}$$

$$\begin{cases}2-b=b\\ \lim_{x\to 0^-}\dfrac{2e^{ax^2}-2}{x}=0\end{cases}\qquad\begin{cases}\forall a\in\mathbb{R}\ \text{(GERARCHIA)}\\ b=1\end{cases}$$

g) $$f(x)=\begin{cases}be^{2x-2}+ax^2+1 & x\ge 1\\ -2ax+3be^{2x-2} & x<1\end{cases}$$

$$\begin{cases}\lim_{x\to 1^-}(-2ax+3be^{2x-2})=\lim_{x\to 1^+}(be^{2x-2}+ax^2+1)\\ \lim_{x\to 1^-}\dfrac{-2ax+3be^{2x-2}+2a-3b}{x-1}=\lim_{x\to 1^+}\dfrac{be^{2x-2}+ax^2+\cancel{1}-b-a-\cancel{1}}{x-1}\end{cases}$$

$$\lim_{x\to 1^-}\frac{-2a(x-1)+3b(e^{2x-2}-1)}{x-1}=\lim_{x\to 1^-}\frac{-2a(x-1)+3b(1+\frac{2x-2}{1}-1)}{x-1}=$$

$$=\lim_{x\to 1^-}\frac{-2a(x-1)+6b(x-1)}{x-1}=-2a+6b$$

$$\lim_{x\to 1^+}\frac{a(x^2-1)+b(e^{2x-2}-1)}{x-1}=\lim_{x\to 1^+}a(x+1)+2b=2a+2b$$

$$\begin{cases}-2a+3b=b+a+1\\-2a+6b=2a+2b\end{cases}\Rightarrow\begin{cases}-3a=-2b+1\\-4a=-4b\to a=b\end{cases}$$

$$\begin{cases}-3a=-2a+1\\b=a\end{cases}\Rightarrow\begin{cases}a=-1\\b=-1\end{cases}$$

h) $f(x)=\begin{cases}\frac{1-\cos(2x)}{x} & x<0\\ e^x(ax+b) & x\geq 0\end{cases}$

$$\lim_{x\to 0^-}\frac{1-\cos(2x)}{x}=\lim_{x\to 0^-}(e^x(ax+b))$$

$$\lim_{x\to 0^-}\frac{1-1+\frac{(2x)^2}{2}}{x}=2x\to 0 \qquad 0=e^0(0+b)\to b=0 \quad f(0)=0$$

$$\lim_{x\to 0^-}\frac{1-\cos(2x)}{x^2}=\lim_{x\to 0^+}\frac{e^x(a\cancel{x})}{\cancel{x}}$$

$$\lim_{x\to 0^-}\frac{\cancel{1}-\cancel{1}+2\cancel{x^2}}{\cancel{x^2}}=2=\lim_{x\to 0^+}e^x a=a\to a=0$$

i) $f(x)=\begin{cases}a\sqrt{x+4}-6 & -4\leq x<0\\ \ln(bx+1)+2b & x\geq 0\end{cases}$

$$\begin{cases}\lim_{x\to 0^-}(a\sqrt{x+4}-6)=\lim_{x\to 0^+}(\ln(bx+1)+2b)\\ \lim_{x\to 0^-}\frac{a\sqrt{x+4}\cancel{-6}-2a\cancel{+6}}{x}=\lim_{x\to 0^+}\frac{\ln(bx+1)\cancel{+2b}-\cancel{2b}}{x}\end{cases}$$

- $2a-6=2b \rightarrow a-3=b$
- $\lim_{x\to 0^-} \frac{a\sqrt{x+4}-2a}{x} = \lim_{x\to 0^-} \frac{2a\sqrt{1+\frac{x}{4}}-2a}{x} =$

$= \lim_{x\to 0^+} \frac{2a(\cancel{1}+\frac{x}{8}+o(x))-\cancel{2a}}{x} = \lim_{x\to 0} \frac{2a\cancel{x}/8}{\cancel{x}} = \frac{a}{4}$

$\lim_{x\to 0^+} \frac{\ln(bx+1)}{x} = \lim_{x\to 0^+} \frac{bx+o(x)}{x} = b$

$\begin{cases} a-3=b \\ b=a/4 \end{cases} \rightarrow \begin{cases} a=4 \\ b=1 \end{cases}$

j) $f(x) = \begin{cases} (2-a)x+1 & x\leq 0 \\ \ln(x-a) & x>0 \end{cases}$

$\lim_{x\to 0^-} (2-a)x+1 = \lim_{x\to 0^+} \ln(x-a)$

$\ln(-a)=1 \Rightarrow a=-e \qquad f(0)=1$

DERIVABILITÀ

$\lim_{x\to 0^-} \frac{(2+e)x}{x} = \lim_{x\to 0^+} \frac{\ln(x+e)-1}{x}$

$2+e = \lim_{x\to 0^+} \frac{\ln(x+e)-\ln(e)}{x} = \lim_{x\to 0^+} \frac{\ln(\frac{x}{e}+1)}{x} = \lim_{x\to 0^+} \frac{x/e}{x} = \frac{1}{e}$

$\Rightarrow 2+e \neq 1/e \Rightarrow$ NON DERIVABILE

k) $f(x) = \begin{cases} ax^2+bx & x<0 \\ cx & x\geq 0 \end{cases}$

$\begin{cases} 0=0 \\ \lim_{x\to 0^-} \frac{ax^2+bx}{x} = \lim_{x\to 0^+} \frac{cx}{x} \end{cases} \Rightarrow \begin{cases} \forall a \in \mathbb{R} \\ b=c \end{cases}$

l) $f(x)=\begin{cases} \arctan(1/x) & x<0 \\ ax^2+bx+c & x\geq 0 \end{cases}$

$$\begin{cases} \lim_{x\to 0^-} \arctan(1/x) = \lim_{x\to 0^+} (ax^2+bx+c) \\ \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = 2ax+b \end{cases}$$

$$\begin{cases} -\pi/2 = c \\ * \end{cases}$$

$$\frac{(n^2-2n)(n+1)}{n} = (n-2)(n+1)$$

$-1 \quad +2$

$$* \begin{cases} \frac{2x}{(1+x^2)^2} & x<0 \\ 2a & x>0 \end{cases}$$

$2a=0$

$$\lim_{x\to 0^-} \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = \lim_{x\to 0^-} -\frac{1}{1+x^2} = \lim_{x\to 0^+} (2ax+b)$$

$-1=b$

m) $f(x)=\begin{cases} (1-x^2)\ln(1-x) & x<1 \\ (2x+b)(ax-1) & x\geq 1 \end{cases}$

CONTINUITÀ

$$\lim_{x\to 1^-} (1-x^2)\ln(1-x) = \lim_{x\to 1^+} (2x+b)(ax-1)$$

$\Rightarrow b=-2 \vee a=1$

$$\lim_{x\to 1^-} (1-x)(1+x)(-x) = (2+b)(a-1) \to (2+b)(a-1)=0$$

$a=1$ DERIVABILITÀ $b=-2$

$$\lim_{x\to 1^-} \frac{(1-x^2)\ln(1-x)}{x-1} = \lim_{x\to 1^+} \frac{(2x+b)(x-1)}{x-1}$$

$2=2+b \qquad (1+x)(x)$

$b=0$

$$\lim_{x\to 1^-} \frac{(1-x^2)\ln(1-x)}{x-1} = \lim_{x\to 1^+} \frac{(2x-2)(ax-1)}{x-1}$$

$2=2(a-1)$

$a=1$, IMPOSSIBILE POICHÈ

$b=-2 \Leftrightarrow a\neq 1 \Rightarrow$ NON DERIVABILE

VERIFICARE CHE LA FUNZIONE SIA PROLUNGABILE PER CONTINUITÀ IN $x=0$ E, IN CASO AFFERMATIVO, SE SIA DERIVABILE IN $x=0$

a) $f(x)=x^2\log(|x|)$

CONTINUITÀ

$$\lim_{x\to 0^-} x^2\log(|x|)=\lim_{x\to 0^+} x^2\log(|x|)$$

$\lim_{x\to 0^+} x^\alpha \log x = 0 \quad \forall \alpha>0$ PER GERARCHIA $\quad \alpha=2$

$0=0 \Rightarrow$ CONTINUA E $f(0)=0$

DERIVABILITÀ

$$\lim_{x\to 0}\frac{f(x)-0}{x-0}=\lim_{x\to 0}\frac{x^2\log(|x|)}{x}=\lim_{x\to 0} x\log(|x|)=0$$

$\Rightarrow$ DERIVABILE

b) $|x|^x$

$$|x|^x=e^{\log(|x|^x)}=e^{x\log(|x|)}$$

CONTINUITÀ

$$\lim_{x\to 0^-} e^{x\log(|x|)}=\lim_{x\to 0^+} e^{x\log(|x|)}$$

$e^0=e^0 \Rightarrow$ CONTINUA E $f(0)=1$

DERIVABILITÀ

$$\cancel{1}+x\log(|x|)+o(x\log(|x|))$$

$$\lim_{x\to 0}\frac{e^{x\log(|x|)}-1}{x}\cdot\frac{\log|x|}{\log|x|}=\lim_{x\to 0}\frac{e^{x\log(|x|)}\cancel{-1}}{x\log(|x|)}\cdot\log(|x|)$$

$=1\cdot(-\infty)=-\infty \Rightarrow$ NON DERIVABILE

DATO $f(x)$ DERIVABILE, DIMOSTRARE CHE:

a) $f(x)$ PARI $\Rightarrow f'(x)$ DISPARI

b) $f(x)$ DISPARI $\Rightarrow f'(x)$ PARI

a) $f(x)$ PARI $f(x) = f(-x)$

$$\frac{d}{dx} f(x) = \frac{d}{dx} f(-x) \Rightarrow f'(x) = f'(-x) \cdot (-1)$$

$f'(x) = -f'(-x) \Rightarrow f'(x)$ DISPARI

b) $f(x)$ DISPARI $f(x) = -f(-x)$

$$\frac{d}{dx} f(x) = \frac{d}{dx} - f(-x) \quad f'(x) = -(-f'(-x)) = f'(-x)$$

$f'(x) = f'(-x) \Rightarrow f'(x)$ PARI

12 - DERIVATE

a) $f(x) = x^5 - 3x^2 + \pi$

$$f'(x) = \frac{d}{dx}(x^5 - 3x^2 + \pi) = \frac{d}{dx}x^5 - \frac{d}{dx}3x^2 - \frac{d}{dx}\pi = 5x^4 - 6x$$

b) $f(x) = x e^{2x}$

$$f'(x) = \frac{d}{dx}x \cdot e^{2x} + x \cdot \frac{d}{dx}e^{2x} = e^{2x} + x \cdot \left(e^{2x} \cdot \frac{d}{dx}2x\right) =$$

$$= e^{2x} + 2xe^{2x} = (1+2x)e^{2x}$$

c) $f(x) = e^{-x^2}$

$$f'(x) = \frac{d}{dx}e^{-x^2} = e^{-x^2} \cdot \frac{d}{dx}(-x^2) = -2xe^{-x^2}$$

d) $f(x) = \dfrac{x^2+3}{x^4-4}$

$$f'(x) = \frac{\frac{d}{dx}(x^2+3)\cdot(x^4-4) - (x^2+3)\cdot\frac{d}{dx}(x^4-4)}{(x^4-4)^2} =$$

$$= \frac{(2x)(x^4-4) - (x^2+3)(4x^3)}{(x^4-4)^2} = \frac{2x^5 - 8x - 4x^5 - 12x^3}{(x^4-4)^2} =$$

$$= \frac{-2x^5 - 12x^3 - 8x}{(x^4-4)^2} = \frac{-2x(x^4+6x^2+4)}{(x^4-4)^2}$$

e) $f(x) = \dfrac{\cos^3 x}{\sin x}$

$$f'(x) = \frac{3\cos^2 x \cdot (-\sin x) \cdot \sin x - \cos^3 x \cdot \cos x}{\sin^2 x} = -\frac{\cos^2 x(3\sin^2 x + \cos^2 x)}{\sin^2 x}$$

P) $f(x) = \dfrac{e^{3x}}{x^2-3x}$

$$f'(x) = \frac{3(x^2-3x)e^{3x} - (2x-3)e^{3x}}{(x^2-3x)^2} = \frac{(3x^2-11x+3)e^{3x}}{(x^2-3x)^2}$$

g) $f(x)=\sqrt{x+2x^2}$

$$f'(x)=\frac{1}{2}(x+2x^2)^{-1/2}(4x+1)=\frac{4x+1}{2\sqrt{2x^2+x}}$$

h) $f(x)=\frac{1}{\sqrt[3]{3+4x^2}}$

$$f'(x)=\frac{d}{dx}(4x^2+3)^{-1/3}=-\frac{1}{3}\cdot(4x^2+3)^{-\frac{1}{3}-1}\cdot 8x=-\frac{8x}{3(4x^2+3)^{4/3}}$$

i) $f(x)=\frac{x^2}{\sqrt{x-3}}$

$$f'(x)=\frac{2x\sqrt{x-3}-\frac{1}{2}(x-3)^{-1/2}\cdot x^2}{x-3}=\frac{2x(x-3)-\frac{1}{2}x^2}{(x-3)^{3/2}}=$$

$$=\frac{4x^2-12x-x^2}{2(x-3)^{3/2}}=\frac{3x(x-4)}{2(x-3)^{3/2}}$$

j) $f(x)=\frac{1}{\tan^3 x}$

$f(x)=\frac{\cos^3 x}{\sin^3 x}$ $\quad f'(x)=\frac{3\cos^2 x(-\sin x)\sin^3 x-3\sin^2 x\cos^4 x}{\sin^6 x}=$

$$=\frac{-3\cos^2 x\sin^{\cancel{4}\,2}x-3\cancel{\sin^2 x}\cos^4 x}{\sin^{\cancel{6}\,4}x}=\frac{-3\cos^2 x\sin^2 x-3\cos^4 x}{\sin^4 x}=$$

$$=\frac{-3\cos^2 x(1+\cancel{\cos^2 x})-\cancel{3\cos^4 x}}{\sin^4 x}=-\frac{3\cos^2 x}{\sin^4 x}$$

k) $f(x)=\ln(x^2-x+3)$ $\qquad (6x+1)(2x+3)$

$$f'(x)=\frac{1}{x^2-x+3}\cdot(2x-1)=\frac{2x-1}{x^2-x+3}$$

l) $f(x)=\frac{\sqrt{2x+3}}{3x^2+x+1}$

$$f'(x)=\frac{\frac{1}{2}(2x+3)^{-1/2}\cdot 2\cdot(3x^2+x+1)-(6x+1)\sqrt{2x+3}}{(3x^2+x+1)^2}=$$

$$=\frac{(2x+3)^{-1/2}\cdot(3x^2+x+1)-(6x+1)(2x+3)^{1/2}}{(3x^2+x+1)^2}=$$

$$=\frac{3x^2+x+1-12x^2-20x-3}{(3x^2+x+1)^2\sqrt{2x+3}}=-\frac{9x^2+19x+2}{(3x^2+x+1)^2\sqrt{2x+3}}$$

m) $f(x) = \arctan(1/x)$

$$f'(x) = \frac{1}{1+(1/x)^2} \cdot (-1/x^2) = -\frac{1/x^2}{1+1/x^2} = -\frac{1}{x^2+1}$$

n) $f(x) = \ln(\cos x)$

$$f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

o) $f(x) = x^3 \ln(5x^2+x)$

$$f'(x) = 3x^2 \ln(5x^2+x) + x^{\cancelto{2}{3}} \cdot \frac{1}{5x^{\cancel{2}}+\cancelto{1}{x}} (10x+1)$$

$$= 3x^2 \ln(5x^2+x) + \frac{x^2(10x+1)}{5x+1}$$

p) $f(x) = \arctan(\sqrt{e^{2x}-1})$

$$f'(x) = \frac{1}{\cancel{1}+\cancel{e^{2x}}-\cancel{1}} \cdot \left(\cancel{\frac{1}{2}} \cdot \frac{1}{\sqrt{e^{2x}-1}}\right) \cdot \cancel{e^{2x}} \cdot \cancel{2} = \frac{1}{\sqrt{e^{2x}-1}}$$

13 - SVILUPPI DI TAYLOR

SVILUPPARE LE SEGUENTI FUNZIONI FINO ALL'ORDINE INDICATO

a) $f(x) = \ln(1+\sin x)$; $n=3$

$$\ln(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} + o(x^3) =$$

$$= x + \frac{x^3}{6} - \frac{x^2}{2} + o(x^3)$$

b) $h(x) = \ln(\cos x)$; $n=4$

$$f(x) = \ln(1 + (\cos x - 1)) = \cos x - 1 - \frac{(\cos x - 1)^2}{2} + o(x^4)$$

$$= 1 - \frac{x^2}{2} - 1 - \frac{\cos^2 x - 2\cos x + 1}{2} + o(x^4)$$

$$= -\frac{x^2}{2} - \frac{(1-\frac{x^2}{2})^2 - 2(1-\frac{x^2}{2}) + 1}{2} + o(x^4)$$

$$= -\frac{x^2}{2} - \frac{1 - x^2 - \frac{x^4}{6} - 2 + x^2 + 1}{2} + o(x^4) = -\frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$$

c) $f(x) = \frac{1}{1+x+x^2}$; $n=4$

$$f(x) = 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 + o(x^4) =$$

$$= 1 - x + x^2 + x^2 + 2x^3 + x^4 - x^3 - 3x^4 + x^4 + o(x^4) =$$

$$= 1 - x + x^3 - x^4 + o(x^4)$$

d) $f(x) = \sqrt{\cosh x}$; $n=4$

$$f(x) = [1 + (\cosh(x) - 1)]^{1/2} = 1 + \frac{1}{2}(\cosh(x) - 1) - \frac{\frac{1}{2}\cdot\frac{1}{2}}{2}(\cosh(x) - 1)^2$$

$$= 1 + \frac{1}{2}\left(\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right) - \frac{1}{8}\left(\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right) = 1 + \frac{x^2}{4} + \frac{x^4}{48}$$

$$- \frac{x^2}{16} - \frac{x^4}{192} + o(x^4) = 1 + \frac{3}{16}x^2 + \frac{3}{192}x^4 + o(x^4)$$

CALCOLARE I SEGUENTI LIMITI SFRUTTANDO LE SERIE DI TAYLOR

a) $$\lim_{x\to 0}\frac{e^{x}-1+\ln(1-x)}{\tan x-x}=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2}+\frac{x^3}{3!}-1+(-x)-\frac{(-x)^2}{2}+\frac{(-x)^3}{3}}{x-\frac{x^3}{3}-x}=$$

$$=\lim_{x\to 0}\frac{-x^3/6}{x^3/3}=-\frac{1}{2}$$

b) $$\lim_{x\to 0}\frac{e^{x^2}-\cos x-\frac{3}{2}x^2}{x^4}=\lim_{x\to 0}\frac{1+x^2+\frac{x^4}{2}-1+\frac{x^2}{2}-\frac{x^4}{4!}-\frac{3}{2}x^2}{x^4}=$$

$$=\frac{1}{2}-\frac{1}{4!}=\frac{1}{2}-\frac{1}{24}=\frac{11}{24}$$

c) $$\lim_{x\to 0}\frac{\ln(1+x\arctan(x))+1-e^{x^2}}{\sqrt{1+2x^4}-1}=$$

$$\lim_{x\to 0}\frac{\left(x\arctan(x)-\frac{x^2\arctan^2(x)}{2}\right)+\theta(x^4)+1-\left(1+x^2+\frac{x^4}{2}\right)+\theta(x^4)}{x^4+\theta(x^4)}=$$

$$=\lim_{x\to 0}\frac{x\left(x-\frac{x^3}{3}\right)+\theta(x^3)-\frac{x^2}{2}\left(x-\frac{x^3}{3}\right)^2-x^2-\frac{x^4}{2}+\theta(x^3)}{x^4}=$$

$$=\lim_{x\to 0}\frac{x^2-\frac{x^4}{3}-\frac{x^4}{2}-x^2-\frac{x^4}{2}}{x^4}=-\frac{1}{3}-1=-\frac{4}{3}$$

d) $$\lim_{x\to \infty}\left(x-x^2\ln\left(1+\sin\tfrac{1}{x}\right)\right)$$

$$g\left(t=\tfrac{1}{x}\right)=\frac{\ln(1+\sin(t))}{t^2}\Rightarrow \lim_{t\to 0}=\frac{\ln(1+\sin(t))}{t^2}=\lim_{t\to 0}\frac{t-\frac{t^2}{2}+\theta(t^2)}{t^2}$$

$$=\lim_{t\to 0}\frac{1}{t}-\frac{1}{2}\Rightarrow g(x)=x-\frac{1}{2}$$

$$\lim_{x\to\infty}\left(x-\left(x-\tfrac{1}{2}\right)\right)=\frac{1}{2}$$

e) $$\lim_{x\to 0}\frac{5^{1+\tan^2 x}-5}{1-\cos x}=\lim_{x\to 0}\frac{5\left(5^{\tan^2 x}-1\right)}{x^2/2+\theta(x^2)}=10\lim_{x\to 0}\frac{e^{\tan^2 x\ln 5}-1}{x^2+\theta(x^2)}=$$

- $$e^{\tan^2 x\ln 5}-1=1+\ln 5\tan^2 x+\theta(x^2)-1=\ln 5\left(x+\frac{x^3}{3}+\theta(x^3)\right)^2$$

$$+\theta(x^2)=\ln 5\cdot x^2+\theta(x^2)=10\lim_{x\to 0}\frac{\ln 5\,\cancel{x^2}+\cancel{\theta(x^2)}}{\cancel{x^2}+\cancel{\theta(x^2)}}=10\ln 5$$