EXTRA: GIVEN $R \subseteq \mathbb{N} \times \mathbb{N}$ DEFINED AS $nRm \Leftrightarrow m = n^2$ :

- IS R A FUNCTION?
- IF THEY EXIST, FIND LEFT-INVERSE AND RIGHT-INVERSE

- $m = n^2$. $\forall n$ THERE ARE DIFFERENT $m \Rightarrow$ R IS A FUNCTION
- $n = \sqrt{m}$

  - INJECTIVITY: $m \neq \Rightarrow n \neq \Rightarrow$ IT IS INJECTIVE
  - SURJECTIVITY: FOR SOME $m, n \notin \mathbb{N} \Rightarrow$ IT IS NOT SURJECTIVE

$\Rightarrow$ THE FUNCTION HAS ONLY RIGHT INVERSE $g(b)$ SUCH THAT

$g(f(a)) = a \quad \forall a \in A$ $\qquad$ $g(f(a)) = a$

IF $f(a) = a^2$ AND $g(b) = \sqrt{b}$

$g(f(a)) = g(a^2) = \sqrt{a^2} = a$

$f(g(b)) = f(\sqrt{b}) = (\sqrt{b})^2 = b$

(ONLY FOR SQUARE NUMBERS, BECAUSE $\sqrt{b}$ IS NOT ALWAYS $\in \mathbb{N}$)

$\Rightarrow$ IT CAN'T BE A LEFT INVERSE

LET $f$ IN THE DIAGRAM BELOW BE A FUNCTION, $f=pm$ ITS CANONICAL FACTORIZATION, $E$ THE KERNEL PAIR OF $f$, $p$ THE PROJECTION ONTO THE QUOTIENT AND $m$ THE INDUCED MONOMORPHISM. PROVE THAT:

$E = ff^{op}$, $m = p^{op}f$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \\ A/E \arrow[ur, "m"'] & \end{tikzcd}$$

1. $$x\,ff^{op}\,y \iff \exists z\,(x f z \wedge z f^{op} y)$$
$$\iff \exists z\,(x f z \wedge y f z)$$
$$\iff \exists z\,(f(x) = z \wedge f(y) = z)$$
$$\iff (f(x) = f(y))$$
$$\iff xEy$$

2. $$\left.\begin{array}{l} p \text{ FUNCTION} \Rightarrow p^{op}p \subseteq 1 \\ p \text{ SURJECTIVE} \Rightarrow 1 \subseteq p^{op}p \end{array}\right\} \Rightarrow p^{op}p = 1$$

$$p^{op}f = p^{op}pm = 1m = m$$

THE MATRICES BELOW REPRESENT FUNCTIONS $f:[m]\to[n]$ FOR SUITABLE $m$ AND $n$. COMPUTE THEIR CANONICAL FACTORIZATION

$f = pm \qquad K = \ker(f) = f f^{op} = p p^{op} \qquad m = p^{op} f \qquad p : A_{/K} \to B$

a) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \qquad f^{op} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$

$K = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} =$

$$= \begin{array}{c} \begin{matrix} [1] & [2] & [3] & [4] \end{matrix} \\ \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \end{array} \qquad [1]=[3];\ [2]=[4]$$

$p = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad p^{op} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}$

$m = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} =$

$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$

$f = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$