- DIVISION $\frac{x}{y} := \exists z(x = zy)$
- $y \neq 0 := \neg(y = 0)$
- QUOTIENT AND REMAINDER $:= y = rx + q$
- $r < y := r \leq y \wedge \neg(r = y)$
- $F(x, y, r, q) := (y = rx + q) \wedge r < y$
- q, r UNIQUE $\Rightarrow$

$$\exists(F(x,y,r,q) \wedge \forall w(F(x,y,w,q) \to w = r) \wedge \forall k(F(x,y,r,k) \to k = q))$$

- $\frac{x}{y} \wedge y \neq 0 \to \exists rq(y = rx + q \wedge r < y \mid \forall wk(F(x,y,w,k) \to ((w = r) \wedge (k = q)))$

3) EVERY EVEN NUMBER GREATER THAN 2 IS THE SUM OF TWO PRIMES

- EVEN NUMBER $e := \exists x(e = 2x)$
- GREATER $> := \neg(\leq)$
- PRIME NUMBER $P(a) := \exists c(a = kc \leftrightarrow (\exists k((k = 1) \vee (k = a))))$

$\forall e > 2 \to \exists a, b(e = P(a) + P(b))$ (?)

4) IF x, y, z ARE POSITIVE AND $x^3 + y^3 = z^3$, THEN x, y, z ARE NOT COPRIME

- $x, y, z > 0 := \neg(x, y, z \leq 0)$
- COPRIME $F(x, y) := \exists k(x = ky)$
- $\forall xyz(x > 0 \wedge y > 0 \wedge z > 0 \wedge x^3 + y^3 = z^3 \to \neg F(x,y) \wedge \neg F(x,z) \wedge \neg F(y,z))$

THE LANGUAGE OF THE INTEGERS IS THE EXPANSION OF THE LANGUAGE OF NATURAL NUMBERS. CONSIDER THE FOLLOWING STATEMENTS IN L:

1) DEFINABILITY OF ORDER: GIVEN TWO NUMBERS, THE FIRST IS LESS THAN OR EQUAL TO THE SECOND IFF THE SECOND CAN BE OBTAINED FROM THE FIRST BY ADDING SOME NUMBERS

2) FIRST CANCELLATION LAW: IF ADDING TWO NUMBERS TO THE SAME NUMBER X PRODUCES THE SAME RESULT, THEN THE TWO GIVEN NUMBERS ARE EQUAL

3) SECOND CANCELLATION LAW: IF THE SUM OF TWO NUMBERS IS ZERO, THEN BOTH SUMMANDS MUST BE ZERO.

FORMALIZE IN L THE ASSERTION THAT TRANSITIVITY OF THE ORDER RELATION IS A CONSEQUENCE OF DEFINABILITY OF ORDER, THE CANCELLATION LAWS AND THE ASSOCIATIVE, COMMUTATIVE AND NEUTRAL ELEMENT PROPERTY OF ADDITION

PREMISES:

- $\forall xy(x \leq y \leftrightarrow \exists z(y = x + z))$ DEFINABILITY
- $\forall xyz(x+y = x+z \rightarrow y = z)$ FIRST CANCELLATION
- $\forall xy(x+y=0 \rightarrow x=0 \wedge y=0)$ SECOND CANCELLATION
- $\forall xyz((x+y)+z = x+(y+z))$ ASSOCIATIVITY
- $\forall xy(x+y = y+x)$ COMMUTATIVITY
- $\forall x(x+0=x)$ NEUTRAL ELEMENT

CONCLUSION: $\forall xyz(x \leq y \wedge y \leq z \rightarrow x \leq z)$

THE LANGUAGE OF THE INTEGERS IS THE EXPANSION OF THE LANGUAGE OF NATURAL NUMBERS BY THE UNARY FUNCTION SYMBOL $-$ REPRESENTING THE OPPOSITE. THUS, THE LANGUAGE OF INTEGERS IS THE FIRST ORDER LANGUAGE WITH EQUALITY GENERATED BY THE SIGNATURE $\langle 0, 1, -, +, \times, \leq \rangle$. FORMULATE THE FOLLOWING ASSERTIONS IN THE LANGUAGE OF INTEGERS:

TWO NONZERO INTEGERS ARE COPRIME (THEIR ONLY COMMON DIVISOR IS 1) IF AND ONLY IF 1 CAN BE WRITTEN AS A LINEAR COMBINATION OF THE TWO

NONZERO $\quad x \neq 0 := \neg(x=0)$

DIVISOR $\quad \frac{x}{y} := \exists z\,(y = zx)$

COPRIME $\quad C(x,y) := \forall z\left(\frac{x}{z} \wedge \frac{y}{z} \rightarrow z = 1\right)$

LINEAR COMBINATION $\quad \exists u,v\,(1 = ux + vy)$

$$\forall xy\,(x \neq 0 \wedge y \neq 0 \wedge C(x,y) \leftrightarrow \exists uv\,(1 = ux + vy))$$