# THEOREM 1: SOUNDNESS AND COMPLETENESS VIA CONSISTENCY

ASSUME $\vdash$ IS AN INFERENCE RELATION ON L.

AN INFERENCE RELATION IS:

- SOUND IF AND ONLY IF EVERY SATISFIABLE SET IS CONSISTENT
- COMPLETE IF AND ONLY IF EVERY CONSISTENT SET IS SATISFIABLE

(SOUNDNESS $\rightarrow F \vdash \varphi \Rightarrow F \vDash \varphi$. COMPLETENESS $\rightarrow F \vDash \varphi \Rightarrow F \vdash \varphi$)

(CONSISTENCY $\rightarrow F, \neg\varphi \vDash \perp$ SATISFIABILITY $\rightarrow F, \neg\varphi \vdash \perp$)

THEREFORE, $\vdash$ IS SOUND AND COMPLETE $\Leftrightarrow$ SATISFIABLE SET $\equiv$ CONSISTENT SET

PROOF

F PROOFS $\varphi$

$\varphi$ CAN BE INFERRED FROM F

$\varphi$ IS A CONSEQUENCE OF F

SOUNDNESS $\Leftrightarrow$ $F \vdash \varphi \Rightarrow F \vDash \varphi$

$$\Leftrightarrow \underbrace{F, \neg\varphi \vdash \perp}_{\text{SATISFIABILITY}} \Rightarrow \underbrace{F, \neg\varphi \vDash \perp}_{\text{CONSISTENCY}}$$

COMPLETENESS $\Leftrightarrow$ $F \vDash \varphi \Rightarrow F \vdash \varphi$

$$\Leftrightarrow \underbrace{F, \neg\varphi \vDash \perp}_{\text{CONSISTENCY}} \Rightarrow \underbrace{F, \neg\varphi \vdash \perp}_{\text{SATISFIABILITY}}$$

SOUNDNESS $\wedge$ COMPLETENESS $\Leftrightarrow ((F, \neg\varphi \vdash \perp \Rightarrow F, \neg\varphi \vDash \perp)$

$\wedge (F, \neg\varphi \vDash \perp \Rightarrow F, \neg\varphi \vdash \perp))$

$\vdash$ IS SOUND AND COMPLETE $\Leftrightarrow$ SATISFIABLE SET $\equiv$ CONSISTENT SET

# THEOREM 2: HINTIKKA LEMMA

## EVERY HINTIKKA SET IS SATISFIABLE

A HINTIKKA SET $H \subseteq L$ HAS THE FOLLOWING PROPERTIES:

- IF $x$ IS A VARIABLE AND $x \in H \Rightarrow \neg x \notin H$
- $\bot, \neg T \notin H$
- $\neg\neg\varphi \in H \Rightarrow \varphi \in H$
- $\alpha \in H \Rightarrow \alpha_1, \alpha_2 \in H$
- $\beta \in H \Rightarrow \beta_1 \in H \vee \beta_2 \in H$

PROOF

CONSIDER HINTIKKA SET $H$. DEFINE VALUATION $\alpha \in \Omega^V \mid \forall x, \alpha(x) = 1 \Leftrightarrow x \in H$

WE PROVE $\alpha(\varphi) = 1 \; \forall \varphi \in H$ BY INDUCTION ON $d(\varphi)$ (DEGREE): $\forall \varphi' \mid d(\varphi') < d(\varphi)$, $\alpha(\varphi') = 1$

- $d(\varphi) = 0 \Rightarrow$
  - $\varphi$ IS A VARIABLE $x \in H \Rightarrow \alpha(\varphi) = \alpha(x) = 1$
  - $\varphi$ IS A CONSTANT $\Rightarrow \varphi = T \in H \Rightarrow \alpha(\varphi) = 1$ ! $\bot \notin H$

- $d(\varphi) > 0 \Rightarrow$ $\quad d(\varphi) = d(\neg\neg\psi) = 2 + d(\psi)$
  - $\varphi = \neg\neg\psi \Rightarrow \psi \in H$. $d(\psi) < d(\varphi) \Rightarrow \alpha(\psi) = 1$ BY INDUCTIVE HYPOTHESIS $\Rightarrow \alpha(\varphi) = \alpha(\neg\neg\psi) = \alpha(\psi) = 1$
  - $\varphi = \alpha \Rightarrow \alpha_1, \alpha_2 \in H$. $\underbrace{d(\alpha_1), d(\alpha_2) < d(\varphi)}_{\text{PROOF IS SIMILAR TO} \bullet} \Rightarrow \alpha(\alpha_1) = \alpha(\alpha_2) = 1$ $\Rightarrow \alpha(\varphi) = \alpha(\alpha) = 1$
  - $\varphi = \beta \Rightarrow \beta_1 \vee \beta_2 \in H$. $d(\beta_i) < d(\varphi) \; \forall i \Rightarrow \alpha(\beta_i) == 1$ $\Rightarrow \alpha(\varphi) = \alpha(\beta) = 1$

# THEOREM 3: MAXIMALITY IN A CONSISTENCY CLASS IMPLIES HINTIKKA

IF $C$ IS A CONSISTENCY CLASS, EVERY MAXIMAL ELEMENT $S \in C$ IS A HINTIKKA SET

## PROOF

CONSISTENCY CLASS:

- $X \in S \Rightarrow \neg X \notin S$
- $\bot, \neg\top \notin S$
- $\neg\neg\varphi \in S \Rightarrow S \cup \{\varphi\} \in C$
- $\alpha \in S$ IS A CONJUNCTIVE FORMULA $\Rightarrow S \cup \{\alpha_1, \alpha_2\} \in C$
- $\beta \in S$ IS A DISJUNCTIVE FORMULA $\Rightarrow S \cup \{\beta_1\} \in C \vee S \cup \{\beta_2\} \in C$

MAXIMAL SET: NO ELEMENT OF $C$ CONTAINS $S$ PROPERLY (EXAMPLE: $S \subseteq T \in C \Rightarrow S = T$)

LETS PROVE THE THEOREM:

$$\left.\begin{array}{l} \bullet\ X \in S \Rightarrow X \notin S \\ \bullet\ \bot, \neg\top \notin S \end{array}\right\} \text{DEFINITIONS OF CONSISTENCY CLASS}$$

- $\neg\neg\varphi \in S \Rightarrow S \cup \{\varphi\} \in C$. HOWEVER, $S \subseteq S \cup \{\varphi\}$. $S$ IS MAXIMAL $\Rightarrow S = S \cup \{\varphi\}$, $\varphi \in S$
- $\alpha \in S \Rightarrow S \cup \{\alpha_1, \alpha_2\} \in C$. HOWEVER, $S \subseteq S \cup \{\alpha_1, \alpha_2\}$. $S$ IS MAXIMAL $\Rightarrow S = S \cup \{\alpha_1, \alpha_2\}$ $\alpha_1, \alpha_2 \in S$
- $\beta \in S \Rightarrow S \cup \{\beta_i\} \in C$ FOR AT LEAST ONE INDEX $i$. SINCE $S \subseteq S \cup \{\beta_i\}$ AND $S$ IS MAXIMAL $\Rightarrow S = S \cup \{\beta_i\}$, $\beta_i \in S$

## THEOREM 4: EXISTENCE OF MAXIMAL ELEMENTS IN CONSISTENCY CLASS

IF $C$ IS A LOCALLY FINITE CONSISTENCY CLASS, THEN EVERY $S \in C$ IS CONTAINED IN A MAXIMAL ELEMENT

PROOF

A CONSISTENCY CLASS $C$ IS LOCALLY FINITE WHEN $S \in C$ IF AND ONLY IF EVERY FINITE SUBSET OF $S \in C$

LET $P = \{T \in C \mid S \subseteq T\}$ BE THE SET OF ELEMENTS OF $C$ CONTAINING $S$, PARTIALLY ORDERED BY INCLUSION. WE PROVE THAT $P$ IS INDUCTIVE. SUPPOSE $S_0 \subseteq S_1 \subseteq \dots$ IS AN ASCENDING CHAIN IN $P$. WE CLAIM THAT $S_\infty = \bigcup_{n=0}^{\infty} S_n \in C$.
SUPPOSE $U \subseteq S_\infty$ IS FINITE. EVERY $x \in U$ MUST BELONG TO SOME $S_i \Rightarrow$ $U \subseteq S_n$ FOR SOME $n$, HENCE $U \in C$ BECAUSE $C$ IS DOWNWARD CLOSED AND THEREFORE $S_\infty \in C$ BECAUSE $C$ IS LOCALLY FINITE. NOW CLEARLY $S \subseteq S_\infty$ AND HENCE $S_\infty \in P$. ALSO, $S_\infty$ IS AN UPPER BOUND FOR THE CHAIN.
THUS, $P$ IS INDUCTIVE AND BY THE ZORN LEMMA, IT HAS A MAXIMAL ELEMENT $\bar{S}$, WHICH IS ALSO A MAXIMAL ELEMENT OF $C$ CONTAINING $S$.

# THEOREM 5: THE CONSISTENCY CLASS OF A REGULAR INFERENCE

FOR A REGULAR INFERENCE, THE CLASS OF CONSISTENT SETS $C = \{F \subseteq L \mid F \nvdash \bot\}$

IS A CONSISTENCY CLASS

PROOF

AN INFERENCE RELATION IS REGULAR WHEN:

- $F \vdash \alpha \Rightarrow F \vdash \alpha_1 \wedge F \vdash \alpha_2$
- $F, \beta_1 \vdash \bot \wedge F, \beta_2 \vdash \bot \Rightarrow F, \beta \vdash \bot$
- $F \vdash \top$

WE HAVE TO VERIFY THE CONDITIONS IN THE DEFINITION OF CONSISTENCY CLASS:

- $x, \neg x \in F \Rightarrow F \vdash \bot \Rightarrow$ IF $x \in F \in C$, $\neg x \notin F$
- $\bot \in F \Rightarrow F \vdash \bot \Rightarrow$ IF $F \in C \Rightarrow \bot \notin F$. SIMILARLY, $\neg \top \notin F$
- FOR DOUBLE NEGATION, $\alpha$ AND $\beta$-FORMULAS WE USE THE DERIVATIONS BELOW

| STEP | CLAIM | REASON |
|---|---|---|
| 1 | $\neg\neg\varphi \in F$ | HYPOTHESIS |
| 2 | $F, \varphi \vdash \bot$ | HYPOTHESIS |
| 3 | $F \vdash \neg\neg\varphi$ | 1, ASSUMPTION |
| 4 | $F, \neg\neg\varphi \vdash \varphi$ | PROPOSITION |
| 5 | $F \vdash \varphi$ | 3, 4 CUT |
| 6 | $F \vdash \bot$ | 5, 2 CUT |

IF $F \cup \{\varphi\} \notin C$, F IS INCONSISTENT $\Rightarrow \neg\neg\varphi \in F \in C \Rightarrow F \cup \{\varphi\} \in C$

| STEP | CLAIM | REASON |
|---|---|---|
| 1 | $\alpha \in F$ | HYPOTHESIS |
| 2 | $F, \alpha_1, \alpha_2 \vdash \bot$ | HYPOTHESIS |
| 3 | $F \vdash \alpha$ | 1, ASSUMPTION |
| 4 | $F \vdash \alpha_1$ | 3, $\alpha$-RULE |
| 5 | $F, \alpha_2 \vdash \bot$ | 4, 2 CUT |
| 6 | $F \vdash \alpha_2$ | 3, $\alpha$-RULE |
| 7 | $F \vdash \bot$ | 6, 5 CUT |

IF $F \cup \{\alpha_1, \alpha_2\} \notin C \Rightarrow F$ IS INCONSISTENT $\Rightarrow \alpha \in F \in C$
$\Rightarrow F \cup \{\alpha_1, \alpha_2\} \in C$

| STEP | CLAIM | REASON |
|---|---|---|
| 1 | $\beta \in F$ | HYPOTHESIS |
| 2 | $F, \beta_1 \vdash \bot$ | HYPOTHESIS |
| 3 | $F, \beta_2 \vdash \bot$ | HYPOTHESIS |
| 4 | $F \vdash \beta$ | 1, ASSUMPTION |
| 5 | $F, \beta \vdash \bot$ | 2, 3 $\beta$-RULE |
| 6 | $F \vdash \bot$ | 4, 5 CUT |

IF $F \cup \{\beta_i\} \notin C$ FOR $i = 1, 2 \Rightarrow$ F IS INCONSISTENT $\Rightarrow \beta \in F \in C$
$\Rightarrow F \cup \{\beta_i\} \in C$ FOR AT LEAST ONE $i$

# THEOREM 6: SOUNDNESS OF RESOLUTION

THE RESOLUTION CALCULUS IS SOUND: $F \vdash \varphi \Rightarrow F \models \varphi$

## PROOF

WE FIRST PROVE THAT THE INFERENCE RULES OF THE CALCULUS ARE SOUND;

IF A VALUATION $\alpha$ SATISFIES THE PREMISES OF A FORMULA IN THE RESOLUTION CALCULUS, IT ALSO SATISFIES THE CONCLUSION:

- **FALSEHOOD** $\rightarrow \bot \in \varphi_i \Rightarrow \varphi_i \equiv \bot \vee \mathcal{X}$, WHERE $\mathcal{X}$ IS A GENERALIZED CLAUSE NOT CONTAINING $\bot$

  BY THE SUBSTITUTION LEMMA, $\varphi_i[\emptyset/\bot] \equiv (\bot \vee \mathcal{X})[\emptyset/\bot] = \mathcal{X} \equiv \bot \vee \mathcal{X} \equiv \varphi_i$

  THEREFORE, $\alpha \models \varphi_i \Rightarrow \alpha \models \varphi[\emptyset/\bot]$

- **DOUBLE NEGATION** $\rightarrow \neg\neg\psi \in \varphi_i \Rightarrow \varphi_i \equiv \neg\neg\psi \vee \mathcal{X}$, WHERE $\mathcal{X}$ IS A GENERALIZED CLAUSE NOT CONTAINING $\neg\neg\psi$

  BY THE SUBSTITUTION LEMMA, $\varphi_i[\psi, \neg\neg\psi] = (\neg\neg\psi \vee \mathcal{X})[\psi/\neg\neg\psi] = \psi \vee \mathcal{X} \equiv \varphi_i$

  THEREFORE, $\alpha \models \varphi_i \Rightarrow \alpha \models \varphi[\psi/\neg\neg\psi]$

- **$\alpha$-EXPANSION** $\rightarrow \alpha \in \varphi_i \Rightarrow \varphi_i \equiv \alpha \vee \mathcal{X}$, WHERE $\mathcal{X}$ IS A GENERALIZED CLAUSE NOT CONTAINING $\alpha$

  $\varphi_i[\alpha_1/\alpha] \wedge \varphi_i[\alpha_2/\alpha] = (\alpha_1 \vee \mathcal{X}) \wedge (\alpha_2 \vee \mathcal{X}) \equiv (\alpha_1 \wedge \alpha_2) \vee \mathcal{X} \equiv \varphi_i$

  THEREFORE, $\alpha \models \varphi_i \Rightarrow \alpha \models \varphi_i[\alpha_i/\alpha]$ FOR $i = 1,2$

- **$\beta$-EXPANSION** $\rightarrow \beta \in \varphi_i \Rightarrow \varphi_i \equiv \beta \vee \mathcal{X}$, WHERE $\mathcal{X}$ IS A GENERALIZED CLAUSE NOT CONTAINING $\beta$

  $\varphi_i[\beta_1, \beta_2/\beta] \equiv (\beta_1 \vee \beta_2) \vee \mathcal{X} = \beta \vee \mathcal{X} \equiv \varphi_i$

  THEREFORE, $\beta \models \varphi_i \Rightarrow \alpha \models \varphi_i[\beta_1, \beta_2/\beta]$

CONTINUE $\longrightarrow$

- **RESOLUTION** → ASSUME $\psi \in \varphi_i$, $\neg\psi \in \varphi_j$ AND $\alpha \models \varphi_i, \varphi_j$. THEN $\varphi_i \equiv \psi \vee \chi_i$ AND $\varphi_j \equiv \neg\psi \vee \chi_j$, WHERE $\chi_i$ AND $\chi_j$ ARE GENERALIZED CLAUSES WITH $\psi \notin \chi_i$ AND $\neg\psi \notin \chi_j$, SO THAT $\varphi_i[\emptyset/\psi] \cup \varphi_j[\emptyset/\neg\psi] \equiv \chi_i \vee \chi_j$.

  THERE ARE TWO POSSIBILITIES:

  - $\alpha \models \psi \Rightarrow \alpha \not\models \neg\psi$. $\alpha \models \varphi_j \Rightarrow \alpha \models \chi_j \Rightarrow \alpha \models \varphi_i[\emptyset/\psi] \cup \varphi_j[\emptyset/\neg\psi]$
  - $\alpha \not\models \psi$. $\alpha \models \varphi_i \Rightarrow \alpha \models \chi_i \Rightarrow \alpha \models \varphi_i[\emptyset/\psi] \cup \varphi_j[\emptyset/\neg\psi]$

# Now:

1. EVERY AXIOM IS TRIVIALLY SATISFIED BY ANY VALUATION
2. SINCE THE INFERENCE RULES ARE SOUND, $\alpha \models F \Rightarrow \alpha$ SATISFIES EVERY EXPANSION OF F
3. F HAS A CLOSED EXPANSION $\Rightarrow$ IT IS UNSATISFIABLE
4. $F \vdash \varphi \Rightarrow F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION, THEREFORE IT IS UNSATISFIABLE
5. $F \vdash \varphi \Rightarrow F \models \varphi$

# THEOREM 7: EXPANSION OF VARIANTS

IF A SET OF FORMULAS $F \subseteq L$ EXPANDS TO $\varphi_i$, THEN $F * \psi$ EXPANDS TO $\varphi * \psi$

## PROOF

WE WILL PROVE SOMETHING MORE DETAILED: IF $E = (\varphi_1, \dots, \varphi_n)$ IS AN EXPANSION OF $F$ ENDING IN $\varphi$, THEN THERE IS AN EXPANSION OF $F * \psi$ OF THE FORM $E' := (\varphi_1', \dots, \varphi_n')$ WHERE $\varphi_i' = \varphi_i * \psi$ | $\varphi_i'$ IS OBTAINED BY APPLYING THE SAME EXPANSION RULE OF $\varphi_k$.

IF $\sigma$ IS A SUBSTITUTION ON $\varphi$ AND $\varphi\sigma = \varphi'$,

$$(\varphi * \psi)\sigma = \begin{cases} \varphi\sigma & \psi \in \varphi \\ \varphi\sigma \cup \{\psi\} & \psi \notin \varphi \end{cases} = \begin{cases} \varphi' & \psi \in \varphi \\ \varphi' \cup \{\psi\} & \psi \notin \varphi \end{cases} = \varphi' * \psi$$

**PROOF BY INDUCTION**: THE CLAIM HOLDS FOR INDICES LESS THAN K

- ASSUMPTION $\rightarrow \varphi_k \in F \Rightarrow \varphi_k' = \varphi_k * \psi \in F * \psi$ ✓
- AXIOM $\rightarrow \varphi_k$ IS AN AXIOM $\Rightarrow \varphi_k' = \varphi_k$
  - IF $\varphi_k$ COMES BEFORE $\varphi_i$ FOR $i < j$ USING RULES FOR $\perp, \neg\neg, \alpha, \beta$
    $\Rightarrow \varphi_k = \varphi_i\sigma$ AND IF $\varphi_k' = \varphi_i'\sigma \Rightarrow \varphi_k' = \varphi_i'\sigma = (\varphi_i * \psi)\sigma = \varphi_i\sigma * \psi = \varphi_k * \psi$
  - IF $\varphi_k = \varphi_i[\emptyset/\pi] \cup \varphi_j[\emptyset/\neg\pi]$ IS OBTAINED BY RESOLUTION FOR SOME INDICES $i, j < k$, SETTING $\sigma = [\emptyset/\pi]$ AND $\tau = [\emptyset/\pi]$,
    $\varphi_k' = \varphi_i'\sigma \cup \varphi_j'\tau = (\varphi_i * \psi)\sigma \cup (\varphi_j * \psi)\tau = (\varphi_i\sigma * \psi) \cup$
    $\cup (\varphi_j\tau * \psi) = (\varphi_i\sigma \cup \varphi_j\tau) * \psi = \varphi_k * \psi$

# THEOREM 8: INCONSISTENCY CRITERION

IF BOTH $F \cup \{\varphi\}$ AND $F \cup \{\neg\varphi\}$ HAVE CLOSED EXPANSION, THEN $F$ HAS A CLOSED EXPANSION

PROOF

IN $F \cup \{\neg\varphi\}$, WE REPLACE $\{\neg\varphi\}$ WITH ITS $\varphi$-VARIANT $\{\varphi, \neg\varphi\}$

SINCE $F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION, $F \cup \{\{\varphi, \neg\varphi\}\}$ EXPANDS

EITHER TO $\perp$ OR TO $\{\perp, \varphi\}$ AND, THEREFORE, TO $\varphi$ BY $\perp$-EXPANSION.

- SINCE $\{\varphi, \neg\varphi\}$ IS AN AXIOM, EITHER $\perp$ OR $\varphi$ BELONG TO AN EXPANSION OF $F$

- SINCE $F \cup \{\varphi\}$ HAS A CLOSED EXPANSION, $\perp$ IS AN EXPANSION OF $F$.

THEREFORE, $F$ HAS A CLOSED EXPANSION

# THEOREM 9: RESOLUTION INDUCES AN INFERENCE RELATION

THE RESOLUTION INFERENCE IS AN INFERENCE RELATION

PROOF

(E → EXPANSION)

- COMPACTNESS → $E \subseteq F$ IS FINITE AND $E \vdash \varphi \Rightarrow E \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION

THIS IS ALSO A CLOSED EXPANSION FOR $F \cup \{\neg\varphi\}$ AND $F \vdash \varphi$. CONVERSELEY, EVERY EXPANSION OF $F \cup \{\neg\varphi\}$ IS FINITE AND IS A CLOSED EXPANSION OF $E \cup \{\neg\varphi\}$ WITH $E \subseteq F$ FINITE

- ASSUMPTION → $\varphi \in F \Rightarrow F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION

| STEP | CLAUSE | RULE |
|---|---|---|
| 1 | $\{\varphi\}$ | ASSUMPTION |
| 2 | $\{\neg\varphi\}$ | ASSUMPTION |
| 3 | $\perp$ | 1,2 RESOLUTION |

- CONSISTENCY → IF $F \vdash \varphi$, $F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION $S$ WHICH IS ALSO A CLOSED EXPANSION FOR $F \cup \{\neg\varphi, \neg\perp\}$. THEREFORE $F, \neg\varphi \vdash \perp$.

CONVERSELEY, $F, \neg\varphi \vdash \perp \Rightarrow F \cup \{\neg\varphi, \neg\perp\}$ HAS A CLOSED EXPANSION $S$. HOWEVER $S$ IS ALSO A CLOSED EXPANSION OF $F \cup \{\neg\varphi\}$ BECAUSE $\{\neg\perp\}$ IS AN AXIOM OF THE CALCULUS. THUS, $F \vdash \varphi$

- CUT → $F$ HAS A CLOSED EXPANSION $\Leftrightarrow F \vdash \perp$ BECAUSE $\neg\perp$ IS AN AXIOM. THUS, IT SUFFICES TO SHOW THAT IF BOTH $F \cup \{\varphi\}$ AND $F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION, SO DOES $F$ (PROVED WITH INCONSISTENCY CRITERION)

# THEOREM 10: REGULARITY OF THE RESOLUTION INFERENCE

INFERENCE RESOLUTION IS REGULAR

PROOF

AN INFERENCE RELATION IS REGULAR IFF IT SATISFIES THE FOLLOWING CONDITIONS:

- $\alpha \vdash \alpha_i \quad i=1,2$
- $\beta, \neg\beta_1, \neg\beta_2 \vdash \bot$
- $\vdash \top$

| STEP | CLAUSE | RULE |
|---|---|---|
| 1 | $\{\alpha\}$ | ASSUMPTION |
| 2 | $\{\neg\alpha_i\}$ | ASSUMPTION |
| 3 | $\{\alpha_i\}$ | 1, $\alpha$-EXPANSION |
| 4 | $\bot$ | 2,3 RESOLUTION |

$\alpha \vdash \alpha_i$

| STEP | CLAUSE | RULE |
|---|---|---|
| 1 | $\{\beta\}$ | ASSUMPTION |
| 2 | $\{\neg\beta_1\}$ | ASSUMPTION |
| 3 | $\{\neg\beta_2\}$ | 1, $\alpha$-EXPANSION |
| 4 | $\{\beta_1, \beta_2\}$ | 1, $\beta$-EXPANSION |
| 5 | $\{\beta_2\}$ | 4,2 RESOLUTION |
| 6 | $\bot$ | 5,3 RESOLUTION |

$\Rightarrow \beta, \neg\beta_1, \neg\beta_2 \vdash \bot$

$\{\neg\top\}$ HAS A CLOSED EXPANSION BY FALSEHOOD RULE

$\Rightarrow \vdash \top$

# THEOREM 11: COMPATIBILITY OF THE PRODUCT WITH THE BOOLEAN STRUCTURE

THE PRODUCT OF RELATIONS IS COMPATIBLE WITH THE BOOLEAN STRUCTURE OF $A \times B$, IN THE SENSE THAT THE FOLLOWING FORMULAS HOLD WHEN THE OPERATIONS ARE DEFINED:

- $\perp R = \perp$ $\qquad$ $R \perp = \perp$
- $\top R \subseteq \top$ $\qquad$ $R \top \subseteq \top$
- $(\bigcup_{i \in I} R_i) S = \bigcup_{i \in I} R_i S$ $\qquad$ $R(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} R S_i$
- $(\bigcap_{i \in I} R_i) S \subseteq \bigcap_{i \in I} R_i S$ $\qquad$ $R(\bigcap_{i \in I} S_i) \subseteq \bigcap_{i \in I} R S_i$

PROOF (LEFT COLUMN; SIMILAR FOR THE RIGHT)

① $\perp R \Leftrightarrow a \perp R c \equiv \exists b. \overbrace{(\overbrace{a \perp b}^{\perp} \land bRc)}^{\perp} \equiv \perp \equiv a \perp c \Rightarrow \perp R = \perp$

② $\top$ IS THE TOTAL RELATION $\Rightarrow$ BY DEFINITION, IT CONTAINS ANY RELATION $\Rightarrow \top R \subseteq \top$

③ $(\bigcup_{i \in I} R_i) S \Leftrightarrow a (\bigcup_{i \in I} R_i) S c \Leftrightarrow \exists b (a (\bigcup_{i \in I} R_i) b \land bSc)$

DEFINITION OF UNION $\qquad$ DISTRIBUTIVITY OF CONJUNCTION OVER DISJUNCTION

$\Leftrightarrow \exists b ((\exists i. a R_i b) \land bSc)) \Leftrightarrow \exists b \exists i (a R_i b \land bSc)$

$\Leftrightarrow \exists i (a R_i S c) \Leftrightarrow \bigcup_{i \in I} R_i S \Rightarrow (\bigcup_{i \in I} R_i) S = \bigcup_{i \in I} R_i S$

④ $(\bigcap_{i \in I} R_i) S \Leftrightarrow a (\bigcap_{i \in I} R_i) S b \Leftrightarrow \exists b (a (\bigcap_{i \in I} R_i) b \land bSc)$

$\Leftrightarrow \exists b (\forall i (a R_i b) \land bSc) \Leftrightarrow \exists b (\forall i (a R_i b \land bSc))$

$\Rightarrow \forall i (a R_i S c) \Leftrightarrow a (\bigcap_{i \in I} R_i S) b \Leftrightarrow \bigcap_{i \in I} R_i S \Rightarrow (\bigcap_{i \in I} R_i) S \subseteq \bigcap_{i \in I} R_i S$

## THEOREM 12: CHARACTERIZATION OF EPIS, MONOS AND ISOS

LET $f: A \to B$ BE A FUNCTION:

1) $f$ IS AN EPIMORPHISM IF AND ONLY IF HAS LEFT INVERSE

2) $f$ IS A MONOMORPHISM IF AND ONLY IF HAS RIGHT INVERSE, PROVIDED $A \neq \emptyset$

3) $f$ IS AN ISOMORPHISM IF AND ONLY IF IT HAS AN INVERSE

**PROOF**

INVERSE IMAGE OF EVERY $b \in B$

- SURJECTIVE EPIMORPHISM $\to |f^*(B)| \geq 1$
- INJECTIVE MONOMORPHISM $\to |f^*(B)| \leq 1$
- ISOMORPHISM $\to |f^*(B)| = 1$

$g: B \to A$

- LEFT INVERSE FOR $f$ IF $gf = B$
- RIGHT INVERSE FOR $f$ IF $fg = A$
- (TWO SIDED) INVERSE FOR $f$ IF $fg = A \wedge gf = B$

1 $\Rightarrow$ $f$ EPIMORPHISM $\Rightarrow$ $\forall b \in B$ WE CAN CHOOSE $a \in f^*(b)$ AND SET $g(b) = a$. $f(g(b)) = f(a) = b \Rightarrow gf = B$ ✓

$\Leftarrow$ $gf = B \Rightarrow \forall b \in B, f(g(b)) = b \Rightarrow a := g(b) \in f^*(b), f^*(b) \neq \emptyset$ ✓

2 $\Rightarrow$ $f$ MONOMORPHISM $\Rightarrow$ SINCE $A \neq \emptyset$ WE CAN CHOOSE $c \in A$

$$g(b) = \begin{cases} a & f^*(b) = \{a\} \\ c & f^*(b) = \emptyset \end{cases}. \quad \forall a \in A, g(f(a)) = g(b) = a$$

BECAUSE $f^*(f(a)) = \{a\} \Rightarrow fg = A$ ✓

$\Leftarrow$ $fg = A \Rightarrow a, a' \in f^*(b) \Rightarrow f(a) = b = f(a')$

$\Rightarrow a = g(f(a)) = g(b) = g(f(a')) = a'$ ✓

3 FOLLOWS FROM 1 AND 2

## THEOREM 13: EXISTENCE OF THE TRANSITIVE CLOSURE

① EVERY RELATION $R: A \to A$ HAS A TRANSITIVE CLOSURE $R^t = \bigcup_{n=1}^{\infty} R^n$

② MOREOVER, $R^t$ IS COMPATIBLE WITH REFLEXIVITY AND ③ SIMMETRY. IN OTHER WORDS, IF $R$ IS REFLEXIVE (SYMMETRIC) SO IS $R^t$

REFLEXIVITY $\Leftrightarrow I \subseteq R$

SYMMETRY $\Leftrightarrow M(R)^T \leq M(R)$

TRANSITIVITY $\Leftrightarrow R^2 \subseteq R$

* $i \geq 2$. REMEMBER THAT, PREVIOUSLY, WE SAW THAT $R^1 \subseteq R^t$. WE ARE LOOKING FOR $\bigcup_i R^i = R^t$

**PROOF**

① CLEARLY $R = R^1 \subseteq R^t$. ALSO, $R^t$ IS TRANSITIVE. IN FACT,

$(R^t)^2 = (\bigcup_i R^i)(\bigcup_j R^j) = \bigcup_{i,j} R^i R^j = \bigcup_{i,j} R^{i+j} = \bigcup_i R^{i^*} = R^t$. IF $S$ IS TRANSITIVE, $R \subseteq S \Rightarrow \forall n > 0, R^n \subseteq S^n = S \Rightarrow \forall n > 0, R^n \subseteq S \Rightarrow \bigcup_{n=1}^{\infty} R^n = R^t = S$

② $R$ REFLEXIVE, $A \subseteq R \Rightarrow A \subseteq R \subseteq \bigcup_{n=1}^{\infty} R^n = R^t \Rightarrow R^t$ REFLEXIVE

③ $R$ SYMMETRIC, $R^{op} = R \Rightarrow (R^t)^{op} = (\bigcup_{n=1}^{\infty} R^n)^{op} = (\bigcup_{n=1}^{\infty} R^{op})^n =$

$= \bigcup_{n=1}^{\infty} R^n = R^t \Rightarrow R^t$ SYMMETRIC

# THEOREM 14: INDUCED ORDER

LET $R: A \to A$ BE A BINARY RELATION AND $T = R^{rt}$ ITS REFLEXIVE AND TRANSITIVE CLOSURE. THEN:

① $E := T \cap T^{op}$ IS AN EQUIVALENCE RELATION ON $A$

② IF $p: A \to A/E$ IS THE PROJECTION ON THE QUOTIENT, THE DIRECT IMAGE OF $T$ ALONG $p$ $p_* T = p^{op} T p$ IS A PARTIAL ORDER ON $A/E$, CALLED THE PARTIAL ORDER INDUCED BY $R$.

③ IF $T$ IS ANTISYMMETRIC AND IS THE GENERATED ORDER, THEN $p_* T$ IS ISOMORPHIC TO $T$

PROOF

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "p"] & A \arrow[d, "p"] \\ A/E \arrow[r, dashed, "u"] & A/E \end{tikzcd}$$

① $E$ IS AN EQUIVALENCE RELATION?

R • $A = A \cap A = A \cap A^{op} \underset{T \text{ IS REFLEXIVE}}{\subseteq} T \cap T^{op} = E \Rightarrow A \subseteq E$ AND $E$ IS REFLEXIVE

S • $E^{op} = (T \cap T^{op})^{op} = T^{op} \cap T^{op\,op} = T^{op} \cap T = T \cap T^{op} = E \Rightarrow E$ IS SYMMETRIC

T • $E^2 = (T \cap T^{op})(T \cap T^{op}) \subseteq TT \cap TT^{op} \cap T^{op}T \cap T^{op}T^{op} \subseteq TT \cap T^{op}T^{op}$
$\subseteq T \cap T^{op} = E \Rightarrow E$ IS TRANSITIVE.

ALSO, $TE = T = ET$ (WE WILL NEED THIS FOR ②) PROOF:

$A \subseteq E \subseteq T \Rightarrow T = TA \subseteq TE \subseteq TT \subseteq T$

② LET $U = p^{op} T p$. WE PROVE THAT $U$ IS A PARTIAL ORDER, SO THAT IT SATISFIES:

| | | | |
|---|---|---|---|
| IDEMPOTENCY | $U^2 = U$ | REMEMBER: | $pp^{op} = E$ |
| STRICT ANTISYMMETRY | $U \cap U^{op} = I$ | | $p^{op}p = 1$ |

- $U^2 = (p^{op}Tp)(p^{op}Tp) = p^{op}T\underbrace{pp^{op}}_{E}Tp = p^{op}TETp = p^{op}TTp \subseteq p^{op}Tp = U$

- TO PROVE BOTH REFLEXIVITY AND ANTISYMMETRISM, IT IS ENOUGH TO SHOW THAT $U \cap U^{op} = 1$

$$V \subseteq U \cap U^{op} \Leftrightarrow (V \subseteq U) \wedge (V \subseteq U^{op})$$

$$\Leftrightarrow (V \subseteq p^{op}Tp) \wedge (V \subseteq p^{op}T^{op}p)$$

//MULTIPLY $p$ LEFT AND $p^{op}$ RIGHT

$$\Leftrightarrow (pVp^{op} \subseteq ETE) \wedge (pVp^{op} \subseteq E^{op}T^{op}p^{op})$$

$$\Leftrightarrow (pVp^{op} \subseteq T) \wedge (pVp^{op} \subseteq T^{op})$$

$$\Leftrightarrow pVp^{op} \subseteq pp^{op} \Leftrightarrow V \subseteq 1 \Rightarrow U \cap U^{op} = 1$$

③ $T$ IS ANTISYMMETRIC. IT IS ALSO REFLEXIVE $\Rightarrow \mathrm{Ker}(p) = E = T \cap T^{op} = 1$

$\Rightarrow p$ IS INJECTIVE. $p$ IS ALSO SURJECTIVE BECAUSE IT IS A PROJECTION

ORDER $\Rightarrow p_* T$ IS ISOMORPHIC TO $T$