Rwanda Maths Olympiad Team Selection Test

Time allowed: 4 hours 30 minutes You may not use calculators.

Please explain and prove your answers as best you can. Each question is worth 7 marks.

Please label all questions clearly. Start each question on a new page.

If there are any words you don't understand or you are confused about what the question is asking you to do, you can write down your query on a separate piece of paper and give it to the invigilator in the first 30 minutes. You will receive a response within 1 hour.

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1. In a triangle ABC, let D and E be the midpoints of AB and AC, respectively, and let F be the foot of the altitude through A. Show that the line DE, the angle bisector of $\angle ACB$ and the circumcircle of ACF pass through a common point.

2. We consider the real sequence (x_n) defined by $x_0 = 0, x_1 = 1$ and $x_{n+2} = 3x_{n+1} - 2x_n$ for $n = 0, 1, ...$ We define the sequence (y_n) by $y_n = x_n^2 + 2^{n+2}$ for every non negative integer n. Prove that for every $n > 0$, y_n is the square of an odd integer

3. A rectangular grid is colored in checkerboard fashion (each square is black and white with no two adjacent squares the same colour), and each cell contains an integer. It is given that the sum of the numbers in each row and the sum of the numbers in each column is even. Prove that the sum of all numbers in black cells is even

4. Find all primes p and q such that $p + q$ and $p + 4q$ are both squares of integers.

5. Show that for all positive real numbers x, y, z, which satisfy $x^2 + y^2 + z^2 + 2xyz = 1$, the inequality $2(x + y + z) \leq 3$ is true.

6. Let ABC be a triangle with H its orthocenter. The circle with diameter $[AC]$ cuts the circumcircle of triangle ABH at K. Prove that the point of intersection of the lines CK and BH is the midpoint of the segment $[BH]$

7. Let M and N be two 9-digit positive integers with the property that if any one digit of M is replaced by the digit of N in the corresponding place, the resulting integer is a multiple of 7.

   (a) Prove that any number obtained by replacing a digit of N with the corresponding digit of M is also a multiple of 7.

   (b) Find an integer $d > 9$ such that the above result remains true when M and N are two d-digit positive integers.

8. Let $f(n)$ be the number of permutations $a_1, \ldots, a_n$ of the integers $1, \ldots n$ such that:

   (i) $a_1 = 1$;

   (ii) $|a_i - a_{i+1}| \leq 2, i = 1, \ldots, n - 1$.

   Determine whether $f(1996)$ is divisible by 3.