Rwanda Mathematics Competition 2020/21 - Final Stage

SOLUTIONS AND MARKING SCHEMES

IMPORTANT NOTICE: The marking schemes are based on the solutions presented. There may be other solutions and then the marking will be decided in consultation with the Problem Captain.

Problem 1.

Solution: Suppose there are k cats. Then $\frac{4}{5}k$ cats answer ”cat”. The number of dogs is $660 - k$ and $\frac{1}{4}(660 - k)$ of them answer ”cat”. In total we get $\frac{2}{5} \cdot 660 = 264$ answers ”cat”.
Thus $\frac{4}{5}k + \frac{1}{4}(660 - k) = 264,$ i.e. $16k + 3300 - 5k = 5280.$
Easy calculation gives $k = 180$. There are 180 cats and $660 - 180 = 480$ dogs in Huye.

Marking scheme:
a). Only correct answer = 2 point.
b). Arriving to the correct equation (or a system of two equations) = 4-5 points, depending on the progress of the soltution-attempt.
c). A solution with some minor miscalculation in the final step = 6 points.
d). Full solution = 7 points.

Problem 2.

Solution: No, it is not possible.
There may be several approaches but one is the following: Consider the ”chain” being placed on a large enough chessboard, with the squares matching the squares of the chain. Then any two neighboring squares of the chain must lie on chessboard squares of different colors. Hence, walking around the chain the colors of underlying chessboard squares alternate. The chain must therefore consist of as many ”white” squares as ”black” squares. So the number of squares in the chain must be even.
Another approach: To make a ”walk” around the chain (where each step means one to neighboring square) one must make as many steps to the right as there must be done to the left. Similarly, same number steps must be done up as it must be done down. Consequently the number of steps must be even.

Marking scheme:
a). Only the answer = 0 points.
b). Any reasonable starting point which may lead to the solution = 0-2 points.
c). An ”almost” solution = 5-6 points.
d). Full solution = 7 points.

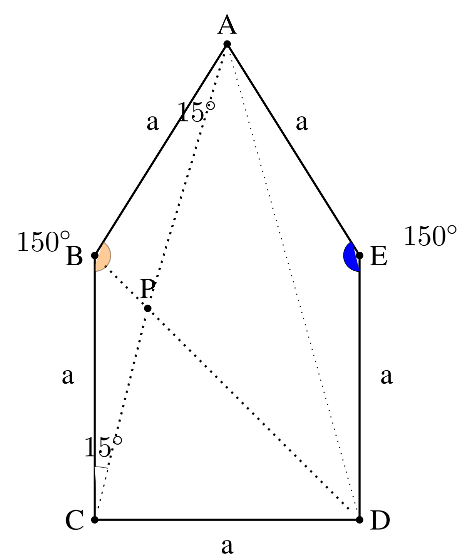
Problem 3.

Solution: The equation reduces to $x^2y^2 + 7xy = 3x^2y + 3xy^2$, which can be written as $xy(xy + 7 - 3x - 3y) = 0$
It is the obvious that all pairs (x, y) of the kind $(0, a)$ and $(b, 0)$, where a, b are arbitrary integers, are solutions to the equation.

A few more solution we may get by solving the equation $xy - 3x - 3y + 7 = 0$. Factorization of the left hand side gives $xy-3x-3y+7 = x(y-3)-3(y-3)-2 = (x-3)(y-3)-2$. Thus we need to solve in integers the equation $(x-3)(y-3) = 2$.
The factor $x - 3$ may then be -2, -1, 1 or 2, while, at the same time, the factor $y - 3$ must be -1, -2, 2 or 1, respectively. Hence, if $x - 3 = -2$ and $y - 3 = -1$ we get $(x, y) = (1, 2)$. The remaining three possibilities give $(x, y) = (2, 1)$, $(x, y) = (4, 5)$ and $(x, y) = (5, 2)$.

Marking scheme:
a). Reduction to the equation in the solution and factoring out xy = 1 point.
b). Finding each one of the infinite families of solutions gives additional 1+1 points.
c) Factorization of the bracket = 2 points
d). Finding final four solutions = remaining 1-2 points, depending if the negative solutions are included or not, or other minor defects.

Problem 4.



Solution: Notice that $BCDE$ is a square (with argument). As $BE = a$, ABE is equilateral and so $\angle\, CBA = 90 + 60 = 150$, and similarly $\angle ADE = 150$. Thus, both $\triangle\, ABC$ and $\triangle\, ADE$ are similars, and $\angle BAC = \angle ACB = \angle DAE = \angle EDA = \frac{180-150}{2} = 15$.
We also have that $\angle PAD = \angle BEA - \angle BAC - \angle DEA = 60 - 15 - 15 = 30$. As the diagonals of a square bisect one another and are perpendicular, $\angle ADP = \angle EDP - \angle EDA = 45 - 15 = 30$. But then $\triangle ADP$ is an isosceles triangle and therefore, $PA = PD$.

Marking scheme:
a). Getting the correct figure and determind the square and the equilateral = 2 point
b). $\triangle\, ABC \sim \triangle\, ADE$ and get the 15's degree angles = 2 point
c) Using the fact that diagonals in the square are bisectors and get $\angle ADP$ = 1 point
d). Arriving to the correct analysis that $\triangle ADP$ is isosceles but missing the final conclusion = 6 point
e). A solution with some minor miscalculation in one of the step = -1 points.
f). A solution without explicitly mentioned arguments = -1 or -2 points.
g). Full solution = 7 points.

Problem 5.
If three of the numbers have the same parity (odd or even) then two of the brackets will be even. The product is then divisible by 4.
If three of the numbers have the same partity then three of the brackets are even. The product is then divisible by 8.

In any case the product is divisible by 4.
Out of the four numbers A, B, C, D at least two (by the Pigeonhole Principe) give the same rest-term when divided by 3 (the rest could be 0, 1 or 2). Their difference is the divisible by 3.
The whole product is then divisible by $4 \cdot 3 = 12$.

Marking scheme:
a). Only some special cases (finite number) = 0 points
b). Only some special infinite families of numbers = 1 point
c). Proving divisibility by 4 = 3 points
d). A reduction should be made of 1-2 points should be made in c). if the arguments are not complete.
e). Proving divisibility by 3 = 4 points
f). A reduction should be made of 1-3 points should be made in e). if the arguments are not complete.

Problem 6.

Solution: a). Note that only one of the statements can be true and not all can be false (otherwise this would contradict statement (6)). Hence the remaining five must be false.Thus the statement 5 is true.

b). Note that if a statement is true then all the previous statements are also true, Hence, if k i the last true statement then this statement says that k statements are false. This implies that there are at least k false statements. The false statements must be statements $k+1, \ldots, 6$. Hence, $6-k \geq k$, which gives $k \leq 3$. But if the number of false statements was greater than 3 the the statement 4 would be true, which contradicts $k \leq 3$. Hence we get that only the first 3 statements are true.

Marking scheme:
a) Noticing that there exist some true statements = 1 point.
Noticing that exactly only one statement can be true = 1 point.
Noticing that statment (i) is false iff statment (6-i) is true = 1 point.
(those two above are additive).
Correct answer with correct argument = 3 points.

b) Only correct answer = 1 point.
Noticing that if a statement is true then all the previous statements are also true = 1 point.
Correct answer with correct argument = 3 points. Reduction by 1-2 points depends on the how long into the solution the student progresses.
One possible misstake is to conclude just after ”The false statements must be statements $k+1, \ldots, 6$” that $k = 3$ instead of $k \leq 3$. This should reduce the points by 1.