Pilot Rwandan Mathematics Olympiad Round 2

Time allowed: 3 hours You may not use calculators.

Please explain your answers as best you can. Each question in section A (questions 1,2,3 and 4) is worth 5 marks. Each question in section B (questions 5,6,7 and 8) is worth 10 marks.

Please label all questions clearly. Start each question on a new page.

You may ask an invigilator to explain any words that you do not understand.

Section 1 questions are taken from Purple Comet High School Competition 2019. Section 2 questions are kindly supplied by the Irish Maths Trust, taken from the Irish Mathematical Olympiad 2019.

Section A

1. (5 marks) Evaluate:

$$\frac{(2+2)^2}{2^2} \times \frac{(3+3+3+3)^3}{(3+3+3)^3} \times \frac{(6+6+6+6+6+6)^6}{(6+6+6+6)^6}$$

2. (5 marks) Figure 1 shows a 12 by 20 rectangle split into four strips of equal width, and an isosceles triangle with the same base as the rectangle. Find the area of the shaded region.

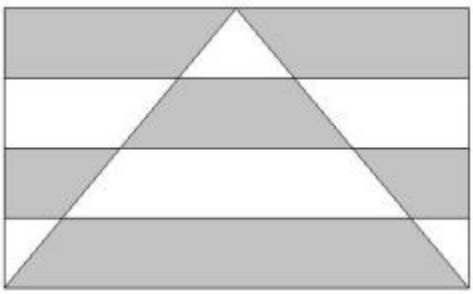

Figure 1

3. (5 marks) Find the number of positive integers less than 2019 that are neither multiples of 3 nor have any digits that are multiples of 3 (0, 3, 6 or 9).

4. (5 marks) The letters AAABBCC are arranged in random order. Find the probability that no two adjacent letters will be the same. (adjacent means next to one another)

Section B

5. (10 marks) The number 2019 has the following nice properties:

   (a) It is the sum of the fourth powers of five distinct positive integers.

   (b) It is the sum of six consecutive positive integers.

   In fact,

   $$2019 = 1^4 + 2^4 + 3^4 + 5^4 + 6^4 \tag{1}$$
   $$2019 = 334 + 335 + 336 + 337 + 338 + 339 \tag{2}$$

   Prove that 2019 is the smallest number that satisfies both (a) and (b).

   - fourth power means to the power of 4. The fourth power of a number b is b^4. The fourth power of 7 is 7^4.
   - distinct means that each of the numbers is different. For example the following is a sum of the fourth powers of five not distinct positive integers:

     $$2004 = 1^4 + 1^4 + 3^4 + 5^4 + 6^4$$

   - Consecutive integers means integers that follow each other - for example here are 6 consecutive integers: 4,5,6,7,8,9.
   - Positive Integer means an integer strictly larger than 0.

6. (10 marks) Find the set of all quadruplets (x, y, z, w) of non-zero real numbers which satisfy

   $$1 + \frac{1}{x} + \frac{2(x+1)}{xy} + \frac{3(x+1)(y+2)}{xyz} + \frac{4(x+1)(y+2)(z+3)}{xyzw} = 0.$$

7. (10 marks) Jenny is going to attend a sports camp for 7 days. Each day, she will play exactly one of three sports: football, basketball or cycling. The only restriction is that in any period of 4 consecutive days, she must play all three sports.

   Find, with proof, the number of possible sports schedules for Jenny's week.

8. (10 marks) A quadrilateral $ABCD$ is such that the sides AB and DC are parallel, and $|BC| = |AB| + |CD|$. Let the angle bisectors of $\angle ABC$ and $\angle BCD$ intersect at point E

   (a) (3 Marks) Prove that $\angle BEC = 90^\circ$

   (b) (7 marks) Prove that the point E lies on the side AD