MATH OLYMPIAD MASTERY

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THE MATH OLYMPIAD MASTERY LEVEL I

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PREFACE

Welcome to "Math Olympiad Mastery: A Comprehensive Guide to Algebra, Number Theory, Geometry, and Combinatorics." This book is a collaborative effort by a team of dedicated authors who are passionate about mathematics and committed to helping you excel in math competitions and problem-solving challenges.

The authors, Nkundayezu Emmanuel, Musangwa Innocent, Warakoze Bel-ami, Abayo Joseph Desire, Munezero Jean Nepo, Akimana Nadine, Korusenge Obed, Hategekimana Hirwa Arnold, Irakarama Alain, Subukino Arnaud, and Kagaba Etienne, have combined their knowledge, experience, and enthusiasm to create a comprehensive resource that will empower you to tackle the intricate and fascinating world of mathematical problem-solving.

In this book, you will embark on a journey through the realms of algebra, number theory, geometry, and combinatorics - key areas of mathematical Olympiad competitions. Each chapter is meticulously crafted to provide a deep understanding of the fundamental concepts, problem-solving techniques, and strategies required to conquer even the most challenging problems.

Key Features of the Book:

Diverse Authorship: With a team of authors hailing from various backgrounds and perspectives, this book offers a rich and diverse range of insights, ensuring a well-rounded approach to problem-solving.

Comprehensive Content: The book is divided into four major sections - Algebra, Number Theory, Geometry, and Combinatorics. Each section is thoughtfully structured, progressing from foundational concepts to advanced problem-solving strategies.

Clear and Concise Explanations: Complex mathematical ideas are explained in a clear and straightforward manner, making even the most intricate topics accessible to readers of all levels.

Abundant Examples and Problems: Throughout the book, you will find numerous examples and carefully selected problems that illustrate the concepts discussed and provide ample practice opportunities.

Challenging Exercises: In each chapter, you'll encounter a range of exercises, from standard problems to challenging ones designed to stretch your problem-solving skills and expand your mathematical thinking.

Problem-Solving Strategies: Alongside the content, you'll discover valuable problem-solving strategies, tips, and techniques that will equip you to approach Olympiad-style problems with confidence and creativity.

Whether you are a novice eager to explore the world of math competitions or a seasoned problem solver seeking to refine your skills, "Math Olympiad Mastery" is designed to be your trusted companion. The authors have poured their expertise into every page, aiming to empower you to overcome obstacles, unravel complex problems, and achieve your fullest mathematical potential.

Happy problem solving!
The Authors

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Part I

MATH OLYMPIAD CONTENT

Chapter 1

ALGEBRA

1.1 SETS AND SUBSETS

1.1.1 History of numbers

The **Babylonians** developed a place-value system based on the numerals 1 (one) and 10 (ten). The ancient **Egyptians** added to this system to include all the powers of 10 up to one million.There is some archaeological evidence that suggests that humans were counting as far back as 50,000 years ago in **South Africa**. The following order of subsets of numbers ranges from Natural numbers to Complex numbers.

Natural numbers($\mathbb{N}$)
People started doing math by counting things and they were able to find solutions to equations like $x+3=5$ in Positive Integers. Eg: 1,2,3,4,..... This set of Natural numbers is the mother of bigger sets of numbers we have in mathematics.

Integer numbers($\mathbb{Z}$)
Everything was okay with Positive integers until people realized they can't get solutions to equations like $x+5=3$ so they had to invent Negative numbers which in union with natural numbers make the whole set of Integers.
Eg: -2, -1,0,1,2.

Rational numbers($\mathbb{Q}$)
There was a quest to find numbers that could satisfy equations like $7x=1$ and this led to the invention of numbers that can be written as the division of two integers.
Eg:$5 = \frac{5}{1} = \frac{15}{3}$, $1.5 = \frac{3}{2}$ and $0.3333333 = \frac{1}{3}$

Irrational numbers($\mathbb{I}$)
Some equations like $x^2 = 2$ led to the invention of numbers that can't be written as division of integers.
Eg: $\sqrt{2}$, π, e, $ln2$ ... i.e: They are Irrational.

Real numbers($\mathbb{R}$) This is the union of Rational and Irrational numbers.

Imaginary numbers
There was no need to invent other numbers till the 17th century René Descartes said: let $\sqrt{-1} = i$ and this was useful in many areas ranging from electricity to Albert Einstein's theories. Now we are able to find solutions to equations like $x^2 = -1$, $log(-1)$.

Complex numbers($\mathbb{C}$)
This is the union of Real numbers and Imaginary numbers. Eg: $2+5i$

Fun fact: It seems like we know all numbers that do exist in the universe, but what is $\frac{x}{0}$?!!!! Do we need another invention?

1.1.2 Sets and subsets

Sets

Examples of sets: Real numbers($\mathbb{R}$), Rational numbers($\mathbb{Q}$), Integers($\mathbb{Z}$), Whole numbers,among others. [1]

Subsets

A subset is a set whose elements are all members of another set. A set A is a subset of another set B if all elements of set A are elements of set B.
The symbol $\subseteq$ means "is a subset of". The symbol "$\subset$" means "is a proper subset of". Since all of the members of set A are members of set D, A is a subset of D. Symbolically this is represented as $A \subseteq D$.
A proper subset is one that contains a few elements of the original set whereas an improper subset contains every element of the original set along with the null set.
A subset which contains all the elements of the original set is called an improper subset. It is denoted by $\subseteq$.

N.B: If a set has **"n" elements**, then the **number of subsets** of the given set is $\mathbf{2^n}$ and the **number of proper subsets** of the given subset is given by $\mathbf{2^n - 1}$. [1]
EXAMPLES:
Subsets of Integers($\mathbb{Z}+$, $\mathbb{Z}-$, even numbers, odd numbers, prime numbers, etc).

Number theory is one of the 4 branches of Olympiad mathematics that deals with integers.
The following are some common subsets of integers:

- **Even numbers**: Those are integers that are divisible by 2 they can be expressed as $\mathbf{2n}$ $\forall\ n \in \mathbb{Z}$
  Example: Prove that $4n^2+6n+8$ is always even $\forall\ n \in \mathbb{Z}$
  Solution: $4n^2 + 6n+8$ can be written as $2(2n^2+3n+4)$ which is a multiple of 2. Hence proven.

- **Odd numbers**: Those are integers that are not divisible by 2 they can be expressed as $2n + 1\ \forall\ n \in \mathbb{Z}$

  Example: Prove that $(2n + 1)^2$ is always odd $\forall\ n \in \mathbb{Z}$
  Solution: $(2n + 1)^2$ can be expanded to be $4n^2 + 4n + 1$ which equals $2(2n^2 + 2n) + 1$ which is an even number plus 1 hence completing the proof.

- **Prime numbers**: Those are natural numbers that have only 2 divisors, that is 1 and itself. Eg: 2,3,5,7,11,13,... There are infinite prime numbers (Hippasus the disciple of Pythagoras proved this by showing its opposite is wrong)

  Example: **Proof:**

  A prime number can either be 2 or greater than 2, so we divide it into cases.

  Case 1: $n = 2$

  When $n = 2$, $n + 1 = 3$, which is divisible by 3.

  Case 2: $n > 2$ and n is odd

  Since all primes greater than 2 are odd, if n is odd, then $n + 1$ is even. An even number is divisible by 2.

  Thus, in both cases, $n + 1$ is either divisible by 2 or divisible by 3 when n is a prime number.

- **Square numbers (perfect squares)**: These are integers that also have integer square roots e.g., 1, 4, 9, 16, ...

There are many properties these kinds of integers possess concerning divisibility. For example, $n^2 - m^2 = (n-m)(n+m)$.

Example: Prove that the square of an even number is divisible by 4.

Solution: Let's say that the even number is $2n$ for some integer n. Then its square will be $4n^2$, which is a multiple of 4, providing a proof.

Subsets of Real numbers

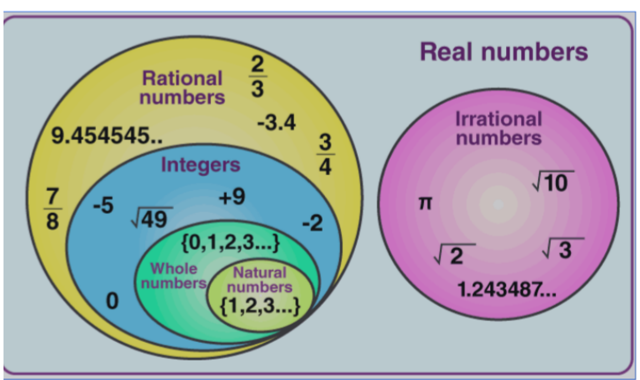
Algebra in Olympiad Mathematics and Real Numbers

Algebra is a branch of Olympiad mathematics that deals especially with manipulating and solving mathematical expressions. One of the fundamental sets of numbers in algebra is the set of **Real Numbers**, denoted by $\mathbb{R}$.

The set of Real Numbers is the union of Rational Numbers ($\mathbb{Q}$) and Irrational Numbers (I). Rational Numbers are numbers that can be expressed as a ratio of two integers, whereas Irrational Numbers cannot be expressed as such. The set of Real Numbers, $\mathbb{R}$, includes both the rational and irrational numbers.

- Rational Numbers ($\mathbb{Q}$) - These are numbers that can be expressed as a ratio of two integers, where the denominator is not zero. Examples of rational numbers include $\frac{1}{2}$, $\frac{3}{4}$, -2, 0, etc.
- Irrational Numbers (I) - These are numbers that cannot be expressed as a simple fraction and have non-terminating, non-repeating decimal expansions. Examples of irrational numbers include π, $\sqrt{2}$, $\sqrt{3}$, etc.
- Integer Numbers ($\mathbb{Z}$) - These are the set of whole numbers, including their negatives and zero. It consists of all positive and negative integers as well as zero. Examples include -3, -2, -1, 0, 1, 2, 3, etc.
- Natural Numbers ($\mathbb{N}$) - These are the set of positive integers, starting from 1 and counting upwards without end. Examples include 1, 2, 3, 4, etc.

The set of Real Numbers, $\mathbb{R}$, is a very large and diverse set, encompassing all possible numbers on the number line.



N.B: The above diagram shows us the Real numbers (as a union of Rational numbers and Irrational numbers.) Note that, from the diagram, you can easily see the other subsets of Rational numbers which are integers, whole numbers, and natural numbers.

EXERCISES

[2]

1. State True or False:

a. All whole numbers are natural numbers

b. All whole numbers are natural numbers

c. All rational numbers are integers.

d. Fractions are non-integers.

2. How many subsets does the set A = {1, 2, 3, 4, 5} have?

3.If $U = \{1, 3, 5, 7, 9, 11, 13\}$, then which of the following are subsets of U

a. B = {2, 4} b, A = {0}

b. C = {1, 9, 5, 13}

c. D = {5, 11, }

d. E= {13, 7, 9, 11, 5, 3, 1}

e. F = {2, 3, 4, 5}

4.Which of the following sets is a universal set for the other four sets?

a. The set of even natural numbers

b. The set of odd natural numbers

c. The set of natural numbers

d. The set of negative numbers

e. The set of integers

5.Write all the subsets for the following

a. {0}

b. {6, 11}

c. {2, 5, 9}

d. {1, 2, 6, 7}

e. {a, b, c}

f. ∅

g. {p, q, r, s}

6.Write down all the possible proper subsets for each of the following

a. {a, b, c, d}

b. {1, 2, 3}

c. {p, q, r}

d. {5, 10}

e. {x}

f. ∅

7. Find the number of subsets for set.

a. containing 36 elements

b. whose cardinal number is 5

8.Find the number of proper subsets for set

a. containing 13 elements

b. whose cardinal number is 17

9.Show with an example that if the number of elements in a set is 'n' , then

a. he number of subsets is 2^n

b. the number of proper subsets is $2^n - 1$

10. Let A = {2, 3, 4, 5, 6, 7} ,B = {2, 4, 7, 8} , C = {2, 4}. Fill in the blanks by $\subset$ or $\not\subset$to make the resulting statements true.

a. B — A

b. C — A

c. B — C

d. $\emptyset$ — B

e. C — C

f. C — B

11.State whether true or false.

a. Quadrilateral $\subseteq$ polygon

b. $\{1\} \leftrightarrow \{0\}$.

c. Whole numbers $\subseteq$ natural numbers

d. a $\in$ d, e,f, a

e. Natural numbers $\subseteq$ whole numbers

f. Integers $\subseteq$ natural numbers

g. $0 \in \emptyset$

h. $\emptyset \in \{1 , 2, 3 \}$

12. Let $Ax : x = n - 2, n < 5$. Find A when

a. n = W, n $\in$ W

b. n = N, n $\in$ N

c. n $\in$ I = I

13. Let $\frac{2}{7}$ and 4 be two rational numbers. Show and explain why their product and division is also rational.

14. Let 4 and $\sqrt{4}$ are rational and irrational numbers. Show and explain why their addition and subtraction results in an irrational number.

15. $fx + 4 = -11 + 2(x + 3)$. In the equation shown, f is a constant. For what value of f does the equation have no real solutions?

16. $4 - 3y = 6y + 4 - 9y$. Which of the following best describes the solution set to the equation shown?

a. The equation has no solutions

b. The equation has exactly one solution, $y = 0$.

c. The equation has exactly one solution, $y = \frac{4}{3}$.

d. The equation has infinitely many solutions

17. If $a = 5 + 2\sqrt{6}$ and $b = \frac{1}{a}$, what will be the value of $a^2 + b^2$?

18. If $a = 2 + \sqrt{3}$, find the value of $a - (\frac{1}{a})$

19. Show that $5 + \sqrt{3}$ is irrational.

20. What is the number of subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\ , 10\}$ having 3 elements?

1.2 FUNCTIONS

1.2.1 Input & Output, Domain, Co-domain and Range, Injective, Surjective and Bijective Functions

1.2.2 Input & Output

An exchange rate calculator inputs a value in USD (1 dollar) and outputs the equivalent value in Rwandan Francs (e.g. 970 RWF). What happens if it inputs 5 dollars instead? Can you describe the function that it is applying?

The above function $f(x) = 970x$ is an example of a function because it takes an input and maps to an output. It is always the same output for a given input. (A different function would be used on a different day with a different exchange rate).

We could also define the function to have two inputs: the number of dollars to change and the exchange rate which will change every day. Then the function would be $f(r, x) = rx$ where r is the exchange rate and x is the number of dollars.

You should think of a function as **a machine** that takes some input(s) and produces some **out-put**. The key thing to remember is that if you give a function the same input multiple times, it will always give the same output. It cannot have multiple possible outputs for an input.

Therefore, a **function** can be defined as a relation between a set of inputs having at most one out-put each.

1.2.3 Domain, Co-domain and Range

A function is only ever considered fully-defined if it has a domain and a co-domain.
Both of these are generally written in set terminology so it can be R (the real numbers), Q (the rational numbers), Z (the integers), N (the natural numbers) or even just an arbitrary set (e.g. 0,1). The set can even be non-numeric such as A,B,C,D or pairs (Z×Z) which would be a pair of integers like (1,-1).

Domain: The domain is the set of values that the input can take.

Co-domain: The co-domain is a set of all possible values (outputs) which can come out as a result.

Range or Image: The range is the set of actual outputs of a function. i.e. it is the set of values which actually comes out.

Examples

1. In the exchange rate function calculator $f(x) = 970x$ above, we would say that the **domain** is the positive real numbers ($\mathbb{R}+$) as any real number can be converted but you can't exchange negative dollars and the **co-domain** would also be the positive real numbers. So a full function definition would be f : $\mathbb{R}+\rightarrow \mathbb{R}+$ where $f(x) = 970x$.

2. Consider the function f : R $\rightarrow$ R such that $f(x) = 0$. The domain of this function is all reals and the co-domain which is all allowed values of the output are reals. But actually, only one value ever comes out which is 0. So the range or image is just $\{0\}$.

In fact, the range is always a subset of the co-domain. In fact, **the range is always a subset of the co-domain** but it is precisely those that are actually taken on while the co-domain can be much larger.

3. In the function $f(r, x) = rx$, we would say that f:$\mathbb{R}+\rightarrow \mathbb{R}+$ to say that the domain is the set of pairs of positive reals (the exchange rate and the amount being converted) and the co-domain is the set of positive reals (the amount in the new currency).

1.2.4 EXERCISES

1. What is the domain, co-domain and range of : $\mathbb{R}+\rightarrow \mathbb{R}+$ where f(x) = 1?

2. What is the domain, co-domain and range of : $\mathbb{R}+\rightarrow \mathbb{R}+$ where $f(x) = 2x$?

3. What is the domain, co-domain and range of : R $\rightarrow$ R where $f(x) = x^2$?

4. What is the domain, co-domain and range of : Z $\rightarrow$ R where $f(x) = x$?

5. What is the domain, co-domain and range of: R $\times$ R $\rightarrow$ R where $f(x, y) = x + y$?

6. What is the domain, co-domain and range of : Q $\rightarrow$ Z where $f(x) = 0$ if x is positive and $f(x) = 1$ if x is negative?

7. What is the domain, co-domain and range of : Z $\rightarrow$ Z where $f(x)$ is the number of positive factors that x has.

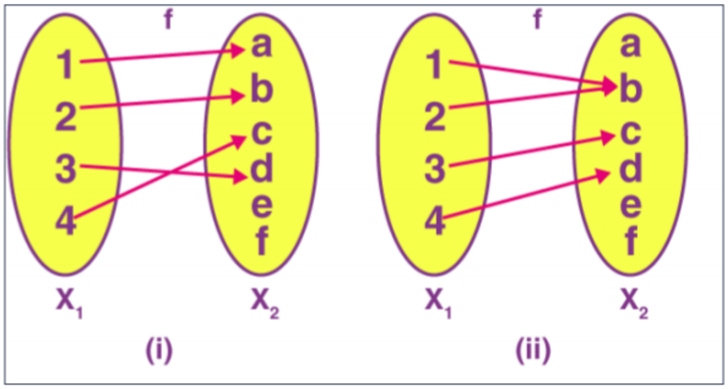
8. What is the domain, co-domain and range of : X $\rightarrow$ Z where X is the set of all possible English words less than 8 letters long and $f(x)$ is the length of the word x

1.2.5 Injective, Surjective and Bijective Functions

There are 3 special types of functions that are important.

Injectivity

[3]
I. A function is defined to be **injective** or **one-to-one** if there are no two elements in the domain that map to the same element in the co-domain.
One-to-One or **Injective** functions define that each element of one set, say Set (A) is mapped with a unique element of another set, say, Set (B).



Example:
a, If function f: R→ R, then $f(x) = 2x$ is injective. b, If function f: R→ R, then $f(x) = 2x + 1$ is injective. c, If function f: R→ R, then $f(x) = x^2$ is not an injective function, because here if$x = -1$, then $f(-1) = 1 = f(1)$. Hence, the element of co-domain is not discrete here.

Surjectivity

II. A function $f : A \to B$ is **surjective** or **onto** if the range (image) is the same as the co-domain, meaning that every element in the co-domain B has a corresponding value in the domain A that maps to it.

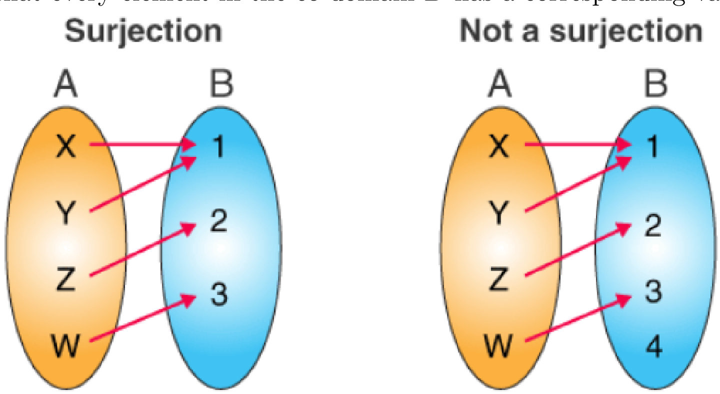


Fig.1 Fig.2

In the **first figure**, you can see that for each element of B, there is a pre-image or a matching element in Set A. Therefore, it is an **onto function**. But if you see in the **second figure**n one element in Set B is not mapped with any element of set A, so it's not an **onto** or **surjective function**.

Examples:

1, Given that the set A = 1, 2, 3, set B = 4, 5 and let the function f = (1, 4), (2, 5), (3, 5). Show that the function f is a surjective function from A to B. Given that the set $A = \{1, 2, 3\}$, set $B = \{4, 5\}$, and the function $f = \{(1, 4), (2, 5), (3, 5)\}$, we can show that f is a surjective function from A to B:

Solution: Domain: $A = \{1, 2, 3\}$ We can see that the element from set A, 1, has an image 4, and both 2 and 3 have the same image 5. Thus, the range of the function is $\{4, 5\}$, which is equal to set B. Therefore, we conclude that $f : A \to B$ is an onto function. Hence, the given function f is a surjective function.

2, Prove if the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is a surjective function or not:

Solution:

For the given function $g(x) = x^2$, the domain is the set of all real numbers, and the range is only the set of square numbers, which does not include all the set of real numbers. Hence, the given function g is not a surjective function.

Bijectivity

III. A function is defined to be **bijective** if it is both injective and surjective.

Examples:

1. For $A = \{-1, 2, 3\}$ and $B = \{1, 4, 9\}$, $f : A \to B$ defined as $f(x) = x^2$ is bijective.

2. Show that the one-to-one function $f : \{1, 2, 3\} \to \{4, 5, 6\}$ is a bijective function:

Solution: The given function $f : \{1, 2, 3\} \to \{4, 5, 6\}$ is a one-to-one function, and hence it relates every element in the domain to a distinct element in the co-domain set. The three elements of the domain set relate to all the three elements of the co-domain set. Also, since the co-domain includes all the elements of the second set, the given function is also an onto function as the range is equal to the co-domain. Therefore, the given function is a bijective function.

Summary:

If there is an injective function from domain X to co-domain Y, then we know that each element in X maps to a unique element in Y, and so Y must be at least as big as X. If there is a surjective function from domain X to co-domain Y, then we know that every element in the co-domain has at least one corresponding element in the domain, which means that Y must be at least as big as X. If a function is bijective, then both of those things must be true, and so the sets X and Y are the same size. This is one way (very useful in combinatorics) of showing that two sets are the same size - by constructing such a function.

1.2.6 EXERCISES

Determine the nature of the following functions:

1. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$
2. $f : \mathbb{Z} \to \mathbb{R}$ where $f(x) = x$
3. $f : \mathbb{R} \to \mathbb{Z}$ where $f(x) = x$
4. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x$
5. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x$
6. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$
7. $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = x^2$
8. $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = \sqrt{x}$
9. $f : \mathbb{R}^+ \to \mathbb{R}^+$ where $f(x) = \sqrt{x}$
10. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $f(x) = x + y$

1.2.7 Linear Functions

Introduction

[4] **Linear functions** are represented in the form $y = f(x) = ax + b$. These functions correspond to straight lines on the Cartesian plane. The independent variable is x and the dependent variable is y. The slope of the linear function is denoted by a, and the constant term or the y-intercept is represented by b.
As a linear function represents a straight line on the Cartesian plane, its equation can be used to find the slope and y-intercept, which provide valuable information about the line's characteristics and behavior.
Please note that linear functions have one independent variable and one dependent variable and can be expressed in the form $y = f(x) = ax + b$.

A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y. The slope of the linear function is "a". The constant term or the y-intercept is "b".

As a linear function represents a straight line on the graph, this means that when we have a linear function then we also have an equation of the line.

Forms of equations of a line

Slope-Intercept	*y = mx + b*	***m*** is the slope ***b*** is the y-intercept
Point-Slope	*y − y1 = m(x − x1)*	***m*** is the slope ***(x1,y1)*** *is a point on the line*
Standard Form	*ax + by = c*	***a*** is positive
Vertical	*x=a*	**Vertical line** with ***a*** as the **x-intercept**
Horizontal	*y=b*	**Horizontal line** with ***b*** as the **y-intercept**

1.2.8 Common terms associated with linear functions.

Slope/Gradient

The slope or gradient of a line represents the change in the y-coordinate with respect to the change in the x-coordinate. It is denoted by "m" and can be calculated as follows:

$$\mathbf{m} = \frac{\mathbf{\Delta y}}{\mathbf{\Delta x}}$$

Where: Δy is the change in the y-coordinate, and Δx is the change in the x-coordinate.
The slope formula between two points P(x1, y1) and Q(x2, y2) on a straight line is given by:

$$\mathbf{m} = \frac{y_2 - y_1}{x_2 - x_1}$$

The equation for the slope of a line passing through the point (x1, y1) is also known as the point-slope form of the equation of a straight line and is given by:

$$y - y_1 = m(x - x_1)$$

Here, "m" represents the slope and (x1, y1) is a point on the line.
The slope-intercept form (general form) of the equation of a line is given by:

$$y = mx + b$$

Where "**m**" is the slope, and "**b**" is the y-intercept.
Note:

1.Slope of Vertical Lines:
Vertical lines have no slope, as they do not have any steepness. In other words, we cannot define the steepness of vertical lines. The slope formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

,
becomes undefined for vertical lines because both x_2 and x_1 have the same value, resulting in a division by zero. Therefore, the slope of vertical lines is undefined.

2. Slope of Horizontal Lines: The slope of a horizontal line is equal to 0, as the y-coordinates of all points on the line are the same. When using the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

, the numerator becomes zero since y_2 and y_1 are equal. Consequently, the slope of horizontal lines is 0.

3. Slope for parallel lines:
For two parallel lines given by l_1 and l_2 with the slopes m_1 and m_2 respectively, their slopes must also be equal, i.e., $m_1 = m_2$.

4. Slope for perpendicular lines:
For two lines l_1 and l_2 to be perpendicular, the product of their slopes must be equal to -1. Thus, $m_1 \times m_2 = -1$.

Examples:
1. Find the slope of a line between $P(-2, 3)$ and $Q(0, -1)$. Solution: Given, $P(-2, 3)$ and $Q(0, -1)$ are the two points. Hence, the slope of the line,

$$m = \frac{-1 - 3}{0 - (-2)} = \frac{-4}{2} = -2$$

2. The line $y = 3x + 1$ is perpendicular to the other line passing through $(3, 4)$. Find the slope and equation of that other line.

Solution:
As the line $y = 3x + 1$ is perpendicular to the line passing through $(3, 4)$, then $m_1 \times m_2 = -1$.
As $m_1 = 3$, then $m_2 = -\frac{1}{3}$.Thus, the slope of the other line passing through $(3, 4)$ is $-\frac{1}{3}$. The equation of the other line passing through (3, 4) is

$$(y - 4) = -\frac{1}{3}(x - 3)$$

or

$$y = -\frac{1}{3}x + 5$$

The x-intercept and y-intercept.

Intercept: The point where the line or curve crosses the axis of the graph. The meaning of intercept of a line is the point at which it intersects either the x-axis or y-axis. If the axis is not specified, usually the y-axis is considered. It is normally denoted by the letter 'b'.

If a point crosses the x-axis, then it is called the **x-intercept**. If a point crosses the y-axis, then it is called the **y-intercept**.

Practice Examples.

1. Find the x-intercept and y-intercept for the line $5x - 8y = 2$.

2. If the y-intercept of a line is -4 and the slope is $\frac{2}{3}$, then write its equation.

3. What is the equation of a line whose x and y-intercepts are given as $\frac{1}{3}$ and -3?

Intersection of lines.

Point of intersection means the point at which two or more lines intersect. By solving the two equations of lines simultaneously, we can then find the solution for the point of intersection of two lines. We can find the point of intersection of three or more lines also.

For example, find out the point of intersection of two lines $x + 2y + 1 = 0$ and $2x + 3y + 5 = 0$.

Solution:
Given straight line equations are: $x + 2y + 1 = 0$ and $2x + 3y + 5 = 0$.
After solving the two equations simultaneously, we then get the intersection point which is $(x, y) = (-7, 3)$.

EXERCISES

1. If the line that passes through the points (2,7) and (a,3a) has a slope of 2, the value of a is

   a. $\frac{5}{2}$

   b. 10

   c. 3

   d. $\frac{11}{5}$

   e. $\frac{12}{5}$

2. If x=3. Which of the following is true?

   a. $2x = 5$

   b. $3x - 1 = 8$

   c. $x + 5 = 3$

   d. $7 - x = 2$

   e. $6 + 2x = 14$

3. What is the intersection point of these two lines, $y = 2x + 6$ and $y = x + 4$.

4. The list 11, 20, 31, 51, 82 is an example of an increasing list of five positive integers in which the first and second integers add to the third, the second and third add to the fourth, and the third and fourth add to the fifth.
How many such lists of five positive integers have 124 as the fifth integer?.

   a. 10

b. 7

c. 9

d. 6

e. 8

5. An expression that produces the values in the second row of the table shown, given the values of n in the first row, is

n	1	2	3	4	5
value	1	3	5	7	9

a. $3n - 2$

b. $2(n - 1)$

c. $n + 4$

d. $2n$

e. $2n - 1$

6.Which equation represents the relationship between the values of x and y in the table

n	1.5	3	3	4
value	1	3	5	7

a. $y = x + 0.5$

b. $y = 1.5x$

c. $y = 0.5x + 1$

d. $y = 2x - 0.5$

e. $x^2 + 0.5$

7. For what value of k is the line through the points $(3, 2k + 1)$ and $(8, 4k - 5)$ parallel to the x-axis?

a. -1

b. 3

c. 2

d. 0

e. -4

8. The y-intercepts of the three parallel lines are 2, 3, and 4. The sum of the x-intercepts of the three lines is 36. What is the slope of these parallel lines?

a. $\frac{1}{3}$

b. -$\frac{2}{9}$

c. -$\frac{1}{6}$

d. -4

e. -$\frac{-1}{4}$

9. The line p is perpendicular to the line with equation $y = x - 3$. Line p has the same x-intercept as the line with equation $y = x - 3$. The y-intercept of line p is

a. -3

b. $\frac{1}{3}$

c. 3

d. -1

e. 0

10.In each row of the table, the sum of the first two numbers equals the third number. Also, in each column of the table, the sum of the first two numbers equals the third number. What is the sum of the nine numbers in the table?

m	4	$m+4$
8	n	8+n
$m+8$	$4+n$	6

a. 18

b. 42

c. -18

d. -6

e. 24

1.3 INEQUALITIES

1.3.1 Introduction

"Which is bigger? The sun or the moon". We are always comparing things in our daily life. When we look at inequalities, we are looking at two expressions that are "inequal" or unequal to each other, as the name suggests. In mathematics, Inequalities usually contain expressions involving the symbols $>, <, \geq$ and $\leq$

1.3.2 Basic properties

For any real numbers a, b, c, d or $(a, b, c, d \in \mathbb{R})$,

1. If If $a \geq b$ and $b \geq c$, then $a \geq c$. This follows directly from the transitive property of inequalities. For example, If $8 \geq 5$ and $5 \geq 3$, then $8 \geq 3$.

   Transtivity: A relation between three elements such that if it holds between the first and second and it also holds between the second and third it must **necessarily** hold between the first and third.

2. If $a \geq b$ and $b \geq a$, then $a = b$. This is also a consequence of the transitive property.

3. If $a \geq b$ and $c \geq 0$, then $ac \geq$ bc. This is because if $a \geq b$ and $c \geq 0$, then we can multiply both sides of the inequality $a \geq b$ by c to get $ac \geq bc$.

4. If $a \geq b$ and $c \leq 0$, then $ac \leq bc$. This is similar to the previous property, but since c is negative, we have to flip the inequality sign.

5. If $a \geq b > 0$, then $0 < \frac{1}{a} \leq \frac{1}{b}$. This property relates to the reciprocal of the numbers a and b. If $a \geq b$ and are both positive, then we can take the reciprocal of both sides to get $\frac{1}{a} \leq \frac{1}{b}$. The inequality is strict since a and b are not equal.

6. If $a \geq b$ and $c \geq d$, then $a + c \geq b + d$. This is a consequence of the transitive property as well, since we can add b and d to both sides of the inequality $a \geq b$ and $c \geq d$ to get $a + c \geq b + d$.

These properties can be very useful when working with inequalities, and they can be used to simplify expressions or to prove other inequalities.

1.3.3 Examples

1. By factorising, show that $x^2 + 2x + 1 \geq 0$ for all real x.

Solution: In other words, this question requires us to show why the LHS(Left Hand Side): $x^2 + 2x + 1$ can't be negative.

By factorising,

$$x^2 + 2x + 1$$
$$= x^2 + x + x + 1$$
$$= x(x + 1) + x + 1$$
$$= (x + 1)(x + 1)$$
$$= (x + 1)^2$$

Notice that, As it is stated in our question that x is a real number, then $x + 1$ is also a real number, hence $(x + 1)^2 \geq 0$ **This is because any square of a real number is always greater than or equal to zero**. This theorem is called the **trivial inequality**.

Trivial inequality: If x is a real number, then $x^2 \geq 0$ with $x^2 = 0$ exactly when $x = 0$

2. For a triangle having sides $2x + 1$, $x + 1$, and $4x - 7$. Find the possible values of x.

Solution: Notice that, it would be possible to find the value of x if and only if the question suggested that we are using a right-angled triangle. Because, then the *pythagoras theorem* would help us. But the question doesn't specify which triangle to use, so we can't use the pythagoras theorem.

Instead, the statement "the possible values of x" tells us that there are many possible values of x, hence we are required to find all of them. To solve our question, we are going to use a theorem called the ***Triangle inequality.***

Triangle inequality: If $\Delta\mathbf{ABC}$ has side lengths $BC = a$, $AC = b$, and $AB = c$, then $a + b > c$ and $a + c > b$ and $b + c > a$. In other words, the sum of any two sides of a triangle is greater than its third side.

By the triangle inequality then,

$$(2x + 1) + (x + 1) > (4x - 7),$$

$$(x + 1) + (4x - 7) > (2x + 1),$$

$$(4x - 7) + (2x + 1) > (x + 1).$$

After simplifying the above three inequalities, we then get $x < 9$, $x > \frac{7}{3}$, $x > \frac{1}{5}$. If we represent these three inequalities on a number line, then we get to see that the possible values of x are in the range between $\frac{7}{3}$ and 9, with $\frac{7}{3}$, 9 not included. In other words, $\frac{7}{3} < x < 9$.

3. Find all integer solutions to the inequality $x^2 - 4x - 3 < 0$

Solution: Let's try to bring the trivial inequality, and to do that we are going to make the LHS(Left Hand Side) of our inequality a square. Hence,

$$x^2 - 4x - 3 < 0$$

$$\implies x^2 - 4x + 4 - 4 - 3 < 0$$

$$\implies (x - 2)^2 - 7 < 0 \;\; As \; (x^2 - 4x + 4) = (x - 2)^2$$

$$\implies (x - 2)^2 < 7$$

From the above inequality, $(x - 2)^2 < 7$ but we know that by the **trivial inequality** $(x - 2)^2 \geq 0$. In other words, it means $(x - 2)^2$ can only be a positive number or zero, because a square of a number is always a positive number or zero. So what numbers can the LHS be? Or, What integer squares are less than 7? The integer squares that are less than 7 are $(-2)^2, (-1)^2, 0^2, 1^2, 2^2$. This means that $(x - 2)$ can be equal to either $-1, -2, 0, 1$ or 2. Therefore, x can take the values 0, 1, 2 , 3 or 4 which are the integer solutions for our inequality.

4. Let a, b and c be lengths of sides of a triangle and suppose that the circumference of this triangle is 2. Show that

$$a^2 + b^2 + c^2 + 2abc < 2.$$

Solution: The circumference or perimeter of $\Delta\mathbf{ABC}$ is 2, $\implies a + b + c = 2$ Since $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 4 - 2(ab + bc + ca)$ as $(a + b + c)^2 = 2^2 = 4$, hence the inequality can be re-written as $4 - 2(ab + bc + ca) + 2abc < 2$, i.e $2 - (ab + bc + ca) + abc < 1$. If we substitute the 2 with $a + b + c$ we then get an equality $(a + b + c) - (ab + bc + ca) + abc < 1$,hence $1 - (a + b + c) + (ab + bc + ca) - abc > 0$. But $1 - (a + b + c) + (ab + bc + ca) - abc = (1 - a)(1 - b)(1 - c)$ and $(1 - a)(1 - b)(1 - c)$ is necessarily > 0, since each side of a triangle is shorter than the circumference.

Note that, we can also prove the above question by using the **Triangle inequality.** Try proving it on your own by using the triangle inequality.

1.3.4 EXERCISE A

[5]

1. What is the smallest integer that can be placed in the box so that $\frac{1}{2} < \frac{\square}{9}$

   (A)7 (B)3 (C)4 (D)5 (E)6

2. Each of a, b, c and d is a positive integer and is greater than 3. If

$$\frac{1}{a-2} = \frac{1}{b+2} = \frac{1}{c+1} = \frac{1}{d-3}$$

   then which of the following is true

   (A)$a < b < c < d$ (B)$c < b < a < d$ (C)$b < a < c < d$
   (D)$d < a < c < b$ (E)$b < c < a < d$

3. If x is a number less than -2, which of the following expressions has the least value?

   (A)x (B)$x + 2$ (C)$\frac{1}{2}x$ (D)$x - 2$ (E)$2x$

4. Kathy owns more cats than Alice and more dogs than Bruce. Alice owns more dogs than Kathy and fewer cats than Bruce. Which of the statements *it must* be true?

   (A) Bruce owns the fewest cats (B) Bruce owns the most cats
   (C)Kathy owns the most cats (D)Alice Kathy owns the most dogs
   (E) Kathy owns the fewest dogs

5. In a group of five friends: Amy is taller than Carla. Dan is shorter than Eric but taller than Bob. Eric is shorter than Carla. Who is the shortest?

   (A)Amy (B)Bob (C)Carla (D)Dan (E)Eric

6. The number of integers n for which $\frac{1}{7} \leq \frac{6}{n} \leq \frac{1}{4}$ is

   (A)17 (B)18 (C)19 (D)20 (E)24

1.3.5 EXERCISE B

[5] [6]
1. By factorising, show that $x^2 - 8x + 20 \geq 0$ for all real x.
2. Show that $x + \frac{1}{x} \geq 2$ for all real and positive x.
3. Determine if it is possible to form a triangle with the given side lengths. If not possible, explain why not.

a. $3km, 4km, 5km$

b. $18m, 20m, 40m$

c. $9cm, 13m, 25cm$

d. $9cm, 13m, 25cm$

4. For a triangle having sides $3x+1$, $2x+1$, and $5x-6$. Find the possible values of x.
5. Find all integer solutions to the inequality $x^2-8x-6<0$
6. If the measures of the sides of a right-angled triangle are a,b and c, where c is the hypotenuse. Show that $a+b \leq c\sqrt{2}$

1.4 PROBLEMS

[5]

1. Barry has three sisters. The average age of the three sisters is 27. The average age of Barry and his three sisters is 28. What is Barry's age?

a. **1**

b. **30**

c. **4**

d. **29**

e. **31**

2. Jack went running last Saturday morning. He ran the first 12 km at 12 km/h and the second 12 km at 6 km/h. Jill ran the same route at a constant speed, and took the same length of time as Jack. Jill's speed in km/h was

a. **8**

b. **9**

c. **6**

d. **12**

e. **24**

3. The regular price for a bicycle is \$320. The bicycle is on sale for 20 % off. The regular price for a helmet is \$80. The helmet is on sale for 10 % off. If Sandra bought both items on sale, what is her percentage savings on the total purchase?

a. **18%**

b. **12%**

c. **15%**

d. **19%**

e. **22.5%**

4. When simplified, $\mathbf{\frac{1}{2+\frac{2}{3}}}$ is equal to

a. $\mathbf{\frac{1}{8}}$

b. $\mathbf{\frac{5}{2}}$

c. $\mathbf{\frac{5}{8}}$

d. $\mathbf{\frac{1}{2}}$

e. $\mathbf{\frac{3}{8}}$

5. The average of 1, 3 and x is 3. What is the value of $\mathbf{x}$?

a. **4**

b. **5**

c. **2**

d. **3**

e. **1**

6. There are two values k of for which the equation $x^2+2xk+7k-10=0$has two equal real roots (that is, has exactly one solution for$x$). The sum of these values of k is

a. **0**

b. **3**

c. **3**

d. **-7**

e. **7**

7. If $x=2y$ and $y\neq 0$, then$(x-y)(2x+y)$ equals

a. $\mathbf{5y}^2$

b. $\mathbf{y}^2$

c. $\mathbf{3y}^2$

d. $\mathbf{6y}^2$

e. $\mathbf{4y}^2$

8. If $x=18$ is one of the solutions of the equation $x^2+12x+c=0$, the other solution of this equation is :

a. $x=\mathbf{216}$

b. $x=-\mathbf{6}$

c. $x=$ **-30**

d. $x=\mathbf{30}$

e. $x=$ **-540**

9. The operation $\otimes$ is defined by $a\otimes b=\frac{a}{b}+\frac{b}{a}$. What is the value of $4\otimes 8$?

a. $\mathbf{\frac{1}{2}}$

b. **1**

c. $\mathbf{\frac{5}{4}}$

d. **2**

e. $\mathbf{\frac{5}{2}}$

10. If $\mathbf{\frac{a}{b}}=\mathbf{3}$ and $\mathbf{\frac{b}{c}}=\mathbf{2}$, then the value of $\frac{a-b}{c-b}$?

a. -4

b. -$\frac{1}{4}$

c. $\frac{2}{3}$

d. 2

e. 6

11. If a and b are positive integers such that $\frac{1}{a} + \frac{1}{2a} + \frac{1}{3a} = \frac{1}{b^2-2b}$, then the smallest possible value of $a+b$

a. 8

b. 6

c. 96

d. 10

e. 50

12. If x and y are positive integers with $x > y$ and $x + xy = 391$, what is the value of $x + y$?

a. 38

b. 39

c. 40

d. 41

e. 42

13. if $x^2 = 8x + y$ and $y^2 = x + 8y$ with $x \neq y$, then the value of $x^2 + y^2$ is

a. 9

b. 49

c. 63

d. 21

e. 56

14. If $\frac{x-y}{z-y} = -10$ then the value of $\frac{x-z}{y-z}$ is

a. 11

b. -10

c. -9

d. 9

e. 10

15. For how many odd integers k between 0 and 100 does the equation

$$2^{4m^2} + 2^{m^2-n^2-4} = 2^{k+4} + 2^{3m^2+n^2+k}$$

have exactly two pairs of positive (m, n) integers that are solutions?

a. 17

b. 20

c. 19

d. 18

e. 21

16. What is the sum of all numbers$q$ which can be written in the form $q = \frac{a}{b}$ where $a$and b are positive integers with $b \leq 10$ and for which there are exactly 19 integers n that satisfy $\sqrt{q} < n < q$?

a. 871.5

b. 743.5

c. 777.5

d. 808.5

e. 1106.5

17. Three distinct integers a, b, and $c$satisfy the following three conditions:

- $abc = 17,995$
- a, b and c form an arithmetic sequence in that order, and
- $(3a + b)$, $(3b + c)$, $(3c + a)$ form a geometric sequence in that order

What is the value of $a + b + c$?
(An arithmetic sequence is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example 3, 5, 7, is an arithmetic sequence with three terms.A geometric sequence is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

a. -63

b. -42

c. -68,229

d. -48

e. 81

18. Four different numbers a, b, c and d are chosen from the list $-1, -2, -3, -4$ and -5 . The largest possible value for the expression $a^b + c^d$ is

a. $\frac{5}{4}$

b. $\frac{7}{8}$

c. $\frac{31}{32}$

d. $\frac{10}{9}$

e. $\frac{26}{25}$

19. Suppose that a, b, and c are integers with $(x - a)(x - 6) + 3 = (x + b)(x + c)$ for all real numbers x. The sum of all possible values of b is

a. -12

b. -24

c. -14

d. -8

e. -16

20. If x and y are integers with $(y-1)^{x+y} = 4^3$, then the number of possible values x is

a. 8

b. 3

c. 4

d. 5

e. 6

Chapter 2

NUMBER THEORY

2.1 INTEGERS: THE BASICS

2.1.1 Introduction

Integers are whole numbers that do not have any fractional or decimal parts. Integers can be positive, negative, or zero. For instance, 10^2 and -$\sqrt{4}$ are integers whereas 3.4,$\frac{1}{2}$ or $\sqrt{2}$ are not integers. The set of integers is denoted as $\mathbb{Z}$ and includes the following numbers: ...,-3,-2,-1,0,1,2,3,..

2.2 TYPES OF INTEGERS

- Positive integers are also called natural numbers or counting numbers. The set of natural numbers is denoted as $\mathbb{N}$ or $\mathbb{Z}^+$.
- Negative Integers are additive inverse of natural numbers. The set of negative natural numbers can be denoted as $\mathbb{Z}^-$.
- Zero is neither positive nor negative. Zero has no sign.

N.B: Whole numbers are natural numbers or positive integers plus zero.

2.2.1 Fundamental arithmetic operations

The operations of integers refer to the basic arithmetic operations that can be performed with whole numbers, including both positive and negative numbers. The four fundamental operations of integers are:

1. **Addition (+)**
   As you learned in school, Addition is the process of combining two or more integers to find their total or sum.

   The rules of addition of integers are as follows,

   When adding two positive integers, the result is a negative integer (e.g 5 + 3 = 8)

   When adding two negative integers, the result is a negative integer (e.g (-7) + (-4) = -11)

   When adding a positive integer to a negative integer, the result can vary (e.g 9 + (-2) = 7 and 5 + (-20)= -15)

2. **Subtraction (-)**
   Subtraction is the process of finding the difference between two integers.

   To subtract an integer, you can add its additive inverse (the negative of the integer) instead.

   Examples:

9 - 4 = 5

(-3) - (-8) = 5

6 - (-2) = 8

3. **Multiplication ($\times$)**
   Multiplication is the process of repeated addition, to get a product.

   Key takeaways:

   When multiplying two integers with the same sign, the result is positive (e.g 3 $\times$ 4 = 12 and (-5) $\times$ (-2) = 10)

   When multiplying two integers with different signs, the result is negative (e.g 6 $\times$ (-3) = -18)

   Any integer multiplied by 0 is 0

   A multiple is the product of that integer with any integer

   For integers m and n, their product mn is both a multiple of m and n

4. **Division ($\div$)**
   This operation divides one integer by another, producing a quotient.

   We say that integer m is divisible y an integer n when $\frac{m}{n}$ is an integer.

   When you divide one integer by another, the result may or may not be an integer (e.g 6 $\div$ 2 = 3 (integer result), 7 $\div$ 2 = 3.5 (non-integer result))

   Similar to multiplication, if both integers have the same sign, then the result is positive. If the integers have different signs, then the result is negative.

2.2.2 Problems

Review problems

1. How many $4's$ must we add to get 44?
2. Is 22 a multiple of 4?
3. Is 63 a multiple of -9?
4. What is the value of $(4 \times 3) + 2$?
5. Find the smallest positive integer that is both a multiple of 4 and 7.
6. Is zero a multiple of 5?
7. Is 100 divisible by 12?
8. Find all positive divisors of 28.
9. 28 less than five times a certain number is 232. What is the number?

Challenging problems

1. The values of r, s, t and u are 2, 3, 4, and 5, but not necessarily in that order. What is the largest value of $r \times s + u \times r + t \times r$?
2. Find the 5 largest negative multiples of 13.
3. Find the smallest natural number that is not a divisor of 5040.
4. Find the smallest integer that is not a divisor of 5040.
5. Is 1111 divisible by 41?

2.3 EVEN AND ODD NUMBERS

2.3.1 Definition

An **even number** is a number which has a remainder of zero upon division by 2, while an **odd number** is a number which has a remainder of 1 upon division by 2.
If the units digit (or ones digit) is 1,3,5,7, or 9, the number is called an **odd number**, and if the units digit is 0,2,4,6,or 8, the number is called an **even number**.
Thus, the set of integers can be partitioned into two sets based on parity:

- the set of even (or parity 0) integers. Given n as an integer,every even number can be written as $2n$
- the set of odd (or parity 1) integers. Given n as an integer, every odd number can be written as $2n + 1$

Parity is a fundamental property of integers, and many seemingly difficult problems can be solved easily using parity arguments.

- **Example 1:** Figure out whether 1729 is an odd or even number.

  Solution:

  Since the remainder obtained on diving 1729 by 2 is 1, 1729 is an odd number **OR** the number 1729 is an odd number because it ends with digit "9."
- **Example 2:** Find out whether -1000 is an even or odd number.

  Solution:

  Since the remainder obtained when -1000 is divided by 2 is 0, -1000 is an even number **OR** -1000 is an even number because it ends with digit "0."

2.3.2 Even and odd properties

The following are the parity properties of odd and even numbers:

- even $\pm$ even $=$ even
- odd $\pm$ odd $=$ even
- even $\pm$ odd $=$ odd
- even $\times$ even $=$ even
- odd $\times$ odd $=$ odd
- even $\times$ odd $=$ even
- **Example 3:** If n is an integer what is the parity of $2n + 2$ (is $2n + 2$ even or odd number?

  Solution:

  Since n is an integer, $n + 1$ is also an integer. Then, $2n + 2 = 2(n + 1)$ shows that the parity of $2n + 2$ is 0, which implies $2n + 2$ is always an even number.
- **Example 4:** Is the number (47630750675+453407032)$\times$549068453 even or odd?

  Solution:

  Here, it would be unwise to actually multiply out these numbers. Instead, we can apply the properties of even and odd numbers.

  Since 47630750675 ends in a 5, it is odd. On the other hand, since 453407032 ends in a 2, it is even. By property 3, even $\pm$ odd= odd, so 47630750675+453407032 is odd. Since that sum is being multiplied by 549068453, which is odd, the entire number is odd since property 6 gives odd $\times$ odd= odd.

- **Problem 5:** If k is an integer, which of the following is always even: A) $2k+1$ B) k^2 C) $4k+4$ D) K^2-1

  Solution:

  A is always odd for any k B is odd whenever k is odd D is odd whenever k is even C can be written as $4(k+1)$, which means the remainder is 0 upon division by 2. The answer is C

2.3.3 Problems

Review problems

1. If a is a negative odd number and b is a positive even number, which of the following must be a positive even number: a) $b-a$ b) $a+b$ c) ab d) $-ab$
2. Given that a and b are integers, the expression $(a^2+a+7)\times(2b+1)$ is: a) always odd b) always even c) even or odd, depending on the values of a and b?
3. Let x and y be integers, which of the following is true about $(x+y)^2+xy$ a) It is even b) It is odd c) It is even if x is even d) It is even if y is even e) It is even if $-xy$ is negative
4. Let P be the product of the first 100 prime numbers, What is the parity of P?
5. If k is an integer, what is the parity of k^2+k?
6. What is the sum of the first and the last even numbers between 1 and 100?

Challenging problems

1. Can an even number divided by another even number, times another even number ever equal to an odd number? If ”yes,” find three numbers that work. If ”no,” then why not?
2. For integers x and y, show that $\frac{x^2+y^2}{2}+\frac{x+y}{2}$ is an integer.
3. The product of the digits in 38 is even because 3×8=24. Similarly, the product of the digits in 57 is odd because 5×7=35. How many 2-digit numbers have an odd product?
4. Prove that the sum of an odd number and an even number is odd.
5. Let n be an odd positive integer. Prove that the sum of three consecutive odd integers, beginning with n, is always divisible by 3.

2.4 PRIMES and COMPOSITES

2.4.1 Introduction

Remember that 1 is a divisor of every natural number and that every natural number is a divisor of itself. This means that every natural number greater than 1 has at least 2 positive divisors.
A prime number is a natural number greater than one that has no positive divisors other than 1 and itself. For example, 5 is a prime number because it has no positive divisors other than 1 and 5.
The first 49 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, and 227.
In contrast to prime numbers, a **composite number** is a positive integer greater than 1 that has more than two positive divisors. For example, 4 is a composite number because it has three positive divisors: 1, 2, and 4.
A composite number c can be written as a product of two integers both between 1 and itself:

$$c = ab$$

where a an b are (not necessarily distinct) divisors of the composite number c.
N.B

- All positive integers greater than 1 are either prime or composite. 1 is the only positive integer that is neither prime nor composite.
- 2 is the smallest and the only even prime number because all other even numbers are divisible by 2. All other prime numbers are odd!

2.4.2 Identifying primes

The simplest way to determine whether a number is a prime is to use **trial division**. It involves testing whether the number is divisible by any integer from 2 up to the square root of the number. If it is not divisible by any of these integers, then it is a prime number.
i.e If n is a composite number, then it must be divisible by a prime p such that $p \leq \sqrt{n}$.

- **Example 1:** Is 211 a prime number?

  solution: If 211 is a prime number, then it must not be divisible by a prime that is less than or equal to $\sqrt{211}$. $\sqrt{211}$ is between 14 and 15, so the largest prime number that is less than $\sqrt{211}$ is 13. It is therefore sufficient to test 2, 3, 5, 7, 11, and 13 for divisibility. 211 is not divisible by any of those numbers, so it must be prime.

- **Example 2:** What is the sum of the two largest two-digit prime numbers?

  Solution: If a two-digit number is composite, then it must be divisible by a prime number that is less than or equal to $\sqrt{100}$ =10. Therefore, it is sufficient to test 2, 3, 5, and 7 for divisibility.

  Counting backward,

  99 is divisible by 3;

  98 is divisible by 2;

  97 is not divisible by 2, 3, 5, or 7, implying it is the largest two-digit prime number;

  96 is divisible by 2;

  95 is divisible by 5;

  94 is divisible by 2;

  93 is divisible by 3;

  92 is divisible by 2;

  91 is divisible by 7;

  90 is divisible by 2;

  89 is not divisible by 2, 3, 5, or 7, implying it is the second largest two-digit prime number.

  The sum of the two largest two-digit prime numbers is 97+89=186

- **Example 3:** What prime number follows 1997/

  Solution: 1998 is divisible by 2. If 1999 is composite, then it must be divisible by a prime number that is less than or equal to $\sqrt{1999}$. $\sqrt{1999}$ is between 44 and 45, so the possible prime numbers to test are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, and 43. 1999 is not divisible by any of those numbers, so it is prime.

Sometimes, testing a number for primality does not involve exhaustively searching for prime factors, but instead making some clever observation about the number that leads to a factorization. The next example demonstrate this:

- **Example 4:** Is 12345 a prime number?

  Solution: As every number that ends in "5" is divisible by 5, We have $\frac{12345}{5} = 2469$. So 12345 is not a prime number.

2.4.3 Prime factorization

A **prime factorization** of a positive integer is that number expressed as a product of powers of prime numbers.
The uniqueness of prime factorization is an incredibly important result, thus earning the name of **Fundamental theorem of arithmetic**: "Any integer greater than 1 is either a prime number, or can be written as a unique product of prime numbers, up to the order of the factors."
This statement implies that if a number is not prime, it has a prime number as its factor. For example, the factors of 10 are 1,2,5, and 10, where 2 and 5 are both prime numbers.
"Up to the order of the factors" means that it does not matter the order in which the product of the prime numbers is written.

- **Example 5:** Give the prime factorization of 48.

  Solution: 48 is divisible by the prime numbers 2 and 3. The highest power of 2 that 48 is divisible by is $16 = 2^4$. The highest power of 3 that 48 is divisible by is $3 = 3^1$. Thus, the prime factorization of 48 is $48 = 2^4 \times 3^1$ No other positive integer has this prime factorization

- **Example 6:** What are the prime factors of 60?

  Solution: The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. The prime factors are 2,3, and 5.

Prime factorization is indeed useful for counting the number of divisors of a given number.
For any natural number $n > 1$, with prime factorization

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} ... p_n^{a_n}$$

where p is a prime number and a is a positive integer, we can find number of divisors using the formula,

$$(a_1 + 1)(a_2 + 1)(a_3 + 1)...(a_n + 1)$$

- **Example 7:** How many divisors does 60 have? z **Solution:** The prime factorization of 60 is $2^2 \times 3 \times 5$. Therefore,60 has $(2 + 1)(1 + 1)(1 + 1) = 3 \times 2 \times 2 = 12$. These factors are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60.

- **Example 8:** If x, y,and z are three distinct prime numbers such that $N = x \times y \times z$, how many positive divisors does N have excluding 1 and itself?

  Solution: Since $N = x \times y \times z$, we can conclude that x, y,and z are factors of N. Since x, y,and z are primes, we can not factor them to get any other number, so that gives us a total of 3 numbers.

  But wait, we know that if x and y are factors of N $x \times y$ is also a factor of N. So a combination of two factors out of the three factors is also a divisor of N In other words, we have $x \times y$, $x \times z$, and $y \times z$ as factors of N, which are another 3 in addition to the 3 above.

  Note that $x \times y \times z$ is also a combination that is a factor of N, but it equals the number itself and is therefore omitted.

  So we have a total of 6 divisors, excluding 1 and the number itself

2.4.4 Problems

Review problems

1. What is the largest 3-digit prime number?

2. In the following sequence, how many prime numbers are present? 121,12321,1234321,123454321,. . .

3. What is the first year in twenty-first century that is a prime number?

4. Is 9409 a prime number?

5. How many prime numbers are multiples of 3?
6. How many prime numbers are multiples of 10?
7. What is the smallest prime factor of 6125?
8. If n has 15 factors (1 and 15 inclusive) and $2n$ has 20 factors, what is the number of factors of $4n$?
9. What is the smallest positive number that has exactly 14 divisors?
10. They are 25 primes less than 100. i stheir sum even or odd?

Challenging problems

1. Find the smallest composite number that has no prime divisors less than 10.
2. A group of 25 pennies is arranged in 3 piles such that each pile contains a different prime number of pennies. What is the greatest number of pennies possible in any of the pile?
3. Consider the expression $2^n - 1$ for n as an integer greater than 1. What are the least two values for which the expression does not produce a prime number?
4. Let p be a prime number greater than 3. Prove that $p^2 - 1$ is always divisible by 6.
5. What is the smallest prime divisor of $5^2 3 + 7^1 7$?

2.5 MULTIPLES and DIVISORS

2.5.1 Greatest common divisor

The **greatest common divisor(GCD)**, also called the **highest common factor(HCF)**, of two numbers is the largest number that divides them both.
For instance, the greatest common factor of 20 and 15 is 5, since 5 divides both 20 and 15 and no larger number has this property.
The concept is easily extended to sets of more than two numbers: the GCD of a set of numbers is the largest number dividing each of them.
The GCD is traditionally notated as gcd(a,b), or when the context is clear, simply (a,b)

2.5.2 Computing the greatest common divisor

- The GCD of several numbers may be computed by **simply listing the factors of each number and determining the largest common one**. While in practice this is terribly inefficient, for particularly small cases it is doable by hand.

  The process may be split up using the method of factor pairs: once one determines a factor a of an integer n, the quotient $\frac{n}{a}$ is a factor of n as well. For instance, since 2 is a factor of 24, $\frac{24}{2} = 12$ is a factor of 24 as well.

  Example 1:

  Find the greatest common divisor of 30, 36, and 24.

  Solution: The divisors of each number are given by;

  30: 1,2,3,5,6,10,15,30; 36: 1,2,3,4,6,9,12,18,36; 24: 1,2,4,6,12,24

  The largest number that appears on every list is 6, so this is the greatest common divisor: gcd(30,36,24)=6

- A somewhat more efficient method is to first compute **the prime factorization** of each number in the set. The resulting GCD is the product of the primes that appear in every factorization, to the smallest exponent seen in the factorizations.

  Example 2:

  Compute gcd(4200,3780,3528)

  Solution: We have

  $$4200 = 2^3 \cdot 3 \cdot 5^2 \cdot 7;\ 3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7;\ 3528 = 2^3 \cdot 3^2 \cdot 7$$

  Since 2 appears in each of these factorizations, it will appear in the GCD as well. It is taken to the smallest power seen in the factorizations, which in this case is 2. So the GCD will contain 2^2 in its factorization. Continuing along these lines, we obtain a GCD of $2^2 \cdot 3 \cdot 7$

- **The euclidean algorithms** Because large numbers are difficult to work with by hand, there are a number of algorithms used to simplify the problem down to a manageable level. Since the GCD has the property that

  $$gcd(a, b, c) = gcd(gcd(a, b), c)$$

  the GCD can be calculated "two at a time", e.g. if we wanted to find the GCD of 20, 28, and 24, we could first find the GCD of 20 and 28 (which is 4) and then the GCD of 4 and 24 (which is also 4). As a result, almost all algorithms focus on the simplest case of determining the GCD of two numbers.

  The euclidean algorithm is based on the following key observation: if d divides a and d divides b, then d also divides $a - b$. This means that the GCD of a and b is the same as the GCD of $a - b$ and b, which is progress since this makes the numbers smaller.

  As a result, we can repeat this process to form an algorithm:

  1. If $a = b$, stop. The GCD of a and a is, of course, a. Otherwise, go to step 2.
  2. If $a > b$, replace a by $a - b$, and go back to step 1.
  3. If $b > a$, replace b by $b - a$, and go back step 1.

  Example 3:

  Determine the greatest common factor of 84 and 132

  Solution: We follow our algorithm,

  1. $a = 84$ and $b = 132$. Since $b > a$, we replace b by $b - a = 132 - 84 = 48$
  2. $a = 84$ and $b = 48$. Since $a > b$, we replace a by $a - b = 84 - 48 = 36$
  3. $a = 36$ and $b = 48$. Since $b > a$, we replace b by $b - a = 48 - 36 = 12$
  4. $a = 36$ and $b = 12$. Since $a > b$, we replace a by $a - b = 36 - 12 = 24$
  5. $a = 24$ and $b = 12$. Since $a > b$, we replace a by $a - b = 24 - 12 = 12$
  6. $a = 12$ and $b = 12$. Since $a = b$, the GCD is 12.

  You may have noticed, for instance, that once we get to 36 and 12 we can "skip" the $a = 24$ step. The Euclidean algorithm is especially useful by hand, since oftentimes the GCD will be become "obvious" by inspection once the numbers get low enough (e.g. you might have noticed at the $a = 36$, $b = 12$ step that the GCD is 12 since $36 = 12 \cdot 3$).

  In practice this can be made rather efficient, as described below:

  1. Let $a = b$ and $b = y$
  2. Given x and y, use division algorithm to write $x = yq + r$, $0 \leq r < y$
  3. if $r = 0$, stop as y is the $gcd(a, b)$
  4. if $r \neq$, replace (x, y) by (y, r) and go to step 2.

Example 4:

What is gcd(16457,1638)?

Solution We apply the euclidean algorithm

16457= 1638×10+77
1638= 77×21+21
77=21×3+14
21=14×1+7
14=7×2+0

he process stops since we reached 0, we obtain
$7 = gcd(7, 14) = gcd(14, 21) = gcd(21, 77) = gcd(77, 1638) = gcd(1638, 16457)$.

Example 5:

Prove that $\frac{21n+4}{14n+3}$ is irreducible for every integer n.

Solution;

$\frac{21n+4}{14n+3}$ is irreducible if and only if the numerator and denominator have no common factor, i.e. their greatest common divisor is 1. Applying the Euclidean algorithm,

$$21n + 4 = (14n + 3) \times 1 + (7n + 1)$$

$$14n + 3 = (7n + 1) \times 2 + 1$$

$$7n + 1 = (7n + 1) \times 1 + 0$$

Hence $gcd(21n + 4, 14n + 3) = 1$, which shows that the fraction is irreducible.

Coprimes

When the only common positive divisor between two integers is 1, we say that those numbers are **relatively prime**. A pair of relatively prime integers is also sometimes called **coprime**, which means the same thing.
i.e if $gcd(a, b) = 1$, where a and b are positive integers, then a and b are coprime.
For instance, 8 and 13 are relatively prime.
Note that neither integers need to be prime in order for them to be relatively prime 8 and 15 are both not prime, yet they are relatively prime.
*Properties of coprimes:

1. 1 is co-prime with every number.

2. Any two prime numbers are co-prime to each other.

3. Any two successive numbers/ integers are always co-prime.

4. The sum of any two co-prime numbers are always co-prime with their product: 2 and 3 are co-prime and have 5 as their sum (2+3) and 6 as the product (2×3). Hence, 5 and 6 are co-prime to each other.

5. Two even numbers can never form a coprime pair as all the even numbers have a common factor as 2.

6. If two numbers have their unit digits as 0 and 5, then they are not coprime to each other. For example 10 and 15 are not coprime since their HCF is 5 (or divisible by 5)

Example 6:

How many positive integers less than 25 are coprime to 25?
Solution:
We need to determine the count of numbers that share no common factors (other than 1) with 25.
The factors of 25, other than 1 and 25, are 5 only since $25 = 5^5$. The numbers less than 25 that share factors with 25 are the multiples of 5 (excluding 25 itself since we are looking for numbers less than 25).
These multiples are: 5, 10, 15, and 20.
Subtract the number of numbers that share factors from the total count of numbers less than 25. Total numbers less than 25 = 1, 2, 3, ..., 24 (24 numbers in total)
Number of numbers less than 25 that share factors with 25 = 4 (5, 10, 15, 20)
Number of numbers less than 25 that are coprime to 25 = Total numbers less than 25 - Number of numbers that share factors = 24 - 4 = 20

2.5.3 Lowest common multiple (LCM)

The lowest common multiple (LCM) of a finite set of non-zero integers is the smallest positive number that is a multiple of each integer in the set. It is a fundamental concept in number theory, and is closely related to the greatest common divisor.

Example 7:

Find the LCM of 30 and 36.
Solution:
Here is a list of the positive multiples of each number:
$30 : 30, 60, 90, 120, 150, 180, \ldots$;

$$36 : 36, 72, 108, 144, 180, \ldots$$

The first number that appears on both lists is 180.

2.5.4 Computing lowest common multiple

Listing multiples is a very ineffective way of computing the LCM in general.
However, if the **prime factorization** of the numbers is known, then computing the least common multiple is much simpler. The primes in the factorization of the LCM are the primes that appear in the factorizations of at least one member of the list, and their exponent is the maximum of the exponents that appear in the individual factorizations.

Example 7:

Compute the $lcm(4200, 3580, 3528)$
Solution:
We have

$$4200 = 2^3 \cdot 3 \cdot 5^2 \cdot 7;\ 3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7;\ 3528 = 2^3 \cdot 3^2 \cdot 7$$

The LCM can be read off from these factorizations by taking the maximum exponent for each prime:
$2^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 = 264600$
Generalizing this example, if the prime factorizations of $m$and n are

$$m = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$$

;

$$n = p_1^{b_1} p_2^{b_2} ... p_k^{b_k}$$

where p_i is distinct prime numbers and a_i and b_i are nonnegative integers then,

$$lcm(a, b) = p_1^{max(a_1,b_1)} p_2^{max(a_2,b_2)} ... p_k^{max(a_k,b_k)}$$

Example 8:

What is the smallest positive integer n for which $lcm(n, 30) = 180$?
Solution:
The prime factorization of 30 is $30 = 2 \times 3 \times 5$ and the prime factorization of 180 is $180 = 2^2 \times 3^2 \times 5$. So n must include the factor of 2^2 and the factor of 3^2. Thus the smallest n is $2^2 \times 3^2 = 36$.

2.5.5 Properties of LCM

- The LCM of a list of numbers divides any other common multiple.
- The LCM of a list of numbers can be computed two at a time, e.g. $lcm(a, b, c) = lcm(lcm(a, b), c)$.

2.5.6 Relationship between GCD and LCM

Recall that the greatest common divisor of a set of integers is the largest (positive) number which is a divisor of each integer in the set. The two concepts are intimately related; in particular, they satisfy the following theorem:

$$gcd(a, b) \times lcm(a, b) = a \times b.$$

2.5.7 Problems

2.5.8 Review problems

1. Without using calculator what is the gcd of 2442 and 171171?
2. What is $\frac{246}{642}$ in lowest terms?
3. what is the largest possible value of the greatest common divisor of numbers $5n + 6$ and $8n + 7$, where n is an arbitrary positive integer?
4. Write any one coprime number to 216 and 215.

2.5.9 Challenging problems

1. What is the $gcd(2015! + 1, 2016! + 1)$?
2. How many ordered pairs of (a, b) such that $100 \leq a, b \leq 1000$
3. Given that two integers a and b, such that $13gcd(a, b) = lcm(a, b)$ and that $a + b = 2016$, what are the values of a and b?
4. the traffic lights at three different locations change every 48 seconds, 72 seconds, and 108 seconds, respectively. If they change simultaneously at 9 am, when is the next time they change simultaneously?

2.6 DIVISIBILITY

2.6.1 Divisibility rules

A **divisibility rule** is a heuristic for determining whether a positive integer can be evenly divided by another (i.e. there is no remainder left over). For example, determining if a number is even is as simple as checking to see if its last digit is 2, 4, 6, 8 or 0.
Multiple divisibility rules applied to the same number in this way can help quickly determine its prime factorization without having to guess at its prime factors.

Basic divisibility rules

A positive integer N is divisible by

- **2**, if the last digit of N is 0, 2, 4, 6, or 8;
- **3**, if the sum of digits of N is a multiple of 3;
- **4**, if the last 2 digits of N are a multiple of 4;
- **5**, if the last digit of N is 0 or 5;
- **6**, if N is divisible by both 2 and 3;
- **7**, if subtracting twice the last digit of N from the remaining digits gives a multiple of 7 (e.g. 658 is divisible by 7 because 65 - 2 x 8 = 49, which is a multiple of 7);
- **8**, if the last 3 digits of N are a multiple of 8;
- **9**, if the sum of digits of N is a multiple of 9;
- **10**, if the last digit of N is 0;
- **11**, if the difference of the alternating sum of digits of N is a multiple of 11 (e.g. 2343 is divisible by 11 because 2 - 3 + 4 - 3 = 0, which is a multiple of 11);
- **12**, if N is divisible by both 3 and 4.

Example 1:

Without performing actual division, show that the number below is an integer: $\frac{1481481468}{12}$
Solution:
From the divisibility rules, we know that a number is divisible by 12 if it is divisible by both 3 and 4. Therefore, we just need to check that 1,481,481,468 is divisible by 3 and 4.
By applying the divisibility text of 3, we get $1+4+8+1+4+8+1+4+6+8=45$which is divisible by 3. Hence 1,481,481,468 is divisible by 3.
Applying the divisibility test for 4, we get that the last two digits, 68, is divisible by 4. Hence 1,481,481,468 is also divisible by 4.
Now, since we know that 1,481,481,468 is divisible by both 3 and 4, it is divisible by 12. Therefore, $\frac{1481481468}{12}$ is divisible by 12.

Example 2:

Find all the possible values of a such that the number $\overline{98a6}$ is a multiple of 3.
solution:
From the rules of divisibility, the number $\overline{98a6}$ is a multiple of 3 if and only if the sum of its digits $9+8+a+6=23+a$ is a multiple of 3. Since $0 \leq a \leq 9$, this implies that $a=1,4,7$ are all the possible values.

N.B

It is also useful to note that the notation $x|y$ is used to say x divides y i.e. the fraction $\frac{y}{x}$ gives an integer. For example, $3|6$ is true, but $2|7$ is false.

2.6.2 Remainders

An integer a is divisible by another integer b (or is a multiple of b) if a can be written as b times another integer:

$$a \times b = (integer)$$

For example, $14 = 7 \times 2 = 2 \times 7$, so 14 is divisible by 2 and 7. However, dividing 14 by any other integer greater than 1 does not produce an integer result. For example, $14 \div 4$ leaves a remainder because 4 does not go evenly into 14. The multiples of the divisor 4 are

$$4, 8, 12, 16, 20, \ldots$$

, so $12 \div 4 = 3$ is the largest number of fours that go into 14. However, since $12 < 14$, we have $14 - 12 = 2$ left over, which we call a "remainder of 2." We say that $14 \div 4 = 3R2$, read as "fourteen divided by four equals three remainder 2."
Note that the remainder will always be less than the divisor; in this case, dividing by 4 will always leave a remainder that is either 0, 1, 2, or 3. To find the remainder of a number n upon division by a divisor d, we first find the largest multiple of d that goes into n, and the remainder r is the amount left over:

$$n = kd + rwithr \leq d$$

Example 3:

If n is an integer that leaves a remainder of 2 upon division by 6, what is the remainder of n upon division by 3?
Solution:
Since n leaves a remainder of 2 upon division by 6, we have $n = 6k + 2$, where $6k$ is the largest multiple of 6 that goes into n Now, this can also be written as $n = 3(2k) + 2$, and since 2¡3, we have $3(2k)$ is the largest multiple of 3 that goes into n. This shows that the remainder of n upon division by 3 is also 2.

2.6.3 Problems

Review problems

1. Without performing division, explain why the number 987654321 is a multiple of 9.
2. Without performing actual division, show that 87456399 is not divisible by 11.
3. If we know an number is a multiple of 5, how many possibilities are there for the last two digits?
4. If n is an integer that leaves a remainder of 4 upon division by 6, what is the remainder of n upon division by 3.
5. How many two-digit integers leave a remainder of 2 when divided by 8?

Challenging problems

1. For what values of a and b is $\overline{12ab}$ a multiple of 99?
2. Is 65973390 divisible by 210?
3. Show that if the last 3 digits of a number N are abc, N is a multiple of 8 if and only if $4a + 2b + c$ is a multiple of 8
4. What is the largest possible remainder when two-digit number is divided by the sum of its digits?
5. prove that the product of three consecutive integers is always divisible by 6.

2.7 PERFECT SQUARES and POWERS

2.7.1 introduction

A perfect square is an integer that can be expressed as the product of two equal integers. For example, 100 is a perfect square because it is equal to 10×10. If N is an integer, then N^2 is a perfect square. Because of this definition, perfect squares are always non-negative.
Similarly, a perfect cube is an integer that can be expressed as the product of three equal integers. For example, 27 is a perfect cube because it is equal to $3 \times 3 \times 3$. Determining if a number is a perfect square, cube, or higher power can be determined from the prime factorization of the number.

2.7.2 Perfect squares

Example 1:

Make a list of 10 perfect squares from the smallest.
Solution:
We have

$$(\pm 0) = 0; (\pm 1) = 1; (\pm 2) = 4; (\pm 3) = 9; (\pm 4) = 16;$$

$$(\pm 5) = 25; (\pm 6) = 36; (\pm 7) = 49; (\pm 8) = 64; (\pm 9) = 81;$$

Thus, the answer is 0,1,4,9,16,25,36,49,64,81.

2.7.3 Properties of perfect squares

1. perfect squares cannot have a units digit of 2, 3, or 7.
2. The square of an even number is even and the square of an odd number is odd.
3. All odd squares are of the form $4n + 1$, hence all odd numbers of the form $4n + 3$, where n is a positive integer, are not perfect squares
4. All even numbers of the form $4n + 2$, where n is a positive integer, are not perfect squares
5. All even squares are divisible by 4.
6. If p divides a^2, then p divides a as well. From this, we can say that a number is a perfect square if its prime factorization contains all primes raised to some even power. i.e perfect squares are of the form

$$p_1^{2a_1} p_2^{2a_2} \dots p_k^{2a_k}$$

   where p_i is distinct prime numbers and a_1 is a positive integer.

2.7.4 Perfect powers

A perfect power is the more general form of squares and cubes. Specifically, it is any number that can be written as the product of some non-negative integer multiplied by itself at least twice. In other words, it is of the form n^m for some integers $n \geq 0$ and $m > 1$.
The set of perfect powers is the union of the sets of perfect squares, perfect cubes, perfect fourth powers, and so on. The perfect powers less than or equal to 100 are 0,1,4,8,9,16,25,27,32,36,49,64,81,100.

- **Example 2:** Which the greatest of the following:

$$2^8, 3^6, 4^6, 5^4, 7^4, 10^3$$

  Solution:

  We have,

$$2^8 = 512, 3^6, = 7294^6 = 4096, 5^4 = 625, 7^4 = 2401, 10^3 = 1000$$

Thus 4^6 is the greatest among these numbers. Do note that there are better ways to determine which is the greatest or least given a set of numbers; the above powers can all be easily computed.

- **Example 3:** Find the number of digits in 64^7 given $6^7 = 279936$ and $7^7 = 823543$.

  Since we are given 6^7 and 7^7, we can easily see that the number of digits of any positive integer between 60 and 70 is in fact 13. Here's how: If n is greater than m, then n raised to k greater than m raised to k, where n, m are real numbers and k is a positive integer.

  Knowing this, we can find the number of digits in 64^7:

  $$60^7 = 6^7 \times 10000000 = 2799360000000, 70^7 = 7^7 \times 10000000 = 8235430000000$$

  Since both 60^7 and 70^7 have an equal number of digits and 60^7 is less than 64^7,but 64^7 is less than 70^7, then 64^7 must also have 13 digits.

- **Example 4:** Find the smallest positive integer n such that n^3 can be written as the sum of three consecutive positive integers.

  Solution:

  Let's assume the four consecutive positive integers are $k, k+1$, and $k+2$. So, we have:
  $n^3 = k + (k+1) + (k+2)$

  Now, let's simplify the right-hand side: $n^3 = 3k + 3$

  Since n^3 is a perfect cube, 3k + 3 must also be a perfect cube.

  Now, let's consider k as a positive integer. We want to find the smallest n, so we should look for the smallest k.

  The smallest value of k that makes $4k + 6$ a perfect cube is $k = 8$. Substituting this value into the equation: $n^3 = 3 \times 8 + 3 = 27$

  Therefore, the smallest positive integer n is 3 (since $3^3 = 27$).

2.7.5 Problems

Review problems

1. What is the positive number a in the following equation: $5^2 + 12^2 = a^2$

2. What are the perfect squares between 301 and 399?

3. Which of the following is not a perfect square?

   a)125 b)144 c)441 d)225

4. Given that n is an integer, is $n^2 + 2n + 1$ a perfect square?

5. What is the smallest n, such that 2^n is a perfect square?

Challenging problems

1. In the following equation, a, b and c are all distinct positive integer: $a^2 + b^2 = c^2$. Determine the smallest possible value of c.

2. Find the unit digit of 7^2023.

2.8 PROBLEMS

1. Which of the following is not a multiple of 3?
   a) 0 b) 3 c) 13 d) 21

2. Determine which of the following is true.
   a) 1260 is divisible by 4, 5, 6, and 7. b) 1260 is divisible by 42. c) 1260 is not divisible by 11. d) none of the above.

3. Problem 3: Consider the expression $2^n - 1$ for $n > 1$. What are the least two values for which the expression does not produce a prime number?
   a) 3, 4 b) 2, 6 c) 4, 5 d) 4, 6

4. How many integers between 1 and 100 that are divisible by 3, 5, or 7?
   a) 55 b) 50 c) 53 d) 54

5. Which of the following is not an even number?
   a) 0 b) 3^7 c) -289756834 d) $\frac{589228650}{3}$

6. What is the smallest composite number that has no prime divisors less than 10?
   a) 100 b) 79 c) 22 d) 121

7. Given that 103041 is a perfect square, how many factors does 254016 have?
   a) 120 b) 105 c) 504 d) 28

8. For what values of a and b is $\overline{12ab}$ a multiple of 99?
   a) 8, 7 b) 8, 9 c) 7, 9 d) 5, 6

9. Which of the following is the smallest prime number?
   a) 0 b) 1 c) 2 d) 3

10. How many two-digit positive integers leave a remainder of 4 when divided by 8?
    a) 10 b) 11 c) 12 d) 13

11. What is the last digit of 49^{25}
    a) 1 b) 9 c) 3 d) 7

12. If n is a positive integer and the $gcd(14n + 3, 12n + 1) = 11$, what is the possible of n?
    a) 11 b) 10 c) 5 b) 14

13. What is the lowest common multiple of 17 and 12?
    a) 0 b) 29 c) 68 d) 204

14. If n is an even number and $S = 3n + 6$, which of the following is false?
    a) S is always divisible by 3. b) S is always divisible by 6. c) S is always divisible by 4. d) none of the above.

15. What is the largest possible remainder when two-digit number is divided by the sum of its digits?
    a) 6 b) 7 c) 8 d) 9

16. Given that two integers a and b, such that $13gcd(a, b) = lcm(a, b)$ and that $a + b = 2016$, what are the values of a and b?
    a) 144, 1872 b) 1, 2015 c) 169, 1847 d) 504, 1512

17. Which of the following is not a perfect power?
    a) 0 b) 81 c) 128 d) 175

18. Evaluate $5 \times (2 \times 3^4) \div +7 - 8$
    a) 162 b) 154 c) 130 d) 125

19. The product of the digits in 38 is even because $3\times8=24$. Similarly, the product of the digits in 57 is odd because $5\times7=35$. How many 2-digit numbers have an odd product?
a) 30 b) 35 c) 25 d) 20

20. What is the value of

$$gcd(1,65)+gcd(2,65)+gcd(3,65)+\cdots+gcd(64,65)+gcd(65,65)$$

a) 225 b) 145 c) 243 d) 130

Chapter 3

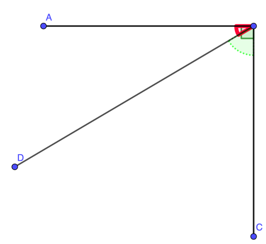
GEOMETRY

3.1 ANGLES AND THEIR PROPERTIES

Def: An angle is the figure formed by two rays, called the sides of the angle and sharing a common endpoint, called the vertex of the angle.

Complementary Angles

Complementary angles are angle pairs whose measures add up to one right angle ($\frac{1}{4}$ turn, 90°, or $\frac{\pi}{2}$ radians). If the two complementary angles are adjacent their non-shared sides form a right angle.



Supplementary Angles

Two angles whose measures add up to a straight angle ($\frac{1}{2}$ turn, 180°, or pi or π radians) are called supplementary angles. If the two supplementary angles are adjacent (i.e., have a common vertex and share just one side), their non-shared sides form a straight line. Such angles are called a linear pair of angles.

Vertically Opposite Angles (VOA)

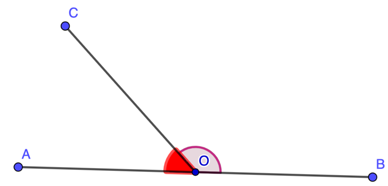
A pair of angles opposite to each other, formed by two intersecting straight lines that form an 'X'-like shape, are called vertical angles or opposite angles or vertically opposite angles. They are abbreviated as vert. opp. $\angle s.Theyarealwaysequal.$

Corresponding Angles Postulate or CA Postulate

If two parallel lines are cut by a transversal, then corresponding angles are congruent.

Alternate Interior Angles Theorem or AIA Theorem

If two parallel lines are cut by a transversal, then alternate interior angles are congruent to each other.



3.1.1 Triangle and its properties

Triangles preliminaries

A triangle is a geometrical shape which has three sides and three angles. It can also be defined as a polygon formed by three non-collinear points. A triangle is also a polygon which has three edges and three vertices. Among the types of triangle which exist includes:

- **Equilateral triangle**. An equilateral is a triangle whose equal sides and length. The angle is always 60 degrees, the sides are all equal.

- **Isosceles triangle**. An isosceles triangle is the one which has two equal sides and angles. the sides b are all equal and the angles are equal.

There are also other types of triangles which can be defined using the property of their angles. A triangle which has one of its internal angles is exactly equal to 90 degrees "**Right triangle**", while one with its internal all are less than 90 is called an "**Acute triangle**" and a triangle whose one of the internal angles is greater than 90 is called "**Obtuse triangle**."

Similarities of triangles

Similarities refers to the property of two or more shapes having the same shape but not necessarily the same size. When two shapes are similar, their corresponding angles are equal, but their corresponding sides are proportional. Similar shapes can be obtained by uniformly scaling one shape to match the size of the other shape. Similarity is denoted by the symbol $\sim$. To prove the similarity of shapes, it is necessary to establish that all corresponding angles are equal and the corresponding sides are in proportion. In geometry, two triangles are considered similar if their corresponding angles are equal, and their corresponding sides are in proportion. When triangles are similar, they have the same shape but may differ in size. This relationship allows us to make comparisons and establish proportional relationships between the corresponding sides and angles.

Key properties and concepts:

- **Angle-Angle (AA) Similarity**: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar. This property is known as the Angle-Angle Similarity Postulate.

- **Side-Side-Side (SSS) Similarity**: If the corresponding sides of two triangles are proportional, then the triangles are similar. This property is known as the Side-Side-Side Similarity Theorem.

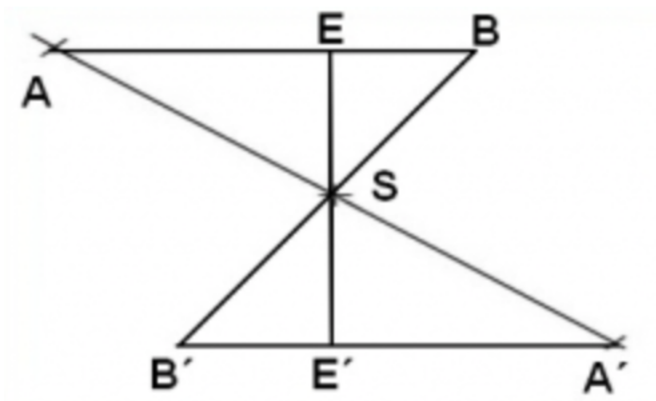
- **Side-Angle-Side (SAS) Similarity**: If one pair of corresponding sides of two triangles is proportional, and the included angles are congruent, then the triangles are similar. This property is known as the Side-Angle-Side Similarity Theorem.

- **Ratio of Corresponding Sides**: In similar triangles, the ratios of corresponding sides are equal. This property is known as the Triangle Proportionality Theorem. For example, if two triangles are similar and their corresponding sides have lengths a and b, then the ratio of their side lengths is a/b.

Examples

1. In the triangle ABC, the point P lies closer to point A in the third of the line AB, the point R is closer to point P in the third of the line P, and the point Q lies on the line BC so that the angles $\angle PCB$ and $\angle RQB$ are identical.

   Determine the ratio of the area of triangles ABC and PQC.



Step-by-step explanation:

$$|AB| = a$$

$$|AP| = \frac{1}{3}a$$

$$|PR| = \frac{1}{3} \cdot \frac{2}{3}a = \frac{2}{9}a$$

. Similarly,

$$|RB| = \left(1 - \frac{1}{3} - \frac{2}{9}\right) a = \frac{4}{9}a$$

Area of triangle ABC is

$$S(ABC) = \frac{1}{2}a \cdot h$$

Area of triangle APC is

$$S(APC) = \frac{1}{2}|AP| \cdot h = \frac{1}{3}a \cdot h = \frac{1}{3} \cdot S(ABC)$$

Area of triangle PCB is

$$S(PCB) = S(ABC) - S(APC) = \frac{2}{3} \cdot S(ABC)$$

Let h_2 be the height of triangle PCB. We have $h_2 = \frac{2}{3}h$
Area of triangle PQB is

$$S(PQB) = \frac{1}{2}|PB| \cdot h_2 = \frac{1}{2} \cdot \frac{9}{4}a \cdot \frac{2}{3}h = \frac{9}{4} \cdot S(ABC)$$

Area of triangle PQC is

$$S(PQC) = S(ABC) - S(APC) - S(PQB) = S(ABC)\left(1 - \frac{1}{3} - \frac{9}{4}\right)$$

Let k be the ratio of the area of triangles ABC and PQC. We have

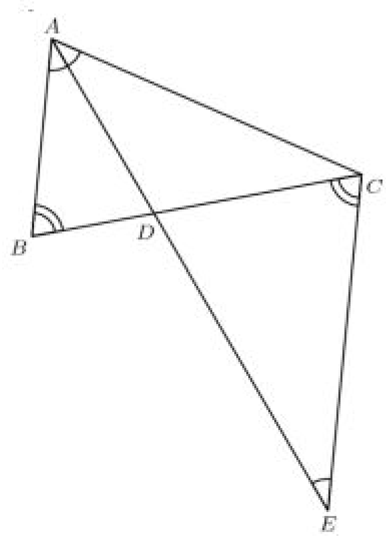
$$k = 1 - \frac{1}{3} - \frac{9}{4} = \frac{9}{2} \approx 0.2222$$

So,

$$p = \frac{S(ABC)}{S(PQC)} = \frac{1}{k} = \frac{1}{\frac{9}{2}} = \frac{2}{9} \approx 0.2222$$

Therefore, $p = 9 : 2$

Solved example. Take $\triangle ABC$. If D $\in$BC so that AD bisects $\angle BAC, thenshowthat$AB$_{\overline{BD=\frac{AC}{CD}}}$
Solution:
Consider this diagram below;



Then

$$\angle ABC = \angle ECD$$

and

$$\angle DEC = \angle DAC$$

so it follows that

$$\triangle ABC \sim \triangle ECD$$

.

Thus,

$$\frac{AB}{BD} = \frac{EC}{CD}$$

.
Finally,

$$\angle CED = \angle DAC$$

so it follows that $\triangle ACE$ is isosceles and AC = EC so we find that

$$\frac{AB}{BD} = \frac{AC}{CD}$$

as desired.

Congruence of triangles

Congruence refers to the property of two or more shapes having the same size and shape. When two shapes are congruent, it means that they can be superimposed onto each other, preserving both their size and shape. The corresponding sides and angles of congruent shapes are equal. Congruence is denoted by the symbol $\cong$. To prove the congruence of shapes, it is necessary to establish that all corresponding angles and sides are equal.

In geometry, congruence refers to the property of two triangles being identical in shape and size. When two triangles are congruent, all corresponding angles and sides of the triangles are equal. This means that they can be superimposed on each other by translations, rotations, and reflections.

Key properties and concepts:

- **Side-Side-Side (SSS) Congruence**: If the three sides of one triangle are congruent to the three sides of another triangle, then the triangles are congruent. This property is known as the Side-Side-Side Congruence Theorem.

- **Side-Angle-Side (SAS) Congruence**: If one pair of corresponding sides of two triangles is congruent, and the included angles are congruent, then the triangles are congruent. This property is known as the Side-Angle-Side Congruence Theorem.

- **Angle-Side-Angle (ASA) Congruence**: If one pair of corresponding angles of two triangles is congruent, and the included side is congruent, then the triangles are congruent. This property is known as the Angle-Side-Angle Congruence Theorem.

- **Angle-Angle-Side (AAS) Congruence**: If two pairs of corresponding angles of two triangles are congruent, and a pair of corresponding sides that includes one of the angles is congruent, then the triangles are congruent. This property is known as the Angle-Angle-Side Congruence Theorem.

- **Right Angle Hypotenuse Side (RHS) Congruence**: Two right triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other triangle.

3.2 QUADRILATERALS

3.2.1 Convex Quadrilaterals

A convex quadrilateral is a four-sided polygon (quadrilateral) where all of its interior angles are less than 180 degrees, and all its vertices point outward. In other words, all the corners of a convex quadrilateral "stick out" and do not fold inward. Properties of Convex Quadrilaterals: Interior Angles: The sum of the interior angles of a convex quadrilateral is always 360 degrees. Diagonals: A convex quadrilateral has two diagonals, i.e., line segments joining non-adjacent vertices. In a convex quadrilateral, both diagonals lie inside the polygon. Opposite Angles: The opposite angles of a convex quadrilateral are equal in measure. For example, if we label the angles as A, B, C, and D, then angle A is equal to angle C, and angle B is equal to angle D. Consecutive Angles: The consecutive angles of a convex quadrilateral are supplementary, meaning the sum of two consecutive angles is 180 degrees. For example, angle A + angle B = 180 degrees, angle B + angle C = 180 degrees, and so on. Parallel Sides: In a convex quadrilateral, opposite sides are parallel. This property is true for parallelograms, which are a special type of convex quadrilateral. Equal Opposite Sides: The opposite sides of a convex quadrilateral are equal in length. If we label the sides as AB, BC, CD, and DA, then side AB is equal to side CD, and side BC is equal to side DA. Symmetry: A convex quadrilateral can have one or more axes of symmetry, depending on its shape. For example, a square has four axes of symmetry, while a rectangle has two. Area: The area of a convex quadrilateral can be calculated using various formulas, depending on the information available about its sides and angles. One common method is by dividing the quadrilateral into triangles and using the formula for triangle area.

Example problem and solution

Points E and F are on side BC of a convex quadrilateral ABCD with BE ¡ BF.Given that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$, prove that $\angle FAC = \angle EDB$.

Proof. Note that

$$\angle EAF = \angle FDE$$

implies that AEF D is cyclic.
It suffices to show that ABCD is cyclic.
Note that

$$\angle ADC = \angle ADF + \angle FDC$$

, so we have

$$\angle ABC + \angle ADC = \angle ABC + \angle ADF + \angle FDC$$

. Then,

$$\angle ABC = \angle AEF - \angle BAE$$

so it follows that

$$\angle ABC + \angle ADC = \angle ABC + \angle ADF + \angle FDC$$

$$= \angle AEF - \angle BAE + \angle ADF + \angle FDC$$

$$= \angle AEF + \angle ADF$$

$$= 180$$

, which shows that ABCD is cyclic, as desired.

3.2.2 Cyclic quadrilaterals

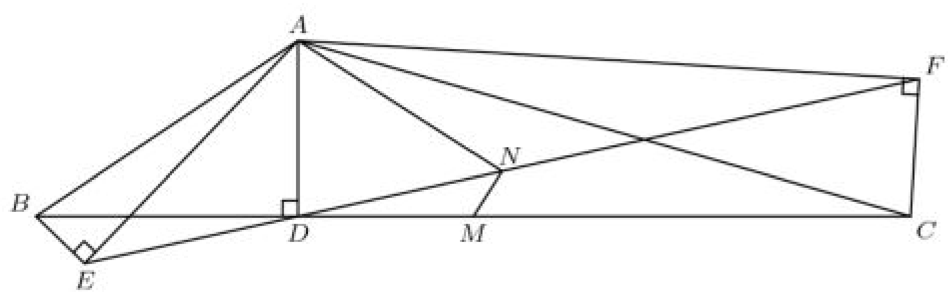
A cyclic quadrilateral is a four-sided polygon (quadrilateral) whose four vertices all lie on a single circle. In other words, if you draw a circle that passes through all four vertices of the quadrilateral, it is said to be a cyclic quadrilateral. Properties of Cyclic Quadrilaterals: Circumcircle: All four vertices of a cyclic quadrilateral lie on the circumference of a single circle, known as the circumcircle of the quadrilateral. Opposite Angles: The opposite angles of a cyclic quadrilateral are supplementary, meaning the sum of two opposite angles is 180 degrees. If we label the angles as A, B, C, and D, then angle A + angle C = 180 degrees, and angle B + angle D = 180 degrees. Cyclic Quadrilateral Theorem: The sum of the measures of any pair of opposite angles in a cyclic quadrilateral is 180 degrees. This theorem is sometimes referred to as the "opposite angles of a cyclic quadrilateral add up to 180 degrees." Diagonals and Circumcircle: The intersection point of the diagonals of a cyclic quadrilateral lies on the circumcircle of the quadrilateral. In other words, the two diagonals meet on the circle that passes through all four vertices. Angle between Side and Tangent: The angle between one side of a cyclic quadrilateral and the tangent to the circumcircle at the opposite vertex is equal to the opposite angle inside the quadrilateral. For example, if we draw the tangent to the circumcircle at vertex A, then angle BCD = angle A. Ptolemy's Theorem: Ptolemy's theorem relates the side lengths and diagonals of a cyclic quadrilateral. For a cyclic quadrilateral with side lengths AB, BC, CD, and DA, and diagonals AC and BD, Ptolemy's theorem states: $AB \cdot CD + BC \cdot DA = AC \cdot BD$. Concyclic Points: The vertices of a cyclic quadrilateral are called concyclic points because they all lie on the same circle. Area: The area of a cyclic quadrilateral can be calculated using various methods, including Brahmagupta's formula for cyclic quadrilaterals: $Area = \sqrt{[(s-a)(s-b)(s-c)(s-d)}$, where s is the semiperimeter, and a, b, c, and d are the side lengths.

3.2.3 Solved example

Let ABC be a triangle and D be the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BC, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

solution

Proof. Note that ABED and AF CD are cyclic quadrilaterals. It follows that $ABC \sim AEF$ since



$$\angle ABD = \angle AED$$

and

$$\angle AFD = \angle ACD$$

. Similarly, we can show that

$$ABM \sim AEN$$

since

$$\frac{AB}{AE} = \frac{BC}{EF} = \frac{2BN}{2EM} = \frac{BN}{EM}$$

Therefore,

$$\angle AND = \angle AMD$$

and it follows that ANMD is cyclic. Therefore

$$\angle ANM = 180 - \angle ADM = 90$$

as desired.

3.3 CIRCLE

3.3.1 Tangents and its properties

In geometry, a tangent is a line that touches a curve or a circle at a single point, without intersecting it. The point where the tangent touches the curve or circle is called the point of tangency. Tangents have

- Tangent Line to a Circle: If a line is tangent to a circle at a particular point, it is perpendicular to the radius of the circle that passes through that point. This property is essential for constructing tangents to circles.
- Tangent and Radius: The tangent to a circle at a given point is perpendicular to the radius drawn to that point. This property follows from the fact that the tangent and radius form a right angle at the point of tangency.
- Unique Tangent: A circle can have only one tangent at any given point on its circumference. This uniqueness property ensures that there is only one line that touches the circle at a specific point without intersecting it.
- Tangent Chord Angle Theorem: The angle between a tangent and a chord drawn from the point of tangency is equal to the angle formed by the chord in the alternate segment of the circle.
- Common Tangents: Two circles can have four types of common tangents: external common tangents, internal common tangents, direct common tangents, and transverse common tangents. External and internal common tangents lie outside and inside both circles, respectively, while direct and transverse common tangents touch one circle internally and the other externally.
- Length of Tangent Segments: If two tangents are drawn to a circle from a point outside the circle, the lengths of the tangent segments are equal.
- Secant-Tangent and Chord-Tangent Theorems: If a secant and a tangent intersect at a point on a circle, the product of the lengths of the whole secant and its external segment is equal to the square of the length of the tangent segment. Similarly, if a chord and a tangent intersect at a point on a circle, the product of the lengths of the whole chord and its external segment is equal to the square of the length of the tangent segment.

3.3.2 Chords and its properties

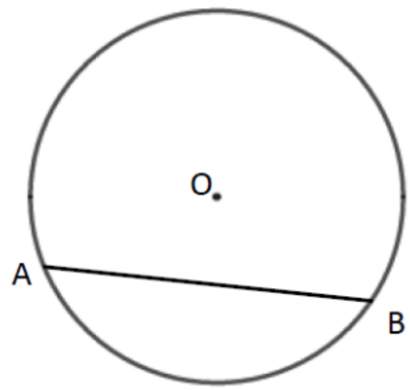
Length of a Chord: The length of a chord can be calculated using the distance formula or by using the properties of right triangles formed by the radius and the chord.

- Diameter as a Special Chord: A diameter is a special type of chord that passes through the center of the circle. It is the longest chord in a circle, and its length is twice the length of a radius.
- Bisecting Chord: The perpendicular bisector of a chord passes through the center of the circle. It divides the chord into two equal parts.
- Intersecting Chords Theorem: In a circle, if two chords intersect, the product of the segments of one chord is equal to the product of the segments of the other chord. This theorem is often used to solve problems involving intersecting chords.

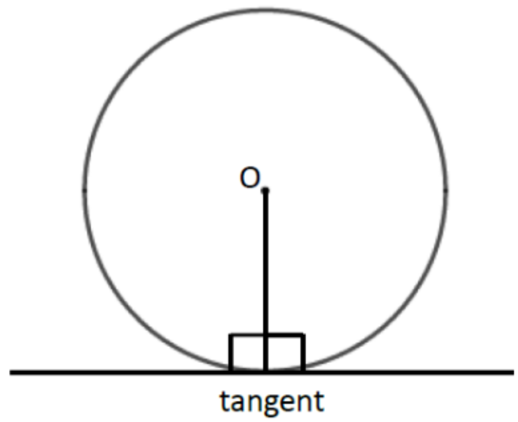
- Perpendicular Chords and Diameter: If two chords in a circle are perpendicular, one of them must be a diameter. This property helps identify diameters when given perpendicular chords.
- Angle with the Center: The angle subtended by a chord at the center of the circle is twice the angle subtended by the same chord at any point on the circumference (inscribed angle).
- Circle Chord Properties: If two chords in a circle are congruent, they are equidistant from the center of the circle. Conversely, if two chords are equidistant from the center, they are congruent.
- Sagitta: The perpendicular distance from the center of the circle to a chord is called the sagitta. In a semicircle, the sagitta is the radius of the circle.
- Circle Theorems with Tangents: There are several properties that relate tangents and chords in a circle. For example, if a tangent and a chord intersect at a point on the circle, the angle between them is equal to the angle formed by the chord in the alternate segment of the circle.

3.3.3 Circle Theorems

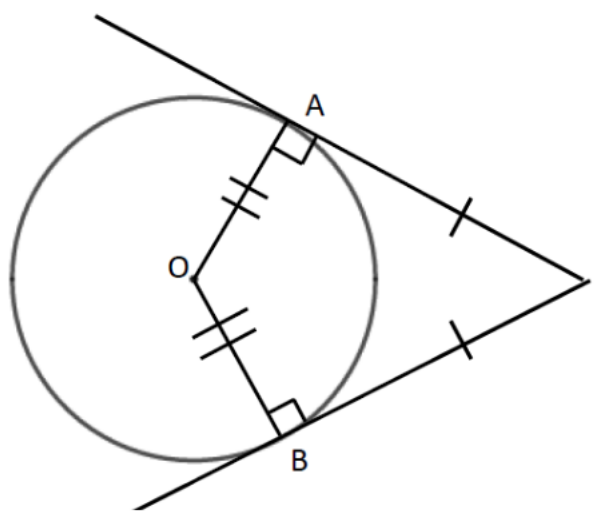
1. A chord is a straight line joining two points on the circumference of a circle. So AB is a chord.



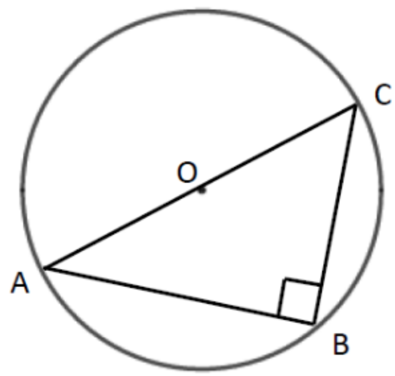
2. A tangent is a straight line that touches the circumference of a circle at only one point. The angle between a tangent and the radius is 90°.



3. Two tangents on a circle that meet at a point outside the circle are equal in length. So AC = BC.

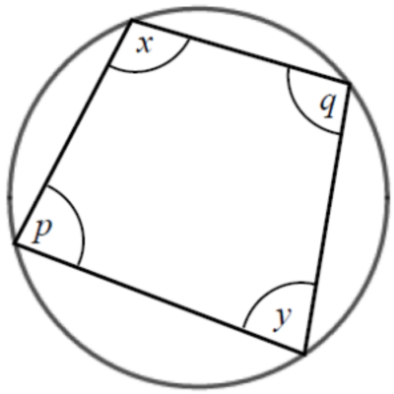


4. The angle in a semicircle is a right angle. So angle ABC = 90°.



5. When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference. So $\angle AOB = 2\angle ACB. Anglessubtendedbythesamearcatthecircumferenceareequal. Thismeansthata$ $\angle ADB and \angle CAD = \angle CBD$.

7. A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle. Opposite angles in a cyclic quadrilateral total 180°. So x + y = 180° and p + q = 180°.



8. The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem. So $\angle BAT = \angle ACB$.

3.4 INTRODUCTION TO CARTESIAN PLANE

3.4.1 Introduction

The Cartesian plane, also known as the coordinate plane or Cartesian coordinate system, is a fundamental concept in mathematics. It provides a way to represent and locate points in a two-dimensional space.

3.4.2 Key Features:

- Axes: The Cartesian plane consists of two perpendicular lines called the x-axis and y-axis. The x-axis is horizontal, and the y-axis is vertical. They intersect at a point called the origin, typically labeled as (0, 0).
- Quadrants: The plane is divided into four quadrants by the x-axis and y-axis. The quadrants are numbered counterclockwise starting from the top right as I, II, III, and IV. Each quadrant has a unique combination of positive and negative x and y values.
- Coordinates: Points in the Cartesian plane are represented by ordered pairs of numbers, written as (x, y), where x is the horizontal distance from the y-axis (positive to the right, negative to the left), and y is the vertical distance from the x-axis (positive upward, negative downward).
- Axes Units: The distance between consecutive units on the x-axis and y-axis is often equal, representing the scale of measurement on the plane. However, the units on each axis can have different intervals.

3.4.3 How to find a point coordinates on cartesian plane

To find a point on the Cartesian plane, you need to know its coordinates, which consist of an x-coordinate and a y-coordinate. Here's how you can find a point on the Cartesian plane:

- Locate the x-coordinate: Determine the horizontal position of the point on the x-axis. If the x-coordinate is positive, move to the right from the origin. If it is negative, move to the left from the origin. The distance you move corresponds to the absolute value of the x-coordinate.
- Locate the y-coordinate: Determine the vertical position of the point on the y-axis. If the y-coordinate is positive, move upward from the origin. If it is negative, move downward from the origin. The distance you move corresponds to the absolute value of the y-coordinate.

- Intersection of x and y: The point on the Cartesian plane is located at the intersection of the horizontal line representing the x-coordinate and the vertical line representing the y-coordinate. Mark this intersection point.
- Label the point: Write the coordinates of the point as an ordered pair, in the form (x, y), where x represents the x-coordinate and y represents the y-coordinate.

3.5 APPROACHES AND BASICS IN SOLVING OLYMPIAD PROBLEMS USING GEOMETRY

3.5.1 Angle chasing

Angle chasing is a problem-solving technique in geometry that involves systematically using angle relationships and properties to find unknown angles in a given geometric figure. This technique is particularly useful when dealing with polygons, circles, and other geometric shapes where angles play a significant role. Angle chasing often requires identifying angles formed by intersecting lines, parallel lines, angles in triangles, angles in quadrilaterals, and angles in circles. The steps involved in angle chasing may vary depending on the specific problem, but the general approach includes the following strategies:

- Identify Known Angles: Start by identifying any angles that are already given or can be easily determined from the problem statement. This may include angle measurements provided in the figure or angles formed by intersecting lines or parallel lines.

- Use Angle Sum Properties: In polygons, such as triangles and quadrilaterals, the sum of the interior angles is constant. For example, in a triangle, the sum of the three interior angles is always 180 degrees. Utilize these angle sum properties to find unknown angles when some angles are given.

- Use Angle Relationships: Identify angle relationships within the figure, such as vertical angles, corresponding angles, alternate interior angles, and supplementary angles. These relationships can help you find missing angles by setting up equations and solving for the unknown values.

- Use Angle Properties in Circles: For problems involving circles, use the properties of central angles, inscribed angles, and angles formed by intersecting chords and tangents. These properties provide relationships between angles and arc measures, which can be used to find unknown angles.

- Extend Lines and Create Parallel Lines: In some cases, you may need to extend lines or create parallel lines to form specific angle relationships that can aid in finding unknown angles.

- Work Methodically: Angle chasing can involve multiple steps and several angle relationships, so it's essential to work methodically and keep track of the angles you find along the way. Draw additional lines or label angles as needed to help you visualize the relationships. Be Patient and Persistent: Angle chasing can sometimes be challenging, and finding all the necessary angle relationships may take time and multiple attempts. Be patient and persistent in your approach, and try different strategies if you get stuck.

Chasing con't

One of the most crucial skills that is necessary to master in order to be a good geometer is the skill of angle chasing. In this meeting, I will go over the skills that are generally lumped under the "angle chasing" category before giving an example as to what angle chasing is. The majority of this meeting will consist of problem-solving, as many of the skills you probably already know.

What exactly is angle chasing?

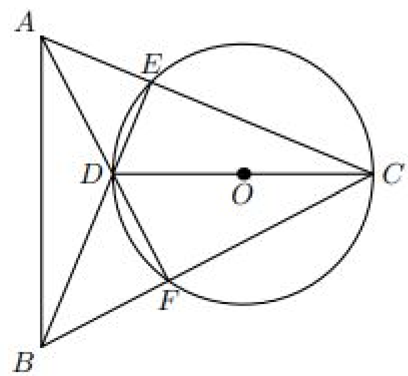
Angle chasing usually consists of any of the following techniques:

- vertical angles
- sum of angles in a triangle
- congruent triangles
- similar triangles
- supplementary angles

- complementary angles
- parallel lines and angles
- perpendicular lines
- angle bisectors
- angles intercepting arcs

However, the above techniques are not necessarily all-inclusive when it comes to angle chasing. An example of this can be shown in the problem below.

Example 1 (MA0 1987). In the figure, O is the center of the circle, $\angle EAD = 40°$, and $\angle EDd = 40°$. Find $\angle DAB$.



Solution. Note that from secant-secant stuff we get that $\angle EAD = \frac{1}{2}(CFd - EDd)$. Since $\angle EAD$ and $\angle EDd$ are both $40°$, we can substitute to get $40° = \frac{1}{2}(CFd - 40°) \Rightarrow CFd = 120°$. Next, note that since CDd is a diameter, we get that $DFd = 180° - CFd = 60°$, so $\angle DCB = 30°$. Finally, since $BE \perp AC$ and $AF \perp BC$, D is the orthocenter of $\triangle ABC$. This means that $CD \perp AB$, so $\angle DAB = 90° - \angle B = \angle DCB = 30°$. (Don't worry, most of the problems in the problem section do not require knowledge of the orthocenter!)

Examples on topic

1. $\triangle$ ABC is an isosceles triangle with AB=BC and $\angle ABD = 60^0$, $\angle DBC = 20^0$ and

$$\angle DCB = 10^0$$

. Find $\angle BDA$.
Solution Let E be the circumcenter of BCD . Then

$$\angle BED = 2\angle BCD = 20^0$$

and

$$\angle DEC = 2\angle DBC = 40^0$$

. Hence

$$\angle BEC = 60^0$$

. This and BE=EC
shows that BEC is equilateral. So BC=BE and

$$\angle CBE = 60^0$$

. By assumption AB=BC , so AB=BE and

$$\angle BEA = 90^0 - \frac{1}{2}\angle ABE = 90^0 - \frac{1}{2} \cdot 140^[0] = 20^0 = \angle BED$$

. Therefore A,D,E are collinear and we find

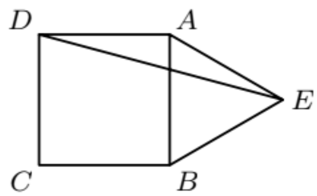
$$\angle BDA = 180^0 - \angle EDB = \angle BED + \angle DBE = 20^0 + 80^0 = 100^0$$

.

Exercises

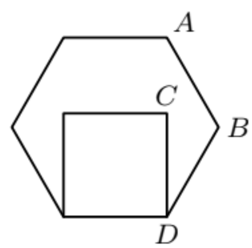
1. In $\triangle ABC$, $AB = AC$ and $\angle A = 40°$. The bisector from $\angle B$ intersects AC at point D. What is $\angle BDC$?

2. In the adjoining figure, $ABCD$ is a square, ABE is an equilateral triangle, and point E is outside square $ABCD$. What is the measure of $\angle AED$?

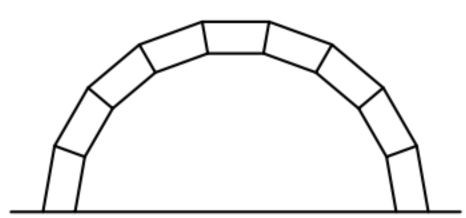


3. (MATHCOUNTS 1986) $\triangle ABC$ is an isosceles triangle such that $AC = BC$. $\triangle CBD$ is an isosceles triangle such that $CB = DB$. BD meets AC at a right angle. If $\angle A = 57°$, what is $\angle D$?

4. A square is located in the interior of a regular hexagon, and certain vertices are labeled as shown. What is the degree measure of $\angle ABC$?



5. The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?



6. [*Math League HS* 2013 − 2014/2009 − 2010/1994 − 1995] In a certain quadrilateral, the three shortest sides are congruent, and both diagonals are as long as the longest side. What is the degree measure of the largest angle of this quadrilateral?

7. Let $\triangle ABC$ be a right triangle with a right angle at C. Let D and E be the feet of the angle bisectors from A and B to BC and CA, respectively. Suppose that AD and BE intersect at point F. Find $\angle AFB$.

8. (AHSME 1990) An acute isosceles triangle $\triangle ABC$ is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at point D. If $\angle ABC = \angle ACB = 2\angle D$, find the measure of $\angle A$.

9. (BMO1 1995) Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find $\angle MCN$.

10. (ELMO SL 2013, Owen Goff) Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y , and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the David Yang triangle of ABC and let XYZ be the Scott Wu triangle of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

11. (IMO 1990) Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines $\overrightarrow{BC}$ and $\overrightarrow{AC}$ at F and G, respectively. If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of t. (Note: You may want to explore the next problem a bit before attempting this one.)

12. An exploration of cyclic quadrilaterals.
(a) Let ABCD be a quadrilateral inscribed inside a circle. Prove that:
i. $\angle ABC + \angle ADC = 180°$.
ii. $\angle ABD = \angle ACD$.
(b) Assume that the converse is true, i.e., if any one of the two properties above holds, then the quadrilateral is cyclic. This is a very useful tool, as it allows you to switch back and forth between the two angle properties/equalities. Using this knowledge, prove the following statements:
i. Let ABC be a right triangle with the right angle at B. Let D be any point on AB, and let E be the foot of the perpendicular from D to AC. Prove that $\angle DBE = \angle DCE$.
ii. [Canada 1986] A chord ST of constant length slides around a semicircle with diameter AB. M is the midpoint of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.
iii. [Sharygin 2009] Given triangle ABC. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

3.5.2 Working backwards

A common stratagem, when trying to prove that a given point has a desired property, is to construct a phantom point with the desired property, then reason backwards to show that it coincides with the original point. We illustrate this point with an example.

Theorems

Suppose the triangles $\triangle ABC$ and $\triangle AB_1C_1$ are directly similar. Then the points A, B, C, $\overline{BB_1} \cap \overline{CC_1}$ lie on a circle.

Since we want to show that $\overline{BB_1} \cap \overline{CC_1}$ lie on the circle through A, B, C, and analogously on the circle through A, B_1, C_1, we define the point P to be the intersection of these two circles. Then

$$\angle APB = \angle ACB, \quad \angle AC_1B_1 = \angle APB_1.$$

So, P lies on the line $\angle BB_1$, and similarly on the line $\angle CC_1$.

EXERCISES ON TOPIC

1. (IMO 1994/2) Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midpoint of BC and O is the point on the line $\overleftrightarrow{AM}$ such that $\overleftrightarrow{OB}$ is perpendicular to $\overleftrightarrow{AB}$;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line $\overleftrightarrow{AB}$ and F lies on the line $\overleftrightarrow{AC}$ such that E, Q, F are distinct and collinear.

Prove that $\overleftrightarrow{OQ}$ is perpendicular to $\overleftrightarrow{EF}$if and only if $QE = QF$.

2. (USAMO 2005/3) Let $\triangle ABC$ be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $\overleftrightarrow{QC_1} \parallel \overleftrightarrow{CA}$, and C_1 and Q lie on opposite sides of line $\overleftrightarrow{AB}$. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $\overleftrightarrow{QB_1} \parallel \overleftrightarrow{BA}$, and B_1 and Q lie on opposite sides of line $\overleftrightarrow{AC}$. Prove that points B_1, C_1, P, and Q lie on a circle.

3. (Morley's theorem) Let $\triangle ABC$ be a triangle, and for each side, draw the intersection of the two angle trisectors closer to that side. (That is, draw the intersection of the trisectors of A and B closer to AB, and so on.) Prove that these three intersections determine an equilateral triangle.

3.6 PROBLEMS

Facts you should know

1. Let ABC be a triangle and extend BC past C to D. Show that $\angle ACD = \angle BAC + \angle ABC$.
2. Let ABC be a triangle with $\angle C = 90$. Show that the circumcenter is the midpoint of AB.
3. Let ABC be a triangle with orthocenter H and feet of the altitudes D, E, F. Prove that H is the incenter of $\triangle DEF$.
4. Let ABC be a triangle with orthocenter H and feet of the altitudes D, E, F. Prove that (i) A, E, F, H lie on a circle with diameter AH, and (ii) B, E, F, C lie on a circle with diameter BC.
5. Let ABC be a triangle with circumcenter O and orthocenter H. Show that $\angle BAH = \angle CAO$.
6. Let ABC be a triangle with circumcenter O and orthocenter H, and let AH and AO meet the circumcircle at D and E, respectively. Show (i) that H and D are symmetric with respect to BC, and (ii) that H and E are symmetric with respect to the midpoint of BC.
7. Let ABC be a triangle with altitudes AD, BE, and CF. Let M be the midpoint of side BC. Show that ME and MF are tangent to the circumcircle of $\triangle AEF$.
8. Let ABC be a triangle with incenter I, A-excenter I_a, and D the midpoint of arc BC not containing A on the circumcircle. Show that $DI = DI_a = DB = DC$.
9. Let ABC be a triangle with incenter I and D the midpoint of arc BC not containing A on the circumcircle. Define E and F similarly. Show (i) that I is the orthocenter of $\triangle DEF$, and (ii) that A, B, C are the reflections of I across EF, FD, DE respectively.
10. Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c. Prove that in triangle $I_aI_bI_c$, A, B, C are the feet of the altitudes and I is the orthocenter.
11. Let ABC be a triangle with incenter I, and whose incircle is tangent to sides BC, AC, AB at D, E, F respectively. Let M, N be midpoints of BC, AC respectively. Prove that EF, BI, MN concur.

12. (Simson Line) Let ABC be a triangle and D a point on its circumcircle. Prove that the feet of the perpendiculars from D to lines AB, AC, and BC are collinear.

13. (Nine Point Circle) Let ABC be a triangle with orthocenter H, altitudes AA_1, BB_1, and CC_1, and midpoints A_2, B_2, C_2. Let the midpoints of AH, BH, CH be A_3, B_3, C_3. Show that $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ lie on a circle.

Problems

1. In parallelogram $ABCD$, let the bisector of $\angle BCD$ intersect lines AB and AD at E and F, respectively. Prove that $BE = AD$ and $DF = AB$.

2. Let ABC be a right triangle, let $\angle C$ be the right angle, and let D be the foot of the altitude from C. Prove that the circumcenters of ACD, CBD, and ABC form a triangle that is similar to ABC.

3. ω_1, ω_2 are two circles intersecting at P and Q. Let A be a variable point on ω_1, and B, C be the intersections of AP, AQ with ω_2. Show that the size of BC is independent of A.

4. (British Math Olympiad 2000) Two intersecting circles $C1, C2$ have a common tangent PQ with P on $C1$ and Q on $C2$. The two circles intersect at M, N, where PQ is nearer to M. The line PN meets the circle $C2$ again at R. Prove that MQ bisects $\angle PMR$.

5. ("Largely Artistic Math Olympiad") Two circles $\omega1, \omega2$ intersect at P, Q. If a line intersects $\omega1$ at A, B and $\omega2$ at C, D such that A, B, C, D lie on the line in that order, and P and Q lie on the same side of the line, compute $\angle APC + \angle BQD$.

6. Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle.

7. (British Math Olympiad 2005) The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q. Show that $BX = CX$ if and only if PQ is a diameter of S.

8. (Own) In triangle ABC, M and N are midpoints of AC and AB respectively. Point D is on BC such that $MD = MC$. Extend lines MD and ND to meet AB and AC at F and E, respectively. Prove that EF is perpendicular to BC.

9. (ELMO 2012) In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

10. (USA(J)MO 2010) Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

11. (IMO Shortlist 2010) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

12. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90$. Prove that AO bisects $\angle BAE$.

13. (IMO 2006) Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$ and that equality holds if and only if $P = I$.

14. (IMO 2002) The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and F. Prove that I is the incenter of the triangle CEF.

15. (Balkan MO 2012) Let A, B, and C be points lying on a circle Γ with center O. Assume that $\angle ABC > 90^\circ$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F.

16. (APMO 2007) Let ABC be an acute-angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

17. In scalene triangle ABC, H, I, and O are the orthocenter, incenter, and circumcenter respectively. Prove that one of the angles of the triangle is 60° if and only if $IH = IO$.

18. (EGMO 2012) Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM, and BC are concurrent.

19. (CGMO 2012) In triangle ABC, $AB = AC$. Point D is the midpoint of side BC. Point E lies outside the triangle ABC such that $CE \perp AB$ and $BE = BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc AD of the circumcircle of triangle ABD such that $MF \perp BE$. Prove that $ED \perp FD$.

20. In triangle ABC, let $D \in BC, E \in AC, F \in AB$ be the points of tangency of the incircle to the sides. Let I be the incenter. The parallel line through A to BC intersects DE and DF at M and N respectively. Let L and T be the midpoints of the segments ND and DM. Show that A, L, I, T lie on a circle.

21. (EGMO 2012) Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that $DE \perp CO$ and $DF \perp BO$. Let K be the circumcenter of triangle AFE. Prove that the lines DK and BC are perpendicular.

22. (IMO 2010) Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the intersection of lines EI and DG lies on Γ.

23. (Romanian TST 1996) Let $ABCD$ be a cyclic quadrilateral and let M be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total). Prove that there are two sets K and L of four parallel lines each, such that every line in $K \cup L$ contains exactly four points of M.

Chapter 4

COMBINATORICS

4.1 COUNTING METHODS

4.1.1 Introduction

Combinatorics is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. It is closely related to many other areas of mathematics and has many applications ranging from logic to statistical physics and from evolutionary biology to computer science.

As an area can be described by the types of problems it addresses, combinatorics is involved with:

- *The enumeration (counting) of specified structures, sometimes referred to as arrangements or configurations in a very general sense, associated with finite systems,*
- *The existence of such structures that satisfy certain given criteria*
- *The construction of these structures, perhaps in many ways, and*
- *Optimization: finding the "best" structure or solution among several possibilities, be it the "largest", "smallest" or satisfying some other optimality criterion.*

In this book, our main interest is to teach you different methods of counting and providing you with an intuition for approaching counting problems.

4.1.2 Fundamental principles of counting

Addition Principle (AP)

Let A_1 and A_2 be disjoint events, that is events having no common outcomes, with n_1 and n_2 possible outcomes for each event , respectively. Then the total number of outcomes for the event "A_1 or A_2" is n_1 +n_2. Note that the events must be disjoint, that is they must not have common outcomes for this principle to be applicable.

Example Suppose there is this list of drinks on the table: Fanta, Juice and Water. How many selections does Kamanzi have for picking one drink? Suppose there is this list of drinks on the table: Fanta, Juice and Water. How many selections does Kamanzi have for picking one drink?

Solution: In this case, an event is "selecting a drink". There is 1 outcome for the Fanta event, 1 outcome for the juice event and 1 outcome for the water event. According to the addition principle there are :1+1+1 = 3 possible selections. This addition principle can be generalized for more than two events.

General Addition Principle:
LetA_1, A_2, ... A_k be disjoint events with n_1,n_2, ... n_k possible outcomes, respectively. Then the total number of outcomes for the event "A_1 or A_2 or ... or A_k" is n_1+n_2+ ... +n_k.

Product principle

Let A_1 and A_2 be events with n_1 and n_2 possible outcomes, respectively. Then the total number of outcomes for the sequence of the two events is $n_1 \times n_2$.

General Multiplication Principle:

Let A_1, A_2, ... A_k be events with n_1,n_2, ... n_k possible outcomes, respectively. Then the total number of outcomes for the sequence of these k events is $n_1 \times n_2 \times ... \times n_2$.
Example: Suppose that there are three major auto routes from Washington DC to Chicago, and five from Chicago to Los Angeles. Then there are 3×5= 15 major routes from Washington DC to Los Angeles.
INCLUSION-EXCLUSION PRINCIPLE

Principle of Inclusion and Exclusion is an approach which derives the method of finding the number of elements in the union of two finite sets. This is used to solve combinations and probability problems when it is necessary to find a counting method, which makes sure that an object is not counted twice.
A frequently occurring problem is to determine the size of the union or intersection of a number of sets, as in the example below:

At a certain university, all second-year science students may choose either mathematics, or physics, or both. The mathematics course is attended by 50 students, the physics course by 30 students. 15 students attend both courses. How many second-year science students are there?

SOLVED EXAMPLE: There are 350 farmers in a large region. 260 farm beetroot, 100 farm yams, 70 farm radish, 40 farm beetroot and radish, 40 farm yams and radish, and 30 farm beetroot and yams. Let B, Y, and R denote the set of farms that farm beetroot, yams and radish respectively. Determine the number of farmers that farm both beetroot, yams, and radish.
Solution:

The letters for denoting the sets have already been provided in the question itself (unlike the above example). We may therefore note the cardinality straight away:
$|U| = 350; |B| = 260; |Y| = 100; |R| = 70; |B \cap R| = 40$; $|Y \cap R| = 40$ and $|B \cap Y| = 30$ We need to determine the cardinality of the intersection of all three sets, which is $|B \cap Y \cap R|$. This is the unknown which we can assign determine algebraically.. Populate a Venn diagram with the given information. Use x to represent $|B \cap Y \cap R|$.
Let x farmers farm beetroot, yams, and radish. That is, let $|B \cap Y \cap R| = x$
Now solve for x algebraically:
$|U| = 350 = 190 + x + (30–x) + x + (40–x) + (40–x) + 30 + x + x–10$
350 = 320 + x
x = 30
Therefore, 30 farmers farm beetroot, yams, and radish.
COUNTING NUMBERS IN A LIST

How many numbers are in the list 2013, 2014, ..., 2497? Rule: In a list of consecutive numbers, a, a+1,...,b there are b-a+1 = b-(a-1) numbers.

4.1.3 Warm-up

1. There are 9 boys and 8 girls in a class. In how many ways can the teacher
   (a) pick a team of 3 students?
   (b) pick a team of 6 students?
   (c) pick a team of 6 students that is of one gender?
   d) pick a team of 3 boys and 3 girls?
   (e) choose a team captain and a vice-captain from the students?
   (f) choose a team captain (of any gender) and a vice-captain (must be a female) from the students?
   (g) line the students up in a line?

2. Alphabet things
   (a) How many ways can you rearrange the letters “AABC”?
   (b) How many ways can you rearrange the letters “AAABBCCC”?
   (c) Can you make a question so that the answer is $\frac{9!}{2!2!3!?}$
   (d) You have the letters “AAABBBCCC”. How many 3 letter words can you make using these letters?
   (e) You have the letters “AAABBBCCC”. How many 4 letter words can you make using these letters?

3. A child has 5 red marbles, 4 white marbles, and 5 green marbles. He wants to bring some of his marbles to school.
   (a) He places 3 marbles in a box, order matters, how many ways can he do this? (Bonus question for later, what if order does not matter?)
   (b) How many different combinations of marbles can he bring to school? (Any number)
   (c) How many combinations if he wants to bring a different number of each color?
   (d) He takes all his marbles and lines them up on the floor, how many patterns can he make?

4. x_1,x_2,x_3,... are all non-negative integers (i.e. from the set 0,1,2,...). How many solutions to the following?
   (a) x_1+x_2= 11
   (b) x_1+x_2= 2
   (c) x_1+x_2= n
   (d)x_1+x_2+x_3=83
   (e) x_1 and x_2 are positive integers, how many solutions to $x_1+x_2 = 10$
   (f) x_1, x_2 and x_3 are positive integers how many solutions to x_1+x_2+x_3=10
   (g) x_1, x_2, x_3 and x_4 are positive integers, how many solutions to x_1+x_2+x_3+x_4= 40
   (h)x_1,x_2 and x_3 are all integers larger than -3. How many solutions to the equation $x_1+x_2+x_3 = 20$?

5. There are 18 boys on a squad. The coach must pick a team of 11 to play on the field. 3 students- Jules, Isaac and Pacifique all refuse to play with each other. (i.e. if Jules is on the field, then Isaac andPacifique aren’t). In how many ways can the coach pick a team?

6. How many binary numbers exist that are made up of 5 digits? (for example 00111 and 10101)

7. I have 8 students, some will get pass and some will get fail. How many different results are possible? (First try: each student gets either a pass or a fail. Then try to count based on number of passes - i.e. case where there is 0 passes, plus when there is 1 pass plus...)

8. a)I have 10 identical soccer balls to give to three students. In how many ways can I do this?
   b)What if the soccer balls were distinct?

9. You have 100 marbles each of colors blue, green, red and yellow. You choose 10 marbles. In how many ways can you do this? (order of color /choosing does not matter)

10. A magic number is a number made up of 10 digits: 0,1,2,3,4,5,6,7,8 and 9. The n first digits makes a number which is divisible by n. For example 1264502783 has 1 which is divisible by 1, 12 which is divisible by 2, 126 which is divisible by 3, 1264 which is divisible by 4 and 12645 which is divisible by 5. But the condition is not satisfied for all remaining numbers as simply this number number is not divisible by 10. How many magic numbers can you make?

11. I have an 8×8 grid, how many paths from the bottom left corner to the top right corner? What about for a 7×6 grid?

12. A cow starting from point (0,0) jumps either to point (x+1,y+2) or to (x+2, y+1). How many paths are there from (0,0) to (15,15) using those steps given above in any order?

4.2 THE PRINCIPLE OF INDUCTION

The principe of induction: *Let a be an integer , and let p(n) be a statement (or proposition) about n for each $n \geq a$. The principle of induction is a way of proving that p(n) is true for all integers $n \geq a$.* It works in three steps:

(a) Base case: Prove that p(a) is true.

(b) Hypothesis step: Assume that p(k) is true for some integer k which is greater than a.

(c) nductive step: Use the fact that p(k) is true to prove that p(k+1) is also true. After confirming that for any p(k) which is true leads to p(k+1) which is true, we can conclude that p(n) is true for all integers $n \geq a$.

This principle of induction is very useful in problem solving, especially when we observe a pattern and want to prove it. The trick to using the principle of induction properly is to spot how to use p(k) to prove p(k+1). Sometimes this must be done rather ingeniously.

Example: Prove that: $1 + 2 + 3 + 4 + 5 + ... + n = \frac{n(n+1)}{2}$ (The sum of n first natural numbers)

PROOF

Step 1: Base case : For n=1;
LHS=1
RHS=$\frac{1(1+1)}{2} = 1$ It is true for n=1 since LHS=RHS=1
Step 2: Hypothesis step : Assume that it is true for some positive integer k, so that p(k)=1+2+3+...+k=$\frac{k(k+1)}{2}$ then we have to prove that p(k+1) is also true.
Step 3: Inductive step
p(k+1)=1+2+3+...+k+(k+1)=p(k)+(k+1)=$\frac{k(k+1)}{2} + (k + 1) = (K + 1)(\frac{k}{2} + 1) = \frac{(k+1)(k+2)}{2}$
Hence p(n) is true for all natural numbers n by the principle of induction.

Example 3: Find the sum of n first odd numbers.
Solution:
We are asked to find the series 1+3+5+7+9+...+(2n-1). *Note that this time we are not told the formula that we have to prove ; We have to find it ourselves ! Let's try some small numbers and see if a pattern emerges.*

1=1; 1+3=4; 1+3+5=9; 1+3+5+7=16; 1+3+5+7+9=25,... We conjecture(guess) that the sum of the first n odd numbers is equal to n^2.
Now , let's prove our statement using induction principle.
p(n)=1+3+5+...+(2n-1)=n^2

Step1: Base case: For n=1, p(1)=1=1^2
Step 2: Hypothesis step: Assume that for some positive integer k, p(k)=1+3+5+...+(2k-1)=k^2 , the we have to prove that p(k+1) is also true.
Step 3: Inductive step: p(k+1)=1+3+5+...+(2k-1)+(2k+1)=k2+2k+1=$(k + 1)^2$

Therefore,by the principle of induction, p(n)=1+3+5+...+(2n-1)=n^2 for all odd n.
Example 3:
Show that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all positive integers n.
SOLUTION
Step 1: Base case: For n=1, p(1)=$2^{1+2} + 3^{2+1} = 35$ Which is divisible by 7.
Step 2: Hypothesis case: Assume p(k) is true for some positive integer k such that$7|2^{k+2} + 3^{2k+1}$ then , let's prove if p(k+1) is also true.
Step 3: Inductive step:

p(k+1)=$2^{(k+1)+2}+3^{2(k+1)+1}$
=$2.2^{k+2}+3^2.3^{2k+1}$
=$2[2^{k+2}+3^{2k+1}]+7.3^{2k+1}$
=$2p(k)+7.3^{2k+1}$
From our assumption in the hypothesis step, we have assumed that $7|p(k)$ therefore, p(k)=7m, m being an integer. It implies that 2p(k)=14m then,
p(k+1)=14m + $7.3^{2k+1}=7(2m+3^{2k+1})$ which is obviously divisible by 7, hence by induction,
We have proved that for all integer n ,$7|2^{n+2}+3^{2n+1}$

Notice: $a|b$ means a divides b or b is divisible by a.

4.2.1 Warm-up

1. Show that :$1^2+2^2+3^2+4^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ $\forall n \in \mathbb{N}$.
2. Prove that 4^n-1 is divisible by 3 $\forall n \in \mathbb{Z}^+$.
3. 1+5+9+...+(4n − 3) = n(2n–1) for all natural numbers n.
4. Show that $\forall n \in \mathbb{N}$:2+4+6+ ... +2n = n^2+n.
5. Verify if $2n<(n+2)!\forall n \in \mathbb{N}$.
6. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}>\frac{13}{24}$ $\forall n \in \mathbb{N}$ and $n>1$.
7. Prove that$\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}}$ for all natural number $n \geq 2$.
8. Show that $\forall n \in \mathbb{N}$: $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number .
9. Prove that $(1-\frac{1}{2^2})\times(1-\frac{1}{3^2})\times(1-\frac{1}{4^2})\times...\times(1-\frac{1}{n^2})=\frac{n+1}{2n}$ for all natural numbers $n \geq 2$.
10. Using induction principle, prove the statement below.
    Let x be a positive real number. Then for any natural number n, we have $(1+x)^n \geq 1+nx$.

4.3 SEQUENCES AND SERIES

4.3.1 Sequences

Definitions

Sequences are ordered lists of numbers (called "terms"), like 2,5,8. Some sequences follow a specific pattern that can be used to extend them indefinitely. For example, 2,5,8 follows the pattern "add 3," and now we can continue the sequence. Sequences can have formulas that tell us how to find any term in the sequence.

MAIN TYPES OF SEQUENCES

- Arithmetic Sequences.
- Geometric Sequence.
- Fibonacci Sequence

ARITHMETIC SEQUENCE/ ARITHMETIC PROGRESSION

Arithmetic Progression (AP) is a sequence of numbers in order, in which the difference between any two consecutive numbers is a constant value. It is also called **Arithmetic Sequence**. For example, the series of natural numbers: 1,2,3,4,5,6,... is an Arithmetic Progression, which has a common difference between two successive terms (say 1 and 2) equal to 1 (2 -1). Even in the case of odd numbers and even numbers, we can see the common difference between two successive terms will be equal to 2.

Finding of the n^{th} term in the AP: Let a_n be the n^{th} term of the Arithmetic Progression.
We have:
$a_1 = a$
$a_2 = a + d$
$a_3 = a + 2d$
.
.
.
$a_n = a + (n-1)d$
[Try to prove it using induction though it is obvious]

4.3.2 Check understanding

1. What is the 20^{th} term in the following sequence: 1,9,17,25,...
2. The first term an AP is 23 and the 16^{th} term of that AP is 890. What is the common difference of that AP.
3. Derive the formula for the sum of n consecutive terms of an AP whose first term is a and common difference is d.

GEOMETRIC PROGRESSION

In Maths, Geometric Progression (GP) is a type of sequence where each succeeding term is produced by multiplying each preceding term by a fixed number, which is called a common ratio. This progression is also known as a geometric sequence of numbers that follow a pattern.For example, 2, 4, 8, 16, 32, 64, ... is a GP, where the common ratio is 2.
Mathematically, a GP is the one whose terms are in this order: a, ar, $ar^2, ar^3, ..., ar^{n-1}$

CHECK UNDERSTANDING

(a) Give two examples of geometric progressions.

(b) Derive the formula for the sum of n terms of the geometric progression whose first term is a and the common ratio is r.

(c) Find the area of the shaded part, knowing that the length of the larger square is 1 unit length. (Not drawn to scale)

4.3.3 Fibonacci sequence

Read carefully this story: ‘’ Fibonacci rabbit never die and can give birth without being married” *Fibonacci had a magic rabbit species. He started with one such a rabbit. This rabbit gives birth to one young-one for the first time two months later after it was born. After the 2 months, the rabbit gives birth to one young-one each month until forever because this rabbit never die. The rabbit doesn’t need to be a male or a female in order to give birth the young-one. The young-ones also behave like their parent rabbit.*

CHECK UNDERSTANDING

(a) Demonstrate mathematically the information narrated in the story using your own approach.

(b) Starting at the birth of the first rabbit of fibonacci, how many rabbits will fibonacci have after 6 months?

In mathematics, the Fibonacci sequence is a sequence in which each number is the sum of the two preceding ones. Numbers that are part of the Fibonacci sequence are known as Fibonacci numbers, commonly denoted F_n. The sequence commonly starts from 0 and 1, although some authors start the sequence from 1 and 1 or sometimes (as did Fibonacci) from 1 and 2. Starting from 0 and 1, the first few values in the sequence are:0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.

Warm-up

(a) The first, the second and the fourth terms of an AP are three first terms of a certain GP. If the sum of the first 5 terms of that GP is 31, what is the sum of the first 7 terms of that AP?

(b) Three lengths of sides of a triangle forms an AP and the area of that triangle is 6. What are the lengths of each side of that triangle? What type is that triangle ?

(c) (IrMO 2021) A sequence whose first term is positive has the property that any given term is the area of an equilateral triangle whose perimeter is the preceding term. If the first three terms form an arithmetic progression, determine all possible values of the first term.

4.4 PIGEON HOLE PRINCIPLE

4.4.1 Definition

If $n + 1$ objects are placed into n boxes, then some box contains at least 2 objects.
Proof: Suppose that each box contains at most one object. Then there must be at most n objects in all. But this is false, since there are n + 1 objects. Thus some box must contain at least 2 objects. This combinatorial principle was first used explicitly by Dirichlet (1805-1859). Even though it is extremely simple, it can be used in many situations, and often in unexpected situations. Note that the principle asserts the existence of a box with more than one object, but does not tell us anything about which box this might be. In problem-solving, the difficulty of applying the pigeonhole principle consists in figuring out which are the ‘objects’ and which are the ‘boxes’.

4.4.2 Examples

1. **Prove that in a group of three people, there must be two of the same sex.**

   Solution: There are only n = 2 sexes, but we have n + 1 = 3 people. Here the sexes are the 'boxes', and the people are the 'objects'. There is no guarantee that two people will both be of the same sex but if we bring the third person, his/her sex will be the same as one of the two people according to the pigeonhole principle.

2. **Prove that among 13 people, there are at least two who were born in the same month.**
   Solution: There are n = 12 months ('boxes'), but we have n+1 = 13 people ('objects'). Therefore two or more people were born in the same month.

There are n people present in a room. Where $n \leq 2$. Prove that among them there are two people who have the same number of acquaintances in the room.
Solution: Suppose that there is at most one person (object) in each room (box).

Case 1: When box 0 contains a person.
This would imply that a person from the remaining n-1 people will belong to one of the boxes labeled from 1 to n-2. Since there are n-1 people who will share n-2 boxes, then by pigeon hole principle there will be at least one box that contains at least 2 objects(people), which contradicts our initial assumption.
Case 2: When box 0 is not occupied.
This would imply that a person from the remaining n people will belong to one of the boxes labeled from 1 to n-1. Then by pigeon hole principle, there will be at least one box that contains at least 2 objects(people), which contradicts our initial assumption again.

Therefore, those two cases are sufficient to prove that among n people in the room, there must be two people who have the same number of acquaintances in that room.

4.4.3 Exercises

1. Nobody has more than 300,000 hairs on his head. The capital of Kimisagara has 300,001 inhabitants. Can you assert with certainity that there are two persons with the same number of hairs on their heads?

2. Show that given a regular hexagon of side 2 cm and 25 points inside it, there are at least two points among them which are at most 1 cm distance apart [7]

3. Show that if 7 points are chosen on the circumference or in the interior of a unit circle, such that their mutual distance apart is greater than or equal to 1, then one of them must be the centre

4. Show that in a party there are always two persons who have shaken hands with the same number of persons.

5. Prove that, among any 52 integers, two can always be found, such that the difference of their squares, is divisible by 100.

6. Show that, for any set of 10 points, chosen within a square, whose side is 3 units, there are two points, in the set, whose distance is at most 2 .

7. There are 7 persons in a group, show that, some two of them, have the same number of acquaintances among them.

8. Six points are given inside an equilateral triangle of area 4. Prove that among the nine points which include the three vertices of the triangle and the six given points, three of these form a triangle of area at most 1.

9. In a tournament with n players, everyone plays with everybody else exactly once. Prove that during the game there are always two players who have played the same number of games.

4.5 GAME THEORY

4.5.1 WINNING STRATEGY

We consider games for two players A and B, who move alternately. A always moves first but otherwise the rules are the same for A and B. A draw cannot occur. We are given the starting state and the set M of legal moves. A player loses if he finds himself in a position from which no legal move can be made. We can think of each position as a vertex of a graph and each move as a directed edge. We consider games with finitely many vertices and no directed circuit (a position can not repeat). This ensures that one of the players will lose. The set P of all positions can be partitioned into the set L of losing positions and the set W of winning positions: $P = L \bigcup W$, $L \bigcap W = 0$. A player finding himself in position L will lose provided his opponent plays correctly. A player finding himself in a position W can force a win whatever his opponent does.

To win, a player must always move so as to force his opponent into a position belonging to L. From each position in L, every move must result in a position in W. From every position in W, a move to a position in L must be possible. L must contain at least one final position f from which there is no move out. The player who leaves his opponent facing such a position has won the game. The problem is to identify the set L of losing positions.

Most of the following problems can be solved by a simple strategy: Divide the set of all positions into pairs, so that there is a move from the first to the second element of the pair. Whenever my opponent occupies one element of a pair, I move to the other element of the pair. Thus, I win, since my opponent runs out of moves first. Initially, if there is one position without a pair, I should occupy it. Otherwise, I should be the second player to win. In more complicated games, a table of losing positions should be used in playing. As a warmup, we will consider some examples with solutions. [8]

4.5.2 Examples

1. **Hirwa and Arnold are playing a game where one has to remove either one, two, or three pens from a box which initially contains 11 pens. The winner is the one who will remove the last pen. If Hirwa plays first, who has the winning strategy?**

   Solution: We want to see the one who will remove the 11^{th} pen no matter what. This person will have to remove the 7^{th} pen because his opponent won't remove the 11^{th} pen when there are 4 pens in the box, and after the opponent's turn, the oppenent will always leave a remainder of 1, 2, or 3 pens in the box, hence giving a winning position to his fellow. Similarly, for the winner to remove the 7^{th} pen, he will have to remove the 3^{rd} pen. This means that the one who will start will have a winning strategy since he has the potential to remove the 3^{rd} pen out of the box no matter what. Therefore, Hirwa has a winning strategy.

2. **Initially there are n checkers on the table. The set of legal moves is the set M = (1, 2, 3,..., k). The winner is the one to take the last checker. Find the losing positions.**

   The set L consists of all multiples of k + 1. Indeed, if n is not a multiple of k+1, then I can always move to a multiple of k + 1. My opponent cannot move to the next multiple of k + 1 since he can only subtract k or less checkers. So he has to move to some number, which is not a multiple of k +1. Then I simply move into a winning position everytime. Thus, I will finally reach 0, which is also a multiple of k +1.

3. **A and B are going to play a game by turns. Before they start, they form a circle with 2001 other persons. At every turn they can remove one of their neighbors from the circle. The winner is the one who gets the other person out of the circle. If A starts, decide who has a winning strategy. Note: The other 2001 persons do not have turns.**

Solution: When the game starts there are 2001 other persons. This means that A and B divide the circle in two arcs, one of which has an odd number of persons and the other an even number. The strategy for A is to remove always a person on the even side. This leaves B with an odd number of persons on each side. When B plays, A has again a side with an odd number of persons and an even number on the other, so he can continue with his strategy. B can never hope to win if A plays this way, since he always has at least one person between himself and A on both sides. Since the game must end after at most 2001 turns, A wins.

In problems involving games, finding a winning strategy means finding a way to play so that, regardless of how the other person plays, one is going to win. The key to solve this kind of problems is to find an invariant in the game and exploit it. The invariant has to be a certain state of the game. We are looking for a state with the following properties:
• A person in that state cannot win the game.
• If the other person played in that state, we can force him back to that state. This kind of state is called a losing position. The positions that can send the other player to a losing position are called winning positions. In the previous example the losing position was having an odd number of persons on each side. In this type of problems, to find the losing positions it is convenient to look at the positions near the end of the game when one cannot win and work backwards in the game. It is also a good strategy to try a few games to look for the invariant.

4.5.3 Exercises

1. Amy and Ben play the game of mis'ere noughts and crosses on a 3×3 square array. On Amy's turn, she can place an X in any vacant square, while on Ben's turn, he can place an O in any vacant square. The players take turns to place their symbol, with Amy going first. Any player who gets three in a row (horizontally, vertically or diagonally) immediately loses the game. The game is considered drawn if there is no winner after all squares have been filled. Which player, if any, has a winning strategy?

2. There are two piles of checkers on a table. A takes any number of checkers from one pile or the same number of checkers from each pile. Then B does the same. The winner is the one to take the last chip. Find the losing positions(L).

3. Start with n=2. Two players A and B move alternatetly by adding a proper divisor of n to the current n. The goal is a number ≥ 1990. Who wins?

4. A and B alternately put white and black knights on the squares of a chessboard, which are unoccupied. In addition a knight may not be placed on a square threatened by an enemy knight (of the other color). The loser is the one who cannot move any more. Who wins?

5. A and B alternately draw diagonals of a reqular 1988-gon.They may connect two vertices if the diagonal does not intersect an earlier one. The loser is the one who cannot move. Who wins?

6. At the start of a game, the numbers 1 and 2 are each written 10 times on a blackboard. Two players take turns to erase two of the numbers, replacing them with a 1 if they are different and with a 2 if they are the same. The first player wins if the last number on the board is 1, while the second player wins if it is 2. Which player has a winning strategy?

7. Two players start with the number 1 and take turns to multiply it by an integer from 2 to 9. The winner is the first player to obtain a number greater than or equal to 1000. Which player has a winning strategy?

8. Two people play a game involving n coins on a table. The first player takes at least one, but not all, of the coins. The players then take turns to take at least one coin, but no more than was taken on the previous move. The player who takes the last coin is considered the winner. For which values of n does the second player have a winning strategy?

9. Consider a chocolate bar in the shape of an equilateral triangle, with sides of length n, divided by grid lines into equilateral triangles of side length 1. Two players take turns to break off a triangular piece along one of the grid lines and pass the remaining block of chocolate to the other player. A player who is unable to move or who leaves an equilateral triangle of side length 1 is declared the loser. For which values of n does the second player have a winning strategy?

4.6 PROBLEMS

1. If a team won 13 games and lost 7 games, its winning percentage was $\frac{13}{13+7} \times 100$

   (a) The Sharks played 10 games and won 8 of these. Then they played 5 more games and won 1 of these. What was their final winning percentage? Show the steps that you took to find your answer.

   (A) 50 (B) 55 (C) 60 (D) 65 (E) 70

   (b) The Emus won 4 of their first 10 games. The team played x more games and won all of these. Their final winning percentage was 70How many games did they play in total? Show the steps that you took to find your answer.

   (A) 11 (B) 15 (C) 13 (D) 20 (E) 18

2. A hat contains six slips of paper numbered from 1 to 6. Amelie and Bob each choose three slips from the hat without replacing any of the slips. Each of them adds up the numbers on his slips.

   (a) Determine the largest possible difference between Amelie's total and Bob's total.

   (A) 8 (B) 9 (C) 7 (D) 6 (E) 10

   (d) If more slips of paper are added to the hat, numbered consecutively from 7 to n, what is the smallest value of n ¿ 6 so that Amelie and Bob can each choose half of the slips numbered from 1 to n and obtain the same total?

   (A) 8 (B) 9 (C) 7 (D) 6 (E) 10

3. Franco and Sarah play a game four times using the following rules:
   (R1) The game starts with two jars, each of which might contain some beans.

   (R2) Franco goes first, Sarah goes second and they continue to alternate turns.

   (R3) On each turn, the player removes a pre-determined number of beans from one of the jars. If neither jar has enough beans in it, the player cannot take their turn and loses. If only one jar has enough beans in it, the player must remove beans from that jar. If both jars have enough beans, the player chooses one of the jars and removes the beans from that jar.

   (R4) Franco must attempt to remove 1 bean on his first turn, 3 beans on his second turn, and 4 beans on his third turn. On each of his following sets of three turns, Franco must continue to attempt to remove 1, 3 and 4 beans in sequence.

(R5) Sarah must attempt to remove 2 beans on her first turn and 5 beans on her second turn. On each of her following sets of two turns, Sarah must continue to attempt to remove 2 and 5 beans in sequence.
(R6) A player is declared the winner if the other player loses, as described in (R3).

(a) At the beginning of the first game, there are 40 beans in one jar and 0 beans in the other jar. After a total of 10 turns (5 turns for each of Franco and Sarah), what is the total number of beans left in the two jars?

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

(b) At the beginning of the second game, there are 384 beans in one jar and 0 beans in the other jar. The game ends with a winner after a total of exactly n turns. What is the value of n?

(A) 110 (B) 114 (C) 118 (D) 120 (E) 124

(c) At the beginning of the third game, there are 17 beans in one jar and 6 beans in the other jar. There is a winning strategy that one player can follow to guarantee that they are the winner. Determine which player has a winning strategy. (A winning strategy is a way for a player to choose a jar on each turn so that they win no matter the choices of the other player.)

(A) Franco (B) Sarah (C) None of them (D) All of them (E) The question is impossible

(d) At the beginning of the fourth game, there are 2023 beans in one jar and 2022 beans in the other jar. Determine which player has a winning strategy.

(A) Franco (B) Sarah (C) None of them (D) All of them (E) The question is impossible

4. Al and Bert must arrive at a town 22.5 km away. They have one bicycle between them and must arrive at the same time. Bert sets out riding at 8 km/h, leaves the bicycle and then walks at 5 km/h. Al walks at 4 km/h, reaches the bicycle and rides at 10 km/h.
For how many minutes was the bicycle not in motion?

(A) 60 (B) 75 (C) 84 (D) 94 (E) 109

5. A deck of 100 cards is numbered from 1 to 100. Each card has the same number printed on both sides. One side of each card is red and the other side is yellow. Barsby places all the cards, red side up, on a table. He first turns over every card that has a number divisible by 2. He then examines all the cards, and turns over every card that has a number divisible by 3. How many cards have the red side up when Barsby is finished?

(A) 83 (B) 17 (C) 66 (D) 50 (E) 49

6. There are 90 cards numbered 10 to 99. A card is drawn and the sum of the digits of the number in the card is noted; if 35 cards are drawn, then, there are at least how many cards whose sum of the digits are identical?

(A) 7 (B) 4 (C) 3 (D) 5 (E) 9

7. Given three points, in the interior of a right angled triangle, At least how many of them are at a distance not greater than the maximum of the lengths of the sides containing the right angle?

   (A) 6 (B) 4 (C) 3 (D) 5 (E) 2

8. Let A be the set of 19 distinct integers, chosen from the Arithmetic Progression 1, 4, 7, 10, ..., 100. There should be at least how many distinct integers in A, such that, their sum is 104?

   (A) 6 (B) 4 (C) 3 (D) 5 (E) 2

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Part II

RWANDA MATH OLYMPIAD GLOSSARY

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Rwanda Math Olympiad Glossary

May 2022

Preface

Glossary is an alphabetical list of words relating to specific dialect with explanations. Rwanda Math Olympiad (RwMO) Glossary seeks to explain to Rwandan students unfamiliar words and symbols that are found in math olympiad competitions to boost their interest in mathematics and problem solving excellence.

Ijambo ry' ibanze

Inkoranyamagambo ni urutonde rw' amagambo ajyanye n' imvugo yihariye hamwe n' ibisobanuro. Iyi nkoranya magambo igamije gusobanurira abanyeshuri bo mu Rwanda amagambo atamenyerewe dusanga mu marushanwa ya olipiyadi (olympiad) y' imibare, hagamijwe kubongerera ubumenyi bujyanye n' imibare no gukora neza ibibazo babajijwe.

Notice: blue-colored words in english definitions section are explained in this glossary.

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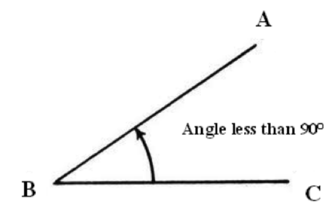
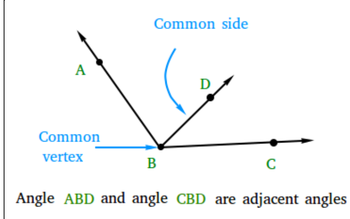
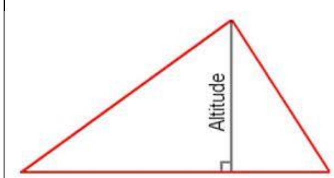
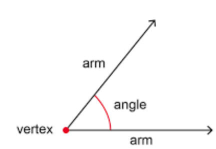
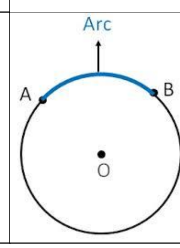
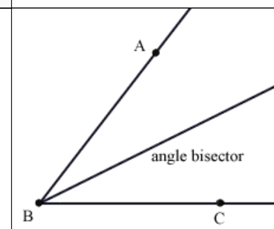
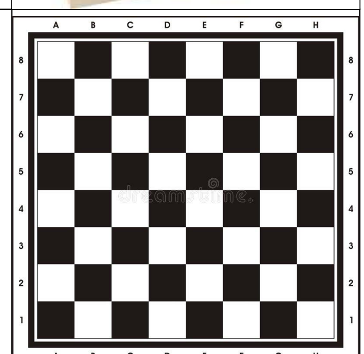
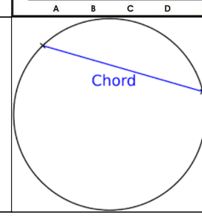
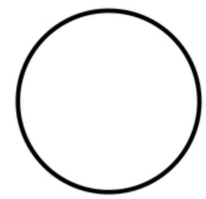
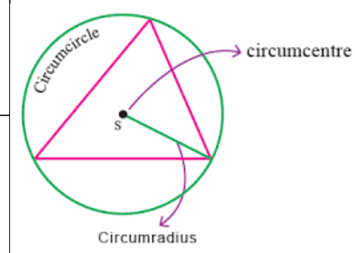
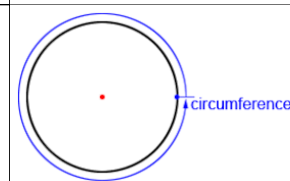
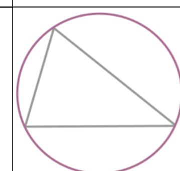
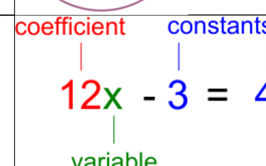
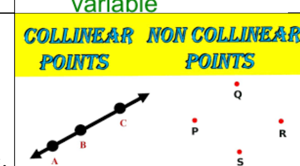
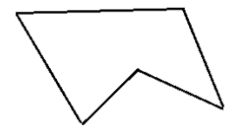
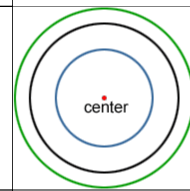
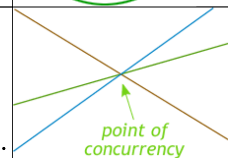
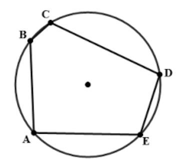
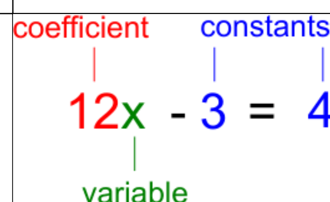
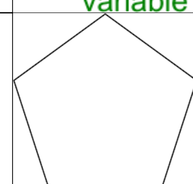
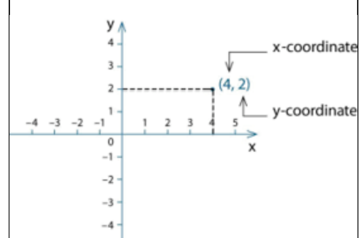
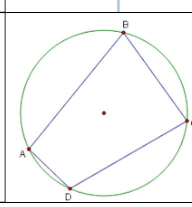
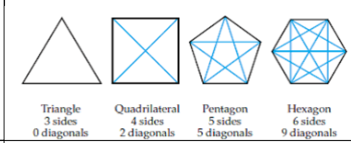
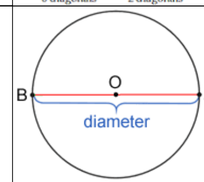
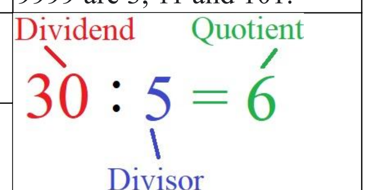
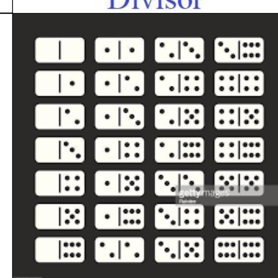
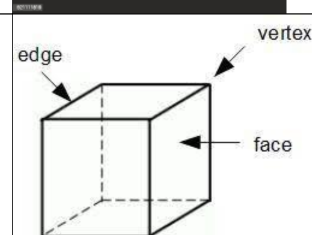
Prepared by Arnold Hategekimana Hirwa

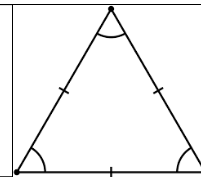
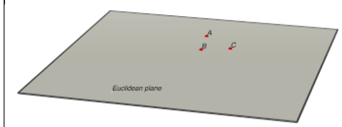
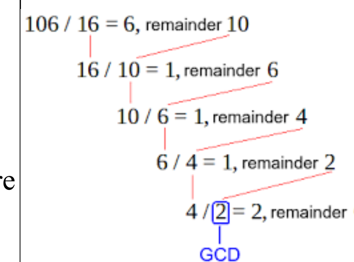
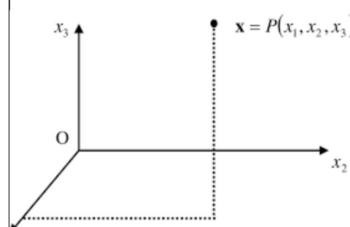
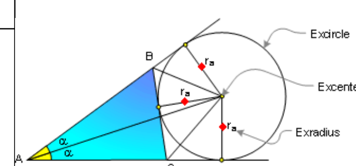
MATHEMATICAL TERMS	ENGLISH DEFINITIONS	KINYARWANDA DEFINITIONS/ TERMS	IMAGE/EXAMPLE
absolute value	the magnitude of a real number without regard to its sign. The absolute value of a number may also be thought as its distance from zero along real number line.	ubunini bwumubare nyawo utitaye ku kimenyetso cyawo.	$\lvert -3 \rvert = 3$
acute-angle	less than 90°	imfuruka iri hagati ya dogere 0 na dogere 90	
adjacent angles	two angles that have a common side and a common vertex.	Inguni ebyiri zihujwe n' uruhande rumwe.	
Altitude	perpendicular line from a vertex to the opposite side of a figure.	uburebure bwumurongo wa perpendicular kuva kuri vertex kugera kuruhande rwigishushanyo.	
angle	the space (usually measured in degrees) between two intersecting lines or surfaces at or close to the point where they meet.	imfuruka	
arc	a part of a curve, especially a part of the circumference of a circle.	igice cy' umurongo, cyane cyane igice cyumuzingi.	

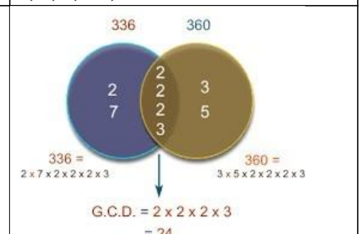
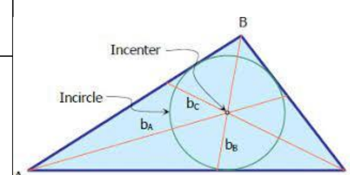
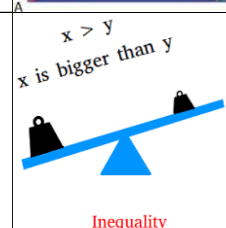
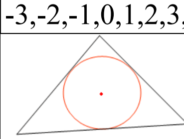
arithmetic progression	a sequence of numbers in which each differs from the preceding one by a constant quantity.	urukurikirane rw'imibare aho buri umwe utandukanywa n' undi hifashishijwe umubare udahinduka.	-6,-4,-2,0,2,4,6,...
arithmetic(mean)	the average of a set of numerical values, as calculated by adding them together and dividing by the number of terms in the set.	impuzandengo y' uruhererekane rw' imibare, ubarwa nyuma yo guteranya imibare yose hanyuma ukagabanya igiteranyo cyayo n' ingano y' iyo mibare.	$\text{arithmetic mean} = \frac{\sum_{n=1}^{k} x_n}{k}$
bisector	The line that divides something into two equal parts.	Umurongo ugabanya ikintu mo ibice bibiri bingana.	
chess	a board game of strategic skill for two players, played on a chessboard on which each playing piece is moved according to precise rules. The object is to put the opponent's king under a direct attack from which escape is impossible (checkmate).	umukino wibibaho byubuhanga bukomeye ku bakinnyi babiri, ukinirwa ku kibaho cyagenzuwe kuri buri gice cyo gukinisha. Intego ni ugushyira umwami wuwo bahanganye mu bitero bitaziguye aho guhunga bidashoboka (cheque).	
chessboard	a square board divided into sixty-four alternating dark and light squares (conventionally called 'black' and 'white'), used for playing chess or draughts (checkers).	ikibaho cya kare kigabanyijemo ibice mirongo itandatu na bine bisimburana byijimye kandi byerurutse (bisanzwe byitwa 'umukara n' umweru'), bikoreshwa mu gukina chess.	
chord	a straight line joining the ends of an arc.	umurongo ugororotse ukora ku mpande zombi z' uruziga.	

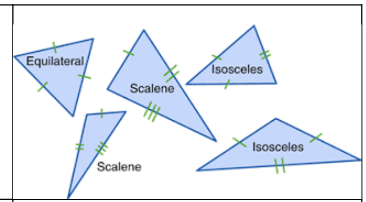
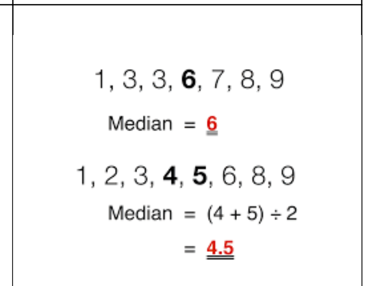
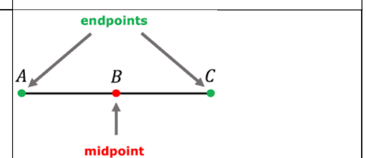
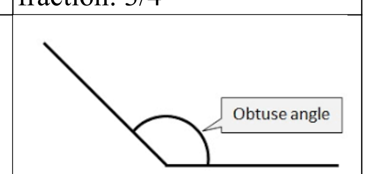
circle	a round plane figure whose boundary (the circumference) consists of points equidistant from a fixed point (the centre).	uruziga	
circumcentre	centre of circumcircle		
circumcircle	a circle touching all the vertices of a triangle or polygon.	uruziga rukora kuri buri nguni y' igishushanyo rurimo.	
circumference	the distance around a circle.	intera ikikije uruziga.	
circumscribe	round another, touching it at points but not cutting it.	kuzenguruka ikindi kintu ugikoraho ariko utakirenga.	
coefficient	a numerical or constant quantity placed before and multiplying the variable in an algebraic expression.	umubare ukuba ikintu gihinduka.	
collinear	lying in the same straight line.	ibintu biri ku murongo umwe.	
complement (set)	the members of a set or class that are not members of a given subset.	ibigize itsinda rinini bitari mu rindi tsinda rito runaka.	Set U = {2,4,6,8,10,12} and set A = {4,6,8}, then the complement of set A, A′ = {2,10,12}
complement (geometry)	the amount in degrees by which a given angle is less than 90°.	ingano muri dogere ibura kugira ngo imfuruka yatanzwe igere kuri 90°.	complement of 50° is 40°
composite number	a number that is a multiple of at least two numbers other than itself and 1.	umubare utarimo ibice ushobora kugabanywa n' imibare ibiri itandukanye.	4,8,9,12,25,...

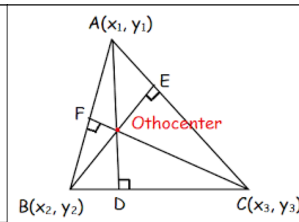
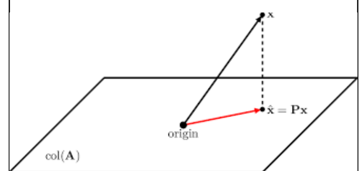
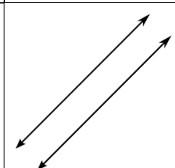
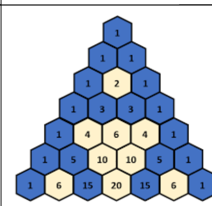
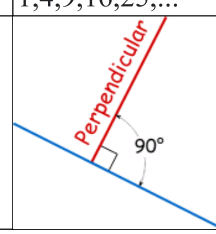
concave polygon	internal angle can be more than 180°.	inguni y' imbere ishobora kuruta dogere 180°.	
concentric circles	circles with a common centre.	inziga zihuje akadomo k' impuzandengo.	
concurrent (of three or more lines)	meeting at or tending towards one point.	guhura kw' imirongo myinshi.	
concyclic points	points that lay on the circumference of a circle.	utudomo turi ku muzingi.	
configuration	an arrangement of parts or elements in a particular form, figure, or combination.	gutondekanya ibintu mu buryo runaka.	
constant	a quantity or parameter that does not change its value whatever the value of the variables, under a given set of conditions.	umubare cyangwa ikintu kidahinduka.	
convex polygon	all interior angles are less than or equal to 180 degrees	inguni z' imbere zose ziri munsi ya dogere 180°	
co-ordinate	each of a group of numbers used to indicate the position of a point, line, or plane.	itsinda ry' imibare rigaragaragaza akadomo, umurongo, cyangwa ikindi kintu.	
cyclic polygon	a polygon having all its vertices lying on a circle.	aho imfuruka z' igishushanyo ziremerwa haba hakora ku ruziga.	

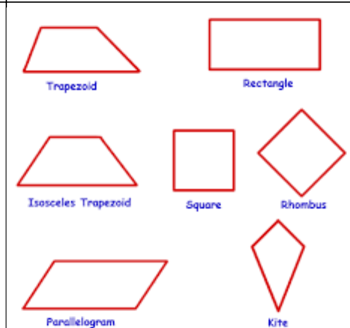
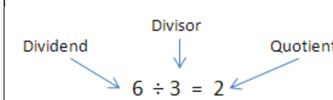
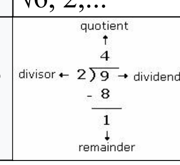
denominator	the number below the line in a vulgar fraction; a divisor.	icyitarusange	2 is denominator in this fraction: 1/2
diagonal	a straight line joining two opposite corners of a square, rectangle, or other straight-sided shape.	umurongo uhuza inguni ebyiri z' ikinyampande runaka.	
diameter	a straight line passing from side to side through the centre of a body or figure, especially a circle or sphere.	umurongo ugabanyamo uruziga cyangwa ikindi gishushanyo mo ibice bibiri bingana.	
digit	any of the numerals from 0 to 9, especially when forming a part of number.	umwe mu mibare uhereye kuri 0 ukageza ku 9.	0,1,2,3,4,5,6,7,8,9
distinct prime numbers	prime numbers without any repeats.		distinct prime factors of 9999 are 3, 11 and 101.
dividend	a number that is being divided.	umubare wagabanyishijwe undi.	
divisor	a number that divides another without returning a remainder.	umubare ugabanya undi ntihagire igisaguka.	
domino	any of 28 small oblong pieces marked with 0–6 pips in each half.	kamwe mu mu duce 28 twerekanwe haruguru.	
edge (geometry)	line segment joining two vertices.	igice cy' umurongo gihuza utudomo tubiri.	
equation	a statement that shows that the values of two mathematical expressions are equal.	inyandiko yerekana ko ibintu bibiri bingana.	2x-2=0

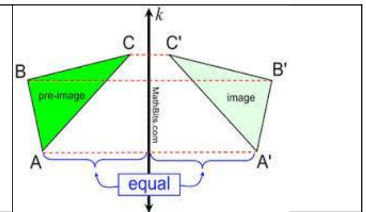
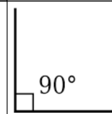
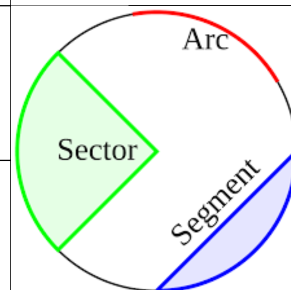
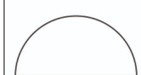
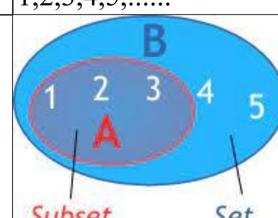
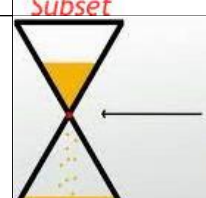
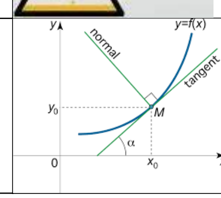
equilateral	having all its sides of the same length.	impande zose ziba zingana.	
euclidean plane	two dimensional part of the euclidean space.		
euclidean algorithm	an efficient method for computing the greatest common divisor (GCD) of two integers (numbers).	uburyo bwo gushaka umubare ugabanya imibare ibiri runaka.	
euclidean space	a space in any finite number of dimensions, in which points are designated by coordinates (one for each dimension) and the distance between two points is given by a distance formula.		
even number	a number that returns zero as a remainder when divided by two.	umubare w'imbangikane.	2:2=1; 4:2=2; 12:2=6,... bivuzeko 2,4,... hakubiyemo na 0 ni imibare y'imbangikane.
excentre	centre of excircle		
excircle	a circle tangent to the extensions of two sides of a triangle and the third side.	uruziga rukora ku kwaguka kw' impande ebyiri za mpandeshatu no ku ruhande rwa gatatu.	
factor	a number or algebraic expression that divides another number or expression evenly—i.e. with no remainder.	umubare ugabanya undi ntihagire igisaguka.	3 and 6 are factors of 12.

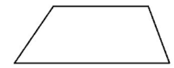
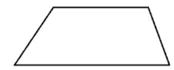
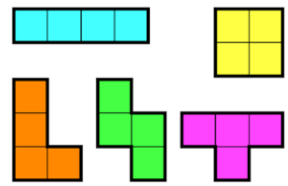
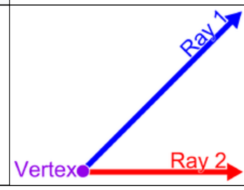
finite set	a set that has a finite number of elements.	itsinda rifite ingano.	For example, {1,3,5,7} is a finite set with four elements.
function	a relation or expression involving one or more variables.	isano iba irimo ikintu kimwe cyangwa byinshi bihinduka.	$y = x + 3$
geometric mean	the central number in a geometric progression.		for a given set of two numbers such as 8 and 1, the geometric mean is equal to $\sqrt{(8\times1)} = \sqrt{8} = 2\sqrt{2}$
geometric progression	a sequence of non-zero numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio.	urukurikirane rw' imibare itari zeru, aho umubare uba wikubyi inshuro zidahinduka runaka uwurinyuma.	2,4,8,16,...
greatest common divisor	the largest integer or the polynomial of highest degree that is an exact divisor of each of two or more integers or polynomials.	umubare cyangwa ikintu kinini gishobora kugabanya ibintu bibiri.	
incentre	the intersection of the three interior angle bisectors of a triangle.		
incircle	an inscribed circle of a polygon that is tangent to each of the polygon's side.		
inequality	difference in size, degree, circumstances, etc.	itandukaniro riri hagati y' ibintu runaka.	
infinite sequence	an infinite ordered set of numerical quantities.	urutonde rutarangira.	-3,-2,-1,0,1,2,3,4,5,...
inscribe	draw (a figure) within another so that their boundaries touch but do not intersect.	gushushanya mw' imbere y' ikindi gishushanyo, ugikoraho ariko utakirenga.	
integer	a number which is not a fraction; a whole number.	umubare utarimo ibice.	...,-2,-1,0,1,2,...

isosceles	having two sides of equal length.	hari impande ebyiri zingana muri iyo mpandeshatu.	
linear factors	The linear factors of a polynomial are the first-degree equations that are the building blocks of more complex and higher-order polynomials.		
median	denoting or relating to a value or quantity lying at the midpoint of a frequency distribution of observed values or quantities, such that there is an equal probability of falling above or below it.	icyakabiri cy' umubare cyangwa ikintu runaka.	
mid-point	a point in the middle of something.	ni muri kimwe cya kabiri.	
multiple	a product that we get when one number is multiplied by another number; a number that may be divided by another a certain number of times without a remainder.	igisubizo tubona iyo umubare umwe wikubye undi.	if we say 4 × 5 = 20, here 20 is a multiple of 4 and 5
natural numbers	the positive [integers]().	imibare yose itarimo ibice uhereye kuri rimwe ukazamuka.	1,2,3,4,5,...
numerator	the number above the line in a vulgar fraction.		3 is numerator in this fraction: 3/4
obtuse angle	[angle]() greater than 90°	inguni iruta dogere 90°	
odd	having one left over as a remainder when divided by two.	Umubare w'igiharwe.	5 ugabanyije 2 = 2.5 bivuze ko 5 ari umubare w' igiharwe.

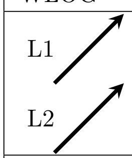
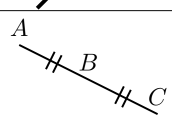
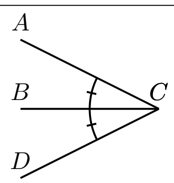
orthocenter of a triangle	the common intersection of the three altitudes of a triangle.		
orthogonal projection	A projection of a figure onto a line or plane so that each element of the figure is mapped onto the closest point on the line or plane.		
pairwise	refers to all subsets of a given set that contain exactly two elements.	bivuze gukora amatsinda mato yose agizwe n' imibare cyangwa ibintu bibiri by' itsinda rinini runaka.	the set {1,2,3} all possible pairs are (1,2),(2,3),(1,3).
pairwise disjoint	If the intersection of two events is the empty set, then the events are called pairwise disjoint events.	n' igihe amatsinda abiri aba adafite aho ahuriye.	{1, 2, 3} and {4,5,6} are pairwise disjoint sets.
parallel	(of lines, planes, or surfaces) side by side and having the same distance continuously between them.	imirongo cyangwa ibintu bibiri runaka biba bifite intera imwe hagati yabyo kuburyo bidashobora guhura.	
pascal traingle	Pascal's triangle, in algebra, is a triangular arrangement of numbers that gives the coefficients in the expansion of any binomial expression.		
perfect square	a number that can be expressed as the square of an integer.	umubare ushobora kwandikwa nk' umubare umwe wikubye inshuro zingana nka wo.	1,4,9,16,25,...
perpendicular	at an angle of 90° to a given line, plane, or surface or to the ground.	gukora imfuruka ya dogere 90° hagati y' ibintu 2 runaka.	

polynomial	an expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s).		2xy-3x
prime number	a number that has exactly 2 factors, i.e: divisible only by itself and unity. N.B: 1 is not a prime number and negative integers can not be prime.	umubare ugabanywa na rimwe cyangwa wo ubwawo utabariyemo rimwe n' imibare iri munsi ya zeru.	7,11,13,17,...
quadrilateral	a four-sided figure	ikinyampande enye	
quotient	the number resulting from the division of one number by another.	umubare uturuka ku kugabanya umubare umwe ukoresheje undi.	
ratio	the quantitative relation between two amounts showing the number of times one value contains or is contained within the other.	isano iri hagati y' imibare ibiri yerekana inshuro umubare umwe wikubye uwundi.	if there is 1 boy and 3 girls you could write the ratio as: 1 : 3 (for every one boy there are 3 girls).
rational number	Any number that can be written as a fraction with integers is called a rational number.	umubare ushobora kugaragazwa n' imibare ibiri, aho umwe uba ugabanya undi.	-1/30,-7/13,1/2, 1/5, 3/4,...
real numbers	a quantity that can be expressed as an infinite decimal expansion.	imibare yose ibaho harimo n' irimo ibice.	-22.22565, 3,13.335451,1/3, √6, 2,...
remainder	the amount left over when one quantity is divided by another.	umubare usigara nyuma yuko umubare umwe ugabanyije undi.	

reflection	a mirror image of a shape or an object, obtained from flipping the image/object.	uburyo bwo kugaragaza ishosha y' kintu uvuye ku murongo cyangwa ahantu runaka.	
right angle	angle that is exactly equal to 90°	inguni ingana nka dogere 90°	
sector(circle)	the plane figure enclosed by two radii of a circle or ellipse and the arc between them.		
segment	a part of a figure cut off by a line or plane intersecting it.	igice cy' igishushanyo cyatandukanyijwe n' ikindi hifashishijwe umurogo.	
semi-circle	a half of a circle.	kimwe cya kabiri cy' uruziga.	
sequence	a list of things (usually numbers) that are in order.	urutonde rw' ibintu (ubusanzwe ni imibare).	muri uru rutonde: 2,4,6,8,... umubare umwe ugenda urusha uwuri imbere ho imibare ibiri.
strictly positive(integer)	The strictly positive integers are the set defined as: $Z>0:=\{x\in Z:x>0\}$ That is, all the integers that are strictly greater than zero: $Z>0$.	imibare itarimo ibice kandi isumba zeru.	1,2,3,4,5,......
subset	a set of which all the elements are contained in another set.	Itsinda rigizwe n' ibintu biri mu rindi tsinda ryisumbuyeho.	
symmetric point	The central point that splits the object or shape into two parts.	akadomo kagabanya ikintu mo ibice bibiri.	
tangent	a straight line or plane that touches a curve or curved surface at a point, but if extended does not cross it at that point.	Umurongo ugororotse ukora ku muzenguruko w' ikintu kiburungushuye (curve, circle).	

trapezium	a four-sided polygon with at least two sides parellel.	Ikinyampande enye gifite nibura impande ebyiri ziteganye.	
trapezoid	a four-sided polygon with at least two sides parellel.	Ikinyampande enye gifite nibura impande ebyiri ziteganye.	
tetramino	a geometric shape composed of four connected squares.	imiterere igizwe na kare enye zihuje.	
vertex	each angular point of a polygon, polyhedron, or other figure.	aho imirongo ibiri ihurira maze ikarema imfuruka.	

SYMBOLS

symbol	meaning	example / explanation
$\forall$	for all	$n+2$ is odd $\forall$n is an odd number
$\exists$	exists	if x$^2 + x - 2 = 0, \exists\, x\; in$ R
ϵ	belongs	$2n+1$ is odd $\forall n \epsilon$N
$\notin$	does not belong	if $x^2+1=0$, x $\notin$R
$\wedge$	intersects	if plane 1 is not parallel to plane 2, then plane 1 $\wedge$ plane 2
$\mid$	divides	5$\mid$ 25 and n $\mid$n^2+n
$\nmid$	does not divide	$2 \nmid 21$
$\approx$	is approximately equal to; is almost equal to.	3.01 $\approx$ 3 and $\pi \approx$ 3.14
$\neq$	is not equal to	$2 \neq 3$
Δ	triangle	we say that ΔABC is equilateral if all sides are equal
gcd	greatest common divisor	gcd$(17,170) = 17$
LCM	lowest common multiple	LCM (15,6)=30
IE	this means that	$2 \nmid n+1$ IE n is an odd number
etc	and so on	4$\mid$ $n+2 \;\forall\; n = 2, 6, 10, 14, 18, 22, etc$
WLOG	without loss of generality	for $x+y-2xy > 1$ suppose WLOG $x \geq y$
	line 1 is parallel to line 2	
	AB=BC	
	angle ACB = angle BCD	
$\therefore$	Therefore	$x+2=5 \quad \therefore$ x=3
Q.E.D.	that which was to be demonstrated	it is often placed at the end of a mathematical proof to indicate its completion
$\cap$	intersection	A= $\{a,b,c,d,e\}$, B= $\{b,c,f\}$, A$\cap B$ ={b,c}
$\cup$	union	A= $\{a,b,c,d,e\}$, B= $\{b,c,f\}$, A$\cup B$ ={a,b,c,d,e,f}
$[x,y[$ or $[x,y)$	real numbers between x and y including x but not y.	
...	and so on	$\frac{1}{3} = 0.3333333...$

$x!$	the product of all positive integers less than or equal to x	4!=4x3x2x1=24
$\lfloor x \rfloor$	the greatest integer less than or equal to x	$\lfloor 3.6 \rfloor = 3$; $\lfloor -2.3 \rfloor = -3$
$\lceil x \rceil$	the least integer greater than or equal to x	$\lceil 3.6 \rceil = 4$; $\lceil -2.3 \rceil = -2$
$\subseteq$	is a subset of	A= $\{a, b, c, d, e\}$, B= $\{b, c\}$ IE B$\subseteq A$
$\nsubseteq$	is not a subset of	A= $\{a, b, c, d, e\}$, B= $\{f, g, h\}$ IE B$\nsubseteq A$
∞	infinity	
$\propto$	is proportional to	$\frac{a}{b} = k$ where k is a positive number, $\therefore a \propto b$
:	ratio	if there are three dogs for every two cats, we would say the ratio of dog to cats is 3:2.
$\equiv$	equivalent to/ is congruent to	$3 \equiv 1 \bmod 2$
$\emptyset$	empty set	A= $\{a, b, c, d, e\}$, B= $\{f, g, h\}$; A$\cap B = \emptyset$
N	natural numbers	
Z	integers	
Q	rational numbers	
R	real numbers	
$$\sum_{n=i}^{k} n \quad (1)$$	$i + ... + k$	given that n$\epsilon\mathbb{Z}$, $$\sum_{n=1}^{3} 2^n = 2 + 2^2 + 2^3 = 14 \quad (2)$$
$$\prod_{n=i}^{k} n \quad (3)$$	$i \times ... \times k$	given that n$\epsilon\mathbb{Z}$, $$\prod_{n=1}^{3} 2^n = 2 \times 2^2 \times 2^3 = 64 \quad (4)$$
π	the ratio of the circumference of a circle to its diameter; it is an irrational number.	$\pi = 3.141592653589793238...$
$\sim$	similar to	$\Delta ABC \sim \Delta DEF$, if their corresponding angles are congruent and their corresponding sides are in proportion.
$\perp$	perpendicular t0	If angle between line L$_1$and line $L_2$is equal to $90^\circ, L_1 \perp L_2$.
$\rightarrow$	implies(if...then...)	p:The triangle PQR is isosceles. q: Two of the angles of the triangle PQR have equal measure. $\therefore p \rightarrow q$
$\surd$	square root	$\sqrt{25} = 5$