Notes on Position Distributions of a Walker

Theophilus Dwamena Frimpong

Contents

1 Simple Diffusion Process

Consider a particle undergoing simple diffusion in one dimension. The probability density function $P(x,t)$ satisfies the diffusion equation:

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{1}$$

with the initial condition:

$$P(x,0) = \delta(x). \tag{2}$$

We apply the Fourier transform:

$$\tilde{P}(k,t) = \int_{-\infty}^{\infty} e^{ikx} P(x,t)\,dx$$

Taking the transform of both sides of equation (1), we get:

$$\frac{\partial \tilde{P}(k,t)}{\partial t} = -Dk^2\tilde{P}(k,t)$$

This is an ordinary differential equation in time t, with solution:

$$\tilde{P}(k,t) = \tilde{P}(k,0)e^{-Dk^2t}$$

Since $P(x,0) = \delta(x)$, it follows that $\tilde{P}(k,0) = 1$, so:

$$\tilde{P}(k,t) = e^{-Dk^2t}$$

Now we apply the inverse Fourier transform to find $P(x,t)$:

$$P(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\tilde{P}(k,t)\,dk \tag{3}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}e^{-Dk^2t}\,dk \tag{4}$$

This integral is a standard Gaussian integral of the form:

$$\int_{-\infty}^{\infty} e^{-ak^2+ibk}\,dk = \sqrt{\frac{\pi}{a}}\,e^{-b^2/(4a)} \quad \text{for } a > 0$$

We identify $a = Dt$, $b = x$. Then:

$$\begin{aligned} P(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dk^2t-ikx}\,dk && (5)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dt\left(k^2+\frac{ikx}{Dt}\right)}\,dk && (6)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dt\left(k+\frac{ix}{2Dt}\right)^2} e^{-x^2/(4Dt)}\,dk && (7) \end{aligned}$$

Now make the substitution $u = k + \frac{ix}{2Dt}$, so $dk = du$, and the limits remain $(-\infty, \infty)$:

$$P(x,t) = \frac{1}{2\pi}e^{-x^2/(4Dt)}\int_{-\infty}^{\infty} e^{-Dtu^2}\,du = \frac{1}{2\pi}e^{-x^2/(4Dt)}\sqrt{\frac{\pi}{Dt}}$$

Thus:

$$\boxed{P(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)}$$

We now plot this analytic against simulations as shown in Figure 1.

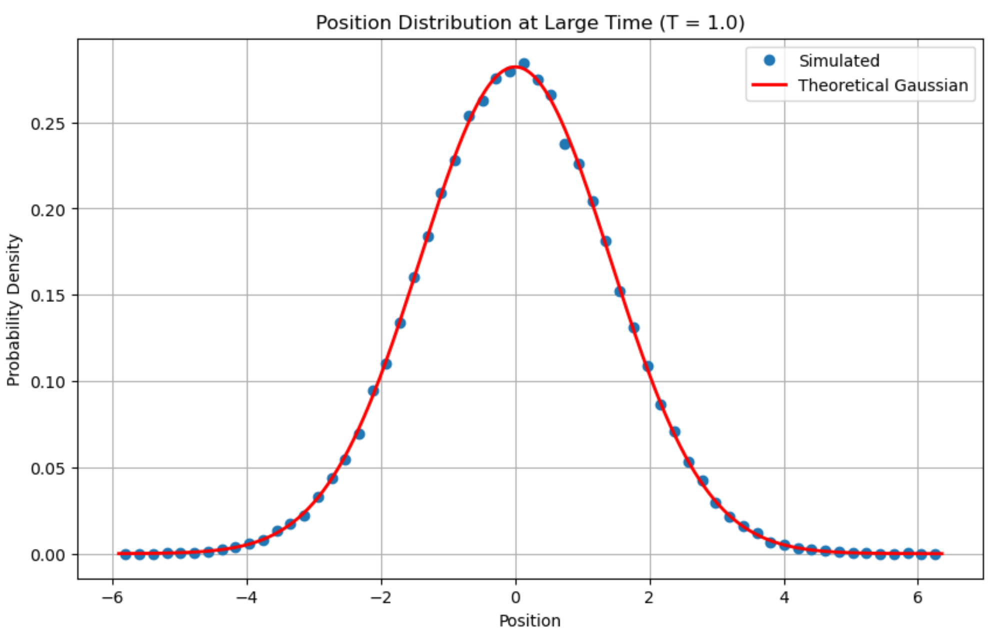

Figure 1: Probability distribution of positions for simple diffusion process with $D = 1$ and large time $t = 500$.

Python notebook.

2 Simple Diffusion with Resetting

We start with the Master Equation (Fokker-Planck equation) for the probability $p(x,t|x_0)$ [2]:

$$\frac{\partial p(x,t|x_0)}{\partial t} = D\frac{\partial^2 p(x,t|x_0)}{\partial x^2} - rp(x,t|x_0) + r\delta(x-x_0) \tag{8}$$

where D is the diffusion constant, r is the resetting rate, and $\delta(x-x_0)$ is the Dirac delta function representing the resetting to the initial position x_0.

To find the stationary position distribution, denoted as $p_{st}(x|x_0)$, we set the time derivative to zero, as the distribution does not change in the steady state:

$$\frac{\partial p(x,t|x_0)}{\partial t} = 0$$

Substituting this into equation (8), we obtain:

$$0 = D\frac{d^2 p_{st}(x|x_0)}{dx^2} - rp_{st}(x|x_0) + r\delta(x-x_0)$$

Rearranging the terms, we get a second-order ordinary differential equation:

$$D\frac{d^2 p_{st}(x|x_0)}{dx^2} - rp_{st}(x|x_0) = -r\delta(x-x_0) \tag{9}$$

Dividing by D, we have:

$$\frac{d^2 p_{st}(x|x_0)}{dx^2} - \frac{r}{D}p_{st}(x|x_0) = -\frac{r}{D}\delta(x-x_0)$$

Let's define a characteristic inverse length scale α_0 as:

$$\alpha_0 = \sqrt{\frac{r}{D}} \tag{10}$$

Substituting this into the differential equation, we get:

$$\frac{d^2 p_{st}(x|x_0)}{dx^2} - \alpha_0^2 p_{st}(x|x_0) = -\alpha_0^2\delta(x-x_0) \tag{11}$$

This is a non-homogeneous linear second-order differential equation. Consider the homogeneous part of the equation:

$$\frac{d^2 p_{st,h}(x)}{dx^2} - \alpha_0^2 p_{st,h}(x) = 0$$

The characteristic equation is $m^2 - \alpha_0^2 = 0$, which yields roots $m = \pm\alpha_0$. The general solution to the homogeneous equation is:

$$p_{st,h}(x) = Ae^{\alpha_0 x} + Be^{-\alpha_0 x}$$

The physical requirement is that the probability distribution $p_{st}(x|x_0)$ must decay to zero as $|x| \to \infty$. This implies that for $x > x_0$, the $e^{\alpha_0 x}$ term must vanish ($A = 0$), and for $x < x_0$, the $e^{-\alpha_0 x}$ term must vanish ($B = 0$). Therefore, we propose a solution of the form:

$$p_{st}(x|x_0) = Ce^{-\alpha_0|x-x_0|}$$

This form ensures that the probability decays exponentially away from the resetting point x_0. We can write this explicitly for the two regions:

$$p_{st}(x|x_0) = \begin{cases} Ce^{-\alpha_0(x-x_0)} & \text{for } x > x_0 \\ Ce^{\alpha_0(x-x_0)} & \text{for } x < x_0 \end{cases}$$

The solution $p_{st}(x|x_0)$ must be continuous at $x = x_0$. For $x = x_0$: $Ce^{-\alpha_0(x_0-x_0)} = Ce^{\alpha_0(x_0-x_0)} \implies C = C$. This condition is satisfied.

The presence of the Dirac delta function on the right-hand side of equation (11) implies a discontinuity in the first derivative of $p_{st}(x|x_0)$ at $x = x_0$. To find this discontinuity, we integrate equation (11) from $x_0 - \epsilon$ to $x_0 + \epsilon$ and then take the limit as $\epsilon \to 0$:

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \left(\frac{d^2 p_{st}}{dx^2} - \alpha_0^2 p_{st} \right) dx = \int_{x_0-\epsilon}^{x_0+\epsilon} (-\alpha_0^2 \delta(x - x_0)) dx$$

Applying the fundamental theorem of calculus to the first term and noting that $\int_{x_0-\epsilon}^{x_0+\epsilon} p_{st} dx \to 0$ as $\epsilon \to 0$ (since p_{st} is continuous), and $\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0) dx = 1$, we get:

$$\left[\frac{dp_{st}}{dx} \right]_{x_0-\epsilon}^{x_0+\epsilon} - \alpha_0^2 \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} p_{st} dx = -\alpha_0^2 \cdot 1$$

$$\left. \frac{dp_{st}}{dx} \right|_{x_0^+} - \left. \frac{dp_{st}}{dx} \right|_{x_0^-} = -\alpha_0^2$$

Now we calculate the derivatives of our proposed solution in the two regions: For $x > x_0$:

$$\frac{dp_{st}}{dx} = \frac{d}{dx} \left(Ce^{-\alpha_0(x-x_0)} \right) = -C\alpha_0 e^{-\alpha_0(x-x_0)}$$

So, at $x = x_0^+$:

$$\left. \frac{dp_{st}}{dx} \right|_{x_0^+} = -C\alpha_0 e^{-\alpha_0(x_0-x_0)} = -C\alpha_0$$

For $x < x_0$:

$$\frac{dp_{st}}{dx} = \frac{d}{dx} \left(Ce^{\alpha_0(x-x_0)} \right) = C\alpha_0 e^{\alpha_0(x-x_0)}$$

So, at $x = x_0^-$:

$$\left. \frac{dp_{st}}{dx} \right|_{x_0^-} = C\alpha_0 e^{\alpha_0(x_0-x_0)} = C\alpha_0$$

Substitute these into the jump condition:

$$(-C\alpha_0) - (C\alpha_0) = -\alpha_0^2$$

$$-2C\alpha_0 = -\alpha_0^2$$

Solving for C:

$$C = \frac{-\alpha_0^2}{-2\alpha_0} = \frac{\alpha_0}{2}$$

Substituting the value of C back into our proposed solution, we obtain the stationary position distribution:

$$\boxed{p_{st}(x|x_0) = \frac{1}{2} \cdot \sqrt{\frac{r}{D}} \cdot \exp\left(-\sqrt{\frac{r}{D}} \cdot |x - x_0| \right)}$$

We now plot this analytic against simulations for $r = \{0, 0.01, 0.025, 0.1\}$ as shown in Figure 2. It is important to note that for r=0, we have a simple diffusion with no resetting hence:

$$P(x, T) = \frac{1}{\sqrt{4\pi DT}} \exp\left(-\frac{(x - x_0)^2}{4DT} \right) \tag{12}$$

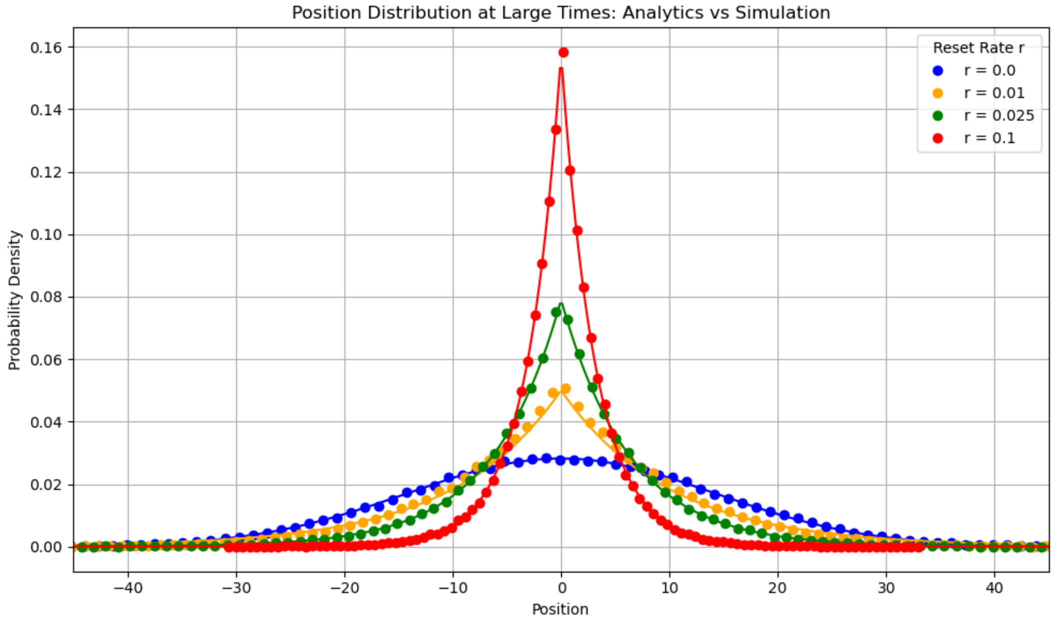

Figure 2: Probability distribution of positions of simple diffusion processes with resetting for different reset probabilities $r = \{0, 0.01, 0.025, 0.1\}$ based on 10^6 simulations.

Python notebook.

3 Sluggish Random Walk

Recall the Fokker–Planck equation with a position-dependent diffusion coefficient [1]:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left[\frac{1}{|x|^\alpha}P(x,t)\right], \tag{13}$$

with the initial condition:

$$P(x,0) = \delta(x - x_0).$$

We propose a scaling form:

$$\boxed{P(x,t) = \frac{1}{t^\mu} H\left(\frac{x}{t^\mu}\right)}, \tag{14}$$

where μ is a scaling exponent to be determined and $z = \frac{x}{t^\mu}$ is the scaling variable.

We begin with LHS of eqn (13):

$$\begin{aligned}
(14) \implies \frac{\partial P(x,t)}{\partial t} &= \frac{d}{dt}\left(\frac{1}{t^\mu} \cdot H\left(\frac{x}{t^\mu}\right)\right) \\
&= -\mu t^{-\mu-1} \cdot H\left(\frac{x}{t^\mu}\right) + t^{-\mu} \cdot H'\left(\frac{x}{t^\mu}\right) \cdot \frac{d}{dt}\left(\frac{x}{t^\mu}\right) && \text{Product and chain rules} \\
&= -\frac{\mu}{t^{\mu+1}} H(z) + t^{-\mu} \cdot H'(z) \cdot \left(x(-\mu)t^{-\mu-1}\right) && \text{since } z = \frac{x}{t^\mu} \\
&= -\frac{\mu}{t^{\mu+1}} H(z) - \frac{\mu z}{t^{\mu+1}} H'(z) \\
\frac{\partial P(x,t)}{\partial t} &= \frac{-\mu}{t^{\mu+1}}\Big[H(z) + zH'(z)\Big].
\end{aligned} \tag{15}$$

For the right-hand side of equation (13):

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{|x|^\alpha}P(x,t)\right] = \frac{\partial^2}{\partial x^2}\left[\frac{1}{|x|^\alpha} \cdot \frac{1}{t^\mu} \cdot H\left(\frac{x}{t^\mu}\right)\right]$$

$$\implies \frac{1}{|x|^\alpha}P(x,t) = \frac{1}{t^{\mu(1+\alpha)}} \cdot \frac{1}{|z|^\alpha} \cdot H(z)$$

For the second derivative with respect to x, we use the chain rule:

$$\frac{d}{dx} = \frac{dz}{dx} \cdot \frac{d}{dz} = \frac{1}{t^\mu} \cdot \frac{d}{dz} \qquad \implies \qquad \frac{d^2}{dx^2} = \frac{1}{t^\mu} \cdot \frac{d}{dz}\left(\frac{1}{t^\mu} \cdot \frac{d}{dz}\right) = \frac{1}{t^{2\mu}} \cdot \frac{d^2}{dz^2}$$

Thus:

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{|x|^\alpha}P(x,t)\right] = \frac{1}{t^{\mu(1+\alpha)}} \cdot \frac{1}{t^{2\mu}} \cdot \frac{d^2}{dz^2}\left[\frac{1}{|z|^\alpha} \cdot H(z)\right] = \frac{1}{t^{\mu(\alpha+3)}} \cdot \frac{d^2}{dz^2}\left[\frac{1}{|z|^\alpha} \cdot H(z)\right] \tag{16}$$

Equating expression (15) with (16), we obtain:

$$-\frac{u}{t^{u+1}}\Big[H(z) + zH'(z)\Big] = \frac{1}{t^{\mu(\alpha+3)}} \cdot \frac{d^2}{dz^2}\left[\frac{1}{|z|^\alpha} \cdot H(z)\right] \tag{17}$$

For this to hold, powers of t must be equal. Thus $\mu + 1 = \mu(\alpha + 3) \implies 1 = \mu(\alpha + 2)$

$$\implies \boxed{\mu = \frac{1}{\alpha + 2}} \tag{18}$$

We now derive the ODE for $H(z)$. Recall:

$$P(x,t) = \frac{1}{t^\mu} H(z), \quad \mu = \frac{1}{\alpha+2}, \quad z = \frac{x}{t^\mu}$$

From equation (17):

$$-\mu\Big[H(z) + zH'(z)\Big] = \frac{d^2}{dz^2}\left[\frac{1}{|z|^\alpha} H(z)\right] \tag{19}$$

Let:

$$Q(z) = \frac{1}{|z|^\alpha} H(z),$$

then:

$$\frac{dQ}{dz} = -\alpha \frac{\operatorname{sgn}(z)}{|z|^{\alpha+1}} H(z) + \frac{1}{|z|^\alpha} H'(z)$$

$$\frac{d^2Q}{dz^2} = \alpha(\alpha-1) \cdot \frac{1}{|z|^{\alpha+2}} \cdot H(z) - 2\alpha \cdot \frac{1}{|z|^{\alpha+1}} \cdot H'(z) + \frac{1}{|z|^\alpha} \cdot H''(z)$$

$\implies$ (19) becomes:

$$-\mu\Big[H(z) + zH'(z)\Big] = \left[\alpha(\alpha-1)\frac{1}{|z|^{\alpha+2}} \cdot H(z) - 2\alpha \cdot \frac{1}{|z|^{\alpha+1}} \cdot H'(z) + \frac{1}{|z|} H'(z) + H''(z)\right]$$

$$-\mu\Big[H(z) + zH'(z)\Big] = \frac{1}{|z|^\alpha} \cdot \frac{d^2}{dz^2}\left[\alpha(\alpha-1) \cdot \frac{1}{|z|^2} \cdot H(z) - 2\alpha \frac{1}{|z|} H'(z) + H''(z)\right]$$

$$\implies -\mu z^\alpha \cdot \mathcal{H}(z) - \mu z^{\alpha+1} \cdot H'(z) = \alpha(\alpha+1) z^{-2} \cdot H(z) - 2\alpha z^{-1} \cdot H'(z) + H''(z)$$

$$H''(z) + \left(\mu z^{\alpha+1} - \frac{2\alpha}{z}\right) H'(z) + \left(\mu z^\alpha + \frac{\alpha(\alpha+1)}{z^2}\right) H(z) = 0. \tag{20}$$

Hence the scaling function $H(z)$ must satisfy the nonlinear second-order ODE in equation (20). To solve this equation, we propose an ansatz for $H(z)$ of the form:

$$H(z) = A|z|^a e^{-B|z|^b}, \quad \text{as } |z| \to \infty,$$

where A, a, b, and B are constants to be determined.
We compute the first and second derivatives for large z (assuming $z > 0$ for simplicity):

$$H'(z) = A\left(a|z|^{a-1}\operatorname{sgn}(z) - Bb|z|^{a+b-1}\operatorname{sgn}(z)\right) e^{-B|z|^b},$$
$$H''(z) = Ae^{-B|z|^b}\left[a(a-1)|z|^{a-2} - 2aBb|z|^{a+b-2} + B^2b^2|z|^{a+2b-2} - Bb(a+b-1)|z|^{a+b-2}\right].$$

For large z, the leading behavior is:

$$H(z) \sim Az^a e^{-Bz^b},$$
$$H'(z) \sim -ABbz^{a+b-1} e^{-Bz^b},$$
$$H''(z) \sim AB^2b^2 z^{a+2b-2} e^{-Bz^b}.$$

At large z, the dominant terms in the ODE (20) are:

$$H''(z) \sim AB^2b^2 z^{a+2b-2} e^{-Bz^b},$$
$$\mu z^{\alpha+1} H'(z) \sim -A\mu Bbz^{a+b+\alpha} e^{-Bz^b}.$$

We match the powers of z in the dominant balance:

$$a + 2b - 2 = a + b + \alpha \quad \Rightarrow \quad b = \alpha + 2.$$

Using the definition $\mu = \frac{1}{\alpha+2}$, we find:

$$b = \frac{1}{\mu}, \quad \text{so} \quad B = \mu^2.$$

To find a, we use:

$$a = \frac{1-2\mu}{\mu} = \alpha.$$

Putting everything together, the scaling function $H(z)$ takes the explicit form:

$$H(z) = A|z|^{\frac{1-2\mu}{\mu}} e^{-\mu^2 |z|^{1/\mu}}.$$

To determine the normalization constant A, we impose:

$$\int_{-\infty}^{\infty} H(z)\, dz = 1.$$

Since $H(z)$ is an even function, we compute:

$$\int_{-\infty}^{\infty} H(z)\, dz = 2A \int_0^{\infty} z^{\frac{1-2\mu}{\mu}} e^{-\mu^2 z^{1/\mu}}\, dz.$$

Let $u = \mu^2 z^{1/\mu} \Rightarrow z = \left(\frac{u}{\mu^2}\right)^{\mu}, \quad dz = \mu \left(\frac{u}{\mu^2}\right)^{\mu-1} \cdot \frac{du}{\mu^2}$. Then,

$$\int_0^{\infty} z^{\frac{1-2\mu}{\mu}} e^{-\mu^2 z^{1/\mu}} dz = \mu^{1-2\mu} \int_0^{\infty} u^{-\mu} e^{-u} du = \mu^{1-2\mu}\Gamma(1-\mu).$$

Hence, the normalization constant is:

$$A = \frac{\mu^{1-2\mu}}{2\Gamma(1-\mu)}.$$

Hence the final expression for the scaling function is

$$\boxed{H(z) = \frac{\mu^{1-2\mu}}{2\Gamma(1-\mu)} |z|^{\frac{1-2\mu}{\mu}} \exp\left(-\mu^2 |z|^{1/\mu}\right)} \tag{21}$$

Substitute equations (18) and (21) into equation (14) to obtain a complete expression for $P(x,t)$. First, we substitute the expression for $H(z)$ from equation (21) into equation (14). We replace z with $\left(\frac{x}{t^\mu}\right)$:

$$P(x,t) = \frac{1}{t^\mu} \left(\frac{\mu^{1-2\mu}}{2\Gamma(1-\mu)} \left|\frac{x}{t^\mu}\right|^{\frac{1-2\mu}{\mu}} e^{-\mu^2 \left|\frac{x}{t^\mu}\right|^{\frac{1}{\mu}}} \right)$$

Next, we simplify the terms involving the absolute value $\left|\frac{x}{t^\mu}\right|$:

$$\left|\frac{x}{t^\mu}\right|^{\frac{1-2\mu}{\mu}} = \frac{|x|^{\frac{1-2\mu}{\mu}}}{(t^\mu)^{\frac{1-2\mu}{\mu}}} = \frac{|x|^{\frac{1-2\mu}{\mu}}}{t^{1-2\mu}} = \frac{|x|^{\frac{1}{\mu}-2}}{t^{1-2\mu}}$$

$$\left|\frac{x}{t^\mu}\right|^{\frac{1}{\mu}} = \frac{|x|^{\frac{1}{\mu}}}{(t^\mu)^{\frac{1}{\mu}}} = \frac{|x|^{\frac{1}{\mu}}}{t}$$

Now, we substitute these simplified terms back into the expression for $P(x,t)$:

$$\begin{aligned}
P(x,t) &= \frac{1}{t^\mu} \frac{\mu^{1-2\mu}}{2\Gamma(1-\mu)} \frac{|x|^{\frac{1}{\mu}-2}}{t^{1-2\mu}} e^{-\mu^2 \frac{|x|^{\frac{1}{\mu}}}{t}} \\
&= \frac{\mu^{1-2\mu}}{2\Gamma(1-\mu)} \frac{|x|^{\frac{1}{\mu}-2}}{t^{1-\mu}} e^{-\mu^2 \frac{|x|^{\frac{1}{\mu}}}{t}} \\
&= \frac{\left(\frac{1}{\alpha+2}\right)^{\frac{\alpha}{\alpha+2}}}{2\Gamma\left(1-\frac{1}{\alpha+2}\right)} \frac{|x|^{\alpha}}{t^{\frac{\alpha+1}{\alpha+2}}} e^{-\left(\frac{1}{\alpha+2}\right)^2 \frac{|x|^{\alpha+2}}{t}} && \text{substituting } \mu = \frac{1}{\alpha+2}
\end{aligned}$$

Hence,

$$\boxed{P(x,t) = \frac{\left(\frac{1}{\alpha+2}\right)^{\frac{\alpha}{\alpha+2}}}{2\Gamma\left(\frac{\alpha+1}{\alpha+2}\right)} \cdot \frac{|x|^{\alpha}}{t^{\frac{\alpha+1}{\alpha+2}}} \cdot \exp\left(-\frac{|x|^{\alpha+2}}{(\alpha+2)^2 t}\right)}$$

We now plot this analytic against simulations for $\alpha = \{0, 0.5, 1.5\}$ as shown in Figure 3.

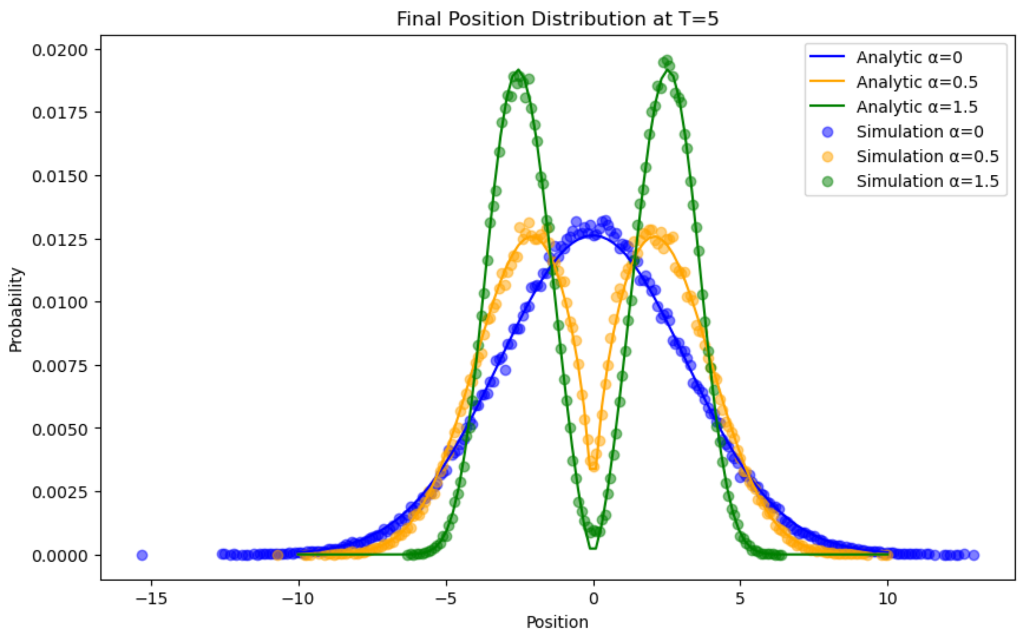

Figure 3: Probability distribution of positions of sluggish random walk for different levels of sluggishness ($\alpha \in \{0, 0.5, 1.5\}$) based on 10^6 simulations.

Python notebook

4 Sluggish Random Walk with Resetting (Novel Approach)

We propose that the probability density $p(x,t|x_0)$ evolves according to:

$$\frac{\partial p(x,t|x_0)}{\partial t} = \frac{\partial^2}{\partial x^2}\left[|x|^{-\alpha} p(x,t|x_0)\right] - rp(x,t|x_0) + r\delta(x-x_0). \tag{22}$$

For the stationary distribution $p_{\mathrm{st}}(x|x_0)$, we set $\partial p/\partial t = 0$:

$$0 = \frac{d^2}{dx^2}\left[|x|^{-\alpha} p_{\mathrm{st}}(x|x_0)\right] - rp_{\mathrm{st}}(x|x_0) + r\delta(x-x_0). \tag{23}$$

Rearranging gives the key equation:

$$\frac{d^2}{dx^2}\left[|x|^{-\alpha} p_{\mathrm{st}}(x|x_0)\right] - rp_{\mathrm{st}}(x|x_0) = -r\delta(x-x_0). \tag{24}$$

For resetting to the origin ($x_0 = 0$), the problem has even symmetry. We solve for $x > 0$ and extend by symmetry. We begin with the homogeneous equation for $x > 0$:

$$\frac{d^2 q}{dx^2} - rx^{\alpha} q(x) = 0, \tag{25}$$

where we defined $q(x) = x^{-\alpha} p_{\mathrm{st}}(x)$.

To reduce this to a known form, we introduce the change of variables:

$$z = \frac{2\sqrt{r}}{\alpha+2} x^{(\alpha+2)/2}, \tag{26}$$

$$q(x) = x^{1/2} f(z). \tag{27}$$

Our goal is to transform equation (25) into a differential equation for $f(z)$.

We first compute derivatives of $q(x)$. Using the product and chain rules:

$$\frac{dq}{dx} = \frac{d}{dx}\left(x^{1/2} f(z)\right) = \frac{1}{2} x^{-1/2} f(z) + x^{1/2} \frac{df}{dz}\frac{dz}{dx}.$$

Next, compute the second derivative:

$$\frac{d^2 q}{dx^2} = \frac{d}{dx}\left(\frac{1}{2} x^{-1/2} f(z) + x^{1/2} \frac{df}{dz}\frac{dz}{dx}\right).$$

Calculate the derivative term-by-term:

$$\frac{d}{dx}\left(\frac{1}{2} x^{-1/2} f(z)\right) = -\frac{1}{4} x^{-3/2} f(z) + \frac{1}{2} x^{-1/2} \frac{df}{dz}\frac{dz}{dx},$$

and

$$\frac{d}{dx}\left(x^{1/2} \frac{df}{dz}\frac{dz}{dx}\right) = \frac{1}{2} x^{-1/2} \frac{df}{dz}\frac{dz}{dx} + x^{1/2} \frac{d}{dx}\left(\frac{df}{dz}\frac{dz}{dx}\right).$$

The last derivative requires the product rule again:

$$\frac{d}{dx}\left(\frac{df}{dz}\frac{dz}{dx}\right) = \frac{d^2 f}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{df}{dz}\frac{d^2 z}{dx^2}.$$

Putting it all together:

$$\begin{aligned}\frac{d^2q}{dx^2} &= -\frac{1}{4}x^{-3/2}f(z) + \frac{1}{2}x^{-1/2}\frac{df}{dz}\frac{dz}{dx} \\ &\quad + \frac{1}{2}x^{-1/2}\frac{df}{dz}\frac{dz}{dx} + x^{1/2}\left[\frac{d^2f}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{df}{dz}\frac{d^2z}{dx^2}\right] \\ &= -\frac{1}{4}x^{-3/2}f(z) + x^{-1/2}\frac{df}{dz}\frac{dz}{dx} + x^{1/2}\frac{d^2f}{dz^2}\left(\frac{dz}{dx}\right)^2 + x^{1/2}\frac{df}{dz}\frac{d^2z}{dx^2}.\end{aligned}$$

Next, compute derivatives of z:

$$z = \frac{2\sqrt{r}}{\alpha+2}x^{\frac{\alpha+2}{2}}.$$

Thus,

$$\frac{dz}{dx} = \frac{2\sqrt{r}}{\alpha+2}\cdot\frac{\alpha+2}{2}x^{\frac{\alpha}{2}} = \sqrt{r}x^{\alpha/2},$$

and

$$\frac{d^2z}{dx^2} = \sqrt{r}\cdot\frac{\alpha}{2}x^{\frac{\alpha}{2}-1} = \frac{\alpha}{2}\sqrt{r}x^{\frac{\alpha-2}{2}}.$$

Substitute these into the expression for d^2q/dx^2:

$$\begin{aligned}\frac{d^2q}{dx^2} &= -\frac{1}{4}x^{-3/2}f(z) + x^{-1/2}\frac{df}{dz}\cdot\sqrt{r}x^{\alpha/2} + x^{1/2}\frac{d^2f}{dz^2}\left(\sqrt{r}x^{\alpha/2}\right)^2 + x^{1/2}\frac{df}{dz}\cdot\frac{\alpha}{2}\sqrt{r}x^{\frac{\alpha-2}{2}} \\ &= -\frac{1}{4}x^{-3/2}f(z) + \sqrt{r}x^{\frac{\alpha-1}{2}}\frac{df}{dz} + rx^{\alpha+1/2}\frac{d^2f}{dz^2} + \frac{\alpha}{2}\sqrt{r}x^{\frac{\alpha-1}{2}}\frac{df}{dz}.\end{aligned}$$

Group terms with $\frac{df}{dz}$:

$$\sqrt{r}x^{\frac{\alpha-1}{2}}\frac{df}{dz} + \frac{\alpha}{2}\sqrt{r}x^{\frac{\alpha-1}{2}}\frac{df}{dz} = \sqrt{r}\left(1+\frac{\alpha}{2}\right)x^{\frac{\alpha-1}{2}}\frac{df}{dz} = \sqrt{r}\frac{\alpha+2}{2}x^{\frac{\alpha-1}{2}}\frac{df}{dz}.$$

Putting everything together:

$$\frac{d^2q}{dx^2} = -\frac{1}{4}x^{-3/2}f(z) + \sqrt{r}\frac{\alpha+2}{2}x^{\frac{\alpha-1}{2}}\frac{df}{dz} + rx^{\alpha+\frac{1}{2}}\frac{d^2f}{dz^2}.$$

Now plug into equation (25):

$$\frac{d^2q}{dx^2} - rx^{\alpha}q = 0.$$

Recall $q = x^{1/2}f(z)$, so

$$rx^{\alpha}q = rx^{\alpha}\cdot x^{1/2}f(z) = rx^{\alpha+1/2}f(z).$$

Thus,

$$-\frac{1}{4}x^{-3/2}f(z) + \sqrt{r}\frac{\alpha+2}{2}x^{\frac{\alpha-1}{2}}\frac{df}{dz} + rx^{\alpha+1/2}\frac{d^2f}{dz^2} - rx^{\alpha+1/2}f(z) = 0.$$

Divide the entire equation by $x^{\alpha+1/2}$:

$$-\frac{1}{4}x^{-\alpha-2}f(z) + \sqrt{r}\frac{\alpha+2}{2}x^{-1}\frac{df}{dz} + r\frac{d^2f}{dz^2} - rf(z) = 0.$$

Rewrite powers of x:

$$-\frac{1}{4}x^{-(\alpha+2)}f(z)+\sqrt{r}\frac{\alpha+2}{2}x^{-1}\frac{df}{dz}+r\frac{d^2f}{dz^2}-rf(z)=0.$$

Recall from the definition of z that

$$x^{\frac{\alpha+2}{2}}=\frac{(\alpha+2)}{2\sqrt{r}}z \implies x^{-(\alpha+2)}=\left(\frac{2\sqrt{r}}{\alpha+2}\right)^2\frac{1}{z^2}.$$

Substitute into the first term:

$$-\frac{1}{4}\left(\frac{2\sqrt{r}}{\alpha+2}\right)^2\frac{1}{z^2}f(z)+\sqrt{r}\frac{\alpha+2}{2}x^{-1}\frac{df}{dz}+r\frac{d^2f}{dz^2}-rf(z)=0.$$

Simplify the first term:

$$-\frac{1}{4}\cdot\frac{4r}{(\alpha+2)^2}\frac{1}{z^2}f(z)=-\frac{r}{(\alpha+2)^2}\frac{1}{z^2}f(z).$$

Next, express x^{-1} in terms of z. From

$$z=\frac{2\sqrt{r}}{\alpha+2}x^{\frac{\alpha+2}{2}},$$

take logarithms and differentiate to find x^{-1} in terms of z, or more simply express the combination:

$$\sqrt{r}\frac{\alpha+2}{2}x^{-1}\frac{df}{dz}=\frac{dz}{dx}x^{-1}\frac{df}{dz}=\frac{dz}{dx}\cdot\frac{1}{x}\frac{df}{dz}.$$

But

$$\frac{dz}{dx}=\sqrt{r}x^{\alpha/2}, \quad \implies \frac{dz}{dx}\cdot\frac{1}{x}=\sqrt{r}x^{\alpha/2-1}.$$

Rearranged powers:

$$\alpha/2-1=\frac{\alpha-2}{2}.$$

But in terms of z, this is

$$x^{\frac{\alpha-2}{2}}=\left(\frac{\alpha+2}{2\sqrt{r}}\right)^{\frac{\alpha-2}{\alpha+2}}z^{\frac{\alpha-2}{\alpha+2}}.$$

Rather than pursue this complicated substitution directly, the standard approach is to realize that the final equation for $f(z)$ will be a modified Bessel equation of order

$$\nu=\frac{1}{\alpha+2}.$$

Indeed, substituting carefully, one obtains

$$z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}-(z^2+\nu^2)f=0. \tag{28}$$

This is the modified Bessel equation of order ν. Its general solution is:

$$f(z)=C_1I_\nu(z)+C_2K_\nu(z),$$

where $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of the first and second kind, respectively.

To ensure that the stationary distribution $p_{\mathrm{st}}(x)$ is normalizable as $x\to\infty$, we must discard the exponentially growing solution $I_\nu(z)$. Hence we set $C_1=0$.

Therefore,

$$q(x) = x^{1/2} K_\nu \left(\frac{2\sqrt{r}}{\alpha+2} x^{(\alpha+2)/2} \right),$$

and using $p_{\rm st}(x) = x^\alpha q(x)$, we obtain:

$$p_{\rm st}(x) = x^{\alpha+1/2} K_\nu \left(\frac{2\sqrt{r}}{\alpha+2} x^{(\alpha+2)/2} \right), \qquad x > 0.$$

Finally, using the even symmetry $p_{\rm st}(-x) = p_{\rm st}(x)$, the solution over the full real line is:

$$p_{\rm st}(x) = C|x|^{\alpha+1/2} K_\nu \left(\frac{2\sqrt{r}}{\alpha+2} |x|^{(\alpha+2)/2} \right), \tag{29}$$

where the constant C is determined from normalization and the jump condition at $x = 0$. We now determine the constant where $\nu = \frac{1}{\alpha+2}$.

We determine C using the jump condition from integrating the stationary Fokker–Planck equation (24) across an infinitesimal region around the origin. Recall:

$$\frac{d^2}{dx^2} \left[|x|^{-\alpha} p_{\rm st}(x) \right] - r p_{\rm st}(x) = -r\delta(x).$$

Integrate both sides from $-\epsilon$ to $+\epsilon$:

$$\int_{-\epsilon}^{\epsilon} \frac{d^2}{dx^2} \left[|x|^{-\alpha} p_{\rm st}(x) \right] dx - \int_{-\epsilon}^{\epsilon} r p_{\rm st}(x)\, dx = -r.$$

As $\epsilon \to 0$, the second term vanishes, and we get:

$$\frac{d}{dx} \left[|x|^{-\alpha} p_{\rm st}(x) \right] \bigg|_{x=0^+} - \frac{d}{dx} \left[|x|^{-\alpha} p_{\rm st}(x) \right] \bigg|_{x=0^-} = -r. \tag{30}$$

By symmetry, the left and right derivatives are equal in magnitude but opposite in sign, so:

$$2\,\frac{d}{dx} \left[x^{-\alpha} p_{\rm st}(x) \right] \bigg|_{x=0^+} = -r.$$

For small argument $z \to 0$, the modified Bessel function has the following behavior:

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2} \left(\frac{z}{2} \right)^{-\nu}, \qquad \nu > 0.$$

Near $x = 0$, the argument of the Bessel function becomes small:

$$z = \frac{2\sqrt{r}}{\alpha+2} x^{(\alpha+2)/2} \to 0.$$

Therefore,

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \cdot \frac{1}{2} \right)^{-\nu} x^{-\frac{(\alpha+2)\nu}{2}} = \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \right)^{-\nu} x^{-1/2}.$$

Then for small $x > 0$,

$$p_{\rm st}(x) \approx C x^{\alpha+1/2} \cdot x^{-1/2} = C x^\alpha \cdot \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \right)^{-\nu} x^{-1/2} = C \cdot \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \right)^{-\nu} x^{\alpha-1/2}.$$

Thus,

$$x^{-\alpha} p_{\mathrm{st}}(x) \sim C \cdot \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \right)^{-\nu} x^{-1/2}.$$

Differentiate:

$$\frac{d}{dx} \left[x^{-\alpha} p_{\mathrm{st}}(x) \right] \sim -\frac{1}{2} C \cdot \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \right)^{-\nu} x^{-3/2}.$$

Then as $x \to 0^+$, the derivative diverges like $x^{-3/2}$, but we can still compute the jump condition in a distributional sense. Using Eq. (30), we match:

$$2 \cdot \left(-\frac{1}{2} C \cdot \frac{\Gamma(\nu)}{2} \left(\frac{2\sqrt{r}}{\alpha+2} \right)^{-\nu} x^{-3/2} \right) = -r \quad \Rightarrow \quad C = \frac{r^{\frac{2\alpha+3}{2(\alpha+2)}}}{\Gamma\left(\frac{\alpha+1}{\alpha+2}\right)} (\alpha+2)^{-\frac{\alpha+1}{\alpha+2}}.$$

Combining everything, the normalized stationary distribution is:

$$\boxed{p_{\mathrm{st}}(x) = \frac{r^{\frac{2\alpha+3}{2(\alpha+2)}}}{\Gamma\left(\frac{\alpha+1}{\alpha+2}\right) (\alpha+2)^{\frac{\alpha+1}{\alpha+2}}} |x|^{\alpha+1/2} K_{\frac{1}{\alpha+2}} \left(\frac{2\sqrt{r}}{\alpha+2} |x|^{(\alpha+2)/2} \right)} \tag{31}$$

We now plot this analytic against simulations for $\alpha = \{0, 0.5, 1.5\}$ and reset probabilities $r = \{0, 0.01, 0.025, 0.1\}$ as shown in Figure 4 and Figure 5.

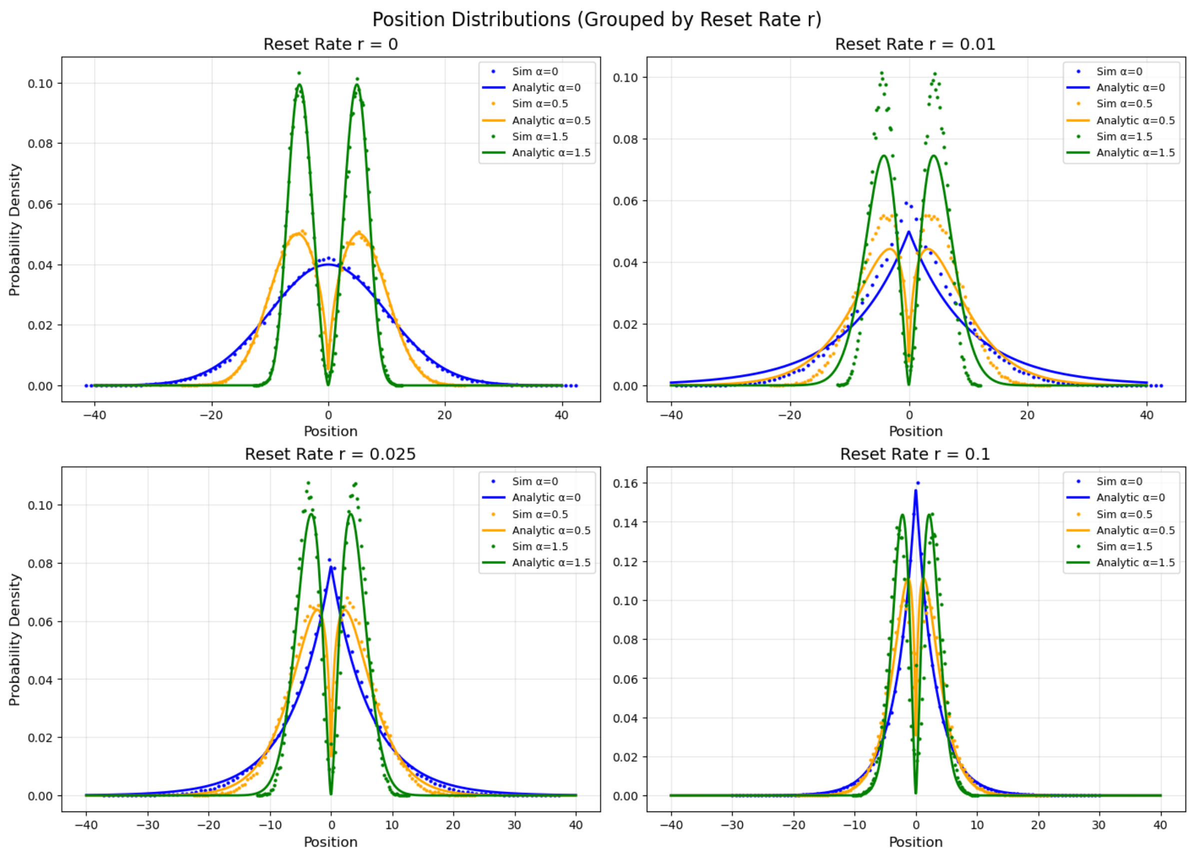

Figure 4: Probability distribution of positions of sluggish random walks with resetting grouped by reset probabilities $r = \{0, 0.01, 0.025, 0.1\}$ for different levels of sluggishness ($\alpha \in \{0, 0.5, 1.5\}$).

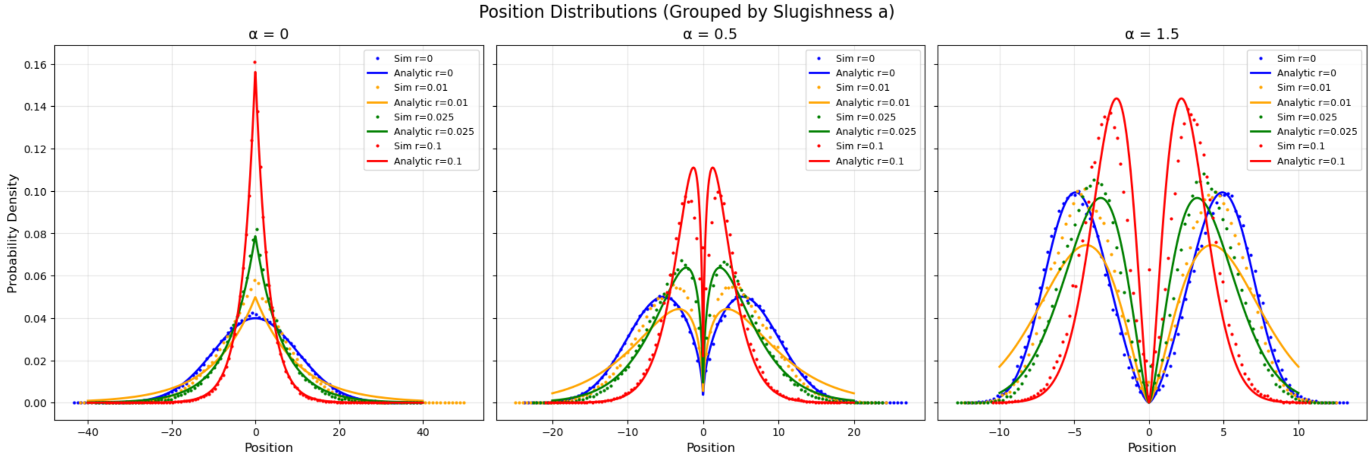

Figure 5: Probability distribution of positions of sluggish random walks with resetting grouped by different levels of sluggishness ($\alpha \in \{0, 0.5, 1.5\}$) for reset probabilities $r = \{0, 0.01, 0.025, 0.1\}$.

Python notebook

References

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