UNIVERSITY OF GHANA

BSc, FIRST SEMESTER EXAMINATIONS: 2018/2019

SCHOOL OF ENGINEERING SCIENCES

FAEN 101: Algebra (4 credits)

INSTRUCTION:

ANSWER ANY 4 OUT OF THE FOLLOWING 6 QUESTIONS

TIME ALLOWED:

THREE (3) HOURS

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1. (a) Solve the equations (i) $4^x - 5(2^x) + 4 = 0$ and (ii) $\log_4(3x+4) = \log_2(2x+1)$.

   (b) Evaluate the following limits:

   $$\text{(i)} \quad \lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \quad \text{(ii)} \quad \lim_{x\to\infty} \frac{-5x+2}{5x^2+2x+9} \quad \text{(iii)} \quad \lim_{x\to 1} \frac{x-1}{2-\sqrt{x+3}}$$

   (c) (i) Find the solution set of the inequality $\left|\dfrac{x+6}{3x-24}\right| < 1$.

   (ii) Illustrate on a diagram the region D in a plane corresponding to all points (x, y) such that $0 < x, \quad |x+y| \leq 1$ and $3x - y \leq 4$.
   Find the least and the greatest values of $2x + 3y$ on D.

2. (a) (i) Expand and simplify $(7+\sqrt{5})(3-\sqrt{5})$.

   (ii) Express $\dfrac{7+\sqrt{5}}{3+\sqrt{5}}$ in the form $a + b\sqrt{5}$, where a and b are integers.

   (b) Find the *domain* and the *co-domain* of the relation

   $$h :\to 4 - \frac{2}{3x+1} + \frac{2}{x+1}$$

   such that h is a *one-to-one* function(prove it). Find the inverse function, h^{-1}, and use it to find the element in the domain with 3 as its image.

   (c) Given

   $$g(x) = \frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6}, \quad x > 3$$

   (i) show that $g(x) = \dfrac{x+1}{x-2}, \quad x > 3$

   (ii) find the range of g

   (iii) find the exact value of a for which $g(a) = g^{-1}(a)$.

3. (a) Expand the function

$$f(x) = \left(\frac{1+x}{2-x}\right)^2$$

in ascending powers of x up to and including the terms in x^3. State the values of x for which the expansion is valid.

(b) In the expansion of $(1+ax)^b$, the coefficients of x and x^2 are -4 and 12 respectively. Find

(i) the values of a and b,

(ii) the range of values of x for which the expansion is valid,

(iii) the terms in x^3 and x^4.

(c) Sketch the graph of

$$f(x) = \frac{x^2+2}{x^2-4}$$

labeling clearly all relevant points and lines.

4. (a) (i) If $2\log(x-y) = \log x + \log y$, then show that $2\log(x+y) = \log 5 + \log x + \log y$.

(ii) If $x = \log_a(bc)$, $y = \log_b(ca)$, and $z = \log_c(ab)$, show that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1.$$

[Hint: Express $x+1$, $y+1$ and $z+1$ as logarithms.]

(iii) Find the value(s) of x satisfying the inequality $\log_2 x - 3\log_x 2 \geq 2$.

(b) Expand $(3+x)^{-2}$ in ascending powers of $1/x$, stating the first four terms only and the values of x for which the expansion is valid.

(c) State the period, domain and range of the trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(i) Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$ without using tables or calculators.

(ii) If α is in the 2nd quadrant with $\sin(\alpha) = \frac{5}{13}$, and β is in the 3rd quadrant with $\tan(\beta) = 2$, find $\sin(\alpha - \beta)$.

(iii) Derive a formula for $\tan(\alpha + \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

5. (a) Express $\dfrac{2-3i}{1+2i}$ in the form $a+bi$ where $a, b \in \mathbb{R}$.

(b) (i) Specify on an Argand diagram the points representing the complex numbers

$$z_1 = 1 - i\sqrt{3}, \quad z_2 = -1 + i$$

and express the arguments $\theta_1, \theta_2 \in (-\pi, \pi]$ of z_1, z_2 respectively as scalar multiples of π.

(ii) Using your results in (i), or otherwise, express the complex number

$$w = \frac{1 - i\sqrt{3}}{-1+i}$$

in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and θ is a principal argument.

(c) (i) Show that the n nth roots of unity are given by

$$w_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), k = 0, 1, 2, \cdots n-1.$$

(ii) Find the n nth roots of unity for $n = 3, 4$, and 5.

6. (a) Given the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$:

(i) evaluate $\mathbf{a} \times \mathbf{b}$, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, and $(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$

(ii) find the angle between a and b

(iii) find the unit vector perpendicular to the plane of the vectors b and c.

(b) Relative to a fixed origin O, the point A has position vector $(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$, the point B has the position vector $(5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k})$, and the point D has the position vector $(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$. The line l passes through the points A and B.

(i) Find the vector $\overrightarrow{AB}$.

(ii) Find a vector equation of the line l.

(iii) Show that the size of the angle BAD is $109°$, to the nearest degree.

(c) The points A, B and D, together with a point C are the vertices of a parallelogram $ABCD$, where $\overrightarrow{AB} = \overrightarrow{DC}$.

(i) Find the position vector of C.

(ii) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures.

(iii) Find the shortest distance from the point D to the line l, giving your answer to 3 significant figures. [Note: i, j, k are unit vectors along the coordinate axes.]