UNIVERSITY OF GHANA

BACHELOR OF SCIENCE IN ENGINEERING
FIRST SEMESTER 2018/2019 EXAMINATIONS
DEPARTMENT OF COMPUTER ENGINEERING
LEVEL 300: FAEN 301: NUMERICAL METHODS
3 CREDITS

TIME ALLOWED: 3 HOURS

INSTRUCTION: Attempt ALL Questions in Section A and ONE question in Section B

SECTION A: Answer all questions in this section

A1. (a) Given the following function $f(x)$ below. Sketch the graph of the following function

$$f(x) = x^2 + x - 3 \qquad x_0 = 1, \quad x_1 = 2$$

[5 marks]

(b) Verify from the graph that the interval endpoints at x_0 and x_1 have opposite signs. Use the bisection method to estimate the root (to 4 decimal places) of the equation that lies between the endpoints given. (Perform only 3 iterations) [5 marks]

(c) Use the secant method to estimate the root (to 4 decimal places) of the equation that lies between the endpoints given. (Perform 2 iterations) [6 marks]

(d) Use Newton-Raphson's method/iterative scheme to solve the following system of nonlinear equations

$$f_1(x_1, x_2) = x_1 + x_2 + x_1^2 + 6x_2^2 - 9$$
$$f_2(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 - 4$$

Use the initial starting point as $x_1 = 1, x_2 = 2$. (Perform only 3 iterations) [12 marks]

(e) Which of the two methods of rootfinding (Secant and Newton-Raphson) would you recommend for solving higher dimensional nonlinear equations and why?

[2 marks]

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A2. Given the matrix below

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(a) Show that the matrix is positive definite. [5 marks]

(b) Compute explicitly the eigenvalues and determine the determinant, [5 marks]

(c) Compute the corresponding eigenvectors of the matrix above [10 marks]

(d) Choose a, b and c in the matrix $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ so that the characteristic polynomial of B is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$. [5 marks]

(e) Based on Cayley-Hamilton's theorem, every matrix fulfills its characteristic polynomial, using the above characteristic polynomial or otherwise, find the inverse of the matrix **B** above. [10 marks]

(f) Find an LU-factorization of the matrix A using elementary row operations, where

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[10 marks]

Use the LU factorization obtained above to solve the following system of equations

$$\begin{aligned} 3x_1 - 2x_2 + 0x_3 &= -1 \\ -2x_1 + 2x_2 + 1x_3 &= 7 \\ 0x_1 + 1x_2 + 2x_3 &= 12 \end{aligned}$$

[10 marks]

A3. Suppose that the current through a resistor is described by the function

$$i(t) = 2t + 1$$

and the resistance is a function of the current:

$$R = 10 + 2i(t)$$

(a) Compute the exact value of the integral given by the expression for the voltage $V(t) = i(t) \times R$ [6 marks]

(b) Compute the average voltage over $t = 0$ to $t = 1$ given as:

$$V_{ave}(t) = \frac{1}{T}\int_0^T V(t)dt$$

by using the composite Simpson's 1/3 rule. Use $n = 5$ divisions and step size $h = 0.2$ *Hint: The voltage is given by* $V(t) = i(t) \times R$ [10 marks]

(c) For the integral above, use the Romberg integration method(based on the trapezoidal rule) to get a more accurate estimate of the integral. Use Romberg integration to estimate $\mathbf{R}(2,1)$. [10 marks]

(d) Compute the true error e_t for the new estimate if any. [2 marks]

(e) Assume that the above integral were to be evaluated using Gauss quadrature method with $n = 3$. Transform the integral limits so that Gauss quadrature method could be used. [10 marks]

A4. (a) Mr. Paul's diet comes from the four basic food groups - chocolate dessert, ice cream, soda, and cheesecake. He checks in a store and finds one of each kind of food, namely, a brownie, chocolate ice cream, Pepsi, and one slice of pineapple cheesecake. Each day he needs at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. Using the table below that gives the cost and nutrition of each item, find out how much he should buy and eat of each of the four items he found in the store so that he gets enough nutrition but spends as little money as possible.

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	\$2.50 / brownie
Chocolate ice cream	200	2	2	4	\$1.00 / scoop
Coke	150	0	4	1	\$1.50 / bottle
Pineapple cheesecake	500	0	4	5	\$4.00 / slice

i. Formulate the standardized linear programming problem based on the information given above. [8 marks]

ii. Find the optimal solution using the Simplex method. [20 marks]

iii. Formulate the dual linear programming problem without solving it. [5 marks]

iv. Write out the MATLAB solution scripts for solving both the primal and dual problems as enumerated above. [7 marks]

(b) Consider the function

$$f(x_1, x_2) = 2x_1^3 + 3x_2^2 + 3x_1^2x_2 - 24x_2$$

over $\Re^2$.

i. Find all the stationary points and classify them. [6 marks]

ii. Find the second order conditions, the Hessian matrix [6 marks]

iii. Check whether the Hessian matrix is **positive definite**, **negative definite** or neither. Comment on your results. [4 marks]

(c) Consider the problem

$$\mathit{Minimize} \quad x_1^2 + x_2^2 + x_3^2$$

subject to:

$$x_1 + 2x_2 + 3x_2 \geq 4$$

$$x_3 \leq 1$$

i. Write down the standardized formulation for this minimization problem by making sure that the constraints are in their proper form. **[4 marks]**

ii. Formulate the Lagrangian function of the primal problem **[2 marks]**

iii. Write down the Karush-Kuhn-Tucker (KKT) conditions. **[6 marks]**

iv. (Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions. **[4 marks]**

v. Find the optimal solution of the problem using the KKT system. **[5 marks]**

SECTION B: Numerical Methods and Applications

B1. Consider the problem

$$\textit{Minimize} \qquad x_1^2 + 2x_2^2 + 2x_1$$

subject to:

$$x_1 + x_2 \leq a$$

where $a \in R$ is a parameter.

(a) Prove that for any $a \in R$, the problem has a unique optimal solution (without actually solving it) by formulating the Lagrangian function and finding the first and second order necessary conditions. By judging from the obtained expressions from the conditions applied, can the resulting expressions be solved or not, make your conclusions. [5 **marks**]

(b) Solve the problem (the solution will be in terms of the parameter a. [6 **marks**]

(c) Let $f(a)$ be the optimal value of the problem with parameter a. Write an explicit expression for f and prove that it is a convex function. [5 **marks**]

(d) Write down the formulas for the forward difference, backward difference and central difference approximations for numerical differentiation. [3 **marks**]

(e) Compute the exact derivative of $f(x) = x\sin(x)$ at $x = \frac{\pi}{3}$ radians. [3 **marks**]

(f) Compute the derivative of $f(x) = x\sin(x)$ at $x = \frac{\pi}{3}$ radians using

i. the central difference approximation. Use the step size $h = 0.1$.

ii. the forward difference approximation and the Richardson extrapolation algorithm. Use $h = 0.1$ and find $\mathbf{D}(2, 1)$

[8 **marks**]

B2. Given the table below obtained from laboratory measurements of bacteria growth

x	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
y	6.0	8.0	11.0	14.0	16.0	19.0	22.0	25.0

(a) Plot a graph of the dependent variable y against the independent variable x

[4 **marks**]

(b) From first principles, fit a straight line to the data of the form $y = a_0 + a_1x$

[8 **marks**]

(c) Compute the value of y when $x = 11$.

[2 **marks**]

(d) Thermistors are used to measure the temperature of hot bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature versus resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

R/Ω	$T/°C$
1101	25
910	30
635	40
450	50

Table 1: Temperature and Corresponding Resistances Values

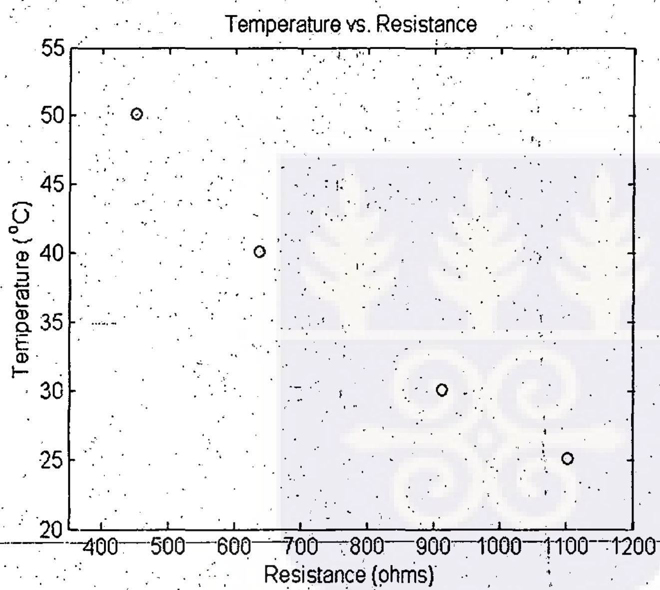

Figure 1: Plot of Temperature versus Resistance

i. Draw the divided difference table for the above data and compute the entries. [4 marks]

ii. Obtain the Newton's interpolating polynomial of order 3 using all the 4 points in the table. [3 marks]

iii. Calculate the value of the interpolating polynomial when the independent variable $R = 800\Omega$ using the polynomials obtained in questions **ii**. [4 marks]

iv. Determine the temperature corresponding to $R = 800\Omega$ using a first order Lagrange polynomial. [5 marks]

B3. (a) Solve exactly the ODE problem given below using the integration factor method or any method of your choice

$$\frac{dy}{dx} = 1 + x^2 + y \qquad y(1) = -4$$

[6 marks]

(b) Use Euler method to solve the following ODE given below:

$$\frac{dy}{dx} = 1 + x^2 + y \qquad y(1) = -4$$

to determine $y(1.01), y(1.02)$ and $y(1.03)$. Use step size of $h = 0.01$. Generate a table and compute the error based on the exact results obtained. [6 marks]

(c) Use second order Taylor series method to solve

$$\frac{dy}{dx} = 1 + x^2 + y \qquad y(1) = -4$$

to determine $y(1.01), y(1.02)$ and $y(1.03)$ with step size $h = 0.01$ [6 marks]

(d) Use Runge-Kutta's method to solve the following ODE

$$\frac{dy}{dx} = 1 + x^2 + y \qquad y(1) = -4$$

to determine $y(1.01), y(1.02)$ and $y(1.03)$ with step size $h = 0.01$ [6 marks]

(e) Transform the following initial value problem (third order equation) into 3 first order linear ODEs

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y(t) = 52t$$

given the following initial conditions

$$\frac{d^2y(0)}{dt^2} = -4, \frac{dy(0)}{dt} = 2, y(0) = 10$$

[6 marks]

Useful Numerical Methods and Corresponding Formulas

Richardson Extrapolation Formulas for improving the accuracy of low order numerical schemes.

Hold $f(x)$, and x fixed:

$$\varphi(h) = \frac{f(x+h) - f(x-h)}{2h} \tag{1}$$

$$\varphi(h) = f'(x) - a_2h^2 - a_4h^4 - a_6h^6 - \dots \tag{2}$$

$$\varphi(\frac{h}{2}) = f'(x) - a_2\left(\frac{h}{2}\right)^2 - a_4\left(\frac{h}{2}\right)^4 - a_6\left(\frac{h}{2}\right)^6 - \dots \tag{3}$$

$$\varphi(h) - 4\varphi(\frac{h}{2}) = -3f'(x) - \frac{3}{4}a_4h^4 - \frac{15}{16}a_6h^6 - \dots \tag{4}$$

$$\rightarrow f'(x) = \frac{4}{3}\varphi(h/2) - \frac{1}{3}\varphi(h) + O(h^4) \tag{5}$$

$D(0,0) = \varphi(h)$			
$D(1,0) = \varphi(h/2)$	$D(1,1)$		
$D(2,0) = \varphi(h/4)$	$D(2,1)$	$D(2,2)$	
$D(3,0) = \varphi(h/8)$	$D(3,1)$	$D(3,2)$	$D(3,3)$

First Column:

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) \tag{6}$$

Others:

$$D(n,m) = \frac{4^m}{4^m-1}D(n,m-1) - \frac{1}{4^m-1}D(n-1,m-1) \tag{7}$$

Recursive Trapezoid Method

$$h = \frac{b-a}{2^n} \tag{8}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{9}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{10}$$

$$h = b - a \tag{11}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{12}$$

$$h = \frac{b-a}{2} \tag{13}$$

$$R(1,0) = \frac{1}{2}R(0,0) + h\left[\sum_{k=1}^{1} f\left(a + (2k-1)h\right)\right] \tag{14}$$

$$h = \frac{b-a}{2^2} \tag{15}$$

$$R(2,0) = \frac{1}{2}R(1,0) + h\left[\sum_{k=1}^{2} f\left(a + (2k-1)h\right)\right] \tag{16}$$

$$h = \frac{b-a}{2^3} \tag{17}$$

$$R(3,0) = \frac{1}{2}R(2,0) + h\left[\sum_{k=1}^{2^2} f\left(a + (2k-1)h\right)\right] \tag{18}$$

$$h = \frac{b-a}{2^n}, R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{19}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{20}$$

$$h = \frac{b-a}{2^n}, \tag{21}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{22}$$

$$R(n,m) = \frac{1}{4^m - 1}\left[\, 4^m \times R(n,m-1) - R(n-1,m-1)\right], n \geq 1, m \geq 1 \tag{23}$$

$R(0,0)$			
$R(1,0)$	$R(1,1)$		
$R(2,0)$	$R(2,1)$	$R(2,2)$	
$R(3,0)$	$R(3,1)$	$R(3,2)$	$R(3,3)$

GAUSS QUADRATURE TABLE OF WEIGHTS AND FUNCTION ARGUMENTS

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$ $c_2 = 1 : 000000000$	$x_1 = -0.577350269$ $x_2 = 0.577350269$
3	$c_1 = 0.555555556$ $c_2 = 0.888888889$ $c_3 = 0.555555556$	$x_1 = -0.774596669$ $x_2 = 0.000000000$ $x_3 = 0.774596669$
4	$c_1 = 0.347854845$ $c_2 = 0.652145155$ $c_3 = 0.652145155$ $c_4 = 0.347854845$	$x_1 = -0.861136312$ $x_2 = -0.339981044$ $x_3 = 0.339981044$ $x_4 = 0.861136312$
5	$c_1 = 0.236926885$ $c_2 = 0.478628670$ $c_3 = 0.568888889$ $c_4 = 0.478628670$ $c_5 = 0.236926885$	$x_1 = -0.906179846$ $x_2 = -0.538469310$ $x_3 = 0.000000000$ $x_4 = 0.538469310$ $x_5 = 0.906179846$
6	$c_1 = 0.171324492$ $c_2 = 0.360761573$ $c_3 = 0.467913935$ $c_4 = 0.467913935$ $c_5 = 0.360761573$ $c_6 = 0.171324492$	$x_1 = -0.932469514$ $x_2 = -0.661209386$ $x_3 = -0.2386191860$ $x_4 = 0.2386191860$ $x_5 = 0.661209386$ $x_6 = 0.932469514$

LAGRANGE INTERPOLATING POLYNOMIALS

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted.

EULER'S METHOD

$$y(x_0 + h) = y(x_0) + h\frac{dy}{dx}|_{x=x_0,y=y_0}| + O(h^2)$$

Notation:

$$x_n = x_0 + nh \qquad y_n = y(x_n)$$

$$\frac{dy}{dx}|_{x=x_i,y=y_i}| = f(x_i, y_i)$$

Euler's Method

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

4th ORDER RUNGE-KUTTA FORMULA FOR SOLVING ODES

$$k_1 = f(x_i, y_i) \tag{24}$$

$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_1 h) \tag{25}$$

$$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_2 h) \tag{26}$$

$$k_4 = f(x_i + h, y_i + k_3 h) \tag{27}$$

(28)

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Local error is $O(h^5)$ and global error is $O(h^4)$