UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2015/2016

SCHOOL OF ENGINEERING

DEPARTMENT OF BIOMEDICAL ENGINEERING

FAEN 201: CALCULUS II (4 credits)

INSTRUCTION:

ANSWER ANY FIVE OUT OF THE FOLLOWING SEVEN QUESTIONS

TIME ALLOWED:

THREE (3) HOURS

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1. (a) Evaluate the limit of the sequence:

$$a_n = \frac{3^n + 4^{n+1}}{3^n - 4^n}$$

for all $n \in \mathbb{N}$. *(8 marks)*

(b) Evaluate the limit of the sequence:

$$a_n = \sqrt{n^2 + 3n} - \sqrt{n^2 - 3n}$$

for all $n \in \mathbb{N} - \{1, 2\}$. *(12 marks)*

(c) Give the $\varepsilon - N$ definition of a convergent sequence. Show, using your definition, that the sequence $a_n = \dfrac{2n+3}{5n-4}$ converges to $\frac{2}{5}$. *(20 marks)*

2. (a) Let p be a real number such that $p > -1$ and $p \neq 0$. Prove by induction that

$$(1+p)^n > 1 + np$$

for every integer $n \geq 2$. *(15 marks)*

(b) Using your results in (a) above, show that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ for all integers $n \geq 1$ is a monotone increasing sequence. *10 marks)*

1c

(c) i. Use the comparison test to test for convergence for

$$\int_3^4 \frac{\ln x}{(x-3)^4}.$$

(7 marks)

ii. Use the ratio criterion test to find the interval of absolute convergence of the power series

$$\sum_{n=1}^{\infty}\left(\frac{x^n}{n^3 \cdot 3^{n-1}}\right).$$

(8 marks)

3. (a) Find f_x, f_y, f_{xx} and f_{yy} of

i. $f(x,y) = xe^{xy}$

ii. $(f(x,y) = \sqrt{x^2+y^2}$ *(12 marks)*

(b) Given that $f(x,y) = x^3 + y^3 - 6xy$ where $x = cost$ and $y = sint$, find

i. $\dfrac{df}{dt}$ *(8 marks)*

ii. the value of $\dfrac{df}{dt}$ when $t = \frac{\pi}{2}$. *(4 marks)*

iii. Find the critical point(s) of $f(x,y)$

(10 marks)

iv. Determine the nature of the critical point(s) of $f(x,y)$.

(6 marks)

4. (a) Give a precise definition of the Gamma function Γ.
Prove that $\Gamma(v+1) = v\Gamma(v)$ where $v > 0$. *(16 marks)*

(b) Using your results (b) above and the fact that $\Gamma(v+1) = v!$ for $v = 1, 2, 3, \ldots$, evaluate

i. $\int_0^{\infty} x^6 e^{-2x}\, dx$

ii. $\int_0^{\infty} \sqrt{y} e^{-y^2}\, dy$.

(12 marks)

(c) Given that

$$\int_0^{\frac{\pi}{2}} \sin^{2u-1}\theta \cos^{2v-1}\theta\, d\theta = \frac{1}{2}\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

evaluate

i. $\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta\, d\theta$

ii. $\int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta$. *(14 marks)*

5. (a) The temperature at the point (x, y, z) in a solid piece of metal is given by $f(x, y, z) = \frac{1}{\sqrt{5}} e^{3x+2y+z}$ degrees.

   i. In what direction at the point $(0, 0, 0)$ does the temperature increase most rapidly?

   ii. Find the rate of increase in temperature in the solid metal. *(12 marks)*

   (b) Find the directional derivative of $f(x, y) = 3xy^3 + y^2z^2$ at the point $(1, -2, -2)$ in the direction from that point towards the origin. *(15 marks)*

   (c) Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2, -1)$ to $(2, 4)$ and ds is a path on C. *(13 marks)*

6. (a) Compute the double integral

$$\iint_R \left[x^2y^2 + \cos(\pi x) + \sin(\pi y)\right] dA,$$

   where $R = [-2, -1] \times [0, 1]$. *(15 marks)*

   (b) Using Green's theorem and polar coordinates or otherwise, evaluate $\oint_C y^3 dx - x^3 dy$, where C is the positively oriented circle of radius 2 centred at the origin. *(13 marks)*

   (c) For the vector function $F = (yz^2)\mathrm{i} + (xy)\mathrm{j} + (yz)\mathrm{k}$, compute $\operatorname{div}(\operatorname{curl} F)$. *(12 marks)*

7. (a) Use Green's theorem to evaluate $\oint_C xy\, dx + x^2y^3\, dy$ where C is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$ positively oriented. *(13 marks)*

   (b) Compute the Laplacian of the function $f(x, y, z) = x^3 - 3x^2y^2 + z^3$. *(12 marks)*

   (c) How much work is accomplished by the force $F(x, y) = (2xy^2)\mathrm{i} + (xy)\mathrm{j}$ in pushing a particle from $(0, 0)$ to $(3, 9)$ along the parabola $y = x^2$? *(15 marks)*