# UNIVERSITY OF GHANA



## Faculty of Engineering Sciences
## Department of Computer Engineering
## First Semester Exams 2012/2013 Academic Year

**Course: FAEN 301 Numerical Methods (3 Credits)**

**TIME: 3 hours**

**INSTRUCTION: Please attempt all questions**

**Question 1(15 marks)**

a. Sketch the graph of the following function **[2marks]**

$$f(x) = x^2 + x - 3 \qquad x_0 = 1, \quad x_1 = 2$$

b. Verify from the graph that the interval endpoints at $x_0$ and $x_1$ have opposite signs. **[1 mark]**

c. Use Newton's method with three (3) to estimate the root to 4 decimal places. **[3 marks]**

d. Use the secant method with three (3) to estimate the root to 4 decimal places **[3 marks]**

e. Use Newton's method to solve the following system of nonlinear equations

$$f_1(x_1, x_2) = x_1 + x_2 + x_1^2 + 6x_2^2 - 9$$

$$f_2(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 - 4$$

Use the initial starting point as $x_1^0 = x_2^0 = 0.5$ (Perform 3 iterations) **[6 marks]**

**Question 2 (15 marks)**

a. Using LU decomposition by hand computations to obtain the lower (L) and upper triangular (U) matrices for the system below. **[10 marks]**

$$-x_1 + 2x_2 - 3x_3 + 5x_4 = 14$$
$$x_1 + 3x_2 + 2x_3 - x_4 = 9$$
$$3x_1 - 3x_2 + 2x_3 + 4x_4 = 19$$
$$4x_1 + 2x_2 + 5x_3 + x_4 = 27$$

b. Carefully outline the solution steps and solve completely for the unknown values of $x_1, x_2, x_3$, and $x_4$ **[5marks]**

**Question 3 (16 marks)**

(a) Derive from first principles the linear least squares regression equations, showing clearly the necessary steps. **[4 marks]**

(b) A level 300 student conducted an experiment in the lab. The student measured the amount of substance of reagents (X mol/dm$^3$) and the corresponding amount of substance of product formed from the chemical reaction output (Y mol/dm$^3$). The result of the experiment is summarized in the table below.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| X | 0.282 | 0.555 | 0.089 | 0.157 | 0.357 | 0.572 | 0.222 | 0.800 | 0.266 | 0.056 |
| Y | 0.685 | 0.563 | 0.733 | 0.722 | 0.662 | 0.588 | 0.693 | 0.530 | 0.650 | 0.713 |

i. Using linear least squares, compute the coefficients that best fits the straight line equation $y = a_0 + a_1 x$ to the observed data. **[4 marks]**

ii. What is the value of the amount of substance of output Y when the amount of substance used is $X = 1.234$? **[1 mark]**

iii. Using linear least squares, compute the coefficients that best fits the second order polynomial $y = a_0 + a_1 x + a_2 x^2$ to the observed data. **[7 marks]**

**Question 4 (24 marks)**

(a) A rectangular prism-type box is to be designed with the largest volume. The sum of its length ($l$), width ($w$), and height ($h$) is limited to a maximum value of 60cm and its length is restricted to a maximum value of 36cm.

(i) Formulate an optimization problem by identifying clearly the *objective function* and the various *constraints* highlighted in the problem.

(ii) Utilizing the itemized conditions of optimality obtained above in question 4(a) above find the dimensions $l$, $w$ and $h$, and the maximum volume possible. **[8 marks]**

(b) Find the dimensions of the box with largest volume if the total surface area is 6000 cm$^2$.
*Hint: Identify the objective function and the constraints regarding the surface area and apply Lagrange multiplier method.* **[8 marks**

(c) Extremize (optimize) the following function subject to the constraints given below using Lagrange multiplier method. Indicate whether the optimal values obtained $x_1$ and $x_2$ minimize or maximize the objective function. State all assumptions necessary for the

$$f(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 14x_1 - 14x_2 + 10$$

Subject to:

$$4x_1^2 + x_1^2 - 25 \leq 0 \leq$$

$$x_1, x_2 \geq 0$$

**[8 marks]**

**Question 5 (25 marks)**

(a) Derive from first principles the ***Simpson's 1/3 rule*** for numerical integration. **[5 marks]**

(b) Given the following integral.

$$I = \int_0^1 \frac{2x+4}{x^2+6x+13}\,dx$$

(i) Compute the exact integral given above. **[3 marks]**

(ii) Approximate the value of the integral using Trapezoidal rule with 10 evenly spaced intervals **[5 marks]**

(iii) Approximate the value of the integral using Simpson's 1/3 rule with 10 evenly spaced intervals **[5 marks]**

(iv) Briefly discuss the differences between the *exact integral calculated* and the numerically evaluated integrals using the two techniques above.

(v) Which of the numerical techniques utilized above is more accurate with respect to the calculation obtained above? **[1 mark]**

(c) Using the two (2)-point Gauss quadrature rule, estimate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by:

$$x = \int_8^{30} \left( 2000 \ln\left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right) dt$$

**[6 marks]**

**Question 6 (15 marks)**

(a) A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right),$$

$$\theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds with step size $h = 240$ seconds using the following:

(i) Euler's method **[4 marks]**

(ii) Fourth (4th) order Runge-Kutta method **[9 marks]**

(b) The exact integral is given by:

$$0.92593 \ln\left(\frac{\theta - 300}{\theta + 300}\right) - 1.8519 \tan^{-1}(0.00333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

(i) Calculate the errors obtained using the above two (2) methods and comment on your results. **[2 marks]**

## TABLE OF GAUSS QUADRATURE POINTS AND WEIGHTS

| Number of Points | Points, $x_i$ | Weights $\omega_i$ |
|---|---|---|
| 1 | $0$ | $2$ |
| 2 | $\pm 1/\sqrt{3}$ | $1$ |
| 3 | $0$<br>$\pm\sqrt{3/5}$ | $8/9$<br>$5/9$ |
| 4 | $\pm\sqrt{\left(3-2\sqrt{6/5}\right)/7}$<br>$\pm\sqrt{\left(3+2\sqrt{6/5}\right)/7}$ | $\frac{18+\sqrt{30}}{36}$<br>$\frac{18-\sqrt{30}}{36}$ |
| 5 | $0$<br>$\pm\frac{1}{3}\sqrt{\left(5-2\sqrt{10/7}\right)}$<br>$\pm\frac{1}{3}\sqrt{\left(5+2\sqrt{10/7}\right)}$ | $128/225$<br>$\frac{322+13\sqrt{70}}{900}$<br>$\frac{322-13\sqrt{70}}{900}$ |
| **Points** | **Weighting Factors** $\omega_i$ | Points, $x_i$ **Function Arguments** |
| 2 | $c_1 = 1.000000000$<br>$c_2 = 1.000000000$ | $x_1 = -0.577350269$<br>$x_2 = 0.577350269$ |
| 3 | $c_1 = 0.555555556$<br>$c_2 = 0.888888889$<br>$c_3 = 0.555555556$ | $x_1 = -0.774596669$<br>$x_2 = 0.000000000$<br>$x_3 = 0.774596669$ |
| 4 | $c_1 = 0.347854845$<br>$c_2 = 0.652145155$<br>$c_3 = 0.652145155$<br>$c_4 = 0.347854845$ | $x_1 = -0.861136312$<br>$x_2 = -0.339981044$<br>$x_3 = 0.339981044$<br>$x_4 = 0.861136312$ |
| 5 | $c_1 = 0.236926885$<br>$c_2 = 0.478628670$<br>$c_3 = 0.568888889$<br>$c_4 = 0.478628670$<br>$c_5 = 0.236926885$ | $x_1 = -0.906179846$<br>$x_2 = -0.538469310$<br>$x_3 = 0.000000000$<br>$x_4 = 0.538469310$<br>$x_5 = 0.906179846$ |
| 6 | $c_1 = 0.171324492$<br>$c_2 = 0.360761573$<br>$c_3 = 0.467913935$<br>$c_4 = 0.467913935$<br>$c_5 = 0.360761573$<br>$c_6 = 0.171324492$ | $x_1 = -0.932469514$<br>$x_2 = -0.661209386$<br>$x_3 = -0.2386191860$<br>$x_4 = 0.2386191860$<br>$x_5 = 0.661209386$<br>$x_6 = 0.932469514$ |