UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2016/2017

SCHOOL OF ENGINEERING

DEPARTMENT OF BIOMEDICAL ENGINEERING

FAEN 201: CALCULUS II (4 credits)

INSTRUCTION:

ANSWER ANY FIVE OUT OF THE FOLLOWING SEVEN QUESTIONS

TIME ALLOWED:

THREE (3) HOURS

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1. (a) Let $a_n = \dfrac{2n-7}{3n+2}$ be a sequence of real numbers for all integers $n \geq 1$.

   i. Show that a_n is a monotone increasing sequence. *(8 marks)*

   ii. Use the $\epsilon - N$ definition of a convergent sequence to show that the sequence a_n converges to $\dfrac{2}{3}$. *(12 marks)*

   (b) Test for convergence or divergence for the series $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\ln n}{2n^3-1}\right]$. *(10 marks)*

   (c) Let p be a real number such that $p > 1$ and $p \neq 0$. Prove by induction that for every integer $n \geq 2$, $(1+p)^n > 1 + np$. *(10 marks)*

2. (a) Evaluate

   i. $\displaystyle\lim_{n\to\infty}\left\{\frac{7^{n+1}+2^n}{7^n-2^n}\right\}$, $n \in \mathbb{N}$; *(5 marks))*

   ii. $\displaystyle\lim_{n\to\infty}\left[\sqrt{n^2+3n}-\sqrt{n^2-3n}\right]$, $n \in \mathbb{N},\ n \geq 3$. *(10 marks)*

   (b) Given that $f(x,y) = x^2\sin(x+y)$,

   i. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. *(4 marks)*

   ii. Hence or otherwise, show that $x\dfrac{\partial f}{\partial x} = [x + 2\tan(x+y)]\dfrac{\partial f}{\partial y}$. *(6 marks)*

   (c) Test for convergence or divergence for each improper integral:

   i. $\displaystyle\int_2^{\infty}\frac{\cos^2 x}{x^2}\,dx$; ii. $\displaystyle\int_3^{\infty}\frac{dx}{x+e^x}\,dx$. *(15 marks)*

3. (a) Let $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

   i. Find the critical points of $f(x, y)$. *(12 marks)*

   ii. Determine the nature of each critical point found in (i). *(8 marks)*

   (b) i. State Green's theorem in the $xy-$plane. *(3 marks)*

   ii. Verify Green's theorem for $\oint_C 4x^2y\,dx + 2y\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,2)$ and $(0,2)$. *(17 marks)*

4. (a) For the series $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

   i. Find a_n, the n^{th} term of the series; *(3 marks)*

   ii. Express a_n in partial fractions; *(8 marks)*

   iii. Find $S_n = \sum_{r=1}^{n} a_r$, the sum of the n^{th} term of the series; *(10 marks)*

   iv. Evaluate $\lim_{n\to\infty} S_n$. *(3 marks)*

   (b) Evaluate $\int_{y=0}^{y=8} \int_{x=0}^{x=4} \frac{1}{16}\left[64 - 8x + y^2\right] dx\,dy$. *(16 marks)*

5. (a) If $\phi = x^2yz^3$ and $A = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find

   i. $\nabla\phi$ ii. $\nabla \times A$ iii. $\text{div}(\phi A)$ *(14 marks)*

   (b) i. Compute the Laplacian $\nabla^2 f(x, y, z)$ of the function

$$f(x, y, z) = 2x^3y - 3y^2z$$

(8 marks)

   ii. Find the directional derivative of $f(x, y, z) = 2x^3y - 3y^2z$ at the point $P(2, 1, -1)$ in the direction from P towards the the point $Q(5, -1, 4)$. *(18 marks)*

6. (a) Give a precise definition of the Gamma function.

Show that $\int_0^M x^v e^{-x} dx = \left[-x^v e^{-x}\right]_0^M - \int_0^M (-e^{-x} v x^{v-1}) dx$ *(10 marks)*

(b) Given that $\dfrac{\Gamma(u)\Gamma(v)}{2\Gamma(u+v)} = \int_0^{\frac{\pi}{2}} \sin^{2u-1}\theta \cos^{2v-1}\theta d\theta$, find in terms of the Gamma function

i. $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^5\theta \, d\theta$;

ii. $\int_0^{\pi} \cos^4\theta \, d\theta$. *(12 marks)*

(c) An ellipse in the xy-plane has centre at the origin with semi-major and semi-minor axes of lengths 4 and 3 respectively.

i. Write down parametric equations for the ellipse in terms of the parameter $0 \le t \le 2\pi$. *(6 marks)*

ii. How much work is done by the force $F = (3x - 4y)\mathbf{i} + (4x + 2y)\mathbf{j}$ in moving a particle once around the ellipse in the counter clockwise direction? *(18 marks)*

7. (a) Use the ratio test to find the interval of convergence for the power series

i. $\sum_{n=1}^{\infty} \left[\dfrac{(x-5)^n}{n^2}\right]$ ii. $\sum_{n=1}^{\infty} n!(2x+1)^n$. *(14 marks)*

(b) Find $\int_C xy^3 \, ds$, where C is the quarter circle $C = [\cos t, \sin t]$, $0 \le t \le \dfrac{\pi}{2}$ and

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$ *(10 marks)*

(c) The temperature $T(°C)$ at position (x, y) in the $xy-$plane is given by $T(x, y) = x^2 e^{-y}$.

i. At the point $(2, 1)$, find the direction in which the temperature increases most rapidly. *(10 marks)*

ii. Find the rate of increase of the temperature in the direction found in (i). *(6 marks)*