UNIVERSITY OF GHANA

FIRST SEMESTER 2016/2017Examination

DEPARTMENT OF COMPUTER ENGINEERING

LEVEL 300: FAEN 301: NUMERICAL METHODS

3 CREDIT HOURS

TIME ALLOWED: 3 HOURS

INSTRUCTION: Attempt All Questions

Q1. (a) Given the following function $f(x)$ below. Sketch the graph of the following function

$$f(x) = x^2 + x - 3 \quad x_0 = 1, \quad x_1 = 2$$

[5 marks]

(b) Verify from the graph that the interval endpoints at x_0 and $x_1$have opposite signs. Use Newton's method to estimate the root (to 4 decimal places) of the equation that lies between the endpoints given. (Perform only 2 iterations)

[5 marks]

(c) Use Newton's method/iterative scheme to solve the following system of nonlinear equations

$$f_1(x_1, x_2) = x_1 + x_2 + x_1^2 + 6x_2^2 - 9$$
$$f_2(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 - 4$$

Use the initial starting point as $x_1 = x_2 = 0$. (Perform only 2 iterations)

[10 marks]

Q2. Thermistors are used to measure the temperature of hot bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature versus resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

(a) Draw the divided difference table for the above data and compute the entries.

[5 marks]

R/Ω	$T/°C$
1101	25
910	30
635	40
450	50

Table 1: Temperature and Corresponding Resistances Values

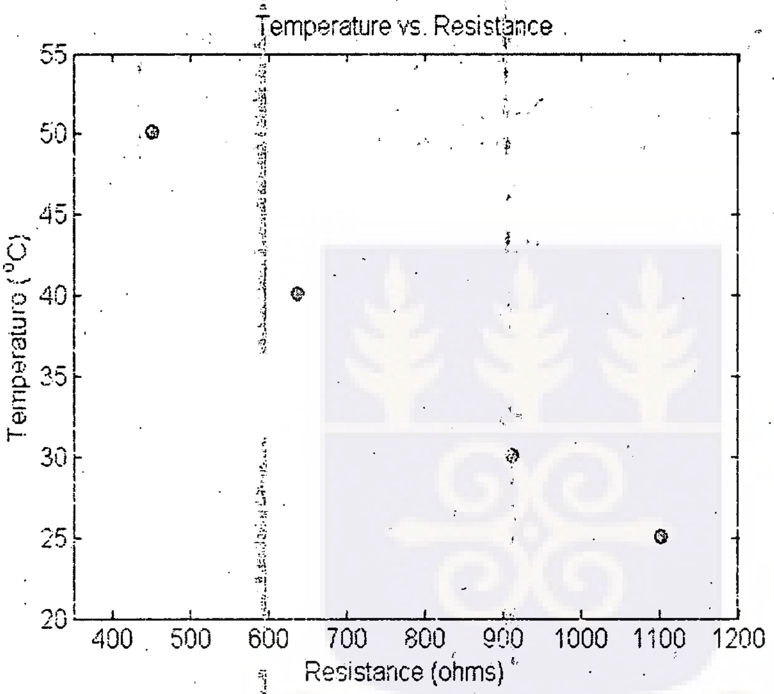

Figure 1: Plot of Temperature versus Resistance

(b) Obtain the Newton's interpolating polynomial of order 2 using the first 3 points in the table.

[3 marks]

(c) Obtain the Newton's interpolating polynomial of order 3 using all the 4 points in the table.

[3 marks]

(d) Calculate the value of the interpolating polynomial when the independent variable $R = 800\Omega$ using the polynomials obtained in questions b and c above.

[4 marks]

(e) Determine the temperature corresponding to $R = 800\Omega$ using a first order Lagrange polynomial. [5 marks]

B

Q3. Given the system of linear equations below

$$\begin{aligned} 2v_1 - v_2 + 3v_3 &= 4 \\ -4v_1 + 3v_2 + 2v_3 &= 1 \\ 3v_1 + v_2 - v_3 &= 3 \end{aligned}$$

(a) Use the Gaussian elimination method to find the solution vector v_1, v_2, and v_3.

[10 **marks**]

(b) Use LU decomposition by hand computations to obtain the lower and upper triangular matrices for the system above. Carefully outline the solution steps to solve completely for the unknown values of v_1, v_2, and v_3.. Hint Use ***elementary row operations*** to perform the calculation and arrive at the final solution.

[10 **marks**]

Q4. (a) Using the graphical method, solve the problem:

$$\text{maximize} \quad f(x_1, x_2) = 2x_1 + x_2$$

$$\text{subject to} \quad \begin{cases} x_2 & \leq 10 \\ 2x_1 + 5x_2 & \leq 60 \\ x_1 + x_2 & \leq 18 \\ 3x_1 + x_2 & \leq 44 \\ x_1, x_2 & \geq 0 \end{cases}$$

[10 **marks**]

(b) Given the following optimization problem, find the minimum and maximum values of the function subject to the constraints given by using the Lagrange multiplier method.

$$f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$$

Subject to

$$x_1^2 + x_2^2 + x_3^2 = 56$$

[10 **marks**]

Q5. (a) Write down the formulas for the forward difference, backward difference and central difference approximations for numerical differentiation. [3 **marks**]

(b) Compute the exact derivative of $f(x) = \sin(x)$ at $x = \frac{\pi}{3}$ radians. [3 **marks**]

(c) Compute the derivative of $f(x) = \sin(x)$ at $x = \frac{\pi}{3}$ radians using

i. the central difference approximation. Use the step size $h = 0.1$.

ii. the forward difference approximation and the Richardson extrapolation algorithm. Use $h = 0.1$ and find the number of rows in the Richardson table required to estimate the derivative with five (5) significant decimal digits.

[8 **marks**]

Q6. (a) Given function $f(x) = e^{-x}$, we would like to study different numerical approximations to the integral

$$\int_0^{0.8} f(x)dx$$

We will use the values of $f(x)$ at the points $x = 0.0, 0.2, 0.4, 0.6, 0.8$. Generate the data set before you start the numerical integration.

i. Write out the trapezoid rule and compute the numerical integration with 5 digits. [5 marks]

ii. Write out the Simpson's 1/3 rule and compute the numerical integration with 5 digits. [5 marks]

iii. What is the exact value of the integral? [2 marks]

iv. What is the absolute error by using trapezoid and Simpson's rules? [4 marks]

v. Which method is better and why? [2 marks]

(b) For the integral above, use the Romberg integration method to combine your two numerical estimates to get a more accurate estimate of the integral. Compute the true error ϵ_t for the new estimate. [10 marks]

Q7. Given the following integral

$$I = \int_{-1}^{1} (x^2 + 1)dx$$

(a) Compute the exact value of the integral analytically. [4 marks]

(b) Find the approximate value of the integral numerically using Gauss Quadrature with $n = 3$. [8 marks]

(c) Comment on the results. [2 marks]

Q8. (a) Use Euler's formula with $h = 0.5$ to obtain an approximation to $y(1.5)$ for the solution of

$$\frac{dy}{dx} = 2x + y \qquad y(0) = 5$$

[5 marks]

(b) Show that the exact solution to the above ODE using the integration factor method is given by

$$y = 7e^x - 2x - 2.$$

[7 marks]

(c) The concentration of salt $x$in a homemade soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is $50g/L$ Using Runge-Kutta 4th order method and a step size of, $h = 1.5$ min, what is the salt concentration after 3 minutes?

[8 marks]

Useful Numerical Methods and Corresponding Formulas

Richardson Extrapolation Formulas for improving the accuracy of low order numerical schemes.

Hold $f(x)$, and x fixed:

$$\varphi(h) = \frac{f(x+h) - f(x-h)}{2h} \tag{1}$$

$$\varphi(h) = f'(x) - a_2h^2 - a_4h^4 - a_6h^6 - \ldots \tag{2}$$

$$\varphi(\frac{h}{2}) = f'(x) - a_2\left(\frac{h}{2}\right)^2 - a_4\left(\frac{h}{2}\right)^4 - a_6\left(\frac{h}{2}\right)^6 - \ldots \tag{3}$$

$$\varphi(h) - 4\varphi(\frac{h}{2}) = -3f'(x) - \frac{3}{4}a_4h^4 - \frac{15}{16}a_6h^6 - \ldots \tag{4}$$

$$\rightarrow f'(x) = \frac{4}{3}\varphi(h/2) - \frac{1}{3}\varphi(h) + O(h^4) \tag{5}$$

$D(0,0) = \varphi(h)$			
$D(1,0) = \varphi(h/2)$	$D(1,1)$		
$D(2,0) = \varphi(h/4)$	$D(2,1)$	$D(2,2)$	
$D(3,0) = \varphi(h/8)$	$D(3,1)$	$D(3,2)$	$D(3,3)$

First Column:

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) \tag{6}$$

Others:

$$D(n,m) = \frac{4^m}{4^m-1}D(n,m-1) - \frac{1}{4^m-1}D(n-1,m-1) \tag{7}$$

Recursive Trapezoid Method

$$h = \frac{b-a}{2^n} \tag{8}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{9}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{10}$$

$$h = b - a \tag{11}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{12}$$

$$h = \frac{b-a}{2} \tag{13}$$

$$R(1,0) = \frac{1}{2}R(0,0) + h\left[\sum_{k=1}^{1} f\left(a + (2k-1)h\right)\right] \tag{14}$$

$$h = \frac{b-a}{2^2} \tag{15}$$

$$R(2,0) = \frac{1}{2}R(1,0) + h\left[\sum_{k=1}^{2} f\left(a + (2k-1)h\right)\right] \tag{16}$$

$$h = \frac{b-a}{2^3} \tag{17}$$

$$R(3,0) = \frac{1}{2}R(2,0) + h\left[\sum_{k=1}^{2^2} f\left(a + (2k-1)h\right)\right] \tag{18}$$

$$h = \frac{b-a}{2^n}, R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{19}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{20}$$

$$h = \frac{b-a}{2^n}, \tag{21}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{22}$$

$$R(n,m) = \frac{1}{4^m - 1}\left[\, 4^m \times R(n, m-1) - R(n-1, m-1)\right], n \geq 1, m \geq 1 \tag{23}$$

$R(0,0)$			
$R(1,0)$	$R(1,1)$		
$R(2,0)$	$R(2,1)$	$R(2,2)$	
$R(3,0)$	$R(3,1)$	$R(3,2)$	$R(3,3)$

GAUSS QUADRATURE TABLE OF WEIGHTS AND FUNCTION ARGUMENTS

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$ $c_2 = 1 : 000000000$	$x_1 = -0.577350269$ $x_2 = 0.577350269$
3	$c_1 = 0.555555556$ $c_2 = 0.888888889$ $c_3 = 0.555555556$	$x_1 = -0.774596669$ $x_2 = 0.000000000$ $x_3 = 0.774596669$
4	$c_1 = 0.347854845$ $c_2 = 0.652145155$ $c_3 = 0.652145155$ $c_4 = 0.347854845$	$x_1 = -0.861136312$ $x_2 = -0.339981044$ $x_3 = 0.339981044$ $x_4 = 0.861136312$
5	$c_1 = 0.236926885$ $c_2 = 0.478628670$ $c_3 = 0.568888889$ $c_4 = 0.478628670$ $c_5 = 0.236926885$	$x_1 = -0.906179846$ $x_2 = -0.538469310$ $x_3 = 0.000000000$ $x_4 = 0.538469310$ $x_5 = 0.906179846$
6	$c_1 = 0.171324492$ $c_2 = 0.360761573$ $c_3 = 0.467913935$ $c_4 = 0.467913935$ $c_5 = 0.360761573$ $c_6 = 0.171324492$	$x_1 = -0.932469514$ $x_2 = -0.661209386$ $x_3 = -0.2386191860$ $x_4 = 0.2386191860$ $x_5 = 0.661209386$ $x_6 = 0.932469514$

EULER'S METHOD

$$y(x_0 + h) = y(x_0) + h\frac{dy}{dx}|_{x=x_0,y=y_0}| + O(h^2)$$

Notation:

$$x_n = x_0 + nh \qquad y_n = y(x_n)$$

$$\frac{dy}{dx}|_{x=x_i,y=y_i}| = f(x_i, y_i)$$

Euler's Method

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

4th ORDER RUNGE-KUTTA FORMULA FOR SOLVING ODES

$$k_1 = f(x_i, y_i) \tag{24}$$

$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_1 h) \tag{25}$$

$$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_2 h) \tag{26}$$

$$k_4 = f(x_i + h, y_i + k_3 h) \tag{27}$$

(28)

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Local error is $O(h^5)$ and global error is $O(h^4)$