UNIVERSITY OF GHANA

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BSc/BA, FIRST SEMESTER EXAMINATIONS: 2017/2018

SCHOOL OF ENGINEERING SCIENCES

FAEN 101: ALGEBRA (4 credits)

INSTRUCTION:

ANSWER ANY 4 OUT OF THE FOLLOWING 7 QUESTIONS

TIME ALLOWED:

TWO HOURS AND THIRTY MINUTES $\left(2\frac{1}{2} \text{ hours}\right)$

NOTE: *The bold letters represent vectors and* **i**, **j** *and* **k** *are unit vectors along the coordinate axes.*

1. (a)(i) List the members of the sets denoted by A, B and C, where

$$A = \{x \in \mathbb{R} : x = 2\,(3a + b)\,, a, b \in \mathbb{Z}\}$$
$$B = \{x \in \mathbb{R} : x^2 - 5x + 6 = 0, x > 2\}$$
$$C = \left\{x \in \mathbb{Z} : x = \frac{1}{n+1}, n \in \mathbb{N}\right\}.$$

(ii) Write each of the sets D, E and F in set-builder notations.

$$D = \{0, 4, 16, 36, 64, 100, 144, \cdots\}$$
$$E = \left\{\cdots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \cdots\right\}$$
$$F = \{-4, -3, -2, -1, 0, 1, 2\}.$$

(iii) Let X and Y be two sets given by

$$X = \{0, 2, 4, 6, 8\} \text{ and } Y = \{8, 6, 5, 4\}.$$

Find $X \cap Y$, $X \cup Y$ and $X \setminus Y$.

[18 Marks]

(b) (i) State the domain of the logarithm function $f\,(x) = \log x$.

Show that if q is in the domain of f and $p \in \mathbb{R}^+$, then

$$\log_{p^n} q = \frac{1}{n} \log_p q.$$

Hence or otherwise, establish a relation between the variables x, y and z given that

$$\log_9\left(x^2+y^2\right)-\log_3 z=2.$$

Find the value of z if $x=3$ and $y=4$.

(ii) Express each of the terms 4^x, 256^x and 32^{1-2x} in the form $2^{f(x)}$, where $f(x)$ is a linear polinomial in x. Hence or otherwise, find the value of x for which

$$\frac{4^x}{4^{x+1}}=\frac{256^x}{32^{1-2x}}$$

[32 Marks]

2. (a)(i) Find the domains of the functions f and g defined by

$$f(x)=\frac{\log(x-3)}{x-1} \quad \text{and} \quad g(x)=\sqrt{\frac{3x-2}{x^2-4x+4}}.$$

(ii) Let $p:[-2,5]\to R_p$ and $q:\mathbb{R}\to R_q$ be defined by

$$p(x)=x^2-2x-3, \quad q(x)=-\frac{1}{2}\left[(x+4)^2+(x-2)^2\right].$$

Show that $R_p=[-4,\infty)$ and $R_q=(-\infty,-9]$, where R_F denotes the range of the function F.

[25 Marks]

(b) (i) Solve the following inequalities:

(α) $\dfrac{(x-2)(x-3)}{x-1}\geq 0$ (β) $\dfrac{|x-7|}{x-2}\leq 1$

(ii) By sketching the graphs of the functions

$$f(x)=4-2x \quad \text{and} \quad g(x)=x^2-4,$$

shade the region in the plane that satisfies the inequalities

$$y+2x\leq 4 \quad \text{and} \quad y\geq x^2-4,$$

indicating clearly the shading that corresponds to the solution set of each inequality.

[25 Marks]

3. (a)(i) Recall that the general term in the expansion of $(a+x)^n$ is given by ${}^nC_r a^{n-r}x^r$ for some $r\leq n$, where $n\in\mathbb{N}$. Find the term in the binomial expansion of $\left(3z-\dfrac{2}{z^2}\right)^{18}$ that is independent of z.

(ii) Write down the first three terms in the binomial expansion of $\left(1-\dfrac{1}{x^2}\right)^{\frac{1}{2}}$. Hence, find, correct to two decimal places, the value of $\dfrac{\sqrt{48}}{7}$.

[Hint: You may use the facts that $(1+x)^n=1+nx+\dfrac{1}{2!}n(n-1)x^2+\dfrac{1}{3!}n(n-1)(n-2)x^3+\cdots$ and $\dfrac{\sqrt{a}}{b}=\sqrt{\dfrac{a}{b^2}}$.]

[18 Marks]

(b)(i) Show that

$$\left(3\sqrt{2}-1\right)^2+\left(3+\sqrt{2}\right)^2 \quad \text{and} \quad \frac{\sqrt{48}+2\sqrt{27}}{\sqrt{12}}$$

are both integers.

(ii) Write the following in the form $a+b\sqrt{c}$, where a, b and c are integers, specifying the values of a, b and c:

$(\alpha)\ \dfrac{5\sqrt{3}-6}{2\sqrt{3}+3}$ $\qquad(\beta)\ \dfrac{\left(3-\sqrt{5}\right)^2}{1+\sqrt{5}}$ $\qquad(\gamma)\ \sqrt{45}+\dfrac{20}{\sqrt{5}}$ $\qquad(\delta)\ \left(4\sqrt{7}-1\right)\left(\sqrt{7}+3\right).$

[32 Marks]

4. (a) (i) State the domain and range of the exponential function $f(x)=a^x$.

In an investment scheme, the formula for computing an investment after t years is given by

$$I(t)=I_0(1+r)^{\frac{t}{2}},$$

where $I_0=I(0)$ is the initial investment and $r\%$ is the rate of interest. If Kaka invests GHC130,000.00 initially, how long will it take for his investment to double, given that the interest rate is 10%? Correct your answer to the nearest whole number.

(ii) Consider the function

$$f(x)=\frac{1}{x^2-x-2}.$$

Find the following, if any:

(α) the domain of f, D_f.

(β) the range of f, R_f.

(γ) the intercepts of f.

(δ) the vertical and horizontal asymptotes of f.

(ε) the minimum and maximum points of f. [25 Marks]

(b) (i) Let $f(x)=\left(1+\dfrac{1}{x}\right)^x$. Find $f(x-1)$.

Given that

$$f(-x)=\frac{xf(x-1)}{x-1},$$

verify that

$$f_e(x)+f_o(x)=f(x),$$

where $f_e(x)=\dfrac{1}{2}[f(x)+f(-x)]$ and $f_o(x)=\dfrac{1}{2}[f(x)-f(-x)]$. How will you call $f_e(x)$ and $f_o(x)$?

(ii) Let $f(x)=e^x$ and define the functions $\cosh:\mathbb{R}\to\mathbb{R}$ and $\sinh:\mathbb{R}\to\mathbb{R}$ by $\cosh x=\dfrac{1}{2}\left(e^x+e^{-x}\right)$ and $\sinh x=\dfrac{1}{2}\left(e^x-e^{-x}\right)$. Prove that

$$\cosh x+\sinh x=e^x,\quad \tanh x=\frac{e^{2x}-1}{e^{2x}+1}\quad\text{and}\quad \operatorname{sech}x=\frac{2e^x}{e^{2x}+1}.$$

Hence, verify that

$$\operatorname{sech}^2 x+\tanh^2 x=1.$$

[25 Marks]

5. (a) (i) Define the radian as a unit of angle measurement, and find the radian equivalence of the following angles:

$$(\alpha)\ 120^\circ \quad (\beta)\ 330^\circ \quad (\gamma)\ 740^\circ.$$

(ii) Let $0 \leq \theta \leq \frac{\pi}{2}$. Show that

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

Hence or otherwise, show that

$$\cos^2\theta + \cos^2\left(\frac{\pi}{2} - \theta\right) = 1.$$

(iii) State Pythagora's Theeorem, explicitly defining every variable used.

Let $\tan\left(\frac{\theta}{2}\right) = t$. By using Pythagora's Theorem, show that

$$\sin\left(\frac{\theta}{2}\right) = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+t^2}}.$$

Hence, by writing $\sin\theta = \sin\left(\frac{\theta}{2} + \frac{\theta}{2}\right)$ and $\cos\theta = \cos\left(\frac{\theta}{2} + \frac{\theta}{2}\right)$, verify that

$$\sin\theta = \frac{2t}{1+t^2}, \quad \text{and} \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

Use this to deduce that $\sin 90^\circ = 1$ and $\cos 90^\circ = 0$. [40 Marks]

[Hint: You may use these identities without proof: $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2\sin x \cos x$.]

(b) (i) What are acute angles?

Let α and β be two acute angles. Given that $\sin\alpha = \frac{12}{13}$ and $\cos\beta = \frac{4}{5}$, find the value of

$$\sin(\alpha - \beta) + \cos(\alpha + \beta).$$

(ii) Write $4\cos x - 3\sin x$ in the form $R\cos(x + \alpha)$, where R and α are positive constants to be determined. Hence, solve the equation

$$4\cos x - 3\sin x = 3,$$

rounding all angles to the nearest degree. [10 Marks]

6. (a) (i) Given that $z = -3 + 4i$, where $i^2 = -1$, find $|z|$ and $\arg(z)$.

Let w be a complex number defined by $w = \dfrac{-14+2i}{z}$. Find w, leaving your answer in the form $a + ib$, where a and b are real numbers.

(ii) Given that $z = 2 - 2i$ and $w = -\sqrt{3} + i$, write the complex number given by wz^2 in the form $r\cos\theta + ir\sin\theta$, where $r = |wz^2|$ and $\theta = \arg(wz^2)$.

Hence, find $w^n z^{2n}$ for some $n \in \mathbb{Z}$. [30 Marks]

(b) (i) Find the fourth roots of unity and represent them on an Argand diagram.

[Hint: You may find the identity $x^4 - 1 = (x^2 - 1)(x^2 + 1)$ useful.]

(ii) Write the complex number $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ in the polar (trigonometric) form and show that the cube roots of unity of z are 1, z and z^2, and that

$$1 + z + z^2 = 0.$$

[20 Marks]

7. (a) Let $\mathbf{a} = \mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{k}$ denote the position vectors of some points A, B and C respectively. Find the following:

   (i) $\mathbf{a} + \mathbf{b}$, $\mathbf{a} + 3\mathbf{b} + \mathbf{c}$, $|4\mathbf{a} + \mathbf{b} - 2\mathbf{c}|$, $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$.

   (ii) the angle between $\mathbf{b}$ and $\mathbf{c}$.

   (iii) the component of $\mathbf{a}$ and $\mathbf{b}$.

   (iv) the unit vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

[30 Marks]

(b)(i) Recall that forces are vectors. A number of n forces $\mathbf{F}_1$, $\mathbf{F}_2$, ..., $\mathbf{F}_n$ are said to be in equilibrium when

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0},$$

where $\mathbf{0}$ is the zero force vector. Find the force G such that the forces $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are in equilibrium, given that

$$\mathbf{E} = 3\mathrm{i} - 2\mathrm{j}, \quad \mathbf{F} = \frac{1}{3}\mathrm{i} + \mathrm{j} \quad \text{and} \quad \mathbf{H} = -\frac{3}{8}\mathrm{j}.$$

(ii) A force of mass 2 kg is acted on by forces $\mathbf{F}_1$ and $\mathbf{F}_2$ simultaneously, where $\mathbf{F}_1 = \langle 2, -3, 1 \rangle$ newtons and $\mathbf{F}_2 = \langle 2, 9, -3 \rangle$ newtons. Given that, the acceleration produced by the actions of these forces is obtained via the equation

$$\mathbf{F} = m\mathbf{a},$$

where $\mathbf{F}$ is the vector sum of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, m is the body's mass and $\mathbf{a}$ is the acceleration, find the magitude and the direction of the acceleration from $\mathbf{F}$. What conclusion can you draw from the angle so obtained?

[20 Marks]