UNIVERSITY OF GHANA

College of Basic and Applied Sciences
School of Engineering Sciences
Bachelor of Science in Engineering
First Semester Exams 2014/2015 Academic Year
LEVEL 300: FAEN 301 Numerical Methods

Time Allowed: 3 hours

Attempt all questions. Use *4 decimal places* for your floating point calculations and answers

1. Consider the following system of linear equations:

$$
\begin{aligned}
4x_1 + 4x_2 + 4x_3 &= 7 \\
12x_1 + 16x_2 + 4x_3 &= 21 \\
4x_1 + 2x_2 + 4x_3 &= 6
\end{aligned}
$$

   (a) Compute the determinant of the above linear system of equations. [3 marks]

   (b) Solve the system of equations using Gaussian elimination with partial pivoting. Show all the necessary steps of your calculation. [10 marks]

   (c) Verify your answer by computing the solution of the given system using *Cramer's rule*. [7 marks]

   (d) Briefly discuss the advantages and disadvantages of direct and iterative methods for solving linear algebraic equations [6 marks]

2. Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature versus resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

Table 1: Temperature as a function of resistance.

R/Ohm	$T/°C$
1101.0	25.0
910.0	30.0
635.0	40.0
450.0	50.0

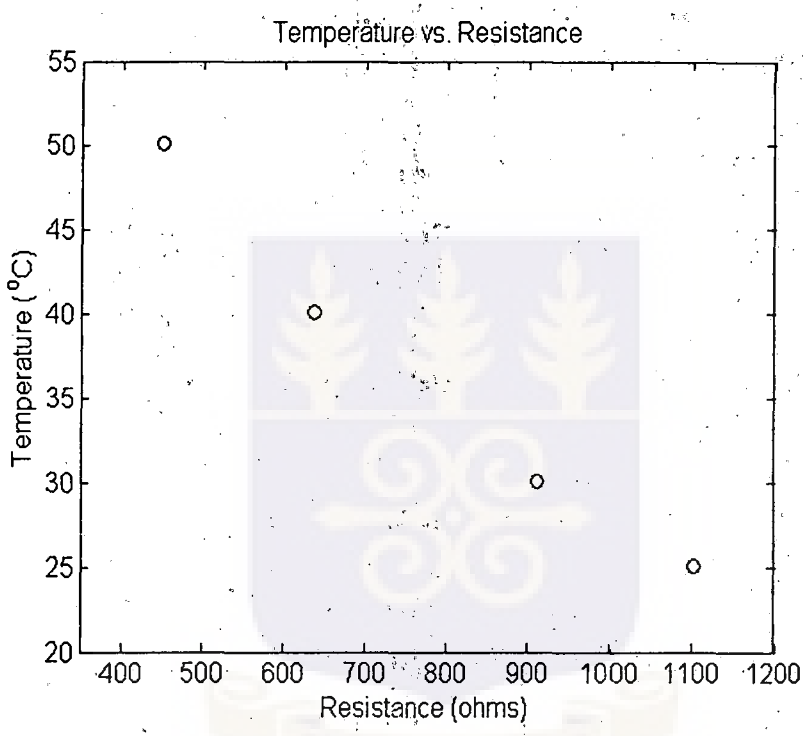

Figure 1: Plot of temperature versus resistance values

(a) Draw the divided difference table for the above data and compute the entries. [5 marks]

(b) Obtain the Newton's interpolating polynomial of order 2 using the first 3 points in the table. [3 marks]

(c) Obtain the Newton's interpolating polynomial of order 3 using all the 4 points in the table. [3 marks]

(d) Calculate the value of the interpolating polynomial when the independent variable $R = 800$ using the polynomials obtained in questions b and c above. [5 marks]

(e) Determine the temperature corresponding to $R = 800$ ohms using a first order Lagrange polynomial. [10 marks]

3. Consider the linear circuit shown in figure 2 below. The resistor R_L represents a fragile component of the electrical system that will melt if it dissipates a power greater than $P_{max} = 5W$. Knowing the value of the different resistors R_1, R_2, R_3, R_4, R_5 and R_L and the voltage applied in V_{source} , we want to compute the power dissipated in resistor R_L given as $P = R_L i_5^2$ to check if this arrangement will cause any damage to this component.

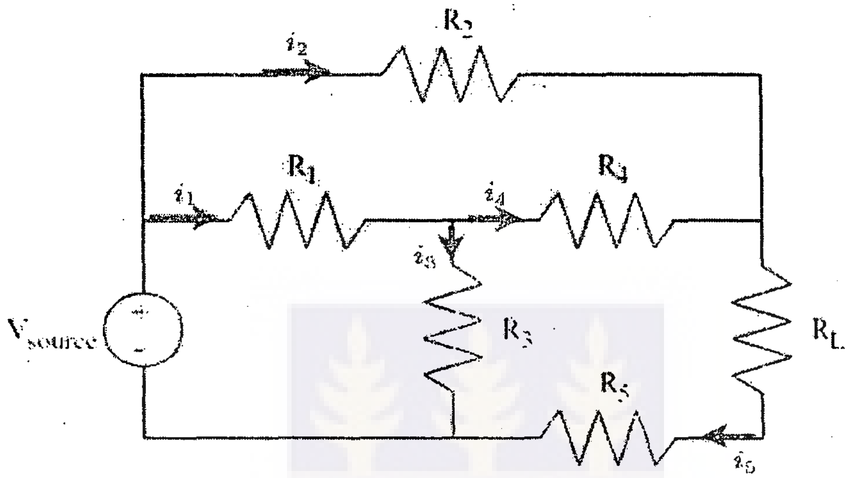

Figure 2: Resistive Network

Use $R_1 = 100\Omega$, $R_2 = 200\Omega$, $R_3 = 300\Omega$, $R_4 = 400\Omega$, $R_5 = 500\Omega$ and $R_L = 500\Omega$, and $V_{source} = 100V$.
Using Kirchhoff's laws for voltages and currents, the following circuit equations are obtained:

$$\begin{aligned}
V_{source} &= R_2 i_2 + R_5 i_5 + R_L i_5 \\
V_{source} &= R_1 i_1 + R_3 i_3 \\
0 &= -R_3 i_3 + R_4 i_4 + R_5 i_5 + R_L i_5 \\
i_1 &= i_3 + i_4 \\
i_5 &= i_2 + i_4
\end{aligned}$$

(a) Substitute the values of the resistances and source voltage given; write the above system of equations using ***matrix-vector notation*** as $\mathbf{Ax} = \mathbf{b}$. Identify your A, x and b variables. [5 marks]

(b) Given that in the solution for the above system of linear equations, the values of the currents namely i_1 and i_2 are given as **0.2403A** and **0.0942A** respectively. Find the values of the other remaining currents that will solve the linear systems of equations obtained above. *[Hint: By substituting the current values; obtain a 3×3 system. Solve the resulting system with your calculator]* [12 marks]

(c) Determine the maximum allowable current that the load resistor can permit without melting and compare with the computed current from application of Kirchhoff's laws. Comment on your results. [8 marks]

4. (a) Given the following 3×3 matrix A below, find the $\boldsymbol{LU}$ decomposition (lower and upper triangular matrices) of the matrix.

$$A = \begin{pmatrix} -2 & -4 & -2 \\ -4 & -1 & 2 \\ 4 & 3 & 2 \end{pmatrix}$$

[12 marks]

(b) Consider the following nonlinear cubic equation

$$x^3 + 2x^2 - 5 = 0$$

Use the Newton-Raphson's method to find a root of the equation given above. Use an initial guess of $x_0 = 0.5$. Perform only three (3) iterations. [8 marks]

5. The Level 300 Engineering Company is a company with only three employees which makes two different kinds of hand-crafted windows: a wood-framed and an aluminium-framed window. They earn 60 GHc profit for each wood-framed window and 30 GHc profit for each aluminium-framed window. Kwasi makes the wood frames, and can make 6 per day. Kofi makes the aluminium frames, and can make 4 per day. Kwame forms and cuts the glass, and can make 48 square feet of glass per day. Each wood-framed window uses 6 square feet of glass and each aluminium-framed window uses 8 square feet of glass. The company wishes to determine how many windows of each type to produce per day to maximize total profit.

(a) Formulate a linear programming model for this problem. [5 marks]

(b) Use the graphical method to solve this model. [10 marks]

6. Given the following objective function below and the corresponding constraints

$$\text{Extremize } f(x) = 20x_1^2 - 10x_2^2,$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ x_1 + 2x_2 &\leq 25 \\ 5x_1 - 2x_2 &\leq 10 \end{aligned}$$

$$\text{and } x_1 \geq 0, x_2 \geq 0.$$

(a) Plot the constraint functions for the optimization problem above on a graph. [5 marks]

(b) Obtain the Lagrangian function for the above system. [2 marks]

(c) Obtain the six (6) Karush-Kuhn-Tucker (KKT) conditions for the above system. [6 marks]

(d) Obtain the first order condition for optimality and solve the resulting system of equations using row reduction techniques or any method of your choice. [12 marks]

(e) Compute the second order optimality conditions and check whether the solution obtained above is minimum or maximum. (Hint: Check for positive or negative definiteness i.e. $\mathbf{x}^{\top}\mathbf{A}\mathbf{x} \geq 0$ or $\mathbf{x}^{\top}\mathbf{A}\mathbf{x} \leq 0$ respectively, where x is any vector and $\mathbf{A}$ is the Hessian of the Lagrangian function or alternatively check for convexity (min) or concavity(max) of the objective function. [8 marks]

(f) Compute the optimal value of $f(x)$ [2 marks]

7. Given the following integral

$$I = \int_{-1}^{1} \frac{4}{1 + x + x^2} dx$$

Hint: You may factorize the denominator and use the integration formula

$$I = \int \frac{a}{a^2 + x^2} dx = arctan\left(\frac{x}{a}\right)$$

Find:

(a) the exact value of the integral analytically using the above hints. [10 marks]

(b) the approximate value of the integral numerically using

i. the trapezoidal rule with an interval (h) of size 0.2 [8 marks]

ii. the Simpson's 1/3 rule; use ten (10) segments and compute the results based on the table data for the trapezium rule. [8 marks]

iii. Use Gauss Quadrature with $n = 3$ to compute the value of the above integral. [12 marks]

8. (a) Derive Euler's formula for solving ODEs from first principles using Taylor series. [5 marks]

(b) State without derivation, the 4th order Runge-Kutta formula for solving ODEs, define all terms used. [5 marks]

(c) Use Euler's formula with $h = 0.5$ to obtain an approximation to $y(1.5)$ for the solution of

$$\frac{dy}{dx} = 2x + y, \; y(0) = 5$$

[6 marks]

(d) Show that the exact solution to the above ODE using the integration factor method is given by $y = 7e^x - 2x - 2$. [12 marks]

[12 marks]

(e) The concentration of salt x in a homemade soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L Using Runge-Kutta 4th order method and a step size of, $h = 1.5$ min, what is the salt concentration after 3 minutes? [10 marks]

GAUSS QUADRATURE TABLE OF WEIGHTS AND FUNCTION ARGUMENTS

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$
5	$c_1 = 0.236926885$	$x_1 = -0.906179846$
	$c_2 = 0.478628670$	$x_2 = -0.538469310$
	$c_3 = 0.568888889$	$x_3 = 0.000000000$
	$c_4 = 0.478628670$	$x_4 = 0.538469310$
	$c_5 = 0.236926885$	$x_5 = 0.906179846$
6	$c_1 = 0.171324492$	$x_1 = -0.932469514$
	$c_2 = 0.360761573$	$x_2 = -0.661209386$
	$c_3 = 0.467913935$	$x_3 = -0.2386191860$
	$c_4 = 0.467913935$	$x_4 = 0.2386191860$
	$c_5 = 0.360761573$	$x_5 = 0.661209386$
	$c_6 = 0.171324492$	$x_6 = 0.932469514$