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UNIVERSITY OF GHANA

FIRST SEMESTER 2017/2018 Examination
DEPARTMENT OF COMPUTER ENGINEERING
LEVEL 300: FAEN 301: NUMERICAL METHODS
3 CREDIT HOURS

TIME ALLOWED: 3 HOURS

INSTRUCTION: Attempt All Questions

Q1. Show that the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

is positive definite by

(a) computing explicitly the eigenvalues,

[8 marks]

(b) only computing determinants of sub-matrices and also,

[5 marks]

(c) compute the corresponding eigenvectors of the matrix above.

[10 marks]

Q2. The upward velocity of a rocket can be computed by the following formula:

$$v = u \ln\left(\frac{m_0}{m_0 - qt}\right) - gt$$

where v = upward velocity,
u = velocity at which fuel is expelled relative to the rocket,
m_0= initial mass of the rocket at time $t = 0$,
q = fuel consumption rate, and
g = downward acceleration of gravity (assumed constant = $9.81 m/s^2$).
If $u = 1850$ m/s, $m_0 = 160,000$ kg , and $q = 2500$ kg/s, determine how high the rocket will fly in 30 s using the following integration methods namely:

(a) Gauss quadrature with $n = 3$

[10 marks]

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(b) Simpson's 1/3 rule with $n = 10$

[10 marks]

Q3. (a) Consider the function

$$f(x_1, x_2) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2$$

over $\Re^2$. Find all the stationary points and classify them.

[12 marks]

(b) Consider the problem

$$\text{Minimize} \qquad f(x) = x_1^2 + x_2^2 + x_3^2$$

subject to:

$$x_1 + 2x_2 + 3x_3 \geq 4$$
$$x_3 \leq 1$$

i. Write down the KKT conditions for the above optimization problem.

[5 marks]

ii. (Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions. Hint: *Show that the system resulting from the application of the first order conditions is invertible.*

[5 marks]

iii. Find the optimal solution of the problem using the KKT system.

[10 marks]

Q4. Consider the problem

$$\text{Minimize} \qquad f(x) = x_1^2 + 2x_2^2 + 2x_1$$

subject to:

$$x_1 + x_2 \leq a$$

where $a \in R$ is a parameter.

(a) Prove that for any $a \in R$, the problem has a unique optimal solution (without actually solving it).

[8 marks]

(b) Solve the problem in terms of the parameter a.

[8 marks]

(c) Let $f(a)$ be the optimal value of the problem with parameter a. Write an explicit expression for f and prove that it is a convex function.

[10 marks]

Q5. (a) Given the following function $f(x)$ below. Sketch the graph of the following function for the interval $-5 \leq x \leq 5$

$$f(x) = x^3 + 2x^2 - x - 2 \qquad x_0 = 0, \qquad x_1 = 2$$

[5 marks]

(b) Find all the turining points and classify them [8 marks]

(c) Verify from the graph that the interval endpoints at x_0 and x_1 have opposite signs. Use Newton's method to estimate the root of the equation that lies between the endpoints given. (Perform only 3 iterations)

[8 marks]

(d) Use Newton's method/iterative scheme to solve the following system of nonlinear equations

$$f_1(x_1, x_2) = x_1 + x_2 + x_1^2 + 6x_2^2 - 9$$
$$f_2(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 - 4$$

Use the initial starting point as $x_1 = x_2 = 0.5$. (Perform only 2 iterations)

[10 marks]

Q6. Given the system of linear equations below

$$2v_1 - v_2 + 3v_3 = 4$$
$$-4v_1 + 3v_2 + 2v_3 = 1$$
$$3v_1 + v_2 - v_3 = 3$$

(a) Use the Gaussian elimination method to find the solution vector v_1, v_2, and v_3..

[10 marks]

(b) Use LU decomposition to obtain the lower and upper triangular matrices for the system above. Carefully outline the solution steps to solve completely for the unknown values of v_1, v_2, and v_3.. Hint Use ***elementary row operations*** to perform the calculation and arrive at the final solution.

[10 marks]

Q7. (a) Write down the formulas for the forward difference, backward difference and central difference approximations for numerical differentiation. [3 marks]

(b) Compute the exact derivative of $f(x) = \cos(x)$ at $x = \frac{\pi}{3}$ radians.

[3 marks]

(c) Compute the derivative of $f(x) = \cos(x)$ at $x = \frac{\pi}{3}$ radians using

i. the central difference approximation. Use the step size $h = 0.1$.

ii. the forward difference approximation and the Richardson extrapolation algorithm. Use $h = 0.1$ and find the number of rows in the Richardson table required to estimate the derivative with five (5) significant decimal digits.

[8 marks]

Q8. (a) Use Euler's formula with $h = 0.5$ to obtain an approximation to $y(1.5)$ for the solution of

$$\frac{dy}{dx} = 4x - 2y \qquad y(0) = 5$$

[5 marks]

(b) Show that the exact solution to the above ODE using the integration factor method is given by

$$y = 6e^{-2x} + 2x - 1.$$

[7 marks]

(c) The concentration of salt x in a homemade soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is $50g/L$ Using Runge-Kutta 4th order method and a step size of, $h = 1.5$ min, what is the salt concentration after 3 minutes?

[8 marks]

Useful Numerical Methods and Corresponding Formulas

Richardson Extrapolation Formulas for improving the accuracy of low order numerical schemes.

Hold $f(x)$, and x fixed:

$$\varphi(h) = \frac{f(x+h) - f(x-h)}{2h} \tag{1}$$

$$\varphi(h) = f'(x) - a_2h^2 - a_4h^4 - a_6h^6 - \dots \tag{2}$$

$$\varphi(\frac{h}{2}) = f'(x) - a_2\left(\frac{h}{2}\right)^2 - a_4\left(\frac{h}{2}\right)^4 - a_6\left(\frac{h}{2}\right)^6 - \dots \tag{3}$$

$$\varphi(h) - 4\varphi(\frac{h}{2}) = -3f'(x) - \frac{3}{4}a_4h^4 - \frac{15}{16}a_6h^6 - \dots \tag{4}$$

$$\rightarrow f'(x) = \frac{4}{3}\varphi(h/2) - \frac{1}{3}\varphi(h) + O(h^4) \tag{5}$$

$D(0,0) = \varphi(h)$			
$D(1,0) = \varphi(h/2)$	$D(1,1)$		
$D(2,0) = \varphi(h/4)$	$D(2,1)$	$D(2,2)$	
$D(3,0) = \varphi(h/8)$	$D(3,1)$	$D(3,2)$	$D(3,3)$

First Column:

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) \tag{6}$$

Others:

$$D(n,m) = \frac{4^m}{4^m - 1}D(n,m-1) - \frac{1}{4^m - 1}D(n-1,m-1) \tag{7}$$

Recursive Trapezoid Method

$$h = \frac{b-a}{2^n} \tag{8}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{9}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{10}$$

$$h = b - a \tag{11}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{12}$$

$$h = \frac{b-a}{2} \tag{13}$$

$$R(1,0) = \frac{1}{2}R(0,0) + h\left[\sum_{k=1}^{1} f\left(a + (2k-1)h\right)\right] \tag{14}$$

$$h = \frac{b-a}{2^2} \tag{15}$$

$$R(2,0) = \frac{1}{2}R(1,0) + h\left[\sum_{k=1}^{2} f\left(a + (2k-1)h\right)\right] \tag{16}$$

$$h = \frac{b-a}{2^3} \tag{17}$$

$$R(3,0) = \frac{1}{2}R(2,0) + h\left[\sum_{k=1}^{2^2} f\left(a + (2k-1)h\right)\right] \tag{18}$$

$$h = \frac{b-a}{2^n}, R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{19}$$

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{20}$$

$$h = \frac{b-a}{2^n}, \tag{21}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right] \tag{22}$$

$$R(n,m) = \frac{1}{4^m - 1}\left[\,4^m \times R(n,m-1) - R(n-1,m-1)\right], n \geq 1, m \geq 1 \tag{23}$$

$R(0,0)$			
$R(1,0)$	$R(1,1)$		
$R(2,0)$	$R(2,1)$	$R(2,2)$	
$R(3,0)$	$R(3,1)$	$R(3,2)$	$R(3,3)$

GAUSS QUADRATURE TABLE OF WEIGHTS AND FUNCTION ARGUMENTS

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$ $c_2 = 1 : 000000000$	$x_1 = -0.577350269$ $x_2 = 0.577350269$
3	$c_1 = 0.555555556$ $c_2 = 0.888888889$ $c_3 = 0.555555556$	$x_1 = -0.774596669$ $x_2 = 0.000000000$ $x_3 = 0.774596669$
4	$c_1 = 0.347854845$ $c_2 = 0.652145155$ $c_3 = 0.652145155$ $c_4 = 0.347854845$	$x_1 = -0.861136312$ $x_2 = -0.339981044$ $x_3 = 0.339981044$ $x_4 = 0.861136312$
5	$c_1 = 0.236926885$ $c_2 = 0.478628670$ $c_3 = 0.568888889$ $c_4 = 0.478628670$ $c_5 = 0.236926885$	$x_1 = -0.906179846$ $x_2 = -0.538469310$ $x_3 = 0.000000000$ $x_4 = 0.538469310$ $x_5 = 0.906179846$
6	$c_1 = 0.171324492$ $c_2 = 0.360761573$ $c_3 = 0.467913935$ $c_4 = 0.467913935$ $c_5 = 0.360761573$ $c_6 = 0.171324492$	$x_1 = -0.932469514$ $x_2 = -0.661209386$ $x_3 = -0.2386191860$ $x_4 = 0.2386191860$ $x_5 = 0.661209386$ $x_6 = 0.932469514$

EULER'S METHOD

$$y(x_0+h)=y(x_0)+h\frac{dy}{dx}\,|_{x=x_0,y=y_0}|+O(h^2)$$

Notation:

$$x_n=x_0+nh \qquad y_n=y(x_n)$$

$$\frac{dy}{dx}\,|_{x=x_i,y=y_i}|=f(x_i,y_i)$$

Euler's Method

$$y_{i+1}=y_i+h\;f(x_i,y_i)$$

4th ORDER RUNGE-KUTTA FORMULA FOR SOLVING ODES

$$k_1=f(x_i,y_i) \tag{24}$$

$$k_2=f(x_i+\frac{h}{2},y_i+\frac{1}{2}k_1h) \tag{25}$$

$$k_3=f(x_i+\frac{h}{2},y_i+\frac{1}{2}k_2h) \tag{26}$$

$$k_4=f(x_i+h,y_i+k_3h) \tag{27}$$

(28)

$$y_{i+1}=y_i+\frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

Local error is $O(h^5)$ and global error is $O(h^4)$