UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2018/2019

DEPARTMENT OF BIOMEDICAL ENGINEERING

FAEN 201: CALCULUS 2 (4 credits)

INSTRUCTION:

ANSWER FIVE(5) QUESTIONS FROM THE FOLLOWING SEVEN(7) QUESTIONS

TIME ALLOWED:

THREE HOURS (3 hours)

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1. a. A sequence $\{a_n\}$ is said to have the limit L if for every $\varepsilon > 0$ there exists a number $N > 0$ such that $|a_n - L| < \varepsilon$ for every integer $n > N$. By this definition, show that the following sequences have limits L indicated

$$\{\frac{5-n}{2+3n}\}; L = -\frac{1}{3}.$$ **15 Marks**

b. Evaluate the sum of the infinite series given by

$$\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}.$$ **20 Marks**

c. For each of the following infinite series, use any preferred test to determine convergence or divergence.

i. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}+1\right)^2$

ii. $\sum_{n=1}^{\infty} \frac{1}{(-1)^n(2^n+3^n)}$ **15 Marks**

2. a. Determine the radius and interval of convergence of the power series given by

$$\sum_{n=1}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(4x-12)^n.$$ **25 Marks**

b. An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration. Consider the following improper integral and evaluate it.

$$\int_{\frac{1}{3}}^{3} \frac{dx}{\sqrt[3]{3x-1}}.$$ **15 Marks**

c. If $n > 0$, the gamma function Γ is defined by the improper integral

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx.$$

Show that

$$\int_0^{\infty} e^{-y^{\frac{1}{m}}} dy = m\Gamma(m).$$ **10 Marks**

3. a. Using the relation between the Beta and Gamma function, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, one can prove the property

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}.$$

Use this property to show that $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta} d\theta = \frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$. **15 Marks**

b. Consider the function $f(x) = \ln(1+x)$

i. Find the first four terms of its Maclaurin's series

ii. Find the n^{th} term of the Maclaurin's series and determine the interval of convergence.

25 Marks

c. Find the value of the directional derivative at the point $(2,0)$ for the function, $f(x,y) = xe^{2y}$ in the direction $\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$. **10 Marks**

4. a. If f is a differentiable function of x and y and $u = f(x,y)$, $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

20 Marks

b. The Laplace's equation in $\mathbb{R}^3$ is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Show that $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ satisfy this equation. **15 Marks**

c. Find the length of the arc having the parametric equations $x(t) = e^{-t}\cos t$ and $y(t) = e^{-t}\sin t$ from $t = 0$ and $t = \pi$. **15 Marks**

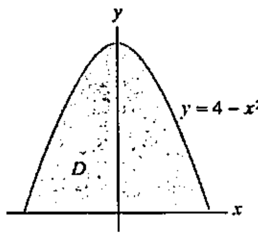

Figure 1: The region D for question 5 a.

5. a. Study the figure 1 above and,
   i. Calulate the area of D using a double integral
   ii. Evaluate the double integral $\int\int_D x^3 dA$. **20 Marks**

   b. Show that if $f(x, y, z) = x^2 \sin y + y^2 \cos z$ then

   $\nabla \times (\nabla f) = 0$ **15 Marks**

   c. Find the function $f(x, y)$ whose gradient is given by

   $(\sin 2x - \tan y)\mathrm{i} - x \sec^2 y\mathrm{j}$ **15 Marks**

6. a. Evaluate the following limits or explain why they fail to exist.
   i. $\lim_{(x,y)\to(0,0), x\neq y} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$
   ii. $\lim_{(x,y)\to(0,0),} \frac{x^2}{x^2+y^2}$ **15 Marks**

   b. If $w = x^2 - 2xy + 3y$, Find dw and use it to approximate the change in w if (x, y) changes from $(1, 2)$ to $(1.03, 1.99)$. **10 Marks**

   c. Consider the function

   $$f(x, y) = x^2 - 6x \cos y + 9, \quad 0 \leq y \leq 2\pi.$$

   i. Find $f_x(x, y)$,$f_y(x, y)$ and hence find the critical points $f(x, y)$.
   ii. Using the second derivative test, classify the critical points. **25 Marks**

7. a. If $\mathbf{x(t)} = (2t + 1, t, 3t - 1)$, $0 \leq t \leq 1$, find $\int_{\mathbf{x}} x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z$. **15 Marks**

   b. Use the Green's theorem to evaluate

   $$\oint_C (x^2 - y^2)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y$$

   where C is the boundary of the square with vertices $(1, 0)$,$(2, 0)$,$(1, 1)$ and $(2, 1)$, oriented counter-clockwise. **15 Marks**

   c. How much work is required to move an object in the vector field $F = y\mathrm{i} + 3x\mathbf{j}$ along the upper part of the ellipse $\frac{x^2}{4} + y^2 = 2$ from $(2, 0)$ to $(-2, 0)$? **20 Marks**