UNIVERSITY OF GHANA

BSc/BA, SECOND SEMESTER EXAMINATIONS: 2015/2016

SCHOOL OF ENGINEERING

DEPARTMENT OF BIOMEDICAL ENGINEERING

FAEN 202: DIFFERENTIAL EQUATIONS (4 credits)

INSTRUCTION:

ANSWER ANY FIVE OUT OF THE FOLLOWING SEVEN QUESTIONS

TIME ALLOWED:

THREE (3) HOURS

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1. (a) Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}.$$

Hence or otherwise, find a general solution to $\frac{dx}{dt} = Ax$, given that x is a function of t.

(30 marks)

(b) Let y be a function of x. Find a general solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$. *(10 marks)*

2. (a) Given that y is a function of x, verify that the following differential equations are exact and solve them.

(i) $(3x^2 + 2y^2) + (4xy + 6y^2)\frac{dy}{dx} = 0$ *(10 marks)*

(ii) $\left(2x - \frac{\ln y}{x^2}\right) + \frac{1}{xy}\frac{dy}{dx} = 0, \ x > 0, y > 0.$ *(15 marks)*

(b) A cold drink at $36°F$ is placed in a sweltering conference room at $90°F$. After 15 minutes, its temperature is $54°F$. Find the temperature $T(t)$ of the drink after t minutes, assuming it obeys Newton's law of cooling. *(15 marks)*

3. (a) Let y be a function of x. By using a suitable integrating factor, find a general solution to

(i) $(x^2+1)\frac{dy}{dx} + xy = x$

(ii) $\frac{dy}{dx} - y = \frac{8}{9}e^{-\frac{x}{3}}$,

(iii) $\frac{dy}{dx} + y\tan x = \sec x$, $x > 0$. *(25 marks)*

(b) The rate of change of the volume Vm^3 of water in a draining tank is proportional to the square root of the depth y metres of water in the tank.

(i) write down a differential equation connecting V and y.

(ii) If the tank is a cylinder with vertical sides and cross sectional area Am^2, show that $\frac{dy}{dt} + h\sqrt{y} = 0$, where h is a constant and related to A.

(iii) solve the differential equation in (ii) above. *(15 marks)*

4. (a) (i) Let $f(t)$ be a real valued function defined on $(0, \infty)$. Define the Laplace transform $F(s)$ of f stating the values of s for which $F(s)$ is defined.

(ii) Use your definition in (i) above to determine the Laplace transform of

$$f(t) = \begin{cases} 2, & 0 < t < 5 \\ 0, & 5 < t < 10 \\ e^{4t}, & t > 10. \end{cases}$$

(15 marks)

(b) Given that y is a function of x, solve the initial value problem:

(i) $\frac{dy}{dx} = 2\sqrt{y+1}\cos x, \quad y(\pi) = 0,$

(ii) $\frac{dy}{dx} + y\tan x = 2x\cos x,\ y\left(\frac{\pi}{4}\right) = -\frac{15\sqrt{2}}{32}\pi^2.$

(25 marks)

5. (a) (i) Show that the Laplace transform of the identity function $f(x) = x$ is $F(s) = \frac{1}{s^2}$, $s > 0$

(ii) Let y be a function of x. Use Laplace transforms to solve $\frac{d^2y}{dx^2} + y = x$, where $y(0) = 0$ and $\frac{dy}{dx} = 2$ at $x = 0$,

Hint: The Laplace transform of $\sin x$ *is* $\frac{1}{s^2+1}$. *(20 marks)*

(b) (b) By first resolving $\frac{4s-2}{s^3-s}$ into partial fractions, find the inverse Laplace transform of $F(s) = \frac{4s-2}{s^3-s}$.

(20 marks)

6. (a) (i) Give the exact definitions of the Fourier series and the Fourier transform of a $2L$-periodic function defined on $(-L, L)$.

   (ii) Expand

$$f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$$

in a Fourier series if the period of f is 10.

(25 Marks)

(b) Determine the Fourier transform of

$$f(x) = \begin{cases} e^{2x}, & x < 0 \\ e^{x}, & x > 0 \end{cases}$$

(15 marks)

7. (a) Use the method of undetermined coefficients to find a particular solution of

   (i) $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 2\cos x$;

   (ii) $\dfrac{d^2y}{dx^2} - 4y = 2e^{2x}$.

(20 marks)

(b) Given that $\dfrac{dy}{dx} - 2xy = 0$, show that $c_0 \displaystyle\sum_{n=0}^{\infty} \left(\frac{x^{2n}}{n!}\right)$ is a power series solution to the differential equation about $x = 0$.

(20 marks)