UNIVERSITY OF GHANA



**BSc/BA, SECOND SEMESTER EXAMINATIONS: 2016/2017**

**SCHOOL OF ENGINEERING SCIENCES**

**DEPARTMENT OF BIOMEDICAL ENGINEERING**

**FAEN 202: DIFFERENTIAL EQUATIONS (4 credits)**

**INSTRUCTION:**

**ANSWER ANY FIVE OUT OF THE FOLLOWING SEVEN QUESTIONS**

**TIME ALLOWED:**

**THREE (3) HOURS**

---

1. (a) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix}$.

   Hence or otherwise, solve $\dfrac{dx}{dt} = Ax$, where $x$ is a function of $t$. (25 marks)

   (b) When a cake is removed from an oven, its temperature is measured at $300°F$. Three minutes later its temperature is $200°F$.

   i. Find the temperature $T(t)$ of the cake after $t$ minutes assuming the room temperature is $30°F$.

   ii. How long will it take for the cake to cool off to $40°F$?

   *[Assume Newton's law of cooling]* (15 marks)

2. (a) Give a precise definition of the Fourier series of a function.

   Compute the Fourier series of $f(x) = \begin{cases} -1, & -3 \leq x < 0 \\ 1 & 0 < x \leq 3 \end{cases}$ for $x \in [-3, 3]$.

   (25 marks)

   (b) A 12 volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Assuming that $E$ is constant,

   i. Determine the current $i$ if the initial current is zero.

   ii. Find the limiting value of the current as $t \to \infty$. (15 marks)

   *[The differential equation for the LR series circuit with voltage $E(t)$ and currents $i(t)$ is given by $L\frac{di}{dt} + Ri = E(t)$, where $L$ and $R$ are constants known as the inductance and resistance respectively].*

3. (a) Let $y$ be a function of $x$. Solve

   i. $\frac{dy}{dx} = \sin 2x \sec y$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, given that $y = \frac{\pi}{6}$ at $x = \frac{\pi}{6}$;

   ii. $(\cos x)\frac{dy}{dx} + y \sin x = \sin x \cos^3 x$, given that $y = 2\sqrt{2}$ at $x = \frac{\pi}{4}$.

(20 marks)

(b) Given that $y$ is a function of $x$, express the solution of $4\frac{d^2y}{dx^2} - y = 0$ in the form of a power series about $x = 0$. (20 marks)

4. (a) Given that $y$ is a function of $x$, verify that the following differential equations are exact. Find the function $F(x, y)$ whose differential corresponds to the left hand side of

   i. $(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0$;

   ii. $(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$.

(20 marks)

(b) i. Let $f(t)$ be a real valued function defined on $(0, \infty)$. Define the Laplace transform $F(s)$ of $f$ stating the values of $s$ for which $F(s)$ is defined.

   ii. Use the definition in (i) above to determine the Laplace transform of

$$f(t) = \begin{cases} 2, & 0 < t < 5 \\ 0, & 5 < t < 10 \\ e^{4t}, & t > 10. \end{cases} .$$

(20 marks)

5. (a) Let $y$ be a function of $t$. Use Laplace transforms to solve the initial value problem $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2e^{3t}$, $y(0) = 0$, $\frac{dy}{dt} = 6$ at $t = 0$. (15 marks)

(b) Find the inverse Laplace transform of $F(s) = \dfrac{s}{s^2 + 6s + 11}$. (5 marks)

(c) Find the general solution of the equation $\frac{dy}{dx} - 2y = 4 - x$.

(20 marks)

6. (a) Use the substitution $y = zx$, where $z$ is a function of $x$, to solve the differential equation $x^2\frac{dy}{dx} = y(x + y)$, $x > 0$. (15 marks)

(b) Determine the fourier transform of $f(x) = \left\{ \begin{array}{ll} e^{2x}, & x < 0 \\ e^{-x} & x > 0 \end{array} \right\}$

(10 marks)

(c) The vertical motion of a weight attached to a spring is modelled by the initial value problem $\frac{1}{4}\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0, x(0) = 4$ and $\frac{dx}{dt} = 2$ at $x = 0$, where the displacement, $x$ is a function of time $t$. Determine the vertical displacement of the spring at time $t$. (15 marks)

7. (a) Given that $x > 1$ and $y > 0$, find the general solution of the differential equation
$\dfrac{dy}{dx} = \dfrac{3y}{x-1}$.
Given that $y = 1$ at $x = \frac{5}{3}$, find the value $y$ at $x = 2$, giving your answer in the form $y = ke^c$ where $k$ and $c$ constants to be found.

(20 marks)

(b) Use the method of undetermined coefficients to solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 54e^{2x}$, given that $y(0) = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$.

(20 marks)

*Brief Table of Laplace Transforms*

| $f(t)$ | $F(s) = \mathcal{L}\{f\}(s)$ |
|---|---|
| $1$ | $\frac{1}{s}, \quad s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| $t^n, n = 1, 2, \ldots$ | $\frac{n!}{s^{n+1}}, s > 0$ |
| $\sin bt$ | $\frac{b}{s^2+b^2}, \quad s > 0$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}, \quad s > 0$ |
| $e^{at}t^n, \ n = 1, 2, \ldots$ | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| $e^{at} \sin bt$ | $\frac{b}{(s-a)^2+b^2}, \quad s > a$ |
| $e^{at} \cos bt$ | $\frac{s-a}{(s-a)^2+b^2}, \quad s > a$ |