Stability condition of the implicit scheme

The finite-difference implicit scheme is given by

$$\boxed{T_i^n + \frac{D\Delta t}{\Delta x^2}\left[-T_{i+1}^n + 2T_i^n - T_{i-1}^n\right] = T_i^{n-1}} \tag{27}$$

Take plane wave solutions of the form $\boxed{T_i^n = e^{an\Delta t} e^{ikj\Delta x}}$

$$e^{an\Delta t}e^{ikj\Delta x} + \frac{D\Delta t}{\Delta x^2}\left[-e^{an\Delta t}e^{ik(j+1)\Delta x} + 2e^{an\Delta t}e^{ikj\Delta x} - e^{an\Delta t}e^{ik(j-1)\Delta x}\right] = e^{a(n-1)\Delta t}e^{ikj\Delta x} \tag{28}$$

Divide by $T_i^n = e^{an\Delta t}e^{ikj\Delta x}$ (to see the growth from one iteration to another)

$$1 + \frac{D\Delta t}{\Delta x^2}\left[2 - e^{ik\Delta x} - e^{-ik\Delta x}\right] = e^{-a\Delta t} \tag{29}$$

$$1 + \frac{4D\Delta t}{\Delta x^2}\sin^2(k\Delta x/2) = e^{-a\Delta t} \tag{30}$$

Therefore,

$$e^{a\Delta t} = \frac{1}{1 + \frac{4D\Delta t}{\Delta x^2}\sin^2(k\Delta x/2)} \tag{31}$$

We require $|e^{a\Delta t}| \leq 1$, which is always true since $1 + \frac{4D\Delta t}{\Delta x^2}\sin^2(k\Delta x/2) \geq 1$

Hence, this scheme is **unconditionally stable**!