Answers to problems

Problem 1.1. Upon differentiating the inverse motion (1.4) with respect to time T, which implies holding the Lagrangian variables X fixed, we find

$$\begin{aligned} 0 &= (\partial_t \Phi^I)(\partial_T t) + (\partial_i \Phi^I)\,(\partial_T \varphi^i) \\ &= \partial_t \Phi^I + F^I{}_i\, v^i \\ &= \partial_t \Phi^I + v^I\,, \end{aligned}$$

and thus we see that the Lagrangian components of the material velocity are given in terms of the inverse motion Φ^I by

$$v^I = -\,\partial_t \Phi^I\,.$$

Problem 1.2. We have

$$\begin{aligned} u^i &= \langle\, \mathbf{e}^i\,,\, \mathbf{u}\,\rangle \\ &= \langle\, \mathbf{e}^i\,,\, u^j\, \mathbf{e}_j\,\rangle \\ &= u^j\,\langle\, \mathbf{e}^i\,,\, \mathbf{e}_j\,\rangle \\ &= u^j\, \delta^i{}_j \\ &= u^i\,, \end{aligned}$$

with a similar result in Lagrangian components.

Problem 1.3. We transform expression (1.3) to Lagrangian coordinates as follows:

$$\begin{aligned} \nabla_I v^J &= F^i{}_I\,(F^{-1})^J{}_j\,\partial_i v^j \\ &= F^i{}_I\,(F^{-1})^J{}_j\,(F^{-1})^L{}_i\,\partial_L(v^K\, F^j{}_K) \\ &= \partial_I v^J + (F^{-1})^J{}_j\,(\partial_I F^j{}_K)\,v^K \\ &= \partial_I v^J + \Gamma^J_{IK}\,v^K\,. \end{aligned}$$

Problem 1.4. In a non-Cartesian Eulerian basis the covariant derivative of a vector has components

$$\nabla_i v^j = \partial_i v^j + \Gamma^j_{ik}\,v^k\,.$$

Upon transforming this expression to Lagrangian coordinates, we obtain

$$\begin{aligned} F^i{}_I\,(F^{-1})^J{}_j\,\nabla_i v^j &= F^i{}_I\,(F^{-1})^J{}_j\,\partial_i v^j + F^i{}_I\,(F^{-1})^J{}_j\,\Gamma^j_{ik}\,v^k \\ &= (F^{-1})^J{}_j\,\partial_I(F^j{}_K\,v^K) + F^i{}_I\,(F^{-1})^J{}_j\,\Gamma^j_{ik}\,F^k{}_K\,v^K \\ &= \partial_I v^J + [(F^{-1})^J{}_j\,\partial_I F^j{}_K + (F^{-1})^J{}_j\,F^i{}_I\,\Gamma^j_{ik}\,F^k{}_K]\,v^K \\ &= \partial_I v^J + \Gamma^J_{IK}\,v^K\,, \end{aligned}$$

which implies relationship (1.49).

Problem 1.5. Using the symmetry of the Lagrangian connection coefficients eq:Lagrangian connection coefficients symmetry, relation (1.49) between Eulerian and Lagrangian connection coefficients implies that

$$\begin{aligned} 0 &= (F^{-1})^K{}_k\,F^i{}_I\,F^j{}_J\,\Gamma^k_{ij} - (F^{-1})^K{}_k\,F^i{}_J\,F^j{}_I\,\Gamma^k_{ij} + (F^{-1})^K{}_k\,(\partial_I F^k{}_J - \partial_J F^k{}_I) \\ &=(F^{-1})^K{}_k\,F^i{}_I\,F^j{}_J\,(\Gamma^k_{ij} - \Gamma^k_{ji}) + (F^{-1})^K{}_k\,(\partial_I\partial_J\varphi^k - \partial_J\partial_I\varphi^k)\,, \end{aligned} \tag{G.231}$$

which, since $\partial_I\partial_J\varphi^k = \partial_J\partial_I\varphi^k$, implies (1.51).

Problem 1.6. Differentiate the relationship $\Gamma^K_{IJ}\,\partial_K\varphi^i = \partial_I\partial_J\varphi^i$ with respect to time T to find the relationship

$$\begin{aligned}
F^i{}_K\,\partial_T\Gamma^K_{IJ} &= \partial_I\partial_J v^i - \Gamma^K_{IJ}\,\partial_K v^i \\
&= \partial_I\partial_J(F^i{}_K\,v^K) - \Gamma^K_{IJ}\,\partial_K\,(F^i{}_L\,v^L) \\
&= \partial_I\left[(\partial_J F^i{}_K)\,v^K + F^i{}_K\,\partial_J v^K\right] - \Gamma^K_{IJ}\,(\partial_K F^i{}_L)\,v^L - \Gamma^K_{IJ}\,F^i{}_L\,\partial_K v^L \\
&= \partial_I\left(F^i{}_L\,\nabla_J v^L\right) - \Gamma^L_{IJ}\,\nabla_L v^K \\
&= F^i{}_K\,(\Gamma^K_{IL}\,\nabla_J v^L + \partial_I\nabla_J v^K - \Gamma^L_{IJ}\,\nabla_L v^K) \\
&= F^i{}_K\,\nabla_I\nabla_J v^K\,.
\end{aligned}$$

Thus follows (1.55).

Problem 1.7. We have

$$\nabla_J v^K = \partial_J v^K + \Gamma^K_{JL}\,v^L\,,$$

and

$$\begin{aligned}
\nabla_I\nabla_J v^K &= \partial_I\partial_J v^K + (\partial_I\Gamma^K_{JL})\,v^L + \Gamma^K_{JL}\,\partial_I v^L \\
&\quad - \Gamma^L_{IJ}(\partial_L v^K + \Gamma^K_{LM}\,v^M) + \Gamma^K_{IL}(\partial_J v^L + \Gamma^L_{JM}\,v^M)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
(\nabla_I\nabla_J - \nabla_J\nabla_I)v^K &= (\partial_I\Gamma^K_{JL} - \partial_J\Gamma^K_{IL})\,v^L + \Gamma^K_{IL}\,\Gamma^L_{JM}\,v^M - \Gamma^K_{JL}\,\Gamma^L_{IM}\,v^M \\
&\quad - (\Gamma^L_{IJ} - \Gamma^L_{JI})\,\nabla_L v^K \\
&= \rho_{IJL}{}^K\,v^L - \sigma_{IJ}{}^L\,\nabla_L v^K\,,
\end{aligned}$$

where we have defined the torsion tensor with Lagrangian elements

$$\sigma_{IJ}{}^K \equiv \Gamma^K_{IJ} - \Gamma^K_{JI}\,,$$

and the curvature tensor with Lagrangian elements

$$\rho_{IJL}{}^K \equiv \partial_I\Gamma^K_{JL} - \partial_J\Gamma^K_{IL} + \Gamma^K_{IM}\,\Gamma^M_{JL} - \Gamma^K_{JM}\,\Gamma^M_{IL}\,.$$

Problem 1.8. We have, according to tensor transformation rules (D.26) and (D.27),

$$\begin{aligned}
(F^{-1})^I{}_i\,F^j{}_J\,\delta^i{}_j &= (F^{-1})^I{}_i\,F^i{}_J \\
&= \delta^I{}_J\,,
\end{aligned}$$

where in the last equality we used (1.18). Similarly,

$$\begin{aligned}
F^i{}_I\,(F^{-1})^J{}_j\,\delta^I{}_J &= F^i{}_I\,(F^{-1})^I{}_j \\
&= \delta^i{}_j\,.
\end{aligned}$$

Problem 1.9. Using definition (D.22), we have

$$D^i{}_j = \tfrac{1}{2}\left[\nabla_j v^i + g_{j\ell}\,(\nabla_k v^\ell)\,g^{ki}\right],$$

and

$$W^i{}_j = \tfrac{1}{2}\left[\nabla_j v^i - g_{j\ell}\left(\nabla_k v^\ell\right) g^{ki}\right] .$$

If we lower the contravariant indices, we have

$$D_{ij} = \tfrac{1}{2}\left[\nabla_j v_i + \nabla_i v_j\right] ,$$

and

$$W_{ij} = \tfrac{1}{2}\left[\nabla_j v_i - \nabla_i v_j\right] ,$$

making the symmetry and antisymmetry of $\mathbf{D}$ and $\mathbf{W}$, respectively, more salient.

Problem 1.10. The trace of the $(1,1)$ tensor $\mathbf{D}$ is given by

$$\begin{aligned}
\mathrm{tr}(\mathbf{D}) &= D^i{}_i \\
&= D^1{}_1 + D^2{}_2 + D^3{}_3 \\
&= \tfrac{1}{2}\left[\nabla_i v^i + g_{i\ell}\left(\nabla_k v^\ell\right) g^{ki}\right] \\
&= \tfrac{1}{2}\left[\nabla_i v^i + \delta^k{}_\ell\left(\nabla_k v^\ell\right)\right] \\
&= \tfrac{1}{2}(\nabla_i v^i + \nabla_k v^k) \\
&= \boldsymbol{\nabla}\cdot\mathbf{v} .
\end{aligned}$$

Similarly, the trace of the $(1,1)$ tensor $\mathbf{W}$ is given by

$$\begin{aligned}
\mathrm{tr}(\mathbf{W}) &= W^i{}_i \\
&= \tfrac{1}{2}(\nabla_i v^i - \nabla_k v^k) \\
&= 0 .
\end{aligned}$$

Problem 1.11. In Cartesian index notation, the left-hand side of equation (1.90) is $W_{ij}\, u^j$. For the right-hand side, we have

$$\begin{aligned}
\tfrac{1}{2}\,\epsilon_{ijk}\,\epsilon^{j\ell m}\left(\partial_\ell v_m\right) u^k &= \tfrac{1}{2}\left(\delta_i{}^m\,\delta_k{}^\ell - \delta_i{}^\ell\,\delta_k{}^m\right)\left(\partial_\ell v_m\right) u^k \\
&= \tfrac{1}{2}\left(\partial_k v_i - \partial_i v_k\right) u^k \\
&= W_{ik}\, u^k ,
\end{aligned}$$

in agreement with the expression for the left-hand side.

Problem 1.12. Upon differentiating the deformation gradient (1.16) with respect to time T , holding the Lagrangian coordinates fixed, we find that

$$\begin{aligned}
\partial_T F^i{}_I &= \partial_T \partial_I \varphi^i \\
&= \partial_I \partial_T \varphi^i \\
&= \partial_I v^i \\
&= (\partial_k v^i)\,\partial_I \varphi^k \\
&= G^i{}_k\, F^k{}_I ,
\end{aligned}$$

where in the last equality we used definitions (1.81) and (1.16).

Problem 1.13. This result may be obtained by differentiating the identity

$$F^i{}_I\,(F^{-1})^I{}_j = \delta^i{}_j$$

with respect to convected time T, and using equation (1.92).

Problem 1.14. Upon differentiating equation (1.66) with respect to convected time T we find

$$\begin{aligned}
\partial_T g_{IJ} &= (\partial_T F^i{}_I)\,F^j{}_J\,\delta_{ij} + F^i{}_I\,(\partial_T F^j{}_J)\,\delta_{ij} \\
&= G^i{}_k\,F^k{}_I\,F^j{}_J\,\delta_{ij} + F^i{}_I\,G^j{}_k\,F^k{}_J\,\delta_{ij} \\
&= F^i{}_I\,F^j{}_J\,(\partial_i v^k\,\delta_{kj} + \partial_j v^k\,\delta_{ik}) \\
&= F^i{}_I\,F^j{}_J\,(\partial_i v_j + \partial_j v_i) \\
&= 2\,F^i{}_I\,F^j{}_J\,d_{ij} \\
&= 2\,D_{IJ}\,.
\end{aligned}$$

Here we used equation (1.91) in the second equality, definition (1.81) in the third equality, and definition (1.85) in the fifth equality.

Problem 1.15. We have $Q(X,T) = q(\varphi(X,T),T)$ and thus

$$\begin{aligned}
\partial_T Q &= \partial_t q + \partial_T\varphi^j\,\partial_j q \\
&= \partial_t q + v^j\,\partial_j q\,.
\end{aligned} \tag{G.232}$$

Problem 1.16. Applying the general expression for the Lie derivative relative to the flow of matter $\mathbf{v}$ in Lagrangian components (1.99) to a vector, we have

$$\begin{aligned}
\partial_T u^I &= \partial_T[(F^{-1})^I{}_i\,u^i] \\
&= u^i\,[\partial_t(F^{-1})^I{}_i + \partial_T\varphi^j\,\partial_j(F^{-1})^I{}_i] + (F^{-1})^I{}_i\,(\partial_t u^i + \partial_T\varphi^j\,\partial_j u^i) \\
&= u^i\,[\partial_t\partial_i\Phi^I + v^j\,\partial_j(F^{-1})^I{}_i] + (F^{-1})^I{}_i\,(\partial_t u^i + v^j\,\partial_j u^i) \\
&= u^i\,(\,-\,\partial_i v^I + v^j\,\partial_j\partial_i\Phi^I) + (F^{-1})^I{}_i\,(\partial_t u^i + v^j\,\partial_j u^i) \\
&= u^i\,\{\,-\,\partial_i[(F^{-1})^I{}_j v^j] + v^j\,\partial_j\partial_i\Phi^I\} + (F^{-1})^I{}_i\,(\partial_t u^i + v^j\,\partial_j u^i) \\
&= \,-\,u^i\,(F^{-1})^I{}_j\,\partial_i v^j + (F^{-1})^I{}_i\,(\partial_t u^i + v^j\,\partial_j u^i) \\
&= (F^{-1})^I{}_i\,(\partial_t u^i + v^j\,\partial_j u^i - u^j\,\partial_j v^i)\,.
\end{aligned}$$

Problem 1.17. According to eqn. (F.141)

$$\mathcal{L}_{\mathbf{v}}\mathbf{u} = \mathsf{d}_{\mathsf{t}}\mathbf{u} + [\mathbf{v},\mathbf{u}] = \mathsf{d}_{\mathsf{t}}\mathbf{u} + \pounds_{\mathbf{v}}\mathbf{u}\,, \tag{G.233}$$

so we see that

$$\pounds_{\mathbf{v}}\mathbf{u} = [\mathbf{v},\mathbf{u}]\,. \tag{G.234}$$

Problem 1.18. Upon differentiating the identity $g^{IK}\,g_{KJ} = \delta^I{}_J$ with respect to the convected time T, using equation (1.98), we find

$$\begin{aligned}
g_{KJ}\,\partial_T g^{IK} &= \,-\,g^{IK}\,\partial_T g_{KJ} \\
&= \,-\,2\,g^{IK}\,D_{KJ}\,,
\end{aligned}$$

which implies upon raising the J index that

$$\partial_T g^{IJ} = -2\, D^{IJ}\,.$$

Problem 1.19. We have, using equations (1.98) and (1.103),

$$\begin{aligned}
\partial_T (\mathbf{t}^t)^I{}_J &= \partial_T (g^{IL}\, t^K{}_L\, g_{KJ}) \\
&= t^K{}_L\, g_{KJ}\, \partial_T g^{IL} + g^{IL}\, g_{KJ}\, \partial_T t^K{}_L + g^{IL}\, t^K{}_L\, \partial_T g_{KJ} \\
&= -2\, t^K{}_L\, g_{KJ}\, D^{IL} + g^{IL}\, g_{KJ}\, \partial_T t^K{}_L + 2\, g^{IL}\, t^K{}_L\, D_{KJ} \\
&= -2\, D^I{}_M\, g^{ML}\, t^K{}_L\, g_{KJ} + g^{IL}\, (\partial_T t^K{}_L)\, g_{KJ} + 2\, g^{IL}\, t^K{}_L\, g_{KM}\, D^M{}_J\,.
\end{aligned}$$

This implies that

$$\mathcal{L}_{\mathbf{v}} \mathbf{t}^t = -2\, \mathbf{D} \cdot \mathbf{t}^t + (\mathcal{L}_{\mathbf{v}} \mathbf{t})^t + 2\, \mathbf{t}^t \cdot \mathbf{D}\,,$$

in agreement with equation (1.104).

Problem 1.20. We have $q(r,t) = Q(\Phi(r,t),t)$ and thus

$$\begin{aligned}
\partial_t q &= \partial_T Q + \partial_t \Phi^J\, \partial_J Q \\
&= \partial_T Q - v^J\, \partial_J Q\,,
\end{aligned}$$

where we used relationship (1.14).

Problem 1.21. Applying the general expression for the Euler time derivative in Eulerian components (1.108) to a vector, we have

$$\begin{aligned}
\partial_t u^i &= \partial_t (F^i{}_I\, u^I) \\
&= u^I\, (\partial_T F^i{}_I + \partial_t \Phi^J\, \partial_J F^i{}_I) + F^i{}_I\, (\partial_T u^I + \partial_t \Phi^J\, \partial_J u^I) \\
&= u^I\, (\partial_I v^i - v^J\, \partial_J F^i{}_I) + F^i{}_I\, (\partial_T u^I - v^J\, \partial_J u^I) \\
&= u^I\, F^i{}_J\, \partial_I v^J + F^i{}_I\, (\partial_T u^I - v^J\, \partial_J u^I) \\
&= F^i{}_I\, (\partial_T u^I - v^J\, \partial_J u^I + u^J\, \partial_J v^I)\,.
\end{aligned}$$

Problem 1.22. We have

$$\begin{aligned}
(\mathrm{d}_t \mathbf{u})^I &= \partial_T u^I - v^J\, \partial_J u^I + u^J\, \partial_J v^I \\
&= \partial_T u^I - v^J\, (\nabla_J u^I - \Gamma^I_{JK}\, u^K) + u^J\, (\nabla_J v^I - \Gamma^I_{JK}\, v^K) \\
&= \partial_T u^I - v^J\, \nabla_J u^I + u^J\, \nabla_J v^I + (\Gamma^I_{JK} - \Gamma^I_{KJ})\, u^J\, v^K \\
&= \partial_T u^I - v^J\, \nabla_J u^I + u^J\, \nabla_J v^I\,,
\end{aligned}$$

provided the connection is torsion free: $\Gamma^I_{JK} = \Gamma^I_{KJ}$.

Problem 1.23. Upon expressing the left-hand side of equation (1.112) in Eulerian components we have $\mathrm{d}_t \mathbf{v} = \partial_t v^i\, \mathbf{e}_i$, whereas upon expressing the right-hand side of equation (1.112) in Lagrangian components we have $\mathcal{L}_{\mathbf{v}} \mathbf{v} = \partial_T v^I\, \mathbf{e}_I$. Thus

$$\begin{aligned}
\mathrm{d}_t \mathbf{v} &= \partial_t v^i\, \mathbf{e}_i \\
&= \mathcal{L}_{\mathbf{v}} \mathbf{v} = \partial_T v^I\, \mathbf{e}_I\,.
\end{aligned}$$

Problem 1.24. Working in Eulerian coordinates, we have according to (1.114) $\mathrm{D}_t\mathrm{q} = \partial_t q + v^j\,\partial_j q$, and according to Problem 1.15 $\mathcal{L}_{\mathbf{v}}\mathrm{q} = \partial_t q + v^j\,\partial_j q$. Hence

$$\mathrm{D}_t\mathrm{q} = \mathcal{L}_{\mathbf{v}}\mathrm{q}\,.$$

Problem 1.25. According to (1.114) we have

$$\begin{aligned}\mathrm{D}_t\mathbf{v} &= \mathrm{d}_t\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\\ &= (\partial_t v^i + v^j\,\nabla_j v^i)\,\mathbf{e}_i\,.\end{aligned}$$

According to Problem 1.23, $\partial_t v^i\,\mathbf{e}_i = \partial_T v^I\,\mathbf{e}_I$, and thus

$$(\partial_t v^i + v^j\,\nabla_j v^i)\,\mathbf{e}_i = (\partial_T v^I + v^J\,\nabla_J v^I)\,\mathbf{e}_I\,.$$

Problem 1.26. We have, using relationship (1.91),

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\mathbf{t})^I{}_J &= \partial_T t^I{}_J\\ &= \partial_T[(F^{-1})^I{}_i\,F^j{}_J\,t^i{}_j]\\ &= F^j{}_J\,t^i{}_j\,\partial_T(F^{-1})^I{}_i + (F^{-1})^I{}_i\,F^j{}_J\,(\partial_t t^i{}_j + v^k\,\partial_k t^i{}_j) + (F^{-1})^I{}_i\,t^i{}_j\,\partial_T F^j{}_J\\ &= (F^{-1})^I{}_i\,F^j{}_J\,[\partial_t t^i{}_j + v^k\,\partial_k t^i{}_j + t^k{}_j\,F^i{}_K\,\partial_T(F^{-1})^K{}_k + t^i{}_k\,(F^{-1})^K{}_j\,\partial_T F^k{}_K]\\ &= (F^{-1})^I{}_i\,F^j{}_J\,(\partial_t t^i{}_j + v^k\,\partial_k t^i{}_j - G^i{}_k\,t^k{}_j + t^i{}_k\,G^k{}_j)\\ &= (F^{-1})^I{}_i\,F^j{}_J\,(\mathrm{D}_t\mathbf{t} - \mathbf{G}\cdot\mathbf{t} + \mathbf{t}\cdot\mathbf{G})^i{}_j\,,\end{aligned}$$

in agreement with equation (1.117).

Problem 1.27. We have

$$g^{IJ} = (F^{-1})^I{}_i\,(F^{-1})^J{}_j\,g^{ij}\,.$$

Taking the determinant of this expression yields

$$\underline{\underline{G}} = \frac{1}{J^2}\,\underline{\underline{g}}\,,$$

and thus

$$\underline{G} = \frac{1}{J}\,\underline{g}\,.$$

Problem 1.28. It is a matter of tedious algebra to derive equation (1.134), performing the sums over J, K, L, M, and N, to obtain the desired expression by investigating the cases $I = P$ and $I \neq P$.

Problem 1.29. Upon multiplying Cramer's rule (1.135) by g_{NQ} and using equation (1.134) on the resulting right-hand side, both sides of the equation reduce to the Kronecker delta $\delta^K{}_Q$.

Problem 1.30. We have, based on equation (1.129),

$$\overline{\overline{G}} = \tfrac{1}{3!}\,\overline{\varepsilon}^{\,JKL}\,g_{JM}\,g_{KN}\,g_{LP}\,\overline{\varepsilon}^{\,MNP}\,,$$

so

$$\partial_I\overline{\overline{G}} = \tfrac{1}{2}\,\overline{\varepsilon}^{\,JKL}\,(\partial_I g_{JM})\,g_{KN}\,g_{LP}\,\overline{\varepsilon}^{\,MNP}\,.$$

Using $\nabla_I g_{JK} = 0$, we see that

$$\begin{aligned}\partial_I \overline{\overline{G}} &= \tfrac{1}{2}\,\overline{\varepsilon}^{\,JKL}\,(\Gamma^Q_{IJ}\,g_{QM} + \Gamma^Q_{IM}\,g_{JQ})\,g_{KN}\,g_{LP}\,\overline{\varepsilon}^{\,MNP} \\ &= \tfrac{1}{2}\,\overline{\varepsilon}^{\,JKL}\,\Gamma^Q_{IJ}\,g_{QM}\,g_{KN}\,g_{LP}\,\overline{\varepsilon}^{\,MNP} + \tfrac{1}{2}\,\overline{\varepsilon}^{\,JKL}\,\Gamma^Q_{IM}\,g_{JQ}\,g_{KN}\,g_{LP}\,\overline{\varepsilon}^{\,MNP} \\ &= 2\,\overline{\overline{G}}\,\Gamma^J_{IJ}\,,\end{aligned}$$

where in the third equality we used the solution to Problem 1.28. We conclude that

$$2\,\overline{G}\,\partial_I\overline{G} \;= 2\,\overline{G}^2\,\Gamma^J_{IJ}\,,$$

or, alternatively,

$$\partial_I\overline{G} \;= \overline{G}\,\Gamma^J_{IJ}\,.$$

Problem 1.31. We have

$$\partial_I\epsilon_{JKL} = \partial_I(\overline{G}\,\underline{\epsilon}_{JKL}) = (\partial_I\overline{G})\,\underline{\epsilon}_{JKL}\,.$$

Now use the solution to Problem 1.30.

Problem 1.32. We have

$$\begin{aligned}\nabla_I\epsilon_{JKL} &= \partial_I\epsilon_{JKL} - \Gamma^M_{IJ}\,\epsilon_{MKL} - \Gamma^M_{IK}\,\epsilon_{JML} - \Gamma^M_{IL}\,\epsilon_{JKM} \\ &= \epsilon_{JKL}\,\Gamma^M_{IM} - \Gamma^M_{IJ}\,\epsilon_{MKL} - \Gamma^M_{IK}\,\epsilon_{JML} - \Gamma^M_{IL}\,\epsilon_{JKM}\,,\end{aligned}$$

where in the second equality we used the solution to Problem 1.31. Choose $J = 1$. Then $K = 2$ and $L = 3$ or $K = 3$ and $L = 2$. Thus for the former

$$\begin{aligned}\nabla_I\epsilon_{123} &= \epsilon_{123}\,\Gamma^M_{IM} - \Gamma^M_{I1}\,\epsilon_{M23} - \Gamma^M_{I2}\,\epsilon_{1M3} - \Gamma^M_{I3}\,\epsilon_{12M} \\ &= \epsilon_{123}\,\Gamma^M_{IM} - \Gamma^1_{I1}\,\epsilon_{123} - \Gamma^2_{I2}\,\epsilon_{123} - \Gamma^3_{I3}\,\epsilon_{123} \\ &= 0\,,\end{aligned}$$

and for the latter

$$\begin{aligned}\nabla_I\epsilon_{132} &= \epsilon_{132}\,\Gamma^M_{IM} - \Gamma^M_{I1}\,\epsilon_{M32} - \Gamma^M_{I2}\,\epsilon_{1M2} - \Gamma^M_{I2}\,\epsilon_{13M} \\ &= \epsilon_{132}\,\Gamma^M_{IM} - \Gamma^1_{I1}\,\epsilon_{132} - \Gamma^3_{I2}\,\epsilon_{132} - \Gamma^2_{I2}\,\epsilon_{132}\,.\end{aligned}$$

Since the choice $J = 1$ is arbitrary, the result is generally valid.

Problem 1.33. For a $(1,1)$ tensor we have according to equation (1.148) in Eulerian components

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\mathbf{t})^i{}_j &= F^i{}_I\,(F^{-1})^J{}_j\,\partial_T\left[(F^{-1})^I{}_k\,F^\ell{}_J\,t^k{}_\ell\right] \\ &= t^k{}_j\,F^i{}_I\,\partial_T(F^{-1})^I{}_k + t^i{}_\ell\,(F^{-1})^J{}_j\,\partial_T F^\ell{}_J + \partial_t t^i{}_j + v^m\,\partial_m t^i{}_j \\ &= \partial_t t^i{}_j + v^k\,\partial_k t^i{}_j + t^i{}_k\,G^k{}_j - G^i{}_k\,t^k{}_j\,,\end{aligned}$$

where we used equation (1.91). These are indeed the Eulerian components of the Lie derivative of a $(1,1)$ tensor. Much simpler, we have in Lagrangian components

$$(\mathcal{L}_{\mathbf{v}}\mathbf{t})^I{}_J = \partial_T t^I{}_J\,,$$

in agreement with definition (1.99) of the Lie derivative in Lagrangian components.
Problem 1.34. From equation (1.162) we deduce immediately that

$$J = \frac{\overline{G}}{\overline{G}_0} = \exp\left[\int_{T_0}^{T} \mathrm{tr}(\mathbf{D})\, \mathrm{d}T'\right].$$

Problem 1.35. Upon multiplying equation (1.98) by g^{IJ}, we have

$$g^{IJ}\,\partial_T g_{IJ} = 2\, g^{IJ}\, D_{IJ} = 2\, D^I{}_I = 2\,\mathrm{tr}(\mathbf{D}).$$

Upon comparing this result with equation (1.161), we conclude that

$$\tfrac{1}{2}\, g^{IJ}\,\partial_T g_{IJ} = \underline{G}\,\partial_T \overline{G}.$$

Problem 1.36. We have

$$\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\epsilon})_{IJK} &= \partial_T \epsilon_{IJK} \\
&= \partial_T(\overline{G}\,\underline{\epsilon}_{IJK}) \\
&= \overline{G}\,(\boldsymbol{\nabla}\cdot\mathbf{v})\,\underline{\epsilon}_{IJK}) \\
&= (\boldsymbol{\nabla}\cdot\mathbf{v})\,\epsilon_{IJK},
\end{aligned}$$

where in the second equality we used equation (1.161).
Problem 1.37. We have

$$\begin{aligned}
\hat{n}_I\, d\mathrm{S} &= F^i{}_I\, \hat{n}_i\, d\mathrm{S} \\
&= F^i{}_I\, \epsilon_{ijk}\, \delta r^j\, dr^k \\
&= F^i{}_I\, \epsilon_{ijk}\, F^j{}_J\, \delta X^J\, F^k{}_K\, dX^K \\
&= \epsilon_{IJK}\, \delta X^J\, dX^K.
\end{aligned}$$

Problem 1.38. The pullback of the spatial surface element $\hat{\mathbf{n}}\, d\mathrm{S}$ is determined by

$$\begin{aligned}
\varphi^*(\hat{\mathbf{n}}\, d\mathrm{S}) &= \epsilon_{ijk}\, \delta r^j\, dr^k\, F^i{}_I\, \mathring{\mathbf{e}}^I \\
&= \epsilon_{IJK}\, \delta X^J\, dX^K\, \mathring{\mathbf{e}}^I \\
&= \overline{G}\,\underline{\varepsilon}_{IJK}\, \delta X^J\, dX^K\, \mathring{\mathbf{e}}^I \\
&= \overline{G}\,\underline{\mathring{G}}\,\overline{\mathring{G}}\,\underline{\varepsilon}_{IJK}\, \delta X^J\, dX^K\, \mathring{\mathbf{e}}^I \\
&= \overline{G}\,\underline{\mathring{G}}\,\mathring{\hat{\mathbf{n}}}\, d\mathring{\mathrm{S}} \\
&= J\,\overline{g}\,\underline{\mathring{G}}\,\mathring{\hat{\mathbf{n}}}\, d\mathring{\mathrm{S}},
\end{aligned}$$

where in the last equality we used (1.129). If the referential metric and the spatial metric are both Cartesian we find the relationship

$$\varphi^*(\hat{\mathbf{n}}\, d\mathrm{S}) = J\,\mathring{\hat{\mathbf{n}}}\, d\mathring{\mathrm{S}}.$$

Problem 1.39. According to Problem 1.4, the relationship between Eulerian and Lagrangian connection coefficients is

$$\Gamma^K_{IJ} = (F^{-1})^K{}_k\, F^i{}_I\, F^j{}_J\, \Gamma^k_{ij} + (F^{-1})^K{}_k\, \partial_I F^k{}_J.$$

Expressed as connection one-forms, we have

$$\begin{aligned}
\mathbf{\Gamma}^K_J &\equiv \Gamma^K_{IJ}\,\mathbf{e}^I \\
&= (F^{-1})^K{}_k\,F^i{}_I\,F^j{}_J\,\Gamma^k_{ij}\,\mathbf{e}^I + (F^{-1})^K{}_k\,(\partial_I F^k{}_J)\,\mathbf{e}^I \\
&= (F^{-1})^K{}_k\,F^j{}_J\,\Gamma^k_{ij}\,\mathbf{e}^i + (F^{-1})^K{}_k\,(\partial_I F^k{}_J)\,\mathbf{e}^I \\
&= (F^{-1})^K{}_k\,(F^j{}_J\,\mathbf{\Gamma}^k_j + \mathrm{d}F^k{}_J)\,.
\end{aligned}$$

From here, it follows that

$$F^i{}_J\,\mathbf{\Gamma}^J_I = F^j{}_I\,\mathbf{\Gamma}^i_j + \mathrm{d}F^i{}_I\,.$$

Problem 1.40. We have

$$\begin{aligned}
\mathrm{d}\boldsymbol{\alpha}^i + \mathbf{\Gamma}^i_j \wedge \boldsymbol{\alpha}^j &= \mathrm{d}(F^i{}_I\,\boldsymbol{\alpha}^I + F^j{}_J\,\mathbf{\Gamma}^i_j \wedge \boldsymbol{\alpha}^J \\
&= F^i{}_I\,\mathrm{d}\boldsymbol{\alpha}^I + (\mathrm{d}F^i{}_J + F^j{}_J\,\mathbf{\Gamma}^i_j) \wedge \boldsymbol{\alpha}^J \\
&= F^i{}_I\,(\mathrm{d}\boldsymbol{\alpha}^I + \mathbf{\Gamma}^I_J \wedge \boldsymbol{\alpha}^J)\,,
\end{aligned}$$

wherein the last equality we used (1.211).
Problem 1.41. To prove expression (1.218), we must show that

$$\mathrm{D}g_{IJ} = 0\,.$$

Using definition (1.213) of the exterior covariant derivative, we have

$$\begin{aligned}
\mathrm{D}g_{IJ} &= \mathrm{d}g_{IJ} - \mathbf{\Gamma}^K_I\,g_{KJ} - \mathbf{\Gamma}^K_J\,g_{IK} \\
&= (\partial_K g_{IJ} - \Gamma^L_{KI}\,g_{LJ} - \Gamma^L_{KJ}\,g_{IL})\,\mathbf{e}^K \\
&= (\nabla_K g_{IJ})\,\mathbf{e}^K \\
&= 0\,.
\end{aligned}$$

Where in the last equality we used the fact that the covariant derivative of the metric tensor vanishes. To prove expression (1.219), we must show that

$$\mathfrak{L}_{\mathbf{v}} g_{IJ} = 0\,.$$

Using the covariant Lie derivative of the metric in Lagrangian coordinates, we have

$$\begin{aligned}
\mathfrak{L}_{\mathbf{v}} g_{IJ} &= \mathcal{L}_{\mathbf{v}} g_{IJ} - g_{KJ}\,\nabla_I v^K - g_{IK}\,\nabla_J v^K \\
&= \partial_T g_{IJ} - g_{KJ}\,\nabla_I v^K - g_{IK}\,\nabla_J v^K \\
&= 2\,D_{IJ} - \nabla_I v_J - \nabla_J v^I \\
&= 0\,,
\end{aligned}$$

where we used relationship (1.98) and definition (1.85).
Problem 1.42. To obtain the Eulerian version of conservation of mass (1.229) from the form expression (1.227), we note that

$$\mathrm{d}_t\boldsymbol{\rho} = \tfrac{1}{3!}\,\partial_t\rho\,\overline{g}\,\underline{\epsilon}_{ijk}\,\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k = \partial_t\rho\,\boldsymbol{\epsilon}\,,$$

whereas

$$\begin{aligned} \mathrm{d}(\mathbf{v}\cdot\boldsymbol{\rho}) &= \tfrac{1}{2}\,\partial_\ell(\rho\, v^i\, \overline{g})\, \underline{\epsilon}_{ijk}\, \mathbf{e}^\ell \wedge \mathbf{e}^j \wedge \mathbf{e}^k \\ &= \underline{g}\, \partial_i(\rho\, v^i\, \overline{g})\, \boldsymbol{\epsilon} \\ &= \nabla_i(\rho\, v^i)\, \boldsymbol{\epsilon}\,. \end{aligned}$$

To obtain the Lagrangian version of conservation of mass (1.230) from the form expression (1.228), we note that

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}\boldsymbol{\rho} &= \tfrac{1}{3!}\,\partial_T(\varrho\,\overline{G})\,\underline{\epsilon}_{IJK}\,\mathbf{e}^I \wedge \mathbf{e}^J \wedge \mathbf{e}^K \\ &= \underline{G}\,\partial_T(\varrho\,\overline{G})\,\boldsymbol{\epsilon} \\ &= J^{-1}\,\partial_T(\varrho\, J)\,\boldsymbol{\epsilon}\,. \end{aligned}$$

In the last equality, we used the fact that according to (1.161) and (1.164), $\underline{G}\,\partial_T \varrho\,\overline{G} = J^{-1}\,\partial_T J$.

Problem 1.43. Upon differentiating equation (1.238) with respect to comoving time we find

$$\begin{aligned} J\,\partial_T P + P\,\partial_T J &= J\,(\partial_T P + P\, J^{-1}\,\partial_T J) \\ &= J\,(\partial_T P + P\,\nabla_I V^I) \\ &= 0\,, \end{aligned}$$

in agreement with equation (1.239).

Problem 1.44. Using the Lagrangian and Eulerian expressions for the Lie derivative of a $(0,0)$ tensor, we have

$$\mathcal{L}_{\mathbf{v}}\rho = \partial_T P = \partial_t\rho + v^i\,\partial_i\rho\,,$$

and thus the Eulerian version of equation (1.240) is

$$\partial_t\rho + v^i\,\partial_i\rho + \rho\,\partial_i v^i = \partial_t\rho + \partial_i(\rho\, v^i) = 0\,,$$

which is in agreement with the Eulerian version of equation (1.223).

Problem 1.45. The mass form is defined as

$$\boldsymbol{\rho} = \rho\,\boldsymbol{\epsilon}\,.$$

Upon taking the Lie derivative of this expression, we obtain

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}\boldsymbol{\rho} &= (\mathcal{L}_{\mathbf{v}}\rho)\,\boldsymbol{\epsilon} + \rho\,\mathcal{L}_{\mathbf{v}}\boldsymbol{\epsilon} \\ &= -\,\rho\,(\boldsymbol{\nabla}\cdot\mathbf{v})\,\boldsymbol{\epsilon} + \rho\,(\boldsymbol{\nabla}\cdot\mathbf{v})\,\boldsymbol{\epsilon} \\ &= \mathbf{0}\,, \end{aligned}$$

where we used expressions (1.240) and Problem 1.240.

Problem 1.46. We must demonstrate that the deformation gradient tensor indeed transforms as a tensor. We have

$$\begin{aligned} (F^{-1})^I{}_i\, F^j{}_J\, F^i{}_j &= (F^{-1})^I{}_i\, F^j{}_J\, F^i{}_K(X,T)\,(F^{-1})^K{}_j(\,\varphi(X,T_0)\,,T_0) \\ &= F^j{}_J\,(F^{-1})^I{}_j(\,\varphi(X,T_0)\,,T_0) \\ &= F^I{}_J\,, \end{aligned}$$

as required.

Problem 1.47. Upon multiplying the deformation gradient tensor (1.242) and its inverse (1.245), we find in Eulerian coordinates

$$F^i{}_I(X,T)\,(F^{-1})^I{}_k(\,\varphi(X,T_0)\,,T_0)F^k{}_J(X,T_0)\,(F^{-1})^J{}_j(\,\varphi(X,T)\,,T) = \delta^i{}_j\,,$$

and upon multiplying the deformation gradient tensor (1.243) and its inverse (1.246), we find in Lagrangian coordinates

$$(F^{-1})^I{}_i(\,\varphi(X,T_0)\,,T_0)\,F^i{}_K(X,T)(F^{-1})^K{}_j(\,\varphi(X,T)\,,T)\,F^j{}_J(X,T_0) = \delta^I{}_J\,.$$

Reversing the order of the deformation gradient and its inverse results in similar identities.

Problem 1.48. Working in Eulerian components, we have

$$\begin{aligned}
F^i{}_k(X,T;T_1)\,F^k{}_j(X,T_1;T_0) &= F^i{}_I(X,T)\,(F^{-1})^I{}_k(\,\varphi(X,T_1)\,,T_1)\,F^k{}_J(X,T_1)\,(F^{-1})^J{}_j(\,\varphi(X,T_0)\,,T_0)\\
&= F^i{}_I(X,T)\,(F^{-1})^I{}_j(\,\varphi(X,T_0)\,,T_0)\\
&= F^i{}_k(X,T;T_0)\,.
\end{aligned}$$

Problem 1.49. Working in Lagrangian coordinates, we have

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}}\mathbf{F} &= (F^{-1})^I{}_i(\,\varphi(X,T_0)\,,T_0)\,\partial_T F^i{}_J(X,T)\,\mathbf{e}_I\otimes\mathbf{e}^J\\
&= (F^{-1})^I{}_i(\,\varphi(X,T_0)\,,T_0)\,F^i{}_K\,G^K{}_J\,\mathbf{e}_I\otimes\mathbf{e}^J\\
&= F^I{}_K\,G^K{}_J\,\mathbf{e}_I\otimes\mathbf{e}^J\\
&= \mathbf{F}\cdot\mathbf{G}\,,
\end{aligned}$$

where in the second equality we used (1.94).

Problem 1.50. We have

$$(\varphi^*\mathbf{g})_{IJ} = F^i{}_I\,F^j{}_J\,g_{ij}\,,$$

and thus

$$(\mathring{\mathbf{g}}^{-1}\cdot\varphi^*\mathbf{g})^K{}_J = \mathring{g}^{KI}\,F^i{}_I\,F^j{}_J\,g_{ij}\,.$$

Pushing this forward again we find

$$\begin{aligned}
\varphi_*\left(\mathring{\mathbf{g}}^{-1}\cdot\varphi^*\mathbf{g}\right)^k{}_\ell &= F^k{}_K\,(F^{-1})^J{}_\ell\,\mathring{g}^{KI}\,F^i{}_I\,F^j{}_J\,g_{ij}\\
&= F^k{}_K\,F^i{}_I\,\mathring{g}^{KI}\,g_{i\ell}\,,
\end{aligned}$$

as required.

Problem 1.51. We have from (1.258),

$$\begin{aligned}
C_{IJ}(X,T;T_0) &= g_{IL}(X,T)\,C^L{}_J(X,T;T_0)\\
&= g_{IL}(X,T)\,g^{LK}(X,T_0)\,g_{KJ}(X,T)\\
&= g_{JK}(X,T)\,g^{KL}(X,T_0)\,g_{LI}(X,T)\\
&= g_{JL}(X,T)\,g^{LK}(X,T_0)\,g_{KI}(X,T)\\
&= C_{JI}(X,T;T_0)\,.
\end{aligned}$$

Problem 1.52. Using the Eulerian components of the deformation gradient tensor (1.242), we have

$$\begin{aligned}
& F^i{}_k\, F^m{}_\ell\, g^{k\ell}(\,\varphi(X,T_0))\, g_{mj} \\
&= F^i{}_I(X,T)\,(F^{-1})^I{}_k(\,\varphi(X,T_0)\,,T_0)\, F^m{}_J(X,T)\,(F^{-1})^J{}_\ell(\,\varphi(X,T_0)\,,T_0)\, g^{k\ell}(\,\varphi(X,T_0))\, g_{mj} \\
&= F^i{}_I\, F^m{}_J\, \mathring{g}^{IJ}\, g_{mj} \\
&= C^i{}_j\,,
\end{aligned}$$

where in the last equality we recognized (1.256).

Problem 1.53. By definition, we have according to equation (1.258)

$$C^I{}_J(X,T;T_0) = g^{IK}(X,T_0)\, g_{KJ}(X,T)\,.$$

Thus

$$C^I{}_J(X,T_0;T) = g^{IK}(X,T)\, g_{KJ}(X,T_0)\,.$$

Upon inspection, this is precisely (1.268).

Problem 1.54. We have

$$(\varphi_*\mathring{\mathbf{g}})_{IJ} = \mathring{g}_{IJ}\,,$$

and thus

$$(\mathbf{C}^{-1})^K{}_J = g^{KI}\,\mathring{g}_{IJ}\,.$$

Problem 1.55. We have

$$(\varphi^*\mathbf{g}^{-1})^{IJ} = g^{IJ}\,,$$

and

$$(\mathring{\mathbf{g}}\cdot\varphi^*\mathbf{g}^{-1})^J{}_K = \mathring{g}_{KI}\, g^{IJ}\,,$$

and thus, in Lagrangian components,

$$(\mathbf{C}^{-1})^J{}_K = \left[\varphi_*\left(\mathring{\mathbf{g}}\cdot\varphi^*\mathbf{g}^{-1}\right)\right]{}^J{}_K = g^{JI}\,\mathring{g}_{IK}\,,$$

in agreement with (1.268).

Problem 1.56. Using equation (1.104) for the Cauchy-Green tensor, we have

$$\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\,\mathbf{C})^t &= \mathcal{L}_{\mathbf{v}}\,\mathbf{C}^t + 2\,\mathbf{D}\cdot\mathbf{C}^t - 2\,\mathbf{C}^t\cdot\mathbf{D} \\
&= \mathcal{L}_{\mathbf{v}}\,\mathbf{C} + 2\,\mathbf{D}\cdot\mathbf{C} - 2\,\mathbf{C}\cdot\mathbf{D} \\
&= 2\,\mathbf{C}\cdot\mathbf{D} + 2\,\mathbf{D}\cdot\mathbf{C} - 2\,\mathbf{C}\cdot\mathbf{D} \\
&= 2\,\mathbf{D}\cdot\mathbf{C}\,,
\end{aligned}$$

where we used the symmetry of the Cauchy-Green tensor (1.260) and the evolution equation (1.277). We conclude that this result corresponds to taking the transpose of the evolution equation (1.277).

Problem 1.57. Using the Zaremba-Jaumann rate identified in equation (1.118) for the Cauchy-Green tensor, we find

$$\begin{aligned}
\mathrm{D_t}\mathbf{C} + \mathbf{C}\cdot\mathbf{W} - \mathbf{W}\cdot\mathbf{C} &= 2\,\mathbf{D}\cdot\mathbf{C} + \mathbf{C}\cdot\mathbf{D} - \mathbf{D}\cdot\mathbf{C} \\
&= \mathbf{C}\cdot\mathbf{D} + \mathbf{D}\cdot\mathbf{C}\,,
\end{aligned}$$

in agreement with equation (1.278).

Problem 1.58. Upon taking the determinant of equation (1.258) we obtain

$$\det(\mathbf{C}) = \overline{G}^2 \, \underline{\mathring{G}}^2 \,.$$

Using the definition of the Jacobian (1.156) then leads to (1.280).

Problem 1.59. The pullback of the spatial metric has referential components

$$(\varphi^* \mathbf{g})_{IJ} = g_{IJ} \,,$$

and thus equation (1.285) has referential components

$$\mathring{E}_{IJ} = \tfrac{1}{2} \left(g_{IJ} - \mathring{g}_{IJ} \right) .$$

Raising the first index of this measure of strain with the referential inverse metric $\mathring{\mathbf{g}}^{-1}$ yields

$$\mathring{E}^I{}_J = \tfrac{1}{2} \left(\mathring{g}^{IK} \, g_{KJ} - \delta^I{}_J \right) ,$$

which are equal to the Lagrangian components of the Lagrangian strain tensor (1.284). Pushing these elements forward yields the Lagrangian strain tensor (1.284) in the spatial manifold.

This sequence of steps may also be expressed as

$$\mathring{\mathbf{g}}^{-1} \cdot \mathring{\mathbf{E}} = \tfrac{1}{2} \left(\mathring{\mathbf{g}}^{-1} \cdot \varphi^* \mathbf{g} - \mathring{\mathbf{I}} \right) \,,$$

followed by

$$\begin{aligned} \varphi_*(\mathring{\mathbf{g}}^{-1} \cdot \mathring{\mathbf{E}}) &= \tfrac{1}{2} \left[\varphi_*(\mathring{\mathbf{g}}^{-1} \cdot \varphi^* \mathbf{g}) - \varphi_*(\mathring{\mathbf{I}}) \right] \\ &= \tfrac{1}{2} \left[\varphi_*(\mathring{\mathbf{g}}^{-1} \cdot \varphi^* \mathbf{g}) - \mathbf{I} \right] \\ &= \mathbf{E}^{\mathrm{L}} \,, \end{aligned}$$

in agreement with equation (1.286).

Problem 1.60. Expressed in Lagrangian components, equation (1.292) is

$$\begin{aligned} \left(E^{\mathrm{E}} \right)^I{}_J &= \tfrac{1}{2} \left[\delta^I{}_J - (C^{-1})^I{}_J \right] \\ &= \tfrac{1}{2} \left(\delta^I{}_J - g^{IK} \, g^0_{KJ} \right) \,, \end{aligned}$$

and lowering the first index yields

$$\left(E^{\mathrm{E}\,\flat} \right)_{IJ} = \tfrac{1}{2} \left(g_{IJ} - g^0_{IJ} \right) \,.$$

Expressed as a tensor equation, this is equation (1.293).

Problem 1.61. We observe that

$$\begin{aligned} \mathbf{C}^2 &= \mathbf{C} \cdot \mathbf{C} \\ &= \mathbf{Q} \cdot \boldsymbol{\Lambda} \cdot \mathbf{Q}^t \cdot \mathbf{Q} \cdot \boldsymbol{\Lambda} \cdot \mathbf{Q}^t \\ &= \mathbf{Q} \cdot \boldsymbol{\Lambda}^2 \cdot \mathbf{Q}^t \,, \end{aligned}$$

and that each power $n = 0, 1, \ldots$ of $\mathbf{C}$ is of the form

$$\mathbf{C}^n = \mathbf{Q} \cdot \mathbf{\Lambda}^n \cdot \mathbf{Q}^t \, .$$

Furthermore, we have

$$(\mathbf{C} - \mathbf{I})^n = \mathbf{Q} \cdot (\mathbf{\Lambda} - \mathbf{I})^n \cdot \mathbf{Q}^t \, .$$

Expressing the logarithmic strain as a Mercator series, as discussed in Appendix D.5, we have based on equation (D.39)

$$\begin{aligned}
\mathbf{E}^{\log} &= \tfrac{1}{2} \log(\mathbf{C}) \\
&= \tfrac{1}{2} \left[(\mathbf{C} - \mathbf{I}) - \tfrac{1}{2} (\mathbf{C} - \mathbf{I})^2 + \tfrac{1}{3} (\mathbf{C} - \mathbf{I})^3 - \tfrac{1}{4} (\mathbf{C} - \mathbf{I})^4 \cdots \right] \\
&= \tfrac{1}{2} \mathbf{Q} \cdot \left[(\mathbf{\Lambda} - \mathbf{I}) - \tfrac{1}{2} (\mathbf{\Lambda} - \mathbf{I})^2 + \tfrac{1}{3} (\mathbf{\Lambda} - \mathbf{I})^3 - \tfrac{1}{4} (\mathbf{\Lambda} - \mathbf{I})^4 \cdots \right] \cdot \mathbf{Q}^t \\
&= \tfrac{1}{2} \mathbf{Q} \cdot \log(\mathbf{\Lambda}) \cdot \mathbf{Q}^t \, .
\end{aligned}$$

Problem 1.62. Upon substituting the diagonalization $E = Q\, f(\Lambda)\, Q^t$ into the corotational material derivative of the strain (1.306), we find that

$$\dot{Q}\, f(\Lambda) Q^t + Q\, f'(\Lambda)\, \dot{\Lambda} Q^t + Q\, f(\Lambda)\, \dot{Q}^t + Q\, f(\Lambda)\, Q^t (W + N) - (W + N) Q\, f(\Lambda)\, Q^t = D \, .$$

Using the rotation-rate tensor R defined by (1.311) and the fact that $QQ^t = Q^t Q = I$, this expression may be rearranged by multiplying from the left with Q^t and from the right with Q to obtain

$$f'(\Lambda)\, \dot{\Lambda} + f(\Lambda)\, (\bar{W} + \bar{N} + R) - (\bar{W} + \bar{N} + R)\, f(\Lambda) = \bar{D} \, ,$$

where $\bar{W}$ and $\bar{D}$ are defined in equation (1.310), and where $\bar{N} \equiv Q^t N Q$.
Problem 1.63. Start with equation (1.320), using the fact that $f(x) = \frac{1}{2} \log x$:

$$\tfrac{1}{2} \left[\log(\lambda_i) - \log(\lambda_j) \right] (\bar{W}^i{}_j + R^i{}_j + \bar{N}^i{}_j) = \bar{D}^i{}_j \, , \qquad i \neq j \quad \text{(no summation)} \, ,$$

and rearrange this in the form

$$\bar{N}^i{}_j = \frac{2}{\log(\lambda_i / \lambda_j)}\, \bar{D}^i{}_j - \bar{W}^i{}_j - R^i{}_j \, , \qquad i \neq j \quad \text{(no summation)} \, .$$

From equation (1.315) we deduce that

$$\bar{W}^i{}_j + R^i{}_j = \frac{(\lambda_i + \lambda_j)}{(\lambda_i - \lambda_j)}\, \bar{D}^i{}_j \, , \qquad i \neq j \quad \text{(no summation)} \, .$$

Substitution of this result gives the desired expression (1.323).