The Cyclotron: Exploration

The purpose of this exploration is to become familiar with how a cyclotron accelerates particles, and in particular to understand the roles of the electric and magnetic fields.

Once you have completed this exploration, you should be able to configure the cyclotron simulation to produce proton beams suitable for isotope creation. (See "Simulate Experiment" under the Resources menu of the applet.)

A note on units

Frequencies used are in the MHZ (megahertz) or GHz (gigahertz) ranges. One hertz (1 Hz) represents one complete cycle per second. (A *cycle* means you return to where you started. So one polarity cycle means the positive and negative reverse, then flip back.) For comparison, radio waves are in the MHz range (think of radio stations). A microwave oven operates at 2.45 GHz, the resonance frequency of water molecules. Times are given in ns (nanoseconds); 1 ns = 10^{-9} s. All distances are in metres.

Magnetic fields are measured in teslas (T). The Earth's magnetic field is 30–50x10^{-6} T. A typical refrigerator magnet is about 5 mT. A coin-sized rare earth magnet is a bit over 1 T, and can lift more than 9 kG and erase credit cards. A field of 16 T can levitate a frog. The largest continuous magnetic field made in a laboratory is 45 T; the largest *pulsed* magnetic field ever made was 2.8 kT, and used explosives. The TRIUMF cyclotron uses a 0.5 T magnetic field. This applet specifies magnetic fields in milliteslas (mT) for reasons of precision.

Energies are given in MeV (mega-electron-volts).

The speed of the particle is given as a fraction of the speed of light (c = 3.00x10^8 m/s); if the graph shows a speed of 0.1, the particle is moving at 10% of the speed of light.

Simplifying assumptions used for this simulation

You can skip this section for now if you like, but come back later to see the small but important differences between this simulation and a real cyclotron.

This simulation uses a number of assumptions to make it simpler to work with (and run faster!). The most obvious is that the simulation assumes the electric potential (the voltage) is always at full strength and simply flips direction twice per cycle. In real cyclotrons the voltage is varied sinusoidally, just like in a radio wave, and is tuned so that the particle crosses the gap at the most positive and most negative points.

Any process involving repeated cycles is very sensitive to frequency; any error is added on again during each cycle. This simulation only records frequencies to three significant figures, but for example the TRIUMF cyclotron uses six. Over many cycles this limitation will cause the polarity of the dees to slip behind or ahead of the motion of the particles. (Though there are real physical effects that do this as well; see *Relativity and the Cyclotron*, below.) There should be enough precision to accelerate the particles all the way to the edge of the cyclotron in the exercises below, but if you watch closely you'll start to see the slippage.

When a charged particle is accelerated, it tends to radiate light — at the cost of some kinetic energy. (This is how radio transmitters work, for example.) In the context of particle accelerators this is known as synchrotron radiation, and causes the particle to lose energy as it travels around the circle. Lighter particles lose more energy in this way than heavier particles. Synchrotron radiation is negligible for protons, but it can be important for muons and (especially!) electrons. There are other reasons why muons and electrons are not used in cyclotrons, however, so this simplification doesn't really impact the results you see.

How Cyclotrons Operate

A cyclotron uses a magnetic field to make the charged particles inside travel in circular paths. The cyclotron is divided into two halves; each half is surrounded by a pair of D-shaped metal plates called, appropriately, "dees". A large potential difference, V_{acc} (the "accelerating voltage") is applied between the dees. Within one half or the other the electric potential is constant, so the electric field is zero and the particle is only affected by the magnetic field. When the particle crosses the small gap to the other dee, the potential difference between the dees causes an electric force to accelerate the particle. If the potential difference were 1 V, the particle gains 1 eV (electron-volt) of kinetic energy. Usually the potential difference between the dees is tens of kV or higher.

Your Turn

Open the Cyclotron applet. Set the magnetic field to about 1000 mT, and the frequency to about 1 MHz. (We'll see what this is the frequency *of* shortly.) Click "Start". Eventually the proton will leave the cyclotron; when this happens, click "Pause".

1. Describe the path you see the proton take inside the cyclotron. What does it look like?

2. Focus for the moment only on the *first* half-circle of the proton's path. Which direction does the magnetic force on the proton point as it goes around this path? (This explains why the proton's path inside a dee is circular.)

3. The proton begins at rest on the positive side of the gap (not shown). It is pushed towards the negative dee and gains energy $e \bullet V_{acc}$. As you've seen, with these settings the proton circles back around and reaches the gap again, still in the *negative* dee. Describe the motion of the proton as it tries to cross the gap towards the positive dee. (The actual gap crossing process is not shown in the simulation.) Describe what happens to the proton's kinetic energy during this process.

4. Each time the proton crosses — or tries to cross — the gap, its energy (after crossing) is added to the *Energy vs Time* graph. Briefly compare this graph with what you'd expect, based on your answers to the previous question.

5. What has to happen in order to increase the proton's energy *every* time it crosses the gap?

The Cyclotron Frequency

Click "Rewind". (If you click "Reset" it will also reset the magnetic field, which is usually not what you want.) Set the frequency to around 9 MHz. Run the simulation again by clicking "Start".

Your Turn

6. What are you changing the frequency of?

7. Look at the *Energy vs Time* graph. What can happen to the proton's energy when it reaches the gap?

8. Try to adjust the frequency until the proton's energy is increasing every time it crosses the gap. Notice that the orbital radius increases each time. Can you keep the proton in a spiral path all the way to the edge of the cyclotron? What is the frequency required to do this? This is called the ***cyclotron frequency***, f_{cyc}. *Hint: At what point in the orbit do you want the sign flip to occur?* (If you couldn't get the proton all the way to the edge, give the best frequency you were able to find. You can check the *Energy vs Radius* graph to see how far you made it.)

9. Considering what you know about how a particle moves in a magnetic field, what will happen if you switch the sign on the magnetic field? ***Write down your prediction***. Then set the field to –1000 mT, run the simulation, and describe what happens.

10. Now increase the magnetic field to its maximum (2000 mT) and find your best frequency again. How does f_{cyc} have to change when you double the magnetic field like this?

11. The TRIUMF cyclotron uses a magnetic field of 460 mT. Set the simulation to this field strength, and find f_{cyc}. What frequency do you find?

12. TRIUMF actually uses a frequency 5 times larger than this, so that the electric field goes through 5 cycles for every full orbit of the particle. Set your frequency 5 times larger and run the simulation. What do you see? Pay particular attention to what the electric field is doing each time the particle crosses the gap. Why does this work? What other frequencies would also work, and why? Why do you think TRIUMF does this instead of using the f_{cyc} that you found?

Energy, Time, and Orbital Radius

Let's look at how the energy is related to the orbital radius. Many modern cyclotrons choose the energy of the proton beam by choosing at what radius to extract the protons. We'll also look at how the proton behaves over time, and show that the proton always takes the same amount of time to go around no matter how much energy it has; this is why we can run the cyclotron at a single frequency through the entire acceleration process.

We've already seen that the proton gains the same amount of kinetic energy ($e \bullet V_{acc}$) each time it crosses the gap (each "half-cycle"). You'll use this below to predict how the energy changes over time.

In a cyclotron, the charged particle moves perpendicular to the magnetic field. The field pulls the particle around in a circle, with a radius r given by

$$r = \frac{mv}{qB}$$

where m is the particle's mass, v is its speed, and q is its charge, and B is the magnitude of the magnetic field. From this and your knowledge of kinetic energy you'll be able to predict the relationship between orbital radius and kinetic energy, below.

Your Turn

1. The proton crosses the gap twice per cycle (orbit), or once per half-cycle. How many half-cycles does the proton go through before it reaches the edge and leaves the cyclotron (or before it reaches as far as you managed to get it)? Use the accelerating voltage shown to determine the final energy of the proton when it leaves the cyclotron, showing your work. Compare your calculation to the highest energy shown in the *Energy vs Time* graph. (You can use the "Evaluate Single Point Tool" (leftmost button on the graph's title bar), to click on a point and find its energy precisely.)

2. Find a formula that relates the time per orbit T to the magnetic field B. From this formula, how does the time per orbit depend on the particle's energy? How does it depend on the orbital radius r? What does this tell you about f_{cyc} as the proton moves outward? Compare this to what you saw in the simulation.

3. From your formula for time per orbit, what happens to T when you increase B? What does this tell you about how f_{cyc} changes when you change B? Compare to what you saw when you did this in the previous section.

4. How does the particle's kinetic energy change with time? (Hint: Consider what is changing the particle's kinetic energy, when it changes, and by how much each time.)

5. Look at the path of the proton in your simulation. How does the radius change on each orbit? Are the orbits getting farther apart, closer together, or are they increasing by the same amount on each pass?

6. Let's try to explain that pattern. First, take the formula for kinetic energy, $E = \frac{1}{2}mv^2$, and solve for v. Now look at the formula for the radius given above to determine the relationship between r and E; specifically, which of these do you find is true?

- $\boldsymbol{r}$ is proportional to $\boldsymbol{E}$
- $\boldsymbol{r}$ is proportional to $\boldsymbol{E^2}$
- $\boldsymbol{r}$ is proportional to $\sqrt{\boldsymbol{E}}$
- $\boldsymbol{r}$ is proportional to $\mathbf{1}/\boldsymbol{E}$

Compare your choice to the way the radius changes on each orbit. Does this relationship predict the pattern you described in the previous question? Explain. Then compare the relationship to the graph of E vs r; is this relationship consistent with what the graph shows?

7. Based on what you know about how the kinetic energy E depends on time t, and about the relationship between E and the proton's speed v, predict the shape of the v vs t graph. Compare your prediction to the graph in the simulation.

Other Beam Particles: Deuterons and Alpha Particles

The most common particle used in cyclotrons is the proton, but heavier particles are sometimes used as well, especially for radioisotope production, including:

Beam Particle	# Protons	# Neutrons
Proton (H^+)	1	0
Deuteron	1	1
Alpha ($^4He^{2+}$)	2	2

For example, technetium-94m is an important isotope for medical physics, used both for PET scanning and for cancer radiotherapy. When produced by a cyclotron it is usually made by bombarding Mo-94 with protons, but it can also be created by bombarding Mo-92 with alpha particles (or by many other production reactions).

For some cyclotrons, operators can switch between different beam particles. Other cyclotrons, such as TRIUMF, can only be used to accelerate a specific particle.

You can switch between particles in the Cyclotron applet in the Options menu, under Particle. When you change particles, the applet changes the scale of the Frequency slider (and sometimes the accelerating voltage) to better demonstrate the operation of the cyclotron.)

Your Turn

Set the applet to use a 1000 mT magnetic field. It doesn't actually matter much, as long as there's a constant field and you don't change it during this section; we're interested in comparisons more than actual numbers.

Just to pick a consistent point for comparison, for this section consider quantities like kinetic energy and orbital radius after the particle has gone through, say, 3 gap-crossings (3 "boosts" of kinetic energy).

1. ***Predict*** how a deuteron's behaviour will differ (or not differ) from a proton. Consider kinetic energy, velocity, orbital radius, and cyclotron frequency.

2. Use the applet to ***test*** your proton-vs-deuteron predictions.

3. ***Predict*** how an alpha particle's behaviour will differ (or not differ) from a deuteron. Again, consider kinetic energy, velocity, orbital radius, and cyclotron frequency.

4. Use the applet to ***test*** your deuteron-vs-alpha predictions.

5. Say a cyclotron of a particular size generates a proton beam of energy E when you dial in the right settings. What settings would you have to change, and by how much, to produce a beam of deuterons with the same energy? How about a beam of alpha particles? (Assume these are all in the same cyclotron so you can't change its size.)

Relativity and the Cyclotron

Cyclotrons are capable of accelerating particles to speeds very close to the speed of light (c). At these speeds, "relativistic effects" significantly change the way the particles behave. These effects usually start to become noticeable at around 10% of the speed of light ($v \approx 0.1c$) or larger.

The problem that affects the operation of the cyclotron is that a particle's speed can never be larger than the speed of light, even though it keeps gaining energy by the same amount every time it crosses the gap between dees! Remember that its energy depends on its *mass* as well as its speed; the boost it gets at the gap also *increases the particle's mass*, especially as the speed approaches the speed of light.

(Here we'll use "mass", m, to refer to the mass seen by an observer in the lab frame, as opposed to its "rest mass", m_0.)

According to relativity, the relationship between an object's mass m and its speed v is:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \gamma m_0$$

where m_0 is the object's "rest mass". (Sometimes m is called the "relativistic mass" to distinguish it from m_0.) The quantity γ is simply called "the relativistic gamma".

There are two important things to remember about γ: At low speeds, $\gamma \approx 1$; and close to the speed of light γ becomes very, very large (infinite when $v = c$) Make sure you can see these things in the formula for γ.

When v is small, v^2/c^2 is tiny, and the particle's speed doesn't really affect its mass; then effectively all the energy goes into increasing the particle's speed. But as the particle's speed gets close to the speed of light, the denominator becomes very small, and so γ grows very large. The effect on the mass usually becomes noticeable at around 10% of the speed of light.

So in the cyclotron, as v approaches c, the energy boost the particle gets through the gap starts to have less and less effect on the *speed* ($v \approx c$) but the particle's *mass* begins to increase dramatically ($m = \gamma m_0$). This has two major effects on the cyclotron's operation: it changes the way the *radius* grows with energy, and it changes the *time per orbit* (and hence the cyclotron frequency f_{cyc}). Let's have a look at the radius, then you'll figure out what happens to f_{cyc}.

First, what happens when we put $m = \gamma m_0$ into the formula for the radius?

$$r = \frac{mv}{qB} = \frac{\gamma m_0 v}{qB}$$

Since γ can become very large at relativistic speeds, the radius starts to grow by bigger and bigger amounts on each pass. (This is the *opposite* of how you saw the radius change at low speeds, remember.)

Your Turn

1. For low speed, you found earlier that the **time per orbit** (T) was *constant*, even though the radius increased on each pass. Following the same reasoning, for high (relativistic) speed does T increase, decrease, or stay the same on each pass? Explain. (You'll need to think about changes in both radius and speed.)

2. Based on what you found for T, does the **cyclotron frequency** f_{cyc} increase, decrease, or stay the same on each pass?

Cyclotrons at Relativistic Energies

[Note: This section does not currently work. There's a bug in the cyclotron code, so that the proton loses sync within 0.1 m for 0.5 m and 1.0 m cyclotron radii (at B=1.5T), even though it works all the way out to the edge of a 0.2 m cyclotron. I've written this up for when the bug is fixed.]

Einstein showed us that $E = mc^2$. This means that, in a sense, mass and energy are equivalent; the c^2 is essentially there to convert between kg and J. For this reason, particle physicists often find it convenient to express even the rest mass of a particle in units of "MeV/c^2". For example, the mass of a proton is about 938 MeV/c^2, and the mass of a muon is about 106 MeV/c^2. An electron's mass is just 0.511 MeV/c^2.

A good rule of thumb is that you'll usually start to notice relativistic effects when the kinetic energy of a particle is at least around 1% of the particle's rest mass energy. (This also means, roughly, that $v \approx 0.1c$, as we mentioned before.) Notice that as you continue to add kinetic energy to the particle, eventually the kinetic energy can exceed the rest mass. The LHC at CERN (a particle accelerator, though not a cyclotron) collides proton beams together; each proton has 7 TeV ("terra-electron-volts"; 7 TeV = $7x10^{12}$ eV) of energy, which is more than 7000 times the mass-energy of the proton.

Relativity is one of several reasons why cyclotrons are never used to accelerate electrons. The electron's mass energy is just 0.511 MeV, so it takes very little kinetic energy before they're moving nearly at the speed of light!

Let's use a proton to see how the relativistic effects impact the cyclotron operation.

Your Turn

Make sure you have the **proton** selected, and that the **cyclotron radius is 0.2 m**. Set the **magnetic field to 1500 mT**.

1. Earlier you found a formula for the *time per orbit*, T, which gives a formula for the cyclotron frequency using $f_{cyc} = 1/T$. Use this formula to find the cyclotron frequency for a proton in this magnetic field, and put that value into the applet. Run the simulation; does the cyclotron continue to accelerate the proton all the way to the edge? If not, what is the maximum speed and kinetic energy of the proton (at the point when it's no longer synchronized with the dees)?

2. Now change the **cyclotron radius to 1.0 m**, and run the simulation again. Does the cyclotron continue to accelerate the proton all the way to the edge? If not, what is the maximum speed and kinetic energy of the proton?

3. Express the maximum energy you reached as a percentage the mass energy of the proton (938 MeV). Is this about what the "rule of thumb" predicts for when relativistic effects start to show up?

4. Now set up the applet as follows:
   • Particle: **Muon**.
   • Cyclotron radius: **0.2 m**.
   • Magnetic field: **1000 mT**.
   Determine f_{cyc} using your previous formula, and run the simulation. How many half-cycles does the muon go through before falling out of sync?

5. What are the maximum speed and kinetic energy of the muon? Compare maximum speed to c, and the maximum energy to the muon's mass energy (106 MeV). Does this fit with our rule of thumb?

Using the Magnetic Field to Correct for Relativity

So relativity is changing the particle's cyclotron frequency. You can see why this is by using the formula you developed earlier for f_{cyc}, which should look something like this:

$$f_{cyc} = \frac{qB}{2\pi m}$$

As the particle moves faster, its mass increases, which reduces f_{cyc}. But if we increase the magnetic field B in the same way that m increases, the effects will cancel and we can continue to accelerate the particle. (We could instead change the frequency at which the dees reverse polarity, but that's very difficult to do because of how the oscillating voltage is generated.)

Recall that the relativistic mass $m = \gamma m_0$, where γ increases as the speed increases. If we set $B = \gamma B_0$, the γ's cancel and we get back a constant f_{cyc}:

$$f_{cyc} = \frac{qB_0}{2\pi m_0}$$

Since the orbital radius r depends on the particle speed, we can directly relate γ to r and use that to find $B(r)$, which describes the shape we need for the magnetic field.

First, recall from an earlier section that $r = mv/qB$. Since $m = \gamma m_0$ and $B = \gamma B_0$, we have

$$r = \frac{m_0 v}{qB_0}$$

which is the same as the non-relativistic relationship between r and v. We can rearrange this to get $v = qB_0 r/m_0$. Substitute this for v in the formula for γ:

$$\gamma(r) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \left(\frac{qB_0 r}{m_0 c}\right)^2}}$$

Then $B(r) = B_0\,\gamma(r)$.

In the applet, when you click the "unlock" button on the magnetic field controls, it uses a simplified version of this formula to find the magnetic field between the minimum and maximum values you specify. The minimum field you set is simply B_0. The maximum field is $B(r_{max})$, the magnetic field at the outer edge of the cyclotron.

Your Turn

Set up the applet as follows:
• Particle: **Muon**.
• Cyclotron radius: **0.2 m**.
• Magnetic field: **1000 mT**.
• "**Unlock**" the magnetic field so that you can set min and max separately.

1. Adjust the **maximum** magnetic field until you can successfully accelerate a muon all the way to the outer edge of the cyclotron. *Then* use the formula for $B(r)$ to determine the "correct" max field value (at r = 0.2 m). How does the prediction compare to the value you found by hand? What are the energy and speed of the muon when it leaves the cyclotron?

2. Now change the applet settings to match what you last used for the proton:
   • Particle: **Proton**.
   • Cyclotron radius: **1 m**.
   • Magnetic field minimum: **1500 mT**.
   Set f_{cyc} to the frequency you used for the proton last time. Recall that with these settings, the proton could not be accelerated all the way to the cyclotron edge. Use the $B(r)$ formula to set the correct value for the max magnetic field. Run the simulation; does this modification to the field do the job?

3. Looking at the formula for r, does the proton's orbit have a **maximum radius**? You don't have to find it, just explain how you know it exists. (Hint: does it have a maximum *speed*?) What does the $B(r)$ formula say the magnetic field needs to be at this radius?

This is one of the factors that limits the highest energies a cyclotron can reach. Accelerators like the LHC are build on the same basic principles, but use different methods and reach much higher energies than any cyclotron. In the LHC, the protons travel in a huge circle, and is given a boost on each orbit; the magnetic field all around the ring is increased at the same time in order to keep the protons in the ring. Once the protons have reached the desired energy, the boosting stops and they continue to orbit the ring for up to a full day, except for the few that collide on each orbit.

This also demonstrates another of the reasons electrons are not accelerated using cyclotrons. We discussed earlier how the electron's low mass means it reaches relativistic speeds with very little energy boost. If we were to use this $B(r)$ method to compensate for this, in a magnetic field that starts at 1 T the electron's maximum radius would be just 2 mm!