A numerical model of a charged particle's motion in a magnetic dipole field

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Abstract

In this project, we have simulated the motion of a charged particle in the earth's magnetic field by modeling earth as a pure magnetic dipole. Results conclude that particles on a trajectory towards the dipole, are bent away. By decreasing the particle's initial position's distance from earth, it is trapped with inconsistent, spiraling motions close to the field source, before accelerated away from the dipole. This simulation assumes no interaction between different particles, and neglect other force fields which may be present.

1 Introduction

In this project we will model how particles' motions depend on their initial velocities, positions and the earth's magnetic field. This model assumes no interaction between different particles, and neglect other electromagnetic and gravitational fields of the sun and earth. By observing a charged particle moving in the magnetic dipole field, we are able to estimate the magnetic forces acting on the particle due to the earth's $\vec{B}$-field, and thereby solve the differential equations of motion numerically.

2 Theory

The electromagnetic force acting on an electrically charged particle is given by Lorentz' Force law[1],

$$\vec{F}_m = qE + q(\vec{v} \times \vec{B}). \qquad (1)$$

While the magnetic induction, or $\vec{B}$-field, of a pure magnetic dipole in vacuum is given by[2],

$$\vec{B}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}]. \qquad (2)$$

Where, $\mu_0 = 4\pi \cdot 10^{-7} \text{H/m}$, is the magnetic permeability in vacuum, $\vec{m}$, is the field source's magnetic dipole moment, and, $\vec{r}$ is the vector from the center of the dipole to the point of observation. Then $\hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}$, is the normalized $\vec{r}$, obviously pointing in the same direction.

To evaluate the particle's motions we apply Euler's method[3]. When a particle happens to be at some point, P, we define the magnetic induction, with strength and direction as $\vec{B}_P$, given by equation 2, at this particular point. The force acting on the particle with a given velocity, is then well defined by 1. By evaluating this force for an infinitely small time interval, dt, we may assume the acceleration of the particle is linear in this interval, and hence the particle's new position will be given by,

$$\vec{s}_{new} = \vec{s}_0 + \vec{v}_0 \cdot dt + \frac{1}{2}\vec{a}(dt)^2. \qquad (3)$$

This process is then repeated for a total of $\frac{T}{dt}$ times, with equally many measurement points. Here, T, is the particle's total fly time, or more accurately put, the overall time where we calculate the particle's motion.

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[1]Griffiths 2008, p.204
[2]Griffiths 2008, p.246
[3]http://en.wikipedia.org/wiki/Euler_method

3 Model

Our model simulates motion in three-dimensional space. The dipole is placed at the origin, at the center of the earth, while an incoming charged particle moves from an initial position, with some initial velocity in the positive direction of the x-axis. The axes have dimensions of, Mm= 10^6m, to simplify the reading of graphs and output values. The magnetic moment of the earth is tilted an angle, $\theta = 11°$, in the xz-plane, from the z-axis. The programming language being applied, is Python.

Assumptions

By studying a single particle's motion at the time, this model assumes no interaction between different particles as they move through earth's $\vec{B}$-field. The magnetic and electric fields of the sun, including the electric field of the earth is also neglected, as well as all gravitational forces. The earth is also assumed to be a pure magnetic dipole, centered at the origin, as well as spherical with a radius, $R_e = 6.370$Mm. Since we're only interested in the *motion* of the particle, the magnetic dipole moment of the earth is set

to, $\vec{m} = \sin(\theta)\hat{x} + \cos(\theta)\hat{z}$. Hence, $|\vec{m}| = 1$A(Mm)^2. The experimentally value according to WolframAlpha[4] is, $m = 7.79 \cdot 10^{22}$Am^2. By having reduced the length of $\vec{m}$, we have reduced the strength of the $\vec{B}$-field, and also the strength, but not direction, of the force acting on the particle. This reduces the amount of needed measurement points per second of the particle and yields a smoother motion curve. This will result in an incorrect model of the *earth*, however, we are mainly interested in how charged particles move in a general magnetic dipole field, so we will accept these premises.

4 Results

The magnetic dipole field

First we present a graphical representation of the magnetic dipole field, shown in Figure 1a. Some might find it inconvenient, or difficult to study this three dimensional figure, and for their leisure, we therefore also present an equivalent graph of the same field, plotted in the two dimensional zx-plane, displayed in Figure 1b.
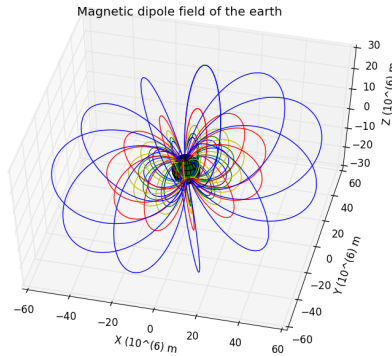
(a) A three-dimensional representation of the earth's magnetic dipole field. The field lines are colored so to ease reading and have no physical meaning regarding the model.
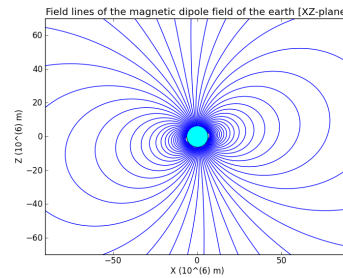
(b) A two-dimensional representation of the earth's magnetic dipole field in the xz-plane, containing $\vec{m}$.

Figure 1: Shows two graphical representations of the earth magnetic dipole field.

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[4]http://www.wolframalpha.com/input/?i=magnetic+dipole+moment+of+the+earth

Particle movement

We have run multiple simulations, each with different initial velocities and positions of the particles. Wikipedia[5] suggests that solar winds of charged particles have velocities approximately equal to 400,000 m/s = 0.4 Mm/s by the time they reach earth's atmosphere. Simulation number 4, shown in Figure 5, models this condition with an initial position equal to earth's diameter from earth's surface. Other simulated initial conditions with corresponding graphs of motion may be found correlated in Table 1.

Table 1: Shows the related particle initial conditions and graphs. Every condition is represented by two equal figures, observing the time development from different angles to make it easier for the reader to study the three dimensional motion. Also note that for every plot, the initial position, P_0, of the particle is marked with a red dot.

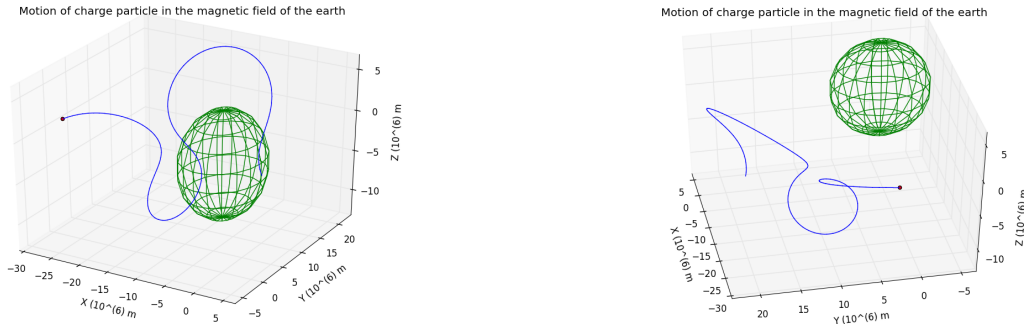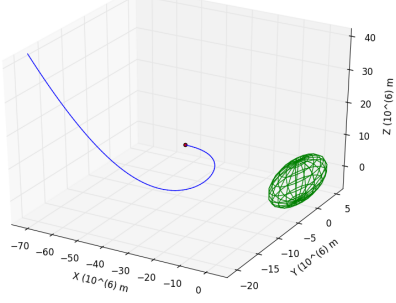| Simulation (#)	$\vec{v}_0$ (Mm/s)	P_0 (Mm)	Figure plot
1	$[02, 08, 00]$	$[-30, 03, 00]$	Figure 2
2	$[10, 00, 00]$	$[-50, 03, 00]$	Figure 3
3	$[10, 00, 00]$	$[-100, 00, 00]$	Figure 4
4	$[0.4, 00, 00]$	$[-3R_e, 00, 00]$	Figure 5

Figure 2: Path of particles with initial conditions, $\vec{v} = [02, 08, 00]$Mm/s, $[x, y, z] = [-30, 03, 00]$Mm.
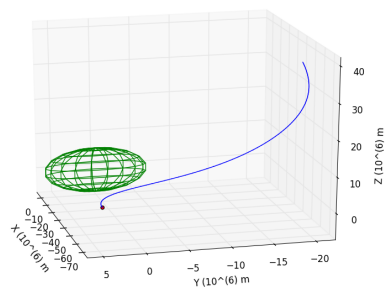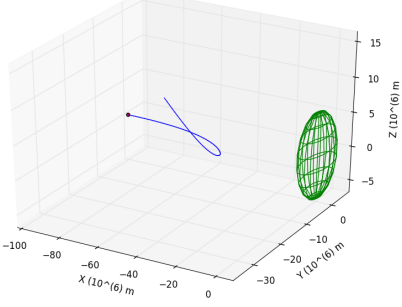
[5]http://en.wikipedia.org/wiki/Aurora_(astronomy)

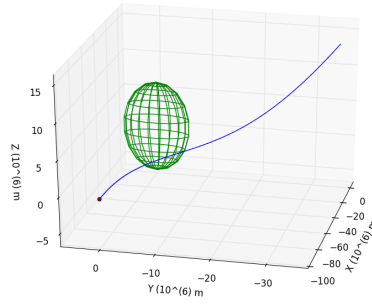Figure 3: Path of particles with initial conditions, $\vec{v} = [10, 00, 00]$Mm/s, $[x, y, z] = [-50, 03, 00]$Mm.

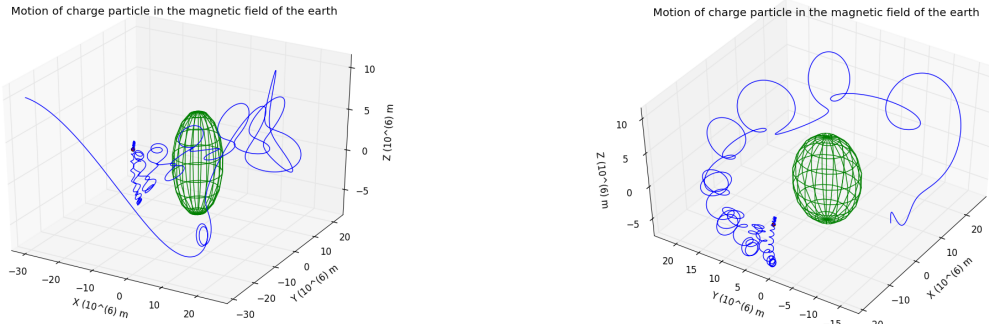Figure 4: Path of particles with initial conditions, $\vec{v} = [10, 00, 00]$Mm/s, $[x, y, z] = [-100, 00, 00]$Mm.

Figure 5: Path of particles with initial conditions, $\vec{v} = [0.4, 00, 00]$Mm/s, $[x, y, z] = [-3R_e, 00, 00]$Mm, where R_e is the radius of the earth.

5 Discussion and conclusion

Due to existing theory and experiments, we would have expected that the particles with high enough initial velocity, would essentially over time end up at, or at least close in on earth's magnetic poles. By applying Euler's method, we obtain an essential error from the divergence of the $\vec{B}$-field. Since the magnetic field is continuously (and non-homogeneous), the magnetic force on the particle will also change continuously as as it moves thorugh space. For an exact simulation, one would need infinitely many measurement points, or find an explicit expression for the particle's path. This means that the real motion of the particle will experience greater curvature than our simmulation by Euler's method is modelling. Hence, the greater the force on the particle, the longer the total time we simulate, and the larger each time interval is between every measurement: the greater the error of the estimated particle path is. We herefrom wish to encourge the reader to improve our simulations by applying some other strategy for solving the particle's equations of motions, or simply improve Euler's method. The complete project, including our python scripts, output files for particle paths, graphs and plots may be found at my NTNU web domain[6].

References

[1] Griffiths, David J., *Introduction to Electrodynamics*. San Francisco, 2008.

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[6]http://folk.ntnu.no/magnud/elmag_teori.html