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Let $\mathbb{B} = \{\lozenge, \blacklozenge\}$ where $\lozenge \neq \blacklozenge$. Let T_{14} be a set and $C_{a_1} : \mathbb{B} \to T_{14}$ and $C_{b_1} : T_{14} \times \mathbb{B} \to T_{14}$ functions such that all of the following conditions are met:

- C_{a_1} and C_{b_1} are injective,
- for every x_{21}, x_{22} and x_1, we have $C_{a_1}(x_1) \neq C_{b_1}(x_{21}, x_{22})$,
- T_{14} is *covered* by C_{a_1} and C_{b_1} (i.e., $\mathit{image}(C_{a_1}) \cup \mathit{image}(C_{b_1}) = T_{14}$)[1].

Then we proceed to define a function over this set.

Definition 1. Let $f_\beta : T_{14} \times \mathbb{N} \to \mathbb{N}$ be the recursive function determined by the following equations (for any a, b and c):

$$f_\beta(C_{a_1}(a), b) = b \tag{1}$$
$$f_\beta(C_{b_1}(a, b), c) = f_\beta(a, Suc(c)) \tag{2}$$

1 The Lemmas

We begin by proving some necessary lemmas.

Lemma 1. *For every $n \in T_{14}$, $p \in T_{14}$ and $q \in \mathbb{N}$*

$$f_\beta(n, Suc(f_\beta(p, q))) = f_\beta(n, f_\beta(p, Suc(q)))$$

Proof. We proceed by induction on p.

For the **base of induction** we need to prove the following statement:

$$\forall x \in \mathbb{B}.\ \forall o \in T_{14}.\ \forall r \in \mathbb{N}.\ \ f_\beta(o, Suc(f_\beta(C_{a_1}(x), r))) = f_\beta(o, f_\beta(C_{a_1}(x), Suc(r)))$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $r \in T_{14}$, $x_2 \in \mathbb{B}$, $o \in T_{14}$ and $s \in \mathbb{N}$, the inductive hypothesis (IH_1) entails the inductive goal (IG_1).

$$\forall t \in T_{14}.\ \forall u \in \mathbb{N}.\ \ f_\beta(t, Suc(f_\beta(r, u))) = f_\beta(t, f_\beta(r, Suc(u))) \tag{IH_1}$$
$$f_\beta(o, Suc(f_\beta(C_{b_1}(r, x_2), s))) = f_\beta(o, f_\beta(C_{b_1}(r, x_2), Suc(s))) \tag{IG_1}$$

[1] This fact allows us to prove theorems about all the elements of T_{14} by induction over the structure given by C_{a_1} and C_{b_1}.

We show this with the following chain of equalities:

$$\begin{aligned} f_\beta(o, Suc(f_\beta(C_{b_1}(r, x_2), s))) &= f_\beta(o, Suc(f_\beta(r, Suc(s)))) && \text{by (2)} \\ &= f_\beta(o, f_\beta(r, Suc(Suc(s)))) && \text{by (IH}_1\text{)} \\ &= f_\beta(o, f_\beta(C_{b_1}(r, x_2), Suc(s))) && \text{by (2)} \end{aligned}$$

Thus we conclude the proof of this lemma. □

2 The Theorem

In this section we prove the main result of this article.

Theorem. *For every $a \in T_{14}$, $b \in T_{14}$ and $c \in \mathbb{N}$*

$$f_\beta(a, f_\beta(b, c)) = f_\beta(b, f_\beta(a, c))$$

Proof. We proceed by induction on a.

For the **base of induction** we need to prove the following statement:

$$\forall x \in \mathbb{B}.\, \forall d \in T_{14}.\, \forall e \in \mathbb{N}.\;\; f_\beta(C_{a_1}(x), f_\beta(d, e)) = f_\beta(d, f_\beta(C_{a_1}(x), e))$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $d \in T_{14}$, $x_2 \in \mathbb{B}$, $e \in T_{14}$ and $f \in \mathbb{N}$, the inductive hypothesis (IH$_2$) entails the inductive goal (IG$_2$).

$$\forall g \in T_{14}.\, \forall h \in \mathbb{N}.\;\; f_\beta(d, f_\beta(g, h)) = f_\beta(g, f_\beta(d, h)) \tag{IH$_2$}$$

$$f_\beta(C_{b_1}(d, x_2), f_\beta(e, f)) = f_\beta(e, f_\beta(C_{b_1}(d, x_2), f)) \tag{IG$_2$}$$

We show this with the following chain of equalities:

$$\begin{aligned} f_\beta(C_{b_1}(d, x_2), f_\beta(e, f)) &= f_\beta(d, Suc(f_\beta(e, f))) && \text{by (2)} \\ &= f_\beta(d, f_\beta(e, Suc(f))) && \text{by Lemma 1} \\ &= f_\beta(e, f_\beta(d, Suc(f))) && \text{by (IH}_2\text{)} \\ &= f_\beta(e, f_\beta(C_{b_1}(d, x_2), f)) && \text{by (2)} \end{aligned}$$

Thus we conclude the proof of this theorem. □