

My paper title

My name

The date

Let $\mathbb{B} = \{\diamond, \blacklozenge\}$ where $\diamond \neq \blacklozenge$. Let T be a set and $C_a : \mathbb{B} \times \mathbb{B} \rightarrow T$ and $C_b : T \rightarrow T$ functions such that all of the following conditions are met:

- C_a and C_b are injective,
- for every x_2, x_{11} and x_{12} , we have $C_a(x_{11}, x_{12}) \neq C_b(x_2)$,
- T is *covered* by C_a and C_b (i.e., $\text{image}(C_a) \cup \text{image}(C_b) = T$).

Then we proceed to define a function over this set.

Definition 1. Let $f_\alpha : T \times T \rightarrow T$ be the recursive function determined by the following equations (for any x, y and z):

$$f_\alpha(C_a(x, y), z) = z \tag{1}$$

$$f_\alpha(C_b(x), y) = C_b(f_\alpha(x, y)) \tag{2}$$

1 The Lemmas

We begin by proving some necessary lemmas.

Lemma 1. *For every $n \in T$ and $p \in T$*

$$f_\alpha(n, C_b(p)) = C_b(f_\alpha(n, p))$$

Proof. We proceed by induction on n .

For the **base of induction** we need to prove the following statement:

$$\begin{aligned} \forall x_1 \in \mathbb{B}. \forall x_{2a} \in \mathbb{B}. \forall q \in T. \\ f_\alpha(C_a(x_1, x_{2a}), C_b(q)) = C_b(f_\alpha(C_a(x_1, x_{2a}), q)) \end{aligned}$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $o \in T$ and $q \in T$, the inductive hypothesis (3) entails the goal (4).

$$\forall r \in T. f_\alpha(o, C_b(r)) = C_b(f_\alpha(o, r)) \tag{3}$$

$$f_\alpha(C_b(o), C_b(q)) = C_b(f_\alpha(C_b(o), q)) \tag{4}$$

We show this with the following chain of equalities:

$$\begin{aligned}
f_\alpha(C_b(o), C_b(q)) &= C_b(f_\alpha(o, C_b(q))) && \text{by (2)} \\
&= C_b(C_b(f_\alpha(o, q))) && \text{by (3)} \\
&= C_b(f_\alpha(C_b(o), q)) && \text{by (2)}
\end{aligned}$$

Thus we conclude the proof of this lemma. \square

2 The Theorem

In this section we prove the main result of this article

Theorem. *For every $a \in T$, $b \in T$ and $c \in T$*

$$f_\alpha(b, f_\alpha(a, c)) = f_\alpha(a, f_\alpha(b, c))$$

Proof. We proceed by induction on b .

For the **base of induction** we need to prove the following statement:

$$\begin{aligned}
&\forall x_1 \in \mathbb{B}. \forall x_{2a} \in \mathbb{B}. \forall d \in T. \forall e \in T. \\
&\quad f_\alpha(C_a(x_1, x_{2a}), f_\alpha(d, e)) = f_\alpha(d, f_\alpha(C_a(x_1, x_{2a}), e))
\end{aligned}$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $d \in T$, $e \in T$ and $f \in T$, the inductive hypothesis (5) entails the goal (6).

$$\forall g \in T. \forall h \in T. f_\alpha(d, f_\alpha(g, h)) = f_\alpha(g, f_\alpha(d, h)) \quad (5)$$

$$f_\alpha(C_b(d), f_\alpha(e, f)) = f_\alpha(e, f_\alpha(C_b(d), f)) \quad (6)$$

We show this with the following chain of equalities:

$$\begin{aligned}
f_\alpha(C_b(d), f_\alpha(e, f)) &= C_b(f_\alpha(d, f_\alpha(e, f))) && \text{by (2)} \\
&= C_b(f_\alpha(e, f_\alpha(d, f))) && \text{by (5)} \\
&= f_\alpha(e, C_b(f_\alpha(d, f))) && \text{by Lemma 1} \\
&= f_\alpha(e, f_\alpha(C_b(d), f)) && \text{by (2)}
\end{aligned}$$

Thus we conclude the proof of this theorem. \square