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Let $\mathbb{B} = \{\lozenge, \blacklozenge\}$ where $\lozenge \neq \blacklozenge$. Let $\mathbb{N}$ be the set of natural numbers, with $\mathtt{Suc}$ the *successor* function. Let T_2 be a set and $C_c : \mathbb{N} \times \mathbb{N} \to T_2$ and $C_d : T_2 \times \mathbb{B} \to T_2$ functions such that all of the following conditions are met:

- C_c and C_d are injective,
- for every x_{21}, x_{22}, x_{11} and x_{12}, we have $C_c(x_{11}, x_{12}) \neq C_d(x_{21}, x_{22})$,
- T_2 is *covered* by C_c and C_d (i.e., $\mathit{image}(C_c) \cup \mathit{image}(C_d) = T_2$)[1].

Then we can proceed to define a function over this set.

Definition 1. Let $f_\alpha : T_2 \times \mathbb{N} \to \mathbb{N}$ be the recursive function determined by the following equations (for any a, b and c):

$$f_\alpha(C_c(a, b), c) = c \tag{1}$$
$$f_\alpha(C_d(a, b), c) = \mathtt{Suc}(f_\alpha(a, \mathtt{Suc}(f_\alpha(a, \mathtt{Suc}(c))))) \tag{2}$$

1 The Theorem

Theorem. *For every $a \in T_2$ and $b \in \mathbb{N}$*

$$\mathtt{Suc}(f_\alpha(a, \mathtt{Suc}(b))) = f_\alpha(a, \mathtt{Suc}(\mathtt{Suc}(b)))$$

Proof. We proceed by induction on a.

For the **base of induction** we need to prove the following statement:

$$\forall x_1 \in \mathbb{N}.\ \forall x_2 \in \mathbb{N}.\ \forall c \in \mathbb{N}.\ \ \mathtt{Suc}(f_\alpha(C_c(x_1, x_2), \mathtt{Suc}(c))) = f_\alpha(C_c(x_1, x_2), \mathtt{Suc}(\mathtt{Suc}(c)))$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $c \in T_2$, $x_2 \in \mathbb{B}$ and $d \in \mathbb{N}$, the inductive hypothesis (IH_1) entails the inductive goal (IG_1).

$$\forall e \in \mathbb{N}.\ \ \mathtt{Suc}(f_\alpha(c, \mathtt{Suc}(e))) = f_\alpha(c, \mathtt{Suc}(\mathtt{Suc}(e))) \tag{IH$_1$}$$
$$\mathtt{Suc}(f_\alpha(C_d(c, x_2), \mathtt{Suc}(d))) = f_\alpha(C_d(c, x_2), \mathtt{Suc}(\mathtt{Suc}(d))) \tag{IG$_1$}$$

[1] This fact allows us to prove theorems about all the elements of T_2 by induction over the structure given by C_c and C_d.

First, notice that the following is true:

$$\begin{aligned}
\texttt{Suc}(f_\alpha(c, \texttt{Suc}(f_\alpha(c, \texttt{Suc}(\texttt{Suc}(d)))))) &= f_\alpha(c, \texttt{Suc}(\texttt{Suc}(f_\alpha(c, \texttt{Suc}(\texttt{Suc}(d)))))) && \text{by } (\mathrm{IH}_1) \\
&= f_\alpha(c, \texttt{Suc}(f_\alpha(c, \texttt{Suc}(\texttt{Suc}(\texttt{Suc}(d)))))) && \text{by } (\mathrm{IH}_1) \qquad (3)
\end{aligned}$$

Ultimately, we can show IG_1 as follows:

$$\begin{aligned}
\texttt{Suc}(f_\alpha(C_d(c, x_2), \texttt{Suc}(d))) &= \texttt{Suc}(\texttt{Suc}(f_\alpha(c, \texttt{Suc}(f_\alpha(c, \texttt{Suc}(\texttt{Suc}(d))))))) && \text{by } (2) \\
&= \texttt{Suc}(f_\alpha(c, \texttt{Suc}(f_\alpha(c, \texttt{Suc}(\texttt{Suc}(\texttt{Suc}(d))))))) && \text{by } (3) \\
&= f_\alpha(C_d(c, x_2), \texttt{Suc}(\texttt{Suc}(d))) && \text{by } (2)
\end{aligned}$$

Thus we conclude the proof of this theorem. $\square$