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Let $\mathbb{B} = \{\lozenge, \blacklozenge\}$ where $\lozenge \neq \blacklozenge$. Let T be a set and $C_a : \mathbb{B} \times \mathbb{B} \to T$ and $C_b : T \to T$ functions such that all of the following conditions are met:

- C_a and C_b are injective,
- for every x_2, x_{11} and x_{12}, we have $C_a(x_{11}, x_{12}) \neq C_b(x_2)$
- T is *covered* by C_a and C_b (i.e., $\mathit{image}(C_a) \cup \mathit{image}(C_b) = T$).
- T is *minimal* (i.e., C_a and C_b generate the whole T).

Then we proceed to define a function over this set.

Definition 1. Let $f_\alpha : T \times T \to T$ be the recursive function determined by the following equations (for any x, y and z):

$$f_\alpha(C_a(x, y), z) = z \tag{1}$$
$$f_\alpha(C_b(x), y) = C_b(f_\alpha(x, y)) \tag{2}$$

1 The Lemmas

We begin by proving some necessary lemmas.

Lemma 1. *For every $n \in T$ and $p \in T$*

$$f_\alpha(n, C_b(p)) = C_b(f_\alpha(n, p))$$

Proof. We proceed by induction on n.

For the **base of induction** we need to prove the following statement:

$$\forall\, x_1 \in \mathbb{B}.\ \forall\, x_{2a} \in \mathbb{B}.\ \forall\, q \in T.$$
$$f_\alpha(C_a(x_1, x_{2a}), C_b(q)) = C_b(f_\alpha(C_a(x_1, x_{2a}), q))$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $o \in T$ and $q \in T$, the inductive hypothesis (3) entails the goal (4).

$$\forall\, r \in T.\ \ f_\alpha(o, C_b(r)) = C_b(f_\alpha(o, r)) \tag{3}$$
$$f_\alpha(C_b(o), C_b(q)) = C_b(f_\alpha(C_b(o), q)) \tag{4}$$

We show this with the following chain of equalities:

$$\begin{aligned} f_\alpha(C_b(o), C_b(q)) &= C_b(f_\alpha(o, C_b(q))) && \text{by (2)} \\ &= C_b(C_b(f_\alpha(o, q))) && \text{by (3)} \\ &= C_b(f_\alpha(C_b(o), q)) && \text{by (2)} \end{aligned}$$

Thus we conclude the proof of this lemma. □

2 The Theorem

In this section we prove the main result of this article

Theorem. *For every $a \in T$, $b \in T$ and $c \in T$*

$$f_\alpha(b, f_\alpha(a, c)) = f_\alpha(a, f_\alpha(b, c))$$

Proof. We proceed by induction on b.

For the **base of induction** we need to prove the following statement:

$$\begin{aligned} &\forall\, x_1 \in \mathbb{B}.\, \forall\, x_{2a} \in \mathbb{B}.\, \forall\, d \in T.\, \forall\, e \in T. \\ &\qquad f_\alpha(C_a(x_1, x_{2a}), f_\alpha(d, e)) = f_\alpha(d, f_\alpha(C_a(x_1, x_{2a}), e)) \end{aligned}$$

This follows trivially from our definitions.

For the **step of induction** we need to show that for every $d \in T$, $e \in T$ and $f \in T$, the inductive hypothesis (5) entails the goal (6).

$$\forall\, g \in T.\, \forall\, h \in T.\;\; f_\alpha(d, f_\alpha(g, h)) = f_\alpha(g, f_\alpha(d, h)) \tag{5}$$
$$f_\alpha(C_b(d), f_\alpha(e, f)) = f_\alpha(e, f_\alpha(C_b(d), f)) \tag{6}$$

We show this with the following chain of equalities:

$$\begin{aligned} f_\alpha(C_b(d), f_\alpha(e, f)) &= C_b(f_\alpha(d, f_\alpha(e, f))) && \text{by (2)} \\ &= C_b(f_\alpha(e, f_\alpha(d, f))) && \text{by (5)} \\ &= f_\alpha(e, C_b(f_\alpha(d, f))) && \text{by Lemma 1} \\ &= f_\alpha(e, f_\alpha(C_b(d), f)) && \text{by (2)} \end{aligned}$$

Thus we conclude the proof of this theorem. □