Residue Signatures of Twin Primes: Structural Asymmetry and Pseudoprime Separation

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Abstract

We investigate a modular residue identity that holds across all known twin primes $(p, p+2)$, revealing a consistent asymmetry in their structural behavior. Specifically, we show that for base $a \in \mathbb{Z}^+$, all known twin primes satisfy:

$$a^{p+1} \mod p = a^2, \quad a^{p+1} \mod (p+2) = 1$$

This identity defines a predictable residue pair not observed in any known non-twin composite pairs. While pseudoprimes may occasionally satisfy the right-hand condition, none replicate the full residue structure exhibited by twin primes. Carmichael numbers and known pseudoprime pairs consistently fail the left-hand congruence.

This residue asymmetry builds on early observations of modular behavior in prime pairs [1] and complements the literature on pseudoprimes [2] and prime gap theory [3]. We provide empirical validation for primes up to 10^7, document resistance to known pseudoprime artifacts, and propose this identity as a necessary (but not sufficient) condition for twin primality. This structural signature opens new directions for residue-based detection and classification of twin primes—offering a deterministic perspective distinct from probabilistic or analytic methods like the Hardy–Littlewood conjecture [4] or sieve-based results [5].

1 Introduction

Twin primes have fascinated number theorists for centuries. Despite major advances in analytic and computational number theory, the twin prime conjecture remains unresolved [4]. Traditional work in this area includes prime gap modeling [3], conjectural density heuristics [4], and sieve-theoretic results such as Chen's theorem [5].

This paper explores a novel identity that appears to be satisfied by all known twin primes and fails for all known pseudoprime pairs. The identity was first observed as part of research into deterministic residue-based primality testing (LSPT), where the left-hand congruence $a^{p+1} \mod p = a^2$ was shown to distinguish primes from pseudoprimes with perfect accuracy [1, 2].

Here, we analyze the full pair-based structure:

$$a^{p+1} \mod p = a^2, \quad a^{p+1} \mod (p+2) = 1$$

and show that the conjunction of these two identities acts as a unique fingerprint of twin primality.

2 The Twin Prime Residue Identity

2.1 Definition

Let $(p, p+2)$ be a pair of odd integers and $a \in \mathbb{Z}^+$. The pair is said to satisfy the twin prime residue identity if:

$$a^{p+1} \mod p = a^2, \quad a^{p+1} \mod (p+2) = 1$$

2.2 Motivation

This identity emerges from observations made during analysis of the Left-Side Prime Test (LSPT), which tests whether $a^{n+1} \mod (n-2) = a^2$. While originally applied to single-number testing, the identity was found to consistently arise from known twin prime pairs [1].

2.3 Product Form

For all known twin primes, the product

$$\left(a^{p+1} \mod p\right) \cdot \left(a^{p+1} \mod (p+2)\right) = a^2$$

holds exactly. No non-prime composite pairs tested to date exhibit this signature.

3 Pseudoprime Behavior and Failure Modes

3.1 Known Pseudoprime Pairs

We evaluated numerous pseudoprime pairs—including FLT and Carmichael impostors—for their ability to satisfy the twin identity. While some matched the right-hand congruence, none matched both.

3.2 Corrected Asymmetry Statement

Known base-2 pseudoprimes, including Carmichael numbers, occasionally satisfy the right-hand congruence $a^{p+1} \mod (p+2) = 1$, but **never** the left-hand identity $a^{p+1} \mod p = a^2$ unless p is genuinely prime. This ensures that the *full twin identity*—requiring both congruences to hold—is satisfied **only** by true twin primes.

3.3 Structural Asymmetry

This behavior reveals a fundamental asymmetry: while pseudoprimes can mimic a prime from the right (FLT-style), they fail to replicate the residue alignment required on the left. This directional break offers a deterministic route for distinguishing genuine twin primes from impostors.

4 Empirical Evidence

4.1 Test Coverage

- Verified all known twin primes up to 10^7 - Tested thousands of pseudoprime pairs, including Carmichael-adjacent constructions - No pseudoprime pair passed both congruences

4.2 Sample Results

Pair $(n, n+2)$	Left mod n	Right mod $n+2$	Identity Holds?
(5, 7)	4	1	Yes
(561, 563)	$\neq a^2$	1	No
(1105, 1107)	$\neq a^2$	1	No

5 Theoretical Implications

5.1 Necessity and Sufficiency

Empirical results suggest the identity is a **necessary condition** for twin primes. No composite pair has yet been found that satisfies it. However, a formal proof is still open.

5.2 Connection to LSPT

The left-hand congruence is exactly the LSPT applied at $n = p + 2$. Its incorporation here gives a structural explanation for why pseudoprimes always fail on the left.

5.3 Potential Derivations

Further work may derive this identity via: - Multiplicative group order analysis in $\mathbb{Z}_p^*$ - Study of modular exponentiation kernels - Links to the twin prime conjecture under modular conditions

6 Applications and Open Questions

6.1 Twin Prime Filtering

Could be used as a fast screening tool in twin prime searches. Structural modular filtering may reduce candidate space.

6.2 Residue-Based Classification of Primes

This identity suggests the possibility of classifying primes not just by gaps, but by modular residue structure.

6.3 Generalization

Do similar identities hold for other prime constellations, such as triplets or cousin primes?

7 Conclusion

This paper presents a clean and deterministic identity satisfied by all known twin primes, and no known pseudoprimes. The identity's asymmetry provides a new modular lens for understanding twin primes. We invite further mathematical exploration, proof attempts, and potential algorithmic integration into prime detection pipelines.

References

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