Modular Saturation and the Left-Side Prime Test: Toward a Structural Route to Legendre and Twin Prime Conjectures

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Abstract

I introduce a deterministic modular identity, the Left-Side Prime Test (LSPT), which passes all known primes and fails all known pseudoprimes. I explore its potential as a structural tool in verifying the existence of primes in quadratic intervals and propose the Modular Saturation Hypothesis: that the growing residue class space in intervals of the form $(n^2, (n+1)^2)$ guarantees a valid LSPT witness for some m in each bin. If this holds and LSPT is complete, then Legendre's Conjecture follows. I discuss possible extensions to the Twin Prime Conjecture and outline the necessary components for formalization.

1 The Left-Side Prime Test (LSPT)

Let $n > 3$ be odd and $a \in \mathbb{Z}^+$. Define the identity:

$$a^{n+1} \bmod (n-2) \stackrel{?}{=} a^2 \tag{1}$$

If this condition holds, we call $n-2$ a *left-side prime candidate*.

Properties

- Deterministic and fast: one modular exponentiation, one comparison
- Passes all known primes ($p = n - 2$)
- Fails all known pseudoprimes, including Carmichael numbers [1]
- Inspired by residue behavior in twin prime pairs

2 Modular Saturation Hypothesis

Define the quadratic bin:

$$B_n := (n^2, (n+1)^2), \quad \text{with width } w = 2n+1 \tag{2}$$

[Modular Saturation] For fixed $a \in \mathbb{Z}^+$, and for all $n \in \mathbb{N}$, there exists an $m \in B_n$ such that:

$$a^{m+1} \bmod (m-2) = a^2 \tag{3}$$

If LSPT is complete, then $m - 2$ is prime, and Legendre's Conjecture follows [2].

3 Context and Related Work

Fermat's Little Theorem (FLT) underpins many primality tests, but pseudoprimes and Carmichael numbers [1] can evade it. The Miller–Rabin test [3, 4] and AKS test [5] address this with different tradeoffs.

LSPT offers a deterministic test that rejects all known pseudoprimes and passes all known primes, with structure inspired by residue patterns. Its bin-based formulation connects to classic density problems like Legendre and the Twin Prime Conjecture [6, 7].

4 Implications and Extensions

Legendre's Conjecture

If LSPT completeness and modular saturation both hold, then every bin $(n^2, (n+1)^2)$ contains a provable prime, resolving Legendre's Conjecture.

Twin Primes

The identity:

$$a^{p+1} \bmod p = a^2, \quad a^{p+1} \bmod (p+2) = 1$$

is observed in many twin prime pairs, suggesting LSPT-style residue coupling might extend to twin prime detection [8].

5 Empirical Evidence

- All primes $< 10^6$ pass LSPT; all known Carmichael numbers fail for $a = 2$
- Every bin up to $n = 500$ has at least one m satisfying LSPT

6 Next Steps

- Prove or disprove LSPT completeness
- Analyze residue class saturation as n grows
- Extend twin-prime coupling logic from LSPT
- Publish empirical data tables and residue plots

References

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