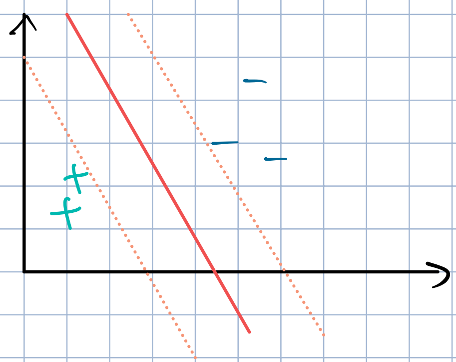
Recap:



Find decision boundary with the maximum margin

$$\text{dist}(\underline{x}_n, l) = \frac{|\underline{w}^T \underline{x}_n + b|}{\|\underline{w}\|} = \frac{y_n(\underline{w}^T \underline{x}_n + b)}{\|\underline{w}\|}$$

$l: \underline{w}^T \underline{x} + b = 0$

<u>WLOG</u>, we can set

$$\min_{1 \le n \le N} y_n(\underline{w}^T \underline{x}_n + b) = 1$$

SVM problem

$$\max_{\underline{w}, b} \frac{1}{\|\underline{w}\|}$$

maximize margin

s.t. $\min_{1 \le n \le N} y_n(\underline{w}^T \underline{x}_n + b) = 1$ (*)

Equivalent Problem

$$\min_{\underline{w}, b} \frac{1}{2} \underline{w}^T \underline{w}$$

$\|\underline{w}\| = \sqrt{\underline{w}^T \underline{w}}$

s.t. $y_n(\underline{w}^T \underline{x}_n + b) \ge 1, \ \forall n$ (**)

This is a relaxation of the original problem

$(*) \Rightarrow (**)$

$(**) \not\Rightarrow (*)$

However, at optimality of this problem, we have

$$\min_{1 \le n \le N} y_n (\underline{w}^T \underline{x}_n + b) = 1$$

Proof: (反证法)

Let $(\underline{w}^*, b^*)$ be an optimal solution

suppose $\theta = \min_n y_n (\underline{w}^{*T} \underline{x}_n + b^*) > 1$

Let $\underline{w} = \frac{\underline{w}^*}{\theta}$, $b = \frac{b^*}{\theta}$

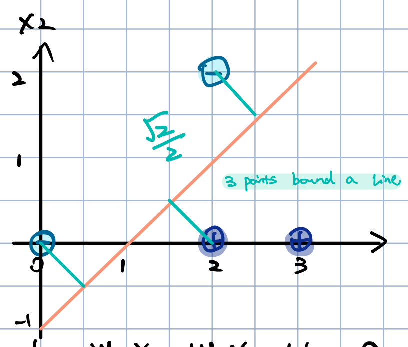
Then $\|\underline{w}\| = \frac{1}{\theta} \|\underline{w}^*\| < \|\underline{w}^*\|$

Also $y_n (\underline{w}^T \underline{x}_n + b) = \frac{1}{\theta} y_n (\underline{w}^{*T} \underline{x}_n + b^*) \ge 1$, $\forall n$

$(\underline{w}, b)$ is a straightly better solution than $(\underline{w}^*, b^*)$

$\Rightarrow$ Contradiction!

E.g.



$\underline{x}_1 = (0,0)$, $y_1 = -1$
$\underline{x}_2 = (2,2)$, $y_2 = -1$
$\underline{x}_3 = (2,0)$, $y_3 = +1$
$\underline{x}_4 = (3,0)$, $y_4 = +1$

l: $w_1 x_1 + w_2 x_2 + b = 0$

$$\min \frac{1}{2} (w_1^2 + w_2^2)$$

$$\text{s.t. } y_n (w_1 x_{n_1} + w_2 x_{n_2} + b) \ge 1, \ \forall n$$

$n=1$: $-b \ge 1$ ①

$n=2$: $-(2w_1 + 2w_2 + b) \ge 1$ ②

$n=3$: $2w_1 + b \ge 1$ ③

$n=4$: $3w_1 + b \ge 1$ ④

① + ② $\Rightarrow$ $w_1 \geq 1$

② + ③ $\Rightarrow$ $-2w_2 \geq 2$, $w_2 \leq -1$

So the best we can have is

$$w_1^* = 1 , \quad w_2^* = -1$$

$$\Rightarrow \frac{1}{2}(w_1^2 + w_2^2) \text{ minimized}$$

Further, setting $b^* = -1$, we satisfy ① - ④

Optimal boundary $x_1 - x_2 - 1 = 0$

The data points that sit on the margin line are called support vectors

Support vectors decide the value of the margin

$x_4 = (3,0)$, $y = +1$ does not affect the margin. Removing it from the dataset would has no impact. $\Leftrightarrow$ ④ not used