18BTMT402: Probability and Queuing Theory

SESSION 1- PROBABILITY

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UNIT I- RANDOM VARIABLES

- Discrete and continuous random variables –
- Probability density function,
- Moments,
- Moments generating functions,
- Binomial, Poisson,
- Geometric,
- Negative binomial,
- Uniform, Exponential,
- Gamma and Normal distribution.

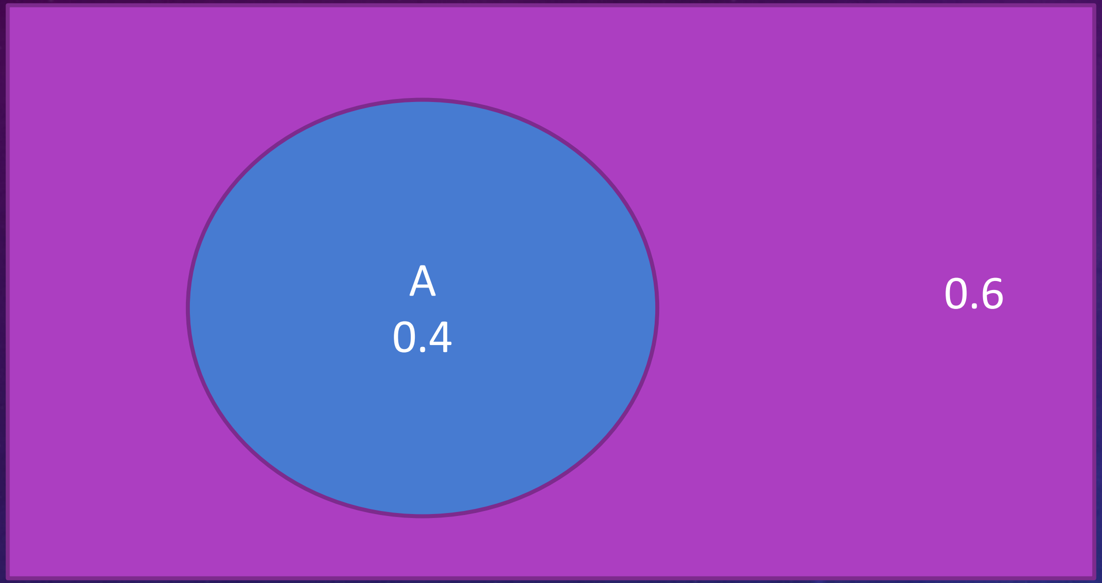
- Probability: Def: Probability is a measure of uncertainty. The **probability** of an event refers to the likelihood that the event will occur.

  - Mathematically, the probability that an event will occur is expressed as a number between 0 and 1. Notionally, the probability of event A is represented by P(A).

•If P(A) equals 0, event A will almost definitely not occur.
•If P(A) is close to zero, there is only a small chance that event A will occur.
•If P(A) equals 0.5, there is a 50–50 chance that event A will occur.
•If P(A) is close to one, there is a strong chance that event A will occur.
•If P(A) equals 1, event A will almost definitely occur.

In a statistical experiment, the sum of probabilities for all possible outcomes is equal to one. This means, for example, that if an experiment can have three possible outcomes (A, B, and C), then P(A) + P(B) + P(C) = 1.

- If P(A) = 0$\Longrightarrow$ Event will “A” not occur
- If P(A) = 1$\Longrightarrow$ Event “A” will occur



All other events

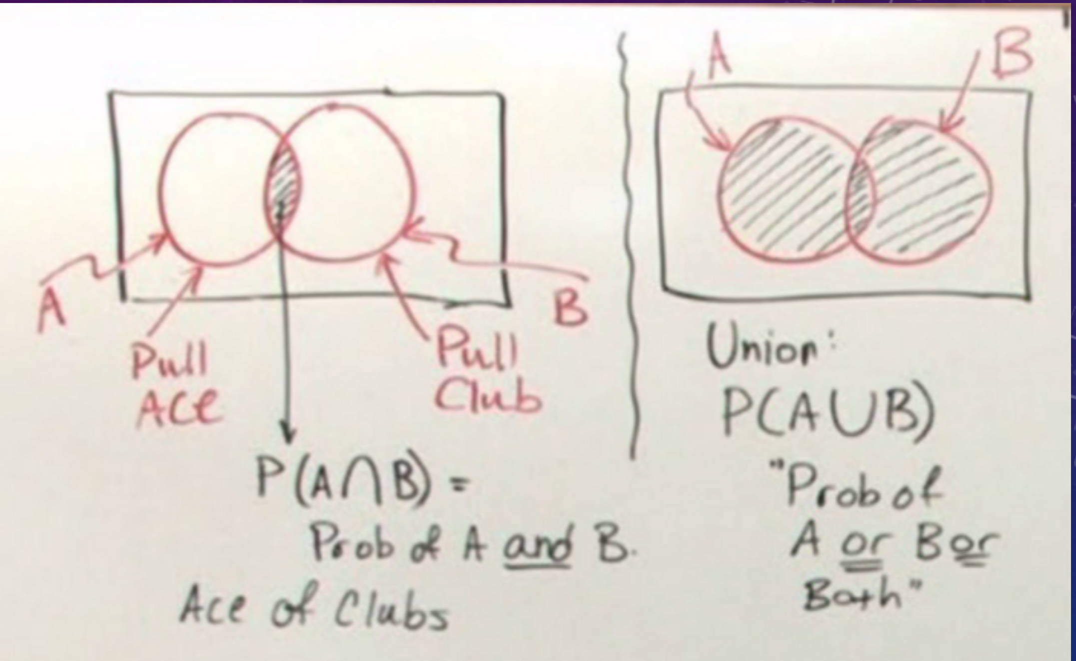
$$\Longrightarrow 0 \leq P(A) \leq 1$$
$$\Longrightarrow P(S) = 1$$
$$\Longrightarrow P(A') \leq 1 - P(A)$$

Terminology

- Die:
- Cards:
- Exhaustive Events or Sample space:
- Experiments:
- Random Experiment:
- Trial and Events:
- Equally Likely Events
- Independent Events
- Mutually Exclusive Events
- Compound Events
- Favorable Events
- Conditional Probability
- Odds in Favor of an Event and Odds against an Event
- Classical Definition of Probability

FEW NOTES:

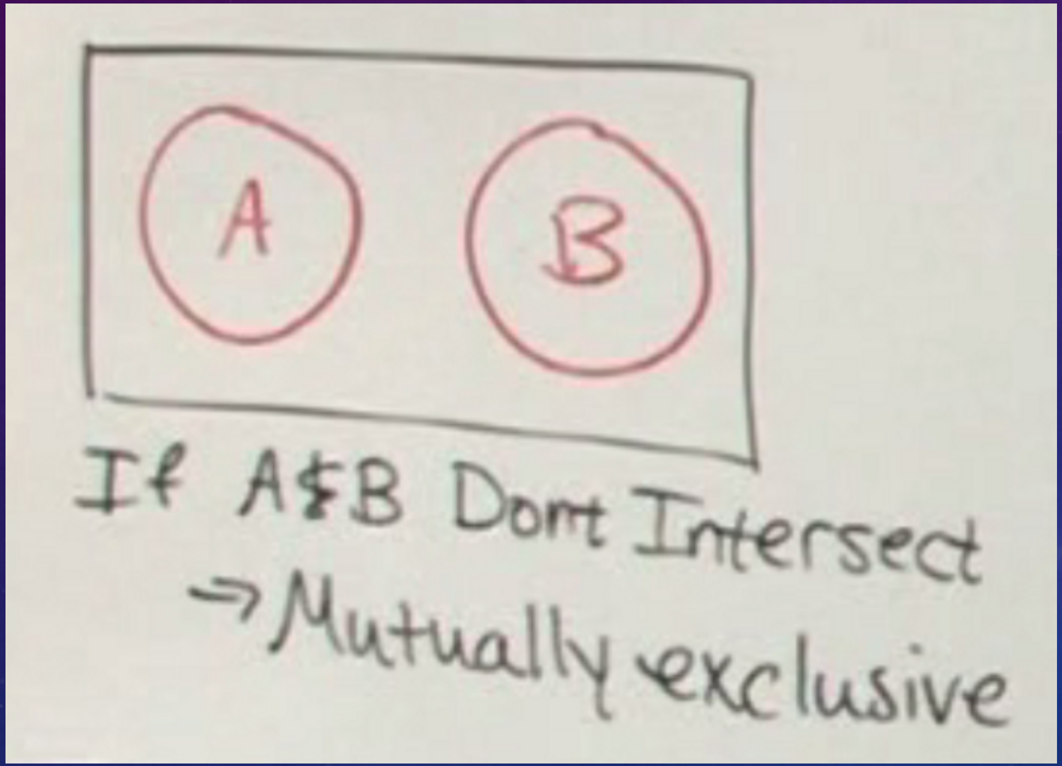
- Two events are **mutually exclusive** or **disjoint** if they cannot occur at the same time.
- If the occurrence of Event A changes the probability of Event B, then Events A and B are **dependent**. On the other hand, if the occurrence of Event A does not change the probability of Event B, then Events A and B are **independent**.
- The **complement** of an event is the event not occurring. The probability that Event A will not occur is denoted by P(A').



The probability that Events A and B ***both*** occur is the probability of the **intersection** of A and B. The probability of the **intersection** of Events A and B is denoted by $P(A \cap B)$.

$P(A \cap B) = P(A|B)\,P(B)$. But then you have to find a way to calculate the conditional probability $P(A|B)$.

The probability that events A **or** B occur is the probability of the **union** of A and B. The probability of the union of events A and B is denoted by P(A ∪ B) .

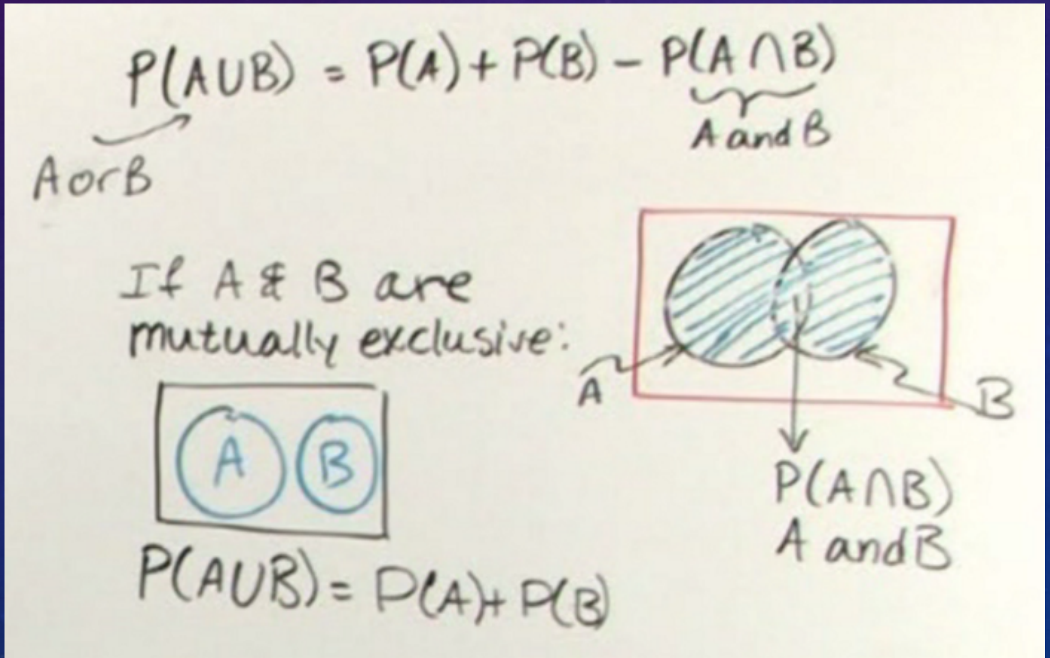


- If Events A and B are mutually exclusive, P(A ∩ B) = 0.

Additive rules of Probability

- The rule of addition applies to the following situation. We have two events, and we want to know the probability that either event occurs.
- **Rule of Addition** The probability that Event A **or** Event B occurs is equal to the probability that Event A occurs plus the probability that Event B occurs minus the probability that both Events A and B occur.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



DEFINITION

- If there are “n” equally likely outcomes of an experiment, of which one is called a success ”S”, then the probability of a success is:

P(A)= $\frac{number\ of\ ways\ “A”\ can\ occur}{Total\ \ number\ of\ outcomes\ possible}$

P(A) = $\frac{S}{n}$

Here A is my success.

Problem 1: What is the probability of drawing an Ace from a 52 card deck?

Sol. A $\longrightarrow$ Pulling an ace.

P(A) = $\frac{S}{n} = \frac{4}{52} = \frac{1}{13} \approx 0.07$

Problem: What is the probability that you guess the day of your friend’s birthday, not including the year?

Sol. P(A) = $\frac{S}{n} = \frac{1}{365} \approx 0.0027$

Problem: You roll a six sided die. What is the probability that you roll a 3?

Sol. P(A) = $\frac{S}{n} = \frac{1}{6}$

Problem: You roll a six sided die. What is the probability that you roll an even number?

Sol. P(A) = $\frac{S}{n} = \frac{3}{6} = \frac{1}{2} = 0.5,$ A $\longrightarrow$ Roll even number

Problem: A car rental company has 18 compact cars and 12 midsize cars. If 4 cars are selected at random , what is the probability of getting 2 cars of each type?

Sol. P(A) = $\frac{S}{n},$ A $\longrightarrow$ get 2 cars of each type

12+18 = 30 Total cars

If pick 4 $\Longrightarrow \binom{30}{4}$ ways to choose cars.

Number of ways the event can occur

Success, there are $\binom{18}{2}$ *ways pick compact &* $\binom{12}{2}$ *ways pick midsize.*

$\binom{18}{2}.\binom{12}{2} = \frac{18!}{(18-2)!2!}.\frac{12!}{(12-2)!2!} = \frac{18!}{16!2!}.\frac{12!}{10!2!} = \frac{18.17}{2.1}.\frac{12.11}{2.1}$ = 10098WAYS FOR “A” Success

Calculating Probability P(A) = $\frac{S}{n} = \frac{10098}{27405} = 0.368$

Problem: In a lot of 20 tires, 3 are defective. If you pick 4 tires at random, what is the probability that you will get 1 defective tire?

Sol.

$$P(A) = \frac{s}{n}$$

$s \rightarrow$ total ways to get 1 bad tire in your pick

$n \rightarrow$ total # ways to pick 4 tires out of 20.

$$n = \binom{20}{4} = \frac{20!}{(20-4)!\,4!} = \frac{20!}{16!\,4!}$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1}$$

$= 4845$ total ways to pick 4 tires out of 20

Number of ways event can occur: When you pick 4 tires, 1 should be defective. Total number of defect tires out of 20 tires is 3.

out of 20.

for "s" $= \binom{3}{1} \cdot \binom{17}{3}$

$$= \frac{3!}{(3-1)!1!} \cdot \frac{17!}{(17-3)!3!}$$

$$= \frac{3!}{2!\,1!} \cdot \frac{17!}{14!\,3!} = \frac{3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1} \cdot 1} \cdot \frac{17 \cdot 16 \cdot 15}{3 \cdot 2 \cdot 1}$$

$$= 3 \cdot \frac{4080}{6} = 2040$$ ways to pick 1 bad tire

- *Calculating the probability:*

$$P(A) = \frac{s}{n} = \frac{2040}{4845}$$

$$P(A) = \frac{8}{19} = 0.421$$

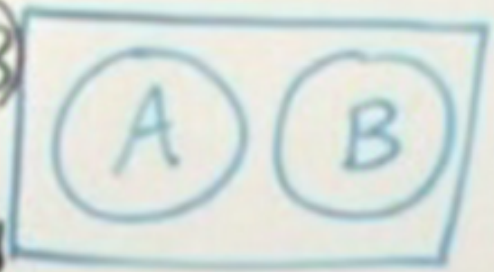
PROBLEM 1: YOU FLIP A COIN. WHAT IS THE PROBABILITY THAT YOU WILL GET A HEADS OR A TAILS?

$A \rightarrow$ heads $\quad B \rightarrow$ tails

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A or B

A and B



$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - 0$

$P(A \cup B) = 1$

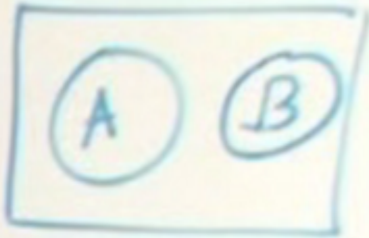
Problem 2: You roll two dice. What is the probability that the sum is 3 or 4?

$A \rightarrow \text{Sum} = 3$ $B \rightarrow \text{Sum} = 4$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A or B

A and B



$P(A) = \frac{2}{6 \cdot 6} = \frac{2}{36}$

$P(B) = \frac{3}{6 \cdot 6} = \frac{3}{36}$

$P(A \cup B) = \frac{2}{36} + \frac{3}{36} - 0$

$P(A \cup B) = \frac{5}{36}$

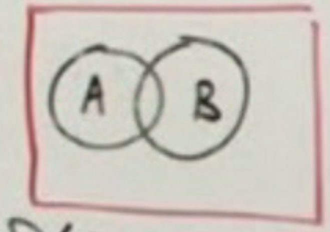
Die1	Die2	
1	2	= 3
2	1	= 3

Die1	Die2	
1	3	= 4
3	1	= 4
2	2	= 4

Problem 3: 1 card is picked from a deck of 52 cards. What is the probability that the card is a heart or a number card greater than 6?

$A \rightarrow$ Get a heart

$B \rightarrow$ Get # Card > 6



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A or B

$P(A) = \frac{13}{52}$ $\quad P(B) = \frac{16}{52}$

$P(A \cap B) = \frac{4}{52}$

$$P(A \cup B) = \frac{13}{52} + \frac{16}{52} - \frac{4}{52}$$

$$= \boxed{\frac{25}{52}}$$

Problem 4: you roll two six-sided dice. What is the probability that the sum of the dice is odd or divisible by 5?

A → Sum is odd ; B → Sum ÷ 5

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or And

Die 1 \ Die 2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A) = \frac{18}{36}$ Sum ODD

$P(B) = \frac{7}{36}$ Sum ÷ 5

$P(A \cap B) = \frac{4}{36}$

$P(A \cup B) = \frac{18}{36} + \frac{7}{36} - \frac{4}{36}$

$= \frac{21}{36} = \boxed{\frac{7}{12}}$

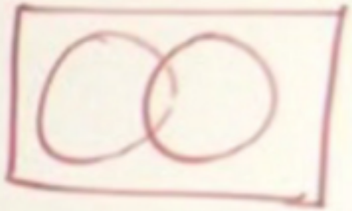
Problem 5: A box has five white balls numbered 1-5 and five green balls numbered 1-5. You choose 1 ball. What is the probability that it is white or odd numbered?

$A \to$ Choose white ; $B \to$ Choose odd

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or

and



$P(A) = \frac{5}{10}$

$P(B) = \frac{6}{10}$

$P(A \cap B) = \frac{3}{10}$

W: 1, 2, 3, 4, 5

G: 1, 2, 3, 4, 5

$P(A \cup B) = \frac{5}{10} + \frac{6}{10} - \frac{3}{10} = \frac{8}{10}$

$= \frac{4}{5}$

CONDITIONAL PROBABILITY

- In probability theory, **conditional probability** is a measure of the probability of an event (some particular situation occurring) given that (by assumption, presumption, assertion or evidence) another event has occurred.

Conditional prob:

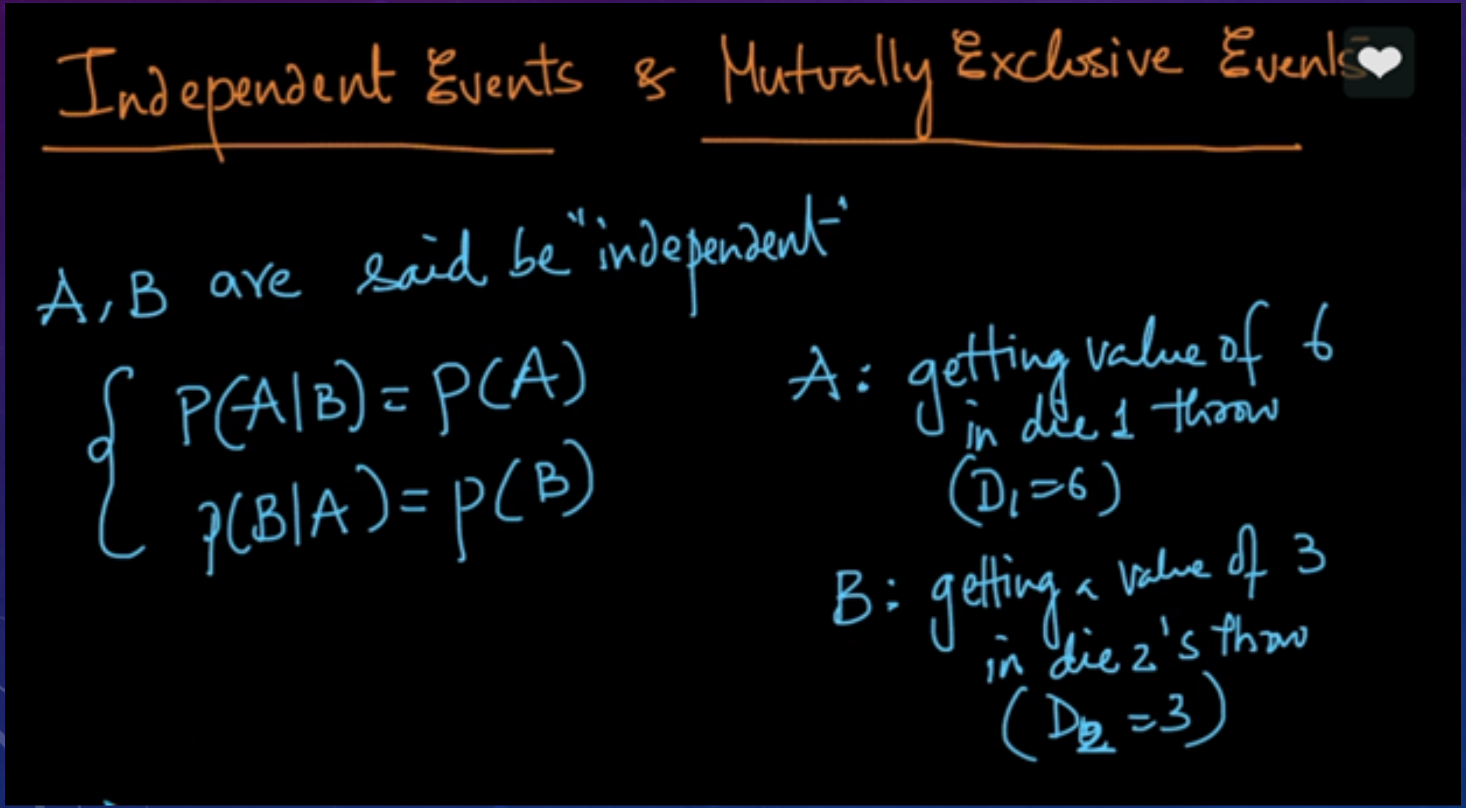
$$P(A|B) = Pr\left(A = \underset{\substack{\uparrow \\ \text{value}}}{a} \;\middle|\; B = \overset{\substack{\text{value} \\ \downarrow}}{b}\right)$$

$$\begin{cases} A : \text{r.v} \\ B : \text{r.v} \end{cases}$$

def: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$; $P(B) \neq 0$

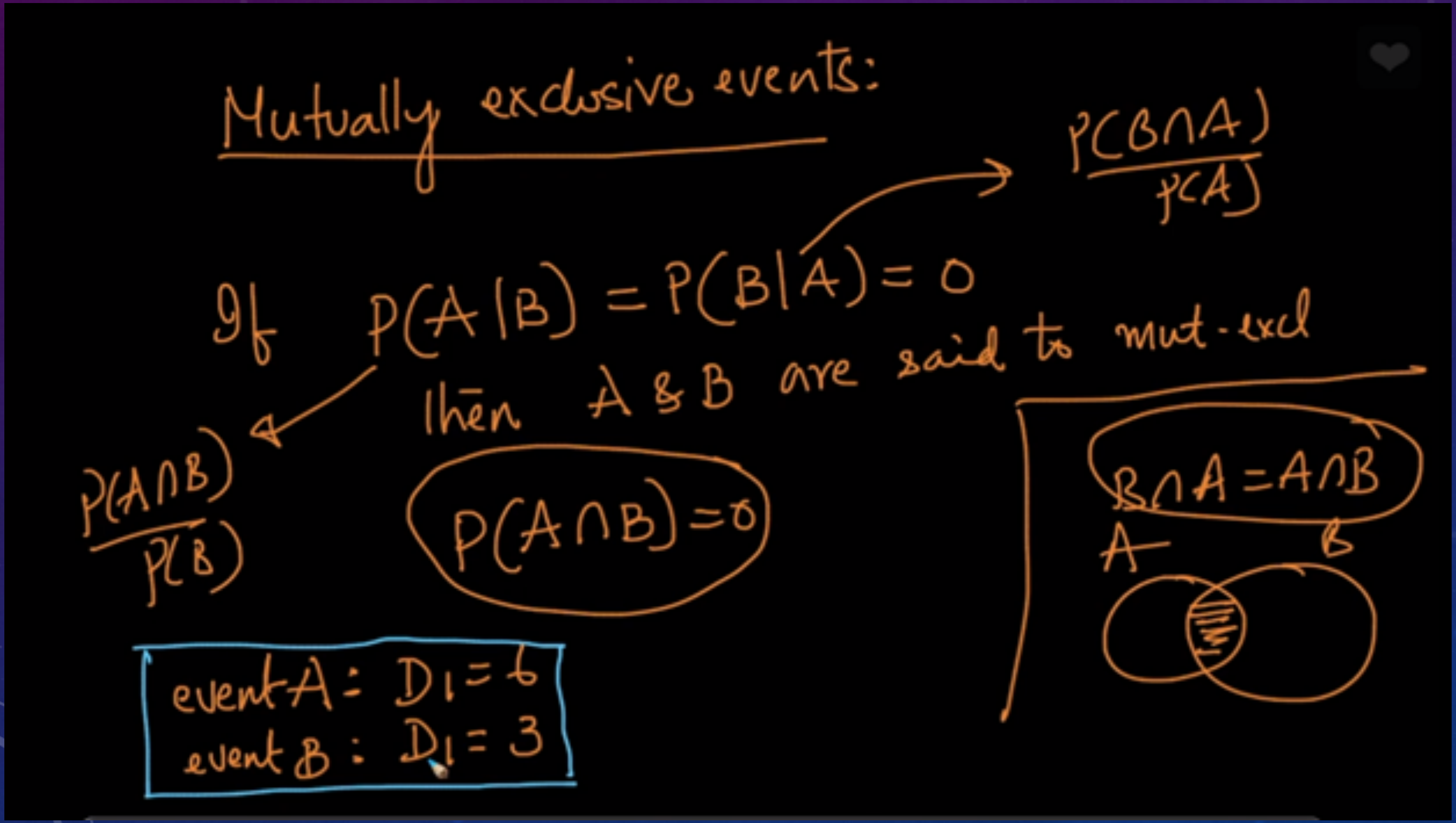
INDEPENDENT & MUTUALLY EXCLUSIVE EVENTS

- Events A and B are independent events if “A” does not influence “B”.

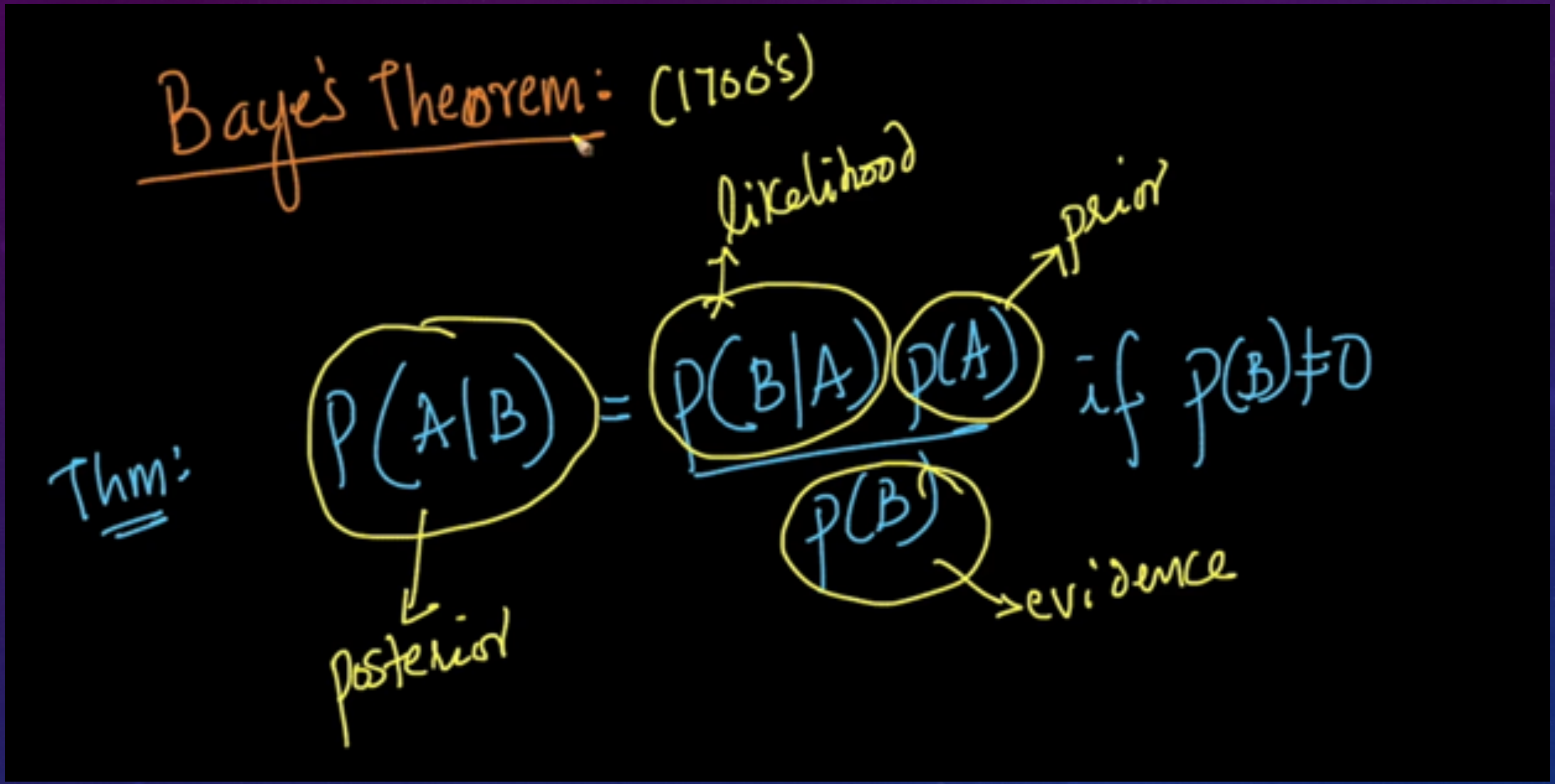


FOR INDEPENDENT EVENTS: (MULTIPLICATION RULE)

$$P(A \cap B) = P(A).P(B)$$

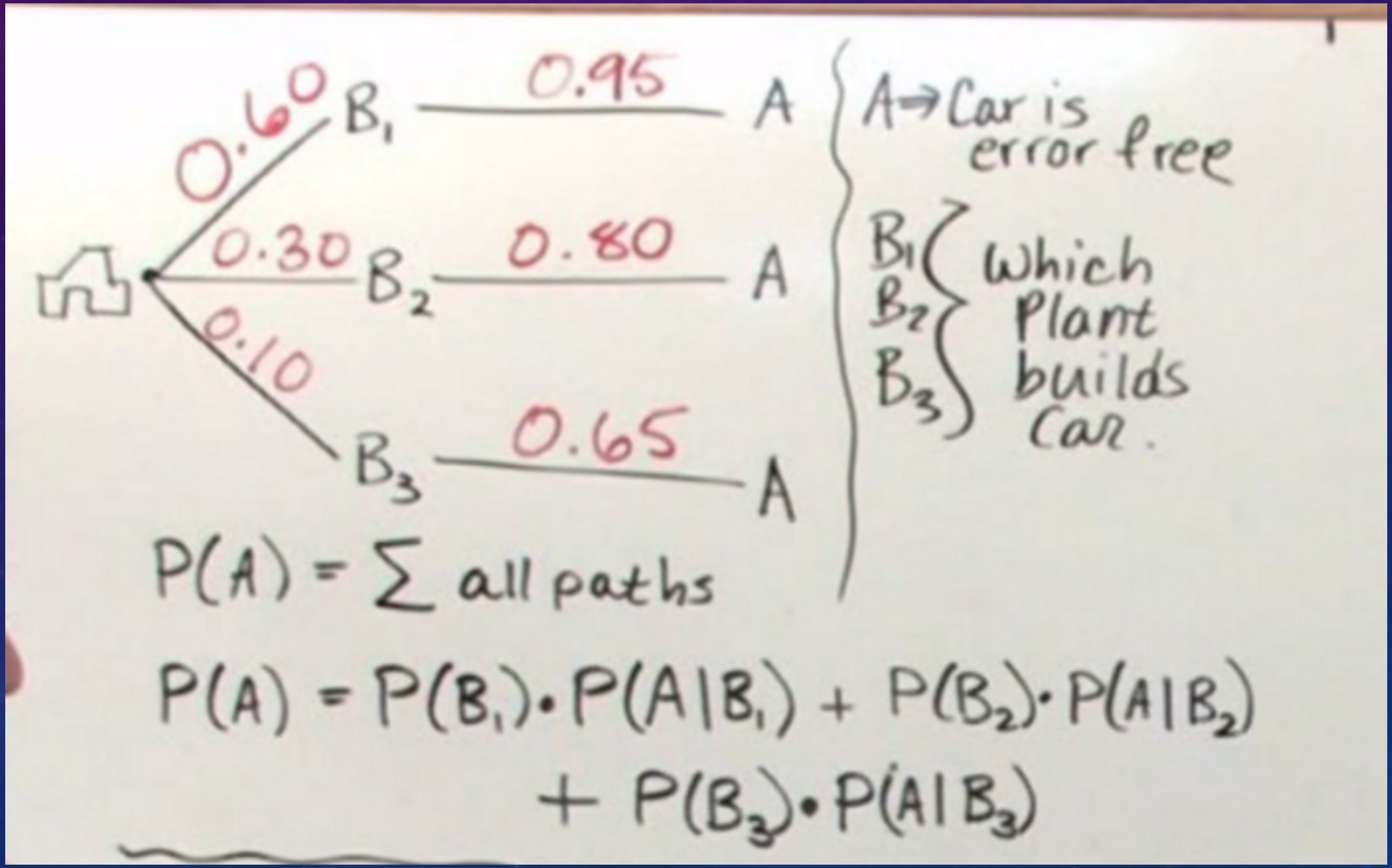


BAYES THEOREM



EXAMPLE:

3 plants make cars. Plant P1 makes 60%, plant P2 makes 30%, plant P3 makes 10%. Further, 95% of cars from plant P1 never fail, 80% of the cars from plant P2 never fail, and 65% of cars from plant P3 never fail. If I buy a car, what is the probability that it will be problem free?



$$P(A) = (0.6)(0.95) + (0.3)(0.8) + (0.1)(0.65)$$

$$P(A) = 0.875$$

$$\boxed{P(B_3 | A)} \Rightarrow P(A \cap B_3) = P(A) \cdot P(B_3 | A)$$

$$P(B_3 | A) = \frac{P(A \cap B_3)}{P(A)} \rightarrow = P(B_3) \cdot P(A | B_3)$$

$$P(B_3 | A) = \frac{P(B_3) \cdot P(A | B_3)}{\sum_{i=1}^{n} P(B_i) \cdot P(A | B_i)}$$

→ Path we are interested in.

→ Prob that Good cars built at all along all Paths

$$P(B_3 | A) = \frac{(0.1)(0.65)}{0.875} = 0.074$$

GENERAL RULE:

Rule of Elimination: If $B_1, B_2, B_3 \ldots B_n$ are mutually exclusive events of which 1 must occur:

$$P(A) = \sum_{i=1}^{n} \left[P(B_i) \cdot P(A \mid B_i) \right]$$

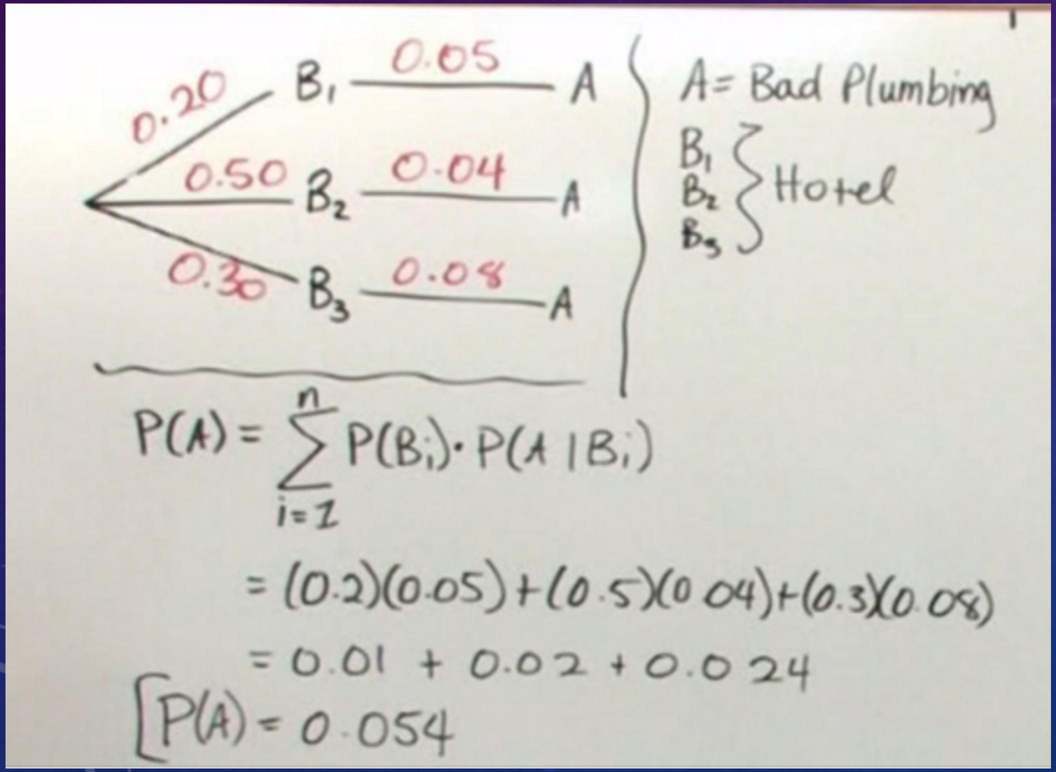
THEOREM:

Bayes' Theorem: If $B_1, B_2, B_3, \ldots B_n$ are mutually exclusive events then:

$$P(B_r|A) = \frac{P(B_r)\cdot P(A|B_r)}{\sum_{i=1}^{n}\left[P(B_i)\cdot P(A|B_i)\right]}$$

PROBLEM 2:

There are 3 hotels in town. 20% of your family stays in hotel H1. 50% stay at hotel H2, and 30% stay at hotel H3. Further, plumbing is faulty in 5% of the rooms in hotel H1, 4% rooms in H2, and 8% room sin H3. What is the probability that a random family member has faulty plumbing?



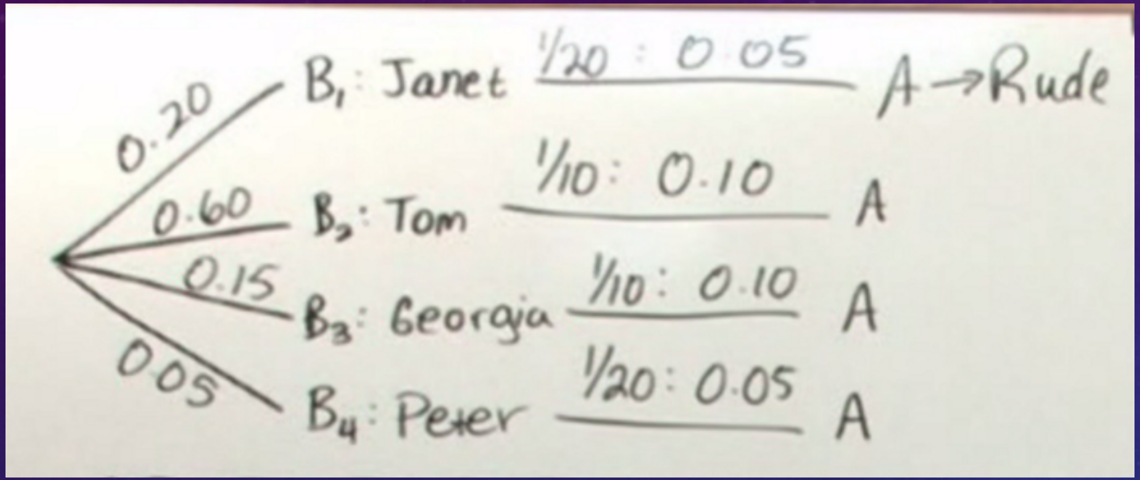
Problem: what is the probability that a person with a room with faulty plumbing was staying at hotel H3?

$$P(B_3|A) = \frac{P(B_3)\cdot P(A|B_3)}{\sum_{i=1}^{n} P(B_i)\cdot P(A|B_i)}$$

$$= \frac{(0.3)(0.08)}{0.054}$$

$$\boxed{P(B_3|A) = 0.44}$$

Problem: Janet, Tom, Georgia, and Peter are doctors. Janet has 20% of patients, Tom has 60%, Georgia has 15%, and Peter has 5%. Janet is rude to 1 in 20 patients, Tom is rude to 1 in 10, Georgia is rude to 1 in 10, and Peter is rude to 1 in 20. If you are treated rudely, what is the probability that you went to see Tom?



$$P(B_2|A) = \frac{P(B_2)\cdot P(A|B_2)}{\sum_{i=1}^{n} P(B_i)\cdot P(A|B_i)}$$

$$= \frac{(0.60)(0.10)}{\left[(0.20)(0.05) + (0.60)(0.10) + (0.15)(0.10) + (0.05)(0.05)\right]}$$

$$= \frac{0.06}{0.0875} = 0.686$$

EXPECTED VALUE

Def: IF $P_1, P_2, P_3, \ldots\ldots P_N$ OF THE PROBABILITIES OF THE EVENTS $X_1, X_2, X_3, \ldots\ldots X_N$ RESPECTIVELY.

$$E(X) = P_1 X_1 + P_2 X_2 + P_3 X_3 + \ldots\ldots P_N X_N$$

$$= \sum_{R=1}^{N} P_R X_R$$

Problem: you buy 1 of 1000 raffle tickets. The grand prize is \$500. What is the mathematical expectation?

$$\text{Expectation} = \frac{1}{1000} \cdot \$500$$

$$= \$\frac{1}{2} = \$0.50$$