The Poisson distribution is a discrete distribution with a single parameter m. As m increases, the distribution shifts to the right. This is illustrated in the following diagram for 4 values from $m = 0.3$ to $m = 4.0$:

The Poisson Distribution

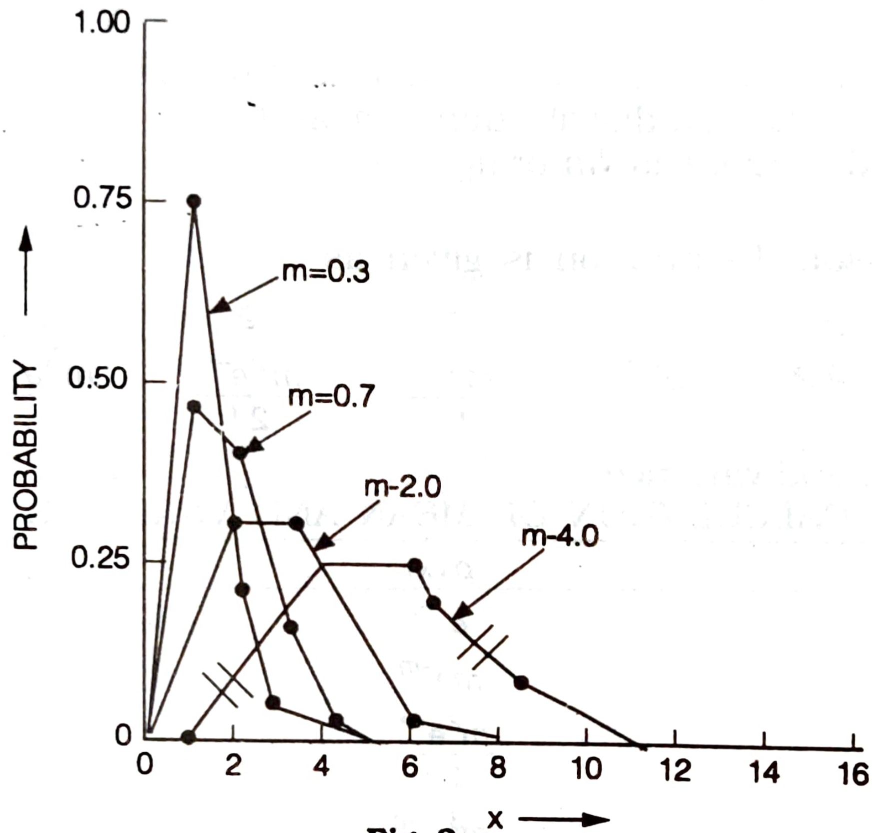

Fig. 2.

All Poisson probability distributions are skewed to the right. This is the reason why the Poisson probability distribution has been called the probability distribution of rare events (the probabilities tend to be high for small numbers of occurrences).

The Poisson probability distribution is concerned with certain processes that can be described by a discrete random variable. The probabilities of 0, 1, 2, ... successes are given by the successive terms of the expansion.

$$e^{-m}\left(1 + m + \frac{m^2}{2\,!} + \frac{m^3}{3\,!} \ldots\ldots + \frac{m^r}{r\,!} \ldots\ldots +\right)$$

This can be written in a tabular form as follows :

No. of successes (x)	Probability $p(x)$	No. of successes (x)	Probability $p(x)$
0	e^{-m}	4	$\frac{m^4 e^{-m}}{4!}$
1	me^{-m}	:	:
2	$\frac{m^2 e^{-m}}{2!}$	r	$\frac{m^r e^{-m}}{r!}$
3	$\frac{m^3 e^{-m}}{3!}$	:	:

The above table gives probabilities. If we want to know the expected number of occurrences for different successes, we have to multiply each term by N_2, *i.e.*, the total number of observations.

Constants of the Poisson Distribution

Since p is very small in case of Poisson distribution, the value of q is almost equal to 1. The constants of the Poisson distribution can, thus, be easily obtained by putting 1 in place of q in the constants of the binomial distribution.

The various constants of the Poisson distribution are :
The mean of the Poisson distribution = m, and
The standard deviations is $\sqrt{m}$ or $\mu_2 = m$.

Proof. The Poisson distribution is given as :

No. of successes (x)	0	1	2	3	4......
Probability $p(x)$	e^{-m}	$\frac{me^{-m}}{1!}$	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^3 e^{-m}}{3!}$	$\frac{m^4 e^{-m}}{4!}$ ......

Find the mean and variance.

CALCULATION OF MEAN AND VARIANCE

x	$p(x)$	$x\,p(x)$
0	e^{-m}	0
1	me^{-m}	me^{-m}
2	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^2 e^{-m}}{2} \times 2$
3	$\frac{m^3 e^{-m}}{3!}$	$\frac{m^3 e^{-m}}{3 \times 2 \times 1} \times 3$
4	$\frac{m^4 e^{-m}}{4!}$	$\frac{m^4 e^{-m}}{4 \times 3 \times 2 \times 1} \times 4$
:	:	:
r	$\frac{m^r e^{-m}}{r!}$	$\frac{m^r e^{-m}}{r!} \times r$
:	:	:

† The rule most often used by statisticians is that the Poisson is a good approximation of the Binomial when n is equal to or greater than 20 and p is equal to or less than 0.05. In cases that meet these conditions, we can substitute (np) in place of mean of the Poisson distribution (m).

$$\text{Mean} = \Sigma\, x\,.\,p\,(x)$$

$$\Sigma\, xp = 0 + me^{-m} + m^2e^{-m} + \frac{m^3e^{-m}}{2\times 1} + \frac{m^4e^{-m}}{3\times 2} + \ldots\ldots$$

$$= me^{-m}\left(1 + m + \frac{m^2}{2\,!} + \frac{m^3}{3\,!} + \ldots\right)$$

$$= me^{-m}\,.\,e^m = m \qquad \left(\because\ e^{-m} = 1 + m + \frac{m^2}{2\,!} + \frac{m^3}{3\,!} + \ldots\right)$$

Hence, the mean of Poisson distribution is m.

Variance or $\quad \mu_2 = \upsilon_2 - \upsilon_1^2$

[where υ_1 and υ_2 denote moment about origin zero]

$$\upsilon_2 = \Sigma\,\{x^2\,.\,p\,(x)\}$$

x	$p\,(x)$	$x^2\,.\,p\,(x)$
0	e^{-m}	0
1	me^{-m}	me^{-m}
2	$\frac{m^2e^{-m}}{2\,!}$	$\frac{m^2e^{-m}}{2}\times 4$
3	$\frac{m^3e^{-m}}{3\,!}$	$\frac{m^3e^{-m}}{3\times2\times1}\times 9$
4	$\frac{m^4e^{-m}}{4\,!}$	$\frac{m^4e^{-m}}{4\times3\times2\times1}\times 16$
$\vdots$	$\vdots$	$\vdots$
r		$\frac{m^r e^{-m}}{r!}$

$$\Sigma\, x^2p\,(x) = 0 + me^{-m} + \frac{2m^2e^{-m}}{1\,!} + 3\,\frac{m^3e^{-m}}{2\,!} + 4\,\frac{m^4e^{-m}}{3\,!} + \ldots\ldots$$

$$= me^{-m}\left(1 + 2m + 3\,\frac{m^2}{2\,!} + 4\,\frac{m^3}{3\,!} + \ldots\ldots\right)$$

Breaking each term within brackets into two parts each we have

$$\Sigma\, x^2p\,(x) = me^{-m}\left(1 + m + \frac{m^2}{2\,!} + \frac{m^3}{3\,!} + \ldots\ldots\right.$$

$$\left.+\left(m + 2\,\frac{m^2}{2\,!} + 3\,\frac{m^3}{3\,!} + \ldots\ldots\right)\right\}$$

$$= me^{-m}\left[e^m + m\left(1 + m + \frac{m^2}{2\,!} + \ldots\ldots\right)\right]$$

$$= me^{-m}\,(e^m + me^m)$$

$$= me^{-m}\,.\,e^m\,(1 + m)$$

$$= m\,(1 + m)$$

$$= m + m^2$$

$$\mu_2 = \upsilon_2 - \upsilon_1^2$$

$$= m + m^2 - (m)^2 = m\ [\therefore\ \upsilon_1 = m]$$

Thus, σ^2 or $\mu_2 = m$ and $\sigma = \sqrt{m}$.

In a similar manner we can show that for Poisson distribution $\mu_2 = m$ and $\mu_4 = m + 3m^2$.

> **Constants of Poisson Distribution**
>
> $\mu_1 = 0$, $\mu_2 = m$, $\mu_3 = m$, $\mu_4 = m + 3m^2$
>
> $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{m^2}{m^3} = \frac{1}{m}$$
>
> $$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{m + 3m^2}{m^2} = 3 + \frac{1}{m}.$$

One great advantage of the Poisson distribution is that we need only the value of mean in order to compute the values of various constants. This shall be clear from the following illustration :

Illustration 13. The mean of the Poisson distribution is 2.25. Find the other constants of the distribution.

Solution. We are given mean or $m = 2.25$

$$\sigma = \sqrt{m} = \sqrt{2.25} = 1.5$$

$$\mu_1 = 0$$

$$\mu_2 = m = 2.25$$

$$\mu_3 = m = 2.25$$

$$\mu_4 = m + 3m^2 = 2.25 + 3\,(2.25)^2$$

$$\doteq 2.25 + 15.1875 = 17.4375 \text{ or } 17.44 \text{ app.}$$

$$\beta_1 = \frac{1}{m} = \frac{1}{2.25} = .444.$$

$$\beta_2 = 3 + \frac{1}{m} = 3 + .444 = 3.444.$$

Role of the Poisson Distribution

The Poisson distribution is used in practice in a wide variety of problems where there are infrequently occurring events with respect to time, area, volume or similar units. Some practical situations in which Poisson distribution can be used are given below :

1. it is used in quality control statistics to count the number of defects of an item,
2. in biology to count the number of bacteria,
3. in physics to count the number of particles emitted from a radio-active substance,
4. in insurance problems to count the number of casualities,
5. in waiting-time problems to count the number of incoming telephone calls or incoming customers,
6. number of traffic arrivals such as trucks at terminals, aeroplanes at airports, ships at docks, and so forth,*

7. in determining the number of deaths in a district in a given period, say, a year, by a rare disease,

8. the number of typographical errors per page in typed material, number of deaths as a result of road accidents, etc.,

9. in problems dealing with the inspection of manufactured products with the probability that any one piece is defective is very small and the lots are very large, and

10. to model the distribution of the number of persons joining a queue (a line) to receive a service or purchase of a product.

In general, the Poisson distribution explains the behaviour of those discrete variates where the probability of occurrence of the event is small and the total number of possible cases is sufficiently large.

Illustration 14. (*a*) Suppose on an average 1 house in 1,000 in a certain district has a fire during a year. If there are 2,000 houses in that district, what is the probability that exactly 5 houses will have a fire during the year ?

Solution. Applying the Poisson distribution

$$\bar{X} = np$$

$$n = 2000,\ p = \frac{1}{1000}$$

$$n = 2000 \times \frac{1}{1000} = 2$$

$$P(r) = e^{-m}\frac{m^r}{r!},\ r = 0, 1, 2, \ldots\ldots$$

Here $m = 2,\ r = 5$ and $e = 2.7183$

$$\therefore \quad P(5) = \frac{2.71823^{-2} \times 2^5}{5!}$$

$$= \frac{\text{Rec. [antilog } (2 \times \log 2.78183)] \times 32}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{\text{Rec. [antilog } (2 \times .4343)] \times 32}{120}$$

$$= \frac{\text{Rec. [antilog } (.8686)] \times 32}{120}$$

$$= \frac{\text{Rec. } [7.389] \times 32}{120} = \frac{.1352 \times 32}{120} = .036.$$

(*b*) Ten per cent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly two will be defective by using (*a*) the binomial distribution, (*b*) the Poisson approximation to the binomial distribution.

Solution. Probability of a defective tool or $p = 0.1$

(*a*) When a binomial distribution is used probability of 2 defectives in 10 is given by

$${}^{10}C_2\,(.1)^2\,(.9)^8 = 0.1937$$

(*b*) When Poisson distribution is used probability of 2 defectives is given by

$$P(2) = \frac{e^{-m} m^2}{2!}$$

where $m = np = 10\,(0.1) = 1.$

$$\frac{e^{-1}}{2!} = \frac{e^{-1}}{2} = \frac{1}{2e} = 0.184.$$

In general the approximation is good if

$$p \le 0.1 \text{ and } m = np < 5.$$

Fitting a Poisson Distribution

The process of fitting a Poisson distribution is very simple. We have just to obtain the value of m, *i.e.*, the average occurrence, and calculate the frequency of 0 success. The other frequencies can be very easily calculated as follows :

$$N(P_0) = Ne^{-m}$$

$$N(P_1) = N(P_0) \times \frac{m}{1}$$

$$N(P_2) = N(P_1) \times \frac{m}{2}$$

$$N(P_3) = N(P_3) \times \frac{m}{3} \text{ etc.}$$

A 'goodness-of-fit' test will confirm whether or not the fit is close enough to justify the belief that the distribution is of the Poisson type.

Illustration 15. (*a*) The following mistakes per page were observed in a book :

No. of mistakes per page	0	1	2	3	4
No. of times the mistake occurred	211	90	19	5	0

Fit a Poisson distribution to fit data.

(M. Com., Bharatidasan Univ., 1997; M. Com., Punjab Univ., 1996; MBA, Kumaun Univ., 2002; M. Com., Jamia Millia, 2005)

Solution. FITTING POISSON DISTRIBUTION

X	f	fX
0	211	0
1	90	90
2	19	38
3	5	15
4	0	0
	$N = 325$	$\Sigma fX = 143$

$$\bar{X} = \frac{\Sigma fX}{N} = \frac{143}{325} = 0.44$$

Mean of the distribution or $m = .44$

$$(P_0) = e^{-m} = 2.7183^{-.44}$$
$$= \text{Rec. [antilog (.44 log 2.7183)]} = \text{Rec. [antilog (.44} \times .4343)]$$
$$= \text{Rec. [antilog 1.1910]} = \text{Rec. } 1.552 = .6443$$
$$N(P_0) = .6443 \times 325 = 209.40$$
$$N(P_1) = N(P_0) \times \frac{m}{1} = 209.4 \times .44 = 92.14$$
$$N_2(P_2) = N(P_1) \times \frac{m}{2} = 92.14 \times \frac{.44}{2} = 92.14 \times .22 = 20.27$$
$$N(P_3) = N(P_2) \times \frac{m}{3} = 20.27 \times \frac{.44}{3} = 6.76 \times .44 = 2.97$$
$$N(P_4) = N(P_3) \times \frac{m}{4} = 2.97 \times \frac{.44}{4} = 2.97 \times .11 = 0.33.$$

The expected frequencies of Poisson distribuion are :

X	0	1	2	3	4	
f	209.40	92.14	20.27	2.97	0.33	= 325.11

Note. A rough check on the accuracy of result is that the total of the expected frequencies should be equal to the total of the observed frequencies. For example, in the above case the total of expected frequencies is 325.11 and the observed total is 325. The slight difference is due to approximation.

(*b*) In a certain factory manufacturing razor blades, there is small chance 1/50 for any blade to be defective. The blades are placed in packets, each containing 10 blades. Using the Poisson distribution, calculate the approximate number of packets containing not more than 2 defective blades in a consignment of 10,000 packets.

Solution. $N = 10{,}000,\ p = \frac{1}{50},\ n = 10$

$$m = np = 10 \times \frac{1}{50} = 0.2$$

$$(P_0) = e^{-m} = e^{-0.07} = 0.8187 \text{ (from the table)}$$

$$N(P_0) = (P_0) \times 10{,}000 = .8187 \times 10{,}000 = 8187$$

$$N(P_1) = N(P_0) \times m = 8187 \times .2 = 1637.4$$

$$N(P_2) = N(P_1) \times \frac{m}{2} = 1637.4 \times \frac{.2}{2} = 163.74$$

The approximate number of packets containing not more than 2 defective blades in a consignment of 10,000 packets is :

$$10{,}000 - [8187 + 1637.4 + 163.74] = [10{,}000 - 9988.14] = 11.86 \text{ or } 12.$$

Illustration 16. Suppose that a manufactured product has 2 defects per unit of product inspected. Using Poisson distribution, calculate the probabilities of finding a product without any defect, 3 defects and 4 defects. (Given $e^{-2} = 0.135$)

(M. Com., Andhra Univ., M.Com., Madurai Univ., 1994)

Solution. Average number of defects or $m = 2$

$$P(r) = \frac{e^{-m} m^r}{r!}, \quad r = 0, 1, 2, \ldots\ldots$$

$$P(0) = e^{-2} = 0.135 \text{ (given)}$$

$$P(1) = (P_0) \times m = .135 \times 2 = .27$$

$$P(2) = (P_1) \times \frac{m}{2} = .27 \times \frac{2}{2} = .27$$

$$P(3) = (P_2) \times \frac{m}{3} = .27 \times \frac{2}{3} = .18$$

$$P(4) = (P_3) \times \frac{m}{4} = .18 \times \frac{2}{4} = .09.$$

Hence, the probability that a product has no defect is 0.135, product has 3 defects is 0.18 and product has 4 defects is 0.09.

Illustration 17. (***a***) The number of defects per unit in a sample of 330 units of manufactured product was found as follows :

No. of defects :	0	1	2	3	4
No. of units :	214	92	20	3	1

Fit a Poisson distribution to the data and test for goodness of fit. (Given $e^{-0.439} = 0.6447$)

(M. Com., Punjab Univ., 1997)

$$\bar{X} = \frac{\Sigma fX}{N} = \frac{145}{330} = 0.439$$

$$P_{(0)} = e^{-m} = e^{-0.439} = 0.6447 \qquad \text{(from the table)}$$

$$N(P_0) = P_{(0)} \times N = .6447 \times 330 = 212.75$$

$$N(P_1) = N(P_0) \times m = 212.75 \times .439 = 93.4$$

$$N(P_2) = N(P_1) \times \frac{m}{2} = 93.4 \times \frac{.439}{2} = 20.5$$

$$N(P_3) = N(P_2) \times \frac{m}{3} = 20.5 \times \frac{.439}{3} = 3.0$$

$$N(P_4) = N(P) \times \frac{m}{4} = 3.0 \times \frac{.439}{4} = 0.33$$

Thus the expected frequencies as per Poisson distribution are :

No. of defects	0	1	2	3	4
No. of units	212.75	93.40	20.50	3.00	0.33

(*b*) The following table gives the number of days in a 50 day period during which automobile accidents occurred in a certain part of a city. Fit a Poisson distribution to the data.

No. of accidents	0	1	2	3	4
No. of days	19	18	8	4	1

(B. Com. (H), Delhi Univ.; M. Com., Sukhadia Univ., 1998; MBA, Kumaun Univ., 2004)

Solution. FITTING OF POISSON DISTRIBUTION

X	f	fx
0	19	0
1	18	18
2	8	16
3	4	12
4	1	4
	$N = 50$	$\Sigma fX = 50$

$$m = \frac{\Sigma fX}{N} = \frac{50}{50} = 1$$

$$(P_0) = e^{-m} = 2.7183^{-1} = 0.36788 \qquad \text{(from the table)}$$

$$N(P_0) = P_{(0)} \times N = .36788 \times 50 = 18.394 \text{ or } 18.4$$

$$N(P_1) = N(P_0) \times m = 18.394 \times 1 = 18.394 \text{ or } 18.4$$

$$N(P_2) = N(P_1) \times \frac{m}{2} = \frac{18.394}{2} = 9.197 \text{ or } 9.2$$

$$N(P_3) = N(P_2) \times \frac{m}{3} = \frac{9.197}{3} = 3.066 \text{ or } 3.1$$

$$N(P_4) = N(P_3) \times \frac{m}{4} = \frac{3.066}{4} = 0.766 \text{ or } 0.8$$

Poisson Distribution as an Approximation of the Binomial Distribution

The Poisson distribution can be a reasonable approximation of the binomial under certain conditions like:

(i) number of trials, *i.e.*, n is indefinitely large, *i.e.*, $n \to \infty$.

(ii) p, *i.e.*, the probability of success for each trial is indenfinitely small. *i.e.*, $p \to 0$.

(iii) $np = m$ (say) is finite.

The rule most often used by statisticians is that the Poisson is a good approximation of the binomial when n is equal to or greater than 20 and p is equal to or less than .05.

If above conditions hold good, we can substitute the mean of the binomial disribution (np) in place of the mean of the Poisson distribution (m), so that the formula becomes :

$$P(r) = \frac{e^{-np}(np)^r}{r!}$$

Proof. In case of binomial distribution the probability of r successes is given by

$$p^{(r)} = {}^nC_r\, q^{n-r} p^r = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!} p^r q^{n-r}$$

Put $p = \dfrac{m}{n}$ $\qquad\therefore\qquad q = 1 - p = 1 - \dfrac{m}{n}.$

We now get $\qquad P(r) = \dfrac{n(n-1\ldots\ldots n-r+1)}{r!}\left(\dfrac{m}{n}\right)^r \times \left(1 - \dfrac{m}{n}\right)^{n-r}$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{r-1}{n}\right)m^r}{r!}\left\{\frac{\left(1-\frac{m}{n}\right)^n}{\left(1-\frac{m}{n}\right)^r}\right\}$$

For fixed r, as $n \to \infty$

$\left(1-\dfrac{1}{n}\right)\ldots\ldots\left(1-\dfrac{r-1}{n}\right)\left(1-\dfrac{m}{n}\right)^r$ all tend to 1 and $\left(1-\dfrac{m}{n}\right)^n$ to $e^{-m}m^r$.

Hence in the limiting case $p^{(r)} = \dfrac{e^{-m}m^r}{r!}$

Needless to point out that use of Poisson as an approximation to binomial probability distribution if the conditions given above are satisfied can simplify calculation work and save time with more or less similar results as one would expect from binomial distribution.

Example: For $n = 3000$, $p = 0.005$, the probability of 18 successes by binomial distribution is given by $b(18; 3000, .005) = 3000_{C_{18}}(.005)^{18}(.995)^{2982}$ which involves prohibitive amount of work. Instead using Poisson distribution as an approximation, we get $\lambda = 3000 \times .005 = 15$. Probability of 18 success $= f(18, 15) = 0.8195$ from table (A8 to A11).

General rule: Poisson approximation to B.D. is used whenever $n \geq 20$ and $p \leq 0.05$. For $n \geq 100$, approximation is excellent provided $\lambda = np \leq 10$.

Worked Out Examples

Poisson distribution

Example 1: A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a particular packet will violate the guarantee.

Solution: The probability of a seed not germinating $= p = \frac{5}{100} = 0.05$

$\lambda =$ mean number of seeds, in a sample of 200, which do not germinate

$= np = 200 \times 0.05 = 10$

Let X = R.V. = number of seeds that do not germinate

A packet will violate guarantee if it contains more than 20 non-germinating seeds.
Probability that the guarantee is violated

$$= P(X > 20) = 1 - P(X \leq 20) = 1 - \sum_{x=0}^{20} \frac{e^{-10}10^x}{x!}$$

$$= 1 - F(20, 10) = 1 - .9984 = 0.0016$$

where cummulative distribution function F is read for x=20 and λ=10 from the tables (A8 to A11).

Example 2: The average number of phone calls/ minute coming into a switch board between 2 and 4 PM is 2.5. Determine the probability that during one particular minute there will be (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 or fewer (f) more than 6 (g) at most 5 (h) at least 20 calls.

Solution: $\lambda = 2.5,\ f(x;\lambda) = f(x;2.5) = \frac{(2.5)^x(e^{-2.5})}{x!}$
Let X = R.V. = number of phone calls/minute during that (odd) 2 and 4 PM.

a. $f(0;2.5) = e^{-2.5} = .08208$

b. $f(1;2.5) = .2052$

c. $f(2;2.5) = .2565$

d. $f(3;2.5) = .2138$

e. $P(X \leq 4) = \sum_{x=0}^{4} f(x;2.5) = F(4;2.5) = .8912$

(read from tables A8 to A11)

f. $P(X > 6) = 1 - P(X \leq 6) = 1 - \sum_{x=0}^{6} f(x;2.5)$

$= 1 - F(6;2.5) = 1 - .9858 = 0.0142$

g. $P(X \leq 5) = \sum_{x=0}^{5} f(x;2.5) = F(5;2.5) = .9580$

h. $P(X \geq 2.0) = 1 - P(X \leq 19) = 1 - \sum_{x=0}^{19} f(x;2.5)$

$= 1 - F(19;2.5) = 1 - 1 = 0.$

Example 3: Suppose that on the average one person in 1000 makes a numerical error in preparing income tax return (ITR). If 10000 forms are selected at random and examined, find the probability that 6, 7 or 8 of the forms will be in error.

Solution: Let X = R.V. = number ITR forms containing a numerical error. Essentially this is a binomial experiment with 10000 trials and probability (of success) $p = \frac{1}{1000} = 0.001$. So by B.D. probability of 6, 7 or 8 error forms = $P(X = 6, 7 \text{ or } 8)$

$$= P(6) + P(7) + P(8) = \sum_{x=6}^{8} b(x; 10000, 0.001)$$

$$= \sum_{x=6}^{8} 10000_{C_x} (.001)^x (.999)^{10000-x}$$

which involves cumbersome lengthy calculations. Since n is large and p is small, approximate the binomial probabilities by Poisson distribution with $\lambda = np = 10000 \times \frac{1}{1000} = 10$.

Probability of 6, 7 or 8 error ITR forms = $P(X = 6, 7 \text{ or } 8)$

$$= \sum_{x=6}^{8} f(x; 10) = \sum_{x=6}^{\infty} \frac{e^{-10}(10)^x}{x!}$$

$$= e^{-10}\left[\frac{10^6}{6!} + \frac{10^7}{7!} + \frac{10^8}{8!}\right] = .2657$$

Instead, using tables A8 to A11, we get the result in a simpler way

$$= F(8; 10) - F(5; 10) = .3328 - .0671 = .2657.$$

Example 4: Fit a Poisson distribution to the following data:

X_i:	0	1	2	3	4
Observed frequencies f_i	30	62	46	10	2

Solution: To fit a Poisson distribution, determine the only parameter λ of the distribution from the given data. Since λ is the arithmetic mean,

$$\lambda = \frac{\sum_{i=0}^{4} f_i X_i}{\sum f_i X_i} = \frac{0 \times 30 + 1 \times 62 + 2 \times 46 + 3 \times 10 + 4 \times 2}{150}$$

$$= \frac{192}{150} = 1.28$$

Thus the Poisson distribution that "fits" to the given data is $P(X) = \frac{e^{-1.28}(1.28)^X}{X!}$.

Here total frequency $N = \sum_{i=0}^{4} f_i = 150$

Expected frequency = (Total frequency) × Probability

X_i:	0	1	2	3	4
$P(X_i)$:	0.27803	.35588	.22776	.09718	.031097
$(N)(P(X_i))$ =Expected frequency	41.7045 ≈ 42	53.382 ≈ 53	34.164 ≈ 34	14.577 ≈ 15	4.6646 ≈ 5

EXERCISE

Poisson distribution

1. Determine the probability that 2 of 100 books bound will be defective if it is known that 5% of books bound at this bindery are defective. (a) use B.D. (b) use Poisson approximation to B.D.

Ans. **a.** $b(2; 100, 0.5) = \binom{100}{2}(0.05)^2(.95)^{98} = 0.081$

b. $f(2; 5) = \frac{5^2 e^{-5}}{2!} = 0.084$ with $\lambda = np = 100(0.05) = 5$

2. Find the probabilities that $0, 1, 2, 3, 4, \ldots$ of 3840 generators fail if the probability of failure is $\frac{1}{1200}$.

Ans.

x:	0	1	2	3	4	5	6	7	8	9	10
$f(x, 3.2)$:	0.041	.130	.209	.223	.178	.114	.06	.028	.011	.004	.002

Hint: $\lambda = 3840 \times \frac{1}{1200} = 3.2$. Use tables (A8 to A11) and the identity $f(x; \lambda) = F(x; \lambda) - F(x-1; \lambda)$.

3. On an average, 1.3 gamma particles/millisecond come out of a radioactive substance, determine (a) mean (b) variance

(c) probability of more than one gamma particles emanate from the substance.

Ans. (a)(b): $\lambda = \sigma^2 = 1.3$ (c) $1 - P(X = 0) = 1 - e^{-1.3} = 0.727$

4. Determine the probability p that there are 3 defective items in a sample of 100 items if 2% of items made in this factory are defective.

Ans. $p = f(3; 2) = \frac{2^3 e^{-2}}{3!} = 0.180$ with $\lambda = np = 100(0.02) = 2$

5. Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability P that a given page contains (i) exactly two misprints (ii) two or more misprints.

Ans. **i.** $f(2; 0.6) = \frac{(0.6)^2 e^{-0.6}}{2!} = 0.0988 \approx 0.1$

ii. $P = 1 - P(0 \text{ or } 1 \text{ misprint})$
$= 1 - (0.549 + 0.329) = 0.122$

6. In a factory producing blades, the probability of any blade being defective is 0.002. If blades are supplied in packets of 10, determine the number of packets containing (a) no defective (b) one defective and (c) two defective blades respectively in a consignment of 10000 packets.

Ans. **a.** $10000 \times P(0) = 10000 \times e^{-0.02}$
$= 10000 \times .9802 = 9802$

i.e., 9802 packets do not have any defective blades.

b. $10000 \times (0.02)(.9802) = 196$

c. $10000 \times \frac{(.02)^2}{2!} \cdot 9802 = 2$

Hint: $\lambda = np = 10 \times 0.002 = 0.02$.

7. A manufacturer of cotter pins knows that 5% of his product is defective. Pins are sold in boxes of 100. He guarantees that not more than 10 pins will be defective. Determine the probability that a box will fail to meet the guarantee.

Ans. $P(X > 10) = 1 - P(X \leq 10)$
$= 1 - \sum_{x=0}^{10} \frac{e^{-5} 5^x}{x!} = 1 - F(10, 5)$
$= 1 - .9863 = 0.0137$

Hint: $\lambda = np = 100 \times 0.05 = 5$

8. On an average 20 red blood cells are found in a fixed volume of blood for a normal person. Determine the probability that the blood sample of a normal person will contain less than 15 red cells.

Ans. $P(X < 15) = \sum_{x=0}^{14} \frac{e^{-20}(20)^x}{x!} = F(14, 20)$
$= 0.105$

Hint: $\lambda = 20$.

9. Two shipments of computers are received. The first shipment contains 1000 computers with 10% defectives and the second shipment contains 2000 computers with 5% defectives. One shipment is selected at random. Two computers are found good. Find the probability that the two computers are drawn from the first shipment.

Ans. 0.183

Hint: $q_1 = 0.1$, $p_1 = 0.9$, $q_2 = 0.05$, $p_2 = 0.95$

$\lambda_1 = n_1 p_1 = (1000)(0.9) = 900$,

$\lambda_2 = n_2\, p_2 = (2000)(.95) = 1900$

C: two computers good, A: first shipment, B: second shipment.

$$P(A/C) = \frac{P(A)P(C/A)}{P(A)P(C/A) + P(B) \cdot P(C/B)}$$

where

$$P(A) = P(B) = \frac{1}{2},\ P(C/A) = \frac{e^{-900}(900)^2}{2!},$$

$$P(C/B) = \frac{e^{-1900}(1900)^2}{2!}.$$

10. Given that the probability of an accident in an industry is 0.005 and assuming the accidents are independent (a) determine the probability that in any given period of 400 days, there will be an accident one day? (b) What is the probability that there are at most three days with an accident?

Ans. (a) $P(X = 1) = e^{-2} 2^1 = 0.271$

(b) $P(X \leq 3) = \sum_{x=0}^{3} \frac{e^{-2} 2^x}{x!} = 0.857$

Hint: $\lambda = np = 400(0.005) = 2.$

11. If one in every 1000 of computers produced is defective, determine the probability that a random sample of 8000 will yield fewer than 7 defective computers?

Ans. $P(X < 7) = \sum_{x=0}^{6} b(x; 8000, 0.001)$

$\simeq \sum_{x=0}^{6} f(x; 8) = 0.3134$

Hint: B.D. calculation is very hard, approximate it by P.D. with $\lambda = np = (8000)(0.001) = 8.$

12. Suppose the average number of telephone calls coming into a telephone exchange between 10 AM to 11 AM is 2, while between 11 AM to 12 noon is 6, determine the probability that more than five calls come in between 10 AM to 12 noon, assuming that calls are independent.

Ans. $P(x > 5) = 1 - P(x \leq 5) = 1 - \sum_{x=0}^{5} \frac{e^{-8} 8^x}{x!} =$ $1 - 0.1912 = 0.8088$

Hint: P.D. is additive: $X = X_1 + X_2$, $\lambda = \lambda_1 + \lambda_2 = 2 + 6 = 8.$

Fitting of Poisson distribution

Fit a Poisson distribution to the following data:

1.

x:	0	1	2	3	4	5	6	7	8
Observed frequency f_i	56	156	132	92	37	22	4	0	1

Ans. 69.6 137.25 135.33 88.95 43.85 17.29 5.68 1.60 0.3942

Hint: $\lambda = \frac{\sum f_i x_i}{N} = \frac{986}{500} = 1.972.$

2.

x:	0	1	2	3	4
f_i:	122	60	15	2	1

Ans. 121 61 15 2 0

Hint:

$\lambda = \sum \frac{f_i x_i}{N} = \frac{60+36+6+4}{200} = 0.5; e^{-.5} = 0.61$

3.

x:	0	1	2	3	4	5
f_i:	142	156	69	27	5	1

Ans. 147.15 147.15 73.58 24.53 6.13 1.23

Hint: $\lambda = \frac{\sum f_i x_i}{\sum f_i} = \frac{400}{400} = 1.$

4. Determine the number of pages expected with 0, 1, 2, 3, and 4 errors in 1000 pages of a book if on the average two errors are found in five pages.

Ans.

x:	0	1	2	3	4
$P(x)$:	.6703	.26812	.053624	.0071	.00071
Expected number of pages	670	268	54	7	1

Hint: $\lambda = 2/5 = 0.4$, $e^{-0.4} = .6703$,

Expected number of pages $= 1000 \times P(x)$.

5.

x:	0	1	2	3	4
f:	109	65	22	3	1

Ans. 108.7 66.3 20.2 4.1 0.7

Hint: $\lambda = \frac{65+44+9+4}{200} = \frac{122}{200} = 0.61.$

Table Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865
4		1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977
5				1.0000	1.0000	1.0000	0.9999	0.9998	0.9997
6							1.0000	1.0000	1.0000

	μ								
r	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**
0	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067
1	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404
2	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247
3	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650
4	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405
5	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622
7	1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666
8		1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319
9			1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682
10				0.9999	0.9997	0.9990	0.9972	0.9933	0.9863
11				1.0000	0.9999	0.9997	0.9991	0.9976	0.9945
12					1.0000	0.9999	0.9997	0.9992	0.9980
13						1.0000	0.9999	0.9997	0.9993
14							1.0000	0.9999	0.9998
15								1.0000	0.9999
16									1.0000

Table (continued) Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

r	μ = 5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
0	0.0041	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001
1	0.0266	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008
2	0.0884	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042
3	0.2017	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149
4	0.3575	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403
5	0.5289	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885
6	0.6860	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649
7	0.8095	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687
8	0.8944	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918
9	0.9462	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218
10	0.9747	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453
11	0.9890	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520
12	0.9955	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364
13	0.9983	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981
14	0.9994	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400
15	0.9998	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665
16	0.9999	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823
17	1.0000	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911
18		1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957
19			1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980
20						0.9999	0.9998	0.9996	0.9991
21						1.0000	0.9999	0.9998	0.9996
22							1.0000	0.9999	0.9999
23								1.0000	0.9999
24									1.0000

Table (continued) Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0	18.0
0	0.0000	0.0000	0.0000						
1	0.0005	0.0002	0.0001	0.0000	0.0000				
2	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000		
3	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000	0.0000
4	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002	0.0001
5	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007	0.0003
6	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021	0.0010
7	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054	0.0029
8	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126	0.0071
9	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261	0.0154
10	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491	0.0304
11	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847	0.0549
12	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350	0.0917
13	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009	0.1426
14	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808	0.2081
15	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715	0.2867
16	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677	0.3751
17	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640	0.4686
18	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550	0.5622
19	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363	0.6509
20	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055	0.7307
21	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469	0.9108	0.8615	0.7991
22	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673	0.9418	0.9047	0.8551
23	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805	0.9633	0.9367	0.8989
24	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888	0.9777	0.9594	0.9317
25		0.9999	0.9997	0.9990	0.9974	0.9938	0.9869	0.9748	0.9554
26		1.0000	0.9999	0.9995	0.9987	0.9967	0.9925	0.9848	0.9718
27			0.9999	0.9998	0.9994	0.9983	0.9959	0.9912	0.9827
28			1.0000	0.9999	0.9997	0.9991	0.9978	0.9950	0.9897
29				1.0000	0.9999	0.9996	0.9989	0.9973	0.9941
30					0.9999	0.9998	0.9994	0.9986	0.9967
31					1.0000	0.9999	0.9997	0.9993	0.9982
32						1.0000	0.9999	0.9996	0.9990
33							0.9999	0.9998	0.9995
34							1.0000	0.9999	0.9998
35								1.0000	0.9999
36									0.9999
37									1.0000