Random Variable: In many areas of science we are interested in quantifying the probability that a certain outcome of an experiment occurs. We can use a random variable to identify numerical events that are of interest in an experiment. In this way, a random variable is a theoretical representation of the physical or experimental process we wish to study. More precisely, a random variable is a quantity without a fixed value, but which can assume different values depending on how likely these values are to be observed; these likelihoods are probabilities.

To quantify the probability that a particular value, or event, occurs, we use a number between 0 and 1. A probability of 0 implies that the event cannot occur, whereas a probability of 1 implies that the event must occur. Any value in the interval (0, 1) means that the event will only occur some of the time. Equivalently, if an event occurs with probability p, then this means there is a p(100)% chance of observing this event.

Conventionally, we denote random variables by capital letters, and particular values that they can assume by lowercase letters. So we can say that X is a random variable that can assume certain particular values x with certain probabilities.

We use the notation $\Pr(X = x)$ to denote the probability that the random variable X assumes the particular value x. The range of x for which this expression makes sense is of course dependent on the possible values of the random variable X. We distinguish between two key cases.

If X can assume only finitely many or countably many values, then we say that X is a discrete random variable. Saying that X can assume only finitely many or countably many values means that we should be able to list the possible values for the random variable X. If this list is finite, we can say that X may take any value from the list $x_1, x_2, \ldots, x_n$, for some positive integer n. If the list is (countably) infinite, we can list the possible values for X as $x_1, x_2, \ldots$. This is then a list without end (for example, the list of all positive integers).

Discrete Random Variables
1. A discrete random variable X is a quantity that can assume any value x from a discrete list of values with a certain probability. 2. The probability that the random variable X assumes the particular value x is denoted by $\Pr(X = x)$. This collection of probabilities, along with all possible values x, is the probability distribution of the random variable X. 3. A discrete list of values is any collection of values that is finite or countably infinite (i.e. can be written in a list).

This terminology is in contrast to a continuous random variable, where the values the random variable can assume are given by a continuum of values. For example, we could define a random variable that can take any value in the interval [1,2]. The values X can assume are then any real number in [1,2]. We will discuss continuous random variables in detail in the second part of this module. For now, we deal strictly with discrete random variables.

We state a few facts that should be intuitively obvious for probabilities in general. Namely, the chance of some particular event occurring should always be nonnegative and no greater than 100%. Also, the chance that something happens should be certain. From these facts, we can conclude that the chance of witnessing a particular event should be 100% less the chance of seeing anything but that particular event.

Discrete Probability Rules

1. Probabilities are numbers between 0 and 1: $0 \leq \Pr(X = x_k) \leq 1$ for all k
2. The sum of all probabilities for a given experiment (random variable) is equal to one: $$\sum_k \Pr(X = x_k) = 1$$
3. The probability of an event is 1 minus the probability that any other event occurs: $$\Pr(X = x_n) = 1 - \sum_{k \neq n} \Pr(X = x_k)$$

Example: Tossing a Fair Coin Once
If we toss a coin into the air, there are only two possible outcomes: it will land as either "heads" (H) or "tails" (T). If the tossed coin is a "fair" coin, it is equally likely that the coin will land as tails or as heads. In other words, there is a 50% chance (1/2 probability) that the coin will land heads, and a 50% chance (1/2 probability) that the coin will land tails. Notice that the sum of these probabilities is 1 and that each probability is a number in the interval [0,1].

We can define the random variable X to represent this coin tossing experiment. That is, we define X to be the discrete random variable that takes the value 0 with probability 1/2 and takes the value 1 with probability 1/2. Notice that with this notation, the experimental event that "we toss a fair coin and observe heads" is the same as the theoretical event that "the random variable X is observed to take the value 0"; i.e. we identify the number 0 with the outcome of "heads", and identify the number 1 with the outcome of "tails". We say that X is a Bernoulli random variable with parameter 1/2 and can write $X \sim \mathrm{Ber}(1/2)$.

Example: Tossing a Fair Coin Twice
Similarly, if we toss a fair coin two times, there are four possible outcomes. Each outcome is a sequence of heads (H) or tails (T):

- HH
- HT
- TH
- TT

Because the coin is fair, each outcome is equally likely to occur. There are 4 possible outcomes, so we assign each outcome a probability of 1/4.

Equivalently, we notice that for any of the four possible events to occur, we must observe two distinct events from two separate flips of a fair coin. So for example, to observe the sequence HH, we must flip a fair coin once and observe H, then flip a fair coin again and observe H once again. (We say that these two events are independent since the outcome of one event has no effect on the outcome of the other.) Since the probability of observing H after a flip of a fair coin is 1/2, we see that the probability of observing the sequence HH should be $(1/2)\times(1/2) = 1/4$.
Observe that again, all of our probabilities sum to 1, and each probability is a number on the interval [0, 1]. Just as before, we can identify each outcome of our experiment with a numerical value. Let us make the following assignments:

- HH -> 0
- HT -> 1
- TH -> 2
- TT -> 3

This assignment defines a numerical discrete random variable Y that represents our coin tossing experiment. We see that Y takes the value 0 with probability 1/4, 1 with probability 1/4, 2 with probability 1/4, and 3 with probability 1/4. Using our general notation to describe this probability distribution, we can summarize by writing

$$\Pr(Y = k) = 1/4, \text{ for } k = 1, 2, 3, 4.$$

Notice that with this notation, the experimental event that "we toss two fair coins and observe first tails, then heads" is the same as the theoretical event that "the random variable Y is observed to take the value 2". We say that Y is a uniform discrete random variable with parameter 4 since Y takes each of its four possible values with equal, or uniform, probability. To denote this distributional relationship, we can write Y ~ Uniform(4).

Random Variables and their Observed Values

We commonly use uppercase letters to denote random variables, and lowercase letters to denote particular values that our random variables can assume.

For example, consider a six-sided die, pictured below.

We could let X be the random value that gives the value observed on the upper face of the six-sided die after a single roll. Then if x denotes a particular value of the upper face, the expression X = x becomes well-defined. Specifically, the notation X = x signifies the event that the random variable X assumes the particular value x. For the six-sided die example, x can be any integer from 1 to 6. So the expression X = 4 would express the event that a random roll of the die would result in observing the value 4 on the upper face of the die.

Probabilities of Discrete Random Variables

We have already defined the notation $\Pr(X = x)$ to denote the probability that a random variable X is equal to a particular value x. Similarly, $\Pr(X \leq x)$ would denote the probability that the random variable X is less than or equal to the value x.

Notation
$\Pr(a \leq X \leq b)$ denotes the probability that the random variable Xlies between values a and b, inclusively.

With this notation, it now makes sense to write, for example, $\Pr(X > a)$, the probability that a random variable assumes a particular value strictly greater than a. Similarly, we can make sense of the expressions $\Pr(X < b)$, $\Pr(X \neq x)$, $\Pr(X = x_1 \text{ or } X = x_2)$, among others.
Notice that this notation allows us to do a kind of algebra with probabilities. For example, we notice the equivalence of the following two expressions: $\Pr(X \geq a \text{ and } X < b) = \Pr(a \leq X < b)$.
An important consequence of this symbolism is the following:

Probabilities of Complimentary Events
$\Pr(X = x) = 1 - \Pr(X \neq x)$
$\Pr(X > x) = 1 - \Pr(X \leq x)$
$\Pr(X \geq x) = 1 - \Pr(X < x)$

Notice that the first identity is simply a restatement of Discrete Probability Rule #3 from the previous page.

These three identities are simple consequences of our notation and of the fact that the sum of all probabilities must always equal 1 for any random variable. The events $X = x$ and $X \neq x$ are called complimentary because exactly one of the events must take place; i.e. both events cannot occur simultaneously, but one of the two must occur. The other expressions above also define complimentary events.
For discrete random variables, we also have the identity:

Algebra of Disjoint Events
If $a \neq b$, then $\Pr(X = a \text{ or } X = b) = \Pr(X = a) + \Pr(X = b)$

Six-Sided Die Example
Using our six-sided die example above, we have the random variable X which represents the value we observe on the upper face of the six-sided die after a single roll. Then the probability that X is equal to 5 can be written as:

$$\Pr(X = 5) = \frac{1}{6}$$

Using our identities for complimentary events and for disjoint events, we find that the probability that X is equal to 1, 2, 3 or 4 can be computed as:

$$
\begin{aligned}
\Pr(1 \le X \le 4) &= \Pr(X = 1, \text{ or } X = 2, \text{ or } X = 3, \text{ or } X = 4) \\
&= 1 - \Pr(X = 5, \text{ or } X = 6) \\
&= 1 - [\Pr(X = 5) + \Pr(X = 6)] \\
&= 1 - (\frac{1}{6} + \frac{1}{6}) \\
&= \frac{2}{3}
\end{aligned}
$$

Notice that X ~ Uniform(6); i.e. X has a uniform distribution on the integers from 1 to 6. Indeed, the probability of observing any one of these integer values (the value on the upper face of the rolled die) is the same for any value. Thus, X must be a uniform random variable.

The Discrete Probability Density Function

Usually we are interested in experiments where there is more than one outcome, each having a possibly different probability. The probability density function of a discrete random variable is simply the collection of all these probabilities.

Discrete Probability Density Function
The discrete probability density function (PDF) of a discrete random variable X can be represented in a table, graph, or formula, and provides the probabilities Pr(X = x) for all possible values of x.

Example: Different Coloured Balls
Although it is usually more convenient to work with random variables that assume numerical values, this need not always be the case. Suppose that a box contains 10 balls:

- 5 of the balls are red
- 2 of the balls are green
- 2 of the balls are blue
- 1 ball is yellow

Suppose we take one ball out of the box. Let X be the random variable that represents the colour of the ball. As 5 of the balls are red, and there are 10 balls, the probability that a red ball is drawn from the box is Pr(X = Red) = 5/10 = 1/2.
Similarly, there are 2 green balls, so the probability that X is green is 2/10. Similar calculations for the other colours yields the probability density function given by the following table.

Ball Colour	Probability
red	5/10
green	2/10
blue	2/10

yellow	1/10

Example: A Six-Sided Die
Consider again the experiment of rolling a six-sided die. A six-sided die can land on any of its six faces, so that a single experiment has six possible outcomes.

For a "fair die", we anticipate getting each of the results with an equal probability, i.e. if we were to repeat the same experiment many times, we would expect that, on average, the six possible events would occur with similar frequencies (we say that such events are uniformly distributed).
There are six possible outcomes: 1, 2, 3, 4, 5, or 6. The probability density function could be given by the following table.

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

The PDF could also be given by the equation $\Pr(X = k) = 1/6$, for $k = 1, 2, 3, \ldots, 6$, where X denotes the random variable associated to rolling a fair die once. Thus we see that uniform random variables have PDFs which are particularly easy to represent.

Some Common Discrete Distributions

A random variable is a theoretical representation of a physical or experimental process we wish to study. Formally, it is a function defined over a sample space of possible outcomes. For our simple coin tossing experiment, where we flip a fair coin once and observe the outcome, our sample space consists of the two outcomes H or T. When tossing two fair coins sequentially, our sample space consists of the four outcomes HH, HT, TH or TT.
Let us fix a sample space of n tosses of a fair coin. Experimentally, we may be interested in studying the number of "heads" observed after tossing the coin n times. Or we could be interested in studying the number of tosses needed to first observe "heads". Or we could be interested in studying how likely a certain sequence of "heads" and "tails" is to be observed. Each of these experiments are defined on the same sample space (the events generated by n tosses of a fair coin), yet each strive to quantify different things. Consequently, each experiment should be associated with a different random variable.

The Binomial Random Variable

Let X_n denote the random variable the counts the number of times we observe "heads" when flipping a fair coin n times. Clearly, X can take on any integer value from 0 to n, corresponding to the experimental outcome of observing 0 to n "heads". How likely is any particular outcome? Notice that we do not care about the order of the observations here, so that if n = 3, the outcome THH is equivalent to the outcomes HTH and HHT. Each of these outcomes contains two "heads".

The likelihood of any particular outcome is what is represented by the probability density function (PDF) of the random variable. Suppose n = 2. Then we see that the PDF of X_2 is given by:

- $\Pr(X_2 = 0) = 1/4$
- $\Pr(X_2 = 1) = 1/2$
- $\Pr(X_2 = 2) = 1/4$

We say that X_2 is a binomial random variable with parameters 2 (the number of times we flip the fair coin) and 1/2 (the probability that we observe heads after a single flip of the coin). We can write $X_2 \sim \text{Bin}(2, 1/2)$.

Binomial PDF
If X is a binomial random variable associated to n independent trials, each with a success probability p, then the probability density function of X is: $$\Pr(X = k) = \frac{n!}{k!(n-k)!}\, p^k(1-p)^{n-k},$$ where k is any integer from 0 to n. Recall that the factorial notation n! denotes the product of the first n positive integers: $n! = 1\cdot 2\cdot 3\cdots(n-1)\cdot n$, and that we observe the convention $0! = 1$.

Just as we did with Bernoulli random variables, we can think of our coin tossing experiment a bit more abstractly. Specifically, we can think of observing "heads" as a success and observing "tails" as a failure. This abstraction will help us apply our coin tossing random variables to more general experiments.

For example, suppose that we know that 5% of all light bulbs produced by a particular manufacturer are defective. If we buy a package of 6 light bulbs and want to calculate the probability that at least one is defective, we can do so by identifying this experiment with a binomial random variable. Here, we can think of observing a defective bulb as a "success" and observing a functional bulb as a "failure". Then our experiment is given by the random variable $X_6 \sim \text{Bin}(6, 1/20)$, since we will observe 6 bulbs in total and each has a probability of 5/100 = 1/20 of being defective.

In general, we can think of observing n independent experimental trials and counting the number of "successes" that we witness. The probability distribution we associate with this setup is the binomial random variable with parameters n and p, where p is the probability of "success." We can denote this random variable by $X_n \sim \text{Bin}(n, p)$.

The Geometric Random Variable

Now consider a slightly different experiment where we wish to flip our fair coin repeatedly until we first observe "heads". Since we can first observe heads on the first flip, the second flip, the third flip, or on any subsequent flip, we see that the possible values our random variable can take are 1, 2, 3,....

Of course, we can consider a more abstract experiment where we observe a sequence of trials until we first observe a success, where the probability of success is p. If we let X denote such a random variable, then we say that X is a geometric random variable with parameter p. We can denote this random variable by X ~ Geo(p).

Letting S denote the outcome of "success" and F denote the outcome of "failure", we can summarize the possible outcomes of a geometric experiment and their likelihoods in the following table.

Experimental Outcome	Value of the Random Variable, X = x	Probability
S	x = 1	p
FS	x = 2	$q \cdot p$
FFS	x = 3	$q^2 \cdot p$
FFFS	x = 4	$q^3 \cdot p$
FFFFS	x = 5	$q^4 \cdot p$
...	...	...

When flipping a fair coin, we see that X ~ Geo(1/2), so that our PDF takes the particularly simple form Pr(X = k) = (1/2)k for any positive integer k.

The Discrete Uniform Random Variable

Now consider a coin tossing experiment of flipping a fair coin n times and observing the sequence of "heads" and "tails". Because each outcome of a single flip of the coin is equally likely, and because the outcome of a single flip does not affect the outcome of another flip, we see that the likelihood of observing any particular sequence of "heads" and "tails" will always be the same. Notice that for n = 2 or 6, we have already encountered this random variable.

We say that a random variable X has a discrete uniform distribution on n points if X can assume any one of n values, each with equal probability. Evidently then, if X takes integer values from 1 to n, we find that the PDF of X must be Pr(X = k) = 1/n, for any integer k between 1 and n.

Expected Value

For an experiment or general random process, the outcomes are never fixed. We may replicate the experiment and generally expect to observe many different outcomes. Of course, in most reasonable circumstances we will expect these observed differences in the outcomes to collect with some level of concentration about some central value. One central value of fundamental importance is the expected value.

The expected value or expectation (also called the mean) of a random variable X is the weighted average of the possible values of X, weighted by their corresponding probabilities. The expectation of a random variable X is the value of X that we would expect to see on average after repeated observation of the random process.

Definition: Expected Value of a Discrete Random Variable
The expected value, $\mathbb{E}(X)$, of a random variable X is weighted average of the possible values of X, weight by their corresponding probabilities: $$\mathbb{E}(X) = \sum_{k=1}^{N} x_k \Pr(\mathrm{x} = \mathrm{x_k})$$ where N is the number of possible values of X.

Note the following:

- Do not confuse the expected value with the average value of a set of observations: they are two different but related quantities. The average value of a random variable Xwould be just the ordinary average of the possible values of X. The expected value of Xis a weighted average, where certain values get more or less weight depending on how likely or not they are to be observed. A true average value is calculated only when all weights (so all probabilities) are the same.
- The definition of expected value requires numerical values for the x_k. So if the outcome for an experiment is something qualitative, such as "heads" or "tails", we could calculate the expected value if we assign heads and tails numerical values (0 and 1, for example).

Example: Grade Distributions

Suppose that in a class of 10 people the grades on a test are given by 30, 30, 30, 60, 60, 80, 80, 80, 90, 100. Suppose a test is drawn from the pile at random and the score X is observed.

1. Calculate the probability density function for the randomly drawn test score.
2. Calculate the expected value of the randomly drawn test score.

Solution

Part 1)

Looking at the test scores, we see that out of 10 grades,

- the grade 30 occurred 3 times
- the grade 60 occurred 2 times
- the grade 80 occurred 3 times
- the grade 90 occurred 1 time
- the grade 100 occurred 1 time

This tells us the probability density function of the randomly chosen test score X which we present formally in the following table.

Grade, x_k	Probability, $\Pr(X = x_k)$

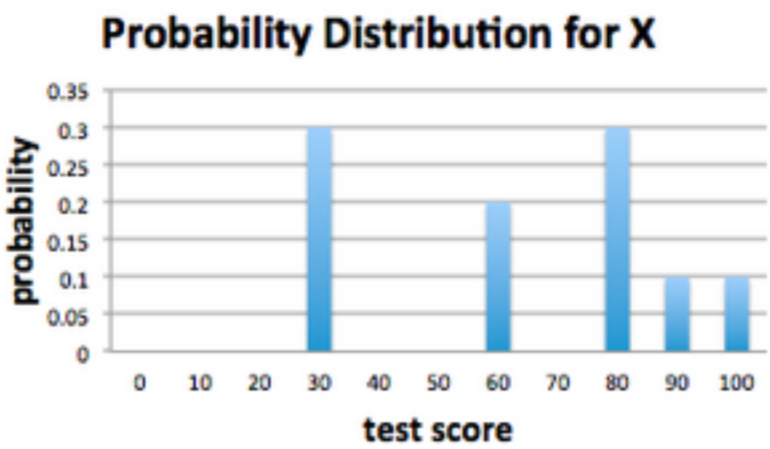
30	3/10
60	2/10
80	3/10
90	1/10
100	1/10

Part 2)
The expected value of the random variable is given by the weighted average of its values:

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k=1}^{N} x_k \Pr(\mathrm{X} = \mathrm{x_k}) \\ &= 30\frac{3}{10} + 60\frac{2}{10} + 80\frac{3}{10} + 90\frac{1}{10} + 100\frac{1}{10} \\ &= 9 + 12 + 24 + 9 + 10 \\ &= 64\end{aligned}$$

Discussion

The PDF Could Be Presented as a Graph
The PDF was listed in a table, but an equivalent representation could be given in a graph that plots the possible outcomes on the horizontal axis, and the probabilities associated to these outcomes on the vertical axis.



We have added points where the probability is zero (test scores of 0, 10, 20, 40, 50, 70). It isn't necessary to have these points displayed, but having these points on a graph of a PDF can often add clarity.

Notice that the expected value of our randomly selected test score, $\mathbb{E}(X)$ = 64, lies near the "centre" of the PDF. There are many different ways to quantify the "centre of a distribution" - for example, computing the 50th percentile of the possible outcomes - but for our purposes we will concentrate our attention on the expected value.

Variance and Standard Deviation

Another important quantity related to a given random variable is its variance. The variance is a numerical description of the spread, or the dispersion, of the random variable. That is, the variance of a random variable X is a measure of how spread out the values of X are, given how likely each value is to be observed.

Definition: Variance and Standard Deviation of a Discrete Random Variable
The variance, Var(X), of a discrete random variable X is $$\mathrm{Var}(X) = \sum_{k=1}^{N} \left(x_k - \mathbb{E}(X)\right)^2 \mathrm{Pr}(\mathrm{X} = \mathrm{x_k})$$ The integer N is the number of possible values of X. The standard deviation, σ, is the positive square root of the variance: $$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

Observe that the variance of a distribution is always non-negative (p_k is non-negative, and the square of a number is also non-negative).
Observe also that much like the expectation of a random variable X, the variance (or standard deviation) is a weighted average of an expression of observable and calculable values. More precisely, notice that

$$\mathrm{Var}(X) = \mathbb{E}\left([X - \mathbb{E}(X)]^2\right)$$

Example: Grade Distributions
Using the grade distribution example of the previous page, calculate the variance and standard deviation of the random variable associated to randomly selecting a single exam.

Solution
The variance of the random variable X is given by

$$
\begin{aligned}
\operatorname{Var}(X) &= \sum_{k=1}^{N}(x_k - \mathbb{E}(X))^2 \Pr(\mathrm{X} = \mathrm{x_k}) \\
&= (30-64)^2\frac{3}{10} + (60-64)^2\frac{2}{10} + (80-64)^2\frac{3}{10} + (90-64)^2\frac{1}{10} + (100-64)^2\frac{1}{10} \\
&= 624
\end{aligned}
$$

The standard deviation of X is then

$$\sigma(X) = \sqrt{624} \approx 24.979992$$

Interpretation of the Standard Deviation

For most "nice" random variables, i.e. ones that are not too wildly distributed, the standard deviation has a convenient informal interpretation. Consider the intervals $S_m = [\mathbb{E}(X) - m\sigma(X),\ \mathbb{E}(X) + m\sigma(X)]$, for some positive integer m. As we increase the value of m, these intervals will contain more of the possible values of the random variable X.

A good rule of thumb is that for "nicely distributed" random variables, all of the most likely possible values of the random variable will be contained in the interval S_3. Another way to say this is that most of the PDF will live on the interval S_3.

For our grade distribution example, notice that all possible values of X are contained in the interval S_3. In fact, all possible values of X are contained in S_2 for this particular example.

Grade Distribution Example

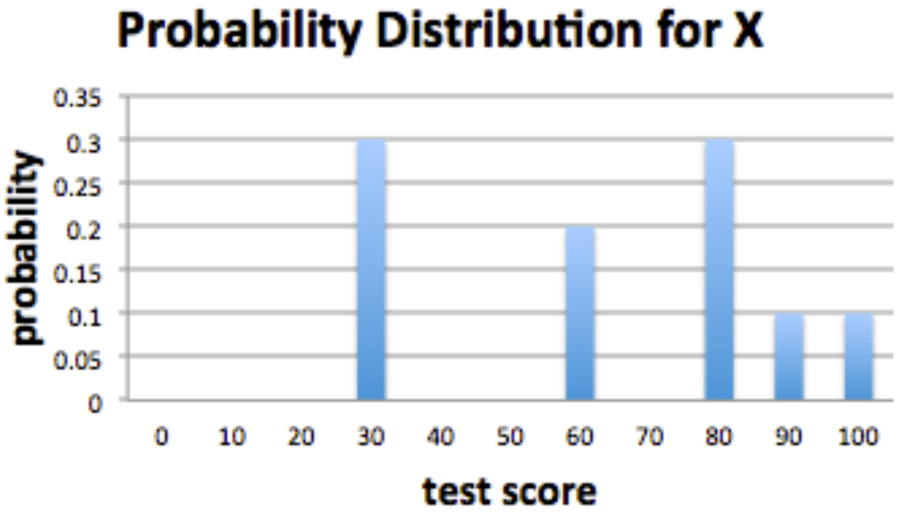
Consider the grade distribution example that we explored earlier: in a class of 10 people, grades on a test were 30, 30, 30, 60, 60, 80, 80, 80, 90, 100. Let X be the score of a randomly drawn test from this collection.

1. Calculate the probability that a test drawn at random has a score less than or equal to 80.
2. Calculate the probability that a test drawn at random has a score less than or equal to x_n, where x_n = 0, 10, 20, 30, ... , 100.

Solution

Part 1)

Recall the probability distribution, calculated earlier:



Let p_k be the probability that the score of a randomly drawn test is $x_k = 10k$. So, for example:

- p_0 is the probability that a randomly drawn test score is 0
- p_1 is the probability that a randomly drawn test score is 10
- p_2 is the probability that a randomly drawn test score is 20
- p_3 is the probability that a randomly drawn test score is 30

and so on. Values for each of these probabilities are given in the above bar graph.

The probability that a test drawn at random has a score of no greater than 80 is exactly the value of the CDF of X at x = 80; i.e.,

$$\begin{aligned}
\Pr(X \leq 80) = F(80) \\
&= \sum_{k=0}^{8} p_k \\
&= p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 \\
&= 0 + 0 + 0 + \frac{\mathbf{3}}{\mathbf{10}} + 0 + 0 + \frac{2}{10} + 0 + \frac{3}{10} \\
&= \frac{8}{10} \\
&= \frac{4}{5}
\end{aligned}$$

The colour blue was used in the above calculation to highlight nonzero probabilities.

Because of the sample space of our experiment, if the randomly selected grade is to be less than or equal to 80, then the this grade can only be 30, 60, or 80. Intuitively, the probability that a randomly selected test has a grade of 30, 60, or 80 is the sum of the probabilities that the score is one of these possibilities, which we note is in agreement with our identity concerning probabilities of disjoint events.

Part 2)

Now we want to calculate the probability that a test drawn at random has a score less than or equal to $x_k = 10k$ for $k = 0,1,...,10$. Again, we identify this as simply finding the value of the CDF of X at each of these x_k values.

$$\Pr(X \leq 0) = F(0) = \sum_{k=0}^{0} p_k = p_0 = 0$$

Similarly, $F(0) = F(10) = F(20) = 0$. F(30) is non-zero:

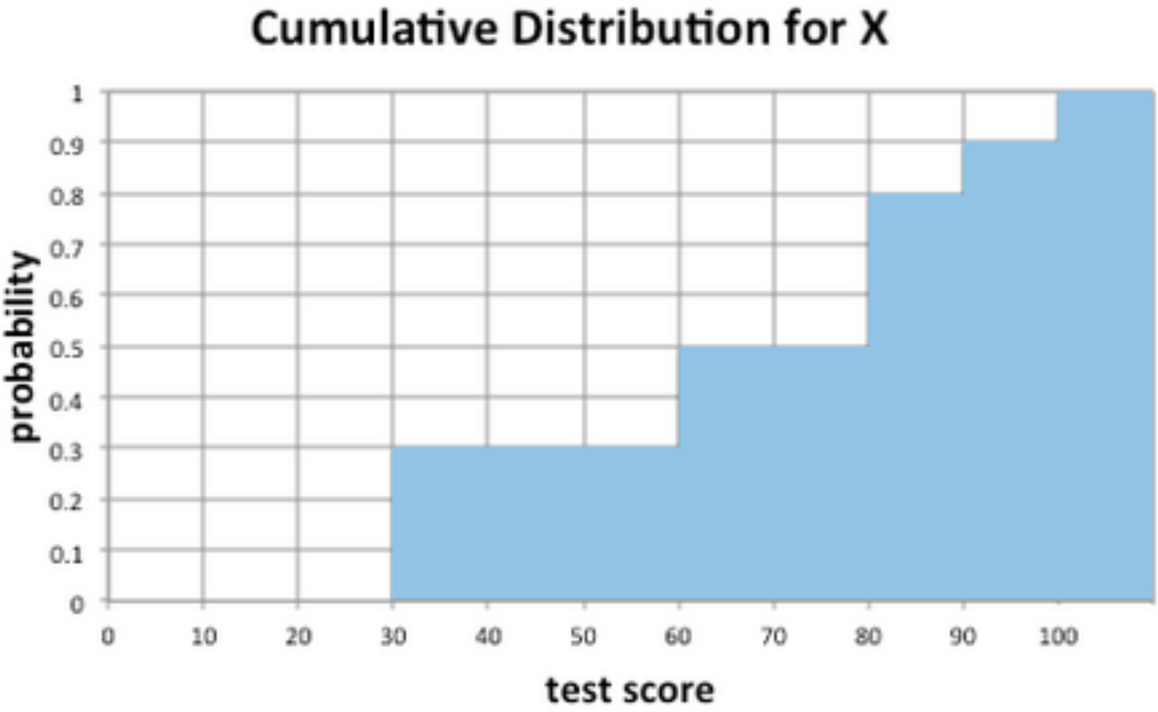
$$\Pr(X \leq 30) = F(30) = \sum_{k=0}^{3} p_k = 0 + 0 + 0 + \frac{3}{10}$$

Notice that F(40) is equal to F(30), since $p_4 = 0$.

Other values of F are calculated in the same way using the definition of the cumulative distribution function. The following table contains the values of the CDF of X for x_k = 0, 10, 20, 30, ... 100.

k	x_k	$F(x_k)$
0	0	0
1	10	0
2	20	0
3	30	0.3
4	40	0.3
5	50	0.3
6	60	0.5
7	70	0.5
8	80	0.8
9	90	0.9
10	100	1.0

The cumulative distribution function is graphed in the figure below.



Expected Value and Variance

The concepts of an expectation and variance are crucial and will be revisited again when we explore continuous random variables. Students should know that

$$\mathbb{E}(X) = \sum_{k=1}^{N} x_k p_k$$

$$\mathrm{Var}(X) = \sum_{k=1}^{N} (x_k - \mathbb{E}(X))^2 p_k$$

The expectation represents the "centre" of a random variable, an expected value of an experiment, or the average outcome of an experiment repeated many times. The variance of a random variable is a numerical measure of the spread, or dispersion, of the PDF of X.

Continuous Random Variables

In the previous lesson, we defined random variables in general, but focused only on discrete random variables. In this lesson, we properly treat continuous random variables.

If for example X is the height of a randomly selected person in British Columbia, or X is tomorrow's low temperature at Vancouver International Airport, then X is a continuously varying quantity.

An Important Distinction Between Continuous and Discrete Random Variables

What is Pr(X = x)? The answer clearly depends on the random variable X. For discrete random variables, we have already seen that if x is a possible value that X can assume, then Pr(X = x) is some positive number. But is this still true if X is a continuous random variable?

In the context of our example above, we may ask what is the probability that the maximum outdoor air temperature in downtown Vancouver on any given day in January is exactly0°C? Since our measurements of the air temperature are never exact, this probability should be zero. If we had instead asked for the probability that the maximum outdoor air temperature was within 0.005° of 0°C, then we would have arrived at a nonzero probability. All practical measurements of continuous data are always approximate. They may be very precise, but they can never be truly exact. Hence, we cannot expect to measure the likelihood of an exact outcome, only an approximate one.

In general, for any continuous random variable X, we will always have Pr(X = x) = 0. We can prove this fact directly by appealing to our basic results about combining probabilities of disjoint events.

Suppose we choose any interval [x , x + Δx]. The probability that the continuous random variable X lies inside of this interval is

$$\Pr(x \leq X \leq x + \Delta x).$$

Using our identity for probabilities of disjoint events, we can write this as the difference

$$\Pr(X \leq x + \Delta x) - \Pr(X \leq x).$$

If we take the limit as Δx goes to zero we obtain

$$\begin{aligned} \lim_{\Delta x \to 0} \Pr(x \leq X \leq x + \Delta x) &= \lim_{\Delta x \to 0} \Big(\Pr(X \leq x + \Delta x) - \Pr(X \leq x)\Big) \\ &= \Pr(X \leq x) - \Pr(X \leq x) \\ &= 0 \end{aligned}$$

Notice that the crucial step in this argument is evaluation of the limit in the second to last line. Since X is a continuous random variable, we are allowed to pass the limit through to the argument of the function F(x) = Pr(X ≤ x). If X were a discrete random variable, this would not be possible and hence the argument would fail in general.

This gives a direct proof of the fact that Pr(X = x) = 0 for any continuous random variable X. We will see that an even simpler proof will come for free for most continuous random variables via the Fundamental Theorem of Calculus. To take advantage of this, we need to relate these probabilities to integration of some appropriate function.

The Probability Density Function

For many continuous random variables, we can define an extremely useful function with which to calculate probabilities of events associated to the random variable.

Definition: The Probability Density Function

Let F(x) be the distribution function for a continuous random variable X. The probability density function (PDF) for X is given by $$f(x) = \frac{dF(x)}{dx}$$ wherever the derivative exists.

In short, the PDF of a continuous random variable is the derivative of its CDF. By the Fundamental Theorem of Calculus, we know that the CDF F(x)of a continuous random variable X may be expressed in terms of its PDF:

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

where f denotes the PDF of X.

Properties of the PDF

This formulation of the PDF via the Fundamental Theorem of Calculus allows us to derive the following properties.

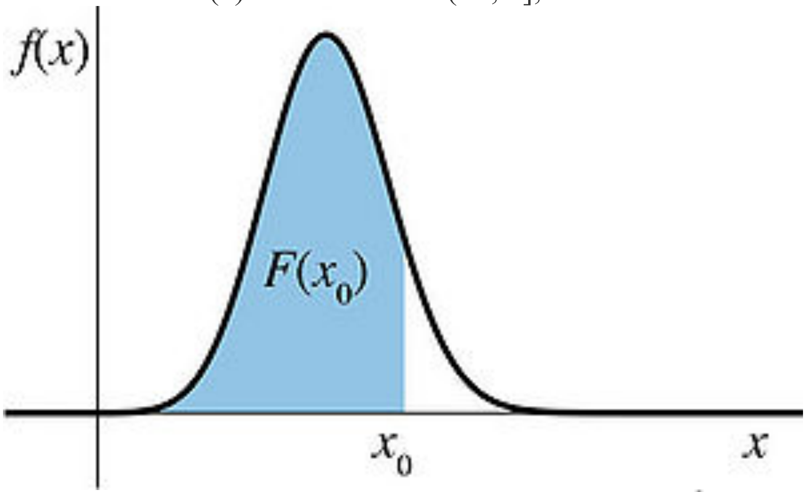
Theorem: Properties of the Probability Density Function
If f(x) is a probability density function for a continuous random variable X then $$1)\ F(b) = \Pr(X \le x) = \int_{-\infty}^{x} f(t)dt$$ $$2)\ f(x) \ge 0 \text{ for any value of } x$$ $$3) \int_{-\infty}^{\infty} f(t)dt = 1$$

The first property, as we have already seen, is just an application of the Fundamental Theorem of Calculus.

The second property states that for a function to be a PDF, it must be nonnegative. This makes intuitive sense since probabilities are always nonnegative numbers. More precisely, we already know that the CDF F(x) is a nondecreasing function of x. Thus, its derivative is f(x) is nonnegative.

The third property states that the area between the function and the x-axis must be 1, or that all probabilities must integrate to 1. This must be true

since $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$; thus 3) follows by the Fundamental Theorem of Calculus.
The PDF gives us a helpful geometrical interpretation of the probability of an event: the probability that a continuous random variable X is less than some value b, is equal to the area under the PDF f(x) on the interval (-∞,b], as demonstrated in the following graph.



Similarly, we have $\Pr(a \le x \le b) = \int_a^b f(x)dx$.

What is Pr(X = x)?
Let's now revisit this question that we can interpret probabilities as integrals. It is now clear that for a continuous random variable X, we will always have Pr(X = x) = 0, since the area under a single point of a curve is always zero. In other words, if X is a continuous random variable, the probability that X is equal to a particular value will always be zero. We again note this important difference between continuous and discrete random variables.

An Important Subtlety
There is an important subtlety in the definition of the PDF of a continuous random variable. Notice that the PDF of a continuous random variable X can only be defined when the distribution function of X is differentiable.
As a first example, consider the experiment of randomly choosing a real number from the interval [0,1]. If we denote this random variable by X, then we see that X is a continuous uniform random variable on [0,1]. Since the likelihood of picking any number is uniform across the interval, we see that the CDF F(x) is given by

$$\Pr(X \le x) = F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

This function is differentiable everywhere except at the points x = 0 and x = 1. So the PDF of X is defined at all points except for these two:

$$\frac{dF(x)}{dx} = f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

Nevertheless, it would still make sense to define the PDF at these points of where the CDF is not differentiable. We know that the integral over a single point is always zero, so we can always change the value of our PDF at any particular point (or at any finite set of points) without changing the probabilities of events associated to our random variable. Thus, we could define

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Both of these are also PDFs of our continuous uniform random variable X. These two formulations have the advantage of being defined for all real numbers.

Not All Continuous Random Variables Have PDFs

We can sometimes encounter continuous random variables that simply do not have a meaningful PDF at all. The simplest such example is given by a distribution function called the Cantor staircase.

The Cantor set is defined recursively as follows:

- Start with the interval [0,1).
- Delete the middle third of this interval. You are now left with two subintervals [0,1/3) and [2/3,1).
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- Repeat this middle third deletion for the new subintervals. Continue indefinitely.

If we take this process to the limit, the set that remains is called the Cantor set. It is extremely sparse in [0,1), yet still contains about as many points as the entire interval itself.

We can define a Cantor random variable to have the distribution function that increases on the Cantor set and remains constant off of this set. We define this function as follows:

- Let F(x) be the CDF of our Cantor random variable X. Define F(x) = 0 for x < 0 and F(x) = 1 for x > 1.

- Define F(x) = 1/2 on [1/3,2/3), i.e. on the first middle third deleted in the construction of the Cantor set.
- Define F(x) = 1/4 on [1/9,2/9) and F(x) = 3/4 on [7/9,8/9).
- Define F(x) = 1/8, 3/8, 5/8, and 7/8 on the deleted middle thirds from the third step in our Cantor set construction.
- Continue indefinitely.

After a limiting argument, this procedure defines a continuous function that begins at 0 and increases to 1. However, since this function is constant except on the Cantor set, we see that its derivative off of the Cantor set must be identically zero. On the Cantor set the function is not differentiable and so has no PDF.
What we see is that, for a Cantor random variable, we cannot make any sensible definition for the PDF. It is either identically zero or not defined.

This is an interesting example of how identifying a random variable with its PDF can lead us astray. Thankfully, for the purposes of Math 105, we will never need to consider continuous random variables that do not have PDFs defined everywhere (except possibly at finitely many points).

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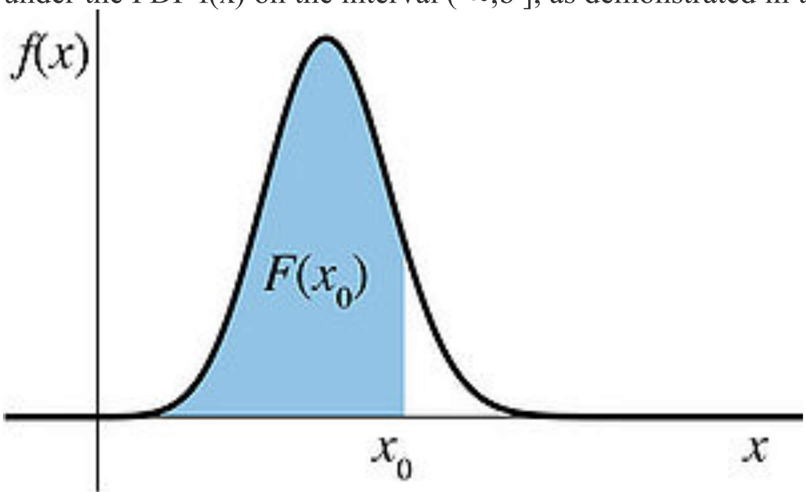

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A Simple PDF Example

Question

Let $f(x) = k(3x^2 + 1)$.

1. Find the value of k that makes the given function a PDF on the interval $0 \leq x \leq 2$.
2. Let X be a continuous random variable whose PDF is f(x). Compute the probability that X is between 1 and 2.
3. Find the distribution function of X.

4. Find the probability that X is exactly equal to 1.

Solution

Part 1)

$$
\begin{aligned}
1 &= \int_0^2 f(x)dx \\
&= \int_0^2 k(3x^2+1)dx \\
&= k(\frac{3x^3}{3}+x)\Big|_0^2 dx \\
&= k(10)
\end{aligned}
$$

Therefore, k = 1/10.

Notice that f(x) ≥ 0 for all x. Also notice that we can rewrite this PDF in the obvious way so that it is defined for all real numbers:

$$
f(x) = \begin{cases} \frac{1}{10}(3x^2+1), & \text{if } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}
$$

Part 2)

Using our value for k from Part 1:

$$
\Pr(1 \le X \le 2) = \int_1^2 \frac{3x^2+1}{10}dx = \frac{x^3+x}{10}\Big|_1^2 = 1 - 2/10 = 4/5
$$

Therefore, Pr(1 ≤ X ≤ 2) is 4/5.

Part 3)

Using the Fundamental Theorem of Calculus, the CDF of X at x in [0,2] is

$$
\begin{aligned}
\Pr(X \le x) = F(x) &= \int_{-\infty}^x f(t)dt \\
&= \int_0^x \frac{1}{10}(3t^2+1)dt \\
&= \frac{1}{10}(t^3+t)|_0^x \\
&= \frac{1}{10}(x^3+x), \text{ for } 0 \le x \le 2
\end{aligned}
$$

We can also easily verify that F(x) = 0 for all x < 0 and that F(x) = 1 for all x > 2.

Part 4)
Since X is a continuous random variable, we immediately know that the probability that it equals any one particular value must be zero. More directly, we compute

$$\Pr(X=1)=\int_1^1 f(t)dt=0$$

Some Common Continuous Distributions

Let us consider some common continuous random variables that often arise in practice. We should stress that this is indeed a very small sample of common continuous distributions.

The Beta Distribution
Suppose the proportion p of restaurants that make a profit in their first year of operation is given by a certain beta random variable X, with probability density function:

$$f(p)=\begin{cases}12p(1-p)^2 & \text{if } 0\leq x\leq 1,\\ 0 & \text{elsewhere.}\end{cases}$$

What is the probability that more than half of the restaurants will make a profit during their first year of operation? To answer this question, we calculate the probability as an area under the PDF curve as follows:

$$\begin{aligned}\Pr(0.5\leq X\leq 1)&=\int_{0.5}^1 f(p)dp\\ &=\int_{0.5}^1 12p(1-p)^2dp\\ &=\int_{0.5}^1\left(12p-24p^2+12p^3\right)dp\\ &=6p^2-8p^3+3p^4\Big|_{0.5}^1\\ &=(6-8+3)-(1.5-1+0.1875)\\ &=0.3125\end{aligned}$$

Therefore, $\Pr(0.5 \leq P \leq 1) = 0.3125$.
The example above is a particular case of a beta random variable. In general, a beta random variable has the generic PDF:

$$f(x) = \begin{cases} kx^{a-1}(1-x)^{b-1} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere} \end{cases}$$

where the constants a and b are greater than zero, and the constant k is chosen so that the density f integrates to 1.
We see that our previous example was a beta random variable given by the above density with a= 2 and b = 3. Let us find the associated cumulative distribution function F(p) for this random variable. We compute:

$$\begin{aligned} F(p) &= \int_{-\infty}^{p} f(t)dt \\ &= \int_{0}^{p} 12t(1-t)^2 dt \\ &= 12\int_{0}^{p} (t - 2t^2 + t^3)dt \\ &= 12\Big(\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\Big)\Big|_0^p \\ &= p^2(6 - 8p + 3p^2), \end{aligned}$$

valid for $0 \leq p \leq 1$.

The Exponential Distribution
The lifespan of a lightbulb can be modeled by a continuous random variable since lifespan - i.e. time - is a continuous quantity. A reasonable distribution for this random variable is what is known as an exponential distribution.
A random variable Y has an exponential distribution with parameter $\beta > 0$ if its PDF is given by

$$f(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta} & \text{if } 0 \leq y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Suppose that the lifespan (in months) of lightbulbs manufactured at a certain facility can be modeled by an exponential random variable Y with parameter $\beta = 4$. What is the probability that a particular lightbulb lasts at least a year? Again, we can calculate this probability by evaluating an integral. Since there are 12 months in one year, we calculate

$$
\begin{aligned}
\Pr(Y \geq 12) &= \int_{12}^{\infty} f(y)dy \\
&= \int_{12}^{\infty} \frac{1}{4} e^{-y/4} dy \\
&= -e^{-y/4}\Big|_{12}^{infty} \\
&= 0 - (-e^{-3}) \\
&\approx 0.04979
\end{aligned}
$$

Thus we can see that it is highly likely we would need to replace a lightbulb produced from this facility within one year of manufacture.

The Continuous Uniform Distribution

Our third example of a common continuous random variable is one that we have already encountered. Consider the experiment of randomly choosing a real number from the interval [a,b]. Letting X denote this random outcome, we say that X has a continuous uniform distribution on [a,b] if the probability that we choose a value in some subinterval of [a,b] is given by the relative size of that subinterval in [a,b]. More explicitly, we have the following: A random variable X has an continuous uniform distribution on [a,b] if its PDF is constant on [a,b]; i.e. its PDF is given by

$$
f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}
$$

The continuous uniform distribution has a particularly simple representation, just as its discrete counterpart does. Nevertheless, this random variable has great practical and theoretical utility. We will explore this distribution in more detail in the exercises.

For the purposes of MATH 105, students are not expected to memorize the formulae for the probability density functions introduced in this section, but may need to use them to complete assigned work.

The Normal Distribution

The most important probability distribution in all of science and mathematics is the normal distribution.

The Normal Distribution
The random variable X has a normal distribution with mean parameter μ and variance parameter $\sigma^2 > 0$ with PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

To express this distributional relationship on X, we commonly write X ~ Normal(μ,σ^2).

This PDF is the classic "bell curve" shape associated to so many experiments. The parameter μ gives the mean of the distribution (the centre of the bell curve) while the σ^2 parameter gives the variance (the horizontal spread of the bell curve). The first of these facts is a simple exercise in integration (see the exercises), while the second requires a bit more ingenuity.
Recall that the standard deviation of a random variable is defined to be the positive square root of its variance. Thus, a normal random variable has standard deviation σ.

This random variable enjoys many analytical properties that make it a desirable object to work with theoretically. For example, the normal density is symmetric about its mean μ. This means that, among other things, exactly half of the area under the PDF lies to the right of the mean, and the other half of the area lies to the left of the mean. More generally, we have the following important fact.

Symmetry of Probabilities for a Normal Distribution

If X has a normal distribution with mean μ and variance σ^2, and if x is any real number, then

$$\Pr(X \le \mu - x) = \Pr(X \ge \mu + x).$$

However, the PDF of a normal distribution is not convenient for calculating probabilities directly. In fact, it can be shown that no closed form exists for the cumulative distribution function of a normal random variable. Thus, we must rely on tables of values to calculate probabilities for events associated to a normal random variable. (The values in these tables are calculated using careful numerical techniques not covered in this course.)

A particularly useful version of the normal distribution is the standard normal distribution, where the mean parameter is 0 and the variance parameter is 1.

The Standard Normal Distribution

The random variable Z has a standard normal distributionif its distribution is normal with mean 0 and variance 1. The PDF of Z is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

For a particular value x of X, the distance from x to the mean μ of X expressed in units of standard deviation σ is

$$z = \frac{x - \mu}{\sigma}.$$

Since we have subtracted off the mean (the centre of the distribution) and factored out the standard deviation (the horizontal spread), this new value z is not only a rescaled version of x, but is also a realization of a standard normal random variable Z.

In this way, we can standardize any value from a generic normal distribution, transforming it into one from a standard normal distribution. Thus we reduce the problem of calculating probabilities for an event from a normal random variable to calculating probabilities for an event from a standard normal random variable.

Theorem: Standardizing a Normal Random Variable
Let X have a normal distribution with mean μ and variance σ^2. Then the new random variable $Z = \frac{X - \mu}{\sigma}$ has a standard normal distribution.

Calculating Probabilities Using a Standard Normal Distribution

Suppose that the test scores for first-year integral calculus final exams are normally distributed with mean 70 and standard deviation 14. Given that $\Pr(Z \leq 0.36) = 0.64$ and $\Pr(Z \leq 1.43) = 0.92$ for a standard normal random variable X, what percentage of final exam scores lie between 75 and 90?

If we let X denote the score of a randomly selected final exam, then we know that X has a normal distribution with parameters $\mu = 70$ and $\sigma = 14$. To find the percentage of final exam scores that lie between 75 and 90, we need to use the information about the probabilities of a standard normal random variable. Thus we must standardize X using the theorem above.

For our particular question, we wish to compute

$$\Pr(75 \leq X \leq 90).$$

We proceed by standardizing the random variable X as well as the particular x values of interest. Thus, since X has mean 70 and standard deviation 14, we write

$$\Pr(75 \leq X \leq 90) = \Pr\left(\frac{75 - 70}{14} \leq \frac{X - 70}{14} \leq \frac{90 - 70}{14}\right).$$

Now we have standardized our normal random variable so that

$$\frac{X-70}{14}=Z,$$

where Z ~ Normal(0,1).
Simplifying the numerical expressions from above, we deduce that we must calculate

$$\Pr(0.36 \leq Z \leq 1.43).$$

Now we can use the information we were given, namely that Pr(Z ≤ 0.36) = 0.64 and Pr(Z ≤ 1.43) = 0.92. Using these values, we find

$$\begin{aligned}\Pr(75 \leq X \leq 90) &= \Pr(0.36 \leq Z \leq 1.43)\\ &= \Pr(Z \leq 1.43) - \Pr(Z \leq 0.36)\\ &= 0.92 - 0.64\\ &= 0.28.\end{aligned}$$

Therefore the percentage of first-year integral calculus final exam scores between 75 and 90 is 28%.

Now suppose we wish to find the percentage of final exam scores larger than 90, as well as the percentage of final exam scores less than 65. To find the percentage of final exam scores larger than 90, we use our knowledge about probabilities of disjoint events:

$$\begin{aligned}\Pr(X > 90) &= 1 - \Pr(X \leq 90)\\ &= 1 - \Pr(Z \leq 1.43)\\ &= 1 - 0.92\\ &= 0.08.\end{aligned}$$

Thus, we find that 8% of exam scores are larger than 90.

To find the percentage of final exam scores less than 65, we must exploit the symmetry of the normal distribution. Recall that our normal random variable X has mean 70. We are given information about the probability of a standard normal random variable assuming a value less than 0.36, which we have already seen corresponds to the probability of our normal random variable X assuming a value less than 75. Now notice that the x value 65 is the reflection of 75 through the mean. That is, both scores 65 and 75 are exactly 5 units from the mean of our random variable X. Thus we should take advantage of the symmetry property of X.
Using the symmetry identity from the top of the page, we find that

$$
\begin{aligned}
\Pr(X < 65) &= \Pr(X < 70 - 5) \\
&= \Pr(X > 70 + 5) \\
&= 1 - \Pr(X \leq 75) \\
&= 1 - \Pr(Z \leq 0.36) \\
&= 1 - 0.64 \\
&= 0.36.
\end{aligned}
$$

Thus, we find that 36% of exam scores are smaller than 65.

Expected Value, Variance, Standard Deviation
Analogous to the discrete case, we can define the expected value, variance, and standard deviation of a continuous random variable. These quantities have the same interpretation as in the discrete setting. The expectation of a random variable is a measure of the centre of the distribution, its mean value. The variance and standard deviation are measures of the horizontal spread or dispersion of the random variable.

Definition: Expected Value, Variance, and Standard Deviation of a Continuous Random Variable
The expected value of a continuous random variable X, with probability density function f(x), is the number given by $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$ The variance of X is: $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f(x) dx$ As in the discrete case, the standard deviation, σ, is the positive square root of the variance: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Simple Example
The random variable X is given by the following PDF. Check that this is a valid PDF and calculate the standard deviation of X.

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

Solution

Part 1

To verify that f(x) is a valid PDF, we must check that it is everywhere nonnegative and that it integrates to 1.

We see that 2(1-x) = 2 - 2x ≥ 0 precisely when x ≤ 1; thus f(x) is everywhere nonnegative.

To check that f(x) has unit area under its graph, we calculate

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_0^1 (1-x)dx = 2\left(x - \frac{x^2}{2}\right)\Big|_0^1 = 1$$

So f(x) is indeed a valid PDF.

Part 2

To calculate the standard deviation of X, we must first find its variance. Calculating the variance of X requires its expected value:

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^1 x\Big[2(1-x)\Big]dx \\ &= 2\int_0^1 \left(x - x^2\right)dx \\ &= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \\ &= 1/3 \end{aligned}$$

Using this value, we compute the variance of X as follows

$$\begin{aligned}
\mathrm{Var}(X) &= \int_{-\infty}^{\infty} \left(x - \mathbb{E}(X)\right)^2 f(x)dx \\
&= \int_0^1 \left(x - 1/3\right)^2 \cdot 2(1-x)dx \\
&= 2\int_0^1 (x^2 - \frac{2}{3}x + \frac{1}{9})(1-x)dx \\
&= 2\int_0^1 \left(-x^3 + \frac{5}{3}x^2 - \frac{7}{9}x + \frac{1}{9}\right)dx \\
&= 2\left(-\frac{1}{4}x^4 + \frac{5}{9}x^3 - \frac{7}{18}x^2 + \frac{1}{9}x\right)\Big|_0^1 \\
&= 2\left(-\frac{1}{4} + \frac{5}{9} - \frac{7}{18} + \frac{1}{9}\right) \\
&= \frac{1}{18}
\end{aligned}$$

Therefore, the standard deviation of X is

$$\begin{aligned}
\sigma &= \sqrt{\mathrm{Var}(X)} \\
&= \frac{1}{3\sqrt{2}}
\end{aligned}$$

An Alternative Formula for Variance

There is an alternative formula for the variance of a random variable that is less tedious than the above definition.

Alternate Formula for the Variance of a Continuous Random Variable
The variance of a continuous random variable X with PDF f(x) is the number given by $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

The derivation of this formula is a simple exercise and has been relegated to the exercises. We should note that a completely analogous formula holds for the variance of a discrete random variable, with the integral signs replaced by sums.

Simple Example Revisited

We can use this alternate formula for variance to find the standard deviation of the random variable X defined above.

Remembering that E(X) was found to be 1/3, we compute the variance of X as follows:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\frac{1}{3}\right)^2 \\
&= 2\int_0^1 x^2(1-x)dx - \frac{1}{9} \\
&= 2\int_0^1 (x^2 - x^3)dx - \frac{1}{9} \\
&= 2\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 - \frac{1}{9} \\
&= 2\left(\frac{1}{3} - \frac{1}{4}\right) - \frac{1}{9} \\
&= \frac{1}{18}
\end{aligned}$$

In the exercises, you will compute the expectations, variances and standard deviations of many of the random variables we have introduced in this chapter, as well as those of many new ones.