(i) What is the average number of customers in the fitting room ?
(ii) How much time a customer is expected to spend in the fitting room ?
(iii) What percentage of the time is the tailor idle ? **[VTU (BE Mech.) 2002]**

## 23.13. Model II (A). General Erlang Queueing Model (Birth-Death Process) [Agra 98]

**(a) To obtain the system of steady state equations**

Let

$$\left.\begin{array}{l}\text{arrival rate } \lambda = \lambda_n \\ \text{service tate } \mu = \mu_n\end{array}\right\} \quad \text{[depending upon } n]$$

Then, by the same arguments as for equations (23.51) and (23.53),

$$P_n(t+\Delta t) = P_n(t)\,[1-(\lambda_n+\mu_n)\,\Delta t] + P_{n-1}(t)\,\lambda_{n-1}\Delta t + P_{n+1}(t)\,\mu_{n+1}\,\Delta t + O(\Delta t),\ n>0; \quad \text{...(23.73)}$$

and $$P_0(t+\Delta t) = P_0(t)\,[1-\lambda_0\,\Delta t] + P_1(t)\,\mu_1\,\Delta t + O(\Delta t),\ n=0. \quad \text{...(23.74)}$$

Now dividing (23.73) and (23.74) by $\Delta t$, taking limits as $\Delta t \to 0$ and following the same procedure as in Model I, obtain

$$\frac{dP_n(t)}{dt} = -(\lambda_n+\mu_n)\,P_n(t) + \lambda_{n-1}\,P_{n-1}(t) + \mu_{n+1}\,P_{n+1}'(t) \quad \text{...(23.75)}$$

and $$\frac{dP_0(t)}{dt} = -\lambda_0\,P_o(t) + \mu_1\,P_1(t) \text{ respectivcly.} \quad \text{...(23.76)}$$

The equations (23.75) and (23.76) are differential-difference equations which could be solved if a set of initial values $P_0(0), P_1(0), \ldots$, is given. Such a system of cquations can be solved if the time dependent solution is required. But, for many problems it suffices to look at the steady state solution.

In the case of steady state,

$$P_n'(t) = 0 \text{ and } P_0'(t) = 0.$$

So the equations (23.75) and (23.76) become,

$$0 = -(\lambda_n+\mu_n)\,P_n + \lambda_{n-1}\,P_{n-1} + \mu_{n+1}\,P_{n+1},\ n>0 \quad \text{...(23.77)}$$

and $$0 = -\lambda_0\,P_0 + \mu_1\,P_1,\ n=0. \quad \text{...(23.78)}$$

The equations (23.77) and (23.78) constitute the system of steady state difference equations for this model. **[Agra 98]**

**(b) To solve the system of difference equations**

Since $P_0 = P_0$

$$P_1 = \frac{\lambda_0}{\mu_1}P_0 \qquad \text{[from equation (23.78)]}$$

$$P_2 = \frac{\lambda_1}{\mu_2}P_1 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}P_0 \qquad [\text{from letting } n = 1 \text{ in equation (23.77) and substituting for } P_1]$$

$$P_3 = \frac{\lambda_2}{\mu_3}P_2 = \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}P_0 \qquad [\text{putting } n = 2 \text{ in equation (23.77)}]$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_1}P_0 \quad [\text{for } n \geq 1] \quad \text{...(23.79)}$$

Now, in order to find $P_0$, use the fact that

$$\sum_{n=0}^{\infty} P_n = 1. \text{ or } P_0+P_1+P_2+\ldots\infty = 1 \text{ or } P_0\left[1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\ldots\infty\right]=1 \text{ or } P_0 = 1/S,$$

where $$S = 1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\ldots\infty. \quad \text{...(23.80)}$$

**Note.** The series $S = 1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\ldots\infty$ is summable and meaningful only when it is convergent.

The result obtained above is a general one and by suitably defining $\mu_n$ and $\lambda_n$ many interesting cases could be studied. Now three particular cases may arise :

**Case 1.** $(\lambda_n = \lambda, \mu_n = \mu)$

In this case, the series $S$ becomes

$$S = 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \ldots\infty = \frac{1}{1-\lambda/\mu} \qquad (\text{when } \lambda/\mu < 1)$$

Therefore, from equations (23.80) and (23.79),

$$P_0 = \frac{1}{S} = 1 - \frac{\lambda}{\mu} \quad \text{and} \quad P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Here, it is observed that this is exactly the case of Model I.

**Case 2.** $\left(\lambda_n = \frac{\lambda}{n+1}, \mu = \mu\right)$

The case, in which the arrival rate $\lambda_n$ depends upon $n$ inversely and the rate of service $\mu_n$ is independent of $n$, is called the case of *"Queue with Discouragement"*.

In this case, the series $S$ becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{2.3\mu^3} + \ldots\infty = 1 + \rho + \frac{\rho^2}{2\,!} + \frac{\rho^3}{3\,!} + \ldots\infty = e^{\rho} \qquad (\rho = \lambda/\mu)$$

The equation (23.80) gives $P_0 = 1/S = e^{-\rho}$.

Also,
$$P_1 = \frac{\lambda}{\mu} P_0 = \rho\, e^{-\rho},$$
$$P_2 = \frac{\lambda^2}{2\mu^2} P_0 = \frac{\rho^2 e^{-\rho}}{2\,!},$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$P_n = \frac{\lambda^n}{n\,!\,\mu^n} P_0 = \frac{\rho^n e^{-\rho}}{n\,!} \text{ for all } n = 0, 1, \ldots, \infty.$$

It is observed in this case that $P_n$ follows the Poisson distribution, where $\lambda/\mu = \rho$ is constant, however $\rho > 1$ or $\rho < 1$ but must be finite. Since, the series $S$ is convergent and hence summable in both the cases.

**Case 3.** $(\lambda_n = \lambda$ **and** $\mu = n\mu)$, *i.e.*, **the case of infinite number of stations.**

In this case, the arrival rate $\lambda_n$ does not depend upon $n$ , but the service rate $\mu_n$ increases as $n$ increases. Here, assume that there are infinite (variable) number of service stations. The word *'infinite'* means that the service stations are available to each arrival. But it does not mean that all the infinite service stations will remain busy every time. In other words, it means that if $n$ customers arrive, then $n$ service stations will be available for all $n = 0, 1, 2, \ldots\infty$. Obviously, no queue will form in this case because each arrival will immediately enter the service facility. For example, in everyday life, it is observed that the telephone (service stations) are always available to all the arriving persons.

In this case, the series $S$ becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2.1\mu^2} + \frac{\lambda^3}{3.2.1\mu^3} + \ldots\infty = 1 + \rho + \frac{\rho^2}{2\,!} + \frac{\rho^3}{3\,!} + \ldots\infty = e^{\rho}, \text{ where } \rho = \lambda/\mu.$$

Therefore,
$$P_0 = e^{-\rho} \quad \text{and} \quad P_n = e^{-\rho} \rho^n / n\,!$$
which again follows the *Poisson* distribution law.

### 23.13-1. Model II (B). $(M \mid M \mid 1) : (\infty \mid SIRO)$

This model is actually the same as *Model* I, except that the service discipline follows the *Service In Random Order* (*SIRO*) rule in place of *FCFS*-rule. Since the derivation of $P_n$ in *Model* I does not depend on any specific queue discipline, we must have

$$P_n = (1-\rho)\,\rho^n, \; n \geq 0$$

for the *SIRO*-rule case also.

Consequently, the average number of customers in the system ($L_s$) will remain the same, whether the queue discipline follows the *SIRO*-rule or *FCFS*-rule. In fact, $L_s$ will not change provided $P_n$ remains unchanged for any queue discipline. Thus $W_s = L_s/\lambda$ under the *SIRO*-rule is also same as under the *FCFS*-rule and it is given by $W_s = 1/(\mu - \lambda)$.

Moreover, this result can be extended to *any queue discipline* so long as $P_n$ remains unchanged. In particular, the result is applicable to the three most common queue disciplines : *FCFS, LCFS* and *SIRO*. These queue disciplines differ only in the distribution of waiting time where the probabilities of long and short waiting times change depending upon the queue discipline used. So we can use the *GD* (*General Discipline*) to represent *FCFS, LCFS* and *SIRO*, whenever the waiting time distribution is not required.

Q. 1. Deduce the difference equations for the queueing model M | M | 1 : (FCFS/∞/∞) with arrival and service rates dependent on system size. Obtain steady state solution. Deduce also the solution for the following special cases : (i) Queues with discouragement, (ii) Queues with ample servers. **[Delhi MA/M.Sc. (OR.) 92, 90]**

2. Derive differential-difference equations for a generalized birth-death queueing model. Obtain steady-state distribution of the system size. **[Delhi MA/M.Sc (State.) 95]**

### 23.13–2. Illustrative Examples on Model II

**Example 26.** *Problems arrive at a computing centre in a Poisson fashion at an average rate of five per day. The rules of the computing centre are that any man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of ⅓ day, and if the average solving time is inversely proportional to the number of people working on the problem, approximate the expected time in the centre for a person entering the line.* **[Rohil. 92]**

**Solution.** Here $\lambda = 5$ problems/day

$\mu = 3$ problems/day (mean service rate with one unsolved problem)

Then, the expected number of persons working at any specified instant is :

$$L_s = \sum_{n=0}^{\infty} nP_n \text{, where } P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu} \text{ [see } \textbf{Case II}]$$

$$= \sum_{n=0}^{\infty} n\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu} \qquad = e^{-\lambda/\mu}\sum_{n=0}^{\infty}\frac{n}{n!}\cdot\left(\frac{\lambda}{\mu}\right)^n$$

$$= e^{-\lambda/\mu}\left\{\frac{\lambda}{\mu} + \frac{2}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \frac{3}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \dots\infty\right\}$$

$$= e^{-\lambda/\mu}\cdot\frac{\lambda}{\mu}\left\{1 + \frac{\lambda}{\mu} + \frac{1}{2!}\cdot\left(\frac{\lambda}{\mu}\right)^2 + \dots\right\}$$

$$= e^{-\lambda/\mu}\cdot\frac{\lambda}{\mu}\cdot e^{\lambda/\mu} = \lambda/\mu.$$

Substituting the values for $\lambda$ and $\mu$, $L_S = 5/3$ persons.

Now, the average solving time which is inversely proportional to the number of people working on the problem is : ⅕ day/problem.

Therefore, expected time for a person entering the line is

$= \frac{1}{5} \times L_S$ days $= \frac{1}{5} \times \frac{5}{3} \times 24$ hours $= 8$ hours. **Ans.**

**Example 27.** *A shipping company has a single unloading berth with ships arriving in Poisson fashion at an average rate of three per day. The unloading time distribution for a ship with the unloading crews is found to be exponential with average unloading time ½ in days. The company has a large labour supply without regular working hours and to avoid long waiting lines the company has a policy of using as many unloading crews as there are ships waiting in line or being unloaded. Under these conditions,* find

(i) *the average number of unloading crews working at any time,* and

(ii) *the probability that more than four crews will be needed.*

**Solution.** Here,

$\lambda = 3$ ships per day

$\mu = 2$ ships per day (mean service rate with one unloading crew)

(*i*) Average number of unloading crews working at any specified instant is :

$$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \frac{e^{-\lambda/\mu}}{n!}\left(\frac{\lambda}{\mu}\right)^n \left[\text{since } P_n = \frac{e^{-\lambda/\mu}}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

$= \lambda/\mu = 3/2$ crews. (*see example 26 above*) **Ans.**

*(ii)* The probability that more than 4 crews will be needed is the same as the probability that there are at least five ships in the system at any specified instant which is given by

$$\sum_{n=5}^{\infty} P_n = \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{4} P_n = 1 - [P_0 + P_1 + P_2 + P_3 + P_4]$$

$$= 1 - \left[e^{-\lambda/\mu} + \left(\frac{\lambda}{\mu}\right)\frac{e^{-\lambda/\mu}}{1!} + \left(\frac{\lambda}{\mu}\right)^2 \frac{e^{-\lambda/\mu}}{2!} + \left(\frac{\lambda}{\mu}\right)^3 \frac{e^{-\lambda/\mu}}{3!} + \left(\frac{\lambda}{\mu}\right)^4 \frac{e^{-\lambda/\mu}}{4!}\right]$$

$$= 1 - e^{-\lambda/\mu}\left[1 + \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2}{2!} + \frac{(\lambda/\mu)^3}{3!} + \frac{(\lambda/\mu)^4}{4!}\right]$$

Now putting values for $\lambda$ and $\mu$, and simplifying, we get

$$\sum_{n=5}^{\infty} P_n = 0.019.$$

**Ans.**

## 23.14. MODEL III. $(M \mid M \mid 1) : (N \mid FCFS)$

Up to this stage, only two models are discussed in which the capacity of the system is infinite. Now consider the case where the capacity of the system is limited, say $N$. In fact the number of arrivals will not exceed the $N$ in any case.

The physical interpretation for this model may be either :

*(i)* that there is only room (capacity) for $N$ units in the system (as in a packing lot),

or *(ii)* that the arriving customers will *go for their service elsewhere* permanently, if the waiting line is too long ($\leq N$).

**(a) To obtain steady state difference equations.** The simplest way of starting this is to treat the model as a special case of *Model* II, where

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, 3, \ldots, N-1 \\ 0, & n \geq N \end{cases} \qquad \ldots(23.81)$$

and $\mu_n = \mu \quad \text{for} \quad n = 1, 2, 3, \ldots \qquad \ldots(23.82)$

Now, following the similar arguments as given for equations (23.53) and (23.51) in Model I, we obtain

$$P_0(t+\Delta t) = P_0(t)[1 - \lambda\Delta t] + P_1(t)\,\mu\Delta t + O(\Delta t), \qquad \text{for } n = 0, \qquad \ldots(23.83)$$

$$P_n(t+\Delta t) = P_n(t)[1 - (\lambda+\mu)\Delta t] + P_{n-1}(t)\,\lambda\Delta t + P_{n+1}(t)\,\mu\,\Delta t + O(\Delta t), \qquad \text{for } n = 1, 2, \ldots, N-1, \qquad \ldots(23.84)$$

and
$$P_N(t+\Delta t) = P_N(t)[1 - (0+\mu)\Delta t] + P_{N-1}(t)\,\lambda\,\Delta t + 0 \times \mu\,\Delta t + O(\Delta t)$$
$$= P_N(t)[1 - \mu\Delta t] + P_{N-1}(t)\,\lambda\Delta t + O(\Delta t) \qquad \text{for } n = N,\ P_{N+1}(t) = 0,\ \lambda = 0 \qquad \ldots(23.85)$$

Now dividing equation (23.83), (23.84), and (23.85) by $\Delta t$, and taking limit as $\Delta t \to 0$, these equations transform into

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \quad \text{for } n = 0 \qquad \ldots(23.83a)$$

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \quad \text{for } n = 1, 2, \ldots, N-1, \qquad \ldots(23.84a)$$

and
$$P_N'(t) = -\mu P_N(t) + \lambda P_{N-1}(t) \quad \text{for } n = N. \qquad \ldots(23.85a)$$

In the case of steady state, when $t \to \infty$, $P_n(t) \to P_n$ (independent of $t$) and hence $P_n'(t) \to 0$. So the system of steady state difference equations is given by

$$0 = -\lambda P_0 + \mu P_1, \text{ for } n = 0 \qquad \ldots(23.83b)$$

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \qquad \text{for } n = 1, 2, \ldots, N-1, \qquad \ldots(23.84b)$$

$$0 = -\mu P_N + \lambda P_{N-1}, \quad \text{for } n = N. \qquad \ldots(23.85b)$$

**(b) To solve the system of difference equations (23.83b), (23.84b) and (23.85b).**

Here

$P_0 = P_0$ (initially)

$P_1 = \dfrac{\lambda}{\mu} P_0$ [from (23.83b)]

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \quad \text{[put } n = 1 \text{ in (23.84b) and substitute value of } P_1\text{]}.$$

Similarly,

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\ldots \qquad \ldots$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0\,,\ n < N \qquad \ldots(23.86)$$

$$P_N = \left(\frac{\lambda}{\mu}\right)^N P_0,\ n = N$$ (because $P_N = (\lambda/\mu)\, P_{N-1}$ and $P_{N-1}$ follows the rule for which $n = 1, 2, \ldots, N-1$).

$P_{N+1} = 0,\ n > N.$

Now, in order to find $P_0$, use the fact that :

$$\sum_{n=0}^{N} P_n = 1$$

or $$P_0\,[1 + (\lambda/\mu) + (\lambda/\mu)^2 + \ldots (\lambda/\mu)^N] = 1 \quad \text{or} \quad P_0\left[\frac{1-\rho^{N+1}}{1-\rho}\right] = 1, \text{ where } \rho = \lambda/\mu$$

or $$P_0 = \frac{1-\rho}{1-\rho^{N+1}}. \qquad \ldots(23.87)$$

Substituting the value of $P_0$ in (23.86),

$$P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n\,, \text{ for } n = 0, 1, 2, \ldots, N. \qquad \ldots(23.88)$$

Thus, the result (23.87) and (23.88) give the required solution for this model which do not require $\lambda < \mu$. That is, in this case, $\rho$ may be greater than 1 also.

**(c) Measures of Model III.**

(i) $$L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n\left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^N$$

or $$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n = P_0 \sum_{n=0}^{N} n\,\rho^n. \qquad \ldots(23.89)$$

Now, four relationships (23.69), (23.70), (23.71) and (23.72) give :

(ii) $L_q = L_s - \lambda/\mu$ ...(23.90)

(iii) $W_s = L_s/\lambda$, and ...(23.91)

(iv) $W_q = W_s - 1/\mu = L_q/\lambda$. ...(23.92)

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**Q. 1.** Obtain the steady state difference equations for the queueing model $(M \mid M \mid 1) : (N \mid FCFS)$ in usual notations and solve them for $P_0$ and $P_1$. Also find the mean queue length for this system. **[Meerut (Stat.) 98; Garhwal (Stat.) 92]**

**2.** For the $(M \mid M \mid 1) : (FCFS\,, K)$ queueing model, show that the steady state probability, $p_n$ is given by

$$p_n = \rho^n \frac{1-\rho}{1-\rho^{K+1}}\,,\ 0 \le n \le K.$$

Also obtain expected number of units in the queue and system separately.

**3.** For the model $(M \mid M \mid 1) : (N \mid FCFS)$ where the notations have their usual meanings, find the following :

(i) The average number of customers in the system. (ii) Average queue length.

**4.** Explain $(M \mid M \mid 1) : (N \mid FCFS)$ system and solve it in steady state. **[Garhwal M.Sc. (Stat.) 96, 95, 93]**

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### 23.14–1. Illustrative Examples on Model III

**Example 28.** *In Example 5 of sec. 23.12–2, if we assume that the line capacity of yard is to admit of 9 trains only (there being 10 lines, one of which is eur marked for the shunting engine to reverse itself from the crest of the hump to the rare of the train). Calculate the following on the assumption that 30 trains, on average, are received in the yard :*

(a) *the probability that the yard is empty,* (b) *average queue length.*

**Solution.** As already computed in *Example 5,* **section 23.12–2,** $\rho = 0.75$.

(a)The probability 'that the queue size is zero' is given by

$$P_0 = (1-\rho)/(1-\rho^{N+1})$$

But given that $N = 9$ so,

$$P_0 = \frac{1-0.75}{1-(0.75)^{10}} = \frac{0.25}{0.90} = 0.28.$$

**Ans.**

(b) Average queue length is given by the formula

$$L_s = \left(\frac{1-\rho}{1-\rho^{N+1}}\right) \sum_{n=0}^{N} n\rho^n$$

$$\therefore \quad L_s = \frac{1-0.75}{1-(0.75)^{10}} \sum_{n=0}^{9} n\,(0.75)^n = 0.28 \times 9.58 = 2.79, \text{ say 3 trains.}$$

**Ans.**

**Example 29.** *If for a period of 2 hours in a day (8—10 A.M.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period :*

(a) *the probability that the yard is empty,* (b) *average queue length, on the assumption that the line capacity of the yard is limited to 4 trains only.*

**Solution.** Here, $\rho = 36/20 = 1.8$ (which is greater than 1) and $N = 4$.

Thus, we obtain

(a) $P_0 = \dfrac{\rho - 1}{\rho^5 - 1} = 0.04$

(b) average queue size $= P_0 \sum_{n=0}^{4} n\rho^n, = .04\,[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4], = .04 \times 72.0 = 2.9$, say 3 trains. **Ans.**

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### EXAMINATION PROBLEMS (ON MODEL III)

1. Discuss the stationary state of the queue system $(M \mid M \mid 1) : (N \mid FCFS)$.

   A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park has negative exponential distribution with mean of 5 hours. How many cars are in the car park on average and what is the probability of a newly arriving customer finding the car park full and having to park his car elsewhere ?

   **[Hint.** Here $N = 5$, $\lambda = 10/60$, $\mu = 1/2 \times 60$, $\rho = \lambda/\mu = 20$. Find $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$, $L_s = P_0 \sum_{n=0}^{N} n\rho^n$.]

2. A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it full he goes to the next shop. Customers randomly arrive at an average rate $\lambda = 10$ per hour and the barber's service time is negative exponential with an average of $1/\mu = 5$ minutes per customer.
   (i) Write recurrence relations for the steady state queueing system (*FCFS*) for above.
   (ii) Determine $P_0$ and $P_n$, probability of having 0 and $n$-customers respectively in the shop.

   **[Hint.** Here $N = 10$, $\lambda = 10/60$, $\mu = 1/5$, $\rho = \lambda/\mu = 5/6$. Find $P_0 = \dfrac{1-\rho}{1-\text{'rho}^{N+1}}$, $P_n = P_0\rho^n$.]

3. Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.
   (i) Find the effective arrival rate at the clinic.
   (ii) What is the probability that an arriving patient will not wait ? Will he find a vacant seat in the room ?
   (iii) What is the expected waiting time until a patient is discharged from the clinic ?

   **[Hint.** Here $N = 14$, $\lambda = 30/60$, $\mu = 20/60$, $\rho = 2/3$. Find $P_n$, $P_0$ and $W_s = \dfrac{P_0}{\lambda} \sum_{n=0}^{N} n\rho^n$.]

4. Customers arrive at a one-window derive-in bank according to a Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The car space in front of the window, including that for the serviced can accomodate a maximum of 3 cars. Other cars can wait outside this space. **[Meerut (MCA) 2000]**
   (a) What is the probability that an arriving customer can drive directly to the space in front of the window ?
   (b) What is the probability that an arriving customer will have to wait outside the indicated space ?
   (c) How long is an arriving customer expected to wait before starting service ?

(d) How, many spaces should be provided in front of the window so that all the arriving customers can wait in front of the window at least 20% of the time.

**[Meerut (MCA) 2000]**

[**Hint.** $\lambda = 10$, $\mu = 60/5 = 12$, $P_0 = 1 - (\lambda/\mu) = 1/6$, (a) $P_0 + P_1 + P_2 = [1 + \lambda/\mu + (\lambda/\mu)^2]\, P_0 = 0.42$,
(b) $1 - (P_0 + P_1 + P_2 + P_3) = 1 - 0.42 - P_3 = 0.58 - (\lambda/\mu)^3\, P_0 + 0.48$. (c) $W_q = \lambda/\{\mu\,(\mu - \lambda)\} = 0.417$,
(d) $P_0 + P_1 = 0.30$. Hence there should be at least one car space for waiting at least 20% of the time.]

**5.** A stenographer has 5 person for whom she performs stenographic work. Arrival rate is Poisson and service times are exponential. Average arrival rate is 4 per hour with an average service time of 10 minutes. Cost of waiting is Rs. 8 per hour while the cost of servicing is Rs. 2.50 each. Calculate :

(i) the average waiting time of an arrival,
(ii) the average length of the waiting line,
(iii) the average time which an arrival spends in the system, and
(iv) the minimum cost service rate.

[**Ans.** (i) 12.4 min., (ii) 0.79 = one stenographer, (iii) 22.4 min.]