Marginal Probability Function. Let (X, Y) be a discrete two-dimensional *r.v.* which takes up countable number of values (x_i, y_j). Then the probability distribution of X is determined as follows :

$$p_X(x_i) = P(X = x_i)$$
$$= P(X = x_i \cap Y = y_1) + P(X = x_i \cap Y = y_2) + \dots \dots + P(X = x_i \cap Y = y_m)$$
$$= p_{i1} + p_{i2} + \dots + p_{ij} + \dots + p_{im} = \sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} p(x_i, y_j) = p_{i\cdot} \qquad \dots (5{\cdot}14a)$$

and is known as *marginal probability mass function or discrete marginal density function* X.

Also $\quad \sum_{i=1}^{n} p_{i\cdot} = p_{1\cdot} + p_{2\cdot} + \dots + p_{n\cdot} = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$

Similarly, we can prove that

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} p(x_i, y_j) = p_{\cdot j} \qquad \dots (5{\cdot}14\,b)$$

which is the *marginal probability mass function of* Y.

Conditional Probability Function

Definition. *Let (X, Y) be a discrete two-dimensional random variable. Then the conditional discrete density function or the conditional probability mass function of X, given Y = y, denoted by $f_{X|Y}(x \mid y)$, is defined as :*

$$p_{X|Y}(x \mid y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \text{ provided } P(Y = y) \neq 0 \qquad \dots (5{\cdot}14\,c)$$

Since for a fixed y,

$$\sum_i \frac{P(X = x_i, Y = y)}{P(Y = y)} = \frac{1}{P(Y = y)} \sum_i P(X = x_i, Y = y) = \frac{1}{P(Y = y)} P(Y = y) = 1,$$

it follows that the conditional discrete density function $f_{X|Y}(x \mid y)$ is a discrete density function, when considered as a function of the values of X.

The conditional probability mass function $p_{Y|X}(y \mid x)$ is similarly defined , *i.e.*,

$$p_{Y|X}(y \mid x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

A necessary and sufficient condition for the discrete random variables X and Y to be *independent* is :

$$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j) \text{ for all values } (x_i, y_j) \text{ of } (X, Y) \qquad \dots (5{\cdot}14\,d)$$

5·5·2. Two-dimensional Distribution Function

Definition. *The distribution function of the two-dimensional random variable* (X, Y) *is a real valued function F defined for all real x and y by the relation :*

$$F_{XY}(x, y) = P(X \leq x, Y \leq y). \qquad \dots (5{\cdot}15)$$

Properties of Joint Distribution Function

1. (*i*) For the real numbers a_1, b_1, a_2 and b_2,

$$P(a_1 < X \leq b_1,\ a_2 < Y \leq b_2) = F_{XY}(b_1, b_2) + F_{XY}(a_1, a_2) - F_{XY}(a_1, b_2) - F_{XY}(b_1, a_2)$$

[For proof see Example 5·46]

(*ii*) Let $a_1 < a_2,\ b_1 < b_2$, then $(X \leq a_1, Y \leq a_2) + (a_1 < X \leq b_1, Y \leq a_2) = (X \leq b_1, Y \leq a_2)$ and the events on the L.H.S. are mutually exclusive.

$\therefore\ F(a_1, a_2) + P(a_1 < X \leq b_1, Y \leq a_2) = F(b_1, a_2) \Rightarrow F(b_1, a_2) - F(a_1, a_2) = P(a_1 < X \leq b_1, Y \leq a_2)$

$\therefore\ F(b_1, a_2) \geq F(a_1, a_2)$ [since $P(a_1 < X \leq b_1, Y \leq a_2) \geq 0$]

Similarly it follows that : $F(a_1, b_2) - F(a_1, a_2) = P(X \leq a_1, a_2 \leq Y \leq b_2)$

$\therefore\ F(a_1, b_2) \geq F(a_1, a_2)$, which shows that $F(x, y)$ is monotonic non-decreasing function.

2. $F(-\infty, y) = 0 = F(x, -\infty), \quad F(-\infty, +\infty) = 1.$

3. If the density function $f(x, y)$ is continuous at (x, y), $\dfrac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$

5·5·3. Marginal Distribution Functions. From the knowledge of joint distribution function $F_{XY}(x, y)$, it is possible to obtain the individual distribution functions, $F_X(x)$ and $F_Y(y)$ which are termed as marginal distribution function of X and Y respectively with respect to the joint distribution function $F_{XY}(x, y)$.

$$F_X(x) = P(X \leq x) = P(X \leq x, Y < \infty) = \lim_{y \to \infty} F_{XY}(x, y) = F_{XY}(x, \infty) \qquad \dots (5{\cdot}16)$$

Similarly, $$F_Y(y) = P(Y \leq y) = P(X < \infty, Y \leq y) = \lim_{x \to \infty} F_{XY}(x, y) = F_{XY}(\infty, y) \qquad \dots (5{\cdot}16\,a)$$

$F_X(x)$ is termed as the marginal distribution function of X corresponding to the joint distribution function $F_{XY}(x, y)$ and similarly $F_Y(y)$ is called marginal distribution function of the random variable Y corresponding to the joint distribution function $F_{XY}(x, y)$.

In the case of jointly discrete random variables, the marginal distribution functions are given as :

$$F_X(x) = \sum_y P(X \leq x, Y = y), \quad \text{and} \quad F_Y(y) = \sum_x P(X = x, Y \leq y)$$

Similarly in the case of jointly continuous random variable, the marginal distribution functions are given as :

$$F_X(x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right\} dx, \qquad F_Y(y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right\} dy$$

5·5·4. Joint Density Function, Marginal Density Function. From the joint distribution function $F_{XY}(x, y)$ of two-dimensional continuous random variable, we get the joint probability density function by differentiation as follows :

$$f_{XY}(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \lim_{\delta x \to 0,\ \delta y \to 0} \frac{P(x \leq X \leq x + \delta x,\ y \leq Y \leq y + \delta y)}{\delta x\ \delta y} \qquad \dots (5{\cdot}17)$$

Or it may be expressed in the following way also :

"The probability that the point (x, y) will lie in the infinitesimal rectangular region, of area $dx\, dy$ is given by

$$P\left(x - \tfrac{1}{2}\,dx \le X \le x + \tfrac{1}{2}\,dx,\; y - \tfrac{1}{2}\,dy \le Y \le y + \tfrac{1}{2}\,dy\right) = dF_{XY}(x, y) \qquad \ldots (5{\cdot}17\,a)$$

and is denoted by $f_{XY}(x, y)\, dx\, dy$, where the function $f_{XY}(x, y)$ is called the joint probability density function of X and Y.

The marginal probability function of X and Y are given respectively as follows :

$$f_X(x) = \begin{cases} \sum\limits_{y} p_{XY}(x, y), & \text{(for discrete variables)} \\ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy, & \text{(for continuous variables)} \end{cases} \qquad \ldots (5{\cdot}17\,b)$$

and $$f_Y(y) = \begin{cases} \sum\limits_{x} p_{XY}(x, y), & \text{(for discrete variables)} \\ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx, & \text{(for continuous variables)} \end{cases} \qquad \ldots (5{\cdot}17\,c)$$

The marginal density functions of X and Y can be obtained in the following manner also :

$$\left.\begin{aligned} f_X(x) &= \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \\ \text{and} \quad f_Y(y) &= \frac{dF_Y(y)}{dy} = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \end{aligned}\right\} \qquad \ldots (5{\cdot}17\,d)$$

5·8·5. The Conditional Distribution Function and Conditional Probability Density Function. For two-diamensional random variable (X, Y), the joint distribution function $F_{XY}(x, y)$ for any real numbers x and y is given by :

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Now let A be the event $(Y \leq y)$ such that the event A is said to occur when Y assumes values upto and inclusive of y. Using conditional probabilities we may now write

$$F_{XY}(x, y) = \int_{-\infty}^{x} P(A \mid X = x)\, dF_X(x) \qquad \ldots (5{\cdot}18)$$

The *conditional distribution function* $F_{Y|X}(y \mid x)$ denotes the distribution function of Y when X has already assumed the particular value x. Hence

$$F_{Y|X}(y \mid x) = P(Y \leq y \mid X = x) = P(A \mid X = x)$$

Using this expression, the joint distribution function $F_{XY}(x, y)$ may be expressed in terms of the conditional distribution function as follows :

$$F_{XY}(x, y) = \int_{-\infty}^{x} F_{Y|X}(y \mid x)\, dF_X(x) \qquad \ldots (5{\cdot}18\ a)$$

Similarly $$F_{XY}(x, y) = \int_{-\infty}^{y} F_{X|Y}(x \mid y)\, dF_Y(y) \qquad \ldots (5{\cdot}18\ b)$$

The *conditional probability density function* of Y given X for two random variables X and Y which are jointly continuously distributed is defined as follows, for two real numbers x and y :

$$f_{Y|X}(y \mid x) = \frac{\partial}{\partial y} F_{Y|X}(y \mid x) \qquad \ldots (5{\cdot}19)$$

integration ...

5·5·7. Generalisation to n-Dimensional Random Variable. The concept of two-dimensional random variables and their joint and marginal distributions is § 5·5 to § 5·5·6 can be easily generalised to the case of n-dimensional random variable.

Joint and Marginal Probability Mass Function.

Let $(X_1, X_2, \ldots, X_n)$ be a discrete n-dimensional $r.v.$, assuming discrete values, in some region, say, R^n of the n-dimensional space. Then the joint $p.m.f.$ of $(X_1, X_2, \ldots, X_n)$ is defined as :

$$p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P\left[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right]$$

$$= P\left[\bigcap_{i=1}^{n} (X_i = x_i)\right] \qquad \ldots(5\cdot21)$$

where,

(i) $p(x_1, x_2, \ldots, x_n) \geq 0, \forall (x_1, x_2, \ldots, x_n) \in R^n$, and

(ii) $\sum\limits_{x_1, x_2, \ldots, x_n} p(x_1, x_2, \ldots, x_n) = 1$

The marginal $p.m.f.$ of any $r.v.$, say, X_i , is obtained on summing $p(x_1, x_2, \ldots, x_n)$, over the values of all other variables except X_i. Thus,

$$P_{X_i}(x_i) = \sum_{\substack{(x_1, x_2, \ldots, x_n) \\ \text{except } x_i}} p(x_1, x_2, \ldots, x_n) \qquad \ldots(5\cdot21\ a)$$

In particular, if $p(x_1, x_2, x_3)$ is the joint $p.m.f.$ of three $r.v.$'s X_1, X_2 and X_3, then the marginal $p.m.f.$ of, say, X_1 is given by :

$$P_{X_1}(x) = \sum_{x_2, x_3} p(x_1, x_2, x_3), \qquad \ldots(5\cdot21\ b)$$

and so on.

As, in the case of two random variables, *the r.v.'s $X_1, X_2, \ldots, X_n$ are independent if and only if their joint p.m.f. is equal to the product of their marginal p.m.f.'s, i.e., iff* :

$$p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = p_{X_1}(x_1) \cdot p_{X_2}(x_2) \cdots p_{X_n}(x_n) \qquad \ldots(5\cdot21\ c)$$

Joint and marginal Probability Density Function.

Let $(X_1, X_2, \ldots, X_n)$ be n-dimensional continuous $r.v.$ assuming all the values in some region, say, $R_1{}^n$ of the n-dimensional space. Then the joint $p.d.f.$ of $(X_1, X_2, \ldots, X_n)$ is given by :

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \lim_{dx_1 \to 0,\, dx_2 \to 0,\, \ldots\, dx_n \to 0} \frac{P\left[\bigcap\limits_{i=1}^{n} (x_i < X_i < x_i + dx_i)\right]}{dx_1 \cdot dx_2, \ldots, dx_n} \qquad \ldots(5\cdot21d)$$

where :

(i) $f(x_1, x_2, \ldots, x_n) \geq 0, \forall (x_1, x_2, \ldots, x_n) \in R_1{}^n$, and

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n = 1$

The marginal *p.d.f.* of any variable, say, X_i, is obtained on integrating the joint *p.d.f.* over the range of all the variables except X_i. Thus,

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_n \quad \ldots(5{\cdot}21\ e)$$

In particular, for three *r.v.'s* X_1, X_2, X_3 with joint *p.d.f.* $f(x_1, x_2, x_3)$, the marginal *p.d.f.* of, say, X_2 is given by :

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} \cdot \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_1\, dx_3 \quad \ldots(5{\cdot}21\ f)$$

and so on.

The necessary and sufficient condition for the independence of r.v.'s $X_1, X_2, \ldots, X_n$ is that their joint p.d.f. is the product of their marginal p.d.f.'s i.e.,

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \quad \ldots(5{\cdot}21\ g)$$

Example 5·30. *In the random placement of three balls in three cells, describe the possible outcomes of the experiment. Let X_i denote the number of balls in cell i; i = 1, 2, 3; and N, the number of cells occupied. Obtain the joint distribution of : (a) (X_1, N)* and *(b) (X_1, X_2).*

Solution. (*a*) Let the three balls be denoted by *a*, *b* and *c*. Then the possible outcomes of placing the three balls in three cells are as follows :

1. {*a* | *b* | *c*}	10. {*ac* | *b* | –}	19. {*b* | *ca* | –}
2. {*a* | *c* | *b*}	11. {*ac* | – | *b*}	20. {*b* | – | *ca*}
3. {*b* | *a* | *c*}	12. {– | *ac* | *b*}	21. {– | *b* | *ca*}
4. {*b* | *c* | *a*}	13. {*bc* | *a* | –}	22. {*c* | *ab* | –}
5. {*c* | *a* | *b*}	14. {*bc* | – | *a*}	23. {*c* | – | *ab*}
6. {*c* | *b* | *a*}	15. {– | *bc* | *a*}	24. {– | *c* | *ab*}
7. {*ab* | *c* | –}	16. {*a* | *bc* | –}	25. {*abc* | – | –}
8. {*ab* | – | *c*}	17. {*a* | – | *bc*}	26. {– | *abc* | –}
9. {– | *ab* | *c*}	18. {– | *a* | *bc*}	27. {– | – | *abc*}

Each of these arrangements represents a sample event, *i.e.*, a sample point. The sample space contains 27 points.

Let *N* denote the number of occupied cells. The favourable cases for *N* = 1 are at numbers 25, 26 and 27, *i.e.*, 3; for *N* = 2 are at numbers 7 to 24, *i.e.*, 18; and for *N* = 3 are at numbers 1 to 6, *i.e.*, 6. Accordingly, the probability distribution of *N* is :

$$P(N=1) = \frac{3}{27}, \quad P(N=2) = \frac{18}{27}, \quad P(N=3) = \frac{6}{27}.$$

Let X_1 denote the number of balls placed in the first cell. Then from the above table of sample points, we get

$$P(X_1=0) = \frac{8}{27}, \quad P(X_1=1) = \frac{12}{27}, \quad P(X_1=2) = \frac{6}{27} \quad \text{and } P(X_1=3) = \frac{1}{27}.$$

Example 5·31. *A random observation on a bivariate population (X, Y) can yield one of the following pairs of values with probabilities noted against them :*

For each observation pair	*Probability*
(1, 1) ; (2, 1) ; (3, 3) ; (4, 3)	$\frac{1}{20}$
(3, 1) ; (4, 1) ; (1, 2) ; (2, 2) ; (3, 2) ; (4, 2); (1, 3) ; (2, 3)	$\frac{1}{10}$

Find the probability that Y = 2 given that X = 4. Also find the probability that Y = 2. Examine if the two events X = 4 and Y = 2 are independent.

Solution. $P(Y=2) = P\{(1\ 2) \cup (2,2) \cup (3,2) \cup (4,2)\} = \frac{4}{10} = \frac{2}{5}$.

$$P(X=4) = P\{(4,1) \cup (4,2) \cup (4,3)\} = \frac{1}{10} + \frac{1}{10} + \frac{1}{20} = \frac{1}{4}$$

$$P(X=4, Y=2) = P\{(4,2)\} = \frac{1}{10}$$

$$P(Y=2 \mid X=4) = \frac{P(X=4 \cap Y=2)}{P(X=4)} = \frac{1/10}{1/4} = \frac{2}{5}$$

Now $\quad P(X=4) . P(Y=2) = \frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10} \quad = P(X=4 \cap Y=2)$

Hence the events $X=4$ and $Y=2$ are independent.

Problem 5.32. *The joint probability distribution of two random variables X and Y is given by : $P(X=0, Y=1) = \frac{1}{3}$, $P(X=1, Y=-1)\ \frac{1}{3}$, and $P(X=1, Y=1) = \frac{1}{3}$.*

Find (i) Marginal distributions of X and Y, and (ii) the conditional probability distribution of X given Y = 1.

Solution. $P(X=-1)$
$= \sum_y P(X=-1, Y=y)$
$= P(X=-1, Y=-1)$
$+ P(X=-1, Y=0)$
$+ P(X=-1, Y=1) = 0$

Similarly $P(X=0) = \frac{1}{3}$

and $\quad P(X=1) = \frac{2}{3}$

Y \ X	−1	0	1	Marginal Y
−1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal (X)	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Thus

Marginal distribution of X is :

Values of X, x :	−1	0	1
$P(X=x)$:	0	$\frac{1}{3}$	$\frac{2}{3}$

Marginal distribution of Y is :

Values of Y, y :	−1	0	1
$P(Y=y)$:	$\frac{1}{3}$	0	$\frac{2}{3}$

(*ii*) The conditional probability distribution of X given Y is :

$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$. Now

$P(X=-1 \mid Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0$, $P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus the conditional distribution of X given Y = 1 is :

Values of X = x	−1	0	1
$P(X=x \mid Y=1)$	0	$\frac{1}{2}$	$\frac{1}{2}$

Example 5·33. *For the adjoining bivariate probability distribution of X and Y, find :*

(i) $P(X \le 1, Y = 2)$,

(ii) $P(X \le 1)$,

(iii) $P(Y \le 3$, *and*

(iv) $P(X < 3, Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(iv) $P(X < 3, Y \le 4) = P(X = 0, Y \le 4) + P(X = 1, Y \le 4) + P(X = 2, Y \le 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}.$$

Example 5·34. *For the joint probability distribution of two random variables X and Y given below :*

X \ Y	1	2	3	4	Total
1	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{8}{36}$
4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{9}{36}$
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1

Find (i) the marginal distributions of X and Y, and (ii) conditional distribution of X given the value of Y = 1 and that of Y given the value of X = 2.

Solution. The marginal distribution of X is defined as :

$$P(X = x) = \sum_y P(X = x, Y = y)$$

$$\therefore \quad P(X=1) = \sum_y P(X=1, Y=y)$$

$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}.$$

Similarly $P(X=2) = \sum_y P(X=2, Y=y) = \frac{9}{36}$; $P(X=3) = \sum_y P(X=3, Y=y) = \frac{8}{36}$

and $P(X=4) = \sum_y P(X=4, Y=y) = \frac{9}{36}$.

Similarly, we can obtain the marginal distribution of Y.

MARGINAL DISTRIBUTION OF X

Values of X, x	1	2	3	4
$P(X=x)$	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

MARGINAL DISTRIBUTION OF Y

Values of Y, y	1	2	3	4
$P(Y=y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

(*ii*) The conditional probability function of X given Y is defined as follows :

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}. \text{ Therefore}$$

$$\therefore \quad P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{4/36}{11/36} = \frac{4}{11}$$

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Hence the conditional distribution of X given $Y=1$ is :

x :	1	2	3	4
$P(X=x \mid Y=1)$:	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

Similarly, we can obtain the conditional distribution of Y for $X=2$ as given below :

y :	1	2	3	4
$P(Y=y \mid X=2)$:	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$

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Example 5·38. *Suppose that two-dimensional continuous random variable* (X, Y) *has joint p.d.f. given by :*

$$f(x, y) = \begin{cases} 6x^2y, 0 < x < 1, 0 < y < 1 \\ 0, \textit{elsewhere} \end{cases}$$

(i) Verify that $\int_0^1 \int_0^1 f(x, y)\, dxdy = 1.$

(ii) Find $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right)$, $P(X + Y < 1)$, $P(X > Y)$ *and* $P(X < 1 \mid Y < 2)$.

Solution. (i)

$$\int_0^1 \int_0^1 f(x, y)\, dxdy = \int_0^1 \int_0^1 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^1 dx = \int_0^1 3x^2 dx = \left|x^3\right|_0^1 = 1$$

$$(ii)\ P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right) = \int_0^{3/4} \int_{1/3}^1 6x^2y\, dxdy + \int_0^{3/4} \int_1^2 0.\, dxdy$$

$$= \int_0^{3/4} 6x^2 \left|\frac{y^2}{2}\right|_{1/3}^1 dx = \frac{8}{9} \int_0^{3/4} 3x^2\, dx = \frac{8}{9}\left|x^3\right|_0^{3/4} = \frac{3}{8}.$$

$$P(X + Y < 1) = \int_0^1 \int_0^{1-x} 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^{1-x} dx = \int_0^1 3x^2(1-x)^2 dx = \frac{1}{10}$$

$$P(X > Y) = \int_0^1 \int_0^x 6x^2y\, dxdy = \int_0^1 3x^2 \left|y^2\right|_0^x dx = \int_0^1 3x^4 dx = \frac{3}{5}.$$

$$P(X < 1 \mid Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)}$$

where $$P(X < 1 \cap Y < 2) = \int_0^1 \int_0^1 6x^2\, y\, dxdy + \int_0^1 \int_1^2 0.\, dxdy = 1$$

and $$P(Y < 2) = \int_0^1 \int_0^2 f(x,y)\, dxdy = \int_0^1 \int_0^1 6x^2\, y\, dxdy + \int_0^1 \int_1^2 0.\, dxdy = 1$$

$\therefore$ $$P(X < 1 \mid Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)} = 1.$$

Example 5·39. *The joint probability density function of a two-dimensional random variable (X, Y) is given by :*

$$f(x, y) = \begin{cases} 2\,; 0 < x < 1, 0 < y < x\,; \\ 0, \textit{elsewhere} \end{cases}$$

(i) Find the marginal density functions of X and Y.

(ii) Find the conditional density function of Y given X = x and conditional density function of X given Y = y.

(iii) Check for independence of X and Y.

Solution. Evidently $f(x,y) \geq 0$ and $\int_0^1 \int_0^x 2\, dx\, dy = 2 \int_0^1 x\, dx = 1.$

(i) The marginal *p.d.f.*'s of X and Y are given by :

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^x 2dy = 2x, 0 < x < 1 \\ 0, \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_y^1 2dx = 2(1-y), 0 < y < 1 \\ 0, \text{ elsewhere} \end{cases}$$

(*ii*) The conditional density function of Y given X, $(0 < x < 1)$ is :

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}, 0 < y < x.$$

The conditional density function of X given Y, $(0 < y < 1)$ is :

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)}, y < x < 1$$

(*iii*) Since $f_X(x) f_Y(y) = 2(2x)(1-y) \neq f_{XY}(x, y)$, X and Y are not independent.

Example 5·40. *The joint p.d.f. of two random variables X and Y is given by :*

$$f(x, y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}; 0 \le x < \infty, 0 \le y < \infty$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for X = x.

Solution. Marginal *p.d.f.* of X is given by :

$$f_X(x) = \int_0^{\infty} f(x, y)\, dy = \frac{9}{2(1+x)^4} \int_0^{\infty} \frac{(1+y)+x}{(1+y)^4} dy$$

$$= \frac{9}{2(1+x)^4} \int_0^{\infty} \{(1+y)^{-3} + x(1+y)^{-4}\}\, dy$$

$$= \frac{9}{2(1+x)^4} \left(\left| \frac{-1}{2(1+y)^2} \right|_0^{\infty} + x \left| \frac{-1}{3(1+y)^3} \right|_0^{\infty} \right)$$

$$= \frac{9}{2(1+x)^4} \cdot \left(\frac{1}{2} + \frac{x}{3} \right) = \frac{3}{4} \cdot \frac{3+2x}{(1+x)^4}; \ 0 < x < \infty$$

Since $f(x, y)$ is symmetric in x and y, the marginal *p.d.f.* of Y is given by :

$$f_Y(y) = \int_0^{\infty} f(x, y)\, dx = \frac{3}{4} \cdot \frac{3+2y}{(1+y)^4}; \ 0 < y < \infty$$

The conditional distribution of Y for X = x is given by :

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \times \frac{4(1+x)^4}{3(3+2x)} = \frac{6(1+x+y)}{(1+y)^4(3+2x)}; 0 < y < \infty$$