Definition. *A random variable X is said to have a continuous rectangular (uniform) distribution over an interval* (a, b), *i.e.,* $(-\infty < a < b < \infty)$, *if its p.d.f. is given by :*

$$f(x\,;a, b) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (9{\cdot}19)$$

Remarks 1. a and b, $(a < b)$ are the two parameters of the distribution. The distributin is called uniform distribution on (a, b) since it assumes a constant (uniform) value for all x in (a, b).

2. The distribution is also known as rectangular distribution, since the curve $y = f(x)$ describes a rectangle over the x-axis and between the ordinates at $x = a$ and $x = b$.

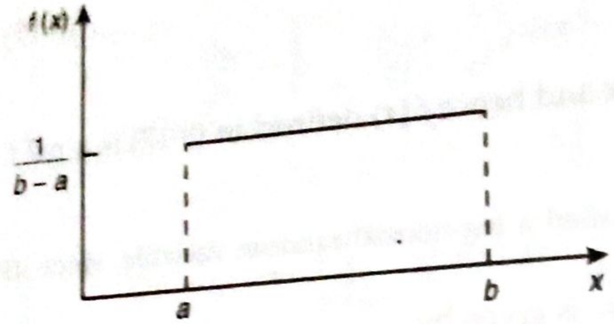
3. A uniform or rectangular variate X on the interval (a, b) is written as : $X \sim U[a, b]$ or $X \sim R[a, b]$.

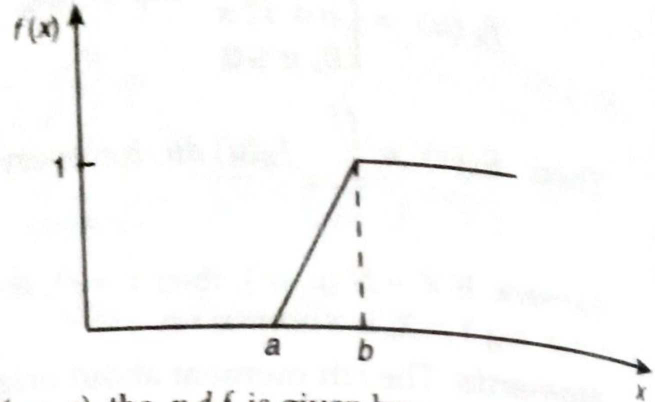
4. The cumulative distribution function $F(x)$ is given by :

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \ge b \end{cases} \qquad \ldots (9{\cdot}19a)$$

Since $F(x)$ is not continuous at $x = a$ and $x = b$, it is not differentiable at these points. Thus $\frac{d}{dx}F(x) = f(x) = \frac{1}{b-a} \neq 0$, exists everywhere except at the points $x = a$ and $x = b$ and consequently p.d.f. $f(x)$ is given by (9·19).

5. The graphs of uniform $p.d.f.$ $f(x)$ and the corresponding distribution function $F(x)$ are given below.





6. For a rectangular or uniform variate X in $(-a, a)$, the $p.d.f.$ is given by :

$$f(x) = \begin{cases} \dfrac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases} \qquad \cdots (9\cdot19b)$$

9·3·1. Moments of Rectangular Distribution. Let $X \sim U[a, b]$.

$$\mu'_r = \int_a^b x^r f(x)\, dx = \frac{1}{b-a} \int_a^b x^r\, dx = \frac{1}{b-a}\left(\frac{b^{r+1} - a^{r+1}}{r+1}\right) \qquad \cdots (9\cdot20)$$

In particular

$$\text{Mean } = \mu_1' = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \boxed{\frac{b+a}{2}} \qquad \cdots (9\cdot20a)$$

and

$$\mu_2' = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{1}{3}(b^2 + ab + a^2)$$

$$\therefore \quad \text{Variance} = \mu_2' - \mu_1'^2 = \frac{1}{3}(b^2 + ab + a^2) - \left\{\frac{1}{2}(b+a)\right\}^2 = \boxed{\frac{1}{12}(b-a)^2} \qquad \cdots (9\cdot20b)$$

9·3·2. M.G.F. of Rectangular Distribution is given by :

$$M_X(t) = \int_a^b e^{tx} f(x)\, dx = \int_a^b \frac{e^{tx}}{b-a}\, dx = \frac{e^{bt} - e^{at}}{t\,(b-a)}, \; t \neq 0 \qquad \cdots (9\cdot20c)$$

9·3·3. Characteristic Function of Rectangular Distribution is given by :

$$\phi_X(t) = \int_a^b e^{itx}\, dx = \frac{e^{ibt} - e^{iat}}{it\,(b-a)}, \; t \neq 0. \qquad \cdots (9\cdot20d)$$

9·3·4. Mean Deviation about Mean, η of Rectangular Distribution is given by :

$$\eta = E\left|X - \text{Mean}\right| = \int_a^b \left|x - \text{Mean}\right| f(x)\, dx$$

$$= \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{1}{b-a}\int_{-(b-a)/2}^{(b-a)/2} |\,t\,|\, dt, \text{ where } t = x - \frac{a+b}{2}$$

$$= \frac{1}{b-a}\cdot 2\int_0^{(b-a)/2} t\, dt = \frac{b-a}{4} \qquad \cdots (9\cdot20e)$$

Example 9·21. *If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$, find* $P(X < 0)$.

Solution. Let $X \sim U[a, b]$, so that $p(x) = \frac{1}{b-a}$, $a < x < b$. We are given :

$$\text{Mean} = \frac{1}{2}(b+a) = 1 \Rightarrow b + a = 2 \text{ and } \text{Var}(X) = \frac{1}{12}(b-a)^2 = \frac{4}{3} \Rightarrow b - a = \pm 4.$$

Solving, we get $a = -1$ and $b = 3$; $(a < b)$. $\therefore \; p(x) = \frac{1}{4}; -1 < x < 3$

$$P(X < 0) = \int_{-1}^{0} p(x)\, dx = \frac{1}{4} \left| x \right|_{-1}^{0} = \frac{1}{4}.$$

Example 9·22. *Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes ?*

Solution. Let the *r.v.* X denote the waiting time (in minutes) for the next train. Under the assumption that a man arrives at the station at random, X is distributed uniformly on (0, 30), with *p.d.f.*,

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$$

The probability that he has to wait at least 20 minutes is given by :

$$P(X \geq 20) = \int_{20}^{30} f(x)\, dx = \frac{1}{30} \int_{20}^{30} 1 . \, dx = \frac{1}{30}(30 - 20) = \frac{1}{3}.$$

Example 9·23. *If X has a uniform distribution in [0, 1], find the distribution (p.d.f.) of* $-2 \log X$. *Identify the distribution also.*

Solution. Let $Y = -2 \log X$. Then the distribution function G of Y is given by :

$$G_Y(y) = P(Y \leq y) = P(-2 \log X \leq y) = P(\log X \geq -y/2) = P(X \geq e^{-y/2})$$

$$= 1 - P(X \leq e^{-y/2}) = 1 - \int_0^{e^{-y/2}} f(x)\, dx = 1 - \int_0^{e^{-y/2}} 1 . \, dx = 1 - e^{-y/2}$$

$$\therefore \quad g_Y(y) = \frac{d}{dy} G(y) = \frac{1}{2} e^{-y/2}, 0 < y < \infty \qquad \ldots (*)$$

[$\because$ as X ranges in (0, 1), $Y = -2 \log X$ ranges from 0 to ∞]

Remark. This example illustrates that if $X \sim U[0, 1]$, then $Y = -2 \log X$, has an exponential distribution with parameter $\theta = \frac{1}{2}$, [*c.f.* § 9·8] or $Y = -2 \log X$, has chi-square distribution with $n = 2$ degrees of freedom [*c.f.* Chapter 15].

Example 9·24. *Show that for rectangular distribution :* $f(x) = \frac{1}{2a}$, $-a < x < a$, *the m.g.f. about origin is* $\frac{1}{at}$ *(sinh at). Also show that moments of even order are given by :*

$$\mu_{2n} = \frac{a^{2n}}{(2n+1)}.$$

Solution. M.G.F. about origin is given by :

$$M_X(t) = E(e^{tX}) = \int_{-a}^{a} e^{tx} f(x)\, dx = \frac{1}{2a}\int_{-a}^{a} e^{tx}\, dx = \frac{1}{2a}\left|\frac{e^{tx}}{t}\right|_{-a}^{a} = \frac{1}{2at}(e^{at} - e^{-at}) = \frac{\sinh at}{at}$$

$$= \frac{1}{at}\left\{at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \ldots + \frac{(at)^{2n+1}}{(2n+1)!} + \ldots\right\} = 1 + \frac{a^2 t^2}{3!} + \frac{a^4 t^4}{5!} + \ldots + \frac{a^{2n} t^{2n}}{(2n+1)!} + \ldots$$

Since there are no terms with odd powers of t in $M(t)$, all moments of odd order about origin vanish, *i.e.*, μ'_{2n+1} (about origin) $= 0$.

In particular, μ_1' (about origin) $= 0$, *i.e.*, mean $= 0$

Thus μ_r' (about origin) $= \mu_r$ (Since mean is origin.)

Hence $\mu_{2n+1} = 0;\ n = 0, 1, 2, \ldots$, *i.e.*, all moments of odd order about mean vanish.

The moments of even order are given by :

$$\mu_{2n} = \text{coefficient of } \frac{t^{2n}}{(2n)!} \text{ in } M(t) = \frac{a^{2n}}{(2n+1)}.$$

Example 9·25. *If X_1 and X_2 are independent rectangular variates on* $[0, 1]$, *find the distributions of* : (*i*) X_1/X_2, (*ii*) $X_1 X_2$, (*iii*) $X_1 + X_2$, *and* (*iv*) $X_1 - X_2$.

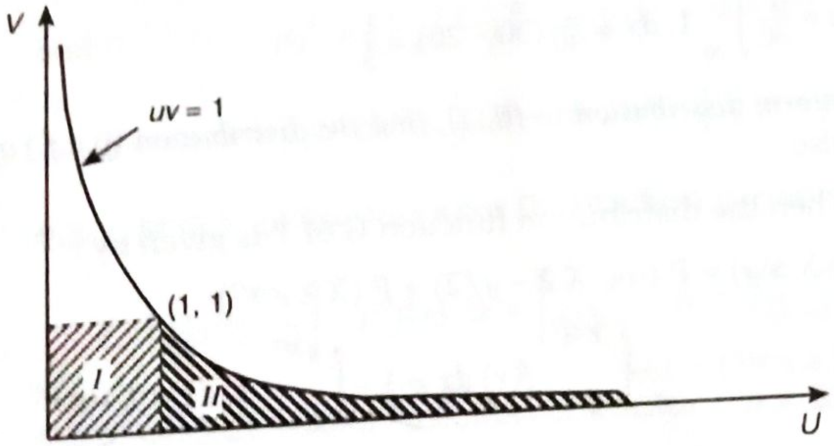
Solution. We are given : $f_{X_1}(x_1) = f_{X_2}(x_2) = 1;\ 0 < x_1 < 1, 0 < x_2 < 1.$

Since X_1 and X_2 are independent, their joint *p.d.f.* is :

$$f(x_1, x_2) = f(x_1) f(x_2) = 1$$

(*i*) Let us transform to : $u = \dfrac{x_1}{x_2}$, $v = x_2$, *i.e.*, $x_1 = uv,\ x_2 = v$

$$J = \frac{\partial(x_1, x_2)}{\partial(u, v)} = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$$



$x_1 = 0$ maps to $u = 0, v = 0$

$x_1 = 1$ maps to $uv = 1$

(Rectangular hyperbola)

$x_2 = 0$ maps to $v = 0$, and

$x_2 = 1$ maps to $v = 1$.

The joint *p.d.f.* of U and V becomes :

$$g(uv) = f(x_1, x_2) \mid J \mid = v;\ 0 < u < \infty, 0 < v < \infty$$

To obtain the marginal distribution of U, we have to integrate out v.

In region (I) : $$g_1(u) = \int_0^1 v\, dv = \left|\frac{v^2}{2}\right|_0^1 = \frac{1}{2},\ 0 \le u \le 1$$

In region (II) : $$g_1(u) = \int_0^{1/u} v\, dv = \left|\frac{v^2}{2}\right|_0^{1/u} = \frac{1}{2u^2},\ 1 < u < \infty$$

Hence the distribution $U = \dfrac{X_1}{X_2}$ is given by : $$g(u) = \begin{cases} \dfrac{1}{2}, & 0 \le u \le 1 \\[2mm] \dfrac{1}{2u^2}, & 1 < u < \infty \end{cases}$$

(ii) Let $u = x_1 x_2,\ v = x_1 \Rightarrow x_1 = v,\ x_2 = \frac{u}{v}$ and $J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$

$x_1 = 0$ maps to $v = 0$, $x_1 = 1$ maps to $v = 1$

$x_2 = 0$ maps to $u = 0$, and $x_2 = 1$ maps to $u = v$

Moreover, $v = \frac{u}{x_2} \Rightarrow v \geq u$ (Since $0 < x_2 < 1$),

The joint p.d.f. of U and V is : $g(u, v) = f(x_1, x_2) \mid J \mid = \frac{1}{v};\ 0 < u \leq v < 1$

$$\therefore \quad g(u) = \int_u^1 g(u, v)\, dv = \int_u^1 \frac{1}{v}\, dv = \left|\log v\right|_u^1 = -\log u,\ 0 < u < 1$$

(iii) and (iv). Let $u = x_1 + x_2,\ v = x_1 - x_2$

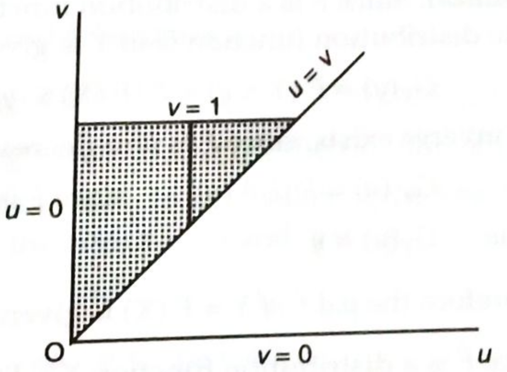
i.e., $x_1 = \frac{u+v}{2}$ $\quad$ $x_2 = \frac{u-v}{2}$

$x_1 = 0 \Rightarrow u + v = 0$ i.e., $v = -u$

$x_2 = 0 \Rightarrow u - v = 0$ i.e., $v = u$

$x_1 = 1 \Rightarrow u + v = 2$

$x_2 = 1 \Rightarrow u - v = 2$



and $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

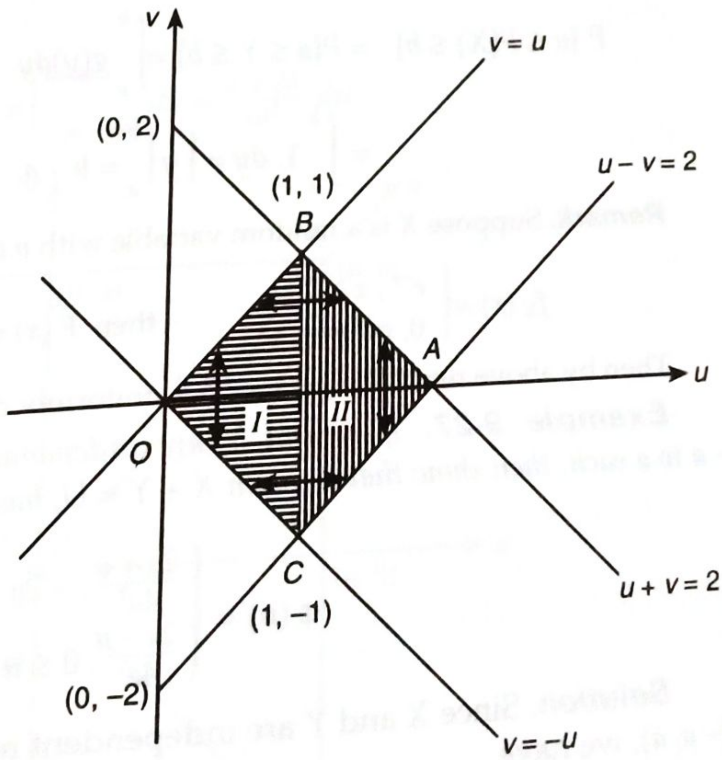
$$\therefore \quad g(u, v) = f(x_1, x_2) \mid J \mid = \frac{1}{2},\ 0 < u < 2,\ -1 < v < 1$$

In region (I)

$$g_1(u) = \int_{-u}^{u} \frac{1}{2}\, dv = \frac{1}{2}\left|v\right|_{-u}^{u} = u$$

and in region (II),

$$g_2(u) = \int_{u-2}^{2-u} \frac{1}{2}\, dv = \frac{1}{2}\left|v\right|_{u-2}^{2-u} = 2 - u$$



Distribution $U = X_1 + X_2$, is given by :

$$g(u) = \begin{cases} u, & 0 < u < 1 \\ 2 - u, & 1 < u < 2 \end{cases}$$

For the distribution of V, we split the region as : OAB and OAC