Unit - III

Markov Process and Markov Chains :-

Classification - Stationary process - Markov process - Poisson process, discrete parameter - Markov chains - Transition probabilities, Chapman Kolmogorov equations - Limiting distributions -

Markov Analysis:- Markov Process (or chain) is a stochastic (or random) process which has the property that the probability of transition from a given state to any future state depends only on the present state and not on the manner in which it was reached.

The nature of Markov processes can be easily understood by considering the situation of a Car-Rental Company. Suppose a Car-Rental Company is running agencies in different cities. A car sent to one city may return to any city where the company's agency is available. If this situation is considered as a Markov process, then the different rental cities would be states. A particular transition probability P_{ij} would be the probability that a car rented to city i would return to city j, where j may be equal to i. The mathematical structure of this problem is to determine expected long term fraction of cars at each city and the mean number of trips a car would make starting from city i, before returning to that location.

Markov process is widely used in examining and predicting the behaviour of customers in terms of their brand loyalty and their switching pattern to the other brands.

Markov processes are also used in the study of equipment maintenance and failure problems analysing accounts recievable that will ultimately become bad debts. It is also used to study the stock market price movements.

Stochastic (Random) Process:-

Defn:- A stochastic (or random) process is defined as a family of random variables $\{X(t_n) : n = 1, 2, 3 --\}$. The random variable $X(t)$ stands for the observation at time t. The number of states n may be finite or infinite depending upon the time range

For example, let us consider the poisson distribution

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 1, 2, 3 ---$$

This distribution represents a stochastic (or random) process with infinite number of states. In this example, the random variable n denotes the number of occurences between the time interval 0 and t (assuming that the system starts at 0 times). Thus the states of the system at any time t are given by $n = 0, 1, 2 --$.

or Stochastic Process (or Chance or Random Process)
It is a family of random variables $\{X(t) | t \in T\}$ defined on a common sample space S and indexed by the parameter t, which varies on an index set T.

The value assumed by the random variables $X(t)$ are called states, and the set of all possible values from the state space of the process is denoted by I.
If the state space is discrete, the stochastic process is known as a Chain. In this case the state space is assumed to be $I = \{0, 1, 2 ---\}$. Thus a (finite) stochastic process consists of a sequence of experiments in which each experiment has a finite number of outcomes with given probabilities.

Example:- Jobs arrive at random points in time, queue for service and depart after service Completion.
If N_K denotes the number of jobs at the time of departure of the K^{th} job (customer) then $\{N_K | K=1,2,--\}$ is a stochastic process.

A Markov (memoryless) process is a stochastic process whose entire past history is summarized in its current (present) state ie, the 'future' is independent of its 'past'.

Markov Chain:- Defn - A stochastic (or random) system is called a Markov process if the occurrence of a future state depends on the immediately preceding state and only on it:

Thus if $t_0 < t_1 < --- < t_n$ represents the points in time scale then the family of random variables $\{X(t_n)\}$ is said to be a Markov process provided it holds the Markovian Property:

$P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1} \cdots X(t_0) = x_0\} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$
for all $X(t_0), X(t_1) \cdots X(t_n)$.

Markov Process is a sequence of n experiments in which each experiment has n possible outcomes $x_1, x_2 \cdots x_n$. Each individual outcome is called a state and the probability (that a particular outcome occurs) depends only on the probability of the outcome of the preceding experiment.
The simplest of the Markov processes is discrete and constant over time. It is used when the sequence of experiments is completely described in terms of its states (possible outcomes). There is a finite set of states numbered $1, 2, \cdots n$. and this process can be only in one state at a prescribed time. A system is said to discrete in time if it is examined at regular intervals e.g. daily, weekly, monthly, or yearly.

Transition Probability :- Defn :- The Probability of moving from one state to another or remaining in the same state during a single time period is called the transition probability.

Mathematically, the probability

$$P_{x-1, x_n} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

is called the transition Probability.

This represents the conditional probability of the system which is now in state x_n at time t_n, provided that it was previously in state x_{n-1} at time t_{n-1}. Sometimes this probability is known as one step transition probability, because it describes the system during the time interval (t_{n-1}, t_n)

Since each time a new result or outcome occur, the process is said to have stepped or incremented one step. Each step represented a time period or any other condition which would result in another possible outcome. The symbol n is used to indicate the number of steps or increments. for example, if $n=0$, then it represents the initial state.

<u>Transition Probability Matrix</u>:- The transition probabilities can be arranged in a matrix form and such a matrix is called a 'one-step transition probability matrix' denoted by

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & & & \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix}$$

The matrix P is a squared matrix whose each element is non-negative and sum of the elements of each row is unity ie,

$$\sum_{j=1}^{m} P_{ij} = 1; \quad i = 1, 2 \cdots m \quad \text{and } 0 \leq P_{ij} \leq 1$$

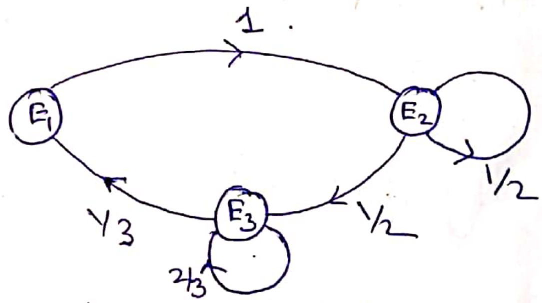
In general, any matrix P, whose elements are non-negative and sum of the elements ~~of each row is unity~~, ie, either in each row or column is unity, is called a transition matrix or a probability matrix. Thus a transition matrix is a square stochastic matrix (since number of rows is equal to the number of columns in the matrix) and therefore, it gives the complete description of the Markov process.

Diagrammatic Representation of Transition Probabilities –
The transition probabilities can also be represented by two types of diagrams:

(1) Transition Diagram :– Transition diagram shows the transition probabilities or shifts that can occur in any particular situation. Such a digram is given in fig.

The arrows from each state indicate the possible states to which a process can move from the given state. The matrix of transition probabilities which corresponds to above diagram is as given below:

$$P = \begin{array}{c c} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \end{bmatrix} \end{array}$$



A zero element in the above matrix indicates that the transition is not possible.

(2) Probability tree diagram :– As the name implies, this diagram emphasizes the probabilities and their movement from one step to another step, along with all possible branches or paths that may connect the outcomes over a period of time. Tree diagram can be explained by the following example,

Q.1 Two manufacturers A and B are Competing with (7)
each other in a restricted market. over the years,
A's customers have exhibited a high degree of loyalty
as measured by the fact that customers using A's
Product 80% of time. Also former customers purchasing
the product from B have switched back to A's 60%
of time.

a) Construct and interpret the state transition matrix
in terms of (i) retention and loss (ii) retention and
gain.

b) Calculate the probability of a customer purchasing
A's product at the end of the second period.

Soln:- (a) The transition probabilities can be arranged
in a matrix form as shown below:

Next purchase $(n=1)$

$$\text{Present } P = \text{Purchase } (n=0) \quad \begin{array}{c|cc} & A & B \\ \hline A & 0.80 & 0.20 \\ B & 0.60 & 0.40 \end{array}$$

Retention and Gains ↓

— Retention and Losses →

obviously, the probability of a customer's purchase
at the next step $n=1$ (next purchase depends upon
the product which a customer is having at present
$(n=0)$. Each probability in the above matrix must
therefore be a Conditional probability for passing
from one state to another.

mathematically, Conditional probabilities in the above
matrix can be stated as.

(i) $$P(A_0|A_1) = P_{11} = 0.80$$

This indicates that the probability that the customers
now using A's product at $n=0$ (present purchase) will
again purchase A's product at $n=1$ (next purchase) is 0.80. This means retention to A's
Product.

(ii) $P(B_0 | A_1) = P_{21} = 0.60$

This indicates that probability that the customer now using B's product at $n=0$ (present purchase) will purchase A's product at $n=1$ (next purchase) is 0.60. This means loss to B's product.

(iii) $P(A_0 | B_1) = P_{12} = 0.20$

This indicates that the probability that the customers now using A's product at $n=0$ (present purchase) will purchase B's product at $n=1$ (next purchase) is 0.20. This means loss to A's product.

(iv) $P(B_0 | B_1) = P_{22} = 0.40$

This indicates that the probability that the customers now using B's product at $n=1$ (present purchase) will purchase B's product at $n=1$ (next purchase) is 0.20. This means retention to B's product.

(b) The transition probabilities can be represented by two types of diagrams:

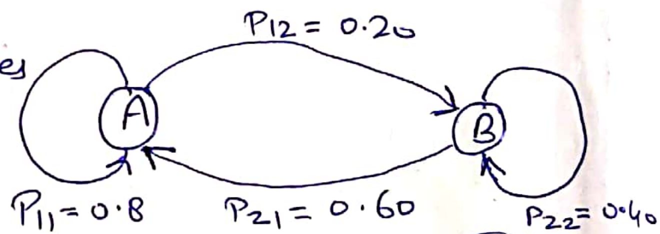
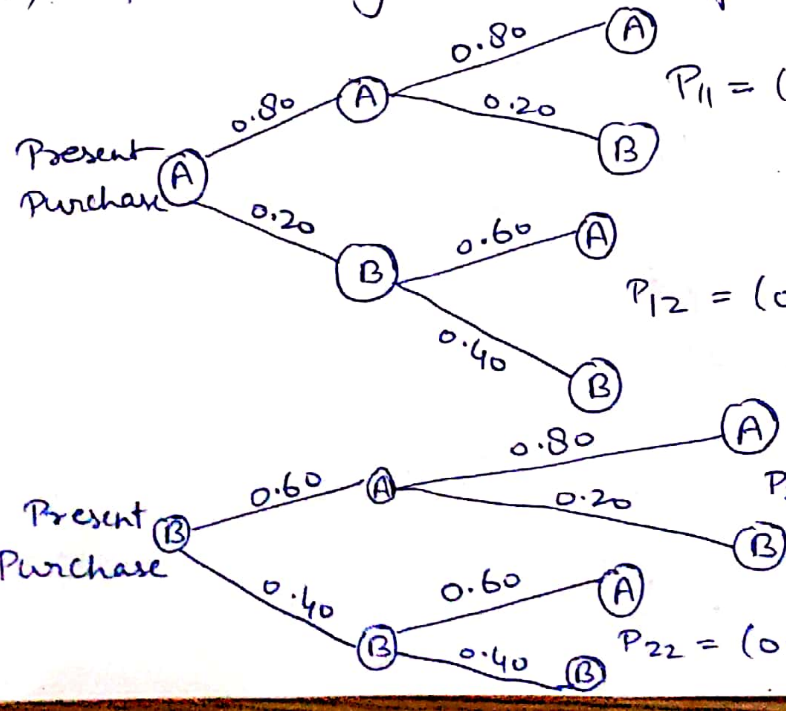


Fig 2

(i) transition diagram as shown in Fig 2 and

(ii) Probability tree diagram as show in Fig (3)



$P_{11} = (0.80)(0.80) + (0.20)(0.60)$
$= 0.76$

$P_{12} = (0.80)(0.20) + (0.20)(0.40)$
$= 0.24$

$P_{21} = (0.60)(0.80) + (0.40)(0.60)$
$= 0.72$

$P_{22} = (0.60)(0.20) + (0.40)(0.40) = 0.28$

Fig. 2, nodes indicates states and arrows represent the transition probabilities between states.

<u>Probability Computations:-</u> If we begin with a customer's purchase of A's product in the state E_1 at $n=0$, then $P_1(0) = 1$ and $P_2(0) = 0$ so

$$R(0) = (1\ 0)$$

After the first transition, the row vector of state probabilities $R(1)$ which describes all possible outcomes at $n=1$ is given by

$$R(1) = R(0)\,P = (1\quad 0)\begin{pmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{pmatrix} = (0.80\quad 0.20)$$

This means that if the present state is E_1, the probability (that the present state is E_1) is $P_1(1) = 0.80$ and (that the next state is E_2) is $P_2(1) = 0.20$.
In other words, the probability of a customer using A's product at the end of step 1 is 80% and there are 20% chances that the customer will switch over to B's product at the end of step 1.

The probability that a customer using A's product in the state E_1 at $n=0$ also uses A's product in the state E_2 at $n=2$ can be obtained by calculating state row vector $R(2)$ of state probabilities which describes all possible outcoms in step $n=2$.

$$R(2) = R(1)\,P = (0.80\quad 0.20)\begin{pmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{pmatrix} = (0.76\quad 0.24)$$

This indicates that if the present state is E_1 at $n=0$, 2-step later (ie, $n=2$) the probability of being in state E_1 is $P_1(2) = 0.76$ and in state E_2 is $P_2(2) = 0.24$. Thus the probability of A's product after the end of 2-step is 76% and that of B's product is 24%.
In a similar manner, if the present state is E_2 then $R(2) = (0.72\quad 0.28)$ as obtained earlier.

First order and Higher order Markov Process –

The first order Markov Process is based on the following three assumptions:

(i) The set of possible outcomes is finite.

(ii) The probability of the next outcome (state) depends only on the immediately preceding outcome

(iii) The transition probabilities are constant over time.

The second order Markov process assumes that the probability of the next outcome (state) may depend on the two previous outcomes. Likewise, a third order Markov process assumes that the probability of the next outcome (state) can be calculated by obtaining and taking account of the past three outcomes. But, in this chapter, we shall discuss only first order Markov Process.

n-step Transition Probabilities :– suppose the system which occupies state E_i at time $t=0$ then we may be interested in finding out the probability that the system moves to state E_j at time $t=n$ (these time periods are sometimes referred to as number of steps). If the n-step transition probability is denoted by $P_{ij}^{(n)}$, then these transition probabilities can be represented in matrix form as given below

$$
\begin{array}{c c}
 & \begin{array}{cccc} E_1 & E_2 & -- & E_m \end{array} \\
\begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_m \end{array} &
\begin{bmatrix}
P_{11}^{(n)} & P_{12}^{(n)} & -- & P_{1m}^{(n)} \\
P_{21}^{(n)} & P_{22}^{(n)} & -- & P_{2m}^{(n)} \\
\vdots & & & \\
P_{m1}^{(n)} & P_{m2}^{(n)} & -- & P_{mm}^{(n)}
\end{bmatrix}
\end{array}
$$

Here $P_{21}^{(n)}$, for example, means the probability that the system which occupies state E_2 will move to state E_1 after n steps.

Let $P_i(n)$ be the probability that the system occupies state E_i will move to state E_j in one transition. It should be noted that the transition probability P_{ij} is independent of time whereas the absolute probability $P_i(n)$ depends on time. If the number of possible states be m, then

$$\sum_{i=1}^{m} P_i(n) = 1 \quad \text{and} \quad \sum_{j=1}^{m} P_{ij} = 1 \text{ for all } i$$

If all the state probabilities are known at time $t=n$ then the state probabilities at time $t=n+1$ can be determined by the equation

$$P_j(n+1) = \sum_{i=1}^{m} P_i(n) P_{ij} \; : n = 0, 1, 2 - -$$

In other words, the probability of being in state E_j at time $t=n+1$ is equal to the probability of being in state E_i at time $t=n$ multiplied by the probability of a transition from state E_i to state E_j for all values of i.

To make the procedure more clear we may rewrite the equations for each state probability at time $t = n+1$ as follows:

$$P_1(n+1) = P_1(n)P_{11} + P_2(n)P_{21} + - - - + P_m(n)P_{m1}$$

$$P_2(n+1) = P_1(n)P_{12} + P_2(n)P_{22} + - - - + P_m(n)P_{m2}$$

$$\vdots$$

$$P_m(n+1) = P_1(n)P_{1m} + P_2(n)P_{2m} + - - - + P_m(n)P_{mm}$$

This system of equations can be written in matrix form as

$$R(n+1) = R(n)P \qquad \text{---} (1)$$

where $R(n+1)$ is the row vector of state probabilities at time $t = n+1$, $R(n)$ is the row vector of state probabilities at time $t=n$ and P is the matrix of transition probabilities

If the state probabilities at time $t=0$ are Known, these can be found at any time by solving the matrix equation (1) that is

$$R(1) = R(0)P \ , \ R(2) = R(1)P = R(0)P^2$$

$$R(3) = R(2)P = R(0) P^3 - -$$

$$R(n) = R(n-1)P = R(0) P^n.$$

Markov Chain: – Let $P_j(0) (j=0,1,2--)$ be the absolute Probability such that the system be in state E_j at time t_0, where E_j $(j=0,1,2---)$ denote the Exhaustive and mutually exclusive outcomes (states) of a system at any time. Also it is assumed that the system is Markovian.

We now define $P_{ij} = P\{X(t_n) = j \mid X(t_{n-1}) = i\}$ as the one-step transition probability of going from state i at time t_{n-1} to state j at time t_n. It is also assumed here that these probabilities from state E_i to state E_j $(i=0,1,2---, j=0,1,2,--)$ are expressed in the matrix form as below.

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & -- \\ P_{10} & P_{11} & P_{12} & -- \\ P_{20} & P_{21} & P_{22} & -- \\ | & | & | & \end{bmatrix}$$

This matrix P is Known as stochastic matrix or homogeneous matrix. The probability P_{ij} must satisfy the boundary conditions:

$$\Sigma P_{ij} = 1 \quad \text{for all } i, \quad \text{and } P_{ij} \geq 0$$
for all i and j.

Definition: Markov Chain:- The transition matrix P as (13)
defined above together with the initial probabilities $\{P_j^{(0)}\}$ associated with the states E_j $(j = 0, 1, 2 \ldots)$ completely define a Markov Chain.

The Markov Chains are of two types

(i) Ergodic (ii) Regular

An Ergodic Markov chain has the property that it is possible to pass from one state to another in a finite number of steps, regardless of present state.

A special type of ergodic Markov Chain is the regular Markov Chain.

A regular Markov chain is defined as a chain having a transition Martix P such that for some power of P it has only non-zero positive probability values.

Thus all Regular Chains must be ergodic chain. The easiest way to 'check if an ergodic chain is regular' is to continue squaring the transition matrix P until all zeros are removed.

Ex.2 Determine if the following transition matrix is ergodic Markov Chain.

Future states

$$\text{Present States}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

Solution:- Here we must check that it is possible to go from every present state to all other states. We observe that from state 1, it is possible to go directly to every other state except state 3. For state 3, it is possible to go from state 1 to state 2 to state 3. Therefore it is possible to go from state 1 to any other state. Similarly, from state 2, it is possible to go to state 3 or state 4, then from state 3 to state 1, or from state 4 to state 3 to state 1. Also, from state 3 it is possible to go directly to state 1.. Finally from state 4, it is possible to go to state 3, then from state 3 to state 1. Hence above transition matrix is an ergodic Markov chain. (14)

Q.3 Test the following transition matrix to see if the Markov Chain is regular and ergodic where x is some positive P_{ij} value.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array}$$

Solution We compute

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array} \quad P^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array} \quad P^8 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array}$$

From this, we observe that P raised to an even number power gives the result as above, while P raised to an odd-number power will give the original matrix, since all the elements are not non-zero positive elements, the above matrix is not regular. But it is an ergodic since it is possible to go from state 1 to state 2 or state 3 from state 2 to state 1 or state 4. From state 2 to state 1. From state 3 to state 1. From state 4 to state 2 or state 1

(14) **Steady State (Equilibrium Condition)** – The determination of steady state conditions in a regular ergodic Markov chain can be accomplished most readily by computing P^n for larger values of n. Therefore there is a limiting probability that the system will reach to steady state equibrium in a finite number of transition. This can be generalized in the following manner.

In eqn (1), if n becomes very large, the values P_{ij} tend to fixed limits and each probability vector $R(n)$ approaches a constant value ie

$$R(n+1) = R(n) = R$$

thus taking limits

$$\lim_{n\to\infty} R(n+1) = \lim_{n\to\infty} R(n)P \quad \text{or} \quad R = RP$$

Therefore as $n\to\infty$, $R(n)$ becomes constant (ie, independent of time) and then the system is said to have reached to a steady state equilibrium.

Q.4 A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as just overhauled good, fair or inoperative. If the item is inoperative, it is overhauled, a procedure that takes one day, Let us denote the four classifications as states 1,2,3 and 4 respectively. Assume that the working condition of the equipment follows a Markov processes with the following transition matrix:

If it costs Rs. 125 to overhaul a machine (including lost time), on the average and Rs 75 in production is lost if a machine is found inoperative. Using steady state probabilities, compute the expected per day cost of maintenance.

∴ solution – The given transition matrix P can be interpreted as indicating 1/4 of the time it is in fair Condition after a day's time, and 3/4 of the time just overhauled machine is in good Condition after a day's use. But a machine which is in good Condition has equal Chances of still being in good Condition or of being in fair Condition after a day's use, while a machine in fair Condition has equal Chances of being in fair or inoperative Condition after a day's use. An inoperative machine will be overhauled the next day, so that at the end of the day it would have been just overhauled.

Tomorrow

$$P = \text{Today} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since the given matrix P is an ergodic regular Markov Process, it will certainly reach to steady state equilibrium. Let the steady state probabilities P_1, P_2, P_3 and P_4 represent the proportion of times that the machine will be in states 1, 2, 3 and 4 respectively.

Now with the help of steady state equations

$R = RP$.

$$(P_1\ P_2\ P_3\ P_4) = (P_1\ P_2\ P_3\ P_4) \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In order to find P_1, P_2, P_3 and P_4, we require to solve the simultaneous equations.

$$P_1 = P_4,\quad P_2 = \frac{3}{4}P_1 + \frac{1}{2}P_2,\quad P_3 = \frac{1}{4}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3$$

$$P_4 = \frac{1}{2}P_3 \quad \text{and} \quad P_1 + P_2 + P_3 + P_4 = 1$$

Solving these equations, we get

$P_1 = \frac{2}{11}$, $P_2 = \frac{3}{11}$, $P_3 = \frac{4}{11}$, $P_4 = \frac{2}{11}$

Thus, on an average, 2 out of every 11 days the machine will be overhauled;, 3 out of every 11 days it will be in good condition; 4 out of every 11 days it will be in fair condition; and 2 out of every 11 days it will be found inoperative at the end of the days.

Hence the expected (average) cost per day of maintenance will be given by

$$(2/11)125 + (2/11)75 = Rs\ 36.36$$

Markov Analysis:- In order to explain the Markov analysis, we present here an example of Brand switching Models which emphasises on the time behaviour of customers who make repeated purchases of a product class, but from time to time may switch over from one brand to another. The basic element of a Markov process has to do with various states. In brand switching models, the state is generally the customer's preference for a particular brand..

Brand Switching Example! - Let us consider a consumer sample distributed over two brands A and B the samples being the representatives of the entire group from the standpoint of their brand loyalty and their switching patterns. The behaviour of the large groups can be better described in probabilistic terms. This probabilistic description can be represented by transition matrix as explained

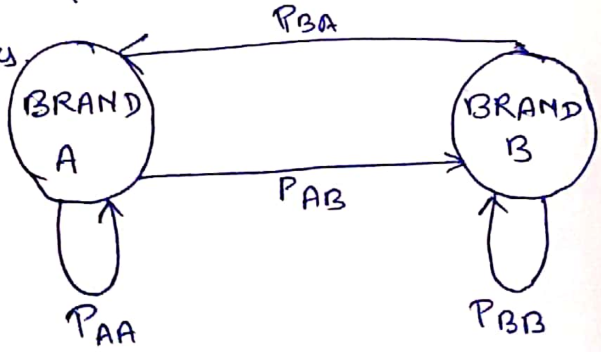


Fig 4 Brand Switching Pattern Diagram

by the following diagram

Transition Matrix

$$\begin{array}{ccc} & & \text{To} \\ & A & B \\ \text{From} \quad A & P_{AA} & P_{AB} \\ B & P_{BA} & P_{BB} \end{array}$$

$$\begin{array}{cccccccc} & & & & \text{To} & & \\ & & A_1 & A_2 & \cdots & A_j & \cdots & A_n \\ & A_1 & P_{11} & P_{12} & \cdots & P_{1j} & \cdots & P_{1n} \\ & A_2 & P_{21} & P_{22} & \cdots & P_{2j} & \cdots & P_{2n} \\ & \vdots & \vdots & & & & & \\ \text{from} & A_i & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots & P_{in} \\ & \vdots & \vdots & & & & & \\ & A_n & P_{n1} & P_{n2} & \cdots & P_{nj} & \cdots & P_{nn} \end{array}$$

In general, for n brands $A_1, A_2, \ldots A_n$ the transition matrix can be represented as follows:

where the probability P_{ij} is such that a customer's preference will switch from brand i to brand j from one period to the next.

The most important characteristic of a transition matrix is that

$$\sum_{j=1}^{n} P_{ij} = 1 \quad ; \quad i = 1, 2, \ldots n$$

meaning thereby a customer must have some preference

Q.5 Suppose there are two market products of brand A and B respectively. Let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below:

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady state.

From \ To	A	B
A	0.9	0.1
B	0.5	0.5

Soln. Here, it is given that the initial state for A and B are 50% each. Then after the promotional efforts made to brands A and B, the transition matrix shows that during second period brand A will retain 90% of its customers and take away 50% of B's so the market share for brand A during the second period will be given by

$$(50\%)(0.9) + (50\%)(0.5) = 70\%$$

The corresponding market share for B during the second period will be $(50\%)(0.1) + (50\%)(0.5) = 30\%$

In matrix form, it can be expressed as

$$(50\% \quad 50\%)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (70\% \quad 30\%)$$

In the same transition matrix holds from one period to the other, then market share of two brands for different periods will be as follows:

Period	Brand A (Market Share	Brand B (Market Share
0	50%	50%
1	70%	30%
2	78%	22%
3	81.2%	18.8%
4	82.48%	17.52%
5	82.992%	17.008%
6	83%	17%

From this table, we observe that sharing with 50%, 50% of the market shares, after 6 time

Periods the resulting market shares are approximately 83% and 17% respectively. So the equilibrium position of market share of A and B will be 5/6 and 1/6 of the total market respectively.

Steady state (or equilibrium) position can be obtained by the matrix equation:

$$(x \ y)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (x \ y)$$

or $\begin{cases} 0.9x + 0.5y = x \\ 0.1x + 0.5y = y \end{cases}$ and $x + y = 1$

The most important aspect of steady state is that - if the transition matrix is same throughout, then it is independent of the initial market shares.

Q.6 Suppose there are three diaries in a town, say A, B and C. They supply all the milk consumed in the town. It is known by all the diaries that consumers switch from dairy to dairy overtime because of advertising, dissatisfaction with service and other reasons. All these diaries maintain records of the number of their customers and the daily from which they obtained each new customers following table illustrates the flow of customers over an observation period of one month, say June

Dairy	June 1 (customers)	Gains from			Losses to			July 1 (Customers)
		A	B	C	A	B	C	
A	200	0	35	25	0	20	20	220
B	500	20	0	20	35	0	15	490
C	300	20	15	0	25	20	0	290

We assume that the matrix of transition probabilities remain fairy stable and that the July market shares are $A = 22\%$, $B = 49\%$, $C = 29\%$

Managers of these diaries are willing to know:

(i) Market share of their diaries on 1st August and 1st september.

(ii) Their market shares in steady state.

Sol^n From the table of the Problem, the matrix of the transition probabilities can be easily obtained as follows:

	A	B	C
A	$\frac{160}{200} = 0.80$	$\frac{20}{200} = 0.10$	$\frac{20}{200} = 0.10$
B	$\frac{35}{500} = 0.07$	$\frac{450}{500} = 0.90$	$\frac{15}{500} = 0.03$
C	$\frac{25}{300} = 0.083$	$\frac{20}{300} = 0.067$	$\frac{255}{300} = 0.85$

Market share of the diaries on the I^{st} August will be

$$\begin{pmatrix}0.22 & 0.49 & 0.29\end{pmatrix}\begin{bmatrix}0.800 & 0.100 & 0.100\\ 0.070 & 0.900 & 0.030\\ 0.083 & 0.067 & 0.850\end{bmatrix} = \begin{pmatrix}0.234 & 0.483 & 0.283\end{pmatrix}$$

Market share of the three diaries on 1^{st} sepetember will be

$$\begin{pmatrix}0.234 & 0.483 & 0.283\end{pmatrix}\begin{bmatrix}0.800 & 0.100 & 0.100\\ 0.070 & 0.900 & 0.030\\ 0.083 & 0.067 & 0.850\end{bmatrix} = \begin{pmatrix}0.245 & 0.477 & 0.278\end{pmatrix}$$

The steady state market shares are given by

$$\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}0.800 & 0.100 & 0.100\\ 0.070 & 0.900 & 0.030\\ 0.083 & 0.067 & 0.850\end{pmatrix} = \begin{pmatrix}x & y & z\end{pmatrix}$$

or

$$\begin{cases}0.800x + 0.070y + 0.083z = x\\ 0.100x + 0.900y + 0.030z = y\\ 0.100x + 0.030y + 0.850z = z\\ x + y + z = 1\end{cases}$$

and solving these four equations we get $x = 0.273$, $y = 0.454$, $z = 0.273$

Q.7 The number of units of an item that are withdrawn from inventory on a day-to-day basis is a Markov Chain Process in which requirements for tomorrow depend on today's requirements. A one day transition matrix is given below:

(i) Construct a tree diagram showing inventory requirements on two consecutive days.

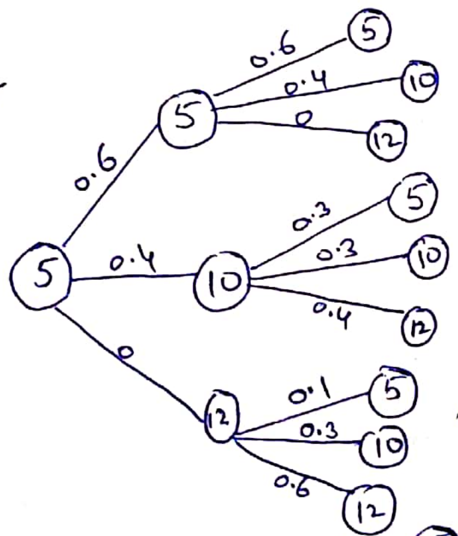
(ii) Develop a two-day transition matrix.

(iii) Comment on how a two day transition matrix might be helpful to a manager who is responsible for inventory management.

Number of units withdrawn from Inventory tomorrow

$$\text{Today}\quad \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array}$$

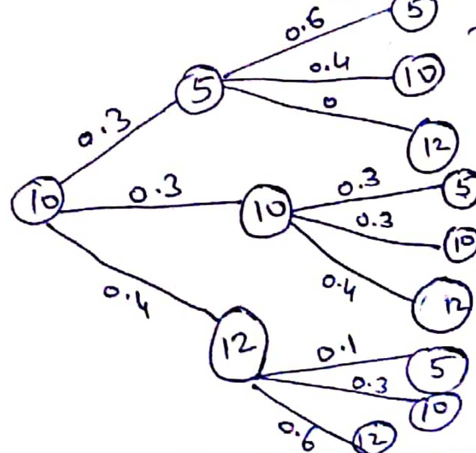
Sol^n



$$P_{11}^{(2)} = (0.6)(0.6) + (0.4)(0.3) + (0)(0.1) = 0.48$$

$$P_{12}^{(2)} = (0.6)(0.4) + (0.4)(0.3) + (0)(0.3) = 0.36$$

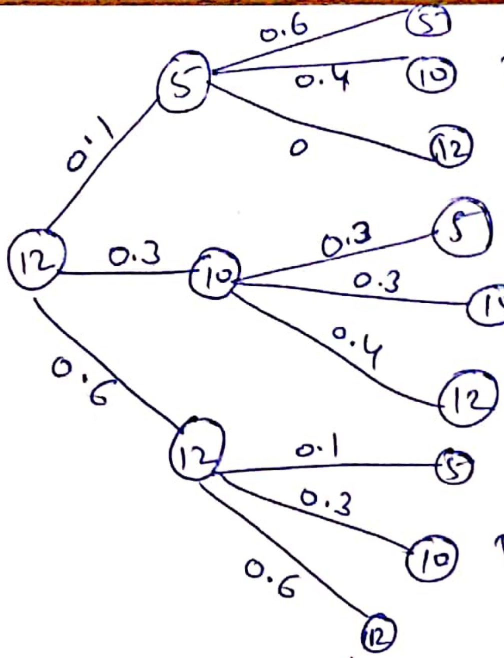
$$P_{13}^{(2)} = (0.6)(0) + (0.4)(0.4) + (0)(0.6) = 0.16$$



$$P_{21}^{(2)} = (0.3)(0.6) + (0.3)(0.3) + (0.4)(0.1) = 0.31$$

$$P_{22}^{(2)} = (0.3)(0.4) + (0.3)(0.3) + (0.4)(0.3) = 0.33$$

$$P_{23}^{(2)} = (0.3)(0) + (0.3)(0.4) + (0.4)(0.6) = 0.36$$



$$P_{31}^{(2)} = (0.1)(0.6) + (0.3)(0.3) + (0.6)(0.1)$$
$$= 0.21$$

$$P_{32}^{(2)} = (0.1)(0.4) + (0.3)(0.3) + (0.6)(0.3)$$
$$= 0.31$$

$$P_{33}^{(2)} = (0.1)(0) + (0.3)(0.4) + (0.6)(0.6)$$
$$= 0.48$$

(ii) If the transition matrix be denoted by P, the two-day transition matrix is given by

$$P^2 = \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{array}{c} 5 \\ 10 \\ 12 \end{array} & \begin{bmatrix} 0.48 & 0.36 & 0.16 \\ 0.31 & 0.33 & 0.36 \\ 0.21 & 0.31 & 0.48 \end{bmatrix} \end{array}$$

(iii) Suppose that every morning a manager must place an order for inventory replenishment. Because of delivery time requirements, an order placed today arrives 2 days later. The two day transition matrix can be used for guiding ordering decisions. For Example, if today the manager Experiences a demand for five units, then two days later (when replenishment stock arrives in response to today's order) the probability of requiring 5 units is 0.48, 10 units is 0.36 and that of 12 is 0.16.