5·7. TRANSFORMATION OF TWO-DIMENSIONAL RANDOM VARIABLE

In this section we shall consider the problem of change of variables in the two-dimensional case. Let *r.v.'s* U and V be transformed to the *r.v's* X and Y by the transformation $u = u(x, y)$, $v = v(x, y)$, where u and v are continuously differentiable functions, for which Jacobian of transformation :

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}, \qquad \ldots (5 \cdot 23)$$

is either > 0 or < 0 throughout the (x, y) plane so that the inverse transformation is uniquely given by $x = x(u, v)$, $y = y(u, v)$.

Theorem 5·5. *The joint p.d.f.* $g_{UV}(u, v)$ *of the transformed variables* U *and* V *is:*

$$g_{UV}(u, v) = f_{XY}(x, y) \, |J|, \qquad \ldots (5 \cdot 24)$$

where $|J|$ *is the modulus value of the Jacobian of transformation and* $f(x, y)$ *is expressed in terms of* u *and* v.

Proof. $P(x < X \leq x + dx, y < Y \leq y + dy) = P(u < U \leq u + du, v < V \leq + dv)$

$$\Rightarrow \qquad f_{XY}(x, y)\, dx\, dy = g_{UV}(u, v)\, du\, dv$$

$$\Rightarrow \qquad g_{UV}(u, v)\, du\, dv = f_{XY}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$$\therefore \qquad g_{UV}(u, v) = f_{XY}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = f_{XY}(x, y)\, |J|$$

Theorem 5·6. (a) *If X and Y are independent continuous r.v.'s, then the p.d.f. of $U = X + Y$ is given by :*

$$h(u) = \int_{-\infty}^{\infty} f_X(v)\, f_Y(u - v)\, dv \qquad \ldots (5{\cdot}25)$$

Proof. Let $f_{XY}(x, y)$ be the joint *p.d.f.* of independent continuous *r.v.*'s X and Y and let us make the transformation : $u = x + y, v = x \quad \Rightarrow \quad x = v, y = u - v.$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

Thus the joint *p.d.f.* of *r.v.*'s U and V is given by :

$$g_{UV}(u, v) = f_{XY}(x, y)\, |J| = f_X(x)\, f_Y(y)\, |J| \qquad \text{(Since } X \text{ and } Y \text{ are independent)}$$
$$= f_X(v)\, f_Y(u - v)$$

The marginal density of U is given by :

$$h(u) = \int_{-\infty}^{\infty} g_{UV}(u, v)\, dv = \int_{-\infty}^{\infty} f_X(v)\, f_Y(u - v)\, dv$$

Remark. The function h (.) is given a special name and is said to be the *convolution of* f_X (.) and f_Y (.) and we write $h(.) = f_X(.) * f_Y(.)$.

and Y, on using the transformation ... variables X

Example 5·48. *Let (X, Y) be a two-dimensional non-negative continuous r.v. having the joint density :*

$$f(x, y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & ; x \geq 0,\ y \geq 0 \\ 0 & , elsewhere \end{cases}$$

Prove that the density function of $U = \sqrt{X^2 + Y^2}$ is :

$$h(u) = \begin{cases} 2u^3 e^{-u^2} & , 0 \leq u < \infty \\ 0 & , elsewhere \end{cases}$$

Solution. Let us make the transformation : $u = \sqrt{x^2 + y^2}$ and $v = x$

$$\Rightarrow \quad v \geq 0, \quad u \geq 0 \text{ and } u \geq v \text{ or } u \geq 0 \text{ and } 0 \leq v \leq u.$$

The Jacobian of transformation J is given by :

$$\frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & 1 \\ \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} = -\frac{y}{\sqrt{x^2+y^2}}$$

The joint $p.d.f.$ of U and V is given by :

$$g(u, v) = f(x, y)\,|\,J\,| = 4xy\, e^{-(x^2+y^2)} \left| -\frac{\sqrt{x^2+y^2}}{y} \right| = 4x\sqrt{x^2+y^2}\, e^{-(x^2+y^2)}$$

$$= \begin{cases} 4vu.\, e^{-u^2} & ; u \geq 0,\ 0 \leq v \leq u \\ 0 & , \text{otherwise} \end{cases}$$

Hence the marginal density function of $U = \sqrt{X^2 + Y^2}$ is :

$$h(u) = \int_0^u g(u, v)dv = 4u\, e^{-u^2} \int_0^u v\,dv = \begin{cases} 2u^3 e^{-u^2}, & u \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Example 5·49. *Let the p.d.f. of the random variable (x, y) be :*

$$f(x, y) = \begin{cases} \alpha^{-2} e^{-(x+y)/\alpha} & ; x, y > 0,\ \alpha > 0 \\ 0 & , elsewhere \end{cases}$$

Find the distribution of $\frac{1}{2}(X - Y)$.

Solution. Consider the transformation :

$$u = \frac{1}{2}(x - y) \text{ and } v = y \quad \Rightarrow \quad x = 2u + v \text{ and } y = v$$

The Jacobian of the transformation is : $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$

Thus, the joint $p.d.f.$ of the random variables (U, V) is given by :

$$g(u, v) = \begin{cases} \frac{2}{\alpha^2} e^{-(2/\alpha)(u+v)} & ; -\infty < u < \infty,\ v > -2u, \text{ if } u < 0; \\ & v > 0 \text{ if } u \geq 0 \text{ and } \alpha > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

The marginal $p.d.f.$ of U is given by :

$$g_U(u) = \begin{cases} \int_{-2u}^{\infty} \frac{2}{\alpha^2} \exp\left\{-(2/\alpha)(u+v)\right\} dv = \frac{1}{\alpha} e^{2u/\alpha}, & u < 0 \\ \int_{0}^{\infty} \frac{2}{\alpha^2} \exp\left\{-(2/\alpha)(u+v)\right\} dv = \frac{1}{\alpha} e^{-2u/\alpha}, & u \geq 0 \end{cases}$$

Hence $\quad g_U(u) = \frac{1}{\alpha}\, exp \left\{-\frac{2}{\alpha} \mid u \mid\right\} - \infty < u < \infty.$