3.1 MOMENTS

Moments are statistical tools, used in statistical investigations. The moments of a distribution are the arithmetic means of the various powers of the deviations of items from some given number.

3.2 MOMENTS ABOUT MEAN (Central Moments)

3.2.1. For an Individual Series

If $x_1, x_2, \ldots, x_n$ are the values of the variable under consideration, the r^{th} moment μ_r about mean $\bar{x}$ is defined as

$$\mu_r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^r}{n} \; ; r = 0, 1, 2, \ldots$$

3.2.2. For a Frequency Distribution

If $x_1, x_2, \ldots, x_n$ are the values of a variable x with the corresponding frequencies $f_1, f_2, \ldots, f_n$ respectively then r^{th} moment μ_r about the mean $\bar{x}$ is defined as

$$\mu_r = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^r}{N} \; ; r = 0, 1, 2, \ldots \quad \text{where} \quad N = \sum_{i=1}^{n} f_i$$

In particular, $$\mu_0 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^0 = \frac{1}{N} \sum_{i=1}^{n} f_i = \frac{N}{N} = 1$$

$\therefore$ For any distribution, $\boxed{\mu_0 = 1}$

For $r = 1$,

$$\mu_1 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x}) = \frac{1}{N} \sum_{i=1}^{n} f_i x_i - \bar{x} \left(\frac{1}{N} \sum_{i=1}^{n} f_i \right) = \bar{x} - \bar{x} = 0$$

$\therefore$ For any distribution, $\boxed{\mu_1 = 0}$

For $r = 2$,

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = (\text{S.D.})^2 = \text{Variance}$$

$\therefore$ For any distribution, μ_2 coincides with the variance of the distribution.

Similarly,
$$\mu_3 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^3, \quad \mu_4 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^4$$

and so on.

Note. In case of a frequency distribution with class intervals, the values of x are the mid-points of the intervals.

EXAMPLES

Example 1. *Find the first four moments for the following individual series:*

x	3	6	8	10	18

Sol. **Calculation of Moments**

S. No.	x	$x - \bar{x}$	$(x - \bar{x})^2$	$(x - \bar{x})^3$	$(x - \bar{x})^4$
1	3	– 6	36	– 216	1296
2	6	– 3	9	– 27	81
3	8	– 1	1	– 1	1
4	10	1	1	1	1
5	18	9	81	729	6561
$n = 5$	$\Sigma x = 45$	$\Sigma(x - \bar{x}) = 0$	$\Sigma(x - \bar{x})^2 = 128$	$\Sigma(x - \bar{x})^3 = 486$	$\Sigma(x - \bar{x})^4 = 7940$

Now,
$$\bar{x} = \frac{\Sigma x}{n} = \frac{45}{5} = 9$$

$$\therefore \quad \mu_1 = \frac{\Sigma(x - \bar{x})}{n} = \frac{0}{5} = 0, \qquad \mu_2 = \frac{\Sigma(x - \bar{x})^2}{n} = \frac{128}{5} = 25.6$$

$$\mu_3 = \frac{\Sigma(x - \bar{x})^3}{n} = \frac{486}{5} = 97.2, \qquad \mu_4 = \frac{\Sigma(x - \bar{x})^4}{n} = \frac{7940}{5} = 1588.$$

Example 2. *Calculate μ_1, μ_2, μ_3, μ_4 for the following frequency distribution:*

Marks	*5–15*	*15–25*	*25–35*	*35–45*	*45–55*	*55–65*
No. of students	*10*	*20*	*25*	*20*	*15*	*10*

Sol. **Calculation of Moments**

Marks	No. of students (f)	Mid-point (x)	fx	$x - \bar{x}$	$f(x-\bar{x})$	$f(x-\bar{x})^2$	$f(x-\bar{x})^3$	$f(x-\bar{x})^4$
5–15	10	10	100	– 24	– 240	5760	– 138240	3317760
15–25	20	20	400	– 14	– 280	3920	– 54880	768320
25–35	25	30	750	– 4	– 100	400	– 1600	6400
35–45	20	40	800	6	120	720	4320	25920
45–55	15	50	750	16	240	3840	61440	983040
55–65	10	60	600	26	260	6760	175760	4569760
	N = 100		Σfx = 3400		$\Sigma f(x-\bar{x})$ = 0	$\Sigma f(x-\bar{x})^2$ = 21400	$\Sigma f(x-\bar{x})^3$ = 46800	$\Sigma f(x-\bar{x})^4$ = 9671200

Now, $\bar{x} = \dfrac{\Sigma fx}{N} = \dfrac{3400}{100} = 34$

$$\therefore \quad \mu_1 = \frac{\Sigma f(x-\bar{x})}{N} = \frac{0}{100} = 0, \qquad \mu_2 = \frac{\Sigma f(x-\bar{x})^2}{N} = \frac{21400}{100} = 214$$

$$\mu_3 = \frac{\Sigma f(x-\bar{x})^3}{N} = \frac{46800}{100} = 468, \qquad \mu_4 = \frac{\Sigma f(x-\bar{x})^4}{N} = \frac{9671200}{100} = 96712.$$

3.3 SHEPPARD'S CORRECTIONS FOR MOMENTS

While computing moments for frequency distribution with class intervals, we take variable x as the mid-point of class-intervals which means that we have assumed the frequencies concentrated at the mid-points of class-intervals.

The above assumption is true when the distribution is symmetrical and the no. of class-intervals is not greater than $\frac{1}{20}^{\text{th}}$ of the range, otherwise the computation of moments will have certain error called **grouping error.**

This error is corrected by the following formulae given by **W.F. Sheppard.**

$$\mu_2 \text{ (corrected)} = \mu_2 - \frac{h^2}{12}$$

$$\mu_4 \text{ (corrected)} = \mu_4 - \frac{1}{2} h^2 \mu_2 + \frac{7}{240} h^4$$

where h is the width of the class-interval while μ_1 and μ_3 require no correction.

These formulae are known as **Sheppard's corrections.**

Example 3. *Find the corrected values of the following moments using Sheppard's correction. The width of classes in the distribution is 10:*

$\mu_2 = 214$, $\mu_3 = 468$, $\mu_4 = 96712$.

Sol. We have $\mu_2 = 214$, $\mu_3 = 468$, $\mu_4 = 96712$, $h = 10$.

Now,
$$\mu_2 \text{ (corrected)} = \mu_2 - \frac{h^2}{12} = 214 - \frac{(10)^2}{12} = 214 - 8.333 = 205.667.$$

$$\mu_3 \text{ (corrected)} = \mu_3 = 468$$

$$\mu_4 \text{ (corrected)} = \mu_4 - \frac{1}{2}h^2\mu_2 + \frac{7}{240}h^4 = 96712 - \frac{(10)^2}{2}(214) + \frac{7}{240}(10)^4$$

$$= 96712 - 10700 - 291.667 = 86303.667.$$

3.4 MOMENTS ABOUT AN ARBITRARY NUMBER (Raw Moments)

If $x_1, x_2, x_3, ..., x_n$ are the values of a variable x with the corresponding frequencies $f_1, f_2, f_3, ..., f_n$ respectively then r^{th} moment μ_r' about the number $x = A$ is defined as

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^r \ ; r = 0, 1, 2, ... \quad \text{where,} \quad N = \sum_{i=1}^{n} f_i$$

For $r = 0$, $$\mu_0' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^0 = 1$$

For $r = 1$, $$\mu_1' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i - \frac{A}{N}\sum_{i=1}^{n} f_i = \bar{x} - A$$

For $r = 2$, $$\mu_2' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^2$$

For $r = 3$, $$\mu_3' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^3 \text{ and so on.}$$

In calculation work, if we find that there is some common factor h (> 1) in values of $x - A$, we can ease our calculation work by defining $u = \frac{x - A}{h}$. In that case, we have

$$\mu_r' = \frac{1}{N}\left(\sum_{i=1}^{n} f_i u_i^r\right) h^r \ ; r = 0, 1, 2, ...$$

Note. For an individual series,

1. $\mu_r' = \frac{1}{n}\sum_{i=1}^{n}(x_i - A)^r \ ; r = 0, 1, 2, ...$

2. $\mu_r' = \frac{1}{N}\left(\sum_{i=1}^{n} u_i^r\right) h^r \ ; r = 0, 1, 2, ...$ | for $u = \frac{x - A}{h}$

3.5 MOMENTS ABOUT THE ORIGIN

If $x_1, x_2, ..., x_n$ be the values of a variable x with corresponding frequencies $f_1, f_2, ..., f_n$ respectively then r^{th} moment about the origin ν_r is defined as

$$\nu_r = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^r \ ; r = 0, 1, 2, ... \quad \text{where,} \quad N = \sum_{i=1}^{n} f_i$$

For $r = 0$, $\quad \nu_0 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^0 = \frac{N}{N} = 1$

For $r = 1$, $\quad \nu_1 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i = \bar{x}$

For $r = 2$, $\quad \nu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2$ and so on.

3.6 RELATION BETWEEN μ_r AND μ_r'

We know that,

$$\mu_r = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^r}{N} = \frac{1}{N}\sum_{i=1}^{n} f_i [(x_i - A) - (\bar{x} - A)]^r$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i [(x_i - A) - \mu'_1]^r \qquad | \because \ \mu_1' = \bar{x} - A$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i [(x_i - A)^r - {}^rc_1 (x_i - A)^{r-1} \mu_1' + {}^rc_2 (x_i - A)^{r-2} \mu_1'^2 - \ldots\ldots + (-1)^r \mu_1'^r]$$

| Using binomial theorem

$$\Rightarrow \quad \mu_r = \mu_r' - {}^rc_1 \mu'_{r-1}\mu_1' + {}^rc_2 \mu'_{r-2}\mu_1'^2 - \ldots\ldots + (-1)^r \mu_1'^r$$

Putting $r = 2, 3, 4$, we get

$$\mu_2 = \mu_2' - 2\mu_1'^2 + \mu_1'^2 = \mu_2' - \mu_1'^2 \qquad | \because \ \mu_0' = 1$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 3\mu_1'^3 - \mu_1'^3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Hence, we have the following relations:

$$\boxed{\mu_1 = 0}$$

$$\boxed{\mu_2 = \mu_2' - \mu_1'^2}$$

$$\boxed{\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}$$

and

$$\boxed{\mu_4 = \mu_4' - 4\mu'_3\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}$$

3.7 RELATION BETWEEN ν_r AND μ_r

We know that,

$$\nu_r = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^r \ ; r = 0, 1, 2, \ldots$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A + A)^r$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\,[(x_i - A)^r + {}^rc_1(x_i - A)^{r-1}\,.\,A + \ldots + A^r]$$

$$= \mu_r' + {}^rc_1\mu_{r-1}\,A + \ldots + A^r$$

If we take, $A = \bar{x}$ (for μ_r) then

$$v_r = \mu_r + {}^rc_1\mu_{r-1}\,\bar{x} + {}^rc_2\mu_{r-2}\,\bar{x}^2 + \ldots + \bar{x}^r \qquad \ldots(1)$$

Putting, $r = 1, 2, 3, 4$ in (1), we get

$$v_1 = \mu_1 + \mu_0\bar{x} = \bar{x} \qquad | \because \mu_1 = 0, \mu_0 = 1$$

$$v_2 = \mu_2 + {}^2c_1\mu_1\bar{x} + {}^2c_2\,\mu_0\bar{x}^2 = \mu_2 + \bar{x}^2$$

$$v_3 = \mu_3 + {}^3c_1\mu_2\bar{x} + {}^3c_2\mu_1\bar{x}^2 + {}^3c_3\mu_0\bar{x}^3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3$$

$$v_4 = \mu_4 + {}^4c_1\mu_3\bar{x} + {}^4c_2\mu_2\bar{x}^2 + {}^4c_3\mu_1\bar{x}^3 + {}^4c_4\mu_0\bar{x}^4$$

$$= \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4$$

Hence we have the following relations:

$$\boxed{v_1 = \bar{x}} \qquad \boxed{v_2 = \mu_2 + \bar{x}^2}$$

$$\boxed{v_3 = \mu_3 + 3\mu_2\,\bar{x} + \bar{x}^3} \quad \text{and} \quad \boxed{v_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4.}$$

3.8 KARL PEARSON'S β AND γ COEFFICIENTS

Karl Pearson defined the following four coefficients based upon the first four moments of a frequency distribution about its mean:

$$\left.\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} \end{aligned}\right\} \qquad (\beta\text{-coefficients})$$

$$\left.\begin{aligned} \gamma_1 &= +\sqrt{\beta_1} \\ \gamma_2 &= \beta_2 - 3 \end{aligned}\right\} \qquad (\gamma\text{-coefficients})$$

The practical use of these coefficients is to measure the skewness and kurtosis of a frequency distribution. These coefficients are pure numbers independent of units of measurement.

EXAMPLES

Example 1. *The first three moments of a distribution, about the value '2' of the variable are 1, 16 and – 40. Show that the mean is 3, variance is 15 and $\mu_3 = -86$.*

Sol. We have $A = 2$, $\mu_1' = 1$, $\mu_2' = 16$, and $\mu_3' = -40$

We know that $\mu_1' = \bar{x} - A \Rightarrow \bar{x} = \mu_1' + A = 1 + 2 = 3$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = 16 - (1)^2 = 15$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -40 - 3(16)(1) + 2(1)^3 = -40 - 48 + 2 = -86.$$

Example 2. *The first four moments of a distribution, about the value '35' are – 1.8, 240, – 1020 and 144000. Find the values of $\mu_1, \mu_2, \mu_3, \mu_4$.*

Sol. $\mu_1 = 0.$

$$\mu_2 = \mu_2' - \mu_1'^2 = 240 - (-1.8)^2 = 236.76$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -1020 - 3(240)(-1.8) + 2(-1.8)^3 = 264.36$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 144000 - 4(-1020)(-1.8) + 6(240)(-1.8)^2 - 3(-1.8)^4 = 141290.11.$$

Example 3. *Calculate the variance and third central moment from the following data:*

x_i	0	1	2	3	4	5	6	7	8
f_i	1	9	26	59	72	52	29	7	1

(U.P.T.U. 2006)

Sol. **Calculation of Moments**

x	f	$u = \frac{x - A}{h}$, $A = 4, h = 1$	fu	fu^2	fu^3
0	1	– 4	– 4	16	– 64
1	9	– 3	– 27	81	– 243
2	26	– 2	– 52	104	– 208
3	59	– 1	– 59	59	– 59
4	72	0	0	0	0
5	52	1	52	52	52
6	29	2	58	116	232
7	7	3	21	63	189
8	1	4	4	16	64
	$N = \Sigma f = 256$		$\Sigma fu = -7$	$\Sigma fu^2 = 507$	$\Sigma fu^3 = -37$

Now, moments about the point $x = A = 4$ are

$$\mu_1' = \left(\frac{\Sigma fu}{N}\right) h = \frac{-7}{256} = -0.02734$$

$$\mu_2' = \left(\frac{\Sigma fu^2}{N}\right) h^2 = \frac{507}{256} = 1.9805$$

$$\mu_3' = \left(\frac{\Sigma fu^3}{N}\right) h^3 = \frac{-37}{256} = -0.1445$$

Moments about mean

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 1.9805 - (-.02734)^2 = 1.97975$$

$\therefore$ Variance = 1.97975

Also, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$$= (-0.1445) - 3\,(1.9805)\,(-.02734) + 2\,(-.02734)^3$$
$$= 0.0178997$$

$\therefore$ Third central moment $= 0.0178997$.

Example 4. *The first three moments of a distribution about the value 2 of the vari are 1, 16 and – 40 respectively. Find the values of the first three moments about the origi*

Sol. We have $A = 2, \quad \mu_1' = 1, \quad \mu_2' = 16, \quad \mu_3' = -40$

$\therefore$
$$\nu_1 = \bar{x} = A + \mu_1' = 2 + 1 = 3$$
$$\nu_2 = \mu_2 + \bar{x}^2 = 15 + (3)^2 = 24$$
$$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = -86 + 3(15)(3) + (3)^3 = 76.$$

For certain theoretical developments, an indirect method for computing moments is used. Th
method depends on the finding of the moment generating function.

3.9.1. In Case of a Continuous Variable x, it is defined as

$$M(t) = \int_a^b e^{tx} f(x)\, dx \quad \text{...(1)}$$

where integral is a function of parameter t only. The limits a, b can be $-\infty$ and ∞ respectively. It is possible to associate a moment generating function with the distribution only when all the moments of the distribution are finite.

Let us see how $M(t)$ generates moments. For this, let us assume that $f(x)$ is a distribution function for which the integral given by (1) exists.

Then e^{tx} may be expanded in a power series and the integration may be performed term by term. It follows that

$$M(t) = \int_a^b \left(1 + tx + \frac{t^2}{2!}x^2 + \ldots\right) f(x)\, dx$$

$$= \int_a^b f(x)\, dx + t\int_a^b xf(x)\, dx + \ldots$$

$$= \nu_0 + \nu_1 t + \nu_2 \cdot \frac{t^2}{2!} + \dots \qquad \dots(2)$$

Obviously, the coefficient of $\frac{t^r}{r!}$ in (2) is the r^{th} moment about the origin.

Also, $$\left| \frac{d^r}{dt^r} M(t) \right|_{t=0} = \left| \frac{\nu_r}{r!} r! + \nu_{r+1} t + \dots \right|_{t=0} = \nu_r \qquad \dots(3)$$

Thus, ν_r about origin = r^{th} derivative of M(t) with t = 0.

Although the moment generating function (m.g.f.) has been defined for the variable x only, the definition can be generalized so that it holds for a variable z where z is a function of x. *e.g.*, if $z = x - m$ (m is mean), the r^{th} moment about z will give r^{th} moment of x about the mean m.

Moment generating function for z will clearly be given as

$$M_z(t) = \int_a^b e^{tz} f(x)\, dx$$

$$M_{x-m}(t) = \int_a^b e^{t(x-m)} f(x)\, dx = e^{-mt} \int_a^b e^{tx} f(x) dx = e^{-mt} M_x(t).$$

3.9.2. In Case of Discrete Distribution of the Variable *x*

We know that, for a variable x,

$$\nu_r = \Sigma x^r . P$$

where P is the probability that the variable takes on the value x.

If z is any function of x, we get r^{th} moment for z by the relation

$$\nu_r = \Sigma z^r P$$

and the moment generating function is given by

$$M_z(t) = \Sigma e^{tz} P \qquad \dots(1)$$

To verify that this function generates moments, we will expand e^{tz} and then sum term by term,

$$\therefore \quad M_z(t) = \sum \left(1 + tz + \frac{t^2}{2!} z^2 + \dots\right) P = \Sigma P + t\Sigma zP + \frac{t^2}{2!} \Sigma z^2 . P + \dots$$

$$= \nu_0 + t\nu_1 + \frac{t^2}{2!} \nu_2 + \dots$$

In this case, we can also show that $\nu_r = \left| \frac{d^r}{dt^r} M_z(t) \right|_{t=0}$

M(t) is clearly the expected value of e^{tx} and hence can be written as E(e^{tx}) which gives the moment generating function incase of discrete as well as continuous variables.

Expectation of any function $\phi(x)$ is given by

$$E\{\phi(x)\} = \sum_i \phi(x_i) f(x_i) \qquad | \text{ for discrete distribution}$$

or, $$E\{\phi(x)\} = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx \qquad | \text{ for continuous distribution}$$

Eqn. (1) can also be rewritten as

$$M_{x-a}(t) = E\,[e^{t(x-a)}] = \sum_i e^{t(x_i-a)}\,P_i = e^{-at}\sum_i e^{tx_i}\,P_i = e^{-at}\,M_0(t)$$

Therefore the moment generating function about the point 'a' is equal to e^{-at} times the moment generating function about the origin.

Note. m.g.f. is not always defined since $E(|\,e^{tx}\,|)$ does not always exist for all values of t.

e.g., if $f(x) = \dfrac{6}{\pi^2 x^2}$, $x = 1, 2, 3, \ldots$ then m.g.f. does not exist.

m.g.f. always exists for $t = 0$ since $M_{x=0}(0) = 1$. (M.T.U. 2013)

3.9.3. Properties of Moment Generating Function

(1) *The moment generating function of the sum of two independent chance variables is the product of their respective moment generating functions.*

Symbolically, $M_{x+y}(t) = M_x(t) \times M_y(t)$ *provided that* x *and* y *are independent random variables.*

Proof. Let x and y be two independent random variables so that $x + y$ is also a random variable.

The m.g.f. of the sum $x + y$ w.r.t. origin is

$$M_{x+y}(t) = E\{e^{t(x+y)}\} = E(e^{tx}\,.\,e^{ty}) = E(e^{tx})\,.\,E(e^{ty})$$

Since x and y are independent variables and so are e^{tx} and e^{ty}.

$$\therefore \qquad M_{x+y}(t) = M_x(t)\,.\,M_y(t)$$

Hence the theorem.

(2) *Effect of change of origin and scale on m.g.f.*

$$\boxed{M_u(t) = e^{-at/h}\,M_x(t/h)} \qquad \text{where } u = \frac{x-a}{h}$$

Proof. Let u be a new random variable given by $u = \dfrac{x-a}{h}$ so that $x = a + hu$

then by definition, the effect of linear transformation on m.g.f. is governed by

$$M_x(t) = E(e^{tx}) = E[e^{t(a+hu)}] = E(e^{at}\,.\,e^{thu})$$

$$= e^{at}\,E(e^{thu}) = e^{at}\,M_u(th)$$

Also, $\qquad M_u(t) = E(e^{tu})$

$$= E\left[e^{t\left(\frac{x-a}{h}\right)}\right] = e^{-\frac{at}{h}}\,M_x\left(\frac{t}{h}\right)$$

(3) $\qquad M_{cx}(t) = M_x(ct)$, *c being a constant.*

Proof. By definition,

$$\text{LHS} = M_{cx}(t) = E(e^{tcx}) = M_x(ct) = \text{RHS}$$

Hence the result.

EXAMPLES

Example 1. *Find the moment generating function of the exponential distribution*

$$f(x) = \frac{1}{c}\, e^{-x/c}\ ;\ 0 \le x \le \infty,\ c > 0 \qquad ([redacted])$$

Hence find its mean and standard deviation.

Sol. Moment generating function about the origin is given by

$$M_x(t) = \int_0^\infty e^{tx} \cdot \frac{1}{c} e^{-x/c}\, dx$$

$$= \frac{1}{c}\int_0^\infty e^{\left(t-\frac{1}{c}\right)x}\, dx = \frac{1}{c}\left[\frac{e^{\left(t-\frac{1}{c}\right)x}}{\left(t-\frac{1}{c}\right)}\right]_0^\infty$$

$$= (1 - ct)^{-1} = 1 + ct + c^2t^2 + c^3t^3 + \ldots$$

$$\therefore \qquad \nu_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = (c + 2c^2t + 3c^3t^2 + \ldots)_{t=0} = c$$

and

$$\nu_2 = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = 2c^2$$

Now, mean $\qquad \bar{x} = \nu_1 = c$

Variance $\qquad \mu_2 = \nu_2 - \bar{x}^2 = \nu_2 - \nu_1^2 = 2c^2 - c^2 = c^2$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = c.$

Example 2. *Obtain the moment generating function of the random variable x having probability distribution*

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

[redacted]]

Also determine mean ν_1, ν_2 and variance μ_2.

Sol. $M_x(t) = E(e^{tx})$

$$= \int_0^1 x \,.\, e^{tx}\, dx + \int_1^2 (2 - x) e^{tx}\, dx + \int_2^\infty 0 \,.\, e^{tx}\, dx$$

$$= \left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right)_0^1 + \left(\frac{2e^{tx}}{t} - \frac{xe^{tx}}{t} + \frac{e^{tx}}{t^2}\right)_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \left[\left(\frac{2e^{2t}}{t} - \frac{2e^{2t}}{t} + \frac{e^{2t}}{t^2}\right) - \left(\frac{2e^t}{t} - \frac{e^t}{t} + \frac{e^t}{t^2}\right)\right] = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$= \left(\frac{e^t - 1}{t}\right)^2 = \frac{\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)^2}{t^2} = 1 + t + t^2 + \ldots$$

$$\text{Mean} = \nu_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = 1$$

Similarly, $\nu_2 = 2$, $\mu_2 = \nu_2 - \bar{x}^2 = \nu_2 - \nu_1^2 = 2 - (1)^2 = 1 =$ Variance.

Example 3. *Find the moment generating function of the random variable whose moments are $\nu_r = (r + 1)!\, 2^r$.*

Sol.
$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \nu_r = \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\,.\,2^r = \sum_{r=0}^{\infty} (r + 1)(2t)^r$$

$$= 1 + 2.2t + 3\,.\,(2t)^2 + ... = (1 - 2t)^{-2}.$$

Example 4. *Find the moment generating function of the probability distribution function*

$$f(z) = e^{-z} (1 + e^{-z})^{-2}, \quad -\infty < z < \infty\,.$$

Sol.
$$M_z(t) = E(e^{tz})$$

$$= \int_{-\infty}^{\infty} e^{tz}\,.\,e^{-z}(1+e^{-z})^{-2}\, dz$$

$$= \int_{1}^{\infty} u^2 (u - 1)^{-t}\, du \quad \text{where } 1 + e^{-z} = u \quad \Rightarrow \quad -e^{-z}\, dz = du$$

$$= \int_{0}^{1} v^{-t} (1 - v)^t\, dv \quad \text{where } v = 1 - \frac{1}{u} \quad \Rightarrow \quad dv = \frac{1}{u^2}\, du$$

$$= \beta(1 - t, 1 + t)\,;\; 1 - t > 0$$

$$= \pi t \operatorname{cosec} \pi t,\; t < 1.$$

Example 9. *The random variable X assuming only non-negative values has a Gamma probability distribution if its probability distribution is given by*

$$f(x) = \begin{cases} \dfrac{\alpha^{\beta}}{\Gamma\beta} x^{\beta-1} e^{-\alpha x} & ; x > 0, \alpha > 0, \beta > 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the moment generating function of Gamma probability distribution.

Sol. $M_x(t) = E(e^{tx})$

$$= \int_0^{\infty} e^{tx} \cdot \frac{\alpha^{\beta}}{\Gamma\beta} \cdot x^{\beta-1} e^{-\alpha x}\, dx = \frac{\alpha^{\beta}}{\Gamma\beta} \int_0^{\infty} x^{\beta-1} e^{-x(\alpha-t)}\, dx$$

$$= \frac{\alpha^{\beta}}{(\alpha-t)^{\beta}\,\Gamma\beta} \int_0^{\infty} y^{\beta-1} e^{-\beta}\, dy \quad | \text{ where } y = x(\alpha - t) \text{ so that } dy = (\alpha - t)\, dx$$

$$= \frac{1}{\left(1 - \dfrac{t}{\alpha}\right)^{\beta}} \cdot \frac{1}{\Gamma\beta} \Gamma\beta = \left(1 - \frac{t}{\alpha}\right)^{-\beta} ; \ | t | < \alpha.$$