2 Introduction to Markovian (Chain) Process

§ 5.2-1. INTRODUCTION

Markov analysis is the method of analysing the current behaviour of some variable to predict its future behaviour.

A Markov analysis looks at a sequence of events, and analyzes the tendency of one event to be followed by another. Using this analysis, we can generate a new sequence of random variable but related events.

As a management tool, Markov analysis as a marketing aid for examining and predicting the behaviour of customers from the one brand to another brand. In a personnel department in determining the future man power requirements of an organization.

We know study of probability has dealt with independent trials processes. These processes are the basis of classical probability theory and of statistics.

In such cases, the possible outcomes for each experiment are the same and occur with the same probability. **The outcomes of the previous experiments does not influence our predictions for the outcomes of the next experiments.**

But in modern probability theory studies chance processes for which the knowledge of previous outcomes influences predictions for future experiments. When we observe a sequence of chance experiments, all the past outcomes could influence our predictions for the next experiments.

For example, this should be the case in predicting a player's performance on a sequence of matches in a series. But it is difficult to prove general results.

The following are the assumptions made in Markov analysis :

- A finite number of possible states.
- Probabilities of changing states remaining the same over time.
- States are collecting exhaustive *i.e.*, all possible states have been identified.
- States are mutually exclusive *i.e.*, only one state at a time is possible.

In 1907, A. A. Markov began the study of an important new type of chance process. In this process, the outcome of a given experiment can affect the outcome of the next experiment. This type of process is called a **Markov chain** (or process).

Thus is random process (sequence of events) in which the current state of system depends only on its the immediately preceding state is known as Markov Process.

§ 5.2-2. MARKOV CHAIN

A **Markov chain** is a process that consists of a finite number of states and some known probabilities P_{ij}, where P_{ij} is the probability of moving from state i to state j. **A state** is the condition or location of an object in the system at a particular time (or each move is called a step). Probability P_{ij} does not depend upon which states the chain was in before the current state.

The probabilities P_{ij}, are called **transition probabilities.**

Thus, Markov chain ⇒ obtain information about the future on the basis of current information.

§ 5.2-3. PROPERTIES OF MARKOV CHAIN (OR PROCESS)

The following properties must be hold for a Markov chain or Process (finite state) :

(a) The process consists of a finite number of states.

(b) For each time period, every object (person) in the system is in exactly one of the defined states. At the end of each time period, each object either moves to a new state or stays in that same state for another period.

(c) The objects move from one state to the next according to the transition probabilities which depend only on the current state (they do not take any previous history into account).

(d) The total probability of movement from a state (movement from a state to the same state does count as movement) must equal one.

(e) The transition probabilities do not change over time (the probability of going from state A to state B today is the same as it will be at any time in the fuuture) *i.e.*, transition probabilities are constant.

In a Markov chain, the probability distribution of next state for a Markov chain depends only on the current state.

§ 5.2-4. TRANSITION PROBABILITY MATRIX

The transition probability can be arranged in a matrix form, such matrix is called a transition probability matrix.

The matrix P whose ijth entry is denoted by P_{ij}, is called the transition matrix associated with the system. The elements in each row add upto 1.

Consider the $m \times n$ transition matirx P (one-step) as :

$$P = \text{state } i \overset{\text{State } j}{\begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & P_{m3} & \cdots & P_{mn} \end{bmatrix}}$$

where P_{ij} = transition probability of moving from state i to state j in one step.

i.e., P_{ij} = Conditional probability of being in state j in the future given the current state of i,

$$P\,(\text{state } j \text{ at time } = 1/\text{state } i \text{ at time } t = 0)$$

For example, P_{12} is the probability of being in state 2 in the future given the event was in state 1 in the period.

Here for each element P_{ij}, i represents the starting location and j represents the ending location for that move. This means that the row is the beginning location, and the column is the ending location after one move.

By the Markov property, we have

$$P_{ij} \in [0, 1] \quad \forall i, j$$

and

$$\sum_{j=1}^{n} P_{ij} = 1 \quad \forall i.$$

The transition matrix used in Markov chain will have the following properties :

(a) Each element of the transition matrix is a probability; therefore, each is a number between 0 and 1, (inclusive both).

(b) The elements of each row of the transition matrix sum to 1. But the columns do not neccesarily sum to 1 *i.e.*, $P_{i1} + P_{i2} + P_{i3} + \ldots + P_{im} = 1$ for all i and $0 \le P_{ij} \le 1$.

(c) The transition matrix must be square because it has a row and a column for each state. The rows identify the current state of the system and the columns identify the alternative states to which in the system can move.

(d) The elements of matrix should be non-negative *i.e.*, $P_{ij} \ge 0$.

For example, for the given transition matrix $P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$.

The row 1 interpretation is :

$$0.8 = P_{11} = P \text{ (in state 1 after being in state 1)}$$
$$0.1 = P_{12} = P \text{ (in state 2 after being in state 1)}$$
$$0.1 = P_{13} = P \text{ (in state 3 after being in state 1).}$$

§ 5.2-5. TRANSITION DIAGRAM (DIAGRAMATICALLY REPRESENTATION OF TRANSITION PROBABILITIES)

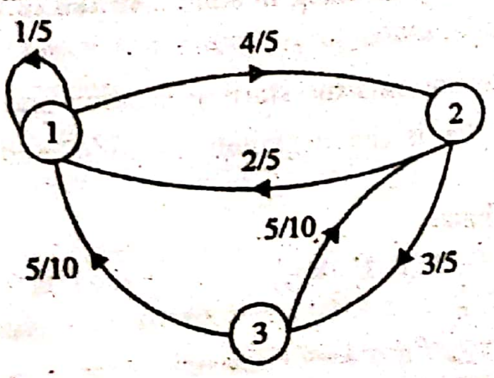
A Markov chain or Markov process can be represented by a state transition diagram, which is a diagram showing all the states and transition probabilities. The state transition diagram identifies all the discrete states of the system and the possible transitions between those states. In a Markov process, the transition frequencies between states depends only on the current state probabilities and the constant transition rates between states. In transition diagram, the states of a Markov chain may be represented by the vertices (or nodes) of the graph and one step transitions between states by directed edges (or arcs).

If $i \to j$ represent an edge from i to j, which is joining vertex V_i to V_j and P_{ij} represents the weight of an edge (or arc).

Consider a matrix of transition probabilities :

$$P = i \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \overset{\displaystyle j}{\begin{array}{c} \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} \\ \begin{bmatrix} 1/5 & 4/5 & 0 \\ 2/5 & 0 & 3/5 \\ 5/10 & 5/10 & 0 \end{bmatrix} \end{array}}$$

Corresponding transition diagram of above matrix is shown as below :



Note that the edge weights are positive and sum of the arc weights of the arcs from each node is unity.

A zero element indicates that the transition is not possible from state i to state j.

5.2-6. n-STEP TRANSITION PROBABILITIES

We have considered one step or unit step transition probabilities (*i.e.*, $P = [P_{ij}]_{m \times n}$ denote the transition matrix of the unit-step transitions).

Now we are interested in finding out the probability that the system will pass from state i to state j at time $t = n$ (*i.e.*, in n-steps). If the n-step transition probability is denoted by $P^{(n)} = [P_{ij}^{(n)}]_{m \times n}$ then these probabilities can be expressed as in matrix form :

$$P^{(n)} = \begin{array}{cc} & \text{State } j \\ \text{State } i & \begin{bmatrix} P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1m}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & \cdots & P_{2m}^{(n)} \\ \vdots & \vdots & & \vdots \\ P_{m1}^{(n)} & P_{m2}^{(n)} & \cdots & P_{mn}^{(n)} \end{bmatrix} \end{array}$$

where $P_{12}^{(n)}$ indicates the probability that the system will pass from state i to state j after in n steps.

The P^n is the n-step transition matrix, which means that the ijth entry in P^n is the probability that the system will pass from state i to state j in n steps.

P^2 is the two-step transition matrix for the system. To obtain, the two-step transition matrix, we calculate repeatedly multiplying the transition matrix P by iteself *i.e.*,

$$P^2 = P \times P.$$

Similarly, $$P^3 = P^2 \times P = P \times P \times P$$

and $$P^n = n\text{-step transition matrix}$$
$$= P^{n-1} \times P.$$

This is also known as **powers of matrix.**

Probability Vector :

A Markov chains probability distribution over its states may be viewed as a probability vectors.

A probability vector or vector of state probabilities is a row vector in which the entries are non-negative in the interval [0, 1] and add upto 1. The entries in probabilities vector can represent the probabilities of finding a system in each of the states. Using probability vector we can determine the state probabilities at a future time.

If V_0 is the state probabilities at time $n = 0$ and P is the transition matrix for a Markov system. Then V_1 vector of state probabilities after 1 step (or at time $n = 1$) is in terms of row matrix as .

$$V_1 = V_0 \times P.$$

Similarly, V_2, vector of state probabilities after 2 steps (or at time $n = 2$) can be obtained by

$$V_2 = V_1 \times P = (V_0 \times P) \times P = V_0 P^2$$

....

....

V_n, vector of state probabilities after n steps can be obtained by

$$V_n = V_{n-1} \times P = V_0 \times P^n.$$

Example 1. *Two manufactures A and B are competing with each other in a restricted market. Over the year, A's customers have exhibited a high degree of loyalty as measured by the fact that customers are*

...sing A's product 80 percent of the time. Also former customers purchasing the product from B ha... switched back to A's product 60 percent of the time.

(a) Construct and interpret the state transimition matrix in terms of (i) retention and loss, and (ii) retention and gain.

(b) Calculate the probability of a customer purchasing A's product at the end of the second period.

Solution. (a) The transition probabilities can be arranged in matrix form as shown below :

$$\begin{array}{c} \\ \text{Present} \\ P = \text{Purchase} \\ (n=0) \end{array}
\begin{array}{c} \\ \\ A \\ B \end{array}
\begin{array}{c} \text{Next purchase} \\ (n=1) \\ \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0\cdot 80 & 0\cdot 20 \\ 0\cdot 60 & 0\cdot 40 \end{bmatrix} \end{array}
\begin{array}{c} \\ \\ \text{Retention} \\ \text{and gain} \end{array}$$

$\leftarrow$ Retention and loss $\rightarrow$

Mathematically, conditional probabilities in the transition matrix can be stated as :

(i) $P(A_0/A_1) = P_{11} = 0\cdot 80$

Probability that customers now using A's product $n = 0$ (present purchase) will again purcha... A's product at $n = 1$ (next purchase) is $0\cdot 80$. This implies retention to A's product.

(ii) $P(B_0/A_1) = P_{21} = 0\cdot 60$

Probability of customers now using B's product at $n = 0$ (present purchase) will purchase A... product at $n = 1$ (next purchase) is $0\cdot 60$. This implies loss to B's product.

(iii) $P(A_0/B_1) = P_{12} = 0\cdot 20$

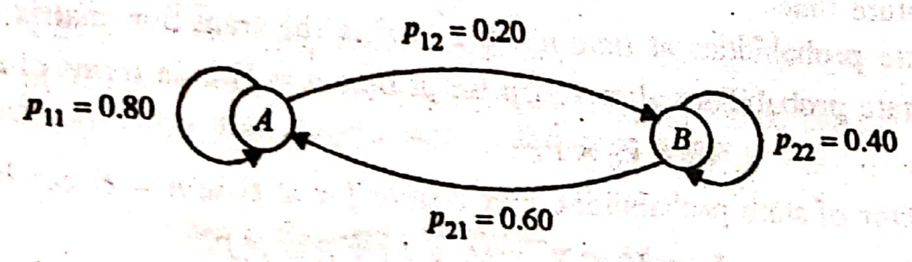
Probability of customers now using A's product at $n = 0$ (present purchase) will purchase B... product at $n = 1$ (next purchase) is $0\cdot 20$. This implies loss to A's product.

(iv) $P(B_0/B_1) = P_{22} = 0\cdot 40$

Probability of customers now using B's product at $n = 0$ (present purchase) will purchase B... product at $n = 1$ (next purchase) is $0\cdot 40$. This implies retention to B's product.

(b) The transition probability from state i at $n = 0$ to another state j at $n = 2$ can be represent... by two diagrams (i) transition diagram and (ii) probability tree diagram :

(i) Transition diagram as shown below :



Fg. Transtion Diagram

(ii) Probability tree diagram :

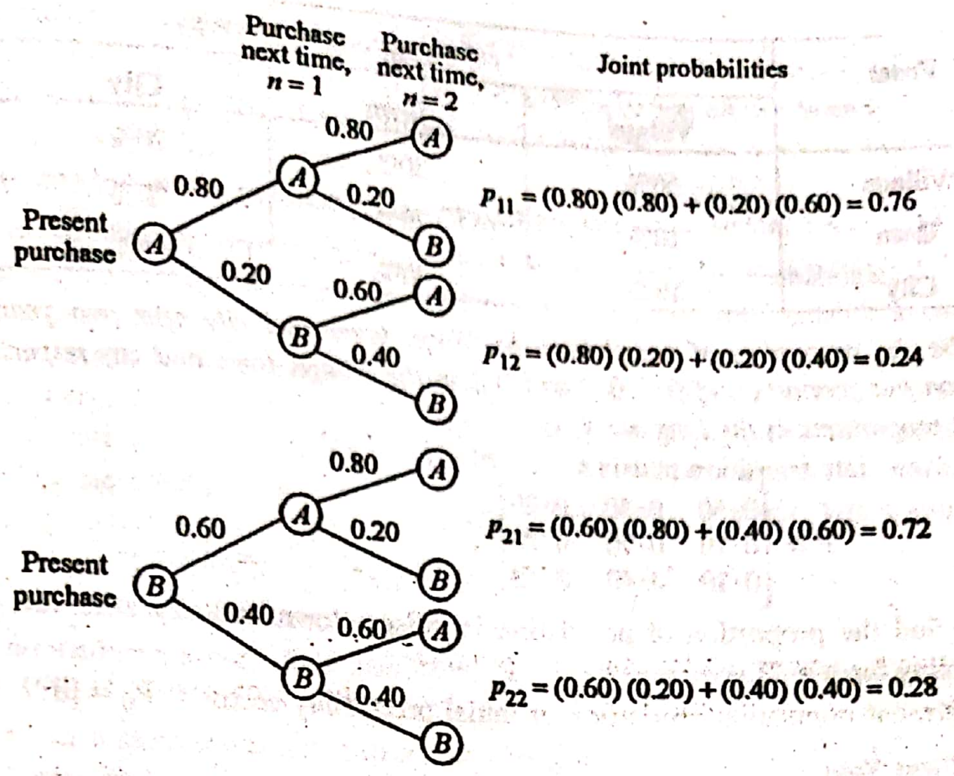


Fig. Probability Tree Diagram

...ability Calculations :

If we start with a customer's purchase of A's product in state 1 at $n = 0$, then $P_{11} = 1$ and $P_{12} = 0$, ...itial distribution vector is

$$V_0 = [P_{11} \;\; P_{12}] = [1 \;\; 0].$$

After the first transition, the state probabilities V_1 at $n = 1$ (*i.e.*, the distribution after one step) is ... by :

$$V_1 = V_0 \times P$$

or
$$= [1 \;\; 0] \begin{bmatrix} 0\cdot80 & 0\cdot20 \\ 0\cdot60 & 0\cdot40 \end{bmatrix}$$
$$= [0\cdot80 \;\; 0\cdot20] = V_1.$$

This implies, the probability of a customer using A's product at the end of state 1 is 80 per cent ... there are 20 per cent chances that the customers will switch over to B's product at the end of ...od 1.

After the second iteration, the state probabilities V_2 at $n = 2$ (*i.e.*, the distribution vector after ...ond step) is given by :

$$V_2 = V_1 \times P$$

or
$$= [0\cdot80 \;\; 0\cdot20] \begin{bmatrix} 0\cdot80 & 0\cdot20 \\ 0\cdot60 & 0\cdot40 \end{bmatrix}$$
$$= [0\cdot76 \;\; 0\cdot24] = V_2.$$

This implies that the probability of A's market share after the end of two periods is 76 per cent ... that of B's market share is 24 per cent.

Example 2. *The school of International studies for population found out by its survey that the ...ility of the population of a state to the village, town and city is in the following percentages :*

From	To		
	Village	Town	City
Village	50%	30%	20%
Town	10%	70%	20%
City	10%	40%	50%

What will be the proportion of population in village, town and city after two years, given that the present population has proportion of $0 \cdot 7, 0 \cdot 2$ and $0 \cdot 1$ in the village, town and city respectively ? What will be the respective proportions in the long run ?

Solution. Given state-transition matrix :

$$P = \begin{bmatrix} 0\cdot50 & 0\cdot30 & 0\cdot20 \\ 0\cdot10 & 0\cdot70 & 0\cdot20 \\ 0\cdot10 & 0\cdot40 & 0\cdot50 \end{bmatrix}.$$

We have to find the proportion of population in village, town and city after two years. [*i.e.*, state transition probability for $n = 2$]

Also given, present population proportion or initial probability vector $= V_0 = [0\cdot7 \;\; 0\cdot2 \;\; 0\cdot1]$.

Calculation of First Year :

The expected proportion of population in a village, town and city after first year is obtained by multiplying the matrix of transition probabilities P by the matrix of present population proportion V. Thus we get

Present proportion $(V_0) \times$ Transition matrix (P) = Proportion after one year

or $$V_0 \times P = V_1$$

$$\Rightarrow \quad [0\cdot70 \;\; 0\cdot20 \;\; 0\cdot10] \times \begin{bmatrix} 0\cdot50 & 0\cdot30 & 0\cdot20 \\ 0\cdot10 & 0\cdot70 & 0\cdot20 \\ 0\cdot10 & 0\cdot40 & 0\cdot50 \end{bmatrix} = [0\cdot38 \;\; 0\cdot39 \;\; 0\cdot23].$$

This gives that the population after one year is

Village $= 38\%$, Town $= 39\%$ and City $= 23\%$.

Calculation for Second Year :

The expected proportion of population in village, town and city after second year is obtained by multiplying the matrix of transition probability P by the matrix of population proportion after first year. Thus we have

Proportion after one year $\times$ transition matrix = proportion after second year

or $$V_1 \times P = V_2$$

$$= [0\cdot38 \;\; 0\cdot39 \;\; 0\cdot23] \times \begin{pmatrix} 0\cdot50 & 0\cdot30 & 0\cdot20 \\ 0\cdot10 & 0\cdot70 & 0\cdot20 \\ 0\cdot10 & 0\cdot40 & 0\cdot50 \end{pmatrix}$$

$$= [0\cdot252 \;\; 0\cdot479 \;\; 0\cdot269].$$

$\therefore$ Proportion of population after second year is obtained as :

Village $= 25\cdot2\%$, Town $= 47\cdot9\%$ and City $= 26\cdot9\%$.

In the long run, the porportion after 'n' years can be determined by using the relation given below :

$$\underset{\text{Initial proportion}}{[0\cdot70 \quad 0\cdot20 \quad 0\cdot10]} \times \underset{\text{Transition matrix}}{\begin{pmatrix} 0\cdot50 & 0\cdot30 & 0\cdot20 \\ 0\cdot10 & 0\cdot70 & 0\cdot20 \\ 0\cdot10 & 0\cdot40 & 0\cdot50 \end{pmatrix}^n} = [\text{Proportion after } n \text{ years}]$$

§ 5.2-7. STEADY STATE (OR EQUILIBRIUM) CONDITIONS

Steady state or equilibrium probabilities are the long term average probabilities for being in each state. After some step, state probabilities may become constant and will remain unchanged. At that point a steady state or equilibrium condition reached to the system.

In other words, steady state (or equilibrium) condition exist if state probability do not change after a large number of periods. At steady state, the system becomes independent of time and state probabilities for the next period equal the state probabilities of current period.

Thus, in steady state condition, if P is a transition matrix for a Markov system and V is the probability vector, then we have

$$V = VP \text{ (independent of time).}$$

Then we refer to V as a steady state vector (or distribution).

Method for determining steady-state condition. The following steps are involved in solving steady-state condition :

Step 1. Construct a state transition matrix from the given data.

Step 2. Predicting Future Market Share. The purpose of Markov analysis is to predict the future value. This is given by using following equation :

$$\begin{bmatrix}\text{Expected market shares} \\ \text{of period 2}\end{bmatrix} = \begin{bmatrix}\text{Market shares} \\ \text{begining of period 1}\end{bmatrix}\begin{bmatrix}\text{State transition} \\ \text{matrix}\end{bmatrix}$$

$$\begin{bmatrix}\text{Expected market shares} \\ \text{of period 3}\end{bmatrix} = \begin{bmatrix}\text{Expected market} \\ \text{shares of period 2}\end{bmatrix}\begin{bmatrix}\text{State transition} \\ \text{matrix}\end{bmatrix}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\begin{bmatrix}\text{Expected market shares} \\ \text{of period } n+1\end{bmatrix} = \begin{bmatrix}\text{Expected market} \\ \text{shares of period } n\end{bmatrix}\begin{bmatrix}\text{State transition} \\ \text{matrix}\end{bmatrix}$$

Step 3. Determine the steady state condition (or equilibrium condition) :

To find a steady state condition for a Markov system with transition matrix P, we solve the system of simultaneous equation by

$$x + y + z + \ldots = 1$$

and
$$\underset{\text{Period } n+1 \text{ share}}{[x \;\; y \;\; z, \ldots]} = \underset{\text{Period } n \text{ share}}{[x \;\; y \;\; z \;\ldots]} \; \underset{\text{State transition matrix}}{P}$$

where $x, y, z, \ldots$ are the unknowns as there are states in the Markov system.

Example 3. *The state-transition matrix for retentions, gains and losses of firms A, B and C is given below. Using this matrix determine the steady state equilibrium conditions :*

From	To A	B	C
A	0·700	0·100	0·200
B	0·100	0·800	0·100
C	0·200	0·100	0·700

Solution. The given state transition matrix is

$$P = \begin{bmatrix} 0\cdot700 & 0\cdot100 & 0\cdot200 \\ 0\cdot100 & 0\cdot800 & 0\cdot100 \\ 0\cdot200 & 0\cdot100 & 0\cdot700 \end{bmatrix}.$$

Let x, y, z denotes the variables of state probabilities. The steady state condition (or equilibrium condition) given by

$$[x \ y \ z] = [x \ y \ z] P$$

i.e.,
$$[x \ y \ z] = [x \ y \ z] \begin{bmatrix} 0\cdot700 & 0\cdot100 & 0\cdot200 \\ 0\cdot100 & 0\cdot800 & 0\cdot100 \\ 0\cdot200 & 0\cdot100 & 0\cdot700 \end{bmatrix}.$$

Solving above matrix equation, we have the following set of simultaneous linear equations :

$$x = 0\cdot700x + 0\cdot100y + 0\cdot200z \qquad ...(1)$$
$$y = 0\cdot100x + 0\cdot800y + 0\cdot100z \qquad ...(2)$$
$$z = 0\cdot200x + 0\cdot100y + 0\cdot700z. \qquad ...(3)$$

Also including, $x + y + z = 1$ ...(4)

as sum of the probability vector [*i.e.*, x, y, z] must be unity.

The values for the steady state variables can be found solving these four equations simultaneous. Writing the above equations as :

$$0\cdot700x - x + 0\cdot100y + 0\cdot200z = 0$$
$$0\cdot100x + 0\cdot800y - y + 0\cdot100z = 0$$
$$0\cdot200x + 0\cdot100y + 0\cdot700z - z = 0$$
$$x + y + z = 1$$

or
$$-0\cdot300x + 0\cdot100y + 0\cdot200z = 0$$
$$0\cdot100x - 0\cdot200y + 0\cdot100z = 0$$
$$0\cdot200x + 0\cdot100y - 0\cdot300z = 0$$
$$x + y + z = 1.$$

Subtracting first equation from third equation, we have

$$\begin{array}{r} 0\cdot200x + 0\cdot100y - 0\cdot300z = 0 \\ -(-0\cdot300x + 0\cdot100y + 0\cdot200z) = 0 \\ \hline 0\cdot500x \qquad\qquad - 0\cdot500z = 0 \end{array}$$

or $0\cdot500x = 0\cdot500z$

or $x = (0\cdot500/0\cdot500)\, z$ ...(5)

or $x = z.$

Multiply second equation by 2 and subtracting from third equation, we have

$$\begin{array}{r} 0\cdot200x + 0\cdot100y - 0\cdot300z = 0 \\ -(0\cdot200x - 0\cdot400y + 0\cdot200z) = 0 \\ \hline 0\cdot500y - 0\cdot500y = 0 \end{array}$$

or $0\cdot500y = 0\cdot500z$ ...(6

$y = z.$

Substituting these values of x and y in equation (4), we have

$$x + y + z = 1$$
$$z + z + z = 1$$
$$3z = 1 \quad \text{or} \quad \boxed{z = 1/3}$$

or

1

Then from equation (5),

$$\boxed{x = 1/3}$$

and from equation (6)

$$\boxed{y = 1/3}$$

Hence, the solution for equilibrium or steady state condition are

$$x = 1/3, y = 1/3, z = 1/3.$$

This means, that the defined system is in its steady state condition when each firm has captured $33\frac{1}{3}$ % (one-third) share of the market.

Example 4. *Suppose there are two market products of brands A and B respectively. Let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below :*

$$\begin{array}{cc} & \textit{To} \\ & \begin{array}{cc} A & B \end{array} \\ \textit{From} \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0{\cdot}9 & 0{\cdot}1 \\ 0{\cdot}5 & 0{\cdot}5 \end{pmatrix} \end{array}$$

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady-state.

Solution. The given transition matrix are as $\begin{pmatrix} 0{\cdot}9 & 0{\cdot}1 \\ 0{\cdot}5 & 0{\cdot}5 \end{pmatrix}$.

Given initial distribution row vector for brands A and B are 50% ~~are~~ and 50% each *i.e.*,

$$V_0 = [0{\cdot}50 \quad 0{\cdot}50].$$

Determine the expected market shares for the next period (*i.e.*, for $n = 1$). This is given by

$$\begin{bmatrix} \text{Market share of} \\ \text{period 1 at beginning} \end{bmatrix} \times \begin{bmatrix} \text{State transition} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{Expected market share} \\ \text{of period 2} \end{bmatrix}$$

i.e., $$V_0 \times P = V_1$$

or $$[0{\cdot}50 \quad 0{\cdot}50] \begin{bmatrix} 0{\cdot}9 & 0{\cdot}1 \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix} = [0{\cdot}7 \quad 0{\cdot}3].$$

Thus, for $n = 1$, we have
$$\begin{array}{lcc} & \text{Brand } A & \text{Brand } B \\ V_1 = [& 0{\cdot}7 & 0{\cdot}3\,] \end{array}$$ 2

Again determine the expected market share for the next period (*i.e.*, for $n = 2$). This is given by

$$\begin{bmatrix} \text{Market share at} \\ \text{beginning of period 2} \end{bmatrix} \times \begin{bmatrix} \text{State transition} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{Expected market} \\ \text{share of period 3} \end{bmatrix}$$

i.e., $$V_1 \times P = V_2$$

$$[0{\cdot}7 \quad 0{\cdot}3] \times \begin{bmatrix} 0{\cdot}9 & 0{\cdot}1 \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix} = \begin{array}{cc} A & B \\ [0{\cdot}78 & 0{\cdot}22] \end{array}$$

In similar way, we have

$$V_2 \times P = V_3$$

or $$[0\cdot78 \quad 0\cdot22] = \begin{bmatrix} 0\cdot9 & 0\cdot1 \\ 0\cdot5 & 0\cdot5 \end{bmatrix} = [0\cdot812 \quad 0\cdot188]$$

$$V_3 \times P = V_4$$

or $$[0\cdot812 \quad 0\cdot188] \begin{bmatrix} 0\cdot9 & 0\cdot1 \\ 0\cdot5 & 0\cdot5 \end{bmatrix} = [0\cdot8248 \quad 0\cdot1752]$$

$$V_4 \times P = V_5$$

$$[0\cdot8248 \quad 0\cdot1752] \begin{bmatrix} 0\cdot9 & 0\cdot1 \\ 0\cdot5 & 0\cdot5 \end{bmatrix} = [0\cdot82992 \quad 0\cdot17008]$$

and $$V_5 \times P = V_6$$

$$[0\cdot82992 \quad 0\cdot17008] \begin{bmatrix} 0\cdot9 & 0\cdot1 \\ 0\cdot5 & 0\cdot5 \end{bmatrix} = [0\cdot83 \quad 0\cdot17].$$

Thus we observe that starting with $[0\cdot50 \quad 0\cdot50]$ of the market shares, after 6 periods the resulting market shares are about $[0\cdot83 \quad 0\cdot17]$ *i.e.*, 83% and 17% respectively. Here, we see that the future amounts approach state values, after 6 period. Hence equilibrium or steady state condition reached at this point.

Thus, the steady-state market shares of A and B will be 5/6 and 1/6 respectively of the total market share.

Remark. Equilibrium (or steady state) position can also be determined by determining x and y that satisfy

$$(x \ \ y)\begin{pmatrix} 0\cdot9 & 0\cdot1 \\ 0\cdot5 & 0\cdot5 \end{pmatrix} = (x \ \ y) \quad \text{and } x + y = 1.$$

This gives : $0\cdot9x + 0\cdot5y = x$, $0\cdot1x + 0\cdot5y$, and $x + y = 1$.

Solving, these we get $x = 5/6$ and $y = 1/6$.

Example 5. *On January 1 (this year), Bakery A had 40% of its local market share while the other two bakeries B and C had 40% and 20% respectively of the market share. Based upon a study by a marketing research firm, the following facts were compiled. Bakery A retains 90% of its customers while gaining 5% of competitor B's customers and 10% of C's customers. Bakery B retains 85% of its customers while gaining 5% of A's customers and 7% of C's customers. Bakery C retains 83% of its customers and gains 5% of A's customers and 10% of B's customers. What will each firm's share be on January 1, next year, and what will each firm's share be on January 1, next year, and what will each firm's market share be at equilibrium ?*

Solution. Using the above given data, the retention and gain and retention and loss, the state transition matrix is formulate as :

$$\begin{array}{cc} & \text{Bakery} \\ P = \text{Bakery} & \begin{pmatrix} 0\cdot90 & 0\cdot05 & 0\cdot05 \\ 0\cdot05 & 0\cdot85 & 0\cdot10 \\ 0\cdot10 & 0\cdot07 & 0\cdot83 \end{pmatrix} \begin{array}{l} \text{Retention} \\ \text{and} \\ \text{gain} \downarrow \end{array} \\ & \rightarrow \text{Retention and loss} \end{array}$$

At initial state, on January 1, the market shares of the three Bakeries are 40, 40 and 20 per cent respectively

i.e., initial distribution row vector is $V_0 = [0\cdot40 \quad 0\cdot40 \quad 0\cdot20]$.

The expected market share for Bakeries A, B and C on January 1 next year (*i.e.*, for $n = 2$) are given as

'..riables ①

'S / 515

$$\begin{pmatrix}\text{Market share on}\\ \text{January 1 this year}\end{pmatrix}\begin{pmatrix}\text{State transition}\\ \text{matrix}\end{pmatrix} = \begin{pmatrix}\text{Expected market share}\\ \text{on January next year}\end{pmatrix}$$

or

$$V_0 \times P = V_1$$

$$[0{\cdot}40 \quad 0{\cdot}40 \quad 0{\cdot}20]\begin{pmatrix}0{\cdot}90 & 0{\cdot}05 & 0{\cdot}05\\ 0{\cdot}05 & 0{\cdot}85 & 0{\cdot}10\\ 0{\cdot}10 & 0{\cdot}07 & 0{\cdot}83\end{pmatrix} = [0{\cdot}400 \quad 0{\cdot}374 \quad 0{\cdot}226].$$

Thus the market shares of Bakeries A, B and C on January 1 next year will be 40%, 37·4% and 22·6% respectively.

Calculation of Equilibrium Condition :

Let x, y, z denotes the variables of state probabilities.

The steady state condition or equilibrium condition given by

$$[x \ \ y \ \ z] = [x \ \ y \ \ z]\,P$$

i.e.,

$$[x \ \ y \ \ z] = [x \ \ y \ \ z]\begin{bmatrix}0{\cdot}90 & 0{\cdot}05 & 0{\cdot}05\\ 0{\cdot}05 & 0{\cdot}85 & 0{\cdot}10\\ 0{\cdot}10 & 0{\cdot}07 & 0{\cdot}83\end{bmatrix}.$$

Solving above matrix equation, we have the following set of simultaneous linear equations

$$x = 0{\cdot}90x + 0{\cdot}05y + 0{\cdot}10z \quad ...(1)$$
$$y = 0{\cdot}05x + 0{\cdot}85y + 0{\cdot}07z \quad ...(2)$$
$$z = 0{\cdot}05x + 0{\cdot}10y + 0{\cdot}83z. \quad ...(3)$$

Also including, $x + y + z = 1$ ...(4)

as sum of the probabilities vector (*i.e.*, x, y, z) must be unity.

The values for the steady state variables can be found by solving these four equations simultaneously-writing the above equations as :

$$-0{\cdot}10x + 0{\cdot}05y + 0{\cdot}10z = 0$$
$$0{\cdot}05x - 0{\cdot}15y + 0{\cdot}07z = 0$$
$$0{\cdot}05x + 0{\cdot}10y - 0{\cdot}17z = 0$$
$$x + y + z = 1$$

Solving these equations, we get

$$x = 0{\cdot}43,\ y = 0{\cdot}28,\ z = 0{\cdot}29.$$

Hence the three firms market shares at equilibrium are :

Bakery A : 43% of the total market

Bakery B : 28% of the total market

Bakery C : 29% of the total market.

Example 6. *The number of units of an item that are withdrawn from inventory on a day-to-day basis is a Markov chain process in which requirements for tomorrow depend on today's requirements. A one-day transition matrix is given below :*

Number of units withdrawn from inventory

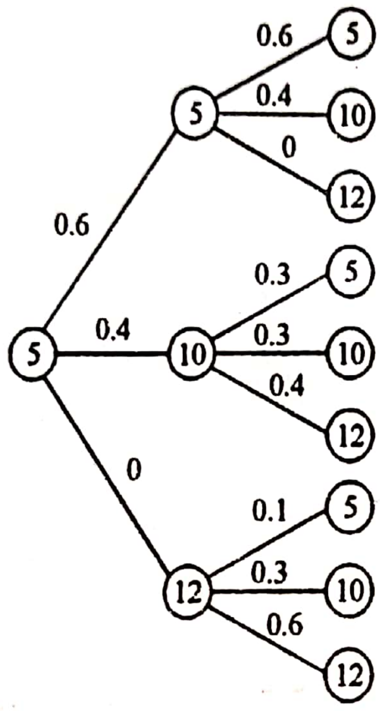
$$\begin{array}{cc} & \textit{Tomorrow} \\ & \begin{array}{ccc}5 & 10 & 12\end{array} \\ \begin{array}{r}5\\ \textit{Today}\ 10\\ 12\end{array} & \begin{bmatrix}0{\cdot}6 & 0{\cdot}4 & 0{\cdot}0\\ 0{\cdot}3 & 0{\cdot}3 & 0{\cdot}4\\ 0{\cdot}1 & 0{\cdot}3 & 0{\cdot}6\end{bmatrix}\end{array}$$

(a) Construct a tree diagram showing inventory requirements on two consecutive days

(b) Develop a two-day transition matrix.

(c) Comment on how a two-day transition matrix might be helpful to a manager who is responsible for inventory management.

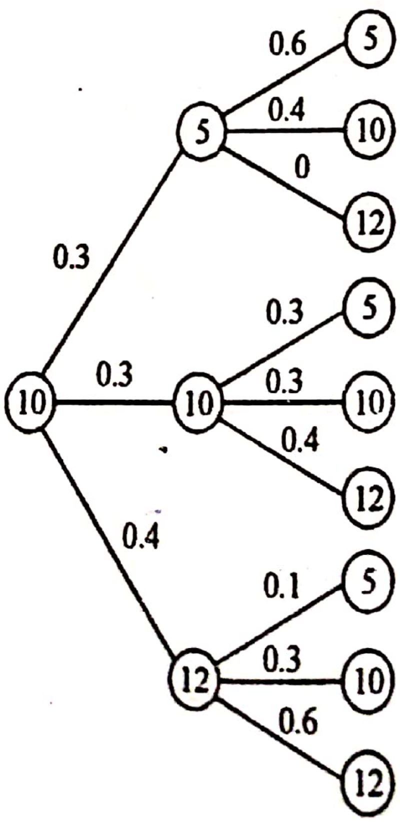
Solution. (a) Tree diagrams showing inventory requirements are shown as :



$p_{11}^{(2)} = (0.6)(0.6) + (0.4)(0.3) + (0)(0.1) = 0.48$

$p_{12}^{(2)} = (0.6)(0.4) + (0.4)(0.3) + (0)(0.3) = \cancel{0.30}$ 0.36.

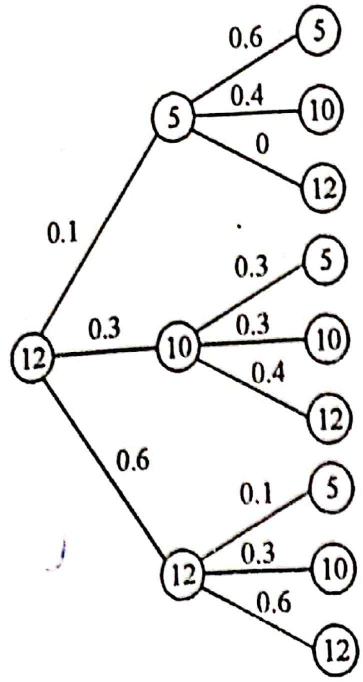
$p_{13}^{(2)} = (0.6)(0) + (0.4)(0.4) + (0)(0.6) = 0.16$



$p_{21}^{(2)} = (0.3)(0.6) + (0.3)(0.3) + (0.4)(0.1) = 0.31$

$p_{22}^{(2)} = (0.3)(0.4) + (0.3)(0.3) + (0.4)(0.3) = 0.33$

$p_{23}^{(2)} = (0.3)(0) + (0.3)(0.4) + (0.4)(0.6) = 0.36$



$p_{31}^{(2)} = (0.1)(0.6) + (0.3)(0.3) + (0.6)(0.1) = \cancel{0.16}$ $= 0.21$

$p_{32}^{(2)} = (0.1)(0.4) + (0.3)(0.3) + (0.6)(0.3) = 0.31$

$p_{33}^{(2)} = (0.1)(0) + (0.3)(0.4) + (0.6)(0.6) = 0.48$

(b) Let P be the transition probabilities matrix. Then a two-day (*i.e.*, $n = 2$) transition matrix is given by

$$P^2 = PP = \begin{bmatrix} 0\cdot6 & 0\cdot4 & 0\cdot0 \\ 0\cdot3 & 0\cdot3 & 0\cdot4 \\ 0\cdot1 & 0\cdot3 & 0\cdot6 \end{bmatrix} \begin{bmatrix} 0\cdot6 & 0\cdot4 & 0\cdot0 \\ 0\cdot3 & 0\cdot3 & 0\cdot4 \\ 0\cdot1 & 0\cdot3 & 0\cdot6 \end{bmatrix} = \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0\cdot48 & 0\cdot36 & 0\cdot16 \\ 0\cdot31 & 0\cdot33 & 0\cdot36 \\ 0\cdot21 & 0\cdot31 & 0\cdot48 \end{bmatrix} \end{array}$$

(c) As a result of delivery time requirements, the two-day transition matrix can be used for guiding ordering decision. For example, if today the manager experiences a demand for five units, then 2 days later, the probability of requiring five units is 0·48, that of requiring ten units 0·36 and twelve units 0·16.

PROBLEM SET

1. What do you understand by Markov process ? In what areas of management can it be applied successfully ?
2. What do you understand by a Markov chain ? Give suitable examples.
3. Explain the following terms :
   (i) Markov process (ii) transition probabilities
   (iii) Matrix of state transition probabilities (iv) Steady-state (equilibrium) condition.
4. Discuss the fundamental properties of Markov chains.
5. Explain with illustrations, how Monto-Carlo methods are useful in Operation Research.
6. Suppose that new razor blades were introduced in the market by three companies at the same time. When they were introduced, each company had an equal share of the number, but during the first year the following changes took place :
   (i) Company A retained 90 percent of its cutomser, lost 3 percent to B and 7 percent to C.
   (ii) Company B retained 80 per cent of its customer, lost 10 per cent to A, and 20 per cent to C.
   (iii) Company C retained 80 per cent of its customers, lost 10 per cent to A and 10 per cent to B.
   Assuming that on changes in the buying habits of the consumer occur,
   (a) What are the market shares of the three companies at the end of the first year ? The second year ?
   (b) What are long-run market shares of the three companies
7. A market survey is made on three brands of breakfast foods X, Y and Z. Every time the customer purchases a new package, he may buy the same brand or switch to another brand. The following estimates are obtained, expressed as decimal fractions :

$$
\begin{array}{cc}
 & \text{Brand just purchased} \\
 & \begin{array}{ccc} X & Y & Z \end{array} \\
\text{Percent brand} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0\cdot7 & 0\cdot2 & 0\cdot1 \\ 0\cdot3 & 0\cdot5 & 0\cdot2 \\ 0\cdot3 & 0\cdot3 & 0\cdot4 \end{bmatrix}
\end{array}
$$

At this time it is estimated that 30 per cent of the people buy brand X, 20 percent brand Y and 50 percent brand Z. What will the distribution of customers be two time periods late and at equilibrium ?

8. In a city there are three T.V. channels fighting it out for the top rating for the 6.00 PM one hour news cast. At the end of each week, the leader is the channel with the highest estimated average fraction of the viewing audience during that time slot. Over a period of time, data have been obtained on the relationship between ratings leadership in successive weekly periods. This information is presented in the form of a transition probability table as follows :

$$
\begin{array}{cc}
 & \text{Next week's leader (station)} \\
 & \begin{array}{ccc} A & B & C \end{array} \\
\text{This week's leader (station)} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0\cdot40 & 0\cdot35 & 0\cdot25 \\ 0\cdot10 & 0\cdot70 & 0\cdot20 \\ 0\cdot15 & 0\cdot25 & 0\cdot60 \end{bmatrix}
\end{array}
$$

Over a long period of time, what proportion of the time will each station lead the weekly ratings ?

9. The number of units of an item that are withdrawn from inventory on a day-to-day basis is a Markov chain process in which requirements for tomorrow depend on today's requirements. A one-day transition matrix is given below :

Number of units withdrawn from inventory

$$
\begin{array}{cc}
 & \text{Tomorrow} \\
 & \begin{array}{ccc} 5 & 10 & 12 \end{array} \\
\text{Today} \begin{array}{c} 5 \\ 10 \\ 12 \end{array} & \begin{bmatrix} 0\cdot6 & 0\cdot4 & 0\cdot0 \\ 0\cdot3 & 0\cdot3 & 0\cdot4 \\ 0\cdot1 & 0\cdot3 & 0\cdot6 \end{bmatrix}
\end{array}
$$

(a) Construct a tree diagram showing inventory requirements on two consecutive days.

(b) Develop a two-day transition matrix.

(c) Comment on how a two-day transition matrix might be helpful to a manager who is responsible for inventory management.

10. Consider the following transition matrix for brand switching :

$$
\begin{array}{cc}
 & \begin{array}{ccc} A & B & C \end{array} \\
\begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0\cdot75 & 0\cdot15 & 0\cdot10 \\ 0\cdot20 & 0\cdot35 & 0\cdot45 \\ 0\cdot35 & 0\cdot20 & 0\cdot45 \end{bmatrix}
\end{array}
$$

The manufactures of brand C are considering a marketing strategy to attract brand B customers. It is estimated that this strategy will : (a) increase probability of customer's switch from B to C by 0·20, (b) decrease probability of customer's switch from C to B by 0·10, and (c) decrease probability of customer's switch from A to C by 0·25. Should the new strategy be used ?

11. A market analysis group studying car purchasing trends in a certain region has concluded that on the average a new car is purchased once very 3 years. The buying pattrens are descirbed by the following matrix A

$$
\begin{array}{cc}
 & \begin{array}{cc} \text{Small} & \text{Large} \end{array} \\
\begin{array}{c} \text{Small} \\ \text{Large} \end{array} & \begin{bmatrix} 80\% & 20\% \\ 40\% & 50\% \end{bmatrix}
\end{array}
$$

The elements of A are to be interpreted as follows. The first row indicate that of the current small cars, 80 per cent will be replaced with a small car, and 20 per cent with a large car. The second new implies that 40 per cent of the current large cars will be replaced with small cars, while 60 per cent will be replaced by large cars. Construct a stochastic matrix P from A that will define a Markov chain model of these burying trends, if there are current 40,000 small cars and 50,000 large cars in the region, what will the distribution be in 12 years ?