Expectation :- The behaviour of a R.V. (either discrete or Continuous) is completely characterized by the distribution function $F(x)$ or density $f(x)$ [$P(x_i)$ in discrete case]. Instead of a function, a more compact description can be made by a single numbers such as mean (expectation), median and mode known as measures of central tendency of the R.V. X.

Expected or mean or expected Value –

~~Expectati~~ Expectation or mean or expected value of a random variable X, denoted by $E(X)$ or μ, is defined as

$$E(X) = \begin{cases} \sum_i x_i f(x_i) & , \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\, dx & , \text{if } X \text{ is Continuous} \end{cases}$$

Note 1: x is median if $P(X < x) \leq \frac{1}{2}$ and $P(X > x) \leq \frac{1}{2}$

Note 2: x is mode for which $f(x)$ or $P(x_i)$ attains its maximum.

Variance:-

Variance characterizes the variability in the distributions, since two distributions with same mean can still have different dispersion of data about their means.

Variance of R.V. X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (X-\mu)^2 f(x), \text{ for } X \text{ discrete}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

for X Continuous.

Standard Deviation:- (S.D.) denoted by σ, is the positive square root of variance.

Result:- $\sigma^2 = E(X^2) - \mu^2$

since

$$\sigma^2 = \sum_x (x-\mu)^2 f(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) f(x)$$

$$= \sum x^2 f(x) - 2\mu \sum x f(x) + \mu^2 \sum f(x)$$

$$= E(X^2) - 2\mu.\mu + \mu^2.1$$

$$= E(X^2) - \mu^2$$

since $\mu = \sum x f(x)$, $\sum f(x) = 1$

similar result follows for Continuous R.V. X, with Σ replaced by integration from $-\infty$ to ∞

Note 1:- In a gambling game, expected value E of the game is considered to be the value of the game to the player, Game is favourable to the player if $E > 0$, unfavourable if $E < 0$, fair if $E = 0$

Note 2:- Mathematical expectation

$$E = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k$$

where the probabilities of obtaining the amounts $a_1, a_2, \cdots$ or a_k are $p_1, p_2, \cdots p_k$ respectively

Continuous Probability Distributions

Ex 1 Suppose a Continuous R.V. x has the probability density

$$f(x) = \begin{cases} K(1-x^2) & \text{for } 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find K (b) Find $P(0.1<x<0.2)$

(c) $P(x>0.5)$

Using distribution function, determine the probabilities that

(d) x is less than 0.3

(e) between 0.4 and 0.6

(f) Calculate mean and variance for the probability density function.

Soln (a) Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$ so

$$\int_{-\infty}^{\infty} f(x) = \int_0^1 K(1-x^2)\,dx = K\left(x-\frac{x^3}{3}\right)_0^1$$

$$\Rightarrow \frac{2}{3}K = 1$$

$$\therefore K = \frac{3}{2}$$

(b) $P(0.1<x<0.2) = \int_{0.1}^{0.2} K(1-x^2)\,dx$

$$= \frac{3}{2}\left(x-\frac{x^3}{3}\right)_{0.1}^{0.2} = 0.1465$$

(c) $P(x>0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{1} f(x)\,dx$

$$= \frac{3}{2}\left(x-\frac{x^3}{3}\right)_{0.5}^{1} = 0.3125$$

(d) Distribution function:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{so}$$

$$F(x) = \int_{0}^{x} \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\left(x - \frac{x^3}{3}\right)$$

$$F(x<0.3) = \int_{-\infty}^{0.3} f(t)\,dt = \frac{3}{2}\left(x - \frac{x^3}{3}\right)_0^{0.3}$$

$$= 0.4365$$

(e) $F(0.4 < x < 0.6) = F(b) - F(a)$

$$= F(0.6) - F(0.4)$$

$$= \frac{3}{2}\left(x - \frac{x^3}{3}\right)_{0.4}^{0.6} = 0.224$$

(f) Mean $= \mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_0^1 x\left\{\frac{3}{2}(1-x^2)\right\}dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 = \frac{3}{8}$$

Variance $= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$$= \int_0^1 \left(x - \frac{3}{8}\right)^2 \frac{3}{2}(1-x^2)\,dx = \frac{19}{320}$$

or Variance $= \int_0^1 x^2\{k(1-x^2)\}dx - \mu^2$

$$= k\left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1 - \mu^2 = \frac{19}{320}$$

Ex 2. The daily consumption of electric power (in millions of KW-hours) is R.V. having the

P.D.F. $f(x) = \begin{cases} \frac{1}{9} x e^{-x/3}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

If the total production is 12 million KW-hours determine the probability that there is power cut (shortage) on any given day.

Solⁿ Probability that the power consumed is between 0 to 12 is

$$P(0 \leq x \leq 12) = \int_0^{12} f(x)\,dx = \int_0^{12} \frac{1}{9} x e^{-x/3}\,dx$$

$$= \left[-\frac{x}{3} e^{-x/3} - e^{-x/3}\right]_0^{12} = 1 - 5e^{-4}$$

Power supply is inadequate if daily consumption exceeds 12 million KW, ie,

$$P(x > 12) = 1 - P(0 \leq x \leq 12)$$

$$= 1 - [1 - 5e^{-4}]$$

$$= 5e^{-4}$$

$$= 0.0915781$$