MAKOV ANALYSIS

16.1. INTRODUCTION

Markov process (or *chain*) is a stochastic (or random) process which has the property that the probability of transition from a given state to any future state depends only on the present state and not on the manner in which it was reached.

The nature of Markov processes can be easily understood by considering the situation of a Car-Rental company. Suppose a Car-Rental company is running agencies in different cities. A car sent to one city may return to any city where the company's agency is available. If this situation is considered as a *Markov process*, then the different rental cities would be the *states*. A particular transition probability p_{ij} would be the probability that a car rented to city i would return to city j, where j may be equal to i. The mathematical structure of this problem is to determine expected long term fraction of cars at each city and the mean number of trips a car would make starting from city i, before returning to that location.

Markov process is widely used in examining and predicting the behaviour of consumers in terms of their brand loyalty and their switching patterns to other brands.

Markov processes are also used in the study of equipment maintenance and failure problems analysing accounts receivable that will ultimately become bad debts. It is also used to study the stock market price movements.

16.2. STOCHASTIC (RANDOM) PROCESS

Definition. A *stochastic* (or *random*) process is defined as a family of random variables $\{X(t_n) : n = 1, 2, 3, \ldots\}$. The random variable $X(t)$ stands for the observation at time t. The number of states n may be *finite* or *infinite* depending upon the time range.

For example, let us consider the poisson distribution

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n\,!},\ n = 1, 2, 3, \ldots .$$

This distribution represents a stochastic (or random) process with infinite number of states. In this example, the random variable n denotes the number of occurrences between the time interval 0 and t (assuming that the system starts at 0 times). Thus the states of the system at any time t are given by n = 0, 1, 2, ...

16.3. MARKOV PROCESS

Definition. *A stochastic* (or *random*) *system is called a* **Markov process** *if the occurrence of a future state depends on the immediately preceding state and only on it.*

Thus if $t_0 < t_1 < \ldots < t_n$ represents the points in time scale then the family of random variables $\{X(t_n)\}$ is said to be a *Markov* process provided it holds the *Markovian* property :

$$P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0\} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

for all $X(t_0), X(t_1), \ldots, X(t_n)$.

Markov process is a sequence of n experiments in which each experiment has n possible outcomes $x_1, x_2, \ldots x_n$. Each individual outcome is called a *state* and the probability (that a particular outcome occurs) depends only on the probability of the outcome of the preceding experiment. The simplest of the *Markov* processes is *discrete* and *constant* over time. It is used when the sequence of experiments is completely described in terms of its states (possible outcomes). There is a finite set of states numbered $1, 2, \ldots, n$, and this process can be only in one state at a prescribed time. A system is said to *discrete* in time if it is examined at regular intervals, *e.g. daily, weekly, monthly,* or *yearly.*

16.4. TRANSITION PROBABILITY

Definition. *The probability of moving from one state to another or remaining in the same state during a single time period is called the **transition probability**.*

Mathematically, the probability

$$p_{x_{n-1}, x_n} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

is called the *transition probability.*

This represents the conditional probability of the system which is now in state x_n at time t_n, provided that it was previously in state x_{n-1} at time t_{n-1}. Sometimes this probability is known as *one step* transition probability, because it describes the system during the time interval (t_{n-1}, t_n).

Since each time a new result or outcome occur, the process is said to have *stepped* or *incremented* one step. Each step represents a time period or any other condition which would result in another possible outcome. The symbol n is used to indicate the number of steps or increments. For example, if $n = 0$, then it represents the initial state.

16.5. TRANSITION PROBABILITY MATRIX

The transition probabilities can be arranged in a matrix form and such a matrix is called a ***one-step transition probability matrix***, denoted by

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

The matrix P is a squared matrix whose each element is non-negative and sum of the elements of each row is unity *i.e.,*

$$\sum_{j=1}^{m} p_{ij} = 1; \quad i = 1, 2, ..., m, \text{ and } 0 \le p_{ij} \le 1.$$

In general, any matrix P, whose elements are non-negative and sum of the elements either in each row or column is unity, is called a ***transition matrix*** or a ***probability matrix***. Thus a transition matrix is a square stochastic matrix (since number of rows is equal to the number of columns in the matrix) and therefore, it gives the complete description of the Markov process.

Diagrammatic Representation of Transition Probabilities :

The transition probabilities can also be represented by two types of diagrams :

(1) Transition Diagram. Transition diagram shows the transition probabilities or shifts that can occur in any particular situation. Such a diagram is given in Fig. 16.1. :

The arrows from each state indicate the possible states to which a process can move from the given state. The matrix of transition probabilities which corresponds to above diagram is as given below :

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & ½ & ½ \\ ⅓ & 0 & ⅔ \end{bmatrix} \end{array}$$

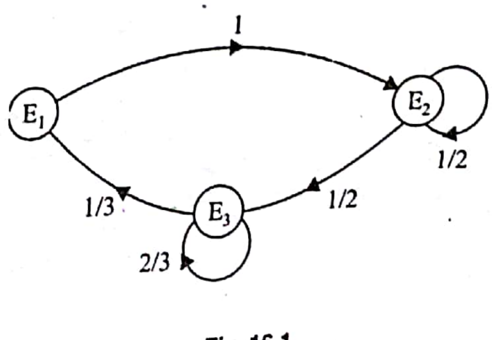

Fig. 16.1

A *zero* element in the above matrix indicates that the transition is not possible.

(2) Probability tree diagram. As the name implies, this diagram emphasizes the probabilities and their movement from one step to another step, alongwith all possible branches or paths that may connect the outcomes over a period of time. Tree diagram can be explained by the following example.

Example 1. *Two manufacturers A and B are competing with each other in a restricted market. Over the years, A's customers have exhibited a high degree of loyalty as measured by the fact that customers using A's product 80% of time. Also former customers purchasing the product from B have switched back to A's 60% of time.*

(a) Construct and interpret the state trasition matrix in terms of (i) retention and loss (ii) retention and gain.

(b) Calculate the probability of a customer purchasing A's product at the end of the second period.

Solution. **(a)** The transition probabilties can be arranged in a matrix form as shown below.

$$\text{Present } P = \textit{Purchase } (n=0) \begin{array}{c|cc} & \text{A} & \text{B} \\ \hline A & 0.80 & 0.20 \\ B & 0.60 & 0.40 \end{array} \quad \textit{Retention} \text{ and } \textit{Gains} \downarrow$$

Next purchase ($n = 1$)

– Retention and Losses →

Obviously, the probability of a customer's purchase at the next step $n = 1$ (next purchase depends upon the product which a customer is having at present ($n = 0$). Each probability in the above matrix must therefore be a conditional probability for passing from one state to another.

Mathematically, conditional probabilities in the above matrix can be stated as

(*i*) $P\,(A_o \mid A_1) = p_{11} = 0.80.$

This indicates that the probability that the customers now using A's product at $n = 0$ (present purchase) will again purchase A's product at $n = 1$ (next purchase) is 0.80. This means ***retention*** to A 's product.

(*ii*) $p\,(B_0 \mid A_1) = p_{21} = 0.60.$

This indicates that probability that the customer now using B's product at $n = 0$ (present purchase) will purchase A's product at $n = 1$ (next purchase) is 0.60. This means ***loss*** to B's product.

(*iii*) $P\,(A_0 \mid B_1) = p_{12} = 0.20.$

This indicates that the probability that the customer now using A 's product at $n = 0$ (present purchase) will purchase B's product at $n = 1$ (next purchase) is 0.20. This means ***loss*** to A 's product.

(*iv*) $P\,(B_0 \mid B_1) = p_{22} = 0.40.$

This indicates that the probability that the customer now using B's product at $n = 1$ (present purchase) will purchase B's product at $n = 1$ (next purchase)is 0.20. This means ***retention*** to B 's product.

(b) The transition probabilities can be represented by two types of diagrams:

(*i*) transition diagram as shown in *Fig.* 16.2, and (*ii*) probability tree diagram as show in *Fig.* 16.3.

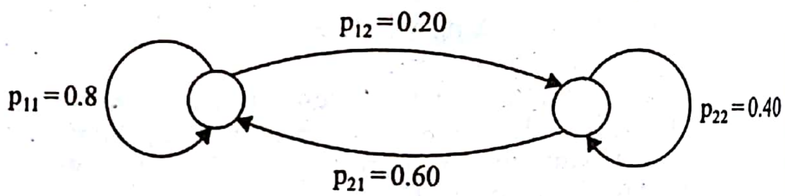

Fig. 16.2

Probability Tree Diagram

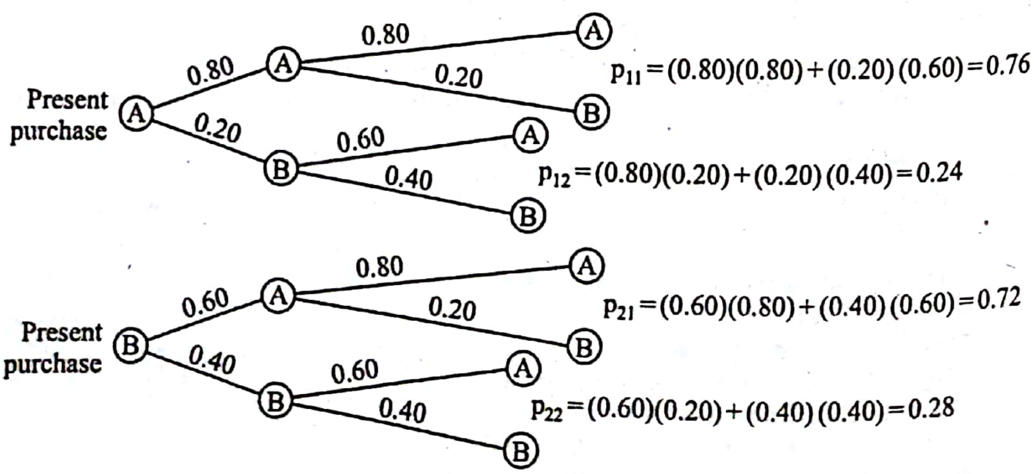

Fig. 16.3

In Fig 16.2, nodes indicate ***states*** and arrows represent the ***transition probabilities*** between states.

Probability Computations :

If we begin with a customer's purchase of A's product in the state E_1, at $n = 0$, then $P_1(0) = 1$ and $P_2(0) = 0$, so

$$R(0) = (1 \quad 0).$$

After the first transition, the row vector of state probabilities $R(1)$ which describes all possible outcomes at $n = 1$ is given by

$$R(1) = R(0)\,P = (1 \quad 0)\begin{pmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{pmatrix} = (0.80 \quad 0.20)$$

This means that if the present state is E_1, the probability (that the present state is E_1) is $P_1(1) = 0.80$, and (that the next state is E_2) is $p_2(1) = 0.20$. In other words, the probability of a customer using A's product at the end of step 1 is 80% and there are 20% chances that the customer will switch over to B's product at the end of step 1.

The probability that a customer using A's product in the state E_1 at $n = 0$ also uses A's product in the state E_2 at $n = 2$, can be obtained by calculating state row vector $R(2)$ of state probabilities which describes all possible outcomes in step $n = 2$.

$$R(2) = R(1)\,P = (0.80 \quad 0.20)\begin{pmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{pmatrix} = (0.76 \quad 0.24)$$

This indicates that if the present state is E_1 at $n = 0$, 2-step later (*i.e.*, $n = 2$), the probability of being in state E_1 is $p_1(2) = 0.76$ and in state E_2 is $p_2(2) = 0.24$. Thus the probability of A's product after the end of 2-step is 76% and that of B's product is 24%.

In a similar manner, if the present state is E_2, then $R(2) = (0.72 \quad 0.28)$ as obtained earlier.

16.6. FIRST ORDER AND HIGHER ORDER *MARKOV* PROCESS

The ***first order Markov process*** is based on the following three assumptions :

(i) The set of possible outcomes is *finite*.

(ii) The probability of the next outcome (state) depends only on the immediately preceding outcome.

(iii) The transition probabilities are constant over time.

The ***second order Markov process*** assumes that the probability of the next outcome (state) may depend on the two previous outcomes. Likewise, a ***third order Markov process*** asumes that the probability of the next outcome (state) can be calculated by obtaining and taking account of the outcomes of the past three outcomes.

But, in this chapter, we shall dicuss only *first order* Markov process.

16.7. *n*-STEP TRANSITION PROBABILITIES

Suppose the system which occupies state E_i at time $t = 0$, then we may be interested in finding out the probability that the system moves to state E_j at time $t = n$ (these time periods are sometimes referred to as number of steps). If the n-step transition probability is denoted by $p_{ij}^{(n)}$, then these transition probabilities can be represented in matrix form as given below.

$$P^{(n)} = \begin{array}{c} \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_m \end{array} \\ \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} & \cdots & p_{1m}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & \cdots & p_{2m}^{(n)} \\ \vdots & \vdots & & \vdots \\ p_{m1}^{(n)} & p_{m2}^{(n)} & \cdots & p_{mm}^{(n)} \end{bmatrix} \end{array}$$

Here $p_{21}^{(n)}$, for example, means the probability that the system which occupies state E_2 will move to state E_1 after n steps.

Let $p_i(n)$ be the probability that the system occupies state E_i will move to state E_j in one transition. It should be noted that the transition probability p_{ij} is independent of time whereas the *absolute probability* $p_i(n)$ depends on time. If the number of possible states be m, then

$$\sum_{i=1}^{m} p_i(n) = 1, \text{ and } \sum_{j=1}^{m} p_{ij} = 1 \text{ for all } i.$$

If all the state probabilities are known at time $t = n$, then the state probabilities at time $t = n + 1$ can be determined by the equation :

$$p_j(n+1) = \sum_{i=1}^{m} p_i(n)\, p_{ij} \; ; n = 0, 1, 2, ..$$

In other words, the probability of being in state E_j at time $t = n + 1$ is equal to the probability of being in state E_i at time $t = n$ multiplied by the probability of a transition from state E_i to state E_j for all values of i.

To make the procedure more clear, we may rewrite the equations for each state probability at time $t = n + 1$ as follows :

$$\begin{aligned} p_1(n+1) &= p_1(n)\, p_{11} + p_2(n)\, p_{21} + \ldots + p_m(n)\, p_{m1} \\ p_2(n+1) &= p_1(n)\, p_{12} + p_2(n)\, p_{22} + \ldots + p_m(n)\, p_{m2} \\ \vdots \quad & \quad\;\; \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ p_m(n+1) &= p_1(n)\, p_{1m} + p_2(n)\, p_{2m} + \ldots + p_m(n)\, p_{mm}. \end{aligned}$$

This system of equations can be written in matrix form as

$$R(n+1) = R(n)\, P, \qquad ...(16.1)$$

where $R(n+1)$ is the row vector of state probabilities at time $t = n + 1$, $R(n)$ is the row vector of state probabilities at time $t = n$, and P is the matrix of transition probabilities.

If the state probabilities at time $t = 0$ are known, these can be found at any time by solving the matrix equation (16.1). that is,

$$R(1) = R(0)\,P,\;\; R(2) = R(1)\,P = R(0)\,P^2,\;\; R(3) = R(2)\,P = R(0)\,P^3 \;\ldots\ldots\ldots,\; R(n) = R(n-1)\,P = R(0)\,P^n.$$

Q. A *Markov* chain with three states α, β and γ defined by the transition matrix

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 0 & 2/3 \\ 1/6 & 1/2 & 1/3 \end{bmatrix}$$

Taking the initial state to be α, determine the *n*th step transition probability $p_{\alpha x}^{(n)}$ and the sbsolute probabilities $p_j^{(n)}$.

16.8. MARKOV CHAIN

Let $p_j^{(0)}$ $(j = 0, 1, 2, ...)$ be the absolute probability such that the system be in state E_j at time t_0, where $E_j (j = 0, 1, 2, ...)$ denote the exhaustive and mutually exclusive outcomes (states) of a system at any time. Also it is assumed that the system is *Markovian.*

We now define

$$p_{ij} = P\{X(t_n) = j \mid X(t_{n-1}) = i\}$$

as the one-step transition probability of going from state i at time t_{n-1} to state j at time t_n. It is also assumed here that these probabilities from state E_i to state E_j $(i = 0, 1, 2, ...; j = 0, 1, 2, \ldots)$ are expressed in the matrix form as below :

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \ldots \\ p_{10} & p_{11} & p_{12} & \ldots \\ p_{20} & p_{21} & p_{22} & \ldots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

This matrix P is known as *stochastic matrix* or *homogeneous matrix*. The probabilities p_{ij} must satisfy the boundary conditions : $\Sigma\, p_{ij} = 1$, for all i, and $p_{ij} \geq 0$ for all i and j

Definition. *Markov chain.* *The transition matrix P as defined above together with the initial probabilities $\{p_j^{(o)}\}$ associated with the states $E_j (j = 0, 1, 2, \ldots)$ completely define a **Markov chain.***

The *Markov* chains are of two types : (i) ergodic, (ii) regular.

An ***ergodic Markov*** chain has the property that it is possible to pass from one state to another in a finite number of steps, regardless of present state.

A special type of *ergodic Markov chain* is the ***regular Markov chain.***

A *regular Markov* chain is defined as a chain having a transition matrix P such that for some hour of P it has only non-zero positive probability values.

Thus all regular chains must be *ergodic* chains. The easiest way to 'check if an *ergodic* chain is *regular*' is to continue squaring the transition matrix P until all zeros are removed.

Example 2. *Determine if the following transition matrix is ergodic Markov chain.*

Future States

$$\textbf{Present States}\quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1/3 & 1/3 & 0 & 1/3 \\ 2 & 0 & 1/2 & 1/4 & 1/4 \\ 3 & 1/4 & 0 & 1/2 & 1/4 \\ 4 & 0 & 0 & 1/3 & 2/3 \end{array}$$

Solution. Here we must check that it is possible to go from every present state to all other states.

We observe that from state 1, it is possible to go directly to every other state except state 3. For state 3, it is possible to go from state 1 to state 2 to state 3. Therefore it is possible to go from state 1 to any other state. Similarly, from state 2, it is possible to go to state 3 or state 4, then from state 3 to state 1, or from state 4 to state 3 to state 1. Also, from state 3 it is possible to go directly to state 1. Finally, from state 4, it is possible to go to state 3, then from state 3 to state 1. Hence above transition matrix is an *ergodic Markov chain.*

Example 3. *Test the following transition matrix to see if the Markov chain is regular and ergodic where x is some positive p_{ij} value.*

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array}$$

Solution. We compute :

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array} \qquad P^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array} \qquad P^8 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array}$$

From this, we observe that P raised to an even-number power gives the result as above, while P raised to an odd-number power will give the original matrix. Since all the elements are not non-zero positive elements, the above matrix is not regular. But, it is an *ergodic* since it is possible to go from state 1 to state 2 or state 3 from state 2 to state 1 or state 4. From state 2 to state 1. From state 3 to state 1. From state 4 to state 2 or state 1.

16.9. STEADY STATE (EQUILIBRIUM) CONDITION

The determination of ***steady state*** conditions in a *regular ergodic Markov chain* can be accomplished most readily by computing *pn* for larger values of n. Therefore, there is a limiting probability that the system will reach to steady state equilibrium in a finite number of transitions. This can be generalized in the following manner.

In equation (16.1), if n becomes very large, the values p_{ij} tend to fixed limits and each probability vector $R(n)$ approaches a constant value *i.e.*, $R(n+1) = R(n) = R$.

Thus, taking limits,

$$\lim_{n \to \infty} R(n+1) = \lim_{n \to \infty} R(n)\,P \quad \text{or} \quad R = RP.$$

Therfore, as $n \to \infty$, $R(n)$ becomes constant (*i.e.* independent of time) and then the system is said to have reached to a steady state equilibrium.

Example 4. *A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as* ***just overhauled good, fair*** *or* ***inoperative.*** *If the item is inoperative, it is overhauled, a procedure that takes one day. Let us denote the four classifications as states 1,2,3 and 4* *respectively. Assume that the working condition of the equipment follows a Markov processes with the following transition matrix :*

If it costs Rs. 125 to overhaul a machine (including lost time), on the average, and Rs. 75 in production is lost if a machine is found inoperative. Using steady state probabilities, compute the expected per day cost of maintenance.

Solution. The given transition matrix P can be interpreted as indicating 1/4 of the time it is in fair condition after a day's time, and 3/4 of the time just overhauled machine is in good condition after a day's use. But, a machine which is in good condition has equal chances of still being in good condition or of being in fair condition after a day's use; while a machine in fair condition has equal chances of being in fair or inoperative condition after a day's use. An inoperative machine will be overhauled the next day, so that at the end of the day it would have been just overhauled.

$$P = \text{Today} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \textbf{Tomorrow} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since the given matrix P is an *ergodic regular Markov process,* it will certainly reach to steady state equilibrium. Let the steady state probabilities $p_1, p_2, p_3,$ and p_4 represent the proportion of times that the machine will be in states 1, 2, 3, and 4 respectively.

Now with the help of steady state equations $R = RP,$ we have

$$(p_1 \; p_2 \; p_3 \; p_4) = (p_1 \; p_2 \; p_3 \; p_4)\begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In order to find p_1, p_2, p_3 and p_4, we require to solve the simultaneous equations :

$$p_1 = p_4,\; p_2 = 3/4\, p_1 + 1/2\, p_2,\; p_3 = 1/4\, p_1 + 1/2\, p_2 + 1/2\, p_3,\; p_4 = 1/2\, p_3 \quad \text{and} \quad p_1 + p_2 + p_3 + p_4 = 1.$$

Solving these equations, we get $p_1 = 2/11, p_2 = 3/11, p_3 = 4/11,$ and $p_4 = 2/11.$

Thus, on an average, 2 out of every 11 days the machine will be *overhauled;* 3 out of every 11 days it will be in *good condition*; 4 out of every 11days it will be in *fair condition;* and 2 out of every 11days it will be found *inoperative* at the end of the days.

Hence the expected (average) cost per day of maintenance will be given by $(2/11)\, 125 + (2/11)\, 75 = \text{Rs.}36.36.$

16.10. MARKOV ANALYSIS

In order to explain the Markov analysis, we present here an example of *Brand Switching Models* which emphasises on the time behaviour of customers who make repeated purchases of a product class, but from time to time may switch over from one brand to another. The basic element of a Markov process has to do with *various states*. In *brand switching models*, the state is generally the customer's preference for a particular brand.

Brand Switching Example. Let us consider a consumer sample distributed over two brands A and B, the samples being the representative of the entire group from the standpoint of their brand loyalty and their switching patterns. The behaviour of the large groups can be better described in probabilistic terms. This probabilistic description can be represented by **transition matrix** as explained by the following diagram

Transition Matrix

$$\text{From} \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \textbf{To} \\ \begin{array}{cc} A & B \end{array} \\ \boxed{\begin{array}{cc} p_{AA} & p_{AB} \\ p_{BA} & p_{BB} \end{array}} \end{array}$$

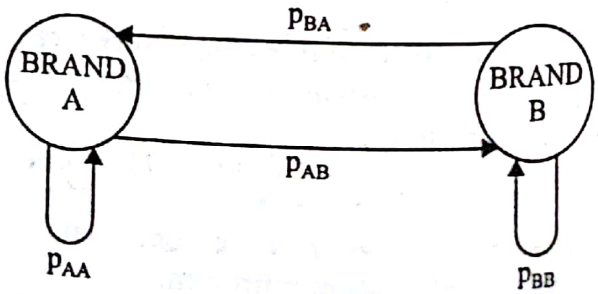

Fig. 16.4 Brand Switching Diagram

In general, for n brands $A_1, A_2, ...A_n$, the transition matrix can be represented as follows :

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where the probability p_{ij} is such that a customer's preference will switch from *brand i to brand j* from one period to the next.

The most important characteristic of a transition matrix is that

$$\sum_{j=1}^{n} p_{ij} = 1;\ i = 1, 2, ..., n$$

meaning thereby a customer must have some preference.

$$\text{From}\ \begin{array}{c|cccccc|} & A_1 & A_2 & \cdots & A_j & \cdots & A_n \\ \hline A_1 & p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\ A_2 & p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ A_i & p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ A_n & p_{n1} & p_{n2} & \cdots & p_{nj} & \cdots & p_{nn} \\ \hline \end{array}$$

(To)

16.10-1. Illustrative Examples

Example 5. *Suppose there are two market products of brand A and B, respectively. Let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below :*

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady state.

From \ To	A	B
A	0.9	0.1
B	0.5	0.5

Solution. Here, it is given that the initial state for A and B are 50% each. Then after the promotional efforts made to brands A and B , the transition matrix shows that during second period brand A will retain 90% of its customers and take away 50% of B's so the market share for brand A during the second period will be given by

$$(50\%)\ (0{\cdot}9) + (50\%)\ (0{\cdot}5) = 70\%$$

The corresponding market share for B during the second period will be (50%) (0.1) + (50%)(0.5) = 30%

In matrix form, it can be expressed as $(50\%\quad 50\%)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (70\%\quad 30\%)$

If the same transition matrix holds from one period to the other, then market share of two brands for different periods will be as follows :

Period	Brand *A* (Market Share)	Brand *B* (Market Share)
0	50%	50%
1	70%	30%
2	78%	22%
3	81.2%	18.8%
4	82.48%	17·52%
5	82.992%	17.008%
6	83%	17 %

From this table, we observe that starting with 50%, 50% of the market shares, after 6 time periods the resulting market shares are approximately 83% and 17% respectively. So the equilibrium position of market share of A and B will be 5/6 and 1/6 of the total market respectively.

Steady state (or equilibrium) position can be obtained by the matrix equation :

$$(x\ \ y)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (x\ \ y) \quad \text{or} \quad \begin{cases} 0.9\,x + 0.5\,y = x \\ 0.1\,x + 0.5\,y = y \end{cases} \quad \text{and} \quad x + y = 1.$$

The most important aspect of the steady state is that—if the transition matrix is same throughout, then it is independent of the initial market shares.

Example 6. *Suppose there are three dairies in a town, say A, B and C. They supply all the milk consumed in the town. It is known by all the dairies that consumers switch from dairy to dairy overtime because of advertising, dissatisfaction with service and other reasons. All these dairies maintain records of the number of their customers and the dairy from which they obtained each new customer. Following table illustrates the flow of customers over an observation period of one month, say June.*

Dairy	June 1 (Customers)	Gains from			Losses to			July 1 (Customers)
		A	B	C	A	B	C	
A	200	0	35	25	0	20	20	220
B	500	20	0	20	35	0	15	490
C	300	20	15	0	25	20	0	290

We assume that the matrix of transition probabilites remain fairy stable and that the July market shares are

$$A = 22\%, B = 49\%, \text{ and } C = 29\%.$$

Managers of these dairies are willing to know :

(i) market share of their dairies on Ist August and Ist September,

(ii) their market shares in steady state.

Solution. From the table of the problem, the matrix of the transition probabilities can be easily obtained as follows :

	A	B	C
A	$\frac{160}{200}=0.80$	$\frac{20}{200}=0.10$	$\frac{20}{200}=0.10$
B	$\frac{35}{500}=0.07$	$\frac{450}{500}=0.90$	$\frac{15}{500}=0.03$
C	$\frac{25}{300}=0.083$	$\frac{20}{300}=0.067$	$\frac{255}{300}=0.85$

Market share of the dairies on the *Ist August* will be

$$(0.22 \quad 0.49 \quad 0.29)\begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{pmatrix} = (0.234 \quad 0.483 \quad 0.283)$$

Market share of the three dairies on Ist september will be

$$(0.234 \quad 0.483 \quad 0.283)\begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{pmatrix} = (0.245 \quad 0.477 \quad 0.278)$$

The steady state market shares are given by

$$(x \quad y \quad z)\begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{pmatrix} = (x \quad y \quad z) \quad \text{or} \quad \begin{Bmatrix} 0.800x + 0.070y + 0.083z = x \\ 0.100x + 0.900y + 0.067z = y \\ 0.100 + 0.030y + 0.850z = z \\ x + y + z = 1 \end{Bmatrix}$$

and solving these four equations, we get $x = 0.273, y = 0.454, z = 0.273$.

EXAMINATION PROBEMS

1. A house-wife buys three kinds of cereals: *A, B,C*. She never buys the same cereal on successive weeks. If she buys cereal *A*, then the next week she buys cereal *B*. However, if she buys *B* or *C*, then next week she is three times as likely to buy *A* as the other brand.Find the transition matrix. In the long run, how often she buys each of three brands ?
2. A salesman's territory consists of three cities. *A,B* and *C*. He never sells in the same city on successive days. If he sells in city *A*, then the next day he sells in city *B*. However, if he sells in either B or *C*, then the next day he is twice as likely to sell in city *A* as in the other city. In the long run how often does he sell in each of the cities ?
3. On January 1 (this year), bakery *A* had 40 per cent of its local market while the other two bakeries *B* and *C* had 40 per cent and 20 per cent respectively of the market. Based upon a study by a marketing research firm, the following facts were compiled. Bakery *A* retains 90 per cent of its customers while gaining 5 per cent of competitor *B's* customers. Bakery B retains 5 per cent of A's customers and 7 per cent of C 's customers. Bakery *C* retains 83 per cent of its customers and gains 5 per cent of *A's* customers and 10 per cent of *B* 's customers. What will each firm's share be on January 1, next years and what will each firm's market share be at equilibrium ?
4. Assume that man's profession cas be classified as professional skilled labourer, or unskilled labourer. Assume that the 80 per cent sons of professional men are professionals, 10 per cent are skilled labourers and 10 per cent are unsikilled labourers. In the case of sons of skilled labourers, 60 per cent are skilled labourers, 20 per cent are professionals, and 20 per cent are unskilled labourers. Finally, in the case of unskilled labourers, 50 per cent of the sons are unskilled labourers, 25 per cent each are in the other two categories. Assume that every man has a son and forms a *Markov* chain by following a given family through several generations. Set-up the matrix of transition probabilities. Find the probability that grandson of an unskilled labourer is a profferssional man.
5. *Honey* Inc. had 35% of the local market for its cosmetics, while the two other manufacturers of cosmetics *Lace Inc.* and *Shalon* Inc. have 40% and 25% shares respectively in the local market, as on Ist April of this year. A study by a market research firm has disclosed the following. *Honey* Inc. retains 86% of its customers, while it gains 4% and 6% of the customers from its two competitors, *Lace* and *Shalon* respectively. *Lace* Inc. retains 90% of its customers, and gains 8% and 9% of customers respectively from *Heney* and *Shalon*. Shalon retains 85% of its customers and gains 6% and 6% of customers from Lace and Honey respectively. What will be the share of each firm on April I next year ? What will be the market share of each firm at equilibrium ?
[Ans. *Honey* Inc. 33.2%, *Lace* Inc. 41.05% and *Shalon Inc.* 25.75%, Market share of each firm at equilibrium will be 25%, 46% and 29%]

probability function $h(x, y)$, (b) $P[(X, Y) \in A]$ where A is the region $\{(x, y)|x + y \le 1\}$, (c) the marginal distributions of X and Y.

Hint: (a) $h(x, y) = ({}^3C_x)({}^2C_y)\left({}^3_{2-x-y}\right)/({}^8C_2)$

(b) $x = 0, 1, 2;\ y = 0, 1, 2;\ 0 \le x + y \le 2$

$P[(x, y) \in A] = P(X + Y \le 1)$

$= h(0, 0) + h(0, 1) + h(1, 0)$

$= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14}$.

(c)

x_i:	0	1	2
$f(x_i)$:	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

,

y_j:	0	1	2
$g(y_j)$:	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

Ans.

Y \ X	0	1	2	Sums
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Sums	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	CHECK 1

8. For the joint distribution in problem 7 (above). (a) Find the conditional distribution of X, given that $Y = 1$, (b) Determine $P(X = 0|Y = 1)$, (c) Show that the random variables X and Y are *not* statistically independent.

Hint:

$$g(1) = \sum_{x=0}^{2} h(x, 1) = \frac{3}{14} + \frac{3}{14} + 0 = \frac{3}{7}$$

$$h(x|1) = \frac{h(x, 1)}{g(1)} = \frac{7}{3} \cdot h(x, 1) \quad \text{for} \quad x = 0, 1, 2$$

Ans.

x_i	0	1	2
(a) $h(x_i\|1)$	$\frac{1}{2}$	$\frac{1}{2}$	0

(b) $p(X = 0|Y = 1) = h(0, 1) = \frac{1}{2}$

(c) $h(0, 1) = \frac{3}{14} \ne \frac{5}{14} \cdot \frac{3}{7} = f(0)g(1)$.

31.2 MARKOV CHAINS

Suppose a box A contains 5 red, 3 white and 8 black marbles while box B contains 3 red and 5 white marbles. A fair die is tossed and if 2 or 5 occurs a marble is chosen from B otherwise from A. Further in box A, two red, one white and 4 black marbles are defective while in box B one red and 2 white marbles are defective. To determine the probability that a marble drawn at random is say a defective red marble, we have to conduct a sequence of experiments in which each experiment has a finite number of outcomes with given probabilities as shown in the tree diagram below (Fig. 31.2).

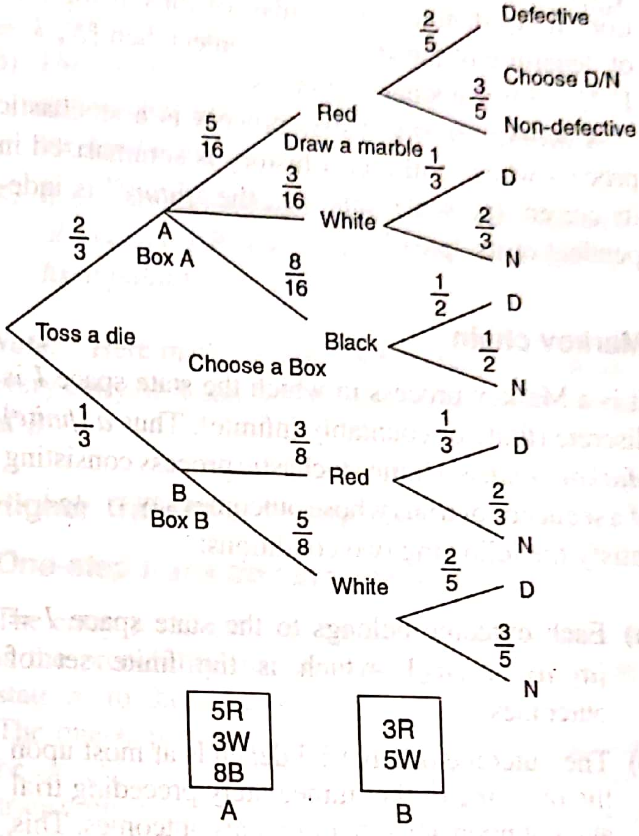


Fig. 31.2

The four experiments are: toss a die, choose a box, draw a marble and decide whether it is defective. Here probability of a defective red marble is $\frac{2}{3} \cdot \frac{5}{16} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{8} \cdot \frac{1}{3}$.

Stochastic Process (or Chance or Random Process)

It is a family of random variables $\{X(t)|t \in T\}$ defined on a common sample space S and indexed by the parameter t, which varies on an index set T.

The values assumed by the random variables $X(t)$ are called *states*, and the set of all possible values from the *state space* of the process is denoted by I. If the state space is discrete, the stochastic process is known as a *chain*. In this case the state space is assumed to be $I = \{0, 1, 2, \ldots\}$. Thus a (finite) stochastic process consists of a sequence of experiments in which each experiment has a finite number of outcomes with given probabilities.

Example: Jobs arrive at random points in time, queue for service and depart after service completion. If N_k denotes the number of jobs at the time of departure of the kth job (customer) then $\{N_k | k = 1, 2, \ldots\}$ is a stochastic process.

A *Markov (memoryless)* process is a stochastic process whose entire past history is summarized in its current (present) state. i.e., the "*future*" is independent of its "*past*".

Markov chain

It is a Markov process in which the state space I is discrete (finite or countably infinite). Thus *a (finite) Markov chain* is a finite stochastic process consisting of a sequence of trials whose outcomes say $x_1, x_2, \cdots$ satisfy the following two conditions:

(a) Each outcome belongs to the state space $I = \{a_1, a_2, \ldots, a_m\}$, which is the finite set of outcomes.

(b) The outcome of any trial depends at most upon the outcome of the immediately preceding trial and not upon any other previous outcomes. This Markov property can be stated as

$$P\left(X_n = i_n \middle| X_0 = i_o, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\right)$$
$$= P\left(X_n = i_n \middle| X_{n-1} = i_{n-1}\right).$$

Now the system is said to be in state 'a_i' at time n or at the nth step if a_i is the outcome on the nth trial. Associated with each ordered pair of states (a_i, a_j), the number p_{ij} gives the probability that system changes from ith state to jth state. In other words, p_{ij} is the probability that a_j occurs immediately after a_i occurs. The numbers p_{ij} are known as transition probabilities.

Transition matrix

P is square matrix of the transition probabilities p_{ij}.

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_m \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mm} \end{pmatrix} \end{array}$$

The ith row of P namely $(p_{i1}, p_{i2}, \ldots, p_{im})$ represents the probabilities of that system will change from a_i to $a_1, a_2, a_3, \ldots, a_m$.

Probability vector

It is a vector $v = (v_1, v_2, \ldots, v_n)$, if $v_i \geq 0$ for every i and $\sum_{i=1}^{n} v_i = 1$.

Note: A vector whose components are non-negative, but their sum is not one, can be converted into a probability vector by dividing each component by the sum of the components.

Stochastic matrix

P is a square matrix with each row being a probability vector. In other words, all the entries of P are non-negative and the sum of the entries of any row is one.

A vector v is said to be a **fixed vector** or a fixed point of a matrix A if $vA = v$ and $v \neq 0$.

Obviously if v is a fixed vector of A, so is kv since $(kv)A = k(vA) = k(v) = kv$.

Theorem 1: *If $v = (v_1 v_2 v_3)$ is a probability vector of a stochastic matrix* $P = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ *then vP is also a probability vector.*

Proof:

$$vP = (v_1 v_2 v_3)_{1\times 3} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}_{3\times 3}$$
$$= (v_1 a_1 + v_2 a_2 + v_3 a_3, v_1 b_1 + v_2 b_2 + v_3 b_3, v_1 c_1 + v_2 c_2 + v_3 c_3).$$

Since a_i, b_i, c_i, v_i are all non-negative for any i, the components of vP are all non-negative. Now the sum of the components of vP is

$$(v_1a_1 + v_2a_2 + v_3a_3) + (v_1b_1 + v_2b_2 + v_3b_3)$$
$$+(v_1c_1 + v_2c_2 + v_3c_3)$$
$$= v_1(a_1+b_1+c_1)+v_2(a_2+b_2+c_2)+v_3(a_3+b_3+c_3)$$
$$= v_1 \cdot 1 + v_2 \cdot 1 + v_3 \cdot 1 = v_1 + v_2 + v_3 = 1,$$ since

p is a stochastic matrix and v is given probability vector.

General result: If $v = (v_1 v_2 v_3 \ldots v_n)$ is a probability vector of a n square stochastic matrix P then vP is also a probability vector.

Theorem 2: *If P and Q are stochastic matrices then their product PQ is also stochastic matrix. Thus P^n is stochastic matrix for all positive integer values of n.*

Proof: The ith row of PQ is the product of ith row of P with matrix Q. Since P and Q are stochastic matrices, ith row of P is a probability vector and by previous Theorem 1, ith row of P with matrix Q is also a probability vector and hence PQ is a stochastic matrix. If $P = Q$, then $PQ = P^2$ is stochastic and in general P^n is stochastic for n positive integer.

Theorem 3: *Let $t = (t_1 t_2 \ldots t_m)$ be a vector and T be a square matrix whose rows are each the same vector t. Then $pT = t$, if $p = (p_1 p_2 \ldots p_m)$ is a probability vector.*

Proof:

$$pT = (p_1 p_2 \cdots p_m)_{1\times m} \begin{pmatrix} t_1 & t_2 & t_3 & \cdots & t_m \\ t_1 & t_2 & t_3 & \cdots & t_m \\ -- & -- & -- & -- & -- \\ t_1 & t_2 & t_3 & \cdots & t_m \end{pmatrix}_{m\times m}$$

$$= (t_1(p_1 + p_2 + \cdots + p_m), t_2(p_1 + p_2 + \cdots + p_m),$$
$$\ldots, t_m(p_1 + p_2 + \cdots + p_m))$$
$$= (t_1 \cdot 1, t_2 \cdot 1, \ldots, t_m \cdot 1) = (t_1 t_2 \ldots t_m) = t$$

since p is a probability vector (then $\sum_{i=1}^{m} p_i = 1$).

Theorem 4: *The transition matrix P of a Markov chain is a stochastic matrix.*

Proof: All the entries of a transition matrix P are non negative because p_{ij} are probabilities. The sum of the elements $(p_{i1}, p_{i2}, \ldots, p_{im})$ of any ith row is one, because they represent the probabilities of all the possible outcomes of transition from state a_i to the states $a_1, a_2, \ldots, a_i, \ldots, a_m$. Thus each row of P is a probability vector. Therefore P is a stochastic matrix.

A Stochastic matrix P is said to be *regular* if all the entries of some power P^m are positive.

Theorem 5: *Let P be a regular stochastic matrix. Then*

(a) *P has a unique fixed probability vector t and the components of t are all positive.*

(b) *The sequence $P, P^2, P^3 \ldots$ of powers of P approaches the matrix T whose rows are each the fixed point t.*

(c) *If p is any probability vector, then the sequence of vectors $pP, pP^2, pP^3, \ldots$ approaches the fixed point t.*

Note: Here matrix A approaches matrix B, means every entry of A approaches the corresponding entry of B.

Higher Transition Probabilities

One-step transition probabilities

The entry p_{ij} in the transition probability matrix P is the probability that the system moves from the state a_i to the state a_j in one step i.e., $a_i \to a_j$. The one-step transition probabilities in P can also be described by a directed graph known as state-transition diagram or simply transition diagram of the Markov chain. A node labelled i of the transition diagram represents state i of the Markov chain. A branch labeled p_{ij} from node i to j represents the conditional probability (or the one-step transition probabilities) defined by

$$p_{ij} = P\left(X_n = j \middle| X_{n-1} = i\right)$$

n-step Transition Probabilities

The probability that a Markov chain will move from state i to state j in exactly n steps, is denoted by $p_{ij}(n)$ or $p_{ij}^{(n)}$ and is given by

$$p_{ij}^{(n)} = p_{ij}(n) = P\left(X_{m+n} = j \middle| X_m = i\right)$$

i.e., $a_i \to a_{k_1} \to a_{k_2} \to \cdots \to a_{k_{n-1}} \to a_j$.

Evaluation of n-step Transition Probability Matrix $P^{(n)}$ or $P(n)$

using Chapman-Kolmogorov equation

$$p_{ij}(m+n) = \sum_k p_{ik}(m)p_{kj}(n).$$

Let $P^{(n)}$ or $P(n)$ represent a matrix whose (i, j)th entry is $p_{ij}^{(n)}$ or $p_{ij}(n)$. Putting $m = 1$ and $n = n - 1$ in the above C-K equation, $P^{(n)}$ or $P(n)$ the n-step transition probabilities matrix can be written as

$$p^{(n)} = P(n) = P \cdot P(n-1) = PPP(n-2) = P^n.$$

Thus the matrix of n-step transition probabilities $P^{(n)}$ is obtained by multiplying the matrix of one step transition probabilities P by itself $n - 1$ times.

Theorem 6: *If P is the transition matrix of a Markov chain, then the n-step transition matrix $P^{(n)}$ is equal to the nth power of P, i.e., $P^{(n)} = P^n$.*

In other words, the problem of finding the n-step transition probabilities is reduced to one of forming powers of a given matrix.

Probability distribution of the system at some arbitrary time is denoted by the probability vector.

$$p = (p_1, p_2, p_i, \ldots, p_m)$$

where p_i denotes the probability that the system is in state a_i. At time $t = 0$, when the process begins, the corresponding probability vector

$$p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots, p_i^{(0)}, \ldots, p_m^{(0)})$$

denotes the initial probability distribution. Similarly, the nth step probability distribution i.e., the distribution after the first n-steps is denoted by

$$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_m^{(n)}).$$

Now the (marginal) pmf of the random variable X_n can be obtained from the n-step transition probabilities and the initial distribution as follows

$$\boxed{p^{(n)} = p^{(0)} P^{(n)} = p^{(0)} P^n}$$

Thus the probability distributions of a homogeneous Markov chain are completely determined from the one-step transition probability matrix P and the initial probability distribution $p^{(0)}$.

Theorem 7: *The probability distribution of the system n-steps later is given by*

$$p^{(n)} = p^{(0)} P^n$$

i.e., $p^{(1)} = p^{(0)} P$, $p^{(2)} = p^{(1)} P = p^{(0)} PP = p^{(0)} P^2$

$$p^{(3)} = p^{(2)} P = p^{(0)} P^2 P = p^{(0)} P^3 \quad \text{etc.}$$

Stationary Distribution of Regular Markov Chains

Theorem 7: *Let P be a regular transition matrix of a Markov chain. Then in the long run, the probability that any state a_j occurs is approximately equal to the component t_j of the unique fixed probability vector t of P.*

Proof: Suppose the Markov chain is regular, i.e., P is regular, then by Theorem 5, the sequence of n-step transition matrices P^n approaches the matrix T, whose rows are each the unique fixed probability vector t of P. Hence the probability $p_{ij}(n)$ that a_j occurs for sufficiently large n is independent of the original state a_i and it approaches the component t_j of t.

Stationary distribution

Stationary distribution of a Markov chain is the unique fixed probability vector t of the regular transition matrix P of the Markov chain because every sequence of probability distributions approaches t.

Absorbing States

A state a_i of a Markov chain is said to be an absorbing state if the system remains in the state a_i once it enters there, i.e., a state a_i is absorbing if $p_{ii} = 1$. Thus once a Markov chain enters such an absorbing state, it is destined there to remain forever. In other words the ith row in P has 1 at the main diagonal (i, i) position and zeros everywhere else.

Theorem 8: *A stochastic matrix P is not regular if a 1 occurs in the principal main diagonal.*

Proof: Suppose a_i is the absorbing state of the given Markov chain whose transition matrix is P. Then 1 occurs in the (i, i) position and the ith row of P is of the form $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$. When powers of P are calculated the ith row of P^n persists to contain $(0\,0\,0\ldots, 1, 0\ldots 0)$. Thus for $i \neq j$ (non diagonal elements), the n-step transition probability $p_{ij}^{(n)} = 0$ for any n. Thus every power of P contains some zero elements. Therefore P is not regular.

Worked Out Examples

Probability vector and stochastic matrix

Example 1: Which vectors are probability vectors

(i) $\left(\frac{1}{4}, \frac{3}{2}, -\frac{1}{4}, \frac{1}{2}\right)$ (ii) $\left(\frac{5}{2}, 0, \frac{8}{3}, \frac{1}{6}, \frac{1}{6}\right)$

(iii) $\left(\frac{1}{12}, \frac{1}{2}, \frac{1}{6}, 0, \frac{1}{4}\right)$ (iv) $(3, 0, 2, 5, 3)$

Solution:

(i) is not a probability vector because negative entry $\left(-\frac{1}{4}\right)$

(ii) is not because the sum of the components do not add up to 1

(iii) is a probability vector because all the entries are non-negative and sum $\frac{1}{12}+\frac{1}{2}+\frac{1}{6}+\frac{1}{4}=\frac{12}{12}=1$.

(iv) Dividing by $3+0+2+5+3=13$, we get the probability vector $\left(\frac{3}{13}, 0, \frac{2}{13}, \frac{5}{13}, \frac{3}{13}\right)$.

Example 2: Which matrices are stochastic

(i) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

(iv) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (v) $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$.

Solution: (i), (iii), (v) are not stochastic because (i) is not square (iii) last row sum is not 1 (v) negative entry (ii) & (iv) are stochastic matrices: each row sum is one, entries non-negative.

Example 3: Which of the stochastic matrices are regular

(i) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$ (ii) $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

(iii) $C = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$

Solution:

(i) not regular since 1 lies on the main diagonal.

(ii) $B^2 = B \cdot B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$,

$$B^3 = B^2 B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7}{16} & \frac{7}{16} & \frac{1}{8} \end{pmatrix}$$

since entries b_{13}, b_{23} are zero, B is not regular

(iii) $C^2 = C \cdot C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$,

$$C^3 = C \cdot C^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$C^4 = C^3 \cdot C = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix},$$

$$C_5 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Since all the entries of some power of C are positive, C is regular stochastic matrix.

Fixed probability vectors

Example 4: Show that $v = (b\ a)$ is fixed point of the stochastic matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$.

Solution: $vP = (b\ a)\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} =$
$(b - ab + ab \quad ba + a - ab) = (b\ a) = v.$

Example 5:

(a) Find the unique fixed probability vector t of

$$P = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) What matrix does P^n approach?

(c) What vector does $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) P^n$ approach?

Solution:

(a) Let $t = (x, y, z)$ be the fixed probability vector. By definition $x + y + z = 1$. So $t = (x, y, 1 - x - y)$, t is said to be fixed vector, if $tP = t$

$$(x\ \ y\ \ 1-x-y)\begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} = (x\ \ y\ \ 1-x-y)$$

Solving

$$\frac{1}{2}y = x$$

$$\frac{3}{4}x + \frac{1}{2}y + 1 - x - y = y$$

$$\frac{1}{4}x = 1 - x - y$$

Solving $\quad y = 2x, x = \frac{4}{13}, y = \frac{8}{13}, z = \frac{1}{13}$

Required fixed probability vector is

$$t = (x, y, z) = \left(\frac{4}{13}, \frac{8}{13}, \frac{1}{13}\right)$$
$$= (0.3077, 0.6154, 0.077)$$

(b)
$$P^2 = P \cdot P = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} & 0 \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^3 = P^2 P = \begin{pmatrix} \frac{5}{16} & \frac{19}{32} & \frac{3}{32} \\ \frac{5}{16} & \frac{5}{8} & \frac{1}{16} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{pmatrix}$$

$$P^4 = P^3 P = \begin{pmatrix} \frac{19}{64} & \frac{40}{64} & \frac{5}{64} \\ \frac{20}{64} & \frac{39}{64} & \frac{5}{64} \\ \frac{20}{64} & \frac{40}{64} & \frac{4}{64} \end{pmatrix}$$

$$P^5 = P^4 \cdot P = \frac{1}{64}\begin{pmatrix} 20 & \frac{196}{4} & \frac{19}{4} \\ \frac{39}{2} & \frac{79}{2} & 5 \\ 20 & 39 & 5 \end{pmatrix}$$

$$= \frac{1}{256}\begin{pmatrix} 80 & 196 & 19 \\ 78 & 158 & 20 \\ 80 & 156 & 20 \end{pmatrix} = \begin{pmatrix} 0.3125 & 0.7656 & 0.0742 \\ 0.304 & 0.61718 & 0.078 \\ 0.3125 & 0.61718 & 0.078 \end{pmatrix}$$

Thus $\quad P^n \to T = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$, where $t = (0.3077, 0.6154, 0.077)$

(c) $\left(\frac{1}{4}\frac{1}{4}\frac{1}{2}\right) P^n = \left(\frac{1}{4}\frac{1}{4}\frac{1}{2}\right)\begin{pmatrix} 0.3125 & 0.7656 & 0.0742 \\ 0.304 & 0.61718 & 0.078 \\ 0.3125 & 0.61718 & 0.078 \end{pmatrix}$

$$= (0.310375, 0.654285, 0.07705) \approx t$$

Finite stochastic process

Example 6: An urn A contains 5 red, 3 white and 8 green marbles while urn B contains 3 red and 5 white marbles (Fig. 31.3). A fair die is tossed; if 3 or 6 appears a marble is chosen from B otherwise from A. Find the probability that (a) a red marble is chosen (b) a white marble is chosen (c) a green marble is chosen.

Solution: $p =$ probability that 3 or 6 appears in the toss of a die

$$p = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$q = 1 - p = \frac{2}{3}$$

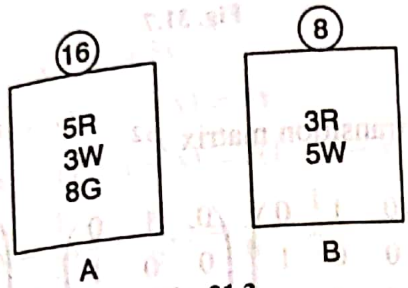


Fig. 31.3

probability of a red marble chosen from urn $A = P(R/A) = \frac{5}{16}$, $P(\text{white from } A) = P(W/A) = \frac{3}{16}$, $P(G/A) = \frac{8}{16}$, probability of a red marble chosen from urn $B = P(R/B) = \frac{3}{8}$, $P(W/B) = \frac{5}{8}$.

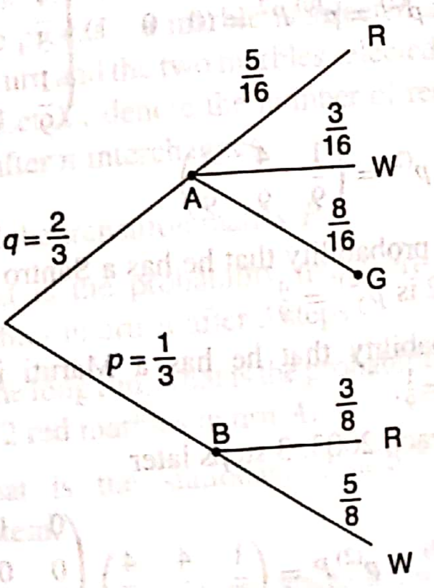


Fig. 31.4

Here we perform a sequence of two experiments. First, toss a die and choose the box. Second, choose a marble from the (chosen) box.

(a) Probability that a red marble is chosen

$$= P(R) = P(A) \cdot P(R/A) + P(B) \cdot P(R/B)$$

$$= \frac{2}{3} \cdot \frac{5}{16} + \frac{1}{3} \cdot \frac{3}{8} = \frac{16}{48} = \frac{1}{3}$$

(b)

$$P(W) = P(A) \cdot P(W/A) + P(B) \cdot P(W/B)$$

$$= \frac{2}{3} \cdot \frac{3}{16} + \frac{1}{3} \cdot \frac{5}{8} = \frac{16}{48} = \frac{1}{3}$$

(c)

$$P(G) = P(A)P(G/A) = \frac{2}{3} \cdot \frac{8}{16} = \frac{1}{3}.$$

Transition matrix and transition diagram

Example 7: Figure 31.5 shows four compartments with door leading from one to another. A mouse in any compartment is equally likely to pass through each of the doors of the compartment. Find the transition matrix of the Markov chain. Draw the transition diagram.

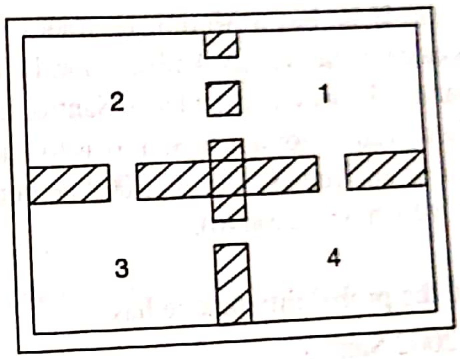


Fig. 31.5

Solution: The 4 rooms are considered as four states say 1, 2, 3, 4. Since mouse is moving, it does not stay in the same room. From room 1 it can go to 4 or 2 with probability $\frac{1}{3}$ or $\frac{2}{3}$. It can not go from 1 to 3. Then the first row consists of 0, $\frac{2}{3}$, 0, $\frac{1}{3}$. Thus the transition matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Figure 31.6 gives the transition diagram:

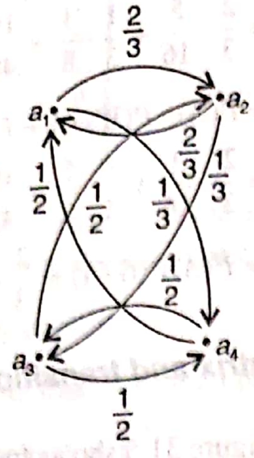


Fig. 31.6

Markov chain

Example 8: Every year, a man trades his car for a new car. If he has a Maruti, he trades it for an Ambassador. If he has an Ambassador, he trades it for a Santro. However, if he has a Santro, he is just as likely to trade it for a new Santro as to trade it for a Maruti or an Ambassador. In 2000 he bought his first car, which was a Santro.

(i) Find the probability that he has

(a) 2002 Santro

(b) 2002 Maruti

(c) 2003 Ambassador

(d) 2003 Santro

(ii) In the long run, how often will he have a Santro.

Solution: (i) Define 3 states a_1, a_2, a_3 as follows a_1: state of having Maruti car, a_2: having Ambassador, a_3: having Santro. Then the transition matrix is

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

2000: Initial state $= P^{(0)} = (0, 0, 1)$ since he has Santro car in 2000 (his first purchase).

(a) To reach 2002 year, (2-steps later) compute the

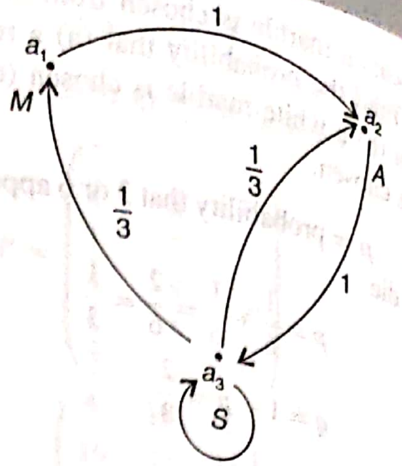


Fig. 31.7

2-step transition matrix P^2

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix}$$

Then

$$p^{(2)} = p^{(0)} P^2 = (0 \quad 0 \quad 1) \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix}$$

$$p^{(2)} = \left(\frac{1}{9} \quad \frac{4}{9} \quad \frac{4}{9} \right)$$

The probability that he has a Santro in the year 2002 is $p_3^{(2)} = \frac{4}{9}$.

(b) Probability that he has a Maruti in 2002 is $p_1^{(2)} = \frac{1}{9}$.

(c) To reach 2003: 3 steps later

$$p^{(3)} = p^{(2)} P = \left(\frac{1}{9} \quad \frac{4}{9} \quad \frac{4}{9} \right) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(or equivalently $p^{(3)} = p^{(0)} P^3$).

$$p^{(3)} = \left(\frac{4}{27} \quad \frac{7}{27} \quad \frac{16}{27} \right)$$

Probability that he has an Ambassador in 2003 is $p_2^{(3)} = \frac{7}{27}$.

(d) Probability that has a Santro in 2003 is

$$p_3^{(3)} = \frac{16}{27}.$$

(ii) To discover what happens in the long run, we must find a fixed probability vector t of P. Let $t = (x, y, 1-x-y)$.

Then $tP = t$

$$(x \quad y \quad 1-x-y)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = (x \quad y \quad 1-x-y)$$

$$1-x-y = 3x$$

or

$$3x + (1-x-y) = 3y$$

$$3y + (1-x-y) = 3(1-x-y)$$

Solving $y = \frac{1}{3}, x = \frac{1}{6}, z = \frac{3}{6} = \frac{1}{2}$

Thus $t = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$

In the long run, he has a Santro 50% $\left(\frac{1}{2}\right)$ of the time.

Example 9: Suppose an urn A contains 2 white marbles and urn B contains 4 red marbles. At each step of the process, a marble is selected at random from each urn and the two marbles selected are interchanged. Let X_n denote the number of red marbles in urn A after n interchanges.

(i) Find the transition matrix P.

(ii) What is the probability that there are 2 red marbles in urn A after 3 steps.

(iii) In the long run, what is the probability that there are 2 red marbles in urn A.

(iv) What is the stationary distribution of the system.

Solution: There are three states a_0, a_1 and a_2 as shown in Fig. 31.8 below:

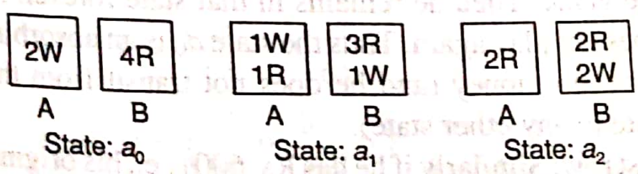


Fig. 31.8

(i) Transition matrix

If the system is in the state a_0, then a white marble from A and a red from B must be selected, so that the system will now move to state a_1. Accordingly the first row of the transition matrix (T.M.) is (0, 1, 0). Now suppose the system is in a_1. It can move to state a_0, iff red from A and white from B with probability $\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$. Thus $p_{10} = \frac{1}{8}$. The system can move from a_1 to a_2, iff white from A and red from B with probability $\frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$ i.e., $p_{12} = \frac{3}{8}$, probability that system will remain in a_1 itself is $1 - \frac{1}{8} - \frac{3}{8} = \frac{1}{2}$. (Note: white from A and white from B with probability $\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$ or red from A and red from B with probability $\frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$. Thus the probability that system will remain in state a_1 itself is $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$). Thus the 2nd row of T.M. is $\left(\frac{1}{8}, \frac{1}{2}, \frac{3}{8}\right)$.

Finally, suppose the system is in state a_2. Note that the system can never move from state a_2 to a_0. However, it may remain in a_2 itself, if a red from A and red from B is chosen. In this case the probability is $\frac{1}{1}\cdot\frac{2}{4} = \frac{1}{2}$. Lastly, if a red from A and white from B is chosen, then system moves from a_2 to a_1 with probability $\frac{2}{4} = \frac{1}{2}$. Thus third row of the T.M. is $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. The Transition Matrix and transition diagram are shown in Fig. 31.9

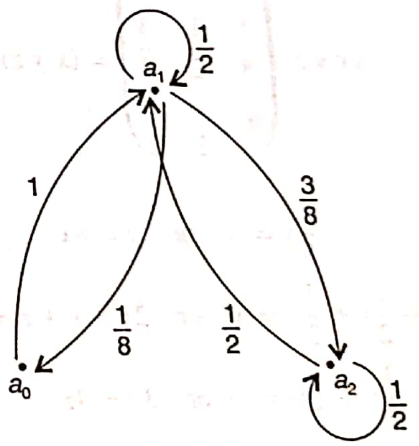


Fig. 31.9

$$\begin{array}{c c} & \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{array}{c} a_0 \\ a_1 \\ a_2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

(ii) The system starts in state a_0, so that $p^{(0)} = (1, 0, 0)$ is the initial state. Now

$$p^{(1)} = p^{(0)}P = (1\ 0\ 0)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (0\ 1\ 0)$$

$$p^{(2)} = p^{(1)}P = (0\ 1\ 0)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{pmatrix}$$

$$p^{(3)} = p^{(2)}P = \begin{pmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{16} & \frac{9}{16} & \frac{6}{16} \end{pmatrix}$$

Probability that there are two red in A i.e., in state a_2 *after* three steps is $\frac{6}{16} = \frac{3}{8}$.

(iii) To study the system in the long run, we should find a unique fixed probability vector t of the transition matrix P. Let t be (x, y, z) or $(x, y, 1 - x - y)$.

Then $$tT = t$$

$$(x\ y\ z)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x\ y\ z)$$

Solving $$\frac{1}{8}y = x \quad \text{or} \quad y = 8x$$

$$x + \frac{1}{2}y + \frac{1}{2}z = y \quad \text{or} \quad 2x - y + z = 0$$

$$\frac{3}{8}y + \frac{1}{2}z = z \quad \text{or} \quad 3y = 4z$$

Now $$3y = 4z = 4(1 - x - y)$$

$$7y = 4 - 4x \quad \text{or} \quad 56x + 4x = 4$$

$$\therefore \quad x = \frac{4}{60},\ y = \frac{8}{15},\ z = \frac{6}{15}$$

Therefore, the fixed vector

$$t = \begin{pmatrix} \frac{1}{15} & \frac{8}{15} & \frac{6}{15} \end{pmatrix}$$

Hence the system in the long run stays in the state a_2, 40% of the time ($\frac{6}{15} = \frac{2}{5}$) (i.e., there will be 2 red in A, 40% of the time).

(iv) The fixed unique probability vector $t = (\frac{1}{15}, \frac{8}{15}, \frac{6}{15})$ is the stationary distribution, since p^n approaches t, in the long run.

Markov chain with absorbing states:

Example 10: A player has Rs. 300. At each play of a game, he losses Rs. 100 with probability $\frac{3}{4}$ but wins Rs. 200 with probability $\frac{1}{4}$. He stops playing if he has lost his Rs. 300 or he has won at least Rs. 300.

(a) Determine the transition probability matrix of the Markov chain.

(b) Find the probability that there are at least 4 plays to the game.

Solution: (a) This is random walk with absorbing barriers at states 0 and 6. The transition probability matrix P is

$$P = \begin{array}{c|ccccccc} & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \hline a_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ a_2 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ a_3 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ a_4 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ a_5 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Explanation for the probability matrix P: There are seven states $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ where a_i indicates the state that he has Rs. i hundreds (i.e., a_3 indicate he has Rs. 300, a_5 he has 500 etc.).

First row: If he is in state a_0 he has zero money (lost his initial amount of Rs. 300) and therefore he stops the game. Then he remains in that state forever. He does not play again. Thus the state a_0 is an absorbing state, no money (and he does not transit from this state to any other state).

Last row: Similarly if he has Rs. 600 (i.e., his original Rs. 300 and winning amount of Rs. 300) he stops the game and remains in the state (he does not play). Thus the state a_6 is also an absorbing state, all money.

Second row: Suppose he has Rs. 100 i.e., he is in state a_1. Then the probability that he will lose Rs. 100 is $\frac{3}{4}$ and therefore transfers to a_0 state (no money). But he can not transfer to states a_2, a_3, a_5, a_6. But by winning Rs. 200 with probability $\frac{1}{4}$ he can have Rs. 300 (Rs. 100 original + Rs. 200 winning). Thus the probability of going from state a_1 to a_3 is $\frac{1}{4}$, and from a_1 to a_0 is $\frac{3}{4}$. Thus the probability vector (second row) is $(\frac{3}{4}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0)$.

Third row: Starting with Rs. 200 (a_2 state) he can lose Rs. 100 with probability $\frac{3}{4}$ thereby go to state a_1 or win Rs. 200 with probability $\frac{1}{4}$, thereby go to state a_4. Thus third row $(0\ \frac{3}{4}\ 0\ 0\ \frac{1}{4}\ 0\ 0)$.

Similarly, other rows of P are obtained.

(b) The initial probability distribution is

$$p^{(0)} = (0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0)$$

because he has started the game with an initial amount of Rs. 300 and is therefore in state a_3. To find the probability that the game has at least 4 plays, we compute $p^{(4)}$, which gives the probability distribution of the system after 4 steps (i.e., 4 games). Now

$$p^{(1)} = p^{(0)} P =$$

$$= (0\ 0\ 0\ 1\ 0\ 0\ 0) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p^{(1)} = (0, 0, \frac{3}{4}, 0, 0, \frac{1}{4}, 0)$$

$$p^{(2)} = p^{(1)} P = \left(0, \frac{9}{16}, 0, 0, \frac{6}{16}, 0, \frac{1}{16}\right)$$

$$p^{(3)} = p^{(2)} P = \left(\frac{27}{64}, 0, 0, \frac{27}{64}, 0, 0, \frac{10}{64}\right)$$

$$p^{(4)} = p^{(3)} P = \left(\frac{27}{64}, 0, \frac{81}{256}, 0, 0, \frac{27}{256}, \frac{10}{64}\right)$$

He plays 4 or more games, if after 4 steps he is not in any one of the absorbing states a_0 or a_6. Thus the probability that there at least 4 plays in the game is

$$0 + \frac{81}{256} + 0 + 0 + \frac{27}{256} = \frac{108}{256} = \frac{27}{64}.$$

EXERCISE

1. Which vectors are probability vectors (i) $(\frac{1}{2}, \frac{1}{3}, 0, -\frac{1}{5})$; (ii) $(3\ 4\ 5\ 0)$; (ii) $(\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4})$.

Ans. (i) not (since negative component); (ii) not (since do not add upto 1); (iii) yes

2. Find a scalar multiple of each vector, which is a probability vector.

   (i) $(2, \frac{1}{2}, 0, \frac{1}{4}, \frac{3}{4}, 0, 1)$; (ii) $(\frac{1}{3}, 2, \frac{1}{2}, 0, \frac{1}{4}, \frac{2}{3})$;
   (iii) $(1\ 2\ 3\ 4\ 5\ 6)$; (iv) $(\frac{1}{2}, \frac{2}{3}, 0, 2, \frac{5}{6})$.

Ans. (i) $\frac{4}{18}$; (ii) $\frac{12}{45}$; (iii) $\frac{1}{21}$. Then required probability vectors are $\frac{4}{18}(2, \frac{1}{2}, 0, \frac{1}{4}, \frac{3}{4}, 0, 1) = (\frac{8}{18}, \frac{2}{18}, 0, \frac{1}{18}, \frac{3}{18}, 0, \frac{4}{18})$;
(iv) multiply 6 : (3, 4, 0, 12, 5). Divide by $3 + 4 + 0 + 12 + 5 = 24$. Then probability vector $(\frac{3}{24} = \frac{1}{8}, \frac{1}{6}, 0, \frac{1}{2}, \frac{5}{54})$.

3. Which matrices are stochastic

   (a) $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{15}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

Ans. (a) not (square); (b) not adding to 1; (c) yes; (d) No (negative entry)

4. Which of the following matrices are regular

   (a) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ (b) $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

   (c) $C = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ (d) $D = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$

Ans. **(a)** Not regular, since 1 appears in the main diagonal.

(b) $B^2 = I$, $B^3 = B$, B is not regular, since 1 appears in the main diagonal

(c) c not regular, 1 appears on diagonal

(d) $D^2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{5}{16} & \frac{9}{16} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$, $D^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{32} & \frac{41}{64} & \frac{13}{64} \\ \frac{1}{8} & \frac{5}{16} & \frac{9}{16} \end{pmatrix}$

D is regular since all the entries of D^3 are positive.

5. Find the unique fixed probability vector of each matrix

(a) $A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$ (b) $B = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}$

(c) $C = \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix}$

Ans. (a) $\left(\frac{6}{11}, \frac{5}{11}\right)$; (b) $\left(\frac{10}{19}, \frac{9}{19}\right)$; (c) $\left(\frac{5}{13}, \frac{8}{13}\right)$.

6. Find the unique fixed probability vector of

(a) $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \end{pmatrix}$, (b) $B = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

Ans. (a) $\left(\frac{2}{9}, \frac{6}{9}, \frac{1}{9}\right)$ (b) $\left(\frac{5}{15}, \frac{6}{15}, \frac{4}{15}\right)$

7. Given $P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (a) find a unique fixed probability vector; (b) what matrix does P^n approach; (c) what vector does $\left(\frac{1}{4} \quad \frac{3}{4}\right) P^n$ approach.

Ans. (a) $\left(\frac{1}{3}, \frac{2}{3}\right)$; (b) $P^5 = \begin{pmatrix} 0.31 & 0.69 \\ 0.34 & 0.66 \end{pmatrix} \to$ (0.33, 0.66); (c) $\left(\frac{43}{128} \quad \frac{85}{128}\right) \approx \left(\frac{1}{3} \quad \frac{2}{3}\right)$

8. (a) Find the unique fixed probability vector t of

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

(b) What matrix does P^n approach

(c) What vector does $\left(\frac{1}{4}\ 0\ \frac{1}{2}\ \frac{1}{4}\right) P^n$ approach

(d) What vector does $\left(\frac{1}{2}\ 0\ 0\ \frac{1}{2}\right) P^n$ approach.

Hint: $t = (x, y, z, 1 - x - y - z)$, $y = 2x$, $x = 2z$, $y = 4z$

Ans. (a) $t = \left(\frac{2}{11}, \frac{4}{11}, \frac{1}{11}, \frac{4}{11}\right) = (0.1818, 0.3636, 0.0909, 0.3636)$

(b) $P^5 = \frac{1}{1024}\begin{pmatrix} 196 & 546 & 136 & 540 \\ 306 & 569 & 100 & 766 \\ 240 & 440 & 96 & 592 \\ 200 & 436 & 80 & 552 \end{pmatrix} \sim t$

(c) $\sim t$

(d) $\sim t$

9. Find the transition matrix

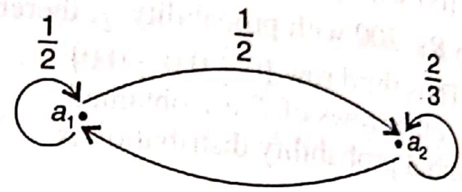


Fig. 31.10

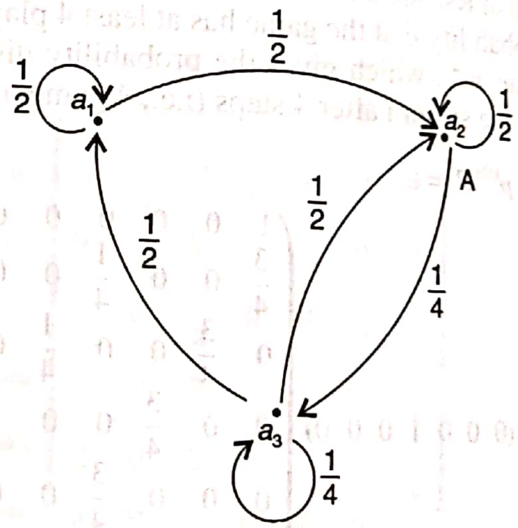


Fig. 31.11

Ans. (i) $\begin{array}{c|cc} & a_1 & a_2 \\ \hline a_1 & \frac{1}{2} & \frac{1}{2} \\ a_2 & \frac{1}{3} & \frac{2}{3} \end{array}$

(ii) $\begin{array}{c|ccc} & a_1 & a_2 & a_3 \\ \hline a_1 & \frac{1}{2} & \frac{1}{2} & 0 \\ a_2 & 0 & \frac{1}{2} & \frac{1}{2} \\ a_3 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}$

10. For a Markov chain, the transition matrix $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ with initial distribution $p^{(0)} = \left(\frac{1}{4} \;\; \frac{3}{4}\right)$. Find

(a) $p_{21}^{(2)}$ (b) $p_{12}^{(2)}$

(c) $p^{(2)}$ (d) $p_1^{(2)}$

(e) the vector $p^{(0)} P^n$ approaches

(f) P^n approaches

Ans. $P^2 = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{pmatrix}$ (a) $p_{21}^{(2)} = \frac{9}{16}$ (b) $p_{12}^{(2)} = \frac{3}{8}$

(c) $p^{(2)} = p^{(0)} P^2 = \left(\frac{37}{64} \;\; \frac{27}{64}\right)$ (d) $p_1^{(2)} = \frac{37}{64}$

(e) $p^{(0)} P^n$ approaches fixed vector $t = \left(\frac{3}{5} \;\; \frac{2}{5}\right)$

(f) $P^n \to T = \begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$

11. A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to nonfilter cigarettes the next week with probability 0.2. On the other hand if he smokes nonfilter cigarettes one week, there is a probability of 0.7 that he will smoke nonfilter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes.

Ans. $P:$

$$\begin{array}{c|cc} & F & NF \\ \hline F & 0.8 & 0.2 \\ NF & 0.3 & 0.7 \end{array}, \quad t = \text{fixed} = (x, 1-x)$$

$$x = \frac{3}{5}, \quad y = \frac{2}{5}, \quad t = \left(\frac{3}{5}, \frac{2}{5}\right)$$

Man smokes filter cigarettes 60% $\left(\frac{3}{5}\right)$ time in the long run.

12. A saleman's territory consists of 3 cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However if he sells in either B or C, then the next day he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities.

Ans.

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \end{array}, \qquad \begin{aligned} \tfrac{2}{3}y + \tfrac{2}{3}z &= x \\ x + \tfrac{1}{3}z &= y \\ \tfrac{1}{3}y &= z \end{aligned}$$

$t = \left(\frac{2}{5}, \frac{9}{20}, \frac{3}{20}\right)$. In the long run he sells 40% of time in city A, 45% in B, 15% of time in C.

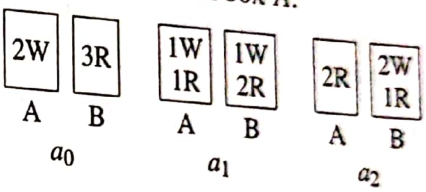
13. There are 2 white marbles in box A and 3 red marbles in box B. At each step of the process a marble is selected from each box and the two marbles selected are interchanged. Let the state a_i of the system be the number i of red marbles in box A.

(a) Find the transition matrix P.

(b) What is the probability that there are 2 red marbles in box A after 3 steps

(c) In the long run, what is the probability that there are 2 red marbles in box A.

Hint:



$$\text{(a) } P = \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

Initial distribution: $p^{(0)} = (1, 0, 0)$

$p^{(1)} = p^{(0)} P = (0, 1, 0)$, $p^{(2)} = p^{(1)} P = \left(\frac{1}{6} \; \frac{1}{2} \; \frac{1}{3}\right)$,

$p^{(3)} = p^{(2)} P = \left(\frac{1}{12} \; \frac{23}{36} \; \frac{5}{18}\right)$.

(b) Probability that there are 2 red marbles in box A after 3 steps is $\frac{5}{18}$.

(c) Fixed probability vector: $t = (0.1, 0.6, 0.3)$.

Ans. (c) In the long run, 30% of the time, there will be 2 red marbles in box A.

Finite stochastic process

14. Box A contains 3 red and 5 white marbles, Box B contains 2 red and 1 white marbles, Box C contains 2 red and 3 white marbles. One box is selected at random and a marble is drawn from the box. If the marble is red, what is the probability that it came from box A.

Hint: $P(A/R) = \frac{P(A \cap R)}{P(R)} = \frac{\frac{1}{8}}{\frac{173}{360}}$

Ans. $\frac{45}{173}$

15. Box A contains cards numbered 1 to 9. Box B contains cards numbered 1 to 5. One box is chosen at random and a card is drawn. If the card is even, another card from the same box is drawn, if odd the card is drawn from the other box. Find

(a) the probability that both cards are even.

(b) if both cards are even, find the probability that they came from box A.

(c) what is the probability that both cards are odd?

Ans. **(a)** $\frac{1}{2} \cdot \frac{4}{9} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{12} + \frac{1}{20} = \frac{2}{15}$

(b) $\frac{\frac{1}{12}}{\frac{2}{15}} = \frac{5}{8}$

(c) $\frac{1}{2} \cdot \frac{5}{9} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{5}{9} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

16. A box contains 3 coins, two of them fair and one two-headed. A coin is selected at random and tossed twice. If head appears both times, what is the probability that the coin is two headed.

Ans. $\frac{\frac{1}{3} \cdot 1 \cdot 1}{\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 \cdot 1} = \frac{2}{3}$

Absorbing states

17. A player has Rs. 200. He bets Rs. 100 at a time and wins Rs. 100 with probability $\frac{1}{2}$. He stops playing if he loses the Rs. 200 or wins Rs. 400.

(a) Find the probability that he has lost his money at the end of at most 5 days.

(b) Determine the probability that the game lasts more than 7 plays.

Hint:

$$\begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$p^{(0)} = (0, 0, 1, 0, 0, 0, 0)$,

$$p^{(1)} = p^{(0)} P = \left(0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right)$$

$$p^{(2)} = p^{(1)} P = \left(\frac{1}{4}\ 0\ \frac{1}{2}\ 0\ \frac{1}{4}\ 0\ 0\right), p^{(5)} = p^{(4)} P$$

$$p^{(5)} = \left(\frac{3}{8}, \frac{5}{32}, 0, \frac{9}{32}, 0, \frac{1}{8}, \frac{1}{16}\right).$$

$p^{(5)}$: probability that he has no money after 5 plays is $\frac{3}{8}$, $p^{(7)} = \left(\frac{29}{64}, \frac{7}{64}, 0, \frac{27}{128}, 0, \frac{13}{128}, \frac{1}{8}\right)$

Ans. (a) $\frac{3}{8}$ (b) $\frac{7}{64} + \frac{27}{128} + \frac{13}{128} = \frac{27}{64}$.