Exponential distribution:- In probability theory and Statistics, the Exponential distribution (also known as the negative Exponential distribution) is the probability distribution that describes the time between events in a Poisson point process, i.e, a process in which events occur continuously and independently at a constant average rate.

The Exponential distribution is one of the widely used continuous distribution. It is often used to model the time elapsed between events. We will now mathematically define the exponential distribution, and derive its mean and expected value. Then we will develop the intuition for the distribution and discuss several interesting properties that it has

Definition:- A continuous random variable X is said to have Negative Exponential distribution with parameter μ if it assumes only real positive values and its probability density function is given by

$$f(x) = \begin{cases} a e^{-ax} & , \text{if } x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

where $a > 0$ is the prescribed parameter.

The Mean and Variance :-

The mean and variance of this distribution are $1/a$ and $1/a^2$ respectively.

This distribution is sometimes simply known as Exponential distribution only.

Properties of Exponential distribution:-

(1) The Exponential distribution is the continuous analog of the geometric distribution.

for Example, if the geometric random variable represents number of trials before the first failure occurs, its equivalence in the geometric case would be the waiting time for failure. Infact, as $P \to 0$, and inter-trial time $\to 0$, the geometric distribution $\to$ Exponential distribution.

(2) There Exists an important relationship between the Poisson distribution and the Exponential distribution. If the Poisson distribution represents the number of failure per unit time, the exponential distribution will describe the time between two successive failures.

Gamma and Exponential Distribution - Although the normal distribution can be used to solve many problems in engineering and science, there are still numerous situations that require different types of density functions. Two such density functions, the gamma and exponential distributions, are discussed in this section.

It turns out that the Exponential distribution is a special case of the gamma distribution. Both find a large number of applications. The Exponential and gamma distributions play an important role in both queuing theory and reliability problems. Time between arrivals at service facilities and time to failure of component parts and electrical systems often are nicely modeled by the exponential distribution. The relationship between the gamma and the exponential allows the gamma to be used in similar types of problems.

The gamma distribution derives its name from the well-known gamma function, studied in many areas of mathematics. Before we proceed to the gamma distribution, let us review this function and some of its important properties.

Defn:- The gamma function is defined as

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx \qquad \text{for } n > 0$$

The following are a few simple properties of the gamma function.

(a) $\Gamma n = (n-1)(n-2)\cdots(1)\Gamma 1$, for a positive integer n.

(b) $\Gamma n = (n-1)!$, for a positive integer n.

(c) $\Gamma 1 = 1$

(d) $\Gamma \frac{1}{2} = \sqrt{\pi}$

<u>The Gamma Distribution</u> – <u>Defn</u> :- A Continuous random variable X defined over the range $x \geq 0$ is said to be gamma distributed if the probability density function is of the form

$$f(x) = \frac{\lambda^r x^{r-1}}{\Gamma r} e^{-\lambda x}, \quad x \geq 0$$

where both r and λ are positive and Γr [to be read as gamma r] is the gamma function, defined by

$$\Gamma r = \int_0^\infty x^{r-1} e^{-x}\, dx$$

Since the gamma function is tabulated, the value of Γr, thus being a constant, can be easily obtained for the given value of r.

If r has an integer value, then $\Gamma r = (r-1)!$ In such a case, the gamma distribution immediately reduces to the Erlang.

<u>The Mean & Variance</u>:- The Expectation and variance of a gamma distributed random variable X are, respectively

$$E(X) = r/\lambda, \quad \& \quad V(X) = r/\lambda^2$$

<u>Applications of Gamma Distribution</u>:- The gamma distribution has many applications in statistical hypothesis testing, but is not often used as a descriptive distribution in modeling applications. For the latter purpose, the Erlang distribution is a much more useful and special case of gamma function.

27.13 THE GAMMA DISTRIBUTION

Consider a system consisting of one original component and $(r-1)$ spare components such that when the original component fails, one of the $(r-1)$ spare components is used. If this component fails, one of the $(r-2)$ spare components is used. System fails only when the original component and all the $(r-1)$ spare components fail. Assume that the lifetimes $X_1, X_2, \ldots, X_r$ of the r duplicates of the essential components have infinite lifetimes (except for the original component). Suppose each of the random variables $X_1, X_2, \ldots, X_r$ have the same exponential distribution with parameter λ and are probabilistically independent. Then the lifetime (time until failure) of the entire system is the sum $Y = \sum_{i=1}^{r} X_i$ having the *gamma distribution* with density function

$$f(y) = \begin{cases} \frac{\lambda^r y^{r-1} e^{-\lambda y}}{\Gamma(r)}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases} \tag{1}$$

(1) is a skewed distribution.

The two parameters of (1) are the positive numbers λ and r (although r need *not* be an integer). If r is a positive integer, then gamma distribution (1) is known as *Erlang distribution*. Introducing $V = \lambda y$, (1) reduces to

$$f(v) = \frac{1}{\lambda} f\left(\frac{v}{\lambda}\right) = \frac{1}{\lambda}\left\{\lambda^r \left(\frac{v}{\lambda}\right)^{r-1} \cdot \frac{e^{-v}}{\Gamma(r)}\right\}$$

$$= \begin{cases} \frac{v^{r-1} e^{-v}}{\Gamma(r)} & \text{if } v \geq 0 \\ 0 & \text{if } v < 0 \end{cases} \tag{2}$$

The probability density function of the random variable V given by (2) is known as the "*standard gamma function*" with parameter r (and is independent of λ). When $r = 1$, the density function (2) reduces to the density function of exponential distribution with the parameter $\lambda = 1$. For large r (say $r \geq 50$) (2) resembles a normal distribution with mean and variance approximately equal to r. The gamma distribution with parameter $\lambda = \frac{1}{2}$ and $r = \frac{v}{2}$ (where v is a positive integer) reduces to the chi-squared distribution with v degrees of freedom credited to Karl Pearson (1857–1936) and F.R. Helmert (1843–1917).

The *incomplete gamma function* defined by

$$F_V(t) = \int_0^t \frac{v^{r-1}e^{-v}}{\Gamma(r)} dv = I_r(t), \qquad t \geq 0 \tag{3}$$

is tabulated in the tables of the appendix for A23 to A28

$r = 1(1)5$, $t = 0.2(0.2)8.0(0.5)15.0$ and for $r = 6(1)10$, $t = 1.0(0.2)8.0(0.5)17.0$.

Now

$$P(Y \geq a) = P(Y > a) = 1 - F(a) = 1 - I_r(\lambda a)$$

and

$$P(a \leq Y \leq b) = F(b) - F(a) = I_r(\lambda b) - I_r(\lambda a)$$

Moments of the Gamma Distribution

For any $k \geq 0$,

$$E(Y^k) = E\left(\frac{V^k}{\lambda^k}\right) = \frac{1}{\lambda^k} E(V^k)$$

$$= \frac{1}{\lambda^k} \int_0^\infty v^k \cdot \left(\frac{v^{r-1}e^{-v}}{\Gamma(r)}\right) dv$$

$$= \frac{1}{\lambda^k \Gamma(r)} \int_0^\infty e^{-v} v^{k+r-1} dv$$

Then $$E(Y^k) = \frac{\Gamma(r+k)}{\lambda^k \Gamma(r)} \tag{4}$$

For $k = 0$ in (4) we have $\int_0^\infty f(y)dy = 1$ so $f(y)$ is a probability density function.

For $k = 1$, in (4) we get the mean $= \mu = E(Y) = \frac{\Gamma(r+1)}{\lambda\Gamma(r)} = \frac{r\Gamma(r)}{\lambda\Gamma(r)}$.

So $\mu = \frac{r}{\lambda}$. (5)

With $k = 2$ in (4), we get

variance $= \sigma^2 = E(Y^2) - \{E(Y)\}^2$

$$= \frac{\Gamma(r+2)}{\lambda^2\Gamma(r)} - \frac{r^2}{\lambda^2}$$

$$\sigma^2 = \frac{r(r+1)\Gamma(r)}{\lambda^2\Gamma(r)} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda^2} \tag{6}$$

Thus the parameter r and λ are determined from (5) and (6) as

$$\lambda = \frac{\mu}{\sigma^2}, \quad r = \frac{\mu^2}{\sigma^2} \tag{7}$$

Relation Between Exponential, Gamma and Poisson Distributions

Suppose the lifetimes of a batch of components each have exponential distribution with parameter λ. Starting at time $t = 0$, the first component is used until its extinction (until it "dies" or "fails"). Replace the component by another instantaneously and wait unit this new component also fails. Continuing this Process, stop at a given time t. Then the number of failed components L is a random variable having the Poisson distribution with $\lambda^* = \lambda t$. Also the lifetime of the entire process $Y = \sum_{i=1}^{k} X_i$ follows gamma distribution with parameters λ and k.

Let $X_1, X_2, \ldots, X_k$ be the lifetimes of the first k components that have failed. Assume that each lifetime X_i has an exponential distribution with parameter λ and are probabilistically independent. Recall that, then $Y = \sum_{i=1}^{k} X_i$ has a gamma distribution with parameters λ and k. If $X_1 + X_2 + \ldots + X_k$ (total lifetime of the process) $\leq t$ then in this case at least k components have all failed, (one after the other) before the time is up.

Now probability of the event that at least k components have failed at the time of termination of the experiment is given by

$$P(L \geq k) = P\{Y \leq t\} = I_k(\lambda t)$$

$$= \int_0^{\lambda t} \frac{v^{k-1}e^{-v}}{\Gamma(k)} dv$$

Integrating by parts, we have

$$= \frac{v^{k-1}}{\Gamma(k)} \cdot \left(\frac{e^{-v}}{-1}\right)\Bigg|_{v=0}^{\lambda t} + \int_0^{\lambda t} \frac{(k-1)v^{k-2} \cdot e^{-v}}{\Gamma(k)} dv$$

$$= -\frac{(\lambda t)^{k-1}e^{-\lambda t}}{(k-1)!} + \int_0^{\lambda t} \frac{v^{k-2} \cdot e^{-v}}{\Gamma(k-1)} dv.$$

Integrating by parts $(k-1)$ more times,

$$P(L \geq k) = -\sum_{i=1}^{k-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!} + \int_0^{\lambda t} \frac{e^{-v}}{\Gamma(1)} dv$$

But $\Gamma(1) = 1$ and $\int_0^{\lambda t} e^{-v} dv = \frac{e^{-v}}{-1}\Big|_0^{\lambda t} = 1 - e^{-\lambda t}$.

$$\text{Thus } P(L \geq k) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

But

$$P(L = k) = P(L \geq k) - P(L \geq k+1)$$

$$= \left[1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}\right] - \left[1 - \sum_{i=0}^{k} \frac{(\lambda t)^i e^{-\lambda t}}{i!}\right]$$

$$P(L = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{\lambda^{*k} \cdot e^{-\lambda^*}}{k!}, \ \lambda^* = \lambda t$$

Thus the probability distribution of the discrete random variable L is a Poisson (discrete) distribution with parameter $\lambda^* = \lambda t$.

The cumulative distribution of the gamma distribution of Y can be calculated in terms of tabulated cumulative distribution of the Poisson distribution from

$$F(t) = P(L \geq k) = P(Y \leq t) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

Worked Out Examples

Example 1: The daily consumption of electric power (in millions if kW-hours) in a certain city is a random variable X having the probability density

$$f(x) = \begin{cases} \frac{1}{9}xe^{-x/3} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Find the probability that the power supply is inadequate on any given day if the daily capacity of the power plant is 12 million kW hours.

Solution: Observe that this is gamma distribution with $r = 2$ and $\lambda = \frac{1}{3}$. The power supply is inadequate when $X > 12$. Now $P(X > 12) = 1 - F(12) = 1 - I_2\left(\frac{1}{3} 12\right) = 1 - I_2(4)$. Using table A23 to A28, we get

$$P(X > 12) = 1 - 0.90892 = 0.09108$$

Example 2: The lifetime X (in months) of a computer has a gamma distribution with mean 24 months and standard deviation 12 months. Find the probability that the computer will

(a) last between 12 and 24 months.

(b) last at most 24 months.

(c) Determine the median lifetime of X.

(d) suppose that the test will actually be terminated after t months. Determine the value of t such that only one-half of 1% of all computers would still be functioning at termination.

Solution: Here $\mu = 24$, $\sigma = 12$. Then from (7)

$\lambda = \frac{24}{12^2} = \frac{1}{6}$, $r = \left(\frac{\mu}{\sigma}\right)^2 = \left(\frac{24}{12}\right)^2 = 4$

(a) $P(12 < X < 24) = F(24) - F(12)$

$$= I_4\left(\frac{1}{6}\cdot 24\right) - I_4\left(\frac{1}{6}\cdot 12\right)$$

$$= I_4(4) - I_4(2)$$

Using table A23 to A28 we get $P(12 < X < 24) = 0.56653 - 0.14288 = 0.42365$.

(b) $P(X \le 24) = I_4\left(\frac{1}{6}\cdot 24\right) = I_4(4) = 0.56653$

(c) median $\hat{x}$ is such that $P(X \le \hat{x}) = \frac{1}{2}$

Then $I_4\left(\frac{\hat{x}}{6}\right) = 0.5$. From table (see entry in the table A23 to A28) we get $\frac{\hat{x}}{6} = 4$) or $\hat{x} = 24$ months

(d) At the termination time t, only one-half of 1% computers are still functioning. So

$$P(X \le t) = 1 - \frac{1}{2}(0.01) = 0.995$$

But $P(X \le t) = I_4\left(\frac{1}{6}\hat{t}\right) = 0.995$ (given). From table A23 to A28, we have $I_4(11) = 0.995$. Therefore $\frac{\hat{t}}{6} = 11$ or $\hat{t} = 66$ months.

EXERCISE

1. Suppose the reaction time X has a standard gamma distribution with $r = 2$. Find (a) $P(3 \le X \le 5)$ (b) $P(X > 4)$.

Ans. (a) $I_2(5) - I_2(3) = 0.95957 - 0.80085 = 0.15872$ (b) $P(X > 4) = 1 - P(X \le 4) = 1 - I_2(4) = 1 - 0.90842 = 0.09158$

2. Suppose that the time (in hours) taken by a home-owner to mow his lawns is a random variable X having a gamma distribution with parameters $r = 2$ and $\lambda = 2$. Find the probability that it takes (a) at most 1 hour (b) at least 2 hours (c) between 0.5 and 1.5 hours to mow the lawn.

Ans. (a) 0.594 (b) 0.092 (c) 0.537

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3. The survival time X (in weeks) of a male mouse exposed to radiation has a gamma distribution with $r = 8$ and $\lambda = \frac{1}{15}$. Find the probability that the mouse survives (a) between 60 and 120 weeks (b) at least 30 week. Find (c) mean (d) variance of X.

Ans. (a) $P(60 \leq X \leq 120) = F\left(\frac{120}{15}, 8\right) - F\left(\frac{60}{15}, 8\right)$
$= I_8(8) - I_8(4) = 0.547 - 0.051 = 0.496$
(b) $P(X \geq 30) = 1 - P(X < 30) =$
$1 - F\left(\frac{30}{15}, 8\right) = 1 - 0.00110 = 0.9989$

4. If Y has gamma distribution with $\lambda = 0.40$ and $r = 5$, find (a) $P(Y > 30)$ (b) $P(15 \leq Y \leq 20)$

Ans. (a) $1 - I_5(12) = 1 - 0.9924 = 0.0076$
(b) $I_5(8) - I_5(6) = 0.90037 - 0.71494 =$
0.18543

5. If a random variable X has the gamma distribution with $r = 2$ and $\lambda = \frac{1}{2}$ find (a) the mean (b) standard deviation (c) the probability that X will take a value less than 4.

Ans. (a) 4 (b) 2.828 (c) 0.5940

6. The gross sales X in thousands of rupees is a random variable having gamma distribution with $\lambda = \frac{1}{2}$ and $r = 80000\sqrt{n}$ where n is the number of employees in the company. If the sales cost is Rs. 8000 per employee, how many employees should the company employ to maximise the expected profit.

Ans. $n = 100$,

Hint: $\mu = \frac{r}{\lambda} = 160000\sqrt{n}$, y = profit = total expected sales − total cost = $160000\sqrt{n} - 8000n$, $\frac{dy}{dn} = 0$ for $n = 100$.

$$\frac{d^2y}{dn^2} = -\frac{40000}{n^{3/2}} < 0$$

Table 9 Values of the Incomplete Gamma Function $I_r(\tau)$ for use in the Computation of the Cumulative Gamma Distribution Function

τ	r = 1	2	3	4	5
0.2	0.18127	0.01752	0.00115	0.00006	0.00000
0.4	0.32968	0.06155	0.00793	0.00078	0.00006
0.6	0.45119	0.12190	0.02312	0.00336	0.00039
0.8	0.55067	0.19121	0.04742	0.00908	0.00141
1.0	0.63212	0.26424	0.08030	0.01899	0.00366
1.2	0.69881	0.33737	0.12051	0.03377	0.00775
1.4	0.75340	0.40817	0.16650	0.05372	0.01425
1.6	0.79810	0.47507	0.21664	0.07881	0.02368
1.8	0.83470	0.53716	0.26938	0.10871	0.03641
2.0	0.86466	0.59399	0.32332	0.14288	0.05265
2.2	0.88920	0.64543	0.37729	0.18065	0.07250
2.4	0.90928	0.69156	0.43029	0.22128	0.09587
2.6	0.92573	0.73262	0.48157	0.26400	0.12258
2.8	0.93919	0.76892	0.53055	0.30806	0.15232
3.0	0.95021	0.80085	0.57681	0.35277	0.18474
3.2	0.95924	0.82880	0.62010	0.39748	0.21939
3.4	0.96663	0.85316	0.66026	0.44164	0.25582
3.6	0.97268	0.87431	0.69725	0.48478	0.29356
3.8	0.97763	0.89262	0.73110	0.52652	0.33216
4.0	0.98168	0.90842	0.76190	0.56653	0.37116
4.2	0.98500	0.92202	0.78976	0.60460	0.41017
4.4	0.98772	0.93370	0.81486	0.64055	0.44882
4.6	0.98995	0.94371	0.83736	0.67429	0.48677
4.8	0.99177	0.95227	0.85746	0.70577	0.52374
5.0	0.99326	0.95957	0.87535	0.73497	0.55951
5.2	0.99448	0.96580	0.89121	0.76193	0.59387
5.4	0.99548	0.97109	0.90524	0.78671	0.62669
5.6	0.99630	0.97559	0.91761	0.80938	0.65785
5.8	0.99697	0.97941	0.92849	0.83004	0.68728
6.0	0.99752	0.98265	0.93803	0.84880	0.71494

$$I_r(\tau) = F_V(\tau) = \int_0^{\tau} \frac{v^{r-1}e^{-v}}{\Gamma(r)}\,dv,\ \tau \geq 0$$

Table 9 *(continued)*

τ	r = 1	2	3	4	5
6.2	0.99797	0.98539	0.94638	0.86577	0.74082
6.4	0.99834	0.98770	0.95368	0.88108	0.76493
6.6	0.99864	0.98966	0.96003	0.89485	0.78730
6.8	0.99889	0.99131	0.96556	0.90719	0.80797
7.0	0.99909	0.99270	0.97036	0.91823	0.82701
7.2	0.99925	0.99388	0.97453	0.92808	0.84448
7.4	0.99939	0.99487	0.97813	0.93685	0.86047
7.6	0.99950	0.99570	0.98124	0.94463	0.87506
7.8	0.99959	0.99639	0.98393	0.95152	0.88833
8.0	9.99966	0.99698	0.98625	0.95762	0.90037
8.5	0.99980	0.99807	0.99072	0.96989	0.92564
9.0	0.99988	0.99877	0.99377	0.97877	0.94504
9.5	0.99993	0.99921	0.99584	0.98514	0.95974
10.0	0.99995	0.99950	0.99723	0.98966	0.97075
10.5	0.99997	0.99968	0.99817	0.99285	0.97891
11.0	0.99998	0.99980	0.99879	0.99508	0.98490
11.5	0.99999	0.99987	0.99920	0.99664	0.98925
12.0	0.99999	0.99992	0.99948	0.99771	0.99240
12.5	1.00000	0.99995	0.99966	0.99845	0.99465
13.0	1.00000	0.99997	0.99978	0.99895	0.99626
13.5	1.00000	0.99998	0.99986	0.99929	0.99740
14.0	1.00000	0.99999	0.99991	0.99953	0.99819
14.5	1.00000	0.99999	0.99994	0.99968	0.99875
15.0	1.00000	1.00000	0.99996	0.99979	0.99914

Table 9 (continued)

τ	r = 6	7	8	9	10
1.0	0.00059	0.00008	0.00001		
1.2	0.00150	0.00025	0.00004		
1.4	0.00320	0.00062	0.00011	0.00002	
1.6	0.00604	0.00134	0.00026	0.00005	0.00001
1.8	0.01038	0.00257	0.00056	0.00011	0.00002
2.0	0.01656	0.00453	0.00110	0.00024	0.00005
2.2	0.02491	0.00746	0.00198	0.00047	0.00010
2.4	0.03567	0.01159	0.00334	0.00086	0.00020
2.6	0.04904	0.01717	0.00533	0.00149	0.00038
2.8	0.06511	0.02441	0.00813	0.00243	0.00066
3.0	0.08392	0.03351	0.01190	0.00380	0.00110
3.2	0.10541	0.04462	0.01683	0.00571	0.00176
3.4	0.12946	0.05785	0.02307	0.00829	0.00271
3.6	0.15588	0.07327	0.03079	0.01167	0.00402
3.8	0.18444	0.09089	0.04011	0.01598	0.00580
4.0	0.21487	0.11067	0.05113	0.02136	0.00813
4.2	0.24686	0.13254	0.06394	0.02793	0.01113
4.4	0.28009	0.15635	0.07858	0.03580	0.01489
4.6	0.31424	0.18197	0.09505	0.04507	0.01953
4.8	0.34899	0.20920	0.11333	0.05582	0.02514
5.0	0.38404	0.23782	0.13337	0.06809	0.03183
5.2	0.41909	0.26761	0.15508	0.08193	0.03967
5.4	0.45387	0.29833	0.17834	0.09735	0.04875
5.6	0.48814	0.32974	0.20302	0.11432	0.05913
5.8	0.52169	0.36161	0.22897	0.13281	0.07084
6.0	0.55432	0.39370	0.25602	0.15276	0.08392
6.2	0.58589	0.42579	0.28398	0.17409	0.09838
6.4	0.61626	0.45767	0.31268	0.19669	0.11420
6.6	0.64533	0.48916	0.34192	0.22044	0.13136
6.8	0.67302	0.52008	0.37151	0.24523	0.14982

Table 9 *(continued)*

τ	r = 6	7	8	9	10
7.0	0.69929	0.55029	0.40129	0.27091	0.16950
7.2	0.72410	0.57964	0.43106	0.29733	0.19035
7.4	0.74744	0.60804	0.46067	0.32435	0.21226
7.6	0.76932	0.63538	0.48996	0.35181	0.23515
7.8	0.78975	0.66159	0.51879	0.37956	0.25889
8.0	0.80876	0.68663	0.54704	0.40745	0.28338
8.5	0.85040	0.74382	0.61440	0.47689	0.34703
9.0	0.88431	0.79322	0.67610	0.54435	0.41259
9.5	0.91147	0.83505	0.73134	0.60818	0.47817
10.0	0.93291	0.86986	0.77978	0.66718	0.54207
10.5	0.94962	0.89837	0.82149	0.72059	0.60287
11.0	0.96248	0.92139	0.85681	0.76801	0.65949
11.5	0.97227	0.93973	0.88627	0.80941	0.71121
12.0	0.97966	0.95418	0.91050	0.84497	0.75761
12.5	0.98510	0.96543	0.93017	0.87508	0.79857
13.0	0.98927	0.97411	0.94597	0.90024	0.83419
13.5	0.99227	0.98075	0.95852	0.92100	0.86474
14.0	0.99447	0.98577	0.96838	0.93794	0.89060
14.5	0.99606	0.98955	0.97606	0.95162	0.91224
15.0	0.99721	0.99237	0.98200	0.96255	0.93015
15.5	0.99803	0.99446	0.98654	0.97121	0.94481
16.0	0.99862	0.99599	0.99000	0.97801	0.95670
16.5	0.99903	0.99712	0.99261	0.98331	0.96626
17.0	0.99933	0.99794	0.99457	0.98741	0.97388

Table 9 (continued)

τ	r = 11	12	13	14	15
4.0	0.00284	0.00091	0.00027	0.00008	0.00002
4.5	0.00667	0.00240	0.00081	0.00025	0.00007
5.0	0.01370	0.00545	0.00202	0.00070	0.00023
5.5	0.02525	0.01099	0.00445	0.00169	0.00060
6.0	0.04262	0.02009	0.00883	0.00363	0.00140
6.5	0.06684	0.03388	0.01603	0.00710	0.00296
7.0	0.09852	0.05335	0.02700	0.01281	0.00572
7.5	0.13776	0.07924	0.04267	0.02156	0.01026
8.0	0.18411	0.11192	0.06380	0.03418	0.01726
8.5	0.23664	0.15134	0.09092	0.05141	0.02743
9.0	0.29401	0.19699	0.12423	0.07385	0.04147
9.2	0.31797	0.21682	0.13926	0.08438	0.05999
9.4	0.34236	0.23743	0.15524	0.09581	0.05590
9.6	0.36705	0.25876	0.17212	0.10815	0.06428
9.8	0.39195	0.28072	0.18988	0.12139	0.07346
10.0	0.41696	0.30322	0.20844	0.13554	0.08346
10.2	0.44197	0.32618	0.22777	0.15055	0.09429
10.4	0.46687	0.34951	0.24779	0.16641	0.10596
10.6	0.49159	0.37310	0.26843	0.18309	0.11847
10.8	0.51603	0.39687	0.28963	0.20054	0.11318
11.0	0.54011	0.42073	0.31130	0.21871	0.14596
11.2	0.56376	0.44459	0.33337	0.23756	0.16090
11.4	0.58690	0.46837	0.35576	0.25702	0.17661
11.6	0.60949	0.49198	0.37839	0.27703	0.19305
11.8	0.63146	0.51535	0.40117	0.29754	0.21019
12.0	0.65277	0.53840	0.42403	0.31846	0.22798
12.2	0.67338	0.56108	0.44690	0.33974	0.24637
12.4	0.69327	0.58331	0.46968	0.36130	0.26531
12.6	0.71239	0.60504	0.49232	0.38307	0.28474
12.8	0.73075	0.62623	0.51475	0.40498	0.30462

Table 9 *(continued)*

τ	r = 11	12	13	14	15
13.0	0.74832	0.64684	0.53690	0.42696	0.32487
13.2	0.76510	0.66681	0.55870	0.44893	0.34543
13.4	0.78108	0.68614	0.58012	0.47084	0.36625
13.6	0.79628	0.70478	0.60110	0.49262	0.38725
13.8	0.81068	0.72273	0.62158	0.51421	0.40838
14.0	0.82432	0.73996	0.64154	0.53555	0.42956
14.2	0.83720	0.75647	0.66094	0.55659	0.45075
14.4	0.84934	0.77225	0.67975	0.57728	0.47188
14.6	0.86076	0.78731	0.69793	0.59756	0.49289
14.8	0.87149	0.80164	0.71549	0.61741	0.51373
15.0	0.88154	0.81525	0.73239	0.63678	0.53435
15.5	0.90388	0.84622	0.77173	0.68292	0.58459
16.0	0.92260	0.87301	0.80688	0.72549	0.63247
16.5	0.93813	0.89593	0.83790	0.76426	0.67746
17.0	0.95088	0.91533	0.86498	0.79913	0.71917
17.5	0.96126	0.93160	0.88835	0.83013	0.75736
18.0	0.96963	0.94511	0.90833	0.85740	0.79192
18.5	0.97635	0.95624	0.92525	0.88114	0.82286
19.0	0.98168	0.96533	0.93944	0.90160	0.85025
19.5	0.98589	0.97269	0.95125	0.91908	0.87427
20.0	0.98919	0.97861	0.96099	0.93387	0.89514
20.5	0.99176	0.98335	0.96897	0.94630	0.91310
21.0	0.99375	0.98710	0.97545	0.95664	0.92843
21.5	0.99528	0.99005	0.98069	0.96520	0.94141
22.0	0.99645	0.99237	0.98488	0.97222	0.95231
22.5	0.99735	0.99418	0.98823	0.97794	0.96140
23.0	0.99802	0.99557	0.99088	0.98275	0.96893
23.5	0.99853	0.99665	0.99297	0.98630	0.97512
24.0	0.99892	0.99748	0.99460	0.98928	0.98018
24.5	0.99920	0.99811	0.99587	0.99166	0.98428