Problem 1: Eulerian Enigma

Find the smallest perfect square > 1 that has a perfect square number of divisors.

Problem 2: Trace Deleter

Consider a sequence of integers $1, 2, 3, ..., n$. So $a_i = i$ for all 1 to n.

In 1 operation, you will remove the numbers at odd indices from the sequence (i.e. delete a_1, a_3, a_5, a_7, ...). The sequence is 1 indexed.

You will keep performing operations, until the sequence is empty.

Let $F(x)$ = the # operations until the number x is removed.
Let $G(n) = \sum_{i=1}^{n} F(i)$. In other words, $G(n)$ is the sum of $F(i)$ for each number in the sequence initially.

$G(10) = 18$.
Find $G(10^{18})$.

Problem 3: K-Constructor

Consider a number N and K. You start with the number 0, and in 1 operation, you can either multiply your total by K or add 1.

Let $F(N, K)$ denote the minimum number of operations to reach N and not exceed it, for a given K.

Let $G(N)$ denote $\sum_{j=1}^{N} F(N, j)$.

For example, $F(13, 3) = 5$ and the total after each operation is: $[1, 3, 4, 12, 13]$.

You are given that $G(13) = 74$.

Compute $G(10^7)$.

Problem 4: Sequence Summation

Let $f(n)$ be defined as follows:

A sequence x is defined as $x_i = i$ for all $1 \leq i \leq n$. If $i > n$ then $x_i = \sum_{j=1}^{i-1} x_j$.
$f(n) = x_{2n}$ modulo $10^9 + 7$.

For example, $f(5) = 240$ because in this case, $x = [1, 2, 3, 4, 5, 15, 30, 60, 120, 240]$, and $x_{10} = 240$.
Find $f(10^{16})$.

Problem 5: GCD MST

Consider a simple, fully connected and undirected graph of nodes from 2 to N.

The weight of the edge between nodes i and j, where $i > j$, is the number of **function calls** the GCD function takes when calculating $GCD(i, j)$.

The GCD function is defined as:

```
function GCD(i, j):
   if j is 0:
       return i
   else:
       return GCD(j, i % j)
```

with the precondition $i > j$.

For instance, the number of calls of $GCD(11, 7)$ is 5.

1) $GCD(11, 7)$
2) $GCD(7, 4)$
3) $GCD(4, 3)$
4) $GCD(3, 1)$
5) $GCD(1, 0)$

Let $F(N)$ denote the sum of the edge weights of the Minimum Spanning Tree.

$F(101) = 209$.

Find $F(10^8)$.

Problem 6: Plex Googol Plex

A Googol is a 1 with 100 zeros, 10^{100}

A Googol Plex is a 1 with a Googol zeros $10^{10^{100}}$

Call a Plex Googol Plex as a power tower of $10^{10^{\cdot^{\cdot^{\cdot^{10^{10}}}}}}$ with a Googol Plex 10's. Define this number to be G.

$G \equiv 0 \bmod 1$ because anything is $0 \bmod 1$

$G \equiv 0 \bmod 2$ because it is a power of 10 so it must be even

$G \equiv 1 \bmod 3$ because it's sum of digits is 1

$G \equiv 1 \bmod 17$

Find $\sum_{m=1}^{10^6} (G \bmod m)$

Note that power towers are evaluated from the topmost number to the bottom

Problem 7: Not a Topological Sort Problem

There are X days in a semester of WWPPC Academy, and N classes numbered 1 to N of which you can choose to enroll in.

Normally, class i will take place on i distinct, randomly chosen days, during the semester. However if there are more classes than days in the semester, the class will have repeat days.

Let the $F(N, X)$ denote the probability that if you enroll in a randomly chosen subset of all N classes, you will only have to attend at most 1 class each day for all X days.

Note that each subset has equal probability and the empty subset is included.

$F(101, 101) = 845069431$ modulo $10^9 + 7$.
Find $F(5 \cdot 10^4, 4 \cdot 10^4)$ modulo $10^9 + 7$.

Note: If the simplified probability of $F(N, X)$ is $\frac{P}{Q}$, then find $P \cdot Q^{-1}$ modulo $10^9 + 7$.

Problem 8: Bitmask Battle

Alice and Bob are playing a game on a bitmask (string of 0/1s) of length n. Alice goes first and neither player is allowed to pass a turn.

In 1 move, a player picks a 1 bit and swaps its with any existing 0 bit between it and the closest 1 bit to the right. Additionally, the player is allowed to swap the rightmost 1 bit (if it exists) with any existing 0 bit to its right.

A player loses if they cannot make a move on their turn.

Let $F(n)$ denote the number of possible bitmasks of length n exists for which Alice wins.

$F(4) = 8$.
$F(6) = 40$.
Find $F(1500)$ modulo $10^9 + 7$.

Problem 9: Subset GCD

Define a pseudo-random number generator r as

$$r_0 = 1$$
$$r_i = (r_{i-1} \cdot 1664525 + 13904216) \bmod 10^9 + 7$$
$$a_i = (r_i \bmod m) + 1$$

Let $f(n, k, m) = \sum_{1 \le i_1 < \cdots < i_k \le n} 2^{\gcd(a_{i_1}, \ldots a_{i_k})}$. In other words, the sum of 2 to the power of the gcd of each subset of size k. Note that m is the modulus used for the pseudo-random number generator a from above.

For example $f(5, 2, 100) = 4194392$ because the sequence a is: $42, 44, 23, 66, 38$. Here are all possible subsets of size 2:

$$(1,2) : \gcd(42, 44) = 2, 2^2 = 4$$
$$(1,3) : \gcd(42, 23) = 1, 2^1 = 2$$
$$(1,4) : \gcd(42, 66) = 6, 2^6 = 64$$
$$(1,5) : \gcd(42, 38) = 2, 2^2 = 4$$
$$(2,3) : \gcd(44, 23) = 1, 2^1 = 2$$
$$(2,4) : \gcd(44, 66) = 22, 2^{22} = 4194304$$
$$(2,5) : \gcd(44, 38) = 2, 2^2 = 4$$
$$(3,4) : \gcd(23, 66) = 1, 2^1 = 2$$
$$(3,5) : \gcd(23, 38) = 1, 2^1 = 2$$
$$(4,5) : \gcd(66, 38) = 2, 2^2 = 4$$

The total sum is 4194392

You are given $f(10^7, 2, 10^7) \equiv 765469928 \bmod 10^9 + 7$

Define $F(n, p, m) = \sum_{i \ge 1} f(n, i^p, m)$

You are given $F(10^7, 20, 10^7) \equiv 250219557 \bmod 10^9 + 7$

Find $F(10^7, 2, 10^7) \bmod 10^9 + 7$

Problem 10: Almost Fibonacci

The Fibonacci series starts of with $F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2}$

Pick a random number $1 \leq x \leq F_N$. Let d be the minimum value of $|F_k - x|$ over all $1 \leq k \leq N$. Let $g(N)$ be this expected value of d for the N'th fibonacci number multiplied by F_N.

You are given

$g(5) = 3$

$g(6) = 9$

$g(10^{18}) \equiv 481968831 \bmod 10^9 + 7$

Find $\sum_{i=1}^{10^6} g(i \cdot 10^{12})$

Note

$$F_1 = 1$$
$$F_2 = 2$$
$$F_3 = 3$$
$$F_4 = 5$$
$$F_5 = 8$$
$$F_6 = 13$$

For example, if we picked $x = 7$ the minimum value of $|F_k - x|$ would be 1 at $k = 5$

Problem 11: Squares and Squares of Squares

Let $f(N, n)$ be the number of solutions to the equation $a^2 + b^2 = c^{2^n}$, where $a, b, c \in \mathbb{Z}^+$, $c \leq N$ and $\gcd(a, b) = n$. Let $G(n) = \sum_{n=1}^{N} f(N, n)$.

$G(13) = 8$.

$G(1337) = 1944$.

Find $G(34567890)$.

Problem 12: Super Summings

Let a be a set of size s where $1 \leq a_1 \ldots a_s \leq n$ and n is a factor of $a_i a_j$ for all $1 \leq i, j \leq s$

Define $f(n, r)$ to be the maximum value of $\sum_{i=1}^{s} \lfloor r \cdot a_i \rfloor$ over all possible sets a

Define $g(n, r) = \sum_{i=1}^{n} f(i, r)$

$g(20, \frac{1}{2}) = 136$
$g(10^6, \frac{1001}{101}) = 14677540520195 \equiv 540417456 \bmod 10^9 + 7$
$g(10^6, 1) = 1480950879222 \equiv 950868862 \bmod 10^9 + 7$

Find $g(10^{14}, \frac{1001}{101}) \bmod 10^9 + 7$