Classification

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Content

Binary Classification

Suppose we want to build an email spam classification software:

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> Hello
> I want to know about different topics that relate to qualitative reinforcement learning and make abstraction&aggregation... to solve problem compactly . I have read some survey to know exactly but sometimes I doubt about some topics are related to or not. for example Qualitative Spatial Representation and Reasoning. can anyone tell me different categorized topics? I need to know the general classification of them.
>
> ...

Binary Classification

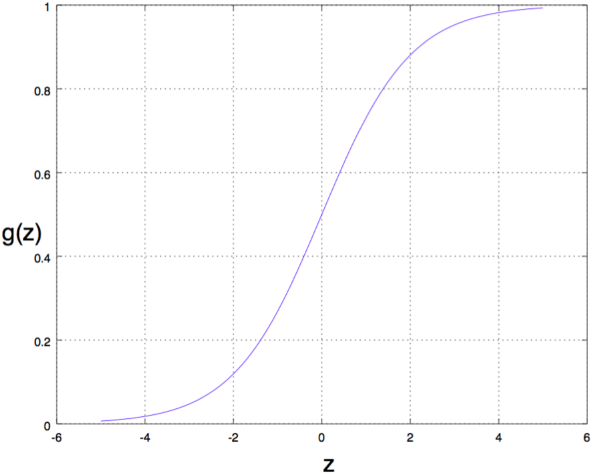
- Classification is the same as regression, the only difference is that the output variable y can take only a small number of discrete values.
- The most simple classification problem is the binary classification, in which $y = \{0, 1\}$.
- In a binary classification problem we have positive examples $y = 1$ (spam) and negative examples $y = 0$ (no spam).
- The $\mathbf{x}$ may be some features of some piece of email.

Binary Classification

- We could approach the classification problem ignoring the fact that y is a discrete value, and use linear regression to try to predict y given $\mathbf{x}$.
- However, it is easy to construct examples showing this method to perform very badly.
- Intuitively, it does not make sense for $h_\theta = (\mathbf{x})$ to take values larger than 1 or smaller than 0, when we know that $y \in \{0, 1\}$
- To fix this we will make $h_\theta(\mathbf{x}) = g(\theta^\top \mathbf{x}) = \frac{1}{1+e^{-\theta^\top \mathbf{x}}}$
- where $g(z) = \frac{1}{1+e^{-z}}$

Binary Classification

$$g(z) = \frac{1}{1+e^{-z}} \qquad g : \mathbb{R} \to (0,1)$$



Derivative of the Sigmoid Function

$$
\begin{aligned}
g'(z) &= \tfrac{d}{dz}\tfrac{1}{1+e^{-z}} \\
&= -\tfrac{1}{\left(1+e^{-z}\right)^2} \cdot \left(- e^{-z} \right) \\
&= \tfrac{1}{\left(1+e^{-z}\right)^2} \cdot e^{-z} \\
&= \tfrac{1}{\left(1+e^{-z}\right)} \cdot \tfrac{e^{-z}}{\left(1+e^{-z}\right)} \\
&= \tfrac{1}{\left(1+e^{-z}\right)} \cdot \tfrac{e^{-z}+1-1}{\left(1+e^{-z}\right)} \\
&= \tfrac{1}{\left(1+e^{-z}\right)} \cdot \left(\tfrac{\left(1+e^{-z}\right)}{\left(1+e^{-z}\right)} - \tfrac{1}{\left(1+e^{-z}\right)} \right) \\
&= g(z)\big(1 - g(z)\big).
\end{aligned}
$$

A Probabilistic Approach

Lets assume that:

$$P(y = 1|x; \theta) = h_\theta(x)$$

$$P(y = 0|x; \theta) = 1 - h_\theta(x)$$

Note that this can be written more compactly as:

$$p(y|x; \theta) = (h_\theta(x))^y (1 - h_\theta(x))^{1-y}$$

Likelihood of the Training Data's Labels

Assuming that the m training examples were generated independently, we can then write:

$$
\begin{aligned}
L(\theta) &= p(\mathbf{y}|\mathbf{X};\theta) \\
&= \prod_{i=1}^{m} p(y_i|\mathbf{x}_i;\theta) \\
&= \prod_{i=1}^{m} (h_\theta(x_i))^{y_i}(1 - h_\theta(x_i))^{1-y_i}
\end{aligned}
$$

Log Likelihood

$$L(\theta) = \prod_{i=1}^{m} (h_\theta(x_i))^{y_i} (1 - h_\theta(x_i))^{1-y_i}$$

Working instead with the log likelihood:

$$\begin{aligned} \ell(\theta) &= \log L(\theta) \\ &= \sum_{i=1}^{m} y_i \log h(x_i) + (1 - y_i) \log(1 - h(x_i)) \end{aligned}$$

Using gradient ascent we get an update rule like this:

$$\theta := \theta + \alpha \nabla_\theta \ell(\theta)$$

Maximizing the Likelihood

Working with one example, the derivatives are as follows:

$$
\begin{aligned}
\frac{\partial}{\partial \theta_j} \ell(\theta) &= \frac{\partial}{\partial \theta_j} \Big[y \log h(x) + (1-y) \log(1 - h(x)) \Big] \\
&= \frac{\partial}{\partial \theta_j} \Big[y \log g(\theta^\top \mathbf{x}) + (1-y) \log(1 - g(\theta^\top \mathbf{x})) \Big] \\
&= \left(y \frac{1}{g(\theta^\top \mathbf{x})} - (1-y) \frac{1}{1 - g(\theta^\top \mathbf{x})} \right) \frac{\partial}{\partial \theta_j} g(\theta^\top \mathbf{x}) \\
&= \left(y \frac{1}{g(\theta^\top \mathbf{x})} - (1-y) \frac{1}{1 - g(\theta^\top \mathbf{x})} \right) g(\theta^\top \mathbf{x}) \Big(1 - g(\theta^\top \mathbf{x}) \Big) \frac{\partial}{\partial \theta_j} \theta^\top \mathbf{x} \\
&= \Big(y(1 - g(\theta^\top \mathbf{x})) - (1-y) g(\theta^\top \mathbf{x}) \Big) x_j \\
&= (y - h_\theta(\mathbf{x})) x_j
\end{aligned}
$$

The LMS Update Rule for Classification

Given that:

$$\frac{\partial}{\partial\theta_j}\ell(\theta) = (y - h_\theta(\mathbf{x}))x_j,$$

our LMS update rule for classification can be written as:

$$\theta_j := \theta_j + \alpha(y_i - h_\theta(\mathbf{x}_i))(x_i)_j,$$

where $h_\theta(\mathbf{x}_i) = g(\theta^\top \mathbf{x}_i) = \frac{1}{1+e^{(-\theta^\top \mathbf{x}_i)}}$ is now defined as a non-linear function of $\theta^\top \mathbf{x}_i$.

So, we end up with the same update rule for a different algorithm and learning problem.

LMS Algorithms for Classification

Batch Gradient Descent

Repeat until convergence $\{$

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left[y_i - g(\theta^\top x_i) \right] (x_i)_j \qquad \text{(for every } j\text{).}$$

$\}$

Stochastic Gradient Descent

Loop $\{$

for $i = 1$ to m $\{$

$$\theta_j := \theta_j + \alpha \left[y_i - g(\theta^\top x_i) \right] (x_i)_j \qquad \text{(for every } j\text{).}$$

$\}$

$\}$

LMS Algorithms for Classification

Mini-Batch Gradient Descent

Repeat until convergence $\{$

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{k} \left[y_i - h_\theta(x_i) \right] (x_i)_j \qquad \text{(for every } j\text{)}.$$

$\}$

Here we use mini-batches containing 10 to 1000 examples. This is $k \in [10, 1000]$.

Error of a Binary Classifier (1/2)

True Positive $=$ A positive example correctly identified as positive.

False Positive $=$ A negative example incorrectly identified as positive.

True Negative $=$ A negative example correctly identified as negative.

False Negative $=$ A positive example incorrectly identified as negative.

Error of a Binary Classifier (2/2)

Precision

It answers the question: How many of the examples identified as positives are indeed positives?

$$\text{Precision} = \frac{TP}{TP+FP}.$$

Recall (Sensitivity)

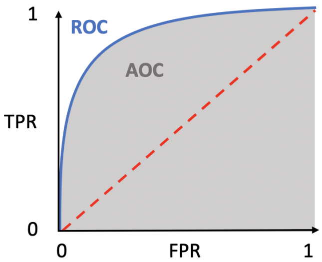
It answers the question: How many of the positive examples were correctly identified as positive?

$$\text{Recall} = \frac{TP}{P} = \frac{TP}{TP+FN}.$$

Receiver Operating Characteristic (ROC) curve

The Receiver Operating Characteristic curve is a graph showing the performance of a binary classifier with all the classification thresholds.

It shows two parameters: the true positive rate (TPR) and the false positive rate (FPR).



$$TPR = Recall = Sensitivity = \frac{TP}{TP+FN}$$

$$Specificity = \frac{TN}{FP+TN}$$

$$FPR = \frac{FP}{FP+TN} = 1 \text{ - specificity}$$

AOC = Area under the ROC curve.

The Perceptron Learning Algorithm

Consider modifying the logistic regression to output either 1 or 0:

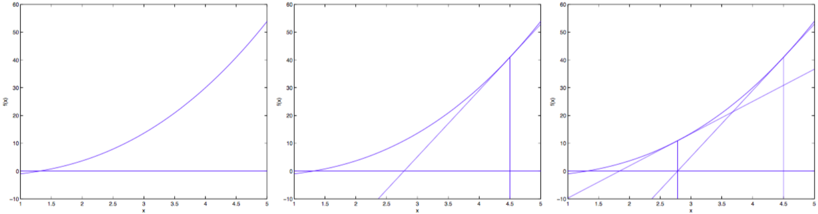
$$g(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathbf{z} \geq 0 \\ 0 & \text{if } \mathbf{z} < 0 \end{cases}$$

By making $h_\theta(x) = g(\theta^\top \mathbf{x})$, then we have the update rule:

$$\theta_j := \theta_j + \alpha(y_i - h_\theta(x_i))(x_i)_j.$$

Newton's Method for Finding a Zero of a function

Suppose we have a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and we want to find a value of θ such that $f(\theta) = 0$, with $\theta \in \mathbb{R}$.



Newton's method performs the following update rule:

$$\theta := \theta - \frac{f(\theta)}{f'(\theta)}.$$

Newton's Method for Finding a Zero of a function

Now, suppose we want to maximize a function ℓ. The maxima of ℓ correspond to points where its first derivative $\ell'(\theta)$ is zero.

So, by letting $f(\theta) = \ell'(\theta)$, we can use the same algorithm to maximize ℓ:

$$\theta := \theta - \frac{\ell'(\theta)}{\ell''(\theta)}.$$

Newton-Raphson Method

In our regression setting θ is vector-valued. The generalization of Newton's method to this multidimensional setting is given by

$$\theta := \theta - H^{-1} \nabla_\theta \ell(\theta),$$

where H is the Hessian matrix

$$H_{ij} = \frac{\partial^2 \ell(\theta)}{\partial \theta_i \partial \theta_j}.$$

Thank you!

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