Convex Optimization in Real Time Signal Processing

Adithya Hosapate
Sushant Meena

IIT Hyderabad

ee16btech11040@iith.ac.in

es16btech11021@iith.ac.in

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Disciplined Convex Programming

Consider the following optimization problem:

$$
\begin{gathered}
\min x^T Qx \\
\text{subject to } |x_i| \leq 1 \\
\sum x_i = 10 \\
Ax \geq 0
\end{gathered}
$$

This problem is not in standard QP form, however it can be easily transformed to standard form by hand.

$$
\begin{gathered}
\min x^T Qx \\
\text{subject to } x_i \leq 1 \\
-x_i \leq -1 \\
\sum x_i = 10 \\
-Ax \leq 0
\end{gathered}
$$

DCP(Continued)

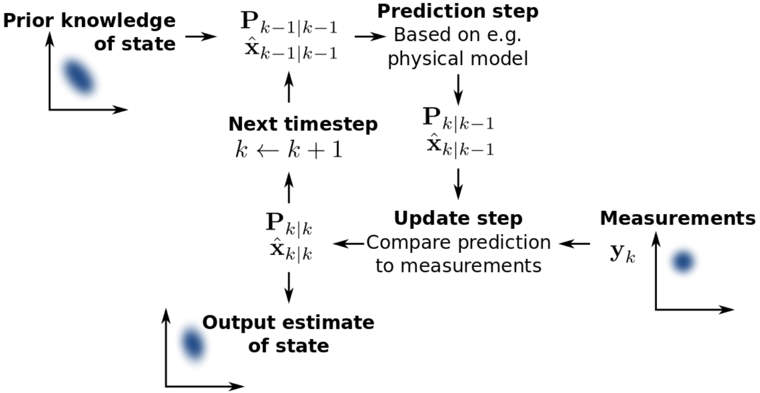
- Recently developed parser-solvers, such as YALMIP, CVX, CVXMOD and Pyomo automate this reduction process.
- The user specifies the problem in a natural form by declaring optimization variables, defining an objective, and specifying constraints.
- The parser can easily verify convexity of the problem and automatically transform it to a standard form, for transfer to the solver.

Example code

$A = [...]$; $b = [...]$; $Q = [...]$;
cvx begin
variable x(5)
minimize (quad form(x, Q))
subjectto
abs(x) $\leq 1; sum(x) == 10; A * x \geq 0$
cvx end
cvx status

Kalman Filter

Kalman filtering is a well-known and widely used method for estimating the state of a linear dynamical system driven by noise. When the process and measurement noises are IID Gaussian, the Kalman filter recursively computes the posterior distribution of the state, given the measurements.



Kalman Filter

We will work with the system: $x_t + 1 = Ax_t + w_t$ and $y_t = Cx_t + v_t + z_t$, where x_t is the state to be estimated and y_t is the measurement available to us at time step t. Process noise w_t is IID N(0, W) and the measurement noise term v_t is IID N(0, V). The term z_t is an additional noise term, which we assume is sparse and centered around zero.
Alternating time and measurement updates in standard Kalman filter are as follows:
The time update is:

$$\hat{x}_{t|t-1} = A\hat{x}_{t-1|t-1}$$

and the measurement update is:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \sum C^T (C \sum C^T + V)^{-1} (y_t - C\hat{x}_{t|t-1}).$$

Robust Kalman Filter

We consider a variation on the Kalman filter, designed to handle an additional measurement noise term that is sparse, i.e., whose components are often zero.

This term can be used to model (unknown) sensor failures, measurement outliers, or even intentional jamming.

Standard Kalman filter requires the solution of a quadratic optimization problem at each step, and has an analytical solution expressible using basic linear algebra operations.

We note that $\hat{x}_{t|t}$ is the solution to the following optimization problem.

$$\min v_t^T V^{-1} v_t + (x - \hat{x}_{t|t-1})^T \Sigma^{-1} (x - \hat{x}_{t|t-1}) + \lambda ||z_t||$$

$$\text{s.t} \quad y_t = Cx + v_t + z_t$$

Robust Kalman Filter

This can be rewritten (with appropriate substitution) as

$$\min_{z_t} (e_t - z_t)^T Q (e_t - z_t) + \lambda ||z_t||$$

Using the cholesky decomposition of $Q = L^T L$, we can rewrite the optimization problem as

$$\min_{z_t} ||L(e_t - z_t)||^2 + \lambda ||z_t||$$

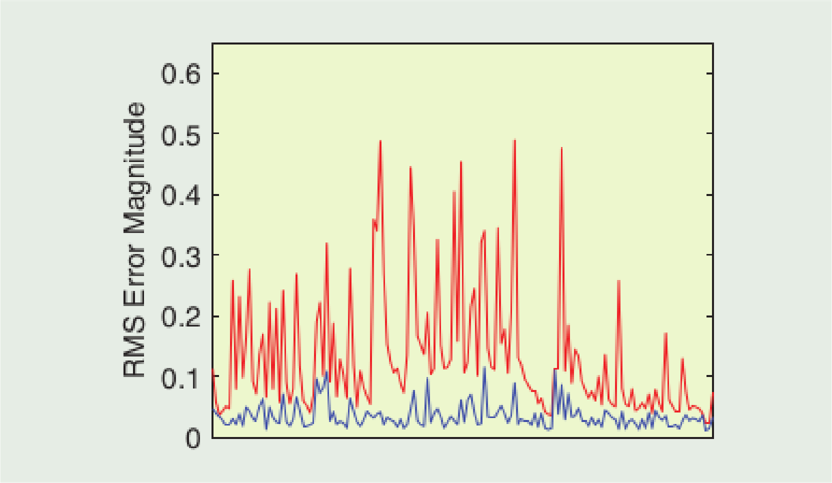
$$\min_{z_t} ||Lz_t - Le_t||^2 + \lambda ||z_t||$$

On inspection, this is similar to LASSO:

$$\min_{x} ||Ax - b||^2 + \lambda ||x||$$

This is very easy to solve using CVXPY as an unconstrained optimization problem.

In the robust Kalman filter we utilize this optimization for the measurement update, and keep the time update the same as before.



[FIG9] The robust Kalman filter (blue) exhibits significantly lower error than the standard Kalman filter (red). The state has RMS magnitude one.