EE5327 : Optimization

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Question

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Find the **Dual** of the following linear programming problem, and compare the computational efficiencies of solving the primal and dual using **Simplex** method :

Maximize: $3x_1 + 5x_2$
Subject to

$$
\begin{aligned}
6x_1 + 6x_2 &\leqslant 55 \\
8x_1 + 2x_2 &\leqslant 30 \\
5x_1 - 3x_2 &\leqslant 18 \\
x_2 &\leqslant 28 \\
x_1 \geqslant 0, x_2 &\geqslant 0
\end{aligned}
$$

Dual form of LPP

Dual Form of an LPP

The Dual for LPP is :

Primal:

Minimize: $\quad \mathrm{f}(\boldsymbol{y}) = \sum_{i=1}^{n} b_i y_i$

subject to $\quad \sum_{i=1}^{n} a_{ij} y_i \geqslant c_j$

. $\quad y_i \geqslant 0$

Dual:

Maximize: $\quad \mathrm{f}(\boldsymbol{x}) = \sum_{j=1}^{n} c_j x_j$

subject to $\quad \sum_{j=1}^{n} a_{ij} x_i \leqslant b_i$

. $\quad x_j \geqslant 0$

The duality gap in LPP is 0, i.e., the optimal values of objective of the primal and the dual are equal.

Dual Form of given LPP

Primal

Maximize: $3x_1 + 5x_2$

$$
\begin{aligned}
\text{Subject to:} \qquad 6x_1 + 6x_2 &\leqslant 55 \\
8x_1 + 2x_2 &\leqslant 30 \\
5x_1 - 3x_2 &\leqslant 18 \\
x_2 &\leqslant 28 \\
x_1 \geqslant 0, x_2 &\geqslant 0
\end{aligned}
$$

Dual:

Minimize: $55y_1 + 30y_2 + 18y_3 + 28y_4$

$$
\begin{aligned}
\text{Subject to:} \qquad 6y_1 + 8y_2 + 5y_3 &\geqslant 3 \\
6y_1 + 2y_2 - 3y_3 + y_4 &\geqslant 5 \\
y_1 \geqslant 0, y_2 \geqslant 0, y_3 \geqslant 0, y_4 &\geqslant 0
\end{aligned}
$$

Comparing Computational Efficiency

(i) See that

. No. of variables in Primal = No. of constraints in Dual

. No. of constraints in Dual = No. of variables in Primal

(ii) Computational Difficulty in Simplex Method depends on number of Row Operations performed.

No. of row operations is mainly associated with number of constraints rather than number of variables.
. $\implies$ The dual problem above has lesser number of constraints than the primal, hence is computationally efficient when using Simplex Method.

Solving Primal using Simplex

Maximize: $3x_1 + 5x_2$
Subject to

$$\begin{aligned} 6x_1 + 6x_2 &\leqslant 55 \\ 8x_1 + 2x_2 &\leqslant 30 \\ 5x_1 - 3x_2 &\leqslant 18 \\ x_2 &\leqslant 28 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

Maximize $3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

We get rid of inequality signs by introducing slack variables and artificial variables(if neccessary). So, the constraints become

$$
\begin{aligned}
6x_1 + 6x_2 + s_1 &= 55 \\
8x_2 + 2x_2 + s_2 &= 30 \\
5x_1 - 3x_2 + s_3 &= 18 \\
x_2 + s_4 &= 28 \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geqslant 0
\end{aligned}
$$

Solving Primal using Simplex

Iteration-1		C_j	3	5	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	MinRatio $\frac{X_B}{x_2}$
S_1	0	55	6	(6)	1	0	0	0	$\frac{55}{6} = 9.1667 \rightarrow$
S_2	0	30	8	2	0	1	0	0	$\frac{30}{2} = 15$
S_3	0	18	5	-3	0	0	1	0	---
S_4	0	28	0	1	0	0	0	1	$\frac{28}{1} = 28$
$z = 0$		Z_j	0	0	0	0	0	0	
		$Z_j - C_j$	-3	-5 ↑	0	0	0	0	

Pivot Element = 6

Entering = x_2, Departing = s_1

By performing below row operations one by one, we get next table.

$$R_1 = \frac{R_1}{6}$$
$$R_2 = R_2 - 2R_1$$
$$R_3 = R_3 + 3R_1$$
$$R_4 = R_4 - R_1$$

Solving Primal using Simplex

Iteration-2		C_j	3	5	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	MinRatio
x_2	5	$\frac{55}{6}$	1	1	$\frac{1}{6}$	0	0	0	
S_2	0	$\frac{35}{3}$	6	0	$-\frac{1}{3}$	1	0	0	
S_3	0	$\frac{91}{2}$	8	0	$\frac{1}{2}$	0	1	0	
S_4	0	$\frac{113}{6}$	-1	0	$-\frac{1}{6}$	0	0	1	
$z = \frac{275}{6}$		Z_j	**5**	**5**	$\frac{5}{6}$	**0**	**0**	**0**	
		$Z_j - C_j$	2	0	$\frac{5}{6}$	0	0	0	

Since all $Z_j - C_j \geqslant 0$, we have reached the optimal solution.

$x_1 = 0, x_2 = \frac{55}{6}$, Optimized cost $= \frac{275}{6}$

Total number of Row Operations used to solve Primal = 4

Solving Dual using Simplex

Solving Dual using Simplex

Minimize: $55x_1 + 30x_2 + 18x_3 + 28x_4$
Subject to

$$\begin{aligned} 6x_1 + 8x_2 + 5x_3 &\geqslant 3 \\ 6x_1 + 2x_2 - 3x_3 + x_4 &\geqslant 5 \\ x_1, x_2, x_3, x_4 &\geqslant 0 \end{aligned}$$

Maximize: $-55x_1 - 30x_2 - 18x_3 - 28x_4 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

We get rid of inequality signs by introducing slack variables and artificial variables(if neccessary). So, the constraints become

$$\begin{aligned} 6x_1 + 8x_2 + 5x_3 - s_1 + a_1 &= 3 \\ 6x_1 + 2x_2 - 3x_3 + x_4 - s_2 + a_2 &= 5 \\ x_1, x_2, x_3, x_4, s_1, s_2, a_1, a_2 &\geqslant 0 \end{aligned}$$

Solving Dual using Simplex

Iteration-1		C_j	-55	-30	-18	-28	0	0	$-M$	$-M$	
B	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	A_1	A_2	MinRatio $\frac{X_B}{x_1}$
A_1	$-M$	3	(6)	8	5	0	-1	0	1	0	$\frac{3}{6} = 0.5 \rightarrow$
A_2	$-M$	5	6	2	-3	1	0	-1	0	1	$\frac{5}{6} = 0.8333$
$z = -8M$		Z_j	$-12M$	$-10M$	$-2M$	$-M$	M	M	$-M$	$-M$	
		$Z_j - C_j$	$-12M + 55 \uparrow$	$-10M + 30$	$-2M + 18$	$-M + 28$	M	M	0	0	

Pivot Element = 6

Entering = x_1, Departing = a_1

By performing below row operations one by one, we get next table.

$$R_1 = \frac{R_1}{6}$$

$$R_2 = R_2 - 6R_1$$

Solving Dual using Simplex

Iteration-2		C_j	-55	-30	-18	-28	0	0	$-M$	
B	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	A_2	MinRatio $\frac{X_B}{S_1}$
x_1	-55	$\frac{1}{2}$	1	$\frac{4}{3}$	$\frac{5}{6}$	0	$-\frac{1}{6}$	0	0	---
A_2	$-M$	2	0	-6	-8	1	(1)	-1	1	$\frac{2}{1} = 2 \rightarrow$
$z = -2M - \frac{55}{2}$		Z_j	-55	$6M - \frac{220}{3}$	$8M - \frac{275}{6}$	$-M$	$-M + \frac{55}{6}$	M	$-M$	
		$Z_j - C_j$	0	$6M - \frac{130}{3}$	$8M - \frac{167}{6}$	$-M + 28$	$-M + \frac{55}{6} \uparrow$	M	0	

Pivot Element = 1

Entering = s_1, Departing = a_2

By performing below row operations, we get the next table.

$$R_1 = R_1 + \frac{R_2}{6}$$

Iteration-3		C_j	-55	-30	-18	-28	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	MinRatio
x_1	-55	$\frac{5}{6}$	1	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	
S_1	0	2	0	-6	-8	1	1	-1	
$z = -\frac{275}{6}$		Z_j	-55	$-\frac{55}{3}$	$\frac{55}{2}$	$-\frac{55}{6}$	0	$\frac{55}{6}$	
		$Z_j - C_j$	0	$\frac{35}{3}$	$\frac{91}{2}$	$\frac{113}{6}$	0	$\frac{55}{6}$	

Since all $Z_j - C_j \geqslant 0$, we have reached the optimal solution.

$x_1 = \frac{5}{6}, x_2 = 0, x_3 = 0, x_4 = 0$, Optimized cost $= \frac{275}{6}$

Total number of Row Operations used to solve Dual = 3

Conclusion: See that since Dual has less number of constraints, it required less number of Row Operations to solve.