Optimization

Problem 4.6

CH18MTECH11008,EEACMTECH11006

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KKT Conditions

KKT conditions are necessary conditions for a solution to be optimal. Consider the following minimizaton problem.

$$\underset{x}{\text{minimize}} \quad f(x)$$
subject to $g_i(x) \leq 0, \; i = 1, \ldots, m,$
$h_j(x) = 0, \; j = 1, \ldots, l.$

where x is the optimization variable, f is the objective function,
g_i $(i = 1, \ldots, m)$ are the inequality constraint functions, and
h_j $(j = 1, \ldots, \ell)$ are the equality constraint functions.
The number of inequality and equality constraints are denoted by m and l respectively.

If x^* is a point at which objective function and constraints are differentiable, then the necessary conditions are as follows..

Necessary conditions

1. Stationarity

$\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) + \sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) = 0$

2.Primal Feasibility

$g_i(x^*) \leq 0,\ \text{for } i = 1, \ldots, m$
$\mathrm{h}_j(x^*) = 0,\ \text{for } j = 1, \ldots, \ell$

3.Dual Feasibility

$\mu_i \geq 0,\ \text{for } i = 1, \ldots, m$

4. Complementary slackness

$\mu_i g_i(x^*) = 0,\ \text{for } \ i = 1, \ldots, m.$

Solve the following problem by converting problem 4.4 into 2 variable convex optimization problem using KKT conditions

$$\underset{x}{\text{minimize}} \quad -x_{11} - 2x_{12} - 5x_{22}$$
$$\text{subject to} \quad 2x_{11} + 3x_{12} + x_{22} = 7,$$
$$x_{11} + x_{12} \geq 1,$$
$$x11, x12, x22 \geq 0$$
$$X \succeq 0$$
$$\mathrm{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

Solution:

$$\underset{x}{\text{minimize}} \quad 9x_1 + 13x_2 - 35$$
$$\text{subject to} \quad \text{-}x_1 - x_2 + 1 \leq 0$$
$$2x_1^2 + x_2^2 + 3x_1x_2 - 7x_1 \leq 0$$

KKT conditions for this problem are:

1. Stationarity condition :

$$\nabla(9x_1+13x_2-35)+\lambda_1\nabla(-x_1-x_2+1)+\lambda_2\nabla(2x_1^2+x_2^2+3x_1x_2-7x_1)=0$$

2. Primal Feasibility :

$-x_1 - x_2 + 1 \leq 0$
$2x_1^2 + x_2^2 + 3x_1x_2 - 7x_1 \leq 0$

3. Dual Feasibility :

$\lambda_1 \geq 0$
$\lambda_2 \geq 0$

4. Complementary Slackness :

$\lambda_1(-x_1 - x_2 + 1) = 0$
$\lambda_2(2x_1^2 + x_2^2 - 7x_1 + 3x_1x_2) = 0$

From stationary condition we will get the following equations.

$9 - \lambda_1 + 4\lambda_2 x_1 + 3\lambda_2 x_2 - 7\lambda_2 = 0 \ldots (1)$

$13 - \lambda_1 + 2\lambda_2 x_2 + 3\lambda_2 x_1 = 0 \ldots (2)$

The another 2 equations from complementary slackness

$\lambda_1(-x_1 - x_2 + 1) = 0 \ldots (3)$

$\lambda_2(2x_1^2 + x_2^2 - 7x_1 + 3x_1 x_2) = 0 \ldots (4)$

case 1: $\lambda_1 = \lambda_2 = 0$
case 2 : $\lambda_1 = 0 \ and \ \lambda_2 \neq 0$
case 3 : $\lambda_1 \neq 0 \ and \ \lambda_2 = 0$
case 4 : $\lambda_1 \neq 0 \ and \ \lambda_2 \neq 0$

solving case 4 :

$$-x_1 - x_2 + 1 = 0$$

$$2x_1^2 + x_2^2 - 7x_1 + 3x_1x_2 = 0$$

$$9 - \lambda_1 + 4\lambda_2 x_1 + 3\lambda_2 x_2 - 7\lambda_2 = 0$$

$$13 - \lambda_1 + 2\lambda_2 x_2 + 3\lambda_2 x_1 = 0$$

Solving these conditions do not give correct solution. We cannot apply KKT conditions to this problem