EE5327 : Optimization

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Question

Question

Solve the following linear programming problem using **Simplex** method :

Minimize: $x_1 + x_2 + 2x_3$
Subject to

$$x_1 + 2x_2 \geqslant 4$$
$$x_2 + 7x_3 \leqslant 5$$
$$x_1 - 3x_2 + 5x_3 = 6$$
$$x_1, x_2 \geqslant 0,\ x_3 \text{ is unrestricted}$$

Question

The dual to the above problem is:

Maximize: $4y_1 + 5y_2 + 6y_3$
Subject to

$$\begin{aligned} y_1 + y_3 &\leqslant 1 \\ 2y_1 + y_2 - 3y_3 &\leqslant 1 \\ 7y_2 + 5y_3 &= 2 \end{aligned}$$

and further subject to:

(A) $y_1 \geqslant 0$, $y_2 \leqslant 0$ and y_3 is unrestricted

(B) $y_1 \geqslant 0$, $y_2 \geqslant 0$ and y_3 is unrestricted

(C) $y_1 \geqslant 0$, $y_3 \leqslant 0$ and y_2 is unrestricted

(D) $y_3 \geqslant 0$, $y_2 \leqslant 0$ and y_1 is unrestricted

Dual form of LPP

Dual Form of an LPP

The Dual for LPP is :

Primal:

Minimize: $\quad f(\boldsymbol{y}) = \sum_{i=1}^{n} b_i y_i$

subject to $\quad \sum_{i=1}^{n} a_{ij} y_i \geqslant c_j$

. $\quad y_i \geqslant 0$

Dual:

Maximize: $\quad f(\boldsymbol{x}) = \sum_{j=1}^{n} c_j x_j$

subject to $\quad \sum_{j=1}^{n} a_{ij} x_i \leqslant b_i$

. $\quad x_j \geqslant 0$

Dual Form of given LPP

Primal (in standard form)

Minimize: $x_1 + x_2 + 2x_4 - 2x_5$

subject to $x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5 \geqslant 4$

. $0x_1 - x_2 + 0x_3 - 7x_4 + 7x_5 \geqslant -5$

. $x_1 - 3x_2 + 0x_3 + 5x_4 - 5x_5 \geqslant 6$

. $-x_1 + 3x_2 + 0x_3 - 5x_4 + 5x_5 \geqslant -6$

. $x_i \geqslant 0$

Dual:

Maximize: $4y_1 - 5y_2 + 6y_3 - 6y_4$

subject to $y_1 + 0y_2 + y_3 - y_4 \leqslant 1$

. $2y_1 - y_2 - 3y_3 + 3y_4 \leqslant 1$

. $0y_1 + -7y_2 + 5y_3 - 5y_4 \leqslant 2$

. $0y_1 + 7y_2 - 5y_3 + 5y_4 \leqslant -2$

. $y_i \geqslant 0$

Converting Dual into form given in question

Current Form of Dual:

Maximize: $4y_1 - 5y_2 + 6y_3 - 6y_4$

subject to $y_1 + 0y_2 + y_3 - y_4 \leqslant 1$

. $2y_1 - y_2 - 3y_3 + 3y_4 \leqslant 1$

. $0y_1 + -7y_2 + 5y_3 - 5y_4 \leqslant 2$

. $0y_1 + 7y_2 - 5y_3 + 5y_4 \leqslant -2$

. $y_i \geqslant 0$

(i) Replace y_2 by $-y_2$, and constraint $y_2 \geqslant 0$ by $y_2 \leqslant 0$

(ii) See that $y_3 - y_4$ is common in all constraints.

. $y_3 \geqslant 0$, $y_4 \geqslant 0 \implies \mathrm{y}_3 - y_4$ unrestricted

. Put $y_3 = y_3 - y_4$, and remove constraints $y_3 \geqslant 0$,$y_4 \geqslant 0$

Converting Dual into form given in question

Final Form of Dual:

Maximize: $4y_1 + 5y_2 + 6y_3$

subject to $y_1 + y_3 \leqslant 1$

. $2y_1 + y_2 - 3y_3 \leqslant 1$

. $7y_2 + 5y_3 = 2$

. $y_1 \geqslant 0$, $y_2 \leqslant 0$

Solving Primal using Simplex

Minimize: $x_1 + x_2 + 2x_3$
Subject to

$$x_1 + 2x_2 \geqslant 4$$
$$x_2 + 7x_3 \leqslant 5$$
$$x_1 - 3x_2 + 5x_3 = 6$$
$$x_1, x_2 \geqslant 0,\ x_3 \text{ is unrestricted}$$

First, we convert the given form to a standard form.
Since,x_3 is unrestricted we rename x_3 as:
$x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$

Maximize $-x_1 - x_2 - 2x_3' + 2x_3'' + 0s_1 + 0s_2 - Ma_1 - Ma_2$

We get rid of inequality signs by introducing slack variables and artificial variables(if neccessary). So, the constraints become

$$
\begin{aligned}
x_1 + 2x_2 - s_1 + a_1 &= 4 \\
x_2 + 7x_3' - 7x_3'' + s_2 &= 5 \\
x_1 - 3x_2 + 5x_3' - 5x_3'' + a_2 &= 6 \\
x_1, x_2, x_3', x_3'', s_1, s_2, a_1, a_2 &\geqslant 0
\end{aligned}
$$

Solving Primal using Simplex

Iteration-1		C_j	-1	-1	-2	2	0	0	$-M$	$-M$	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	A_1	A_2	MinRatio $\frac{X_B}{x_3'}$
A_1	$-M$	4	1	2	0	0	-1	0	1	0	---
S_2	0	5	0	1	(7)	-7	0	1	0	0	$\frac{5}{7} = 0.7143 \rightarrow$
A_2	$-M$	6	1	-3	5	-5	0	0	0	1	$\frac{6}{5} = 1.2$
$Z = -10M$		Z_j	$-2M$	M	$-5M$	$5M$	M	0	$-M$	$-M$	
		$Z_j - C_j$	$-2M+1$	$M+1$	$-5M+2 \uparrow$	$5M-2$	M	0	0	0	

Pivot Element = 7

Entering = x_3' Departing = s_2

$r_2 = r_2/7$

$r_3 = r_3 - 5r_2$

Iteration-2		C_j	-1	-1	-2	2	0	0	$-M$	$-M$	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	A_1	A_2	MinRatio $\frac{X_B}{x_1}$
A_1	$-M$	4	1	2	0	0	-1	0	1	0	$\frac{4}{1}=4$
x_3'	-2	$\frac{5}{7}$	0	$\frac{1}{7}$	1	-1	0	$\frac{1}{7}$	0	0	---
A_2	$-M$	$\frac{17}{7}$	(1)	$-\frac{26}{7}$	0	0	0	$-\frac{5}{7}$	0	1	$\frac{\frac{17}{7}}{1}=\frac{17}{7}=2.4286 \rightarrow$
$Z=-\frac{45M}{7}-\frac{10}{7}$		Z_j	$-2M$	$\frac{12M}{7}-\frac{2}{7}$	-2	2	M	$\frac{5M}{7}-\frac{2}{7}$	$-M$	$-M$	
		$Z_j - C_j$	$-2M+1 \uparrow$	$\frac{12M}{7}+\frac{5}{7}$	0	0	M	$\frac{5M}{7}-\frac{2}{7}$	0	0	

Pivot Element = 1

Entering = x_1' Departing = a_2

$r_1 = r_1 - r_3$

Solving Primal using Simplex

Iteration-3		C_j	-1	-1	-2	2	0	0	-M	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	A_1	MinRatio $\frac{X_B}{x_2}$
A_1	-M	$\frac{11}{7}$	0	$\left(\frac{40}{7}\right)$	0	0	-1	$\frac{5}{7}$	1	$\frac{\frac{11}{7}}{\frac{40}{7}} = \frac{11}{40} = 0.275 \rightarrow$
x_3'	-2	$\frac{5}{7}$	0	$\frac{1}{7}$	1	-1	0	$\frac{1}{7}$	0	$\frac{\frac{5}{7}}{\frac{1}{7}} = 5$
x_1	-1	$\frac{17}{7}$	1	$-\frac{26}{7}$	0	0	0	$-\frac{5}{7}$	0	---
$Z = -\frac{11M}{7} - \frac{27}{7}$		Z_j	-1	$-\frac{40M}{7} + \frac{24}{7}$	-2	2	M	$-\frac{5M}{7} + \frac{3}{7}$	-M	
		$Z_j - C_j$	0	$-\frac{40M}{7} + \frac{31}{7} \uparrow$	0	0	M	$-\frac{5M}{7} + \frac{3}{7}$	0	

Pivot Element = 40/7

Entering = x_2' Departing = a_1

$r_1 = 7/40 r_1$

$r_2 = r_2 - 1/7 r_1$

$r_3 = r_3 + 26/7 r_1$

Solving Primal using Simplex

Iteration-4		C_j	-1	-1	-2	2	0	0	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	MinRatio $\frac{X_B}{S_2}$
x_2	-1	$\frac{11}{40}$	0	1	0	0	$-\frac{7}{40}$	$\left(\frac{1}{8}\right)$	$\frac{\frac{11}{40}}{\frac{1}{8}} = \frac{11}{5} = 2.2 \rightarrow$
x_3'	-2	$\frac{27}{40}$	0	0	1	-1	$\frac{1}{40}$	$\frac{1}{8}$	$\frac{\frac{27}{40}}{\frac{1}{8}} = \frac{27}{5} = 5.4$
x_1	-1	$\frac{69}{20}$	1	0	0	0	$-\frac{13}{20}$	$-\frac{1}{4}$	---
$Z = -\frac{203}{40}$		Z_j	-1	-1	-2	2	$\frac{31}{40}$	$-\frac{1}{8}$	
		$Z_j - C_j$	0	0	0	0	$\frac{31}{40}$	$-\frac{1}{8} \uparrow$	

Pivot Element = 1/8

Entering = s_2' Departing = x_2

$r_1 = 8r_1$

$r_2 = r_2 - 1/8r_1$

$r_3 = r_3 + 1/4r_1$

Iteration-5		C_j	-1	-1	-2	2	0	0	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	MinRatio
S_2	0	$\frac{11}{5}$	0	8	0	0	$-\frac{7}{5}$	1	
x_3'	-2	$\frac{2}{5}$	0	-1	1	-1	$\frac{1}{5}$	0	
x_1	-1	4	1	2	0	0	-1	0	
$Z = -\frac{24}{5}$		Z_j	-1	0	-2	2	$\frac{3}{5}$	0	
		$Z_j - C_j$	0	1	0	0	$\frac{3}{5}$	0	

All $z_j - c_j$ are positive, hence we have reached optimal solution.
$x_1 = 4, x_2 = 0, x_3' = 2/5, x_3'' = 0$
Optimized cost $= -4.8$

Solving Dual using Simplex

Maximize $4x_1 - 5x_2 + 6x_3' - 6x_3'' + 0s_1 + 0s_2 - Ma_1$

We get rid of inequality signs by introducing slack variables and artificial variables(if neccessary). So, the constraints become

$$\begin{aligned} x_1 + x_3' - x_3'' + s_1 &= 1 \\ 2x_1 - x_2 - 3x_3' + 3x_3'' + s_2 &= 1 \\ -7x_2 + 5x_3' - 5x_3'' + a_1 &= 2 \\ x_1, x_2, x_3', x_3'', s_1, s_2, a_1 &\geqslant 0 \end{aligned}$$

Solving Dual using Simplex

Iteration-1		C_j	4	-5	6	-6	0	0	$-M$	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	A_1	MinRatio $\frac{X_B}{x_3'}$
S_1	0	1	1	0	1	-1	1	0	0	$\frac{1}{1} = 1$
S_2	0	1	2	-1	-3	3	0	1	0	---
A_1	$-M$	2	0	-7	(5)	-5	0	0	1	$\frac{2}{5} = 0.4 \rightarrow$
$Z = -2M$		Z_j	0	$7M$	$-5M$	$5M$	0	0	$-M$	
		$Z_j - C_j$	-4	$7M+5$	$-5M-6 \uparrow$	$5M+6$	0	0	0	

Pivot Element = 5

Entering = x_3' Departing = a_1

$r_3 = 1/5 r_3$

$r_1 = r_1 - r_3$

$r_2 = r_2 + 3r_3$

Solving Dual using Simplex

Iteration-2		C_j	4	-5	6	-6	0	0	
B	C_B	X_B	x_1	x_2	x_3'	x_3''	S_1	S_2	MinRatio $\frac{X_B}{x_1}$
S_1	0	$\frac{3}{5}$	(1)	$\frac{7}{5}$	0	0	1	0	$\frac{\frac{3}{5}}{1} = \frac{3}{5} = 0.6 \rightarrow$
S_2	0	$\frac{11}{5}$	2	$-\frac{26}{5}$	0	0	0	1	$\frac{\frac{11}{5}}{2} = \frac{11}{10} = 1.1$
x_3'	6	$\frac{2}{5}$	0	$-\frac{7}{5}$	1	-1	0	0	---
$Z = \frac{12}{5}$		Z_j	0	$-\frac{42}{5}$	6	-6	0	0	
		$Z_j - C_j$	-4 $\uparrow$	$-\frac{17}{5}$	0	0	0	0	

Pivot Element = 1

Entering = x_1' Departing = a_2

$r_2 = r_2 - 2r_1$

Solving Dual using Simplex

Iteration-3		C_j	4	-5	6	-6	0	0	
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3'**	**x_3''**	**S_1**	**S_2**	**MinRatio**
x_1	4	$\frac{3}{5}$	1	$\frac{7}{5}$	0	0	1	0	
S_2	0	1	0	-8	0	0	-2	1	
x_3'	6	$\frac{2}{5}$	0	$-\frac{7}{5}$	1	-1	0	0	
$Z = \frac{24}{5}$		**Z_j**	**4**	**$-\frac{14}{5}$**	**6**	**-6**	**4**	**0**	
		$Z_j - C_j$	0	$\frac{11}{5}$	0	0	4	0	

All $z_j - c_j$ are positive, hence we have reached optimal solution.

$x_1 = 3/5, x_2 = 0, x_3' = 2/5, x_3'' = 0$

Optimized cost $= 4.8$