Optimization

MA17BTECH11002,MA17BTECH11007

Convexity of functions.

Problem 1.5

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Convex functions

Definition

A real-valued function defined on an n-dimensional interval is said to be convex if the line segment between any two points on the graph of the function lies above or on the graph.

A single variable function f is said to be convex if:

$$f[\lambda \mathrm{x}+(1-\lambda)y] \quad \leq \quad \lambda \mathrm{f}(\mathrm{x})+(1-\lambda)f(y)$$

$$\text{for } 0<\lambda<1$$

Examples: x^2,x^3 in R^+,e^x etc.

Concave functions

Definition

A real-valued function defined on an n-dimensional interval is said to be concave if the line segment between any two points on the graph of the function lies below or on the graph.

A single variable function f is said to be concave if:

$$f[\lambda \mathrm{x}+(1\text{-}\lambda)y] \quad \geq \quad \lambda \mathrm{f}(\mathrm{x})+(1\text{-}\lambda)f(y)$$

$$\text{for } 0<\lambda<1$$

Examples: x^3 in R^-,lnx etc.

Problem

Let f(z) $=$xy , z$\in R^2$

Sketch f(z) and deduce wheather it is convex.Theoritically explain your observation.

Python Code

Example

```
import numpy as np
import matplotlib.pylab as plt
from mpl_toolkits.mplot3d import Axes3D
def fnc(X):
    return (X[0]* X[1])
fig = plt.figure()
ax = fig.add_subplot(111, projection=Axes3D.name)
x = y = np.linspace(-50,50,100)
```

Python Code

Example

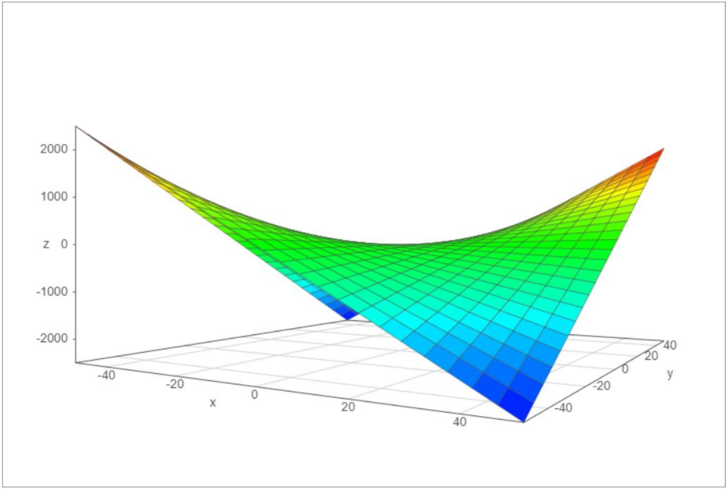
```
X, Y = np.meshgrid(x, y)
Z = fnc([X,Y])
ax.plot_surface(X, Y, Z)
ax.set_xlabel('X Label')
ax.set_ylabel('Y Label')
ax.set_zlabel('Z Label')
ax.view_init(elev=15, azim=-118)
plt.show()
```

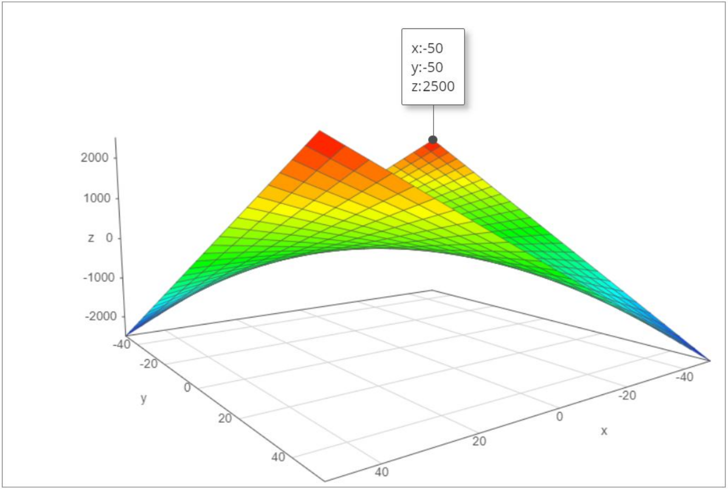
Solution

The function f(z) is neither convex nor concave as a whole because it behaves as a convex function in certain regions and concave in certain regions.

If we look at the two quadrants where $x > 0$, $y > 0$ and $x < 0$, $y < 0$ we can clearly observe by the below figure that at some points a straight line drawn between two points is always above the graph.

Similarly,if we look at the two quadrants where $x > 0$, $y < 0$ and $x < 0$, $y > 0$ we can clearly observe by the below figure that at some points a straight line drawn between two points is always below the graph.


2000
1000
z 0
-1000
-2000
-40
-20
x
0
20
40
40
20
0
y
-20
-40


x:-50
y:-50
z:2500
2000
1000
z 0
-1000
-2000
-40
-20
0
y
20
40
40
20
0
x
-20
-40

The notion of mathematical definition for single variable can be extended to n-dimentional variables where x=$(x_1,x_2,x_3,....x_n)$;y=$(y_1,y_2,y_3,....y_n)$

from the definition, consider $f[\lambda x+(1-\lambda)y] - [\lambda f(x)+(1-\lambda)f(y)]$

consider two points $(x_1, y_1)$and(x_2, y_2).

.

calculating $f[\lambda(x_1, y_1)+(1-\lambda)(x_2, y_2)] - [\lambda f(x_1, y_1) + (1-\lambda)f(x_2, y_2)]$

$\Rightarrow f[\lambda x_1+(1-\lambda)x_2,\lambda y_1+(1-\lambda)y_2]$-[$\lambda f(x_1, y_1) + (1-\lambda)f(x_2, y_2)]$

$\Rightarrow \lambda^2 x_1 y_1+(1-\lambda)^2 x_2 y_2+\lambda(1-\lambda)(x_1 y_2 + x_2 y_1)$-$[\lambda x_1 y_1+(1-\lambda)x_2 y_2]$

$\Rightarrow(\lambda^2 - \lambda)(x_1 y_1 + x_2 y_2)+(\lambda - \lambda^2)(x_1 y_2 + x_2 y_1)$

$\Rightarrow(\lambda^2 - \lambda)(x_1 y_1 + x_2 y_2 - x_1 y_2 - x_2 y_1)$

$\Rightarrow(\lambda^2 - \lambda)[x_1(y_1 - y_2) + x_2(y_2 - y_1)]$

$\Rightarrow(\lambda^2-\lambda)(x_1-x_2)(y_1-y_2)$
for those points where $(x_1-x_2)(y_1-y_2)\leq 0$ it takes a negative value, hence it shows that at those points

Hence,from the above equation we can conclude that for those points where $(x_1-x_2)(y_1-y_2)\geq 0$ it takes a negative value, hence it shows that at those points
$f[\lambda$x+(1-$\lambda)y] \leq\lambda$f(x)+(1-$\lambda)f(y)$

Which concludes that it is convex at those points.

For those points where $(x_1-x_2)(y_1-y_2)\leq 0$ it takes a positive value, hence it shows that at those points
$f[\lambda$x+(1-$\lambda)y] \geq\lambda$f(x)+(1-$\lambda)f(y)$

Which concludes that it is concave at those points.

Final Conclusion:

The function is convex where:
$x_1 > x_2, y_1 > y_2$ and $x_1 < x_2, y_1 < y_2$
The function is convex where:
$(x_1 < x_2), (y_1 > y_2)$ and $(x_1 > x_2), (y_1 < y_2)$

The End
Thank You!