Implementation of Low Density Parity Check (LDPC) codes

Theresh Babu Benguluri, Sandeep Kumar Khyalia and G V V Sharma*

Contents

1 Introduction 1

2 Encoding 1

3 Decoding 2

3.1 Useful Calculations for proceeding LDPC Decoding . . 2

3.2 Message Passing Algorithm using min-sum Approximation 4

3.3 Simulation Results 4

4 LDPC Decoding for Higher Order Mapping Schemes 4

4.1 General Expression for Calculation of Log Likelihood Ratio(LLR) 4

4.2 Approximated LLR's for QPSK Mapping Scheme . . 5

4.3 Approximated LLR's for 8-PSK Mapping Scheme . . . 5

References 5

***Abstract*—A brief description about the design and implementataion of LDPC codes using (7,4) Hamming parity check matrix. Calculation of LLR's for higher order mapping schemes are decribed by taking examples of QPSK and 8-SPK.**

1. Introduction

Let the Channel model be,

$$Y_k = X_k + V_k, \quad k = 0, \ldots, 6 \tag{1.1}$$

where X_k is the transmitted symbol in the kth time slot using the BPSK modulation and $V_k(m) \sim \mathcal{N}\left(0, \sigma^2\right)$.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

2. Encoding

LDPC codes are popular linear block codes with closest shannon limt channel capacity [1]. As an example Lets take (7,4) Hamming parity check matrix.

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

Above H matrix having the parameters, information bits $k = 4$ i.e $m = m_0, \ldots, m_3$, parity bits are $m = n - k = 3$ i.e $p = [p_0, p_1, p_2]$ and code word length $n = 7$ i.e $c = [m\ p]$. Encoding can be carried

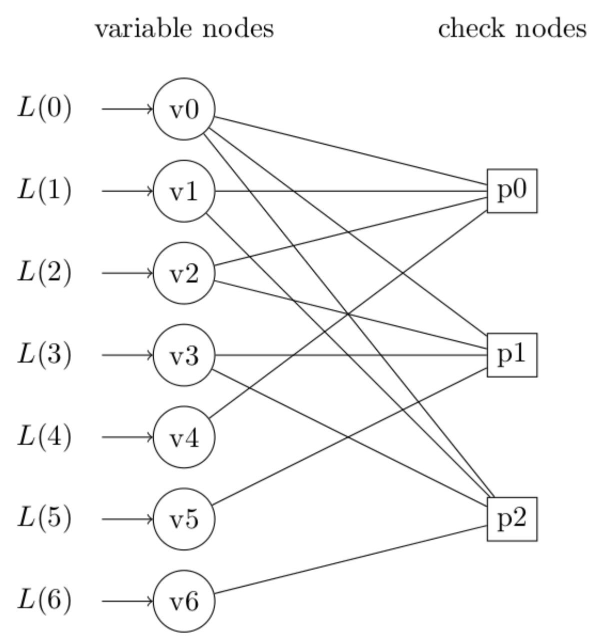

Fig. 1: Tanner Graph Representation for (7,4) Hamming parity check matrix

out by using

$$H \times c^T = 0 \tag{2.2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ p_0 \\ p_1 \\ p_2 \end{bmatrix} = 0 \tag{2.3}$$

solving we get

$$p_0 = m_0 \oplus m_1 \oplus m_2 \tag{2.4}$$
$$p_1 = m_0 \oplus m_2 \oplus m_3 \tag{2.5}$$
$$p_2 = m_0 \oplus m_1 \oplus m_3 \tag{2.6}$$

This is called Systematic Encoding.i.e Encoder will ensures information bits followed by parity bits.

3. Decoding

A. *Useful Calculations for proceeding LDPC Decoding*

1) Calculation of Input Channel Log Likelihood Ratio LLR

$$L(x_j) = \log\left(\frac{Pr(x_j = 1|y)}{Pr(x_j = -1|y)}\right) \quad X = 1 - 2c \tag{3.1}$$

$$= \log\left(\frac{f(y|x_j = 1)Pr(x_j = 1)}{f(y|x_j = -1)Pr(x_j = -1)}\right) \tag{3.2}$$

$$= \log\left(\frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y_j-1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y_j+1)^2}{2\sigma^2}}}\right) \tag{3.3}$$

$$= \log\left(e^{\frac{2y_j}{\sigma^2}}\right) \tag{3.4}$$

$$L(x_j) = \frac{2y_j}{\sigma^2} \tag{3.5}$$

2) Check Node Operation :
Lets assume that we have initilized all LLR values to variable nodes and we sent to check nodes. V_j represents all the variable nodes which are connected to j^{th} check node. Using the min-sum approximation [2], the message from j^{th} check node to i^{th} variable node given by, since parity node equation for the first check node is $p_0 = m_0 + m_1 + m_2 + m_4$. we need to calculate

$$L_{ext0,0} = \log\left(\frac{Pr(x_0 = 0|y_1, y_2, y_4)}{Pr(x_0 = 1|y_1, y_2, y_4)}\right) \tag{3.6}$$

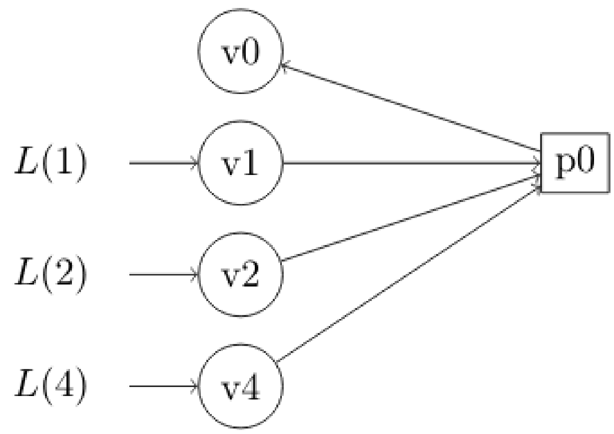

Fig. 2: Check node operation

Defining,

$$L_1 = \log\left(\frac{Pr(x_1 = 0|y_1)}{Pr(x_1 = 1|y_1)}\right) = \log\left(\frac{p_1}{1 - p_1}\right) \tag{3.7}$$
$$L_2 = \log\left(\frac{Pr(x_2 = 0|y_2)}{Pr(x_2 = 1|y_2)}\right) = \log\left(\frac{p_2}{1 - p_2}\right) \tag{3.8}$$
$$L_4 = \log\left(\frac{Pr(x_4 = 0|y_4)}{Pr(x_4 = 1|y_4)}\right) = \log\left(\frac{p_4}{1 - p_4}\right) \tag{3.9}$$

(3.10)

Using Table. I we can find the

c0	c1	c2	c4
0	0	0	0
1	0	0	1
1	0	1	0
0	0	1	1
1	1	0	0
0	1	0	1
0	1	1	0
1	1	1	1

TABLE I: Probability of a varibale node from other check nodes

$$p_0 = Pr(c_0 = 0|c_1, c_2, c_4) \tag{3.11}$$

$$\begin{aligned} p_0 &= p_1 p_2 p_4 + p_1(1 - p_2)(1 - p_4) \\ &\quad + (1 - p_1)p_2(1 - p_4) + (1 - p_1)(1 - p_2)p_4 \\ 1 - p_0 &= p_1 p_2 (1 - p_4) + p_1(1 - p_2)p_4 \\ &\quad + (1 - p_1)p_2 p_4 + (1 - p_1)(1 - p_2)(1 - p_4) \end{aligned}$$

by rearranging above equations,

$$p_0-(1-p_0) = p_1-(1-p_1)+p_2-(1-p_2)+p_4-(1-p_4) \tag{3.12}$$

Where p_i is the probability. getting message from check to variable node by taking all variable node informations.

$$\frac{1-\frac{1-p_0}{p_0}}{1+\frac{1-p_0}{p_0}} = \frac{1-\frac{1-p_1}{p_1}}{1+\frac{1-p_1}{p_1}} \times \frac{1-\frac{1-p_2}{p_2}}{1+\frac{1-p_2}{p_2}} \times \frac{1-\frac{1-p_4}{p_4}}{1+\frac{1-p_4}{p_4}} \tag{3.13}$$

$$\frac{1-e^{-L_{ext0,0}}}{1+e^{-L_{ext0,0}}} = \frac{1-e^{-L_1}}{1+e^{-L_1}} \times \frac{1-e^{-L_2}}{1+e^{-L_2}} \times \frac{1-e^{-L_4}}{1+e^{-L_4}} \tag{3.14}$$

$$-\tanh\left(\frac{L_{ext0,0}}{2}\right) = \left(-\tanh\left(\frac{L_1}{2}\right)\right)\left(-\tanh\left(\frac{L_2}{2}\right)\right)\left(-\tanh\left(\frac{L_4}{2}\right)\right) \tag{3.15}$$

$$\tanh\left(\frac{L_{ext0,0}}{2}\right) = \left[\prod_{k\in V_j\backslash i}\alpha_{k,0}\right]\left[\prod_{k\in V_j\backslash i}\tanh\left(\frac{\beta_{k,0}}{2}\right)\right] \tag{3.16}$$

$$L_{ext0,0} = \left(\prod_{k\in V_j\backslash i}\alpha_{k,0}\right)2tanh^{-1}\left(\prod_{k\in V_j\backslash i}\tanh\left(\frac{\beta_{k,0}}{2}\right)\right) \tag{3.17}$$

$$= \left(\prod_{k\in V_j\backslash i}\alpha_{k,0}\right) \tag{3.18}$$

$$2tanh^{-1}\log^{-}\log\left(\prod_{k\in V_j\backslash i}\tanh\left(\frac{\beta_{k,0}}{2}\right)\right) \tag{3.19}$$

$$= \left(\prod_{k\in V_j\backslash i}\alpha_{k,0}\right) \tag{3.20}$$

$$2tanh^{-1}\log^{-1}\left(\sum_{k\in V_j\backslash i}\log\tanh\left(\frac{\beta_{k,0}}{2}\right)\right) \tag{3.21}$$

$$L_{ext0,0} = \left(\prod_{k\in V_j\backslash i}\alpha_{k,0}\right)f\left(\sum_{k\in V_j\backslash i}f\left(\beta_{k,0}\right)\right) \tag{3.22}$$

Where,

$$\alpha_{k,j} = sign(L_{k,j}) \tag{3.23}$$

$$\beta_{k,j} = \left|L_{k,j}\right| \tag{3.24}$$

$$f(x) = -\log\left(\tanh\frac{x}{2}\right) \tag{3.25}$$

Using the Fig 3 and using its 45° symmetricity,

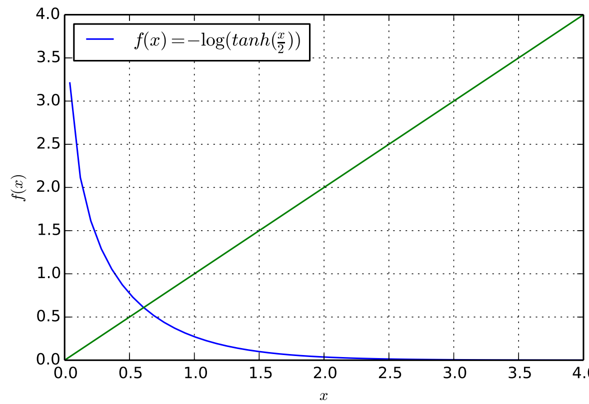

Fig. 3: Plot of function $f(x)$

we can approximate the above equation as, given by minimum sum approximation [2]

$$f\left(\sum_{k\in V_j\backslash i}f\left(\beta_{k,0}\right)\right) \approx f\left(f\left(\min_{k\in V_j\backslash i}\left(\beta_{k,0}\right)\right)\right) \tag{3.26}$$

$$= \min_{k\in V_j\backslash i}\left(\beta_{k,0}\right) \tag{3.27}$$

Combining (3.27) in (3.22),

$$L(r_{j=0,i=0}) = \left(\prod_{k\in V_j\backslash i}\alpha_{k,0}\right)\left(\min_{k\in V_j\backslash i}\left(\beta_{k,0}\right)\right) \tag{3.28}$$

3) Variable Node Operation :
Let C_i denotes all the check nodes connected to i^{th} variable node. The message from i^{th} variable node to j^{th} check node given by,

$$L(q_{i=0,j=0}) = \log\left(\frac{Pr(x_j = 1|y_0, y_1, y_2)}{Pr(x_j = -1|y_0, y_1, y_2)}\right) \quad X = 1-2c \tag{3.29}$$

$$= \log\left(\frac{f(y_0, y1, y_2|x_j = 1)Pr(x_j = 1)}{f(y_0, y_1, y_2|x_j = -1)Pr(x_j = -1)}\right) \tag{3.30}$$

$$= \log\left(\frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^3 e^{\frac{-(y_0-1)^2}{2\sigma^2}} e^{\frac{-(y_1-1)^2}{2\sigma^2}} e^{\frac{-(y_2-1)^2}{2\sigma^2}}}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^3 e^{\frac{-(y_0+1)^2}{2\sigma^2}} e^{\frac{-(y_1+1)^2}{2\sigma^2}} e^{\frac{-(y_2+1)^2}{2\sigma^2}}}\right) \tag{3.31}$$

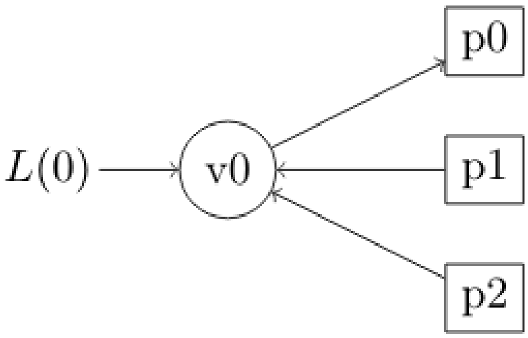

Fig. 4: Variable node operation

$$= \log\left(e^{\frac{2(y_0+y_1+y_2)}{\sigma^2}}\right) \tag{3.32}$$

$$L(q_{i=0,j=0}) = \frac{2(y_0+y_1+y_2)}{\sigma^2} = L(x_i) + \sum_{k\in C_i \setminus j} L(r_{ki}) \tag{3.33}$$

B. *Message Passing Algorithm using min-sum Approximation*

Transmitted frames = N, Total number of bits = N × 7 and Total number of information bits = N × 4. For Each Frame,

1) Initialize $L(q_{ij})$ using (3.5) for all i, j for which $h_{ij} = 1$ with channel LLR's.
2) Update $\left\{L(r_{ji})\right\}$ using (3.28)
3) Update $\left\{L(q_{ji})\right\}$ using (3.33).
4) Update $\{L(V_i)\}$ using,

$$L(V_i) = L(x_i) + \sum_{k\in C_i} L(r_{ki}) \quad i = 0, \ldots, 6. \tag{3.34}$$

5) Proceed to step 2.
   After maximum specified iterations,

Decoding can be done using,

$$\hat{c}_i = \begin{cases} 1 & L(V_i) < 0 \\ 0 & else \end{cases} \tag{3.35}$$

C. *Simulation Results*

For frames N=10000. Fig 5 Shows the Comparison of Probability error with channel coding and without channel coding. Since the parity check matrix taken was not much sparse, we are not getting near shannon limit performance. (Good sparse matrix i.e number of entries in H << $m \times n$)

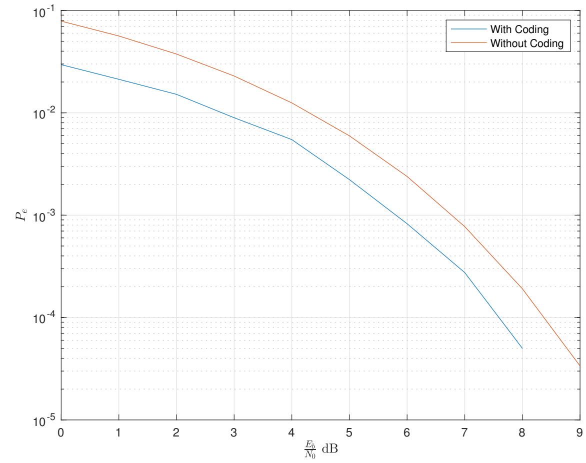

Fig. 5: SNR vs BER curves using LDPC channel coding and no channel coding

4. LDPC Decoding for Higher Order Mapping Schemes

In the case of higher order mapping schemes, calculation of Log Likelihood Ratio's are crutial. A generalized and approximated Log Likelihood Ratio given by [3]

A. *General Expression for Calculation of Log Likelihood Ratio(LLR)*

$$LLR(b_j) = \log\left(\frac{Pr(b_j = 0|y)}{Pr(b_j = 1|y)}\right) \tag{4.1}$$

$$LLR(b_j) = \log\left(\frac{\sum_{i\in\left\{K_{b_j=0}\right\}} \frac{1}{2\sigma^2} e^{\frac{-|y-x_i|^2}{2\sigma^2}}}{\sum_{i\in\left\{K_{b_j=1}\right\}} \frac{1}{2\sigma^2} e^{\frac{-|y-x_i|^2}{2\sigma^2}}}\right) \tag{4.2}$$

Where the alphabet $\left\{K_{b_j=b}\right\}$ is the set of all symbols representing the j^{th} bit equals to $b = 0$ or $b = 1$. Using the maximum logarithmic Approach,

$$\log\left(e^a + e^b\right) \approx \max(a, b) \tag{4.3}$$

The LLR expression can be written as,

$$LLR(b_j) \approx \max_{i\in\left\{K_{b_j=0}\right\}} \left(e^{\frac{-|y-x_i|^2}{2\sigma^2}}\right) - \max_{i\in\left\{K_{b_j=1}\right\}} \left(e^{\frac{-|y-x_i|^2}{2\sigma^2}}\right) \tag{4.4}$$

B. Approximated LLR's for QPSK Mapping Scheme

For the QPSK mapping scheme showed in Fig. 6 is grey code constellation and b_1, b_0 are the MSB and LSB of the mapped symbols.

$$LLR(b_1) \approx \max(L_0, L_1) - \max(L_3, L_2) \tag{4.5}$$
$$LLR(b_0) \approx \max(L_0, L_2) - \max(L_1, L_3) \tag{4.6}$$

Where,

$$L_i = e^{-\frac{|y-x_i|^2}{2\sigma^2}} \quad i = 0, 1, 2, 3 \tag{4.7}$$

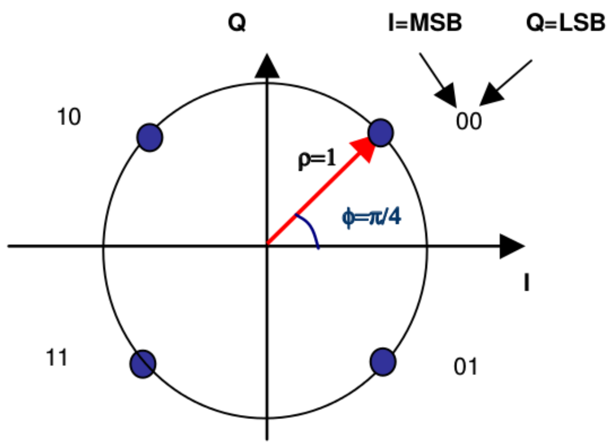

Fig. 6: Constellation Diagram of QPSK

C. Approximated LLR's for 8-PSK Mapping Scheme

For the 8-PSK mapping scheme showed in Fig. 7 is grey code constellation and b_2, b_0 are the MSB and LSB of the mapped symbols.

$$LLR(b_2) \approx \max(L_0, L_1, L_3, L_2) - \max(L_6, L_7, L_5, L_4) \tag{4.8}$$
$$LLR(b_1) \approx \max(L_0, L_1, L_5, L_4) - \max(L_3, L_2, L_6, L_7) \tag{4.9}$$
$$LLR(b_0) \approx \max(L_0, L_2, L_6, L_4) - \max(L_1, L_3, L_7, L_5) \tag{4.10}$$

Where,

$$L_i = e^{-\frac{|y-x_i|^2}{2\sigma^2}} \quad i = 0, 1, \ldots, 7 \tag{4.11}$$

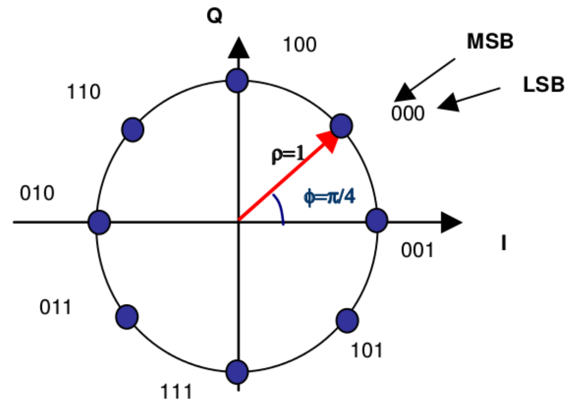

Fig. 7: Constellation Diagram of 8-PSK

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[2] N. Wiberg, "Codes and Decoding on General Graphs," *Phd Dissertation,Linkoping University,Sweden*, 1996.

[3] V. B. Olivatto, R. R. Lopes, and E. R. de Lima, "Simplified LLR calculation for DVB-S2 LDPC decoder," in *2015 IEEE International Conference on Communication, Networks and Satellite (COMNESTAT)*, Dec 2015, pp. 26–31.