Hand-In Exercise 1: Modeling and Control of Robot Arm

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1 System Modeling

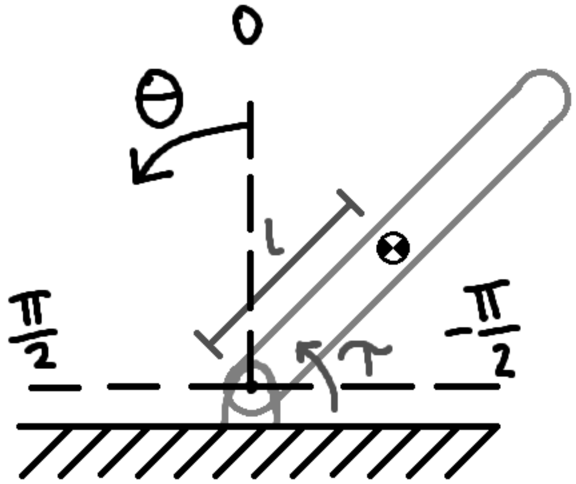

Figure 1: Sketch of the system that is processed in this report

1.1 Model in Terms of nth-Order Differential Equation

1.1.1 Prerequisites for modelling of the system

In this report, it is assumed that the robot arm in question is an infinitely thin (and rigid) rod of length L and mass m, perpendicular to the axis of rotation, rotating about one end. Thus the moment of inertia is defined as

$$I = \frac{1}{3} \cdot m \cdot l^2 \quad [kg \cdot m^2] \tag{1}$$

Newton's second law for rotation defines the sum of torques on a rotating system about a fixed axis as

$$\sum \tau = I \cdot \alpha \quad [N \cdot m] \tag{2}$$

or as we usually note it

$$\tau_{net} = I \cdot \ddot{\theta} \; [N \cdot m] \tag{3}$$

Gravity on an object with mass m in a translational system is defined as

$$\boldsymbol{F_g} = -m \cdot \boldsymbol{g} \quad [N] \tag{4}$$

Where the negative sign is applied to suit the net-torque equation. The torque produced by a force applied at a given position L on an arm is

$$\tau = \boldsymbol{F} \times L \quad [N \cdot m] \tag{5}$$

the gravitational torque can be derived as

$$\tau_g = \boldsymbol{F_g} \times L \quad [N \cdot m] \tag{6}$$

For this report, a viscous friction model is used, thus the torque of friction is defined as

$$\tau_f = -b \cdot \dot{\theta} \quad [N \cdot m] \tag{7}$$

1.1.2 Modelling of the system

Utilizing the previously defined formulas, the system is modeled using a second order differential equation.

$$I \cdot \ddot{\theta} = \tau_{net} = \tau + \tau_f + \tau_g \tag{8}$$

$\Updownarrow$

$$\tau - b \cdot \dot{\theta} + (\boldsymbol{F_g} \times \boldsymbol{L}) \tag{9}$$

To further simplify this model, the length of the arm is needed. $\theta = 0$ is defined as the arm pointing completely vertical. From this, the length of the arm is found using trigonometry

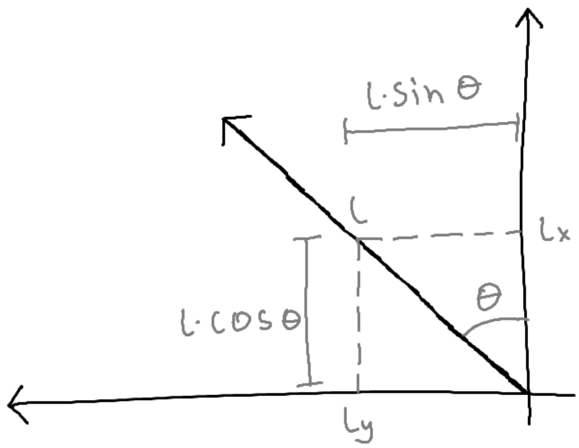

Figure 2: Sketch of how the two components of l was found using trigonometry

$$l_x = \sin\theta \cdot l \tag{10}$$

$$l_y = \cos\theta \cdot l \tag{11}$$

$$l = \begin{bmatrix} l_x \\ l_y \end{bmatrix} = \begin{bmatrix} \sin\theta \cdot l \\ \cos\theta \cdot l \end{bmatrix} \tag{12}$$

This is plugged into the model of the system

$$I \cdot \ddot{\theta} = \tau - b \cdot \dot{\theta} + \begin{bmatrix} 0 \\ m \cdot g \end{bmatrix} \times \begin{bmatrix} \sin\theta \cdot l \\ \cos\theta \cdot l \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta \cdot m \cdot g \cdot l \end{bmatrix} \tag{13}$$

$\Updownarrow$

$$\tau - b \cdot \dot{\theta} + m \cdot g \cdot l \cdot \sin\theta \tag{14}$$

Thus, the final model of the system (with variables seperated) is

$$\ddot{\theta} = \frac{1}{I} \cdot \tau - \frac{b}{I} \cdot \dot{\theta} + \frac{m \cdot g \cdot l}{I} \cdot \sin\theta \tag{15}$$

The dynamics of the system is given by

$$\ddot{\theta} = \frac{1}{I} \cdot \tau - \frac{b}{I} \cdot \dot{\theta} + \frac{m \cdot g \cdot l}{I} \cdot \sin\theta \ [{}^{rad}\!/_{s^2}] \tag{16}$$

where

$$\begin{gathered}
\theta\ [rad] \text{ is the angle} \\
\dot{\theta} = \omega\ [{}^{rad}\!/_{s}] \text{ is the angular velocity} \\
\ddot{\theta} = \dot{\omega} = \alpha\ [{}^{rad}\!/_{s^2}] \text{ is the angular acceleration} \\
I\ [kg \cdot m^2] \text{ is the moment of inertia} \\
\tau\ [N \cdot m] \text{ is the input torque} \\
b \text{ is the viscous friction constant} \\
m\ [kg] \text{ is the mass} \\
g\ [{}^{m}\!/_{s^2}] \text{ is the acceleration of gravity} \\
l\ [m] \text{ is the length to center of mass}
\end{gathered}$$

1.2 Model in Terms of System of First-Order Differential Equations

Eqn. (16) is rewritten as a system of first-order differential equations.

$$\ddot{\theta} = \dot{\omega} \tag{17}$$

$\Updownarrow$

$$\dot{\omega} = \frac{1}{I} \cdot \tau - \frac{b}{I} \cdot \dot{\theta} + \frac{l \cdot g}{I} \cdot \sin\theta \tag{18}$$

$\Updownarrow$

$$\dot{x} = f(x,u) = \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\omega} = \frac{1}{I} \cdot \tau - \frac{b}{I} \cdot \dot{\theta} + \frac{l \cdot g}{I} \cdot \sin\theta \end{bmatrix} \tag{19}$$

The dynamics of the system (16) can be rewritten as a system of first-order differential equations

$$\dot{x} = f(x,u) \ \begin{matrix} [{}^{rad}\!/_{s}] \\ [{}^{rad}\!/_{s^2}] \end{matrix} \tag{20}$$

where the state x, the input u, and the function f are

$$x = \begin{bmatrix} \theta\ [rad] \\ \dot{\theta}\ [{}^{rad}\!/_{s}] \end{bmatrix} \tag{21}$$

$$u = \begin{bmatrix} 0\ [N \cdot m] \\ \tau_{in}\ [N \cdot m] \end{bmatrix} \tag{22}$$

$$f(x,u) = \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega\ [{}^{rad}\!/_{s}] \\ \frac{1}{I} \cdot \tau - \frac{b}{I} \cdot \dot{\theta} + \frac{m \cdot l \cdot g}{I} \cdot \sin\theta\ [{}^{rad}\!/_{s^2}] \end{bmatrix} \tag{23}$$

1.3 Linearized Model of Robot

In order to control the system, the system needs to be put on a linear and time-invariant form. An approximation of the linearized state space model of the system is computed utilizing a first order Taylor approximation defined as:

$$\dot{x} \approx f(\bar{x}, \bar{u}) + \left.\frac{\partial f}{\partial x}\right|_p \cdot \hat{x} + \left.\frac{\partial f}{\partial u}\right|_p \cdot \hat{u} \tag{24}$$

$\Updownarrow$

$$f(\bar{x}, \bar{u}) + \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_p \hat{x} + \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}\right|_p \hat{u} \tag{25}$$

Where the variables are defined as:

$$\begin{aligned} f_1 &= \omega \\ f_2 &= \tfrac{1}{I} \cdot \tau - \tfrac{b}{I} \cdot \dot{\theta} + \tfrac{m \cdot l \cdot g}{I} \cdot \sin\theta \\ x_1 &= \omega \\ x_2 &= \dot{\omega} \\ u_1 &= 0 \\ u_2 &= \tau_{in} \\ \hat{x} &= x - \bar{x} \\ \hat{u} &= u - \bar{u} \end{aligned} \tag{26}$$

Linearization happens arround the working point $\bar{\theta} = \frac{\pi}{3}$. The point is analysed at a standstill, which means that the angular- velocity and acceleration is zero.

$$\begin{aligned} \bar{\theta} &= \tfrac{\pi}{3} \\ \bar{\omega} &= 0 \\ \bar{\dot{\omega}} &= 0 \end{aligned} \tag{27}$$

For the pendulum to stay at a standstill at the point $\bar{\theta}$, the size of the actively applied torque needs to be the same size as the gravitational torque τ_g in the opposite direction.

$$\bar{\tau} = -\tau_g = -\sin\bar{\theta} \cdot m \cdot g \cdot l \tag{28}$$

These points are used in the approximation

$$\dot{x} \approx \begin{bmatrix} \bar{\omega} \\ \frac{1}{I} \cdot \bar{\tau} - \frac{b}{I} \cdot \bar{\omega} + \frac{m \cdot l \cdot g}{I} \cdot \sin\bar{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \frac{m \cdot g \cdot l}{I} \cdot \cos\bar{\theta} & \frac{-b}{I} \end{bmatrix} \cdot \hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{I} \end{bmatrix} \cdot \hat{u} \tag{29}$$

$\dot{\hat{x}}$ is defined as $\dot{\hat{x}} = \dot{x} - \dot{\bar{x}}$, which allows for further simplification

$$\dot{\hat{x}} = \dot{x} - f(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & 1 \\ \frac{m \cdot g \cdot l}{I} \cdot \cos\bar{\theta} & \frac{-b}{I} \end{bmatrix} \cdot \hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{I} \end{bmatrix} \cdot \hat{u} \tag{30}$$

The output measurement is defined as the joint angle θ in the assignment, thus the state space output $\hat{y}$ is given by:

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{bmatrix} + 0 = \hat{\theta} \tag{31}$$

As such, the approximated state space model of the system is given by:

A linearized state space model of the system is given by

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{m \cdot g \cdot l}{I} \cdot \cos\bar{\theta} & \frac{-b}{I} \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{I} \end{bmatrix} \hat{u} \tag{32}$$

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{bmatrix} \hat{x} \tag{33}$$

For the specified choice of parameters, the state space model is

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -7.3650 & -0.3000 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \hat{u} \tag{34}$$

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{bmatrix} \hat{x} \tag{35}$$

The transfer function is calculated using the formula

$$G(s) = C(sI - A)^{-1}B + D \tag{36}$$

$\Downarrow$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(sI - \begin{bmatrix} 0 & 1 \\ -\frac{m \cdot g \cdot l}{I} \cdot \cos\bar{\theta} & \frac{-b}{I} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{I} \end{bmatrix} + 0 \tag{37}$$

$\Downarrow$

$$G(s) = \frac{1}{Is^2 + bs - glm\cos\hat{\theta}} \tag{38}$$

A transfer function of the linearized model is given by

$$G(s) = \frac{1}{I \cdot s^2 + b \cdot s - g \cdot l \cdot m \cdot \cos(\bar{\theta})} \tag{39}$$

For the specified choice of parameters, the transfer function is

$$G(s) = \frac{1}{\frac{1}{3}s^2 + 0.1s - 2.455} \tag{40}$$

2 Performance Specification

2.1 Performance Specification in Time-Domain

For this report the following specifications have been set for the time domain:

- M_p is overshoot. We have choosen an overshoot of less than 5o%.
- t_r is rise time. We have choosen a rise time of 6 seconds.
- t_s is setteling time. We have choosen a setteling time of 12 seconds.
- $\alpha\%$ is the value required for setteling to be achived. We have choosen an α of 1%.

In figure 3 below, the different symbolic values of the specification can be seen:

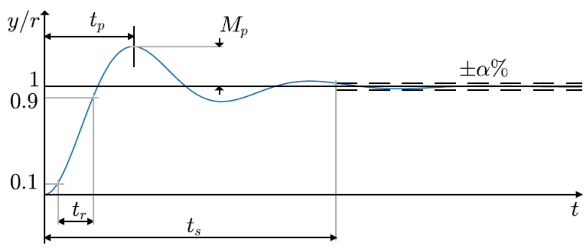

Figure 3: Time domain specification graph, taken from lecture 2 slides.

2.2 Performance Specification in Frequency-Domain

We transcribe these specifications to frequency domain, in order to check if we achive a controller that meets the requirements. First we calculate the undampend natural frequency, to obtain a rise shorter than t_r. We do this using the following formula:

$$\omega_n \geq \frac{1.8}{t_r} \Rightarrow \omega_n \geq \frac{1.8}{6} \;\Leftrightarrow\; \omega_n \geq 0.3 \tag{41}$$

Then we calculate the dampening ratio, to obtain an overshoot that is smaller than M_p. We do this using the following formula:

$$\zeta \geq \sqrt{\frac{(\frac{log(M_p)}{-\pi})^2}{1+(\frac{log(M_p)}{-\pi})^2}} \Rightarrow \zeta \geq \sqrt{\frac{(\frac{log(0.05)}{-\pi})^2}{1+(\frac{log(0.05)}{-\pi})^2}} \Leftrightarrow \zeta \geq 0.5169 \tag{42}$$

We recalculate this to an angle:

$$\arcsin(0.51693) = 0.5433 \Rightarrow \frac{0.5433 \cdot 180}{\pi} = 31.13^\circ \tag{43}$$

Lastly we calculate our σ, to obtain an α%-setteling time shorter than t_s. We do this using the following formula:

$$\sigma \geq \frac{-log(\frac{\alpha}{100})}{t_s} \Rightarrow \sigma \geq \frac{-log(\frac{1}{100})}{12} \Leftrightarrow \sigma \geq 0.3838 \tag{44}$$

In figure 4 we can see the perfomance specification illustrated in the S-plane:

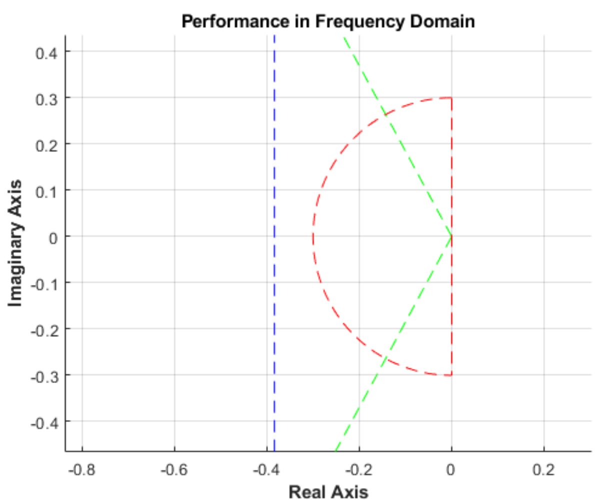

Figure 4: S-domain specification.

3 Controller Design

3.1 Design of PID Controller

When designing a control-system, the simplest solution should always be considered first.

3.1.1 Controller 1: P-Controller

The following considers the design of a P-controller.

$$K(s) = K_p \tag{45}$$

Where K_p is the proportional gain. The characteristic equation of the closed-loop system on standard form is

$$1 + K(s) \cdot G(s) = 0 \tag{46}$$

To illustrate where the closed-loop poles are placed in the s-plane when the gain K_p is changed, a root locus plot of the transfer function

$$L_1(s) = K(s)G(s) = K_p \frac{1}{\frac{1}{3}s^2 + 0.1s - 2.455} \tag{47}$$

is shown in Figure 5.

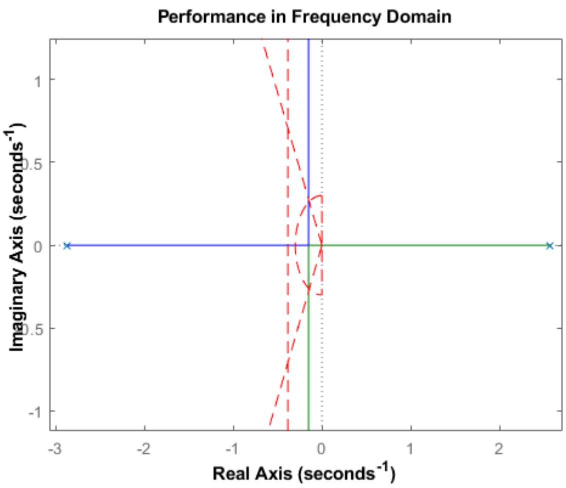

Figure 5: Root Locus for P-Controller, with frequency specifications.

The left-most possible pole placement has a real part $Re(s) = -0.15$ which will not reach the settling-time specifications where the real part should be $\sigma \geq -0.28$. Almost any other gain-value will result in a pole in the right-side plane and make the system unstable. Thus, it is not possible to reach the system requirements using only a P-Controller.

3.1.2 Controller 2: PI-Controller

A PI-controller design is analyzed as it should be able to eliminate the steady state error.

$$K(s) = K_p + K_i \cdot \frac{1}{s} \tag{48}$$

Where K_p is the proportional gain, and K_i is the integral gain.
This can be reformulated to

$$K(s) = Kp \cdot (1 + \frac{1}{s \cdot Ti}) \tag{49}$$

$$K(s) = Kp \cdot \frac{s + 1/Ti}{s} \tag{50}$$

$$K(s) = K_p \frac{s - z}{s} \tag{51}$$

Where $Ti = Kp/Ki$ and $z = -1/Ti$ is a zero of $K(s)$.
The characteristic equation of the closed-loop system on standard form is

$$1 + K(s) \cdot G(s) = 0 \tag{52}$$

Which means

$$L_2(s) = K(s)G(s) = K_p \frac{s - z}{s} \cdot \frac{1}{\frac{1}{3}s^2 + 0.1s - 2.455} \tag{53}$$

To illustrate where the closed-loop poles are placed, in the s-plane when the gain K_p is changed, root locus plots of the transfer function is created.

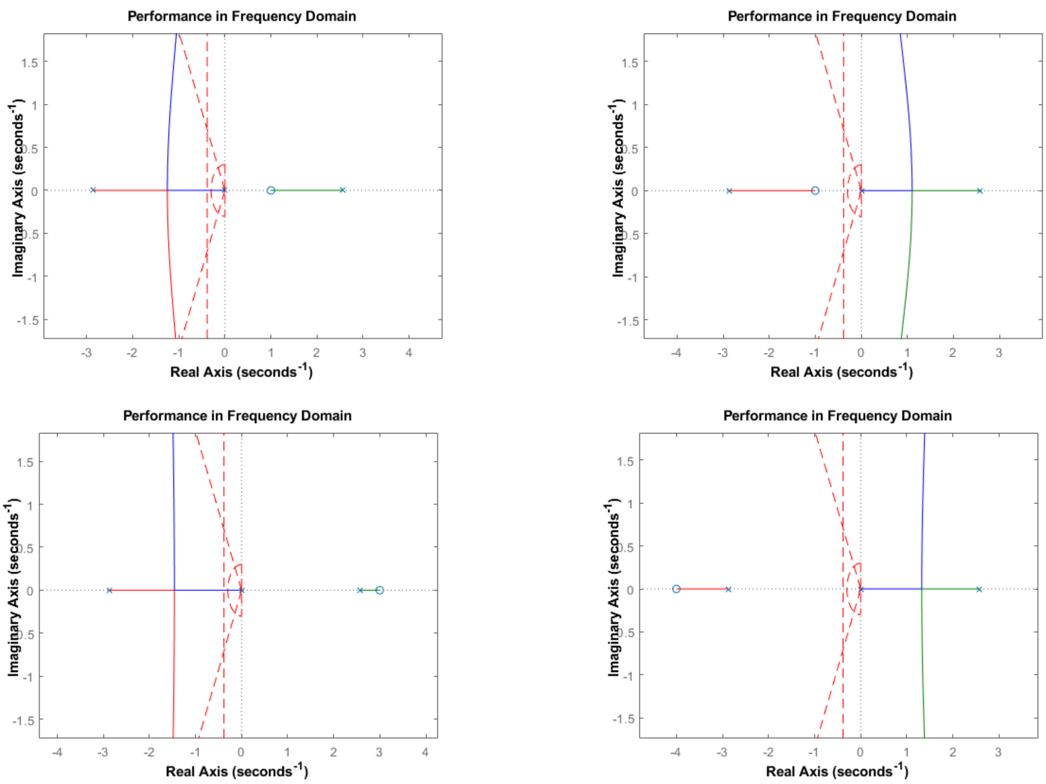

Figure 6: Root locus plots with different poles in respectively $z = 1$, $z = -1$, $z = 3$ and $z = -4$.

The system is unstable no matter where the zero is located in relation to the poles. This means a PI-controller can not be a solution to the problem.

3.1.3 Controller 3: PD-Controller

We want to add a zero in origo, to see if we get a stable system, that also meets the set requirements. The following considers the design of a PD-controller

$$K(s) = K_p + K_d \cdot s$$

where K_p is the proportional gain, and K_d is the differential gain.
Noise can be a problem for a PD-controller, so a high-pass filter is added to the controller.

$$K(s) = K_p \frac{sT_d}{1 + \frac{sT_d}{N}} = K_p \frac{sN}{N/T_d + s} \tag{54}$$

$$K(s) = K_p \frac{sN}{s - p} \tag{55}$$

Where $p = -N/T_d$ is a pole of the controller
The characteristic equation of the closed-loop system on standard form is

$$1 + K(s) \cdot G(s) = 0 \tag{56}$$

To illustrate where the closed-loop poles are placed in the s-plane when the gain K_p is changed, a root locus plot of the transfer function

$$L_3(s) = K_p \frac{sN}{s - p} \frac{1}{\frac{1}{3}s^2 + 0.1s - 2.455} \tag{57}$$

is shown in Figure 7.

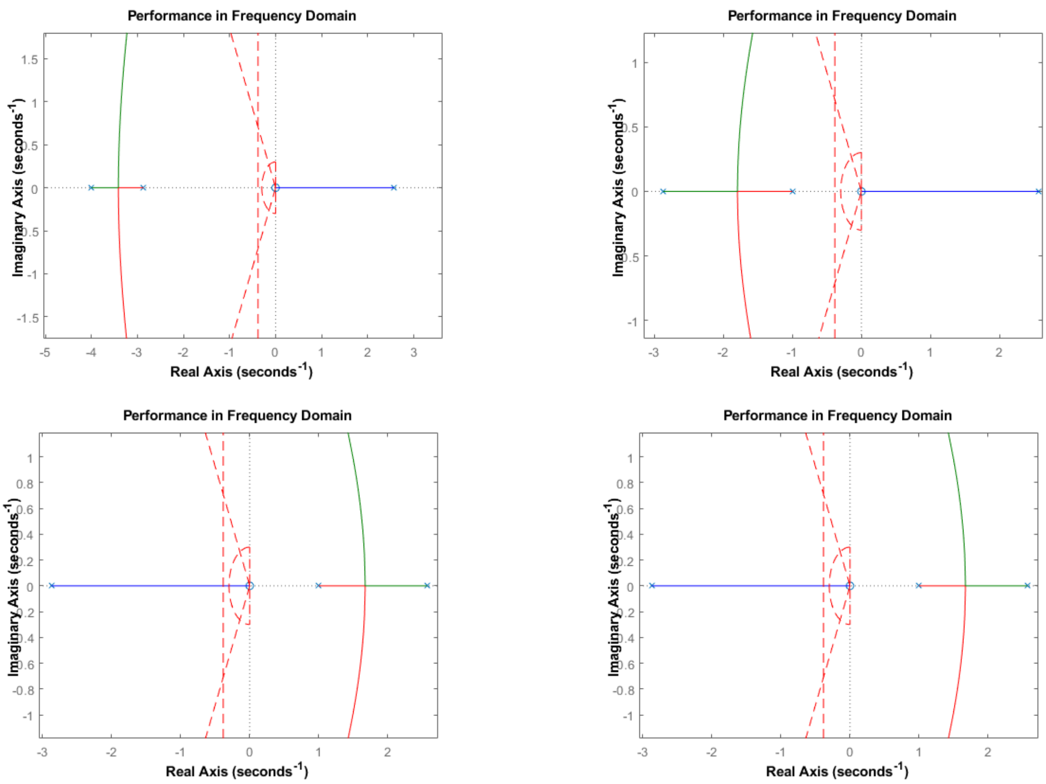

Figure 7: PD-controller Root locus plot with the filter constant $N = 1$. One pole is moved throughout the plots, the rest remain at the same places. From top left to bottom right, the pole values are respectively $P = -4$, $P = -1$, $P = 1$, $P = 3$.

It is seen that with a pole at 1 and 3 the system is unstable. At pole placements -4 and -1, the system is marginal stable at gain equal to infinity. Because a Root Locus plot only is precise when used on a second-degree system, it could be, that a very high gain is actually stable. This is tested on figure 8 with a step response of the system. It turns out the system is stable with a gain of 300 or higher, but trying to change the parameters never achieved a system close to reaching the specifications.

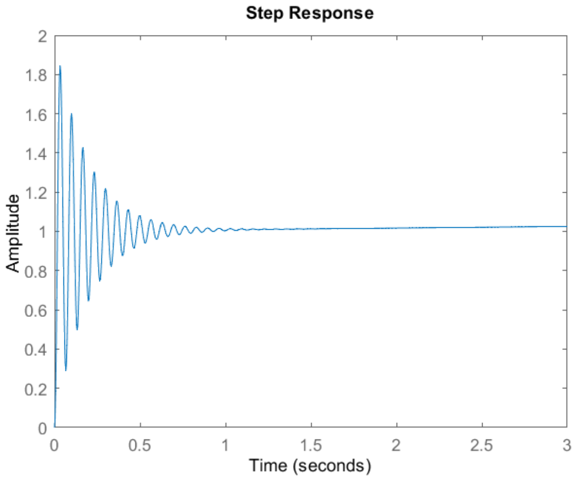

Figure 8: Step response of the system with a PD-Controller with gain $Kp = 300$

3.1.4 Controller 4: PID-Controller

Since the PD-controller couldn't match the performance specifications, a PID-controller is considered. The following considers the design of a PID-controller

$$K(s) = K_p + K_i \cdot \frac{1}{s} + K_d \cdot s \tag{58}$$

Where K_p is proportional gain, K_i is the integral gain, and K_d is the differential gain.
When including a first order low-pass filter for differential gain, this can be reformulated as such.

$$K(s) = K_p\left(1 + \frac{1}{sT_i} + \frac{sT_d}{1 + sT_d/N}\right) \tag{59}$$

The characteristic equation of the closed-loop system on standard form is

$$1 + K(s) \cdot G(s) = 0 \tag{60}$$

To illustrate where the closed-loop poles are placed in the s-plane when the gain K_p is changed, a root locus plot of the transfer function

$$L_4(s) = K_p\left(1 + \frac{1}{sT_i} + \frac{sT_d}{1 + sT_d/N}\right)\frac{1}{\frac{1}{3}s^2 + 0.1s - 2.455} \tag{61}$$

is shown in Figure 9.

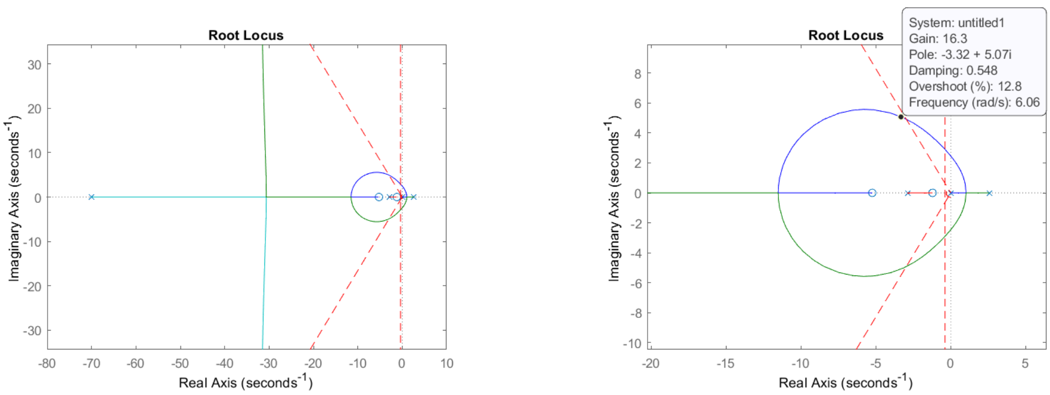

Figure 9: Root locus with $T_d = 1/7$, $T_i = 1$ and $N = 10$

It is seen that all performance specifications should be met at a gain of 16 or higher. Indeed a gain of 16 is stable, but it does not meet the requirements, so a gain of 38 is chosen.

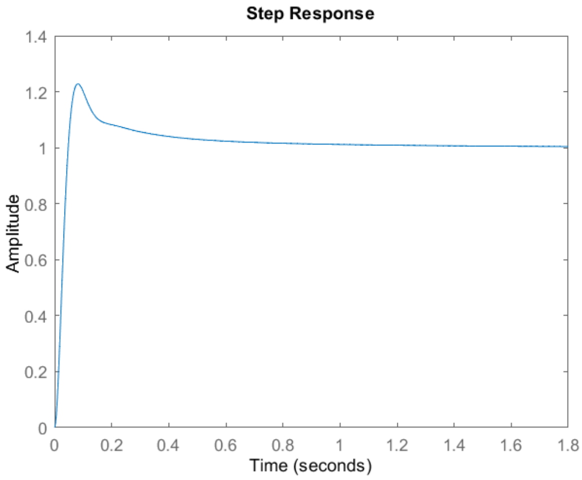

Figure 10: step response of the system with gain $K_p = 85$ has the characteristics: $M_p = 22.82$ $t_s = 0.6705$ $t_r = 0.0327$

The step response on figure 10 shows that rise time and settling time is well within the requirements, but the overshoot is drastically higher. However for this project we were unable to find better fit. Therefore this will be the final controller.

The final controller is given by

$$K(s) = 85\left(1 + \frac{1}{s\frac{1}{7}} + \frac{s\frac{1}{7}}{1 + s\frac{1/7}{10}}\right) \tag{62}$$

The closed-loop system has the following poles

$$\begin{array}{cc} P_1 = 0.0000 + 0.0000i & P_2 = -70.0000 + 0.0000i \\ P_3 = -31.5134 + 36.8720i & P_4 = -31.5134 - 36.8720i \\ P_5 = -6.0110 + 0.0000i & P_6 = 2.5680 + 0.0000i \\ P_7 = -2.8680 + 0.0000i & P_8 = -1.2622 + 0.0000i \end{array} \tag{63}$$

and the following zeros

$$\begin{array}{cc} Z_1 = 0 & Z_1 = -70.0000 \\ Z_1 = -5.2401 & Z_1 = 2.5680 \\ Z_1 = -2.8680 & Z_1 = -1.2144 \end{array} \tag{64}$$

4 Simulation

Both the linearized and non-linearized system is modeled and simulated with a step input of $\frac{\pi}{3}$.

4.1 Simulation of Linearized System Model

The linearized control system is modelled and simulated

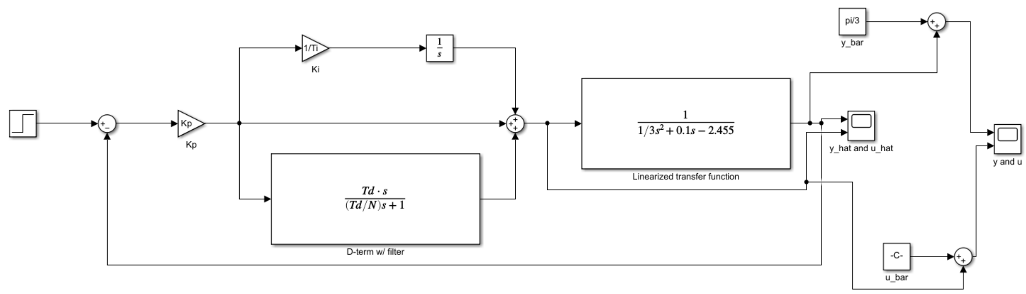

Figure 11: Linearized PID-Control System modelled in Simulink

And the step responses to a step-input of value $\frac{\pi}{3}$ is simulated

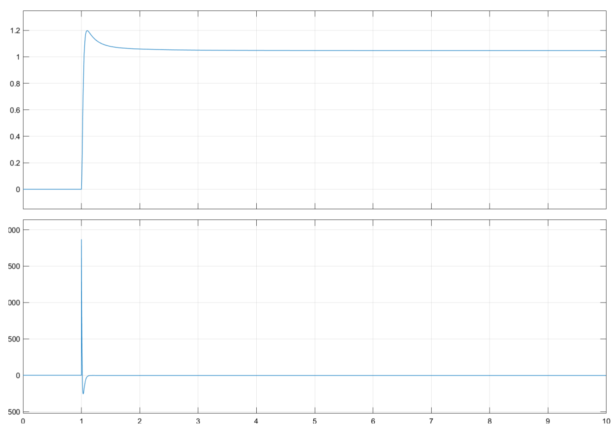

Figure 12: Step response of respectively $\hat{y}$ and $\hat{u}$ with step-input $\frac{\pi}{3}$.

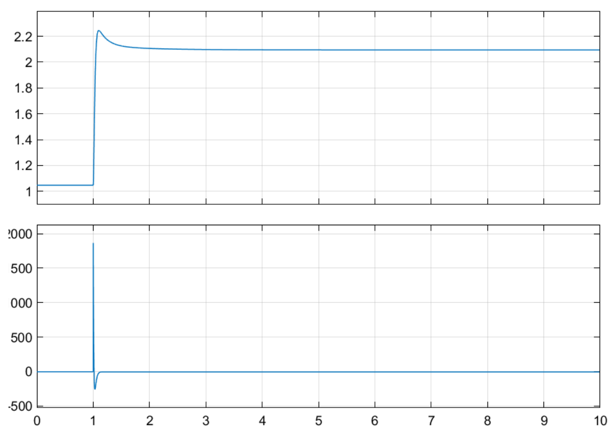

Figure 13: Step response of respectively y and u with step-input $\frac{\pi}{3}$.

4.2 Simulation of Nonlinear System Model

The non-linearized system is modelled

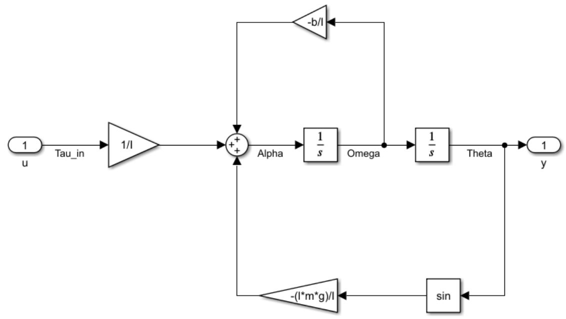

Figure 14: Non-linearized System modelled in Simulink

Then, the non-linearized control system is modelled and simulated

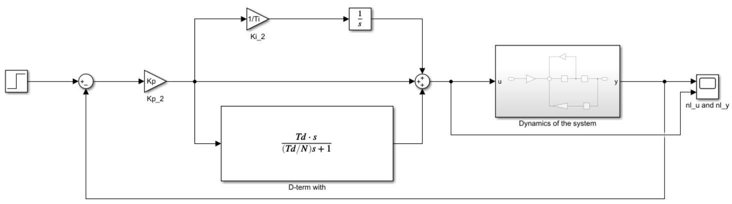

Figure 15: Linearized PID-Control System modelled in Simulink

Step response of non-linear system model

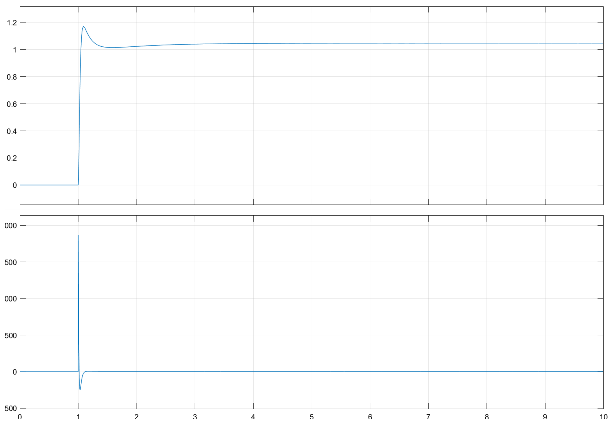

Figure 16: Step response of respectively y and u with step-input $\frac{\pi}{3}$.