Hand-In Exercise 1: Modeling and Control of Robot Arm

Kristoffer Rosenkjær (krros17)
Johan Hammer-Jakobsen (Jhamm18)
Mads Willum Christiansen (Mach018)

1 System Modeling

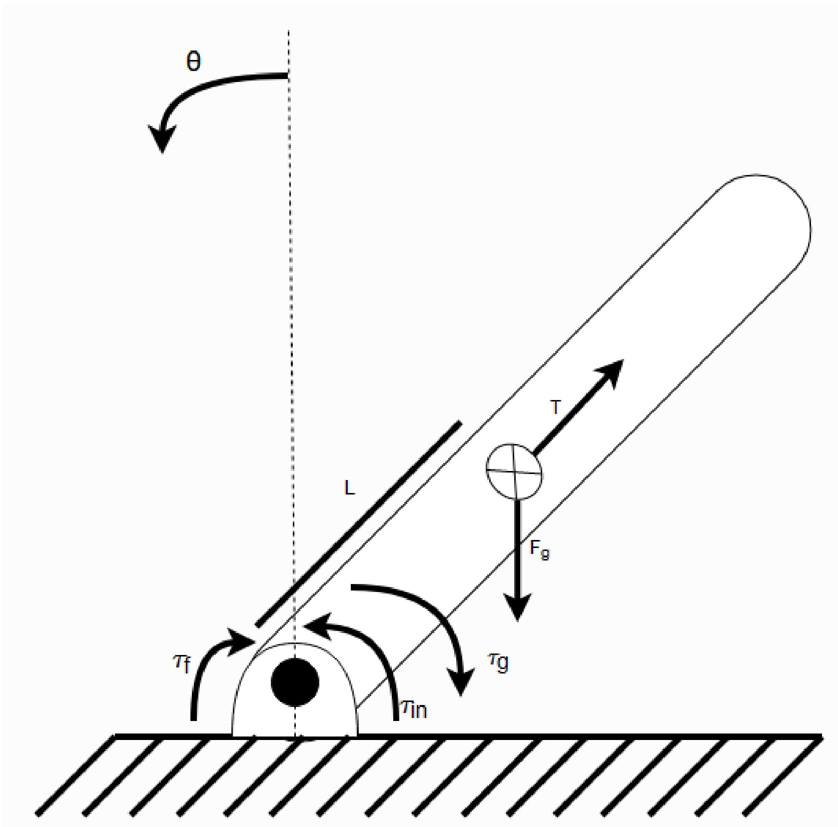

Figure 1: Free body diagram

1.1 Model in Terms of nth-Order Differential Equation

To derive a model of the physics of the system, it is determined that $\theta = 0$ is when the rod is in the upright position. The physical model is derived by using Newtons second law of motion.

$$\ddot{\theta} I = \tau \text{ [Nm]} \tag{1}$$

Where α is the angle acceleration, I the inertia, and τ the collective torque. The collective torque is the sum of all acting torques in the system contributing to the rotation of the rod.

$$\tau_{NET} = \tau_{in} - \tau_f - \tau_g \tag{2}$$

Where τ_{in} is the input torque from the motor, τ_f is the friction torque, and τ_g is the torque induced by gravity. To calculate τ_g the formula of the moment of inertia is used.

$$\tau_g = l \times F_g = \begin{bmatrix} \sin(\theta) \cdot l \\ \cos(\theta) \cdot l \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -m \cdot g \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sin(\theta) \cdot l \cdot m \cdot g \end{bmatrix} \tag{3}$$

To calculate τ_f We see that it is the angular speed $\dot{\theta}$ times the friction coefficient b. such that:

$$\tau_f = \dot{\theta} \cdot b[Nm] \tag{4}$$

So the final equation for the dynamics of the system is all of these together expressed as $\ddot{\theta}$:

$$\ddot{\theta} = \frac{\tau_{in} - \tau_f - \tau_g}{I} \tag{5}$$

The dynamics of the system is given by

$$\ddot{\theta} = \frac{1}{I} \cdot \tau_{in} + \frac{l \cdot m \cdot g}{I} \cdot \sin(\theta) - \frac{b}{I} \cdot \dot{\theta} \tag{6}$$

where

$$\begin{aligned} \ddot{\theta} &= \text{angular acceleration of the link.} \\ \dot{\theta} &= \text{angular velocity of the link.} \\ \theta &= \text{angular position of the link.} \\ I &= \text{moment of inertia of the link.} \\ b &= \text{friction coefficient.} \\ \tau_{in} &= \text{input torque of the system.} \\ \tau_f &= \text{friction torque.} \\ T &= \text{tensional force.} \\ l &= \text{distance from link the center of mass of the rod.} \\ m &= \text{mass of the link.} \\ g &= \text{gravitational acceleration on earth} \end{aligned}$$

1.2 Model in Terms of System of First-Order Differential Equations

We will be looking at the x values to be able to find to first order differential equation $\dot{x} = f(x, u)$. Equation (6) is a second order differential equation of θ where we define the to states as: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and the input: $u = \tau_{in}$. These values are now substituted into equation (6):

$$f(x, u) = \begin{bmatrix} x_2 \\ \frac{1}{I} \cdot u + \frac{l \cdot m \cdot g}{I} \cdot \sin(x_1) - \frac{b}{I} \cdot x_2 \end{bmatrix}$$

The dynamics of the system (6) can be rewritten as a system of first-order differential equations

$$\dot{x} = f(x, u) \tag{7}$$

where the state x, the input u, and the function f are

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ u &= \tau_{in} \\ f(x, u) &= \begin{bmatrix} x_2 \\ \frac{1}{I} \cdot u + \frac{l \cdot m \cdot g}{I} \cdot \sin(x_1) - \frac{b}{I} \cdot x_2 \end{bmatrix} \end{aligned} \tag{8}$$

1.3 Linearized Model of Robot

We want to linearize the system around the operating point $\bar{\theta} = \frac{\pi}{3}$ and the rotary velocity will be $\bar{\dot{\theta}} = 0$ since the linerization is worked out at a stationary point.

$$\bar{x} = \begin{bmatrix} \bar{\theta} \\ \bar{\dot{\theta}} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{3} \\ 0 \end{bmatrix} \tag{9}$$

From equation 8 we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{x_2} \\ \frac{1}{I} \cdot \bar{u} + \frac{l \cdot m \cdot g}{I} \cdot \sin(\bar{x_1}) - \frac{b}{I} \cdot \bar{x_2} \end{bmatrix} \tag{10}$$

This we will use to isolate for the unknown $\bar{u}$

$$0 = \frac{1}{I} \cdot \bar{u} + \frac{l \cdot m \cdot g}{I} \cdot \sin(\bar{x_1}) - \frac{b}{I} \cdot \bar{x_2} \tag{11}$$

$$\frac{1}{I} \cdot \bar{u} = \frac{b}{I} \cdot \bar{x_2} - \frac{l \cdot m \cdot g}{I} \cdot \sin(\bar{x_1}) \tag{12}$$

$$\bar{u} = b \cdot \bar{x_2} - l \cdot m \cdot g \cdot \sin(\bar{x_1}) \tag{13}$$

Inserting the value for $\hat{x_1}$ we get

$$\bar{u} = b \cdot 0 - l \cdot m \cdot g \cdot \sin\left(\frac{\pi}{3}\right) \tag{14}$$

$$\bar{u} = -\frac{l \cdot m \cdot g \cdot \sqrt{3}}{2} \tag{15}$$

Now that we have all the components we can approximate a linearized expression for $\dot{x} = f(x, u)$ which is from a Taylor approximation has the form

$$f(x, u) \approx \hat{\dot{x}} = f(\bar{x}, \bar{u}) + \left.\frac{\partial f}{\partial x_1}\right|_{\bar{x}} \cdot \hat{x_1} + \left.\frac{\partial f}{\partial x_2}\right|_{\bar{x}} \cdot \hat{x_2} + \left.\frac{\partial f}{\partial u}\right|_{\bar{x}} \cdot \hat{u} \tag{16}$$

$f(\bar{x}, \bar{u})$ evaluates to 0 by definition since that is what we used to calculate $\bar{u}$. Calculating the remaining partial derivatives we get:

$$f(x, u) \approx \hat{\dot{x}} = \begin{bmatrix} 0 \\ \frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1}) \end{bmatrix} \cdot \hat{x_1} + \begin{bmatrix} 1 \\ -\frac{b}{I} \end{bmatrix} \cdot \hat{x_2} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \cdot \hat{u} \tag{17}$$

To get to state space, the matrices at $\bar{x_1}$ and $\bar{x_2}$ are combined to be the system matrix A

$$A = \begin{bmatrix} 0 & 1 \\ \frac{l \cdot m \cdot g}{I} \cdot \cos\left(\bar{x_1}\right) & -\frac{b}{I} \end{bmatrix} \tag{18}$$

where the state matrix is

$$\bar{x} = \begin{bmatrix} \bar{x_1} \\ \bar{x_2} \end{bmatrix} \tag{19}$$

And B matrix

$$B = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \tag{20}$$

Since we are looking for angular velocity, the C matrix will be $\bar{x_1}$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x_1} \\ \bar{x_2} \end{bmatrix} \tag{21}$$

A linearized state space model can then be written as:

$$\begin{aligned}
\dot{\hat{x}} &= A \cdot \hat{x} + B \cdot \hat{u} \\
\dot{\hat{x}} &= \begin{bmatrix} 0 & 1 \\ \frac{l \cdot m \cdot g}{I} \cdot \cos\left(\frac{\pi}{3}\right) & -\frac{b}{I} \end{bmatrix} \cdot \hat{x} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \cdot \hat{u} \\
\hat{y} &= C \cdot \hat{x} + D \cdot \hat{u} \\
\hat{y} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \hat{x}
\end{aligned}$$

The parameters in the state space model are given in the assignment($m = 1kg$, $l = 0.5m$, and $b = 0.1\frac{Nm}{\frac{rad}{s}}$). Along with the gravitational acceleration $g = 9.82\frac{m}{s^2}$ and calculating the moment of inertia $I = \frac{1}{3} \cdot m \cdot l^2 = \frac{1}{3} \cdot 1 \cdot \frac{1}{2}^2 = \frac{1}{12}$ we get the values in the state space model:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 29.46 & -1.2 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 12 \end{bmatrix} \hat{u} \tag{22}$$

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x} \tag{23}$$

A linearized state space model of the system is given by

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ \frac{l \cdot m \cdot g}{I} \cdot \cos\left(\bar{x_1}\right) & -\frac{b}{I} \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \hat{u} \tag{24}$$

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x} \tag{25}$$

For the specified choice of parameters, the state space model is

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 29.46 & -1.2 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 12 \end{bmatrix} \hat{u} \tag{26}$$

$$\hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x} \tag{27}$$

The next step is to derive a transfer function of the model $G(s)$. This is done by Laplace transforming the state space model from equation 26 and 27

$$
\begin{aligned}
G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} &= C \cdot (s \cdot I_d - A)^{-1} \cdot B && (28)\\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1}) & -\frac{b}{I} \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} && (29)\\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s & -1 \\ -\frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1}) & s + \frac{b}{I} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} && (30)\\
&= \frac{1}{s^2 + \frac{b}{I} \cdot s - \frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1})} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s + \frac{b}{I} & 1 \\ \frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1}) & s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} && (31)\\
&= \frac{1}{s^2 + \frac{b}{I} \cdot s - \frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1})} \cdot \begin{bmatrix} s + \frac{b}{I} & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} && (32)\\
&= \frac{1}{s^2 + \frac{b}{I} \cdot s - \frac{l \cdot m \cdot g}{I} \cdot \cos(\bar{x_1})} \cdot \frac{1}{I} && (33)\\
G(s) &= \frac{1}{I \cdot s^2 + s \cdot b - l \cdot m \cdot g \cdot \cos(\bar{x_1})} && (34)
\end{aligned}
$$

A transfer function of the linearized model is given by

$$G(s) = \frac{1}{I \cdot s^2 + b \cdot s - l \cdot m \cdot g \cdot \cos(\bar{x_1})} \tag{35}$$

For the specified choice of parameters, the transfer function is

$$G(s) = \frac{1}{0.08 \cdot s^2 + 0.1 \cdot s - 2.46} \tag{36}$$

1.3.1 Poles

The poles of the system are calculated from the state space model. The poles of the system are equal to the eigenvalues of matrix A. Eigenvalues are calculated:

$$
\begin{aligned}
\det(\lambda I - A) &= 0\\
\begin{vmatrix} \lambda & -1 \\ -29.46 & \lambda - (-1.2) \end{vmatrix} &= 0\\
\lambda^2 + 1.2\lambda - 29.46 &= 0\\
\lambda_1 &= -6.06\\
\lambda_2 &= 4.86
\end{aligned}
$$

Therefore the system has two poles in $\{-6.06, 4.86\}$.

1.3.2 Zeros

The system has no zeros, because the numerator of the transfer function in equation 36 does not consist of a polynomial dependent on s to any other power than s^0.

2 Performance Specification

2.1 Performance Specification in Time-Domain

In Figure 2 we see the different symbolical values of the specification.

- M_p is overshoot. We have chosen a max overshoot of 0.2
- t_p is peak time. We have chosen a peak time of 1.2

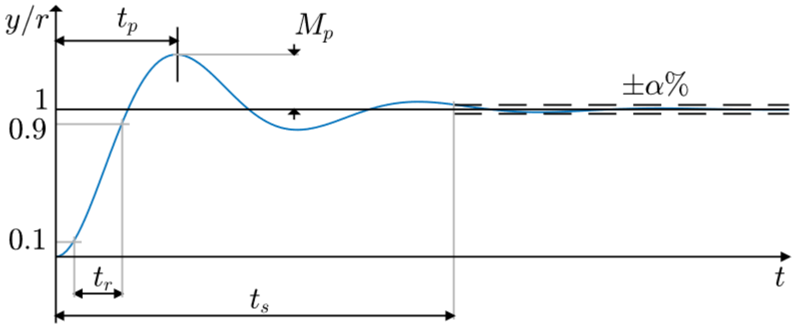

Figure 2: Time-Domain Specification Diagram, taken from slides in lecture 2

- t_r is rise time. we have chosen a rise time of 2 seconds.
- t_s is settling time. we have chosen a settling time of 5 seconds
- $\alpha\%$ is the value required for settling to be achieved. we have chosen a $\alpha\%$ of 1.

for the transfer function found, we get the following step response:

2.2 Performance Specification in Frequency-Domain

In order for these specifications to work, we will use the following formulas to design our controller:

$$\omega_n \geq \frac{1.8}{t_r} = \frac{1.8}{2} = 0.9 \tag{37}$$

$$\tag{38}$$

To achieve an overshoot smaller than M_p, ζ must be:

$$\zeta \geq \sqrt{\frac{(\frac{log(M_p)}{-\pi})^2}{1 + (\frac{log(M_p)}{-\pi})^2}} = \sqrt{\frac{(\frac{log(0.2)}{-\pi})^2}{1 + (\frac{log(0.2)}{-\pi})^2}} = 0.4559 \tag{39}$$

$$\tag{40}$$

To achieve a settling time shorter than t_s, σ must be:

$$\sigma \geq \frac{4.6}{t_s} = \frac{4.6}{5} = 0.92 \tag{41}$$

These three values are used to show where the closed loop poles should be placed.

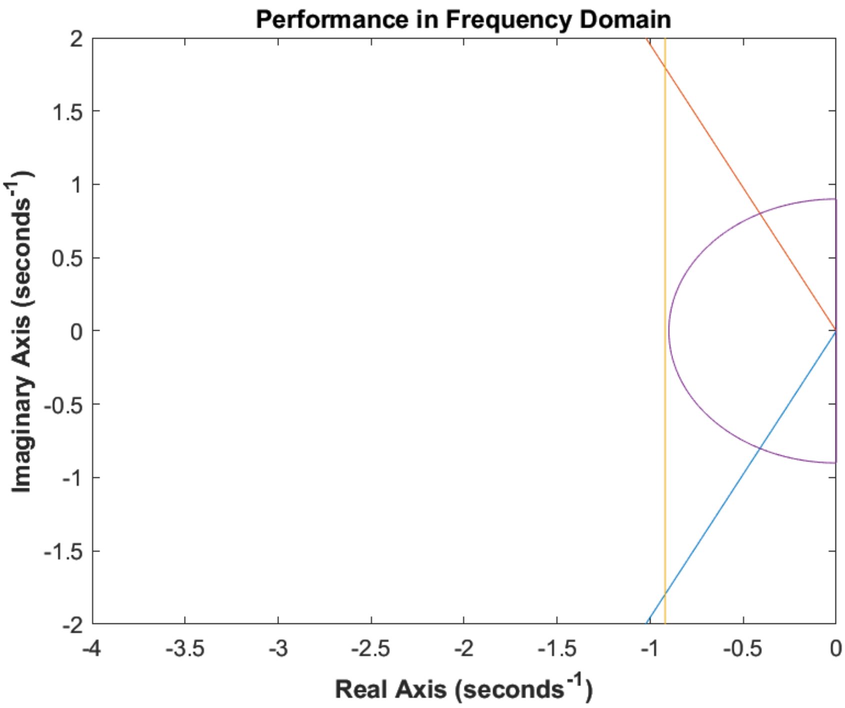

Figure 3: Performance specification in Frequency domain.

3 Controller Design

3.1 Design of PID Controller

3.1.1 Controller 1: P-Controller

The following considers the design of a P-controller

$$K(s) = K_p \tag{42}$$

where K_p is the proportional gain. The characteristic equation of the closed-loop system on standard form is

$$1 + K(s)G(s) = 0 \tag{43}$$

To illustrate where the closed-loop poles are placed in the s-plane when the gain K_p is changed, a root locus plot of the transfer function

$$L_1(s) = K_p \cdot G(s) \tag{44}$$

And insert our G(s)

$$L_1(s) = K_p \cdot \frac{1}{0.08 \cdot s^2 + 0.1 \cdot s - 2.46} \tag{45}$$

Is shown in Figure 4.

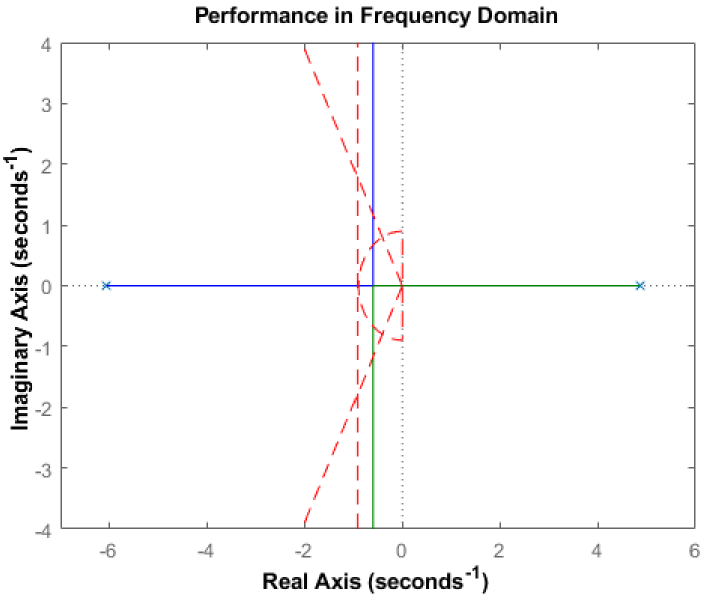

Figure 4: Root Locus plot of open loop system with P-controller.

From the root locus plot we can see that it is possible to tune the P-controller to make the system stable, by raising the gain and thereby moving both poles into the left half plane. It is not possible though to tune the system such that it is possible to meet the design requirements. While it is possible to tune the controller to meet the requirements for rise time and overshoot, it is not possible to do so at the same time, and it is neither possible to meet the requirement for settling time.

3.1.2 Controller 2: PD-Controller

We would like to place a zero in the system, so that we are capable of staying inside our performance specification. The following considers the design of a PD-controller

$$K(s) = K_p + K_d \cdot s$$

where K_p is the proportional gain and K_d is the differential gain. The characteristic equation of the closed-loop system on standard form is

$$1 + K(s)G(s) = 0 \tag{46}$$

To illustrate where the closed-loop poles are placed in the s-plane when the gain K_p is changed, a root locus plot of the transfer function

$$L_2(s) = (K_p + K_d \cdot s) \cdot \frac{1}{0.08 \cdot s^2 + 0.1 \cdot s - 2.46} \tag{47}$$

By introducing the derivative part of the controller, we add a zero to the system because we multiply an s into the numerator of the transfer function. The position of the zero is determined with T_d when we rewrite the controller and out factor K_p.

$$K(s) = K_p\,(1 + T_d s) \tag{48}$$

To figure out where the zero is placed, we put $K(s) = 0$. Because K_p is a variable in tuning, we disreguard it.

$$0 = 1 + T_d s \tag{49}$$

$$\Rightarrow s = -\frac{1}{T_d} \tag{50}$$

substituting s in equation 50 we get.

$$0 = 1 - \frac{T_d}{T_d} \tag{51}$$

From this we can see that we are free to choose the value of T_d that we want, and thereby the placement of the zero that we want, the equation will always be true.
A root locus of the PD control system is shown in Figure 5.

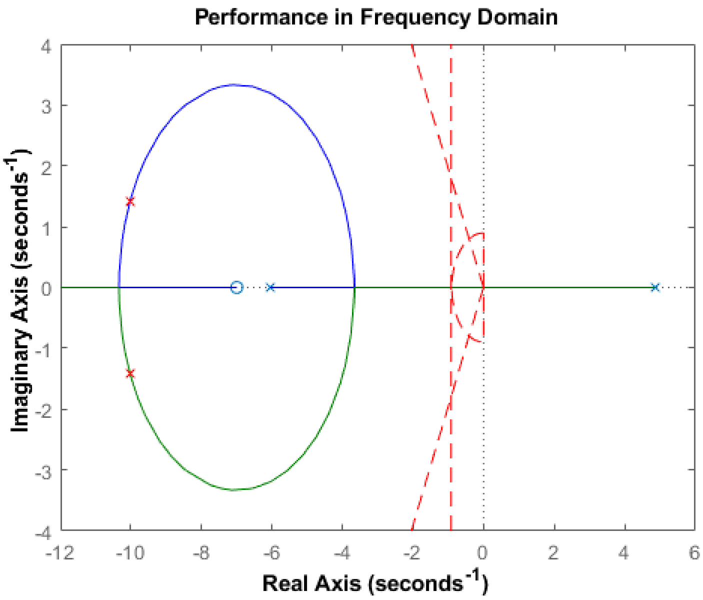

Figure 5: Root Locus plot of open loop system with PD-controller with $T_d = 1/7$. Poles placed with $K_p = 11$ marked with $\times$.

Since the two poles are placed on the left hand side of the red dotted lines, indicating our performance specifications, the system will comply with the performance specifications. .

3.1.3 Controller 3: PID-Controller

Since the step response of the PD controller showed a steady state error, we introduced the K_i gain. The following considers the design of a PID-controller.

$$K(s) = K_p + K_i \cdot \frac{1}{s} + K_d \cdot s$$

where K_p is the proportional gain, K_i is the integral gain, and K_d is the differential gain. The characteristic equation of the closed-loop system on standard form is

$$1 + K(s)G(s) = 0 \tag{52}$$

To illustrate where the closed-loop poles are placed in the s-plane when the gain K_p is changed, a root locus plot of the transfer function

$$L_3(s) = (K_p + K_i \cdot \frac{1}{s} + K_d \cdot s) \cdot \frac{1}{0.08 \cdot s^2 + 0.1 \cdot s - 2.46} \tag{53}$$

Similarly to analyzing the zero placement with the addition of the derivative part for the PD-controller we analyze what happens when we add the integral part for the PID-controller

$$K(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right) = K_p \frac{T_d s^2 + s + 1/T_i}{s} \tag{54}$$

We see that with the inclusion of the integral part we add a pole at 0 and now we have not one but two zeroes we can place, which are defined as the roots of the second order polynomial in the numerator of $K(s)$.

A root locus of the PID control system is shown in Figure 6.

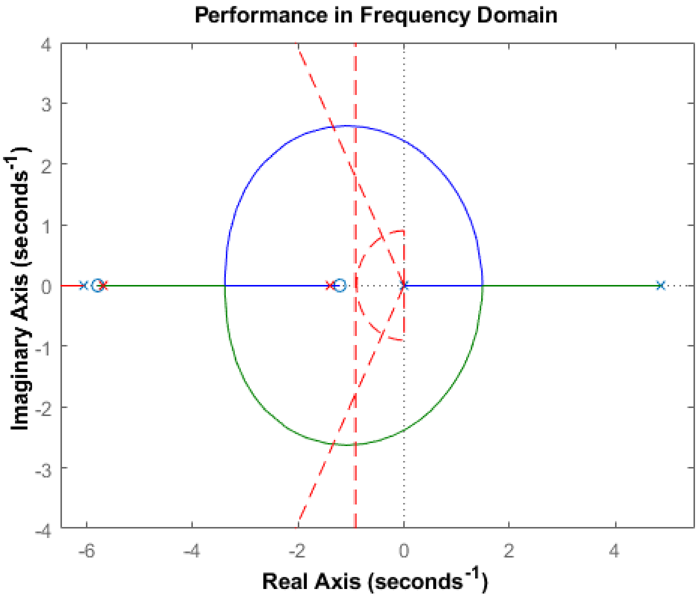

Figure 6: Root Locus plot of open loop system with PID-controller with $T_d = \frac{1}{7}, T_i = 1$. Poles placed with $K_p = 30$ marked with $\times$. There is an additional pole placed at $-45.54 + 0j$.

The controller is stable, since the three poles are placed on the left hand side of the red dotted lines, indicating our performance specification. Therefore the system will comply with the performance specifications. The 3rd pole is too far out in the negative plan to show.

The final controller is given by

$$K(s) = 30 + 4.29 \cdot s + 30 \cdot \frac{1}{s} \tag{55}$$

The closed-loop system has the following poles

$$\begin{aligned} p_1 &= 0 \\ p_2 &= -45.5446 \\ p_3 &= 4.8608 \\ p_4 &= -6.0608 \\ p_5 &= -5.6963 \\ p_6 &= -1.3876 \end{aligned}$$

and the following zeros

$$\begin{aligned} z_1 &= 0 \\ z_2 &= 4.8608 \\ z_3 &= -6.0608 \\ z_4 &= -5.7913 \\ z_5 &= -1.2087 \end{aligned}$$

4 Simulation

4.1 Simulation of Linearized System Model

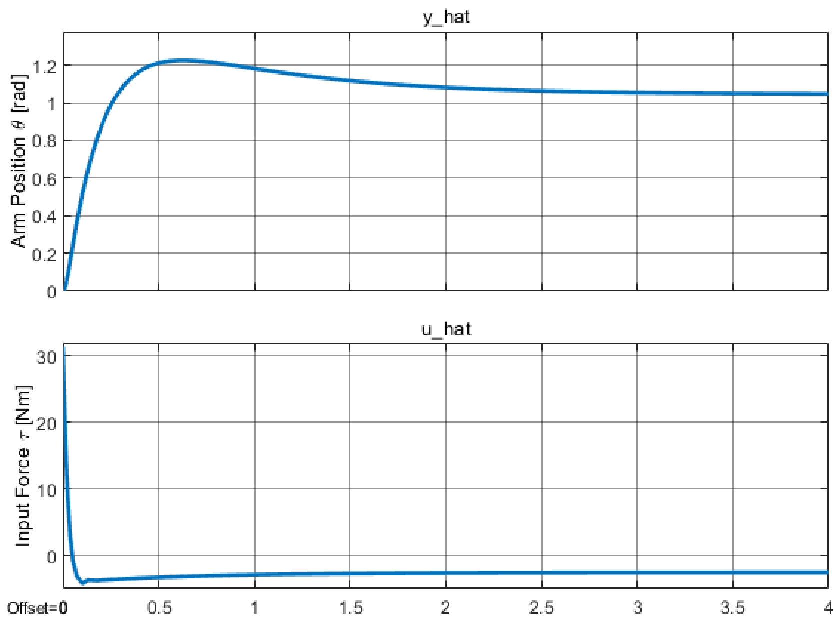

Figure 7: $\hat{u}$ and $\hat{y}$ outputs from the linear system model in simulink with a step input of $\pi/3$.

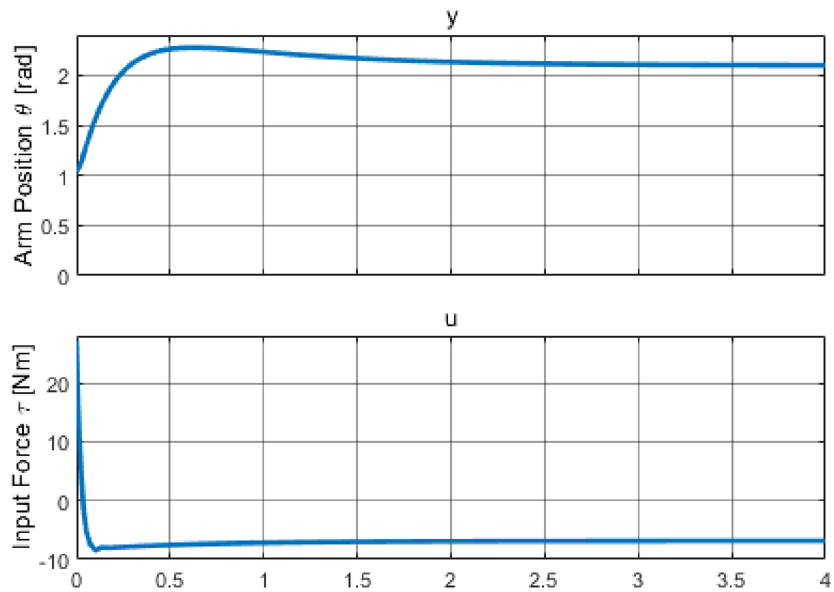

Figure 8: u and y outputs from the linear system model in simulink with a step input of $\pi/3$.

4.2 Simulation of Nonlinear System Model

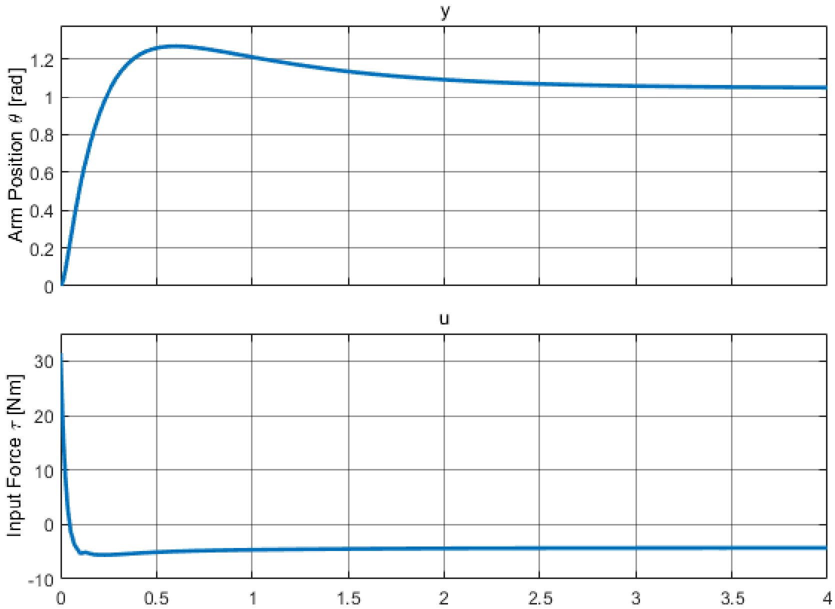

Figure 9: $\hat{u}$ and $\hat{y}$ outputs from the nonlinear system model in simulink with a step input of $\pi/3$.

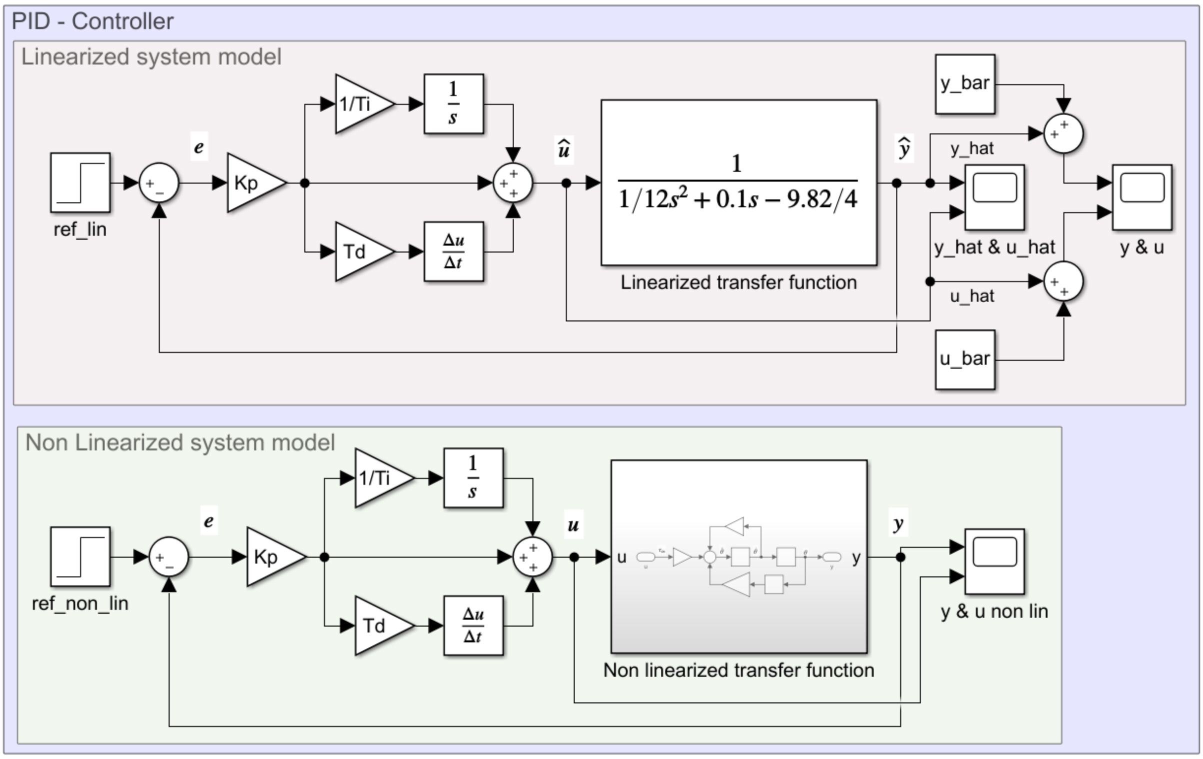

Figure 10: Block diagram of the system and PID-controller as implemented in simulink. The references are defined to step to the value $\pi/3$

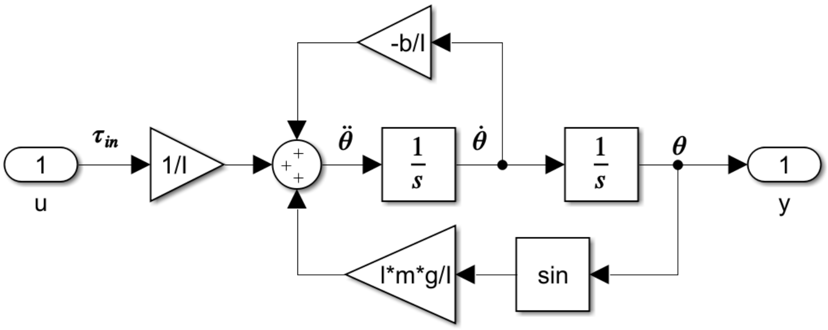

Figure 11: Block diagram of the subsystem in the nonlinearized model as implemented in simulink.