Exercise sheet 2 on Discrete Mathematics

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Let A be a finite alphabet, such as the English alphabet, and consider the following simple programming language **WHILE** whose well-formed programs are those obtained by applying the rules of the following grammar:

Numbers:

$$\mathit{Num} ::= d \mid d\mathit{Num} \qquad \text{where } d \in \{0, 1, \ldots, 9\}$$

Identifiers:

$$\begin{aligned} \mathit{Id} &::= a\mathit{Id}' && \text{where } a \in A \\ \mathit{Id}' &::= \lambda \mid a\mathit{Id}' \mid \mathit{Num}\mathit{Id}' \end{aligned}$$

Numeric expressions:

$$\begin{aligned} \mathit{Exp} ::= {} & \mathit{Num} \mid \mathit{Id} \\ & \mid \mathit{Exp} + \mathit{Exp} \mid \mathit{Exp} - \mathit{Exp} \mid \mathit{Exp} * \mathit{Exp} \mid \mathit{Exp}/\mathit{Exp} \end{aligned}$$

Boolean expressions:

$$\begin{aligned} \mathit{BExp} ::= {} & \mathbf{true} \mid \mathbf{false} \mid \mathit{Exp} = \mathit{Exp} \\ & \mid \neg \mathit{BExp} \mid \mathit{BExp} \wedge \mathit{BExp} \mid \mathit{BExp} \vee \mathit{BExp} \end{aligned}$$

Programs:

$$\begin{aligned} P ::= {} & \mathbf{skip} \mid \mathit{Id} := \mathit{Exp} \mid P; P \\ & \mid \mathbf{if}\ \mathit{BExp}\ \mathbf{then}\ P\ \mathbf{else}\ P\ \mathbf{fi} \\ & \mid \mathbf{while}\ \mathit{BExp}\ \mathbf{do}\ P\ \mathbf{done} \end{aligned}$$

Exercise 2.1. *Show, by providing the appropriate functions, that the set of identifiers has the same cardinality as* $\mathbb{N}$.

Hint. Make use of results of the Exercise 1.2.2 in the lecture notes.

Exercise 2.2. *Show that the set of all well-formed* **WHILE** *programs is countable.*

Exercise 2.3. *Show the following statements:*

1. *If* A *is an uncountable set and* B *is a countable set, then* $|A \setminus B| = |A|$.
2. *Let* $\mathbb{I}$ *be the set of irrational numbers, i.e.,* $\mathbb{I} = \{ r \in \mathbb{R} \mid r \notin \mathbb{Q} \}$. *Then* $|\{0,1\}^\omega \setminus \{0,1\}^* 0^\omega| = |\mathbb{I} \cap [0,1]|$.
3. $|\{0,1\}^\omega| = |[0,1]|$.
4. $|[0,1]| = |[0,1] \times [0,1]| = |[0,1]^n|$ *for each* $n \geq 1$.

Hint. For point 3, make use of points 1 and 2; for point 4, make use of point 3.

Exercise 2.4. *Consider the semantic bracket operator presented in Definition 2.1.6 of the lecture notes. Show that* $\models \varphi \rightarrow \psi$ *iff* $[\![\varphi]\!] \subseteq [\![\psi]\!]$.

Exercise 2.5. *Let* $\mathfrak{Q}_1 \subseteq FOF$ *be the set of FOL formulas* φ *such that each quantifier operator* Qx *(with* $Q \in \{\forall, \exists\}$ *and* $x \in VS$*) appears at most once in* φ.

Provide a function Scope, including its type, such that, given a quantifier operator Qx *(with* $Q \in \{\forall, \exists\}$ *and* $x \in VS$*) and a formula* $\varphi \in \mathfrak{Q}_1$*, it returns the formula corresponding to the scope of* Qx.

Exercise 2.6. *Given* $\varphi, \psi, \eta \in FOF$,

1. *provide a function* $\#$*, including its type, that returns how many times* ψ *occurs in* φ *as sub-formula;*
2. *provide a function* R_∞*, including its type, that replaces each occurrence of* φ *in* η *with* ψ*;*
3. *provide a function* R_1*, including its type, that replaces exactly one occurrence of* φ *in* η *with* ψ *if* φ *occurs in* η*, and that returns* η *if* φ *does not occur in* η. *If* φ *occurs in* η *multiple times, then there is no requirement on the particular instance to be replaced.*

Exercise 2.7. *Prove the following:*

1. *For any predicate* P *with arity* 2, $\forall x \forall y P(x,y) \vdash \forall y \forall z P(y,z)$.
2. *Assume* x *is not free in* φ*, then* $\varphi \rightarrow \forall x \psi$ *and* $\forall x(\varphi \rightarrow \psi)$ *are syntactically equivalent.*

3. *We say a formula φ has* repeated occurrences *of a bound variable x, if Qx appears more than once in the sub-formulas of φ (recall $Q \in \{\forall, \exists\}$). Prove that there exists a formula φ' which has no repeated occurrences of any bound variable such that φ and φ' are syntactically equivalent.*

Exercise 2.8. *Prove that a Hintikka set Γ is consistent, and moreover, for each formula φ, either $\varphi \notin \Gamma$, or $\neg\varphi \notin \Gamma$.*

Exercise 2.9. *Consider the formula $\varphi = fa \wedge \forall x(gxy) \wedge \forall ygfxy$. Here a is a constant, x, y are variables, f is a function symbol with arity one and g is a function with arity two. Show that φ is satisfiable by providing a corresponding Tarski structure and an assignment.*

Exercise 2.10. *Assume Γ is consistent, and assume $\Gamma \nvdash \neg\varphi$ with $\varphi = \neg\forall x\psi$. Moreover, assume that the constant a does not occur in $\Gamma \cup \{\varphi\}$. Then, show $\Gamma \cup \{\varphi, \neg S^x_a \psi\}$ is consistent.*