Homework1

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$$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{2}\times\frac{1}{2}x+\frac{1}{2}\times\frac{1}{2}\times\frac{3}{2}x-...... = 1\times x^0+\binom{\frac{1}{2}}{1}x^1+\binom{\frac{1}{2}}{2}x^2+....... = \sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}x^i$$

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2.1

$$\sum_{i=1}^{n}\frac{1}{i+1}\binom{n}{i} = \frac{1}{i+1}\frac{n!}{i!(n-i)!} = \frac{n!}{(i+1)!(n-i)!} = \sum_{i=1}^{n}\frac{1}{n+1}\binom{n+1}{i+1} = \frac{2^{n+2}-n-2}{n+1}$$

2.2

1，q，q^2是三次单位根,$1+q+q^2=0$

$$2^n = \sum_{k\geq 0}^{\infty}\binom{n}{k} \quad (1+q)^n = \sum_{k\geq 0}^{\infty}\binom{n}{k}q^k \quad (1+q^2)^n = \sum_{k\geq 0}^{\infty}\binom{n}{k}q^2k$$

记$a=\sum_{k\geq 0}^{\infty}\binom{n}{3k}$ $\quad b=\sum_{k\geq 0}^{\infty}\binom{n}{3k+1}$ $\quad c=\sum_{k\geq 0}^{\infty}\binom{n}{3k+2}$ 于是我们得到

$3b = 2^n+(1+q)^n\times q^2+(1+q^2)^n\times q \quad 3a = 2^n+(1+q)^n+(1+q^2)^n \quad 3c = 2^n+(1+q)^n\times q+(1+q^2)^n\times q^2$

解方程，得

$a=\frac{1}{3}(2^n+2cos(\frac{n\pi}{3})) \quad b=\frac{1}{3}(2^n+2cos(\frac{(n-2)\pi}{3})) \quad c=\frac{1}{3}(2^n+2cos(\frac{(n+2)\pi}{3}))$

2.3

$$\sum_{k\geq 0}^{n}k^2\binom{n}{k} = \sum_{k\geq 0}^{n}k(k-1)\binom{n}{k}+\sum_{k\geq 0}^{n}k\binom{n}{k} = \sum_{k\geq 0}^{n}n(n-1)\binom{n-2}{k}+\sum_{k\geq 0}^{n}n\binom{n-1}{k}$$
$$= n(n-1)2^{n-2}+n2^{n-1} = n(n+1)2^{n-2}$$

2.4

$$\sum_{i\geq 0}^{n} i^3 = \sum_{i\geq 0}^{n}[i(i-1)(i-2)+3i(i-1)+i] = \sum_{i\geq 0}^{n}[6\binom{i}{3}+6\binom{i}{2}+\binom{i}{1}] = \sum_{i\geq 0}^{n}[6\binom{i+1}{3}+\binom{i}{1}] = [\frac{n(n+1)}{2}]^2$$

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数学归纳法：
n=1时

$$x^1 = x^{\underline{1}}$$

假设n=k 时成立
n=k+1时，有

$$x^{k+1} - x(x-1)(x-2)...(x-k) = ax^k + bx^{k-1}... + zx$$

由归纳假设，等式右边可以表示为$\{x^{\underline{1}}, x^{\underline{2}}, ..., x^{\underline{k}}\}$的线性组合。Q.E.D.

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n为奇数时，

$$\binom{n}{0} < \binom{n}{1} < ... < \binom{n}{\frac{n}{2}-1} < \binom{n}{\frac{n}{2}} > \binom{n}{\frac{n}{2}+1} > ... > \binom{n}{n}$$

n为偶数时，

$$\binom{n}{0} < \binom{n}{1} < ... < \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}} > ... > \binom{n}{n}$$

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$$(\frac{n}{k})^k = \frac{n}{k}\frac{n}{k}...\frac{n}{k} \quad \binom{n}{k} = \frac{n}{k}\frac{n-1}{k-1}...\frac{n-k+1}{1} \quad (\frac{en}{k})^k = \frac{en}{k}\frac{en}{k}...\frac{en}{k}$$

因$\frac{n-i}{k-i} \geq \frac{n}{k}, 0 \leq i \leq k$,所以有$(\frac{n}{k})^k \leq \binom{n}{k}$成立
若有$\frac{n-i}{k-i} \leq \frac{en}{k}$,i需要满足$n-i \leq \frac{en}{k}(k-i)$,化简得$i \leq \frac{ek-k}{en-k}n \leq n$成立
综上，原式成立

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不妨认为圆桌存在位置编号1至12号，即有起始顺序
排男士，有$4^{\underline{4}}$种；排女士，有$8^{\underline{8}}$种
故总计有$4^{\underline{4}} \times 8^{\underline{8}}$种

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圆桌依旧有顺序。将AB看作一个个体，由逆事件可得到种数为$15^{\underline{15}}-2\times 14^{\underline{14}}$
若只不坐在右边，则为$15^{\underline{15}}-14^{\underline{14}}$

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先挑10位男士，后挑10位女士，然后配对，故为

$$\binom{15}{10}\binom{20}{10}10^{\underline{10}}$$

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易知有

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}$$

若不关心是谁拿到了哪份牌，则为

$$\frac{\binom{52}{13}\binom{39}{13}\binom{26}{13}}{4^{\underline{4}}}$$

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总计有六种情况：3a4b3c,3a3b4c,3a2b5c,2a3b5c,2a4b4c,1a4b5c,故总计为

$$\frac{\binom{5}{3}10^{\underline{10}}}{3^{\underline{3}}4^{\underline{4}}3^{\underline{3}}}+\frac{\binom{5}{4}\binom{4}{3}10^{\underline{10}}}{3^{\underline{3}}3^{\underline{3}}4^{\underline{4}}}+\frac{\binom{4}{2}10^{\underline{10}}}{3^{\underline{3}}2^{\underline{2}}5^{\underline{5}}}+\frac{\binom{3}{2}\binom{4}{3}10^{\underline{10}}}{2^{\underline{2}}3^{\underline{3}}5^{\underline{5}}}+\frac{\binom{3}{2}\binom{5}{4}10^{\underline{10}}}{2^{\underline{2}}4^{\underline{4}}4^{\underline{4}}}+\frac{\binom{3}{1}10^{\underline{10}}}{4^{\underline{4}}5^{\underline{5}}}$$

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从自然数中挑选k个，从ab中挑选n-k个，则有

$$\sum_{k\geq 0}^{n}\binom{n+1}{k}(n-k+1)=\sum_{k\geq 0}^{n}(n+1)\binom{k+1}{k}-\sum_{k\geq 0}^{n}(n+1)\binom{n}{k-1}=(n+1)\times 2^{n}$$