计算机科学导论

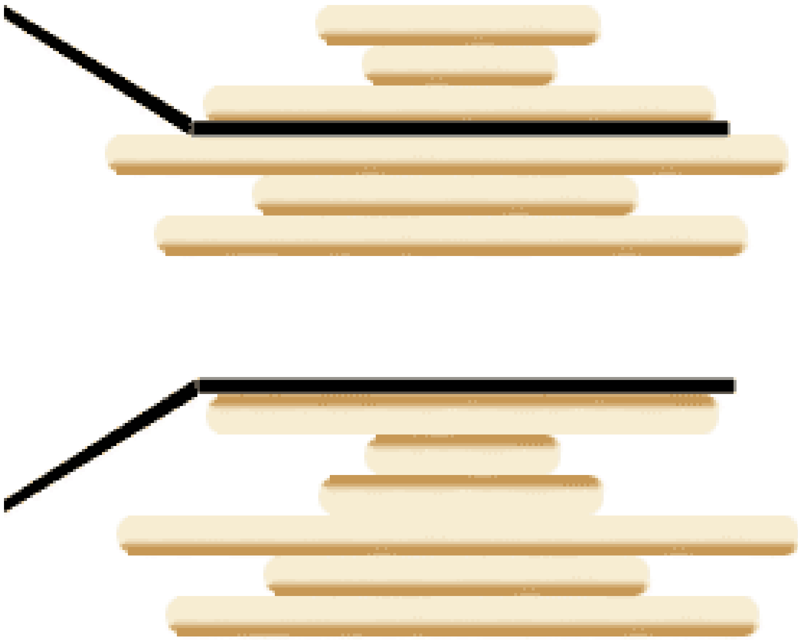
张家琳
中国科学院计算技术研究所
zhangjialin@ict.ac.cn

2018-5-11

思考题

- 翻煎饼问题(Pancake Sorting)

一个厨师做了一叠大小不同的煎饼，他要不断从上面拿起几个煎饼翻到下面。假设有**n**个煎饼，厨师需要翻动多少次，才能把煎饼按从小到大排好？



翻煎饼问题

- 2n次翻动一定可以做到
- 盖茨的答案
  - 5/3n (1979, "Bounds for Sorting by Prefix Reversal")
  - 下界：17/16n
- 后续发展
  - 18/11n (2009)
  - 下界：15/14n (2009)
- 变种：burnt pancake problem

思考题

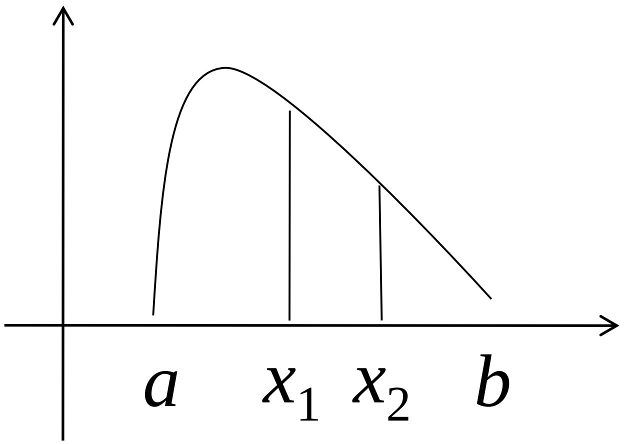
- $2^{\Theta(n)}$ 和 $\Theta(2^n)$ 一样吗？

算法思维

- 算法例子：排序
  - 冒泡排序、快速排序
- 大O符号
- 分治思想
- P=NP?问题

分治算法

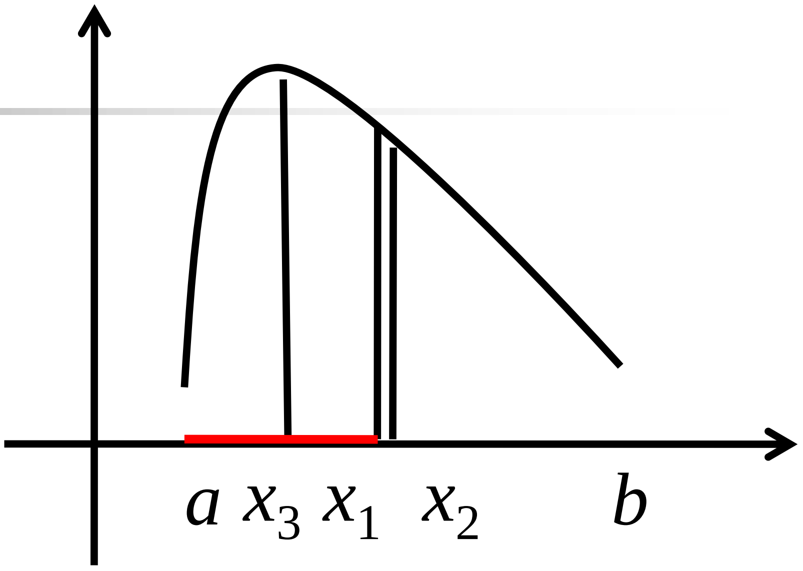
- 单因素优选法: $f(x)$在$[a, b]$上先递增后递减，找出$\max f(x)$.



- idea:
- 选取点x_1, x_2
- If $f(x_1) > f(x_2)$, 舍去$[x_2, b]$
- If $f(x_1) < f(x_2)$, 舍去$[a, x_1]$
- If $f(x_1) = f(x_2)$, 舍去$[a, x_1]$和$[x_2, b]$

- 方案一：

$x_1 \approx x_2 \approx (a+b)/2$

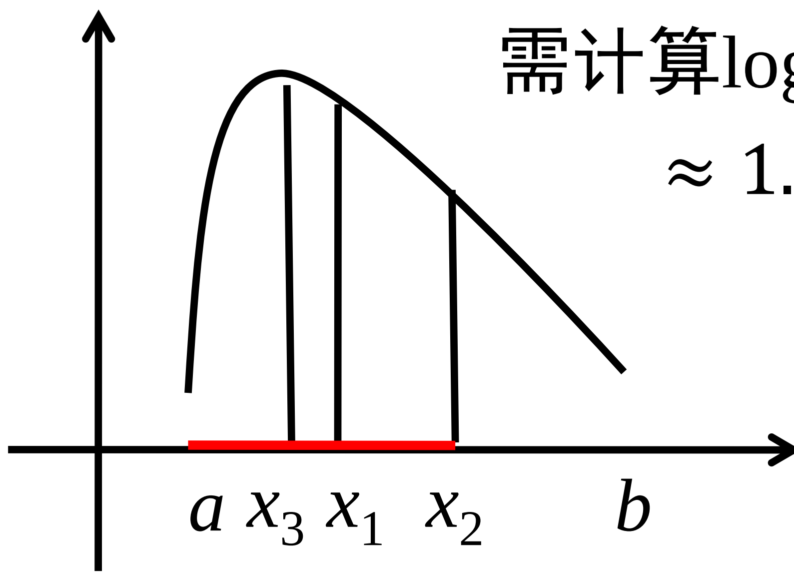


假定将$[a, b]$离散化成n个点

- $T(n) = T(n/2) + 2$
- $T(2) = 2$

需计算 $2\log_2 n$次函数值

- 方案二： $x_1 = ta+(1\text{-}t)b,\ x_2 = (1\text{-}t)a+tb$，其中 $t=\frac{\sqrt{5}-1}{2}\approx 0.618$



需计算$\log_{1.618} n$次函数值

$\approx 1.44\log_2 n$

乘法(1)

n^2次乘法运算

- 输入：$X=x_n x_{n-1}\ldots x_1$, $Y=y_n y_{n-1}\ldots y_1$
- 输出：$Z = XY$

$$
\begin{array}{r}
123 \\
\times 321 \\
\hline
123 \\
246 \\
369 \\
\hline
39483
\end{array}
$$

- idea:

$X = X_1 \times 10^{n/2} + X_2$,

$Y = Y_1 \times 10^{n/2} + Y_2$

$Z = XY =$

$$X_1Y_1 \times 10^{n} + (X_1Y_2 + X_2Y_1) \times 10^{n/2} + X_2Y_2$$

乘法(2)

- 分别计算： $X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2$
- 乘法次数：$T(n) = 4\,T(n/2), T(1) = 1$
- $T(n) = n^2 \ldots$
- $\boxed{X_1Y_1} \times 10^n + \boxed{(X_1Y_2+X_2Y_1)} \times 10^{n/2} + \boxed{X_2Y_2}$
- **计算：** X_1Y_1, X_2Y_2 , $\mathbf{(X_1+X_2)(Y_1+Y_2)}$

$$X_1Y_2+X_2Y_1=\mathbf{(X_1+X_2)(Y_1+Y_2)}-X_1Y_1-X_2Y_2$$

乘法(3)

- 乘法次数：$T(n) = 3\,T(n/2),\ T(1) = 1$
- $T(n) = n^{\log_2 3} \approx n^{1.59}$

思考题

- 能不能更快？

- 可以！快速傅里叶变换（FFT）
- 通过把X, Y各等分成3段，来演示FFT的思想

$$X = X_2 \times 10^{2n/3} + X_1 \times 10^{n/3} + X_0,$$

$$Y = Y_2 \times 10^{2n/3} + Y_1 \times 10^{n/3} + Y_0,$$

$$Z = XY =$$

$$X_2 Y_2 \times 10^{4n/3} + (X_1 Y_2 + X_2 Y_1) \times 10^{n}$$

$$+ (X_0 Y_2 + X_1 Y_1 + X_2 Y_0) \times 10^{2n/3}$$

$$+ (X_1 Y_0 + X_0 Y_1) \times 10^{n/3} + X_0 Y_0$$

- 尝试一
  - 分别计算X_0Y_0，X_1Y_1，X_2Y_2，$(X_0+X_1)(Y_0+Y_1)$ $(X_0+X_2)(Y_0+Y_2)$, $(X_1+X_2)(Y_1+Y_2)$
  - $T(n) = 6T(n/3), T(1) = 1$
  - $T(n) = n^{\log_3 6} \approx n^{1.631}$
- 能否用更少的乘法次数实现

$$\omega = e^{i\frac{2\pi}{3}}$$

$$A_0 = X_2 + X_1 + X_0, \qquad B_0 = Y_2 + Y_1 + Y_0,$$

$$A_1 = X_2 + \omega X_1 + \omega^2 X_0, \quad B_1 = Y_2 + \omega Y_1 + \omega^2 Y_0,$$

$$A_2 = X_2 + \omega^2 X_1 + \omega X_0, \quad B_2 = Y_2 + \omega^2 Y_1 + \omega Y_0.$$

$$A_0B_0 + A_1B_1 + A_2B_2 = 3(\boxed{X_2Y_2} + \boxed{X_1Y_0 + X_0Y_1})$$

$$A_0B_0 + \omega A_1B_1 + \omega^2 A_2B_2 = 3(\boxed{X_0Y_2 + X_1Y_1 + X_2Y_0})$$

$$A_0B_0 + \omega^2 A_1B_1 + \omega A_2B_2 = 3(\boxed{X_2Y_1 + X_1Y_2} + \boxed{X_0Y_0})$$

- 记两个 n 位数相乘的乘法次数为$T(n)$

乘法次数：$T(n) = 5\,T(n/3),\ T(1) = 1$

$T(n) = n^{\log_3 5} \approx n^{1.465}$

- 思考题：能否更快？
- **快速傅立叶变换(FFT)** **$O(n \log n)$**

矩阵乘法(1)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & \ddots & \\ b_{n1} & & & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \ddots & & \\ \vdots & & \ddots & \\ c_{n1} & & & c_{nn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$O(n^3)$次**乘法**，$O(n^3)$次加法

矩阵乘法(2)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$m_1 = (a_{12} - a_{22})(b_{21} + b_{22})$$
$$m_2 = (a_{11} + a_{22})(b_{11} + b_{22})$$
$$m_3 = (a_{11} - a_{21})(b_{11} + b_{12})$$
$$m_4 = (a_{11} + a_{12})b_{22}$$
$$m_5 = a_{11}(b_{12} - b_{22})$$
$$m_6 = a_{22}(b_{21} - b_{11})$$
$$m_7 = (a_{21} + a_{22})b_{11}$$

$$c_{11} = m_1 + m_2 - m_4 + m_6$$
$$c_{12} = m_4 + m_5$$
$$c_{21} = m_6 + m_7$$
$$c_{22} = m_2 - m_3 + m_5 - m_7$$

$O(n^{\log 7 \approx 2.81})$次**乘法**

矩阵乘法(3)

- Strassen algorithm'69 $O(n^{2.81})$
- $O(n^{2.79})$, $O(n^{2.55})$, $O(n^{2.48})$ …
- Coppersmith–Winograd algorithm'89 $O(n^{2.376})$
- Stothers'10 $O(n^{2.374})$
- Williams'11 $O(n^{2.373})$
- Le Gall'14 $O(n^{2.3729})$

算法思维

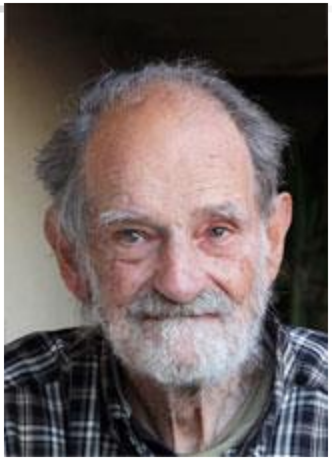
- 分治思想
  - 单因素优选法
  - 大整数乘法
  - 矩阵乘法
- 贪心、递归、穷举、回溯、动态规划……
- 随机算法、近似算法、在线算法、流算法……
- 后续课程：数据结构、理论计算机基础、算法设计与分析、高等算法……

思考题

- 汉诺塔（Hanoi）　　2^n-1
- 如果有4根柱子怎么办？

2012年经济学诺贝尔奖

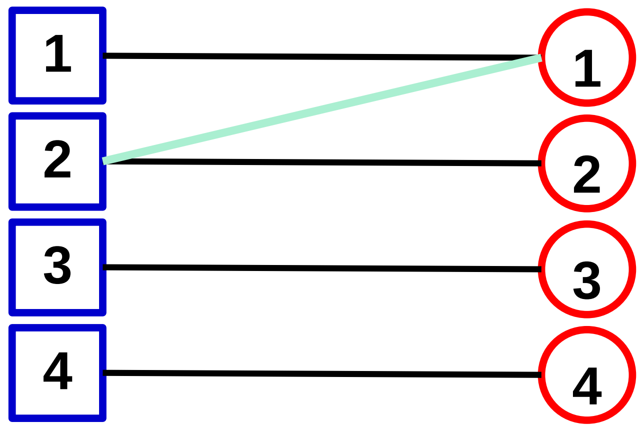
Alvin E. Roth



Lloyd S. Shapley

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley "for the theory of stable allocations and the practice of market design"

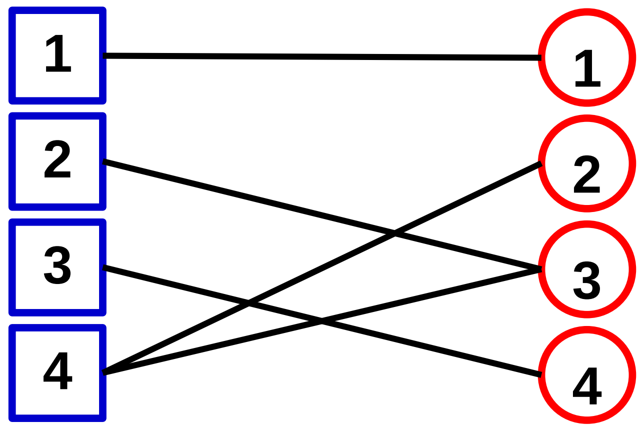
稳定婚姻问题



1	1	3	2	4
2	3	4	1	2
3	4	2	3	1
4	3	2	1	4

1	2	1	3	4
2	4	1	2	3
3	1	3	2	4
4	2	3	1	4

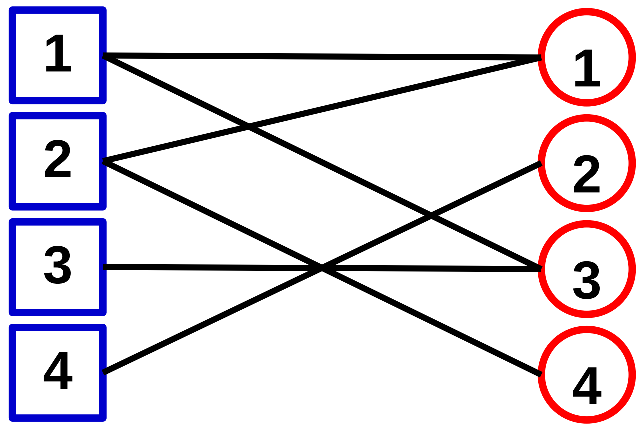
Gale-Shapley Algorithm (1962)



1	1	3	2	4
2	3	4	1	2
3	4	2	3	1
4	3	2	1	4

1	2	1	3	4
2	4	1	2	3
3	1	3	2	4
4	2	3	1	4

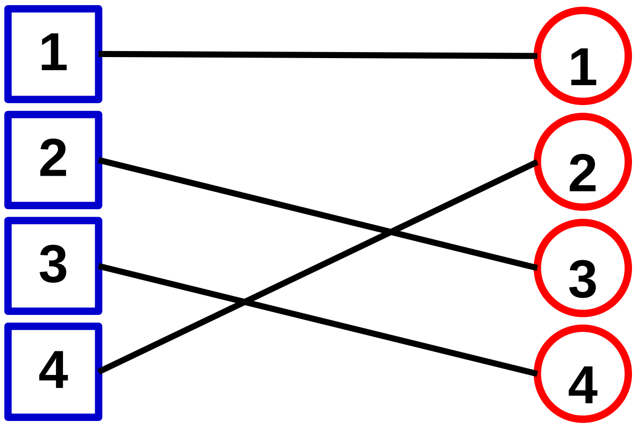
Gale-Shapley Algorithm (1962)

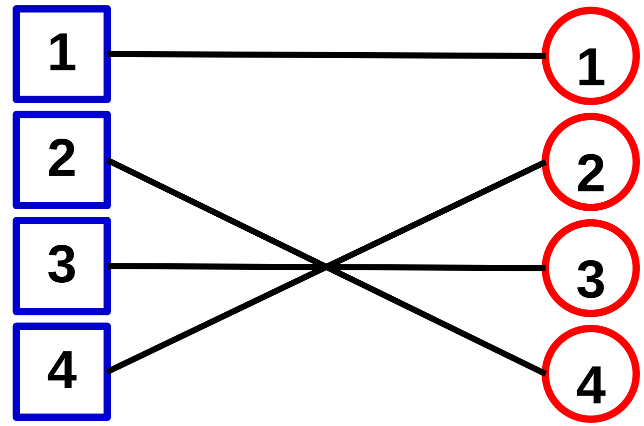


1	1	3	2	4
2	3	4	1	2
3	4	2	3	1
4	3	2	1	4

1	2	1	3	4
2	4	1	2	3
3	1	3	2	4
4	2	3	1	4

Gale-Shapley Algorithm (1962)





1	1	3	2	4
2	3	4	1	2
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4	3	2	1	4

1	2	1	3	4
2	4	1	2	3
3	1	3	2	4
4	2	3	1	4

稳定婚姻问题

- 稳定婚姻并不唯一
- Gale-Shapley 算法
  - 可以得到一种稳定婚姻匹配
  - 又称 men propose 算法
  - 对求婚者一方有利

算法思维

- 算法例子：排序
  - 冒泡排序、快速排序
- 大O符号
- 分治思想
- P=NP?问题

- 思考题：4根柱子的汉诺塔

谢谢！