1 Answer to Exercise 1.4

To show that $\{\neg, \wedge\}$ is functionally complete, it suffices to prove that any propositional logic formula φ has a semantically equivalent expression which only contains $\neg$ and $\wedge$. We conduct induction on the formula φ.

- $\varphi = a \in AP$. Trivial.
- I.H.: φ has a semantically equivalent expression which only contains $\neg$ and $\wedge$, namely φ'. Then $\neg\varphi \equiv \neg\varphi'$, and $\neg\varphi'$ only contains $\neg$ and $\wedge$.
- I.H.: φ and ψ both have a semantically equivalent expression which only contains $\neg$ and $\wedge$, namely φ' and ψ'. Then $\varphi \to \psi \equiv \varphi' \to \psi' \equiv (\neg\varphi') \vee \psi' \equiv \neg(\varphi' \wedge (\neg\psi'))$, and $\neg(\varphi' \wedge (\neg\psi'))$ only contains $\neg$ and $\wedge$.

Here we complete the proof.

Following a similar method, we can prove that $\{\neg, \vee\}$ is also functionally complete. Naturally, $\{\neg, \to\}$ is functionally complete. Now we claim that $\{\wedge, \vee\}$ is NOT functionally complete. Let $a \in AP$, and we show that $\neg a$ does not have a semantically equivalent expression which only contains $\wedge$ and $\vee$. Let the assignment $\sigma = AP$, and naturally $\sigma(\neg a) = \mathbf{F}$. Then we prove by induction that, for any formula φ which only contains $\wedge$ and $\vee$, $\sigma(\varphi) = \mathbf{T}$.

- $\varphi = a' \in AP$. $\sigma(a') = \mathbf{T}$.
- I.H.: $\sigma(\varphi) = \mathbf{T}$ and $\sigma(\psi) = \mathbf{T}$. Then $\sigma(\varphi \wedge \psi) = \mathbf{T}$ and $\sigma(\varphi \vee \psi) = \mathbf{T}$.

Therefore, $\neg a$ does not have a semantically equivalent expression φ which only contains $\wedge$ and $\vee$.

2 Proof of Lemma 1.3.30

(1) If Γ is consistent, and there is some finite subset $\Gamma' \subset \Gamma$ which is not consistent, then there exists a formula φ such that $\Gamma' \vdash \varphi$ and $\Gamma' \vdash \neg\varphi$, and we have $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, which contradicts with the assumption that Γ is consistent.

If every finite subset of Γ is consistent and Γ is not consistent, then there exists a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. From (**Fin**) we know that there exists finite subsets $\Gamma', \Gamma'' \subseteq \Gamma$ such that $\Gamma' \vdash \varphi$ and $\Gamma'' \vdash \neg\varphi$. Then $\Gamma' \cup \Gamma''$ is also a finite subset of Γ, and it is consistent. However, we can see $\Gamma' \cup \Gamma'' \vdash \varphi$ and $\Gamma' \cup \Gamma'' \vdash \neg\varphi$. Contradiction!

(2) Notice that Γ is consistent if and only if Γ is satisfiable. It is obvious from (1).

In the following is the wrong proof I mentioned in the morning. It does not work because AP is countable (i.e. likely to be infinite) in our settings, and this will make the set $\{\varphi_\sigma \mid \sigma \text{ is an assignment}\}$ infinite. However, it is still worth considering how to proof (2) directly without using the fact that Γ is consistent if and only if Γ is satisfiable.

If Γ is satisfiable, then there exists an assignment σ, s.t. $\sigma(\Gamma) = \{\mathbf{T}\}$. Then obviously for any finite subset $\Gamma' \subseteq \Gamma$, $\sigma(\Gamma') = \{\mathbf{T}\}$, and Γ' is satisfiable.

If every finite subset of Γ is satisfiable and Γ is not satisfiable, then for any assignment σ, there exists a formula $\varphi_\sigma \in \Gamma$ such that $\sigma(\varphi_\sigma) = \mathbf{F}$. Then $\{\varphi_\sigma \mid \sigma \text{ is an assignment}\}$ is a finite subset of Γ, and it is not satisfiable. Contradiction!

3 Proof of Lemma 1.3.34

We prove it by induction on φ.

- $\varphi = a$ or $\varphi = \neg a$, where $a \in AP$. Trivial.
- I.H.: φ is satisfiable iff it has a consistent tableau. We consider $\neg\neg\varphi$.

$$\begin{aligned} & \neg\neg\varphi \text{ is satisfiable} \\ \text{iff } & \varphi \text{ is satisfiable} \\ \text{iff } & \varphi \text{ has a consistent tableau} \\ \text{iff } & \neg\neg\varphi \text{ has a consistent tableau.} \end{aligned}$$

- I.H.: φ, as well as ψ, $\neg\varphi$, $\neg\psi$, is satisfiable iff it has a consistent tableau. Then we consider $\varphi \to \psi$.

$$\begin{aligned} & \varphi \to \psi \text{ is satisfiable} \\ \text{iff } & \neg\varphi \text{ is satisfiable or } \psi \text{ is satisfiable} \\ \text{iff } & \neg\varphi \text{ has a consistent tableau or } \psi \text{ has a consistent tableau} \\ \text{iff } & \varphi \to \psi \text{ has a consistent tableau.} \end{aligned}$$

  Finally we consider $\neg(\varphi \to \psi)$.

$$\begin{aligned} & \neg(\varphi \to \psi) \text{ is satisfiable} \\ \text{iff } & \varphi \text{ is satisfiable and } \neg\psi \text{ is satisfiable} \\ \text{iff } & \varphi \text{ has a consistent tableau and } \neg\psi \text{ has a consistent tableau} \\ \text{iff } & \neg(\varphi \to \psi) \text{ has a consistent tableau.} \end{aligned}$$

Here we complete the proof.

4 Proof of Lemma 1.3.39

It suffices to show that for any assignment σ, $\sigma \Vdash \varphi$ implies $\sigma \Vdash \varphi \cup \{R\}$, since the other side is trivial. We only need to show that $\sigma \Vdash \varphi$ implies $\sigma \Vdash \{R\}$. Let the assignment σ satisfy $\sigma \Vdash \varphi$. We have $\sigma(C_1 \wedge C_2) = \mathbf{T}$. Then there exist literals $L_1 \in C_1$ and $L_2 \in C_2$ such that $\sigma(L_1) = \sigma(L_2) = \mathbf{T}$. Notice that $\{L_1, L_2\} \neq \{L, \neg L\}$, so at least one of L_1 and L_2 is in $R = (C_1 \setminus \{L\}) \cup (C_2 \setminus \{\neg L\})$, and we have $\sigma(\{R\}) = \mathbf{T}$.