Exercise sheet 1 on Discrete Mathematics

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To be submitted on March 20, 2018.

Recall the following normal forms.

Conjunctive Normal Form (CNF):

- Every formula φ is a conjunction of clauses $\mathcal{C}_1, \ldots, \mathcal{C}_n$.
- A clause $\mathcal{C}$ is the disjunction of literals $\mathtt{l}_1, \ldots, \mathtt{l}_m$.
- A literal $\mathtt{l}$ is an atomic proposition or the negation of an atomic proposition.

$$\begin{aligned} \varphi &::= \mathcal{C} \mid \mathcal{C} \wedge \varphi \\ \mathcal{C} &::= \mathtt{l} \mid \mathtt{l} \vee \mathcal{C} \\ \mathtt{l} &::= p \mid \neg p && \text{(where } p \in AP\text{)} \end{aligned}$$

Disjunctive Normal Form (DNF):

- Every formula φ is a disjunction of clauses $\mathcal{C}_1, \ldots, \mathcal{C}_n$.
- A clause $\mathcal{C}$ is the conjunction of literals $\mathtt{l}_1, \ldots, \mathtt{l}_m$.
- A literal $\mathtt{l}$ is an atomic proposition or the negation of an atomic proposition.

$$\begin{aligned} \varphi &::= \mathcal{C} \mid \mathcal{C} \vee \varphi \\ \mathcal{C} &::= \mathtt{l} \mid \mathtt{l} \wedge \mathcal{C} \\ \mathtt{l} &::= p \mid \neg p && \text{(where } p \in AP\text{)} \end{aligned}$$

Negation-free Normal Form (NNF):

- Negation may appear only in front of atomic propositions.

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \qquad \text{(where } p \in AP\text{)}$$

Exercise 1.1 (Relations between normal forms)**.**

1. *Prove formally that for each CNF formula* φ_c *there is a DNF formula* φ_d *such that* $\varphi_c \equiv \varphi_d$ *and vice-versa.*
2. *Given the Boolean formula* $\varphi = \neg(\neg(a \vee (b \wedge c)) \to (b \leftrightarrow \neg c))$*, transform it in NNF, by showing all performed steps.*
3. *Given the Boolean formula* $\varphi = \neg(\neg(a \vee (b \wedge c)) \to (b \leftrightarrow \neg c))$*, transform it in CNF, by showing all performed steps.*

Exercise 1.2. *Show, by applying the rules of the deduction system or their properties presented in Section 1.3, the following statements:*

1. $\vdash (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
2. $\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$
3. $\vdash \neg(\varphi \to \psi) \to (\psi \to \varphi)$

Exercise 1.3. *By means of the semantics, show the following semantical equivalences:*

1. $(\varphi_1 \to \varphi_3) \vee (\varphi_2 \to \varphi_3) \equiv (\varphi_1 \wedge \varphi_2) \to \varphi_3$
2. $\varphi \leftrightarrow \psi \equiv (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$
3. $\neg(\varphi \leftrightarrow \psi) \equiv \varphi \leftrightarrow \neg\psi$

Exercise 1.4. *Let* $\mathbb{B}$ *denote the Boolean set. An* n*-ary Boolean function* φ *is a function* $\varphi\colon \mathbb{B}^n \to \mathbb{B}$*. Intuitively, an* n*-ary Boolean function* φ *corresponds to a Boolean formula with* n *variables/atomic propositions.*

A set of logical connectives is called functionally complete *if any* n*-ary Boolean function is definable with it. Show that* $\{\neg, \wedge\}$*,* $\{\neg, \vee\}$ *are both funtionally complete. How about* $\{\neg, \to\}$*, or* $\{\wedge, \vee\}$*?*

Exercise 1.5. *Provide detailed proofs for the compactness theorem (Theorem 1.3.30), tableau theorem (Theorem 1.3.34), and Lemma 1.3.39.*

Moreover, consider the formula φ *given below:*

$$(a \vee \neg b) \wedge (a \vee \neg c) \wedge (\neg a \vee c) \wedge (\neg a \vee \neg b) \wedge (c \vee \neg b) \wedge (\neg c \vee b)$$

Use one of the approaches (tableau, resolution or DPLL) to check whether φ *is satisfiable.*

Exercise 1.6 (Special exercise)**.** *Take one paper accepted at one of the previous 2 editions of SAT (*`http://sat2016.labri.fr/` *and* `http://sat2017.gitlab.io/`*), study it, and present it.*

This exercise is alternative to the others, i.e., by solving this exercise you don't need to solve the other exercises; moreover you will receive a secret present from the instructor.