Chapter 3

First Order Logic (FOL)

3.1 Syntax of FOL

Propositional logic is a coarse language, which only concerns about propositions and boolean connectives. Practically, this logic is not powerful enough to describe important properties we are interested in.

Example 3.1.1 (Syllogism of Aristotle)**.** *Consider the following assertions:*

1. *All men are mortal.*
2. *Socrates is a man.*
3. *So Socrates would die.*

$$\forall x(Man(x) \to Mortal(x))$$

Definition 3.1.2. *First order logic is an extension of proposition logic:*

1. *To accept parameters, it generalizes propositions to predicates.*

2. *To designate elements in the domain, it is equipped with functions and constants.*

3. *It also involves quantifiers to capture infinite conjunction and disjunction.*

Definition 3.1.3. *We are given:*

- *an arbitrary set of variable symbols* $VS = \{x, y, x_1, \dots\}$*;*
- *an arbitrary set (maybe empty) of function symbols* $FS = \{f, g, f_1, \dots\}$*, where each symbol has an arity;*
- *an arbitrary set (maybe empty) of predicate symbols* $PS = \{P, Q, P_1, \dots\}$*, where each symbol has an arity;*
- *an equality symbol set* ES *which is either empty or one element set containing* $\{\approx\}$*.*

Let $L = VS \cup \{(,), \rightarrow, \neg, \forall\} \cup FS \cup PS \cup ES$*. Here* $VS \cup \{(,), \rightarrow, \neg, \forall\}$ *are referred to as* logical symbols*, and* $FS \cup PS \cup ES$ *are referred to as* non-logical symbols.

We often make use of the

- *set of constant symbols, denoted by* $CS = \{a, b, a_1, \dots\} \subseteq FS$*, which consist of function symbols with arity* 0*;*
- *set of propositional symbols, denoted by* $PS = \{p, q, p_1, \dots\} \subseteq FS$*, which consist of predicate symbols with arity* 0*.*

Definition 3.1.4 (FOL terms)**.** *The terms of the first order logic are constructed according to the following grammar:*

$$t ::= x \mid f t_1 \dots t_n$$

where $x \in VS$*, and* $f \in FS$ *has arity* n*.*

Accordingly, the set T *of terms is the smallest set satisfying the following conditions:*

- *each variable* $x \in VS$ *is a term.*
- *Compound terms:* $ft_1 \dots t_n$ *is a term (thus in* T*), provided that* f *is a* n*-arity function symbol, and* $t_1, \dots, t_n \in T$*. Particularly,* $a \in CS$ *is a term.*

We often write $f(t_1, \dots, t_n)$ *for the compound terms.*

Definition 3.1.5 (FOL formulas)**.** *The well-formed formulas of the first order logic are constructed according to the following grammar:*

$$\varphi ::= Pt_1 \dots t_n \mid \neg\varphi \mid \varphi \to \varphi \mid \forall x\varphi$$

where $t_1, \dots, t_n$ *are terms,* $P \in PS$ *has arity* n*, and* $x \in VS$*.*

We often write $P(t_1, \dots, t_n)$ *for clarity. Accordingly, the set* FOF *of first order formulas is the smallest set satisfying:*

- $P(t_1, \dots, t_n) \in FOF$ *is a formula, referred to as the atomic formula.*
- *Compound formulas:* $(\neg\varphi)$ *(negation),* $(\varphi \to \psi)$ *(implication), and* $(\forall x\varphi)$ *(universal quantification) are formulas (thus in* FOF*), provided that* $\varphi, \psi \in FOF$*.*

We omit parentheses if it is clear from the context.

As syntactic sugar, we can define $\exists x\varphi$ *as* $\exists x\varphi := \neg\forall x\neg\varphi$*. We assume that* $\forall$ *and* $\exists$ *have higher precedence than all logical operators.*

Definition 3.1.6 (Sub-formulas)**.** *For a formula* φ*, we define* *the sub-formula function* $Sf : FOF \to 2^{FOF}$ *as follows:*

$$\begin{aligned}
Sf(P(t_1,\ldots,t_n)) &= \{P(t_1,\ldots,t_n)\}\\
Sf(\neg\varphi) &= \{\neg\varphi\} \cup Sf(\varphi)\\
Sf(\varphi \to \psi) &= \{\varphi \to \psi\} \cup Sf(\varphi) \cup Sf(\psi)\\
Sf(\forall x\varphi) &= \{\forall x\varphi\} \cup Sf(\varphi)\\
Sf(\exists x\varphi) &= \{\exists x\varphi\} \cup Sf(\varphi)
\end{aligned}$$

Definition 3.1.7 (Scope)**.** *The part of a logical expression to which a quantifier is applied is called the scope of this quantifier. Formally, each sub-formula of the form* $Qx\psi \in Sf(\varphi)$*, the scope of the corresponding quantifier* Qx *is* ψ*. Here* $Q \in \{\forall, \exists\}$*.*

Substitution for Terms

Definition 3.1.8 (Sentence)**.** *We say an occurrence of* x *in* φ *is free if it is not in scope of any quantifiers* $\forall x$ *(or* $\exists x$*). Otherwise, we say that this occurrence is a bound occurrence. If a variable* φ *has no free variables, it is called a* closed formula*, or a* sentence*.*

Definition 3.1.9 (Substitution)**.** *The substitution of* x *with* t *within* φ*, denoted as* $S^x_t\varphi$*, is obtained from* φ *by replacing each free occurrence of* x *with* t*.*

We would extend this notation to $S^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}\varphi$.

Remark 3.1.10. *It is important to remark that* $S^{x_1,\ldots,x_n}_{t_1,\ldots,t_n}\varphi$ *is not the same as* $S^{x_1}_{t_1}\ldots S^{x_n}_{t_n}\varphi$*: the former performs a simultaneous substitution.*

For example, consider the formula $P(x,y)$*: the subsitution* $S^{x,y}_{y,x}P(x,y)$ *gives* $S^{x,y}_{y,x}P(x,y) = P(y,x)$ *while the substitutions* $S^x_yS^y_xP(x,y)$ *give* $S^x_yS^y_xP(x,y) = S^x_yP(x,x) = P(y,y)$*.*

Remark 3.1.11. *Consider* $\varphi = \exists y(x < y)$ *in the number theory. What is* $S^x_t \varphi$ *for the special case of* $t = y$*?*

Definition 3.1.12 (Substitutable on Terms)**.** *We say that* t *is substitutable for* x *within* φ *iff for each variable* y *occurring in* t*, there is no free occurrence of* x *in scope of* $\forall y / \exists y$ *in* φ*.*

Definition 3.1.13 (α-β condition)**.** *If the formula* φ *and the variables* x *and* y *fulfill:*

1. *y has no free occurrence in φ, and*

2. *y is substitutable for x within φ,*

then we say that φ*,* x *and* y *meet the α-β condition, denoted as* $C(\varphi, x, y)$*.*

Lemma 3.1.14. *If* $C(\varphi, x, y)$*, then* $S^y_x S^x_y \varphi = \varphi$*.*