3.2 The Axiom System: the Hilbert's System

As for propositional logic, also FOL can be axiomatized.

Definition 3.2.1 (Axioms).

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))$
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
4. $\underline{\forall x \varphi \to S^x_t \varphi}$ $\qquad \underbrace{P t_1 \dots t_n}$

   if t is substitutable for x within φ
5. $\forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$
6. $\varphi \to \forall x \varphi$ $\qquad x^2 = y \to \forall z\, x^2 = y$

   if x is not free in φ $\qquad \boxed{\approx xy}$
7. $\forall x_1 \dots \forall x_n \varphi$

   if φ is an instance of (one of) the above axioms

MP Rule: $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$

Definition 3.2.2 (Syntactical Equivalence)**.** *We say φ and ψ are syntactically equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$.*

Theorem 3.2.3. *(Gen): If x has no free occurrence in Γ, then $\Gamma \vdash \varphi$ implies $\Gamma \vdash \forall x \varphi$.*

Solution. Suppose that $\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$ is the deductive sequence of φ from Γ.

$\forall x \varphi_0 \quad \forall x \varphi_1 \quad \cdots \quad \forall x \varphi_n$

- If φ_i is an instance of some axiom, then according to (AS7), $\forall x \varphi_i$ is also an axiom.
- If $\varphi_i \in \Gamma$, since x is not free in Γ, we have $\vdash \varphi_i \to \forall x \varphi_i$ according to (AS6). Therefore, we have $\Gamma \vdash \forall x \varphi_i$ in this case.

$$\frac{\varphi_j \quad \overbrace{\varphi_j \to \varphi_i}^{\varphi_k}}{\varphi_i}$$

$j, k < i$

- If φ_i is obtained by applying (M-P) to some φ_j and $\varphi_k = \varphi_j \to \varphi_i$. By induction, we have $\Gamma \vdash \forall x \varphi_j$ and $\Gamma \vdash \forall x(\varphi_j \to \varphi_i)$. With (AS5) and (MP), we also have $\Gamma \vdash \forall x \varphi_i$ in this case.

Thus, we have $\Gamma \vdash \forall x \varphi_n$, i.e., $\Gamma \vdash \forall x \varphi$.

Exercise 3.2.4. *Prove that* Gen

$1 \Leftarrow \forall x \forall y \varphi \vdash \forall x \varphi$

$\Leftarrow \forall x \forall y \varphi \vdash \varphi$

$(T) \begin{cases} \forall y \varphi \vdash \varphi \\ \Leftarrow \forall x \forall y \varphi \vdash \forall y \varphi \end{cases}$

1. $\forall x \forall y \varphi \vdash \forall y \forall x \varphi$,
2. $\exists x \forall y \varphi \vdash \forall y \exists x \varphi$.

$\Uparrow \exists x \forall y \varphi \vdash \exists x \varphi$

$\underbrace{\exists x \varphi \equiv \neg \forall x \neg \varphi}$ $\neg \forall x \neg \forall y \varphi \vdash \neg \forall x \neg \varphi$

A3 $\Uparrow$ $\forall x \neg \varphi \vdash \forall x \neg \forall y \varphi$

G $\Uparrow$ $\forall x \neg \varphi \vdash \underline{\neg \forall y \varphi}$

$\Uparrow \{\forall x \neg \varphi, \forall y \varphi\}$ incons

$\Gamma \nvdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ cons

$\Leftrightarrow \Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ incon

Exercise 3.2.5. *Prove that*

1. $\forall x(\varphi \to \psi) \vdash \forall x(\neg \psi \to \neg \varphi)$,
2. $\forall x(\varphi \to \psi) \vdash \exists x \varphi \to \exists x \psi$.

Exercise 3.2.6. *Prove that*

1. *If* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \neg \psi$, *then* $\Gamma \vdash \neg(\varphi \to \psi)$, $\equiv \varphi \wedge \neg \psi$

$\Leftrightarrow$

$\varphi, \neg \psi \vdash \neg(\varphi \to \psi)$

$(\to^+)$ (A3) $\varphi \vdash (\varphi \to \psi) \to \psi$

$= \varphi \wedge \neg\exists x\psi$

2. $\forall x \neg(\varphi \to \psi) \vdash \neg(\varphi \to \exists x\psi)$.

$\Leftrightarrow \forall x \neg(\varphi \to \psi) \vdash \varphi$

$\neg\exists x\psi$

Lemma 3.2.7. (Ren): *If $C(\varphi, x, y)$, then $\forall x\varphi$ and $\forall y S^x_y \varphi$ are syntactical equivalent. That is,*

1. $\forall x\varphi \vdash \forall y S^x_y \varphi$. $\Leftarrow \forall x \varphi \vdash S^x_y \varphi$

2. $\forall y S^x_y \varphi \vdash \forall x\varphi$.

ψ

$\Rightarrow \forall y \psi \vdash \forall x S^y_x \psi$

$S^y_x (S^x_y \varphi)$

Lemma 3.2.8. (RS): *Let η^φ_ψ denote the formula obtained by replacing (some or all) φ inside η by ψ.*

If $\varphi \vdash \psi$ and $\psi \vdash \varphi$ then $\eta \vdash \eta^\varphi_\psi$ and $\eta^\varphi_\psi \vdash \eta$.

Solution. By induction on the structure of η.

Lemma 3.2.9. *If* $C(\varphi, x, y)$ *and* $\Gamma \vdash \psi$, *then* $\Gamma \vdash \psi^{\forall x\varphi}_{\forall y S^x_y \varphi}$.

Solution. An immediate result of (Ren) and (RS).

Theorem 3.2.10. *(GenC) If* $\Gamma \vdash S^x_a\varphi$ *where* a *does not occur in* $\Gamma \cup \{\varphi\}$, *then* $\Gamma \vdash \forall x\varphi$.

$\exists\ \varphi_0\ \varphi_1\ \dots\ \varphi_n = S^x_a\varphi$

Γ_0: set of assumptions used above

$\Gamma_0 \vdash S^x_a\varphi$

Pick y not occurring in Γ_0, φ

Consider: $S^a_y\varphi_0\ S^a_y\varphi_1\ \dots\ S^a_y\varphi_n = (S^a_y S^x_a\varphi) = S^x_y\varphi$

Gen $\Rightarrow$ $\forall y S^x_y\varphi$

3.3 Semantics of FOL

To give semantics of terms/formulas of first order logic, we need an appropriate structure in which interpret the functions and predicates of FOL.

Definition 3.3.1. *A Tarski structure is a pair* $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$, *where:*

- $\mathcal{D}$ *is a non-empty set, called the domain.*

- *For each n-ary function f, we have $\mathcal{I}(f) \in \mathcal{D}^n \to \mathcal{D}$.*
- *For each n-ary predicate P, we have $\mathcal{I}(P) \in \mathcal{D}^n \to \{0, 1\}$.*

$f\, t_1, t_2$

Thus, for each constant a, we have $\mathcal{I}(a) \in \mathcal{D}$.

$\mathcal{I}(+): N^2 \to N$ $\quad + x\, y$

$\mathcal{I}(\cdot)$ . .

$\mathcal{I}(s\cdot): N \to N$

Definition 3.3.2. *Given a Tarski structure $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$, an assignment σ under $\mathscr{I}$ is a mapping $\sigma\colon VS \to \mathcal{D}$.*

We use $\Sigma_{\mathscr{I}}$ to denote the set consisting of assignments under $\mathscr{I}$.

Definition 3.3.3. *Let $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ and $\sigma \in \Sigma_{\mathscr{I}}$.*

Each term t is interpreted to an element $\mathscr{I}(t)(\sigma)$ belonging to $\mathcal{D}$:

- *If $t = x$ is a variable, then $\mathscr{I}(t)(\sigma) = \sigma(x)$.*

- *If* $t = f(t_1, \ldots, t_n)$ *where* f *is an* n*-ary function, then* $\mathscr{I}(t)(\sigma) = \mathcal{I}(f)(\mathscr{I}(t_1)(\sigma), \ldots, \mathscr{I}(t_n)(\sigma))$.

Thus, if $t = a$ *is a constant, then* $\mathscr{I}(t)(\sigma) = \mathcal{I}(a)$.

Definition 3.3.4. *Each formula* φ *has a truth value* $\mathscr{I}(\varphi)(\sigma) \in \{0, 1\}$*:*

- *If* $\varphi = P(t_1, \ldots, t_n)$*, where* P *is an* n*-ary predicate, then* $\mathscr{I}(\varphi)(\sigma) = \mathcal{I}(P)(\mathscr{I}(t_1)(\sigma), \ldots, \mathscr{I}(t_n)(\sigma))$.

- *If* $\varphi = \neg\psi$*, then* $\mathscr{I}(\varphi)(\sigma) = 1 - \mathscr{I}(\psi)(\sigma)$.

- *If* $\varphi = \psi \to \eta$*, then*

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma) = 0 \text{ or } \mathscr{I}(\eta)(\sigma) = 1, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma) = 1 \text{ and } \mathscr{I}(\eta)(\sigma) = 0. \end{cases}$$

- *If* $\varphi = \boxed{\forall x \psi,}$ *then*

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 1 \text{ for each } d \in \mathcal{D}, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 0 \text{ for some } d \in \mathcal{D} \end{cases}$$

$$\varphi = \exists x \psi \begin{cases} 1 & \text{if } \ldots = 1 \text{ for some } d \in D \\ 0 & \text{if } \quad = 0 \text{ for each } d \in D \end{cases}$$

where $\sigma[x/d]$ is a new assignment defined as

$$\sigma[x/d](y) = \begin{cases} \sigma(y) & \text{if } y \neq x, \\ d & \text{if } y = x. \end{cases}$$

We write $(\mathscr{I}, \sigma) \Vdash \varphi$ if $\mathscr{I}(\varphi)(\sigma) = 1$.

• $\boxed{\varphi = \forall y \psi}$ $S^x_t \varphi \equiv \forall y\, S^x_t \psi$

$\forall d \in D: (\mathcal{I}, \sigma[y/d]) \Vdash S^x_t \psi$

IH $\Updownarrow$ $(\mathcal{I}, \sigma[y/d][x/\mathcal{I}(t)\sigma[y/d]]) \Vdash \psi$

$=$

$\forall d \in D: (\mathcal{I}, \sigma[x/\mathcal{I}(t)(\sigma)][y/d]) \Vdash \psi$

Theorem 3.3.5 (Theorem of Substitution). *Suppose that t is substitutable for x within φ, then*

$$(\mathscr{I}, \sigma) \Vdash S^x_t \varphi \text{ if and only if } (\mathscr{I}, \sigma[x/\mathscr{I}(t)(\sigma)]) \Vdash \varphi.$$

• $\varphi = P(t_1 \cdots t_n)$

$S^x_t \varphi = P(S^x_t t_1, \cdots, S^x_t t_n)$

$I(P)(\mathcal{I}(\ )(\sigma) \ \cdots \ ) = 1$

• $\varphi = \neg \psi$ $S^x_t \varphi = \neg(S^x_t \psi)$

We say that $\mathscr{I}$ is a model of φ, denoted as $\mathscr{I} \Vdash \varphi$, if $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\sigma \in \Sigma_{\mathscr{I}}$.

In particular, we say that $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ is a frugal model of φ if $|\mathcal{D}|$ is $\leq$ not more than the cardinality of the language.

Recall that φ is a sentence, if there is no free variable occurring in φ.

Theorem 3.3.6. *If φ is a sentence, then*

- *$\mathscr{I} \Vdash \varphi$ iff $(\mathscr{I}, \sigma) \Vdash \varphi$ for some $\sigma \in \Sigma_{\mathscr{I}}$.*

Definition 3.3.7. *Let φ, ψ be FOL formulas and Γ be a set of FOL formulas. Then we define:*

- *$(\mathscr{I}, \sigma) \Vdash \Gamma$ if for each $\eta \in \Gamma$, $(\mathscr{I}, \sigma) \Vdash \eta$;*
- *$\Gamma \models \varphi$ if for each $\mathscr{I}$ and $\sigma \in \Sigma_{\mathscr{I}}$, $(\mathscr{I}, \sigma) \Vdash \Gamma$ implies $(\mathscr{I}, \sigma) \Vdash \varphi$;*
- *φ and ψ are equivalent if $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$;*
- *φ is valid if $\emptyset \models \varphi$.*

Definition 3.3.8 (Tautology for FOL)**.** *For a formula $\varphi \in FOF$, we construct φ' as follows:*

- *for each sub-formula ψ of φ which is either an atomic formula, or a formula of the form $\forall x \eta$, we replace it with a corresponding propositional variable p_ψ.*

If φ' is a tautology in propositional logic, then we say φ is a tautology for FOL.

Lemma 3.3.9 (Löwenheim–Skolem theorem (1919))**.** *If a set of sentences in a countable first-order language has an infinite model M then it has at least one model of each infinite cardinality greater than $|M|$.*

The original goal of Hilbert was to produce axiomatic theories for *all parts of mathematics.* The Löwenheim–Skolem theorem shows that this goal cannot be established, as one cannot characterize the natural numbers, the real numbers, or any other infinite structure up to *isomorphism*, using a set of first-order axioms.

3.4 A Sound and Complete Axiomatization for FOL without Equality $\approx$ Gödel 1929, Henkin 1949

3.4.1 The Axiom System: Soundness

Similarly to propositional logic, for FOL we have the soundness property:

Theorem 3.4.1. *If* $\Gamma \vdash \varphi$*, then* $\Gamma \models \varphi$*.*

Hint. For proving the theorem, show and make use of the following results: (T, 6)

- $\{\forall x(\varphi \to \psi), \forall x\varphi\} \models \forall x\psi$;
- if x is not free in φ, then $\vdash \varphi \to \forall x\varphi$. □

Corollary 3.4.2. *If* $\vdash \varphi$*, then* $\models \varphi$*.*

3.4.2 The Axiom System: Completeness

A Hintikka set Γ is a set of FOL formulas fulfilling the following properties:

1. For each atomic formula φ (i.e, $\varphi = P(t_1, \ldots, t_n)$, where $n \geq 0$), either $\varphi \notin \Gamma$ or $\neg\varphi \notin \Gamma$.

2. $\varphi \to \psi \in \Gamma$ implies that either $\neg\varphi \in \Gamma$ or $\psi \in \Gamma$.

   $\varphi \wedge \neg\psi$

3. $\neg\neg\varphi \in \Gamma$ implies that $\varphi \in \Gamma$.

4. $\neg(\varphi \to \psi) \in \Gamma$ implies that $\varphi \in \Gamma$ and $\neg\psi \in \Gamma$.

5. $\forall x \varphi \in \Gamma$ implies that $S^x_t \varphi \in \Gamma$ for each t which is substitutable for x within φ.

6. $\exists x \neg\varphi$ $\neg\forall x \varphi \in \Gamma$ implies that there is some t with $C(\varphi, x, t)$ such that $\neg S^x_t \varphi \in \Gamma$.

Note: $C(\varphi, x, t)$ iff $C(\varphi, x, y)$ for all y occurring in t.

Lemma 3.4.3. *A Hintikka set Γ is consistent, and moreover, for each formula φ, either $\varphi \notin \Gamma$, or $\neg\varphi \notin \Gamma$.*

Theorem 3.4.4. *A Hintikka set Γ is satisfiable, i.e, there is some interpretation $\mathscr{I}$ and some $\sigma \in \Sigma_{\mathscr{I}}$ such that $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\varphi \in \Gamma$.*

Theorem 3.4.5. *If Γ is a set of FOL formulas, then "Γ is consistent" implies that "Γ is satisfiable".*

Particularly, if Γ consists only of sentences, then Γ has a frugal model.

Proof. Let us enumerate[1] the formulas as $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots,$ and subsequently define a series of formula sets as follows. Let $\Gamma_0 = \Gamma$, and

$\exists x \neg\psi$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\neg\varphi_i\} & \text{if } \Gamma_i \vdash \neg\varphi_i \\ \Gamma_i \cup \{\varphi_i\} & \text{if } \Gamma_i \nvdash \neg\varphi_i \text{ and } \varphi_i \neq \neg\forall x\psi \\ \Gamma_i \cup \{\varphi_i, \neg S_a^x \psi\} & \text{if } \Gamma_i \nvdash \neg\varphi_i, \text{ and } \varphi_i = \neg\forall x\psi \end{cases}$$

Above, for each formula $\forall x\psi$, we pick and fix the constant a which does not occur in $\Gamma_i \cup \{\varphi_i\}$. Finally let $\Gamma^* = \lim\limits_{i\to\infty} \Gamma_i$.

$\exists x (x > 5)$

$(I, \sigma) \Vdash \neg\forall x \psi$

[1]We assume the language to be countable, yet the result can be extended to languages with arbitrary cardinality.

D: set of terms

$I(f)(t_1 \cdots t_n) = f t_1 \cdots t_n$

$I(P)(t_1 \cdots t_n) = 1$ iff $P t_1 t_2 \cdots t_n \in \Gamma^*$

$\sigma(x) = x$

If Γ is consistent, the set Γ^* is maximal and consistent, and is referred to as the Henkin set. Thus, a Henkin set is also a Hintikka set. $\square$

Theorem 3.4.6. *If* $\Gamma \models \varphi$, *then* $\Gamma \vdash \varphi$.

Corollary 3.4.7. *If* $\models \varphi$, *then* $\vdash \varphi$.

Theorem 3.4.8. Γ *is consistent iff each of its finite subset is consistent. Moreover,* Γ *is satisfiable iff each of its finite subsets is satisfiable.*

3.5 A Sound and Complete Axiomatization for FOL with Equality $\approx$

The axiomatization based on the Hilbert's systems seen in the previous section can be extended to the case of first order logic with the equality $\approx$. To do this, two additional axioms have to be included in the Hilbert's system:

$A_{\approx}$: $x \approx x$;

$A'_{\approx}$: $(x \approx y) \rightarrow (\alpha \rightarrow \alpha^x_y)$, where α is an atomic formula.

The soundness and completeness results can be proved similarly in the extended Hilbert's system; note that for the completeness one, a variation of the Tarski structure is required, namely, the domain considered in the construction modulo the relation $\approx$. This allows us so manage correctly the formulas that are equivalent under $\approx$.

The actual details about the above construction are omitted; the interested reader is invited to formalize them.