Relation, I

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Outline

1. Definitions
2. Relational database
3. Representations of relations
4. Exercises

General view

- Relation is a concept basic to all the sciences
- Relations as mathematical objects can be mathematically studied
- Mathematical definition of relations is the model for the study of relations in other disciplines
- There are wide applications, especially relational database for CS
  Question Is this sufficient for big data?
- New directions: discovering and predicting links from data, that are noisy, structured or unstructured, big data

Binary Relation

Definition 1

Let A, B be two sets. A **binary relation** from A to B is a subset of $A \times B$, i.e.

$$R \subseteq A \times B.$$

Remark

- Relation R is a set of **ordered pairs** (a, b) for $a \in A$ and $b \in B$.
- Relation is **directed**.

Relations as Extensions of Functions

Given two sets A, B and a binary relation R from A to B, we define a *partial function*

$$f_R : A \to B \tag{1}$$

as follows:
For every $a \in R$, if there is a $b \in B$ such that $(a, b) \in R$, then choose a unique such b and define $f_R(a) = b$.
Then,

(i) R is an extension of f_R.

(ii) f_R is a partial function from A to B.

We thus call f_R a *selection function* of R.

Intuition and background

Therefore, relations extend functions or maps in two ways:

1) Partial functions are allowed
2) One element may link to many elements

Why?

- Reasonable
- The extensions allow wider applications in all areas of sciences

Relations of Functions

Given two sets A, B, let $f : A \to B$ be a function from A to B. Define the **relation of function f** as follows:

$$R_f = \{(a, f(a)) \mid a \in A\}. \tag{2}$$

Relations on Sets

Definition 2

Let A be a set. A relation on A is a relation from A to A, i.e.,

$$R \subseteq A \times A.$$

Reflexivity

Definition 3

A relation R on a set A is called **reflexive**, if for every $a \in A$, $(a, a) \in R$ holds.

Examples:

- $\leq$ for numbers, $<$ is not
- $$(a, b) \in R \iff a|b.$$

Symmetric relations

Definition 4

1. A relation R on a set A is called **symmetric**, if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
2. A relation R on a set A is called **antisymmetric** if for all $a, b, c \in A$,

$$[(a, b) \in R \,\&\, (b, a) \in R] \Rightarrow a = b.$$

Transitivity

Definition 5

A relation R on a set A is called **transitive**, if for all $a, b, c \in A$,

$$[(a, b) \in R \ \& \ (b, c) \in R] \Rightarrow (a, c) \in R.$$

Operations of Relations as Sets

Relations are sets, for which the usual operations of sets apply, including, for instance:

- $R_1 \cup R_2$
- $R_1 \cap R_2$
- $\bar{R}$
- $R_1 \setminus R_2$
- $R_1 \oplus R_2$

Composition of Relations

Definition 6

Let A, B, and C be three sets. $R \subseteq A \times B$, $S \subseteq B \times C$ be two relations. Define the **composition** of R and S by:

$$S \circ R = \{(a, c) \mid (\exists b \in B)[(a, b) \in R, \ \& \ (b, c) \in S]\}. \tag{3}$$

Power of Relations

Definition 7

Let R be a relation on a set A. Define the power of relations by:

$$\begin{aligned} R^1 &= R, \\ R^{n+1} &= R^n \circ R. \end{aligned}$$

Transitivity Characterisation

Theorem 8

Let A be a set and R be a relation on A. Then, R is transitive if and only if $R^2 \subseteq R$.

Trivial.

n-ary Relations

Definition 9

Let $A_1, A_2, \cdots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$, i.e.,

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n. \tag{4}$$

The sets $A_1, A_2, \cdots, A_n$ are called the *domains* of R and n is called the *dimension* (degree, in the textbook) of R.

Database

A database consists of records, which are n-tuples, made up of fields. The fields are the entries of the n-tuples.
Example:
A database of the form:

(Name, ID number, Age, Sex, Profession)

The relations are represented by a Table, in which

- the entries are the *attributes*
- Some entries are unique to an n-tuple, and called **primary key**

Relational Data Model

A database that are expressed by relations supports the following operations for data:

- Inserting records
- Deleting records
- Updating records
- Searching a record
- Combining records
- Classifying records, etc

Selection Operator

Given an n-ary relation $R \subseteq A_1 \times A_2 \times \cdots \times A_n$ and a condition C, the **selection operator** $S_C(R)$ finds the set of all the n-tuples in R that satisfy the condition C.

Projection Operator

Definition 10

The **projection operator** $P_{i_1 \cdots i_k}$ where $1 \le i_1 < \cdots i_k \le n$, maps the n-tuple $(a_1, a_2, \cdots, a_n)$ to $(a_{i_1}, \cdots, a_{i_k})$.

The Join Operator

Definition 11

Let R and S be m- and n-dimensional relations, respectively. For $p \leq m, n$, the *join operator* $J_p(R, S)$ is defined as follows:

$$(a_1, \cdots, a_{m-p}, c_1, \cdots, c_p, b_1, \cdots, b_{n-p}) \in J_p(R, S). \qquad (5)$$

if:

$$(a_1, \cdots, a_{m-p}, c_1, \cdots, c_p) \in R,$$

and

$$(c_1, \cdots, c_p, b_1, \cdots, b_{n-p}) \in S.$$

More materials: Database theory

New Challenge

Real-world data are dynamically evolving, high-dimensional and noisy. That is, for an item of the form

$$(a_1, a_2, \cdots, a_n)$$

it is possible that

- a_i's are noisy
- a_i's are dynamically evolving
- n is large

The classic database theory apparently fails to support the analysis of the data.
This urgently calls for a new database theory that supports real-world data analysis.

Remarks

- Binary relations
- List of ordered pairs
- Table
- Why new representations?

Matrix Representation

Let A, B be finite sets and R be a relation from A to B. Suppose that A, B are ordered as they are listed as follows:

$$A = \{a_1, a_2, \cdots, a_m\}$$

$$B = \{b_1, b_2, \cdots, b_n\}.$$

The *matrix representation* of R, written M_R, is defined by a matrix with entry m_{ij} given by

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R, \\ 0, & \text{o.w.} \end{cases} \tag{6}$$

Matrices of Relations on a Set

Theorem 12

Given a set A and a relation R on A, let M_R be the matrix representation of R. Then,

(1) *R is symmetric if and only if M_R is a symmetric matrix.*

(2) *If R is an antisymmetric relation, then for any $i \neq j$,*

$$m_{ij} \cdot m_{ji} = 0.$$

Operations

Theorem 13

Let R_1, R_2 be two relations on a set A. Then,

1. $$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}.$$
2. $$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}.$$

Composition

Theorem 14

Given sets A, B and C, and relations $R \subseteq A \times B$ and $S \subseteq B \times C$, we have

$$M_{S \circ R} = M_R \odot M_S.$$

Proof.

Let $M_{S \circ R} = (t_{ij})$, $M_R = (r_{ij})$ and $M_S = (s_{ij})$.
Then,
$t_{ij} = 1$ is and only if there is a k such that $r_{ik} = s_{kj} = 1$. $\square$

Questions

1. What is the algebraic theory of relations?
2. What is the algebraic representation of high-dimensional relations?

Directed Graphs

Definition 15

A directed graph or digraph is a pair $G = (V, E)$ of vertices V and directed edges E.
For a pair $(a, b) \in E$,
– we assume there is a direction from a to b
– a is called initial vertex
– b is called terminal vertex.

A binary relation $R \subseteq A \times B$ is naturally represented by a directed bipartite graph $G = (A, B, E)$ where $(a, b) \in E$ if $(a, b) \in R$, i.e., $E = R$. Therefore,

Binary relations $=$ Directed bipartite graphs

Questions

- What is the theory of binary relations from the graphic approach study?
- Again, what is the graphic representation for high-dimensional relations?

Exercises

1. How many relations are there on a set with n elements that are
   1.1 symmetric?
   1.2 antisymmetric?
   1.3 asymmetric?
   1.4 irreflexive?
   1.5 reflexive and symmetric?
   1.6 neither reflexive n irreflexive?
2. Given the directed graph representations of two relations, how can the directed graph of the union, intersection, symmetric difference, difference and composition of these relations be found?

谢谢！