Chapter 2

Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

2.1 Sets and Functions

Definition 2.1.1.

- *Fix an universal set U. Set operations: union $\cup$, intersection $\cap$, complement $\overline{A}$.*
- *Set inclusion: $A \subseteq B$ iff for all $a \in A$ it holds $a \in B$. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.*
- *Given a set S, the* power set *of S is the set of all subsets of the set S. The power set is denoted by $\mathcal{P}(S)$, or 2^S.*
- *The* Cartesian product *of sets $A_1, A_2, \ldots, A_n$ is defined by: $A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}$.*
- *The* cardinality *of finite set A, denoted by $|A|$, is the number of its elements. The* principle of inclusion-exclusion*:*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Definition 2.1.2. *Let A and B be nonempty sets. A function $f\colon A \to B$ from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if $b \in B$ is assigned by f to the element $a \in A$. We say that*

- *A is the* domain *of f,*
- *B is the* codomain *of f.*
- *If $f(a) = b$, we say that b is the* image *of a and a is a* preimage *of b.*
- *The* range, *or image, of f is the set of all images of elements of A.*

Definition 2.1.3. *Let A and B be two sets. The function $f\colon A \to B$ is called*

- one-to-one, *or an* injunction, *if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.*
- onto, *or a* surjection, *if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.*
- one-to-one correspondence, *or a* bijection, *if it is both one-to-one and onto.*

Definition 2.1.4. *Let A, B, and C be three sets.*

- *Let $f\colon A \to B$ be bijective. The* inverse function *of f, denoted by f^{-1}, is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.*
- *Let $g\colon A \to B$ and let $f\colon B \to C$. The* composition *of the functions f and g, denoted $f \circ g$, is defined by*

$$(f \circ g)(a) = f(g(a))$$

Definition 2.1.5 (Some Notations)**.** *Let A and B be two sets.*

- *For a function $f\colon A \to B$, and a set $D \subseteq A$, we use $f|_D\colon D \to B$ to denote the function f with domain restricted to the set D.*
- *A* partial function *f from a set A to a set B is an assignment to each element $a \in D \subseteq A$, called the* domain of definition *of f, of a unique element $b \in B$. We say that f is undefined for elements in $A \setminus D$. When $D = A$, we say that f is a total function.*

Definition 2.1.6. *Consider the set $U = 2^{AP}$ of all assignments. The semantic bracket is a function $[\![\,\cdot\,]\!]\colon WFF \to 2^U$ defined by:*

- $[\![p]\!] = \{\sigma \in U \mid p \in \sigma\}$,
- $[\![\neg\varphi]\!] = \overline{[\![\varphi]\!]}$,
- $[\![\varphi \to \psi]\!] = \overline{[\![\varphi]\!]} \cup [\![\psi]\!]$.

Is $[\![\,\cdot\,]\!]$ injective, surjective, or bijective?

2.2 Cardinality, Diagonalization Argument

Definition 2.2.1. *Let A and B be two sets.*

- *The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.*
- *If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.*

Definition 2.2.2. *A set that is either finite or has the same cardinality as the set of positive integers is called* countable. *A set that is not countable is called* uncountable. *When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$. We write $|S| = \aleph_0$ and say that S has cardinality aleph null.*

Lemma 2.2.3.

- *If $A \subseteq B$, then $|A| \leq |B|$.*
- *If $A \subset B$, then $|A| \leq |B|$.*

Assuming $B \subset A$, can it be the case that $|A| \leq |B|$?

Theorem 2.2.4 (SCHRÖDER-BERNSTEIN THEOREM)**.** *If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Lemma 2.2.5. *If $|A| = |B|$, and $|A| \leq |C| \leq |B|$, then $|A| = |C|$.*

Lemma 2.2.6. *Prove that*

1. *The union, intersection of countable sets is countable.*
2. *The set $\mathbb{Z}$ of integer numbers is countable.*
3. *The set $\mathbb{N}^2$ is countable.*
4. *The set $\mathbb{Q}$ of rational numbers is countable.*
5. *The set $\mathbb{N}^c$ with $c \in \mathbb{N}$ is countable.*
6. *The countable union of countable sets is countable.*
7. *The set $\mathbb{N}^*$ is countable.*

For a set Σ, define Σ^ω the set of infinite strings $\{a_0a_1a_2\ldots \mid a_i \in \Sigma, i \in \mathbb{N}\}$.

Lemma 2.2.7. *Prove that*

1. $|[0,1]| = |(0,1]| = |[0,1)| = |(0,1)|$.
2. $|(0,1]| = |[1,\infty)|$.
3. $|[0,1]| = |[0,k]| = |[0,\infty)| = |\mathbb{R}|$.
4. $|2^{\mathbb{N}}| = |\{\ f \mid f\colon \mathbb{N} \to \{0,1\}\ \}|$.
5. $|\{0,1\}^{\omega}| = |[0,1]|$.
6. $|2^{\mathbb{N}}| = |\{0,1\}^{\omega}|$.

Lemma 2.2.8 (Cantor diagonalization argument)**.**

- *The set $\mathbb{R}$ of real numbers is uncountable.*
- *For a set A, it holds: $|A| < |2^{A}|$.*