Chapter 3

First Order Logic (FOL)

3.1 Syntax of FOL

Propositional logic is a coarse language, which only concerns about propositions and boolean connectives. Practically, this logic is not powerful enough to describe important properties we are interested in.

Example 3.1.1 (Syllogism of Aristotle)**.** *Consider the following assertions:*

1. *All men are mortal.*
2. *Socrates is a man.*
3. *So Socrates would die.*

$$\forall x(Man(x) \to Mortal(x))$$

Definition 3.1.2. *First order logic is an extension of proposition logic:*

1. *To accept parameters, it generalizes propositions to predicates.* $P(x,y)$ $<xy$
2. *To designate elements in the domain, it is equipped with functions and constants.* $\boxed{f(x,y) \quad +xy \quad \cdot xy}$
3. *It also involves quantifiers to capture infinite conjunction and disjunction.* $\forall$ 全称 $\exists$ 存在

Definition 3.1.3. *We are given:*

- *an arbitrary set of variable symbols* $VS = \{x, y, x_1, \ldots\}$*;*
- *an arbitrary set (maybe empty) of function symbols* $FS = \{f, g, f_1, \ldots\}$*, where each symbol has an arity;*
- *an arbitrary set (maybe empty) of predicate symbols* $PS = \{P, Q, P_1, \ldots\}$*, where each symbol has an arity;*
- *an equality symbol set* ES *which is either empty or one element set containing* $\{\approx\}$*.*

Let $L = VS \cup \{(,), \to, \neg, \forall\} \cup FS \cup PS \cup ES$. *Here* $\underbrace{VS \cup \{(,), \to, \neg, \forall\}}$ *are referred to as* logical symbols*, and* $FS \cup PS \cup ES$ *are referred to as* non-logical symbols.

We often make use of the

- *set of constant symbols, denoted by* $CS = \{a, b, a_1, \ldots\} \subseteq FS$*, which consist of function symbols with arity* 0*;*

- *set of propositional symbols, denoted by* $PS = \{p, q, p_1, \ldots\} \subseteq FS$*, which consist of predicate symbols with arity* 0*.*

Definition 3.1.4 (FOL terms)**.** *The terms of the first order logic are constructed according to the following grammar:*

$$t ::= x \mid f t_1 \ldots t_n$$

where $x \in VS$*, and* $f \in FS$ *has arity* n*.*

Accordingly, the set T *of terms is the smallest set satisfying the following conditions:*

- *each variable $x \in VS$ is a term.*
- *Compound terms: $ft_1 \ldots t_n$ is a term (thus in T), provided that f is a n-arity function symbol, and $t_1, \ldots, t_n \in T$. Particularly, $a \in CS$ is a term.*

We often write $f(t_1, \ldots, t_n)$ for the compound terms.

$h\,(g\,t_1)\,(h\,t_2 t_3)$

Definition 3.1.5 (FOL formulas)**.** *The well-formed formulas of the first order logic are constructed according to the following grammar:*

$$\varphi ::= Pt_1 \ldots t_n \mid \neg\varphi \mid \varphi \to \varphi \mid \forall x \varphi$$

where $t_1, \ldots, t_n$ are terms, $P \in PS$ has arity n, and $x \in VS$.

We often write $P(t_1, \ldots, t_n)$ for clarity. Accordingly, the set FOF of first order formulas is the smallest set satisfying:

- *$P(t_1, \ldots, t_n) \in FOF$ is a formula, referred to as the atomic formula.*

$\text{PassDM}(\ ) \equiv \text{PassExam}(\ ) \wedge \text{PassE}(\ )$

$\text{PassExam}(\cdot)$ 1-arity

$\text{PassExer}(\cdot)$ ..

$\text{ShowInClass}(\ )$ ..

$\forall x(\neg \text{ShowIn}(x) \to \neg \text{PassDM}(x))$

$\text{Copy}(x, y)$ 2-arity

$\forall x \exists y\, (\text{Copy}(x,y) \to \neg \text{PassDM}(x) \wedge \neg \text{PassDM}(y))$

$\forall x \forall y$

$\text{SolutionEx}(\cdot)$

$\text{copy}(x,y) \leftrightarrow (\text{SolutE}(x) \approx \text{SolutE}(y))$

0 – 0 arity function

$S(\cdot)$ – 1 arity function

$Sx \not\approx 0$

$Sx \approx Sy \to x \approx y$

$+)$ $x + 0 \approx x$
$x + Sy \approx S(x+y)$

$\cdot)$ $x \cdot 0 \approx 0$
$x \cdot Sy \approx x + x \cdot y$

$\cdot)\ \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\, \varphi(x)$

- *Compound formulas:* $(\neg\varphi)$ *(negation),* $(\varphi \to \psi)$ *(implication), and* $(\forall x\varphi)$ *(universal quantification) are formulas (thus in FOF), provided that* $\varphi, \psi \in FOF$.

We omit parentheses if it is clear from the context.

As syntactic sugar, we can define $\exists x\varphi$ *as* $\exists x\varphi := \neg\forall x\neg\varphi$. *We assume that* $\forall$ *and* $\exists$ *have higher precedence than all logical operators.*

Definition 3.1.6 (Sub-formulas)**.** *For a formula* φ*, we define the sub-formula function* $\mathit{Sf} : FOF \to 2^{FOF}$ *as follows:*

$$
\begin{aligned}
\mathit{Sf}(P(t_1, \dots, t_n)) &= \{P(t_1, \dots, t_n)\}\\
\mathit{Sf}(\neg\varphi) &= \{\neg\varphi\} \cup \mathit{Sf}(\varphi)\\
\mathit{Sf}(\varphi \to \psi) &= \{\varphi \to \psi\} \cup \mathit{Sf}(\varphi) \cup \mathit{Sf}(\psi)\\
\mathit{Sf}(\forall x\varphi) &= \{\forall x\varphi\} \cup \mathit{Sf}(\varphi)\\
\mathit{Sf}(\exists x\varphi) &= \{\exists x\varphi\} \cup \mathit{Sf}(\varphi)
\end{aligned}
$$

辖

Definition 3.1.7 (Scope)**.** *The part of a logical expression to which a quantifier is applied is called the scope of this quantifier. Formally, each subformula of the form $Qx\psi \in Sf(\varphi)$, the scope of the corresponding quantifier Qx is ψ. Here $Q \in \{\forall, \exists\}$.*

$\forall x\ \varphi$

$\forall y \forall x (x \wedge y)\ y \quad \forall x (\ ) \quad \forall y (\ )\ x$

Substitution for Terms

Definition 3.1.8 (Sentence)**.** *We say an occurrence of x in φ is free if it* 自由 *is not in scope of any quantifiers $\forall x$ (or $\exists x$). Otherwise, we say that this occurrence is a bound occurrence. If a ~~variable~~ formula φ has no free variables, it is called a* closed formula, *or a* sentence.

$\forall x (x > 1 + y) \rightarrow\ > x(1+y)$

free

Definition 3.1.9 (Substitution)**.** *The substitution of x with t within φ, denoted as $S^x_t\varphi$, is obtained from φ by* *replacing each free occurrence of x with t.*

$\varphi \quad (\forall x.\, \neg x) \wedge \forall y(y \wedge x)$

$S^x_y \varphi$

We would extend this notation to $S^{x_1,\dots,x_n}_{t_1,\dots,t_n}\varphi$.

Remark 3.1.10. *It is important to remark that $\boxed{S^{x_1,\dots,x_n}_{t_1,\dots,t_n}\varphi}$ is not the same as $S^{x_1}_{t_1}\dots S^{x_n}_{t_n}\varphi$: the former performs a simultaneous substitution.*

For example, consider the formula $P(x,y)$: the subsitution $\boxed{S^{x,y}_{y,x}P(x,y)}$ gives $S^{x,y}_{y,x}P(x,y) = P(y,x)$ while the substitutions $S^x_y S^y_x P(x,y)$ give $S^x_y \boxed{S^y_x P(x,y)} = S^x_y P(x,x) = P(y,y)$.

Remark 3.1.11. *Consider $\varphi = \exists y(x < y)$ in the number theory. What is $S^x_t\varphi$ for the special case of $t = y$?*

Definition 3.1.12 (Substitutable on Terms)**.** *We say that t is substitutable for x within φ iff for each variable y*

$\forall y(\exists x(t \cdots)) \quad \exists y(\ ) \quad y \quad x$

no free x!

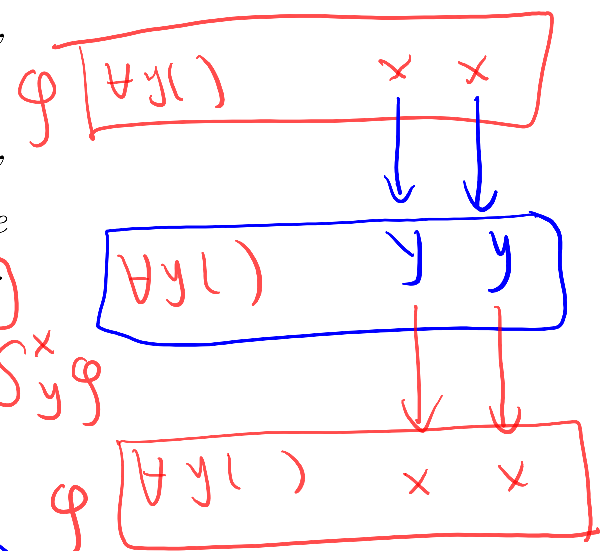
occurring in t, there is no free occurrence of x in scope of $\forall y/\exists y$ in φ.

Definition 3.1.13 (α-β condition)**.** *If the formula φ and the variables x and y fulfill:*

1. *y has no free occurrence in φ, and*

2. *y is substitutable for x within φ,*

then we say that φ, x and y meet the α-β condition*, denoted as $C(\varphi, x, y)$.*



Lemma 3.1.14. *If $C(\varphi, x, y)$, then $S^y_x S^x_y \varphi = \varphi$.*

3.2 The Axiom System: the Hilbert's System

As for propositional logic, also FOL can be axiomatized.

Definition 3.2.1 (Axioms). *1.* $\varphi \to (\psi \to \varphi)$

2. $(\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))$

3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

4. $\forall x\varphi \to S^x_t\varphi$

if t is substitutable for x within φ

5. $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$

6. $\varphi \to \forall x\varphi$

if x is not free in φ

7. $\forall x_1 \ldots \forall x_n\varphi$ $\qquad Sx = Sy \to x = y$

if φ is an instance of (one of) the above axioms

MP Rule: $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$

Definition 3.2.2 (Syntactical Equivalence)**.** *We say φ and ψ are syntactically equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$.*

Theorem 3.2.3. *(Gen): If x has no free occurrence in Γ, then $\Gamma \vdash \varphi$ implies $\Gamma \vdash \forall x \varphi$.*

Solution. Suppose that $\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$ is the deductive sequence of φ from Γ.

- If φ_i is an instance of some axiom, then according to (AS7), $\forall x \varphi_i$ is also an axiom.

- If $\varphi_i \in \Gamma$, since x is not free in Γ, we have $\vdash \varphi_i \to \forall x \varphi_i$ according to (AS6). Therefore, we have $\Gamma \vdash \forall x \varphi_i$ in this case.

- If φ_i is obtained by applying (M-P) to some φ_j and $\varphi_k = \varphi_j \to \varphi_i$. By induction, we have $\Gamma \vdash \forall x \varphi_j$ and $\Gamma \vdash \forall x(\varphi_j \to \varphi_i)$. With (AS5) and (MP), we also have $\Gamma \vdash \forall x \varphi_i$ in this case.

Thus, we have $\Gamma \vdash \forall x \varphi_n$, i.e., $\Gamma \vdash \forall x \varphi$.

Exercise 3.2.4. *Prove that*

1. $\forall x \forall y \varphi \vdash \forall y \forall x \varphi$,
2. $\exists x \forall y \varphi \vdash \forall y \exists x \varphi$.

Exercise 3.2.5. *Prove that*

1. $\forall x(\varphi \to \psi) \vdash \forall x(\neg\psi \to \neg\varphi)$,
2. $\forall x(\varphi \to \psi) \vdash \exists x \varphi \to \exists x \psi$.

Exercise 3.2.6. *Prove that*

1. *If* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \neg\psi$, *then* $\Gamma \vdash \neg(\varphi \to \psi)$,

2. $\forall x \neg(\varphi \to \psi) \vdash \neg(\varphi \to \exists x \psi)$.

Lemma 3.2.7. *(Ren): If $C(\varphi, x, y)$, then $\forall x \varphi$ and $\forall y S_y^x \varphi$ are syntactical equivalent. That is,*

1. $\forall x \varphi \vdash \forall y S_y^x \varphi$.

2. $\forall y S_y^x \varphi \vdash \forall x \varphi$.

Lemma 3.2.8. *(RS): Let η_ψ^φ denote the formula obtained by replacing (some or all) φ inside η by ψ.*

If $\varphi \vdash \psi$ and $\psi \vdash \varphi$ then $\eta \vdash \eta_\psi^\varphi$ and $\eta_\psi^\varphi \vdash \eta$.

Solution. By induction on the structure of η.

Lemma 3.2.9. *If* $C(\varphi, x, y)$ *and* $\Gamma \vdash \psi$, *then* $\Gamma \vdash \psi^{\forall x\varphi}_{\forall y S^x_y \varphi}$.

Solution. An immediate result of (Ren) and (RS).

Theorem 3.2.10. *(GenC) If* $\Gamma \vdash S^x_a \varphi$ *where* a *does not occur in* $\Gamma \cup \{\varphi\}$, *then* $\Gamma \vdash \forall x \varphi$.

3.3 Semantics of FOL

To give semantics of terms/formulas of first order logic, we need an appropriate structure in which interpret the functions and predicates of FOL.

Definition 3.3.1. *A Tarski structure is a pair* $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$, *where:*
设

- $\mathcal{D}$ *is a non-empty set, called the domain.*

$+$

$+\ 3\ 5 = 8$

- *For each n-ary function f, we have $\mathcal{I}(f) \in \mathcal{D}^n \to \mathcal{D}$.*
- *For each n-ary predicate P, we have $\mathcal{I}(P) \in \mathcal{D}^n \to \{0, 1\}$.*

Thus, for each constant a, we have $\mathcal{I}(a) \in \mathcal{D}$.

Definition 3.3.2. *Given a Tarski structure $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$, an assignment σ under $\mathscr{I}$ is a mapping $\sigma\colon VS \to \mathcal{D}$.*

We use $\Sigma_{\mathscr{I}}$ to denote the set consisting of assignments under $\mathscr{I}$.

$\sigma(x)$: the value of x

Definition 3.3.3. *Let $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ and $\sigma \in \Sigma_{\mathscr{I}}$.*

$x+y$

Each term t is interpreted to an element $\mathscr{I}(t)(\sigma)$ belonging to $\mathcal{D}$:

- *If $t = x$ is a variable, then $\mathscr{I}(t)(\sigma) = \sigma(x)$.*

$f(t_1, t_2, t_3)$

$+ x\ y$

- *If* $t = f(t_1, \dots, t_n)$ *where* f *is an* n*-ary function, then* $\mathscr{I}(t)(\sigma) = \mathcal{I}(f)(\mathscr{I}(t_1)(\sigma), \dots, \mathscr{I}(t_n)(\sigma))$.

Thus, if $t = a$ *is a constant, then* $\mathscr{I}(t)(\sigma) = \mathcal{I}(a)$.

Definition 3.3.4. *Each formula* φ *has a truth value* $\mathscr{I}(\varphi)(\sigma) \in \{0, 1\}$*:*

- *If* $\varphi = P(t_1, \dots, t_n)$*, where* P *is an* n*-ary predicate, then* $\mathscr{I}(\varphi)(\sigma) = \mathcal{I}(P)(\mathscr{I}(t_1)(\sigma), \dots, \mathscr{I}(t_n)(\sigma))$.
- *If* $\varphi = \neg\psi$*, then* $\mathscr{I}(\varphi)(\sigma) = 1 - \mathscr{I}(\psi)(\sigma)$.
- *If* $\varphi = \psi \to \eta$*, then*

$\neg\psi \quad \vee \quad \eta$

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma) = 0 \text{ or } \mathscr{I}(\eta)(\sigma) = 1, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma) = 1 \text{ and } \mathscr{I}(\eta)(\sigma) = 0. \end{cases}$$

- *If* $\varphi = \forall x \psi$*, then*

$$\mathscr{I}(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 1 \text{ for each } d \in \mathcal{D}, \\ 0 & \text{if } \mathscr{I}(\psi)(\sigma[x/d]) = 0 \text{ for some } d \in \mathcal{D} \end{cases}$$

$\forall x (Sx \neq 0)$

where $\sigma[x/d]$ *is a new assignmen-t defined as*

$$\sigma[x/d](y) = \begin{cases} \sigma(y) & \textit{if } y \neq x, \\ d & \textit{if } y = x. \end{cases}$$

We write $(\mathscr{I}, \sigma) \Vdash \varphi$ *if* $\mathscr{I}(\varphi)(\sigma) = 1$.

Theorem 3.3.5 (Theorem of Substitution). *Suppose that* t *is substitutable for* x *within* φ, *then*

$(\mathscr{I}, \sigma) \Vdash S^x_t\varphi$ *if and only if* $(\mathscr{I}, \sigma[x/\mathscr{I}(t)(\sigma)]) \Vdash \varphi$.

We say that $\mathscr{I}$ is a model of φ, denoted as $\mathscr{I} \Vdash \varphi$, if $(\mathscr{I}, \sigma) \Vdash \varphi$ for each $\sigma \in \Sigma_{\mathscr{I}}$.

In particular, we say that $\mathscr{I} = \langle \mathcal{D}, \mathcal{I} \rangle$ is a frugal model of φ if $|\mathcal{D}|$ is not more than the cardinality of the language.

Recall that φ is a sentence, if there is no free variable occurring in φ.

Theorem 3.3.6. *If φ is a sentence, then*

- $\mathscr{I} \Vdash \varphi$ *iff* $(\mathscr{I}, \sigma) \Vdash \varphi$ *for some* $\sigma \in \Sigma_{\mathscr{I}}$.

$\forall y \underline{\forall x (x < y)}$

Definition 3.3.7. *Let φ, ψ be FOL formulas and Γ be a set of FOL formulas. Then we define:*

- $(\mathscr{I}, \sigma) \Vdash \Gamma$ *if for each* $\eta \in \Gamma$, $(\mathscr{I}, \sigma) \Vdash \eta$;
- $\Gamma \models \varphi$ *if for each* $\mathscr{I}$ *and* $\sigma \in \Sigma_{\mathscr{I}}$, $(\mathscr{I}, \sigma) \Vdash \Gamma$ *implies* $(\mathscr{I}, \sigma) \Vdash \varphi$;
- φ *and* ψ *are equivalent if* $\{\varphi\} \models \psi$ *and* $\{\psi\} \models \varphi$;
- φ *is valid if* $\emptyset \models \varphi$.