Graphs, IV

李昂生

Discrete Mathematics
U CAS
June, 2018

Outline

1. Euler path
2. Hamilton path
3. Shortest paths
4. Planar graphs
5. Colouring
6. Research projects

General view

- Finding paths is fundamental to all graph algorithms
- Designing maps
- Network algorithms

The Questions

1) (**Euler**) Can we travel along the edges of a graph starting at a vertex and returning to it by **traversing each edge of the graph exactly once**?
2) (**Hamilton**) Can we travel along the edges of a graph starting at a vertex and returning to it while **visiting each vertex of the graph exactly once**?

Euler Circuit

Definition 1

An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

Theorem 2

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Necessity

Let $G = (V, E)$ be a multigraph that has an Euler circuit.
Let a be the starting vertex and $\mathcal{E}$ be an Euler circuit starting from a and arriving at a.
Assume that

$$\mathcal{E}: \ a = x_0, x_1, \cdots, x_{m-1}, x_0 = a,$$

where (x_i, x_{i+1}) is an edge and $\{(x_i, x_{i+1}) \mid i = 0, 1, \cdots, m-1\}$ is the set of edges E.
Note that a vertex $v \in V$ may occur several times in $\mathcal{E}$.
For every vertex $v \neq a$, for $v = x_i$ for some $i \neq 0, m$, then the edges (x_{i-1}, x_i) and (x_i, x_{i+1}) contribute 2 to the degree $d(v)$ of v.
For $v = a$, then if $a = x_i$ for $i \neq 0, m$, the edges (x_{i-1}, x_i) and (x_i, x_{i+1}) contribute 2 to $d(a)$, and the edges (x_0, x_1) and (x_{m-1}, x_m) contribute 2 to $d(a)$.
Therefore, for every vertex x, the degree $d(x)$ of x in G is even.

Sufficiency

Let $G = (V, E)$ be a multigraph such that for every $v \in V$, the degree $d(v)$ of v in G is even.
Fix $a \in V$.
Let H be a cycle

$$C: \; a = x_0, x_1, \cdots, x_{L-1}, x_L = a.$$

If $\{(x_i, x_{i+1}) \mid 0 \le i < L\} = E$, then H is an Euler circuit of G.
If H is not an Euler circuit of G, let I be the largest connected component of the graph obtained from G by deleting all the edges in H.
Let $H = (V_H, E_H)$, $I = (V_I, E_I)$.
Since G is connected, $V_H \cap V_I \neq \emptyset$.
Let $b \in V_H \cap V_I$. Assume $b = x_i$ for some i.
By the assumption, for every $y \in V_I$, the degree of y in I is even. Let Q be a path from b to b in I.

Sufficiency - continued

Assume

$$Q: \; b = y_0, y_1, \cdots, y_{M-1}, y_M = b.$$

Let P be the following expanded path

$$P: \; a = x_0, x_1, \cdots, x_{i-1}, x_i, y_1, \cdots, y_M, x_{i+1}, \cdots, x_L.$$

- Set

$$H \leftarrow P$$

- repeat the process above.

Euler Path

Theorem 3

A connected multigraph G has an Euler path not an Euler circuit if and only if G has exactly two vertices of odd degrees.

(For $\Rightarrow$): Trivial.
(For $\Leftarrow$): Let a, b be the two vertices of odd degree.
Let P be a path starting from a and arriving at b in G.
If P is an Euler path, the result holds.
Otherwise. Then let I be the largest connected component of the graph obtained from G by deleting the edges in P.
Then every vertex in $I = (V_I, E_I)$ has even degree, and $V_I \cap V_P \neq \emptyset$.
Find an Euler circuit of V_I and combine P with the Euler circuit.

Chinese Postman Problem

管梅谷, 1962

Definition 4

Given a graph $G = (V, E)$, find a circuit of G with the minimum number of edges that traverses every edge at least once.

Euler Circuit for Directed Graphs

Theorem 5

A directed multigraph G has an Euler circuit if and only if:

(i) *G is weakly connected, and*

(ii) *For every $v \in V$, $d_{\text{in}}(v) = d_{\text{out}}(v)$.*

Necessity

Suppose that

$$\mathcal{E} : x_0, x_1, \cdots, x_{m-1}, x_0$$

is an Euler circuit.
Let G^* be the underlying undirected graph of G.
Clearly,
(i) G^* is connected, and
(ii) for any $v \in V$, for every occurrence of v in $\mathcal{E}$, the in- and out-degree of v is increased exactly by 1. This shows that

$$d_{\mathrm{in}}(v) = d_{\mathrm{out}}(v).$$

Sufficiency

Fix $a \in V$.
Let

$$H: \; a = x_0, x_1, \cdots, x_L, x_0 = a$$

be a directed path from a to a.
If H contains all the edges of G, then H is the desired Euler circuit.
Otherwise. Let G' be the graph obtained from G by deleting all the edges in H. Let I be the graph obtained from G' by removal all the isolated vertices.

Sufficiency - continued

Let $I = (V_I, E_I)$.
By the assumption, there is an i such that $1 \leq i \leq L$ and $x_i \in V_I$.
First, I satisfies the conditions of the theorem. Find a path staring from x_i and ending at x_i in I. Let Q be such a path.
Replace x_i in P by Q.
Repeating the process above, we have constructed an Euler circuit for G.

Hamilton Paths and Circuits

Definition 6

- A path P in a graph G is called a **Hamilton path**, if it passes through every vertex exactly once.
- A circuit C in a graph G is a **Hamilton circuit**, if it passes through every vertex exactly once.

Sufficient Conditions

Theorem 7

(Dirac's Theorem) If G is a simple graph with n vertices ($n \geq 3$) such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton circuit.

Theorem 8

(ORE's Theorem) If G is a simple graph with n vertices for $n \geq 3$ such that for every two vertices u, v such that $(u, v) \notin E$, $d(u) + d(v) \geq n$, then G has a Hamilton circuit.

Weighted Graphs - Examples

Let V be the set of cities, E be the set of flights between cities. For each flight $e = (u, v)$, e is associated with different weights, such as:

- the distance between city u and city v
- the flight time between city u and city v
- the travel fares between city u and city v.

Generally, a weighted graph is a graph $G = (V, E)$ such that there is a weight function w from E to $\mathbb{R}^+$.
In this case, we use $G = (V, E, w)$ to denote the weighted graph.

Distance in Weighted Graphs

Definition 9

Let $G = (V, E, w)$ be a weighted graph.

1) For a path $P: \ x - 0, x_1, \cdots, x_l$ in G, the *length* of P in G is

$$\sum_{i=0}^{l-1} w(x_i, x_{i+1}).$$

2) For u, $v \in V$, the *distance* between u and v in G is the minimum length of all the paths between u and v.
We use $\mathrm{dist}(u, v)$ to denote the distance between u and v in G.

Traveling Salesman Problem

Definition 10

(Traveling Salesman Problem) (TSP) Let $G = (V, E)$ be a complete graph with a weight function $w: E \to \mathbb{R}^+$. Find a path P in G such that

i) Every vertex $v \in V$ appears in P exactly once, and
ii) The length of P is minimised among all the paths that visit each vertex of G exactly once.

Finding the Shortest Path

Let $G = (V, E, w)$ be a connected simple graph. How to find the shortest path between an arbitrarily given pair of vertices?
First, consider the special case for unit weight, i.e. $w \equiv 1$.
Fix $a \in V$.
We can find the set of vertices x such that $\mathrm{dist}(x, a) = k$ for each k as follows.

$$
\begin{aligned}
S_0 &= \{a\}, \quad T_0 = S_0 \\
S_1 &= \{x : \ x \notin T_0, \ \ d(x, S_0) = 1\}, \quad T_1 = T_0 \cup S_1 \\
S_2 &= \{x : \ x \notin T_1, \ \ d(x, S_1) = 1\}, \quad T_2 = T_1 \cup S_2 \\
S_3 &= \{x : \ x \notin T_2, \ \ d(x, S_2) = 1\}, \quad T_3 = T_2 \cup S_3 \\
S_{k+1} &= \{x : \ x \notin T_k, \ \ d(x, S_k) = 1\}, \quad T_{k+1} = T_k \cup S_{k+1}.
\end{aligned}
$$

Special Case

Given a vertex x and a set Y of vertices with $x \notin Y$, $d(x, Y)$ is the minimum of $d(x, y)$ for all $y \in Y$.
According to the definition, S_k is the set of all the vertices x such that the distance between a and x is exactly k.
For any $b \in V$, find the least k such that $b \in S_k$, then the distance between a and b is k.
The time complexity:

$$O(n^2).$$

Dijkstra's Algorithm - Key

Let $G = (V, E, w)$ be a weighted graph. Find the distance between arbitrarily given two vertices.
Dijkstra, 1959
(Turing Award 1972)
Observation: The u, v-part of a shortest u, z-path must be a shortest u, v-path.

Dijkstra's Algorithm - Idea

Input: A graph with nonnegative edge weights and a starting vertex u. The weight of edge xy is $w(x, y)$, let $w(x, y) = \infty$ if there is no xy-edge.

Idea:

- Maintain the ordered set S of vertices to which a shortest path from u is known, enlarging S to include all vertices.
- Define a tentative distance $l(z)$ from u to each $z \notin S$ to be the length of the shortest u, z-path currently found.
- Once z is added in S, $l(z)$ has reached its final value.

Dijkstra's Algorithm

Initialisation: Set $S = \{u\}$, $l(u) = 0$, $l(z) = w(u, z)$ for $z \neq u$.
Iteration:

(1) Let v be the vertex such that

$$l(v) = \min\{l(z) \mid z \notin S\}.$$

(2) Add v to S

(3) (Updating rule) For every edge vz with $z \notin S$, set

$$l(z) = \min\{l(z),\ l(v) + w(v, z)\}.$$

The Algorithm - continued

The algorithm terminates if $S = V_G$.
At the end of the execution of the algorithm, for every $v \in V$, set

$$d(u, v) = l(v).$$

The time complexity:

$$O(n^2).$$

Theorem

Theorem 11

Given a graph G and a vertex u, Dijkstra's algorithm computes $d(u, z)$ *for every* $z \in V$*.*

We prove that at each iteration,

(1) for each $z \in S$, $l(z) = d(u, z)$, and

(2) for each $z \notin S$, $l(z)$ is the least length of a u, z-path reaching z directly from S.

Proof

We probe the results by induction on $k = |S|$.
Basis step: $k = 1$
From the initialisation step, $S = \{u\}$, $d(u, u) = l(u) = 0$, and the least length of a u, z-path reaching z from S is $l(z) = w(u, z)$, which is ∞ when uz is not an edge.
Induction step: Suppose that when $|S| = k$, (1) and (2) hold. Let v be the vertex chosen at iteration $k + 1$. Let $S' = S \cup \{v\}$.

Proof - continued

(1) for S':
First, we prove $d(u, v) = l(v)$.
A shortest u, v-path must exit S before reaching v. The inductive hypothesis states that the length of the shortest path going directly to v from S is $l(v)$. The inductive hypothesis and the choice of v also ensure that a path reaching v has length at least $l(v)$. hence $d(u, v) = l(v)$ and (1) holds for S'.

Proof - continued

(2) for S':
Let z be a vertex outside S other than v. By the hypothesis, the shortest u, z-path reaching z directly from S has length $l(z)$ (it is ∞, if no such a path). When we add v to S, we must also consider paths reaching z from v. Since we have now computed $d(u, v) = l(v)$, the shortest such path has length $l(v) + w(v, z)$ and we compare this with the previous value of $l(z)$ to find the shortest path reaching z directly from S'.
This completes the inductive proof.

Applications

Edmonds and Johnson, 1973:
Gave a way to solve the Chinese Postman Problem by using the Dijkstra's algorithm

The Travelling Salesman Problem

Given a weighted graph, find an order for a salesman to visit every city exactly once with minimum cost, returning back to the starting vertex.

Planar Representation

Definition 12

A graph is called *planar* if it can be drawn in the plane without any edge crossing.
A *planar representation* of a graph is a drawing of the graph without edge crossing.

Examples:

- K_4
- $K_{2,3}$

$K_{3,3}$ Is Not Planar

Let $V = \{v_1, v_2, v_3\}$ and $U = \{u_1, u_2, u_3\}$. For every i and j, there is an edge $v_i u_j$.

In any planar representation of $K_{3,3}$, the edges $v_1 u_1$, $v_1 u_2$, $v_2 u_1$ and $v_2 u_2$ form a closed curve that splits the plane into two regions R_1 and R_2.

The vertex v_3 is either in R_1 or R_2. In either case, v_3 splits one of the R_1 or R_2 into two subregions.

In any case, now, there is no place for u_3 any more.

K_5 Is Not Planar

Any planar representation of four vertices splits the plane into four regions. However, in this case, there is no place for the fifth vertex anymore.

Applications of Planar Graphs

- Electronic circuits
- Road maps

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Theorem 13

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then,

$$r = e - v + 2. \tag{1}$$

Proof

By induction on e.
Basis step: $e = 1$.
In this case, $e = 1$, $v = 2$ and $r = 1 = e - v + 2$.
Let $G = (V, E)$ be a simple graph.
Pick an edge uv.
Let $V' = V$, $E' = E \setminus \{uv\}$.
By inductive hypothesis, for $G' = (V', E')$, G' is planar, and

$$r' = e' - v' + 2.$$

Suppose that R' is a planar representation of G'.
Now adding the edge uv into R' such that the resulting graph is still planar.

Proof - continued

Then the edge uv must be in a region with which both endpoints u and v are on the boundary of the region. This increases the number of regions by 1.
Therefore, $r = r' + 1$, $e = e' + 1$, $v' = v$.
By the inductive hypothesis,

$$r' - e' - v' + 2,$$

implying that

$$r = e - v + 2.$$

Degree of a Region

Definition 14

Let $G = (V, E)$ be a simple and connected graph, and P be a planar representation of G.
For a region R of P, the *degree* of R in P is the number of edges that are on the boundary of the region R, denoted $d(R)$.

Lemma 15

1. *For any region R,*

$$d(R) \geq 3.$$

2.

$$\sum_{\text{all regions } R} d(R) = 2e.$$

Proof

By observation the planar representation P of G.

Basic Properties

Proposition 1

(1) *If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then*

$$e \leq 3v - 6.$$

(2) *If G is a connected planar simple graph, then G has a vertex of degree not exceeding* 5*.*

(3) K_5 *is not planar.*

(4) *If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no cycle of length* 3*, then* $e \leq 2v - 4$*.*

(5) $K_{3,3}$ *is not planar.*

Proof

For (1).

$$\begin{cases} 2e = \sum d(R) \geq 3r \\ e - v + 2 = r. \end{cases}$$

This implies that

$$e \leq 3v - 6.$$

For (2).
By (1), $e \leq 3v - 6$.
If for any x, $d(x) \geq 6$, then

$$2e = \sum d(x) \geq 6v,$$

implying that $e \geq 3v$. A contradiction

Proof - continued

For (3). For K_5, $e = 10$, $v = 5$, $3v - 6 = 9 < 10$, a contradiction.
For (4). Each region has degree ≥ 4.
For (5). Using (4).

Elementary Subdivision

Definition 16

Let $G = (V, E)$ be a graph. An *elementary subdivision* is an operation of the following form:
To replace an edge uv of G by two new edges incident to a newly created vertex:
uw and wv, where w is a newly created vertex.
Let G_1, G_2 be two graphs. We say that G_1 and G_2 are *homeomorphic*, if there is a graph G such that G_1 and G_2 are both the resulting graphs of the elementary divisions from the same graph G.

Characterisation of Planar Graphs

Theorem 17

(Kuralowski's Theorem) A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to $K_{3,3}$ *or* K_5*.*

Planarity Test

Hopcroft-Tarjan, 1974
A linear time algorithm for decide whether or not a given graph is planar.
Turing award, 1986

Graph Coloring

Map $\Leftrightarrow$ Dual graph, planar.

Definition 18

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph such that no two adjacent vertices are assigned the same color.

Definition 19

The *Chromatic number* of a graph is the least number of colors needed for a coloring of the graph.
We use $\chi(G)$ to denote the Chromatic number of G.

The Four Color Theorem

Theorem 20

Every planar graph can be colored by at most four colors.

- A conjecture in 1850
- First proof, 1976, a computer proof
- No simple proof yet

Coloring for General Graphs

It is hard to find the Chromatic number for a general graph.

Local algorithms

Algorithms in time $\mathrm{poly}(\log n)$.

Dynamical Algorithms

A research project:
Algorithms for maintaining the structure when the graphs are dynamically evolving.

Exercises - 1

(1) Let G be a weighted graph.
The Floyd's algorithm proceeds as follows:
Stage 1. For $i = 1, 2, \cdots, n$,
for $j = 1, 2, \cdots, n$, define

$$d(i,j) = \begin{cases} w_{i,j}, \text{if there is an edge between } i \text{ and } j, \\ \infty, \text{otherwise.} \end{cases} \tag{2}$$

Stage 2:
For each i from 1 to n,
For each j from 1 to n,
For each k from 1 to n,
If $d(j,i) + d(i,k) < (j,k)$, then
set

$$d(j,k) \leftarrow d(j,i) + d(i,k).$$

Prove that the Floyd's algorithm finds the length between

Exercises - 2

(2) We define the crossing number of a simple graph to be the minimum number of crossings that occur when the graph is drawn in a plane, where no three arcs representing edges are allowed to cross at the same point.

(2a) Find the crossing numbers of each of the nonplanar graphs:

(i) K_5
(ii) K_6
(iii) K_7
(iv) $K_{3,4}$
(v) $K_{4,4}$
(vi) $K_{5,5}$

(2b) Show that if m and n are even positive integers, then the crossing number of $K_{m,n}$ is less than or equal to $\frac{m \cdot n \cdot (m-2) \cdot (n-2)}{16}$.

Exercises - 3

(3) Show that every planar graph G can be colored by using 5 or fewer colors.
Let G be a simple connected graph. The diameter of G is the maximum distance between two vertices of the graph.

(4) Show that if the diameter of G is at least four, then the diameter of its complement $\bar{G}$ is no more than two.

(5) Show that if the diameter of a graph G is at least three, then the diameter of its complement $\bar{G}$ is no more than three.

谢谢！