Chapter 2

Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

2.1 Sets and Functions

Definition 2.1.1.

- *Fix an universal set* U*. Set operations: union* $\cup$*, intersection* $\cap$*, complement* $\overline{A}$*.*

- *Set inclusion:* $A \subseteq B$ *iff for all* $a \in A$ *it holds* $a \in B$*.* $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$*.*

- *Given a set* S*, the* power set *of* S *is the set of all subsets of the set* S*. The power set is denoted by* $\mathcal{P}(S)$*, or* 2^S*.*

- *The* Cartesian product *of sets* $A_1, A_2, \ldots, A_n$ *is defined by:* $A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}$.

$\Sigma^2 := \Sigma \times \Sigma$

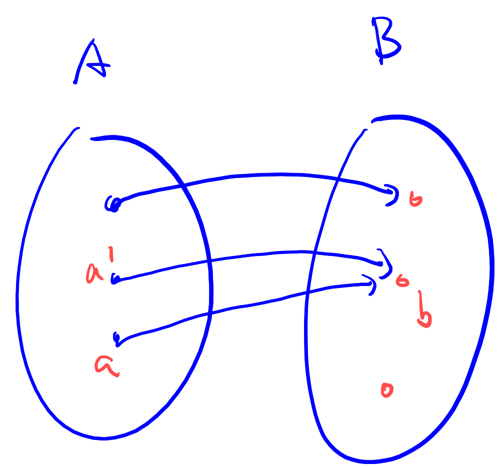
$c \geq 1 \quad \Sigma^c$

- *The* cardinality *of finite set* A*, denoted by* $|A|$*, is the number of its elements. The* principle of inclusion-exclusion*:*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Definition 2.1.2. *Let* A *and* B *be nonempty sets. A function* $f \colon A \to B$ *from* A *to* B *is an assignment of exactly one element of* B *to each element of* A*. We write* $f(a) = b$ *if* $b \in B$ *is assigned by* f *to the element* $a \in A$*. We say that*

- A *is the* domain *of* f*,*
- B *is the* codomain *of* f*.*
- *If* $f(a) = b$*, we say that* b *is the* image *of* a *and* a *is a* preimage *of* b*.*



$f : A \to B$

- *The* range*, or image, of* f *is the set of all images of elements of* A.

Definition 2.1.3. *Let* A *and* B *be two sets. The function* $f\colon A \to B$ *is called*

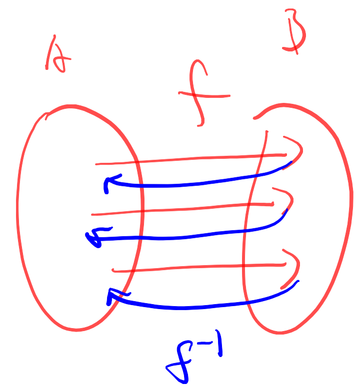
- one-to-one*, or an* injunction*, if and only if* $f(a) = f(b)$ *implies that* $a = b$ *for all* a *and* b *in the domain of* f. 单

- onto*, or a* surjection*, if and only if for every element* $b \in B$ *there is an element* $a \in A$ *with* $f(a) = b$. 满

- one-to-one correspondence*, or a* bijection*, if it is both one-to-one and onto.* 双 一一映射

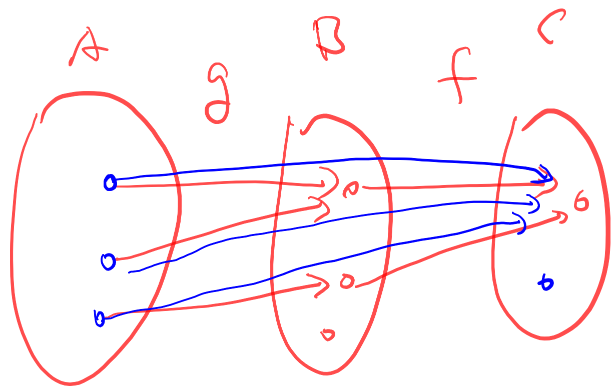
Definition 2.1.4. *Let A, B, and C be three sets.*

- *Let $f\colon A \to B$ be bijective. The* inverse function *of f, denoted by f^{-1}, is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.*



- *Let $g\colon A \to B$ and let $f\colon B \to C$. The* composition *of the functions f and g, denoted $f \circ g$, is defined by*

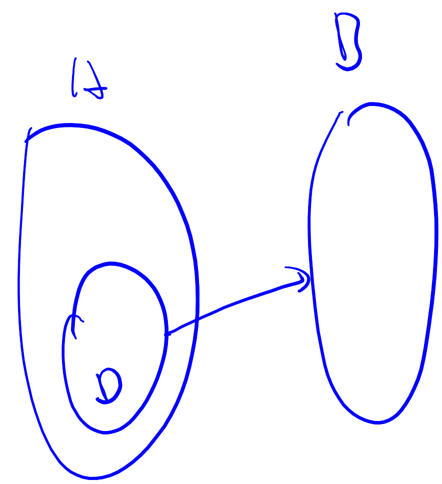
$$(f \circ g)(a) = f(g(a))$$



Definition 2.1.5 (Some Notations)**.** *Let A and B be two sets.*

- *For a function $f\colon A \to B$, and a set $\boxed{D \subseteq A,}$ we use $f|_D\colon D \to B$ to denote the function f with domain restricted to the set D.*

- *A* partial function *f from a set A to a set B is an assignment to each element $a \in D \subseteq A$, called* *the* domain of definition *of f, of a unique element $b \in B$. We say that f is undefined for elements in $A \setminus D$. When $D = A$, we say that f is a total function.*



$x \longrightarrow f(x)$

Definition 2.1.6. *Consider the set $U = 2^{AP}$ of all assignments. The semantic bracket is a function $[\![\cdot]\!] : WFF \to 2^U$ defined by:*

$f(\varphi)$

$\varphi \longrightarrow [\![\varphi]\!]$

- $[\![p]\!] = \{\sigma \in U \mid p \in \sigma\}$,
- $[\![\neg\varphi]\!] = \overline{[\![\varphi]\!]}$, 补
- $[\![\varphi \to \psi]\!] = \overline{[\![\varphi]\!]} \cup [\![\psi]\!]$.

$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$

$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$

Is $[\![\cdot]\!]$ injective, surjective, or bijective?

$\varphi \to \circ$

$\neg\neg\varphi \to \circ$

AP finite $\{p_1, p_2 \cdots p_n\}$

$\{\{p_1\}\}$

$\overline{p_1 \wedge \neg p_2 \cdots \wedge \neg p_n}$

AP infinite

$\Downarrow$

2^U not countable

2.2 Cardinality, Diagonalization Argument

Definition 2.2.1. *Let A and B be two sets.*

- *The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.* 等势 *When A and B have the same cardinality, we write $|A| = |B|$.*

- *If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.*

Definition 2.2.2. *A set that is either finite or has the same cardinality as the set of positive integers is called*

$$\begin{array}{cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ & | & | & | & | & | & & \\ A: & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots & \end{array}$$

countable. *A set that is not countable is called* uncountable. *When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$. We write $|S| = \aleph_0$ and say that S has cardinality aleph null.*

Lemma 2.2.3.
- *If $A \subseteq B$, then $|A| \leq |B|$.*

$a \to a$

- *If $A \subset B$, then $|A| \leq |B|$.*

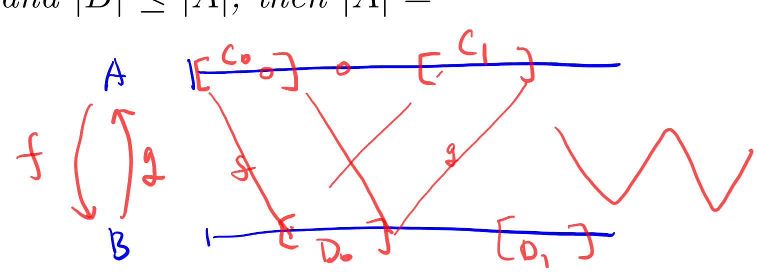
Assuming $B \subset A$, can it be the case that $|A| \leq |B|$?

[illegible] $\subset \mathbb{N}$

$2n \longrightarrow n$

$[0,1]$ $(0,1)$

Theorem 2.2.4 (SCHRÖDER-BERNSTEIN THEOREM)**.** *If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*



•) If g is surjec, ✓

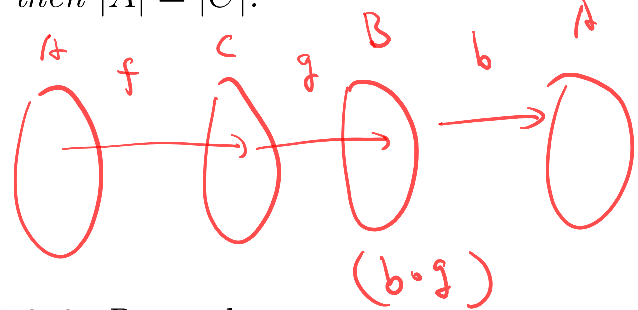
•) $C_0 := A \setminus g(B) = \{g(b) \mid b \in B\}$

$D_i = f(C_i)$

$h : A \to B$

$h(x) = \begin{cases} f(x) & \text{if } x \in C_i \text{ for some } i \\ g^{-1}(x) & \text{otherwise} \end{cases}$

Lemma 2.2.5. *If* $|A| = |B|$, *and* $|A| \leq |C| \leq |B|$, *then* $|A| = |C|$.



Lemma 2.2.6. *Prove that*

1. *The union, intersection of countable sets is countable.*
2. *The set* $\mathbb{Z}$ *of integer numbers is countable.*
3. *The set* $\mathbb{N}^2$ *is countable.*
4. *The set* $\mathbb{Q}$ *of rational numbers is countable.*

$A \quad a_0 \quad a_1 \quad a_2 \quad a_3 \quad \ldots$

$B \quad b_0 \quad b_1 \quad b_2 \quad b_3 \quad \ldots$

$-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\begin{array}{cccc} (0,0) & (0,1) & (0,2) & \ldots \\ (1,0) & (1,1) & (1,2) & \ldots \\ (2,0) & (2,1) & (2,2) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

$|\mathbb{N}| \leq |\mathbb{Q}| \leq |\mathbb{N}^2|$

5. *The set* $\mathbb{N}^c$ *with* $c \in \mathbb{N}$ *is countable.*
6. *The countable union of countable sets is countable.*
7. *The set* $\mathbb{N}^*$ *is countable.*

$N^{n+1} \equiv N^n \times N$

$$\begin{array}{llll} A_0 & a_{00} & a_{01} & a_{02} \ldots \\ A_1 & a_{10} & a_{11} & a_{12} \ldots \\ A_2 & a_{20} & a_{21} & a_{22} \ldots \\ A_3 & \vdots & \vdots & \vdots \\ \vdots & & & \end{array}$$

$= \{ a_0 a_1 \cdots a_n \mid n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{N} \}$

$N^* = N^0 \cup N^1 \cup N^2 \ldots$

$= \bigcup_{i=0}^{\infty} N^i$

For a set Σ, define Σ^ω the set of infinite strings $\{a_0a_1a_2\ldots \mid a_i \in \Sigma, i \in \mathbb{N}\}$.

Lemma 2.2.7. *Prove that*

$x \to \frac{x}{2}+\frac{1}{2}$

1. $|[0,1]| = |(0,1]| = |[0,1)| = |(0,1)|$.
2. $|(0,1]| = |[1,\infty)|$. $x \to \frac{1}{x}$

   $x \to kx$
3. $|[0,1]| = |[0,k]| = |[0,\infty)| = |\mathbb{R}|$.
4. $|2^{\mathbb{N}}| = |\{\, f \mid f\colon \mathbb{N} \to \{0,1\}\,\}|$.
5. $|\{0,1\}^\omega| = |[0,1]|$. $(0,1)$ $\quad 0.10\ldots$ $\quad 0.0111$
6. $|2^{\mathbb{N}}| = |\{0,1\}^\omega|$.

0 1 2 3 4 - - - 0 0

0 0 1 1 0 - - 1 0 .. 1 . .

$A \in 2^N$

N $\quad (0,1)$

Lemma 2.2.8 (Cantor diagonalization argument)**.**

- *The set $\mathbb{R}$ of real numbers is uncountable.*
- *For a set A, it holds: $|A| < |2^A|$.*

0 — $0.d_{00}d_{01}d_{02}$ - - -

1 — $0.d_{10}d_{11}d_{12}$ - -

2 — $0.d_{20}d_{21}d_{22}$ . .

$\vdots$

$0.\bar{d}_{00}\,\bar{d}_{11}\,\bar{d}_{22}$ - - -

Assume a bijection

$f: A \to 2^A$

$\boxed{\bar{0} = 1 \quad \bar{1} = 0}$

$B := \{a \in A \mid a \notin f(a)\}$

Assume: $f(b) = B$ $\qquad b \in B$?