ECS504

2nd Assignment

Candidate 34

Contents

List of Figures

1 Question 1

1.1 FOCs and MRS

Before finding the Marshallian labour supply and consumption demand functions, I calculate the necessary first-order conditions and marginal rate of substitution. Please note that I assume an interior solution for this simple problem.

Let c be the consumption level, ℓ be the amount of leisure (both assumed to be normal goods), L^s be Marshallian labour supply, p be the price on consumption (normalized to 1 later), w be the real wage rate (and hence also the price of leisure from an opportunity cost perspective), m be non-labour income, $\beta \in [0;1]$ be a constant, and A a time endowment (of 24 hours).

We then have a utility maximization problem, in which we try to maximize $u(c,\ell) = c^{\beta}\ell^{1-\beta}$ under a budget constraint $pc \leq wL^s + m$, of which the cost of consumption is listed on the left-hand side and the total income is listed on the right-hand side.

As we have leisure ℓ in the utility function, it is useful to rewrite the constraint into having leisure instead of labour. Substituting $L^s = A - \ell$ gives:

$$pc \leq wL^s + m \iff pc \leq wA - w\ell + m \iff pc - wA + w\ell - m \leq 0$$

The Lagrangian is then:

$$\mathscr{L}(c,\ell,\lambda) = c^{\beta}\ell^{1-\beta} - \lambda(pc - wA + w\ell - m)$$

1.1.1 $\frac{\partial \mathscr{L}}{\partial c} = 0$

The first first-order condition is:

$$\frac{\partial \mathscr{L}}{\partial c} = 0$$

Substituting gives:

$$\frac{\partial}{\partial c}\left[c^{\beta}\ell^{1-\beta} - \lambda(pc - wA + w\ell - m)\right] = 0$$

Which equals:

$$\beta c^{-1} \cdot \left(c^{\beta} \ell^{1-\beta}\right) - \lambda p = 0$$

With $c^{\beta} \ell^{1-\beta} =: u(c, \ell)$, we can rewrite to:

$$\beta \frac{u(c, \ell)}{c} = \lambda p$$

However, $\beta \frac{u(c,\ell)}{c} = \frac{\partial u}{\partial c}$, so we can also rewrite to:

$$\frac{\partial u}{\partial c} = \lambda p$$

1.1.2 $\frac{\partial \mathscr{L}}{\partial \ell} = 0$

The second first-order condition is:

$$\frac{\partial \mathscr{L}}{\partial \ell} = 0$$

Substituting gives:

$$\frac{\partial}{\partial \ell}\left[c^{\beta} \ell^{1-\beta} - \lambda (pc - wA + w\ell - m)\right] = 0$$

Which equals:

$$(1-\beta)\ell^{-1} \cdot \left(c^{\beta} \ell^{1-\beta}\right) - \lambda w = 0$$

With $c^{\beta} \ell^{1-\beta} =: u(c, \ell)$, we can rewrite to:

$$(1-\beta) \frac{u(c, \ell)}{\ell} = \lambda w$$

However, $(1-\beta) \frac{u(c,\ell)}{\ell} = \frac{\partial u}{\partial \ell}$, so we can also rewrite to:

$$\frac{\partial u}{\partial \ell} = \lambda w$$

1.1.3 $\frac{\partial \mathscr{L}}{\partial \lambda} = 0$

The third first-order condition is:

$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0$$

Substituting gives:

$$\frac{\partial}{\partial \lambda}\left[c^{\beta}\ell^{1-\beta} - \lambda(pc - wA + w\ell - m)\right] = 0$$

Which equals:

$$-(pc - wA + w\ell - m) = 0$$

Which can also be rewritten to matching our budget constraint:

$$pc - wA + w\ell - m = 0 \tag{1}$$

1.1.4 Marginal Rate of Substitution

We have:

$$MRS = \frac{\partial u/\partial \ell}{\partial u/\partial c}$$

Substituting our results from section 1.1.1 and 1.1.2 gives:

$$MRS = \frac{1-\beta\frac{u(c,\ell)}{\ell}}{\beta\frac{u(c,\ell)}{c}}$$

Which can be rewritten to:

$$MRS = \frac{1-\beta}{\beta}\frac{c}{\ell} \tag{2}$$

But we also have that $\frac{\partial u}{\partial c} = \lambda p$ and $\frac{\partial u}{\partial \ell} = \lambda w$. Hence:

$$\frac{\partial u/\partial \ell}{\partial u/\partial c} = \frac{\lambda w}{\lambda p}$$

Simplifying:

$$\frac{\partial u/\partial \ell}{\partial u/\partial c} = \frac{w}{p}$$

Substituting for $MRS = \frac{\partial u/\partial \ell}{\partial u/\partial c}$ in (2) ultimately gives:

$$\frac{1-\beta}{\beta}\frac{c}{\ell} = \frac{w}{p} \tag{3}$$

1.2 Marshallian demand function

Rewriting (3) to:

$$\ell = \frac{1-\beta}{\beta}\frac{p}{w}c$$

We can substitute ℓ in the budget equation (1). This gives:

$$pc - wA + w\frac{1-\beta}{\beta}\frac{p}{w}c - m = 0$$

Moving full income to the right-hand side and factorizing the remaining left-hand side gives:

$$\left(p + \frac{1-\beta}{\beta}p\right)c = m + wA$$

$p + \frac{1-\beta}{\beta}p = \frac{\beta+1-\beta}{\beta}p$, which can be simplified to $\frac{1}{\beta}pc$. Substituting:

$$\frac{1}{\beta}pc = m + wA$$

Multiplying both sides with $\frac{\beta}{p}$ and specifying that $c = c(p,w,m)$ gives us the Marshallian consumption demand function:

$$c(p,w,m) = \beta \frac{m + wA}{p} \tag{4}$$

1.3 Marshallian labour supply

Rewriting (3) to:

$$c = \frac{\beta}{1-\beta} \frac{w}{p} \ell$$

We can substitute c in the budget equation (1). This gives:

$$p \frac{\beta}{1-\beta} \frac{w}{p} \ell - wA + w\ell - m = 0$$

Moving full income to the right-hand side and factorizing the remaining left-hand side gives:

$$\left(\frac{\beta}{1-\beta} + 1 \right) w\ell = m + wA$$

$\frac{\beta}{1-\beta} + 1 = \frac{\beta+1-\beta}{1-\beta}$, which can be simplified to $\frac{1}{1-\beta}$. Substituting:

$$\frac{1}{1-\beta} w\ell = m + wA \tag{5}$$

We can derive the Marshallian leisure demand function by multiplying both sides with $\frac{w}{1-\beta}$ and specifying that $\ell = \ell(p,w,m)$:

$$\ell(p,w,m) = (1-\beta) \frac{m + wA}{w} \tag{6}$$

From (6), we can also derive the Marshallian labour supply function. Substitution then gives us $L^s = A - \ell \iff \ell = A - L^s$ gives:

$$A - L^s = (1-\beta)\frac{m+wA}{w}$$

Moving over gives:

$$L^s = A - (1-\beta)\frac{m+wA}{w}$$

Splitting up the fraction gives:

$$L^s = \big(A - (1-\beta)A\big) - (1-\beta)\frac{m}{w}$$

Since $\big(A-(1-\beta)A\big) = A - A + \beta A$, which simplifies to βA, and specifying that $L^s = L^s(p,w,m)$, ultimately gives us the Marshallian labour supply function:

$$L^s(p,w,m) = \beta A - (1-\beta)\frac{m}{w} \tag{7}$$

The labour supply is thus dependent on how much the non-labour income m is relative to the real wage w—the greater the ratio, the less the consumer is willing to work, all else held equal.

2 Question 2

2.1 Production function in (L^s, y) space

Let F represent a constant, labour L^s be represented on the x-axis and output $y = \max\{F \ln L^s; 0\}$ be represented on the y-axis. Then:

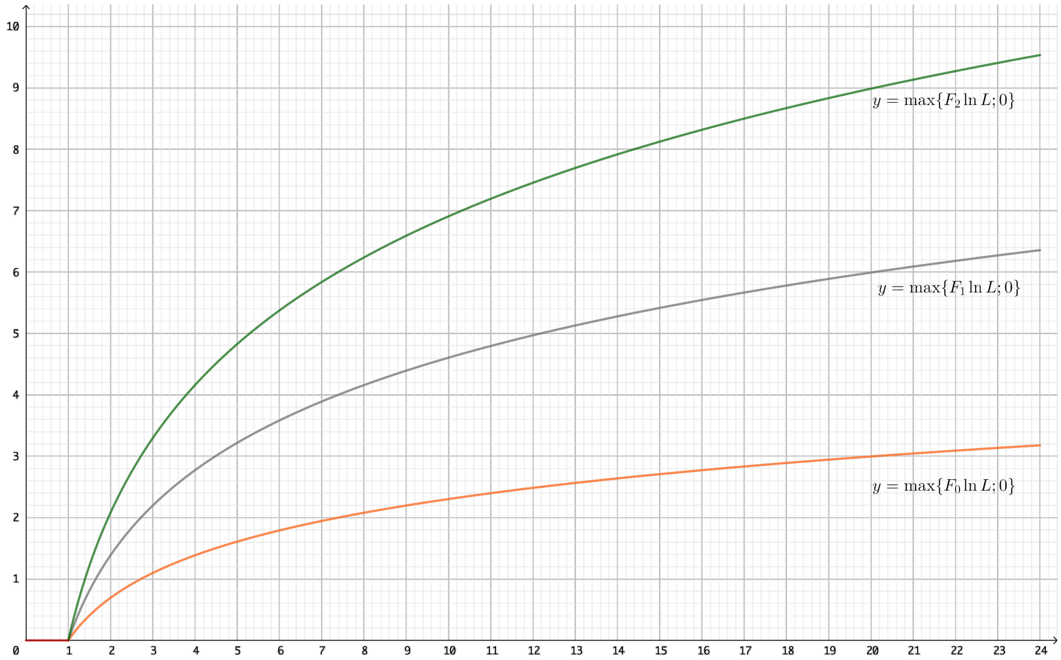

Figure 1: Production function in (L^s, y) space

As we can see from figure 4, we have production functions for various levels of F, with $F_0 < F_1 < F_2$.

We will not produce anything unless we have at least 1 hour of labour available for production.

Additionally, please note that I for practical purposes ended the production function at $L = 24$ since we have a time endowment of $A = 24$.

2.2 Elasticity of scale

Whereas returns to scale studies scale in the full domain (a global value), elasticity of scale allows us to study scale in sections of the domain.

Let μ represent elasticity of scale for labour. Then:

$$\mu := \frac{\partial y}{\partial L^s} \frac{L^s}{y}$$

Substituting $y = F \ln L^s \quad \forall L^s > 1$:

$$\mu = \frac{\partial}{\partial L^s} \left[F \ln L^s \right] \frac{L^s}{F \ln L^s} \bigg|_{L^s > 1}$$

Deriving with respect to L^s gives:

$$\mu = \frac{F}{L^s}\frac{L^s}{F\ln L^s}\bigg|_{L^s>1}$$

Which ultimately gives us:

$$\mu = \frac{1}{\ln L^s}\bigg|_{L^s>1} \tag{8}$$

2.2.1 $\mu > 1$

We begin with (8):

$$\mu = \frac{1}{\ln L^s}\bigg|_{L^s>1}$$

We want to know in which section of the domain:

$$\frac{1}{\ln L^s} > 1\bigg|_{L^s>1}$$

Multiplying by $\ln L^s$ preserves the inequality:

$$1 > \ln L^s\bigg|_{L^s>1}$$

$e^{LHS} > e^{RHS}$ preserves the inequality:

$$e^1 > e^{\ln L^s}\bigg|_{L^s>1}$$

Which ultimately gives us that the elasticity of scale is larger than 1 for:

$$\mu > 1\bigg|_{L^s>1} \iff L^s < e\bigg|_{L^s>1} \tag{9}$$

We enjoy increasing returns to scale for the first e (excluding e) hours of labour ($L^s > 1$).

2.2.2 $\mu < 1$

We begin with (8):

$$\mu = \frac{1}{\ln L^s}\bigg|_{L^s>1}$$

We want to know in which section of the domain:

$$\frac{1}{\ln L^s} < 1\bigg|_{L^s>1}$$

Multiplying by $\ln L^s$ preserves the inequality:

$$1 < \ln L^s\Big|_{L^s>1}$$

$e^{LHS} < e^{RHS}$ preserves the inequality:

$$e^1 < e^{\ln L^s}\Big|_{L^s>1}$$

Which ultimately gives us that the elasticity of scale is larger than 1 for:

$$\mu < 1\Big|_{L^s>1} \iff L^s > e\Big|_{L^s>1} \tag{10}$$

We experience decreasing returns to scale for the all hours of labour strictly after the initial e hours.

2.3 Elasticity of scale in (L^s, y) space

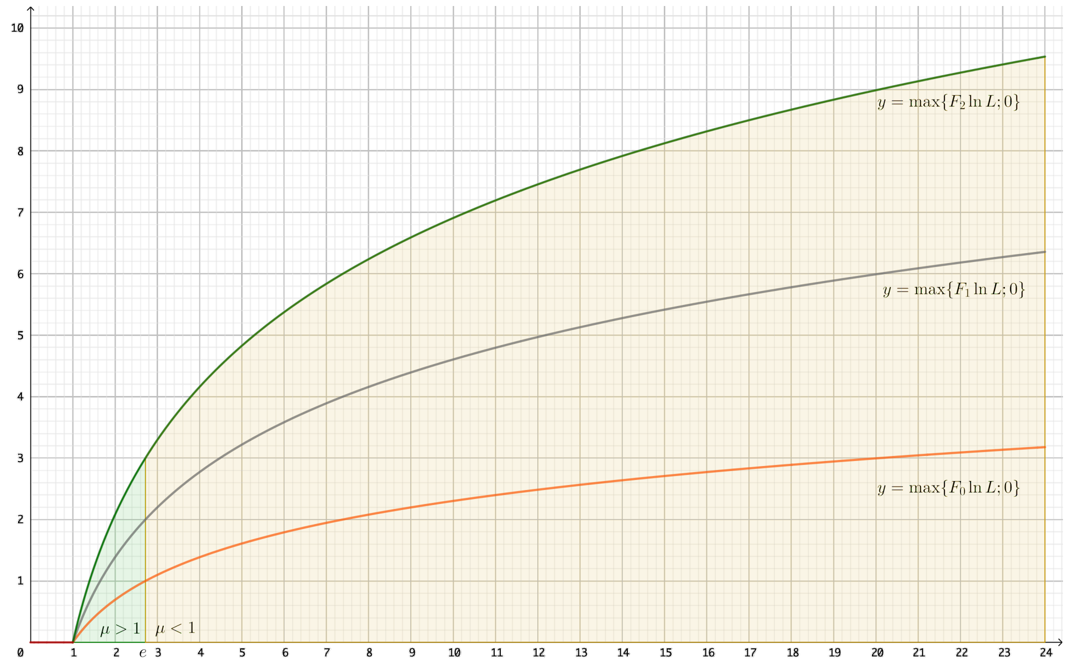

Figure 2: Elasticity of scale in (L^s, y) space

3 Question 3

3.1 Labour demand

Let π denote profit and L^d denote labour demand. Then the profit function is:

$$\pi(p,w) = py - wL^d$$

Substituting for y gives:

$$\pi(p,w) = pF \ln L^s - wL^d \Big|_{L^d > 1}$$

That gives us a profit maximization problem. Substituting $L^s = L^d$ gives $\max_{L^d} pF \ln L^d - wL^d$ such that $L^d \geq 0$ (the $L^d > 1$ solution is treated at a later point only if needed).

Assuming an interior solution (which is reasonable, considering the simplicity of the problem) gives:

$$\frac{\partial}{\partial L^d}\left[PF\ln L^d - wL^d\right] = 0$$

Deriving gives:

$$\frac{pF}{L^d} - w = 0$$

Moving over and multiplying both sides with $\frac{L^d}{w}$ and specifying that $L^d = L^d(p,w)$ gives us the labour demand function:

$$L^d(p,w) = \frac{pF}{w}\bigg|_{L^d(p,w)>1} \tag{11}$$

We can also verify that this is indeed the maximum profit, instead of minimum profit, with the second-order condition:

$$\frac{\partial}{\partial L^d}\frac{\partial}{\partial L^d}\left[PF\ln L^d - wL^d\right] \leq 0$$

Which gives:

$$\frac{-pF}{(L^d)^2} \leq 0 \tag{12}$$

(12) holds for all situations assuming $F > 0$, and so this is indeed the profit-maximizing labour demand. For the rest of the assignment, I assume that the second-order condition holds because of the simplicity of the optimization models.

3.2 Output supply

We begin with the production function:

$$y := \max\left\{F\ln L; 0\right\}$$

Substituting L with $L^d(p,w) = \frac{pF}{w}$, in which $L^d > 1$ is implied from the max function, gives us the output supply function:

$$y(p,w) := \max\left\{F\ln\left(\frac{pF}{w}\right);0\right\} \tag{13}$$

3.3 Maximal profit function

Starting with the profit function:

$$\pi(p,w) = py - wL^d$$

Substituting $y = y(p,w)$ with the optimal output supply, $L^d = L^d(p,w)$ with the optimal labour demand, and $\pi(p,w)$ with $\Pi(p,w)$ for all $L^d > 1 \iff \frac{pF}{w} > 1$ to denote the maximal profit function, we get:

$$\Pi(p,w) = pF\ln\left(\frac{pF}{w}\right) - w\frac{pF}{w}\bigg|_{\frac{pF}{w}>1}$$

$w\frac{pF}{w}$ can be simplified to pF, giving:

$$\Pi(p,w) = pF\ln\left(\frac{pF}{w}\right) - pF\bigg|_{\frac{pF}{w}>1}$$

Factorizing ultimately gives the maximal profit function:

$$\Pi(p,w) = pF\left(\ln\left(\frac{pF}{w}\right) - 1\right)\bigg|_{\frac{pF}{w}>1} \tag{14}$$

The requirement $\frac{pF}{w} > 1$ applies for production to even take place, but the marginal costs may be too high for small production levels to make it reasonable from a profit perspective to produce. Hence, to achieve non-negative profits, we must require that $\ln\left(\frac{pF}{w}\right) - 1 \geq 0 \iff \frac{pF}{w} \geq e$.

4 Question 4

The cost-minimizing production plan satisfies $\min_{L^d} wL^d$ such that $y(L^d) \geq y$[1] and $L^d \geq 0$. With a one-variable input and assuming an interior solution, solving the optimization model is redundant. Specifically, because we know $y(L^d) = y$ in optimum, and because of a one-variable input, we can use inversion instead.

$$y(L^d) = y\Big|_{y>0}$$

Substituting $y(L^d) = F \ln L^d$ gives:

$$F \ln L^d = y\Big|_{y>0}$$

Dividing by F and transforming to $e^{LHS} = e^{RHS}$ gives:

$$e^{\ln L^d} = e^{y/F}\Big|_{y>0}$$

Setting $e^{\ln L^d} = L^d$ and letting $L^d = \bar{L}^d(w,y)$ be the labour demand from the cost-minimization problem gives:

$$\bar{L}^d(w,y) = e^{y/F}\Big|_{y>0} \tag{15}$$

Please note that (15) equals $L^d(p,w)$ from (11) in optimum.

Furthermore, we can plug (15) into the cost function to get the minimal cost function wL^d:

$$C(w,y) = we^{y/F}\Big|_{y>0} \tag{16}$$

[1]Please note that I specified $y = y(p,w)$ earlier, which is still correct, but it was derived from $y = y(L^d)$.

Dividing (16) by y gives us the average cost function:

$$AC(w,y) = \frac{we^{y/F}}{y}\bigg|_{y>0} \tag{17}$$

And derivating (16) with respect to y gives us the marginal cost function. Proceeding in steps:

$$\frac{\partial C(w,y)}{\partial y} = \frac{\partial}{\partial y}\left[we^{y/F}\right]\bigg|_{y>0}$$

Applying the product rule on the right-hand side gives:

$$\frac{\partial C(w,y)}{\partial y} = \frac{\partial}{\partial y}\left[w\right]e^{y/F} + w\frac{\partial}{\partial y}\left[e^{y/F}\right]\bigg|_{y>0}$$

Deriving the right-hand side and setting $\frac{\partial C(w,y)}{\partial y}$ to the marginal cost function $MC(w,y)$ ultimately gives us:

$$MC(w,y) = \frac{we^{y/F}}{F}\bigg|_{y>0} \tag{18}$$

5 Question 5

Please note that we must have $y > 0$, which means that $F \ln\left(\frac{pF}{w}\right) > 0$, which in turn requires that $\frac{pF}{w} > 1$. Since $p = 1$, we must require that $F > w$.

Additionally, instead of using arbitrary values for F and w, am I instead using the values from the upcoming section Question 8 on page 17.

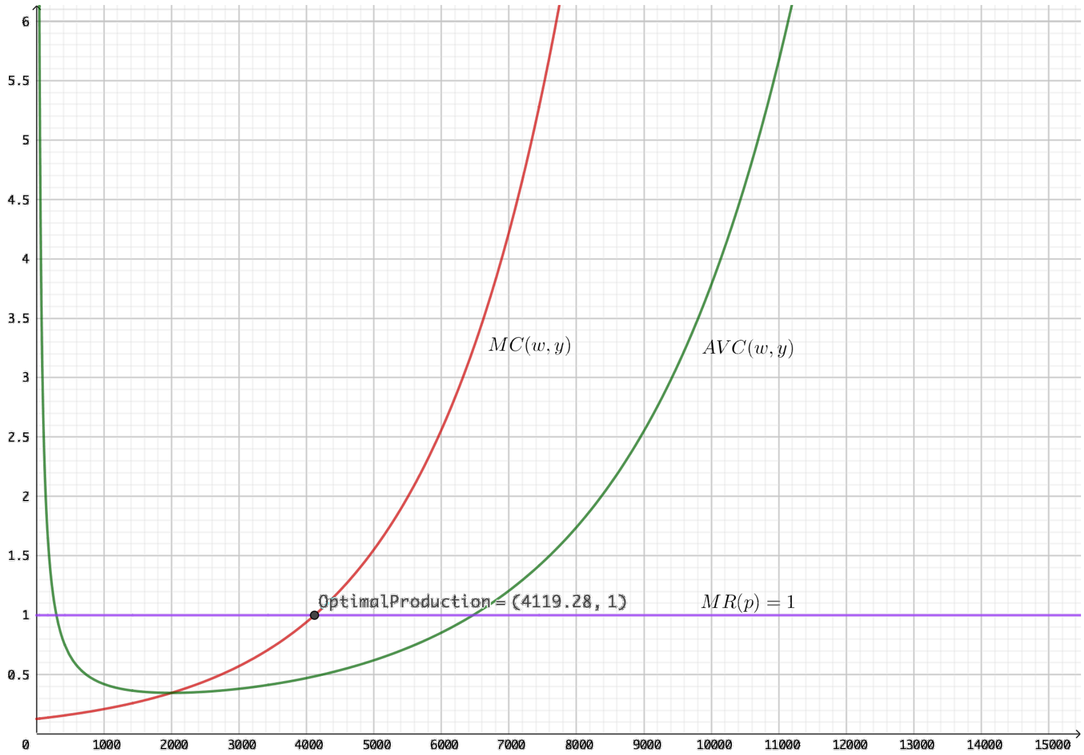

Figure 3: The average and marginal cost functions, together with marginal revenue and the optimal production level.

6 Question 6

6.1 Labour demand

From (11) on page 11, we have:

$$L^d(p,w) = \frac{pF}{w}\bigg|_{L^d(p,w)>1}$$

Substituting $p = 1$ and $F = 2000$ gives us

$$L^d(1,w) = \frac{2000}{w}\bigg|_{L^d(1,w)>1}$$

Please note that we must have $L^d(1,w) > 1$ for production to take place, which means that we must have $w < 2000$. Also, we need that $w > 0$.

Additionally, in section 3.3 on page 12, I specified that we must achieve $\frac{pF}{w} \geq e$ for profit to be non-negative. Substituting for $p = 1$ and $F = 2000$, and rearranging, we get that $w \leq \frac{2000}{e}$, which gives approximately $w \leq 736$.

Hence:

$$L^d(1,w) = \frac{2000}{w}, \quad w \in \left\langle 0; \frac{2000}{e} \right] \tag{19}$$

6.2 Labour supply

From 7 on page 6, we have:

$$L^s(p,w,m) = \beta A - (1-\beta)\frac{m}{w}$$

Non-labour income m consists only of the firm's profit $\Pi(p,w) = pF\left(\ln\left(\frac{pF}{w}\right) - 1\right)$. Subsituting $m = \Pi(p,w)$ gives:

$$L^s(p,w,m = \Pi(p,w)) = \beta A - (1-\beta)\frac{pF\left(\ln\left(\frac{pF}{w}\right) - 1\right)}{w} \tag{20}$$

Substituting $\beta = \frac{1}{2}$, $p = 1$, $F = 2000$, and $A = 24$ gives:

$$L^s(1,w,m = \Pi(1,w)) = \frac{1}{2}24 - \left(1 - \frac{1}{2}\right)\frac{2000\left(\ln\left(\frac{2000}{w}\right) - 1\right)}{w}$$

Which ultimately gives us:

$$L^s(1,w,m = \Pi(1,w)) = 12 - \frac{1000\left(\ln\left(\frac{2000}{w}\right) - 1\right)}{w} \tag{21}$$

Please note that we need $w > 0$ and that $\ln\left(\frac{2000}{w}\right) > 0 \iff \frac{2000}{w} > 1 \iff w < 2000$.

Hence:

$$L^s(1,w,m=\Pi(1,w)) = 12 - \frac{1000\left(\ln\left(\frac{2000}{w}\right)-1\right)}{w}, \quad w\langle 0;2000\rangle \tag{22}$$

7 Question 7

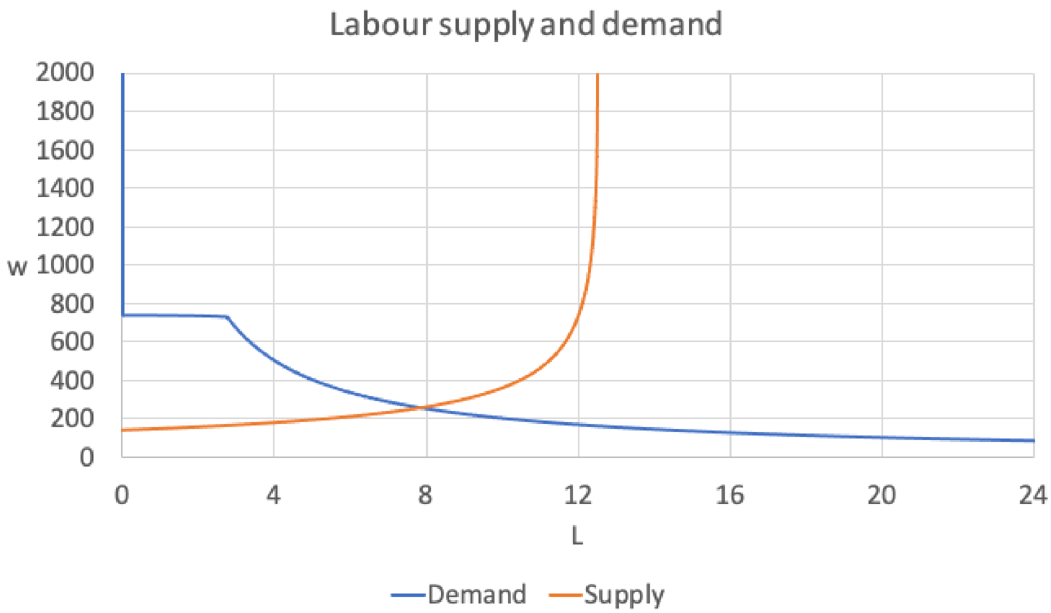

Figure 4: Labour supply and demand. In section 3.3 on page 12, I specified that we must achieve $\frac{pF}{w} \geq e$ for profit to be non-negative, which gives $w \leq \frac{2000}{e}$. This requirement is what causes the kink in the demand graph. Equilibrium is achieved at $w \approx 255$, which gives $L^* \approx 7.84$ hours of work per day.

8 Question 8

To find the equilibrium wage rate, we start with:

$L^d(p,w) = L^s(p,w,m=\Pi(p,w))$

Substituting from (11) on page 11 and (20) on page 16 gives:

$$\frac{pF}{w} = \beta A - (1-\beta)\frac{pF}{w}\left(\ln\left(\frac{pF}{w}\right)-1\right)$$

The properties of this equation is similar to a product–log function. Hence, I will seek out the Lambert W function to identify the equilibrium wage rate.

First, substitute $\frac{pF}{w} = x$, which gives:

$$x = \beta A - (1-\beta)x(\ln x - 1)$$

Moving over, in addition to substituting $\beta = \frac{1}{2}$ and $A = 24$, gives:

$$x + \frac{1}{2}x(\ln x - 1) = 12$$

The left-hand side can be rewritten to $\frac{1}{2}x + \frac{1}{2}\ln x$. Substituting:

$$\frac{1}{2}x + \frac{1}{2}x\ln x = 12$$

And multiplying both sides by 2 gives:

$$x + x\ln x = 24$$

Now, let us divide both sides by $x \neq 0$ (an assumption we know holds):

$$1 + \ln x = \frac{24}{x}$$

Substituting $x = \frac{1}{u}$:

$$1 + \ln\left(\frac{1}{u}\right) = 24u$$

We can rewrite $\ln\left(\frac{1}{u}\right)$ to $\ln 1 - \ln u = -\ln u$. Substituting:

$$1 - \ln u = 24u$$

Moving over:

$$\ln u + 24u = 1$$

Because Lambert W has the form $e^{W(z)}W(z) = z$, we transform both sides $e^{LHS} = e^{RHS}$:

$$e^{\ln u + 24u} = e^{1}$$

$e^{\ln u + 24u}$ can be rewritten to $e^{\ln u}e^{24u} = ue^{24u}$. Substituting then gives:

$$ue^{24u} = e$$

Multiplying both sides by 24:

$$24ue^{24u} = 24e \tag{23}$$

Now we have transformed the equilibrium into the form of the Lambert W function—$e^{W(z)}W(z) = z$—with $W(z) = 24u$ and $z = 24e$. We must now substitute for z, u and then for x to reach the equilibrium price. We begin with:

$$W(z) = 24u \tag{24}$$

Substituting $z = 24e$ gives:

$$W(24e) = 24u$$

Substituting $u = \frac{1}{x}$ gives:

$$W(24e) = \frac{24}{x}$$

Multiplying both sides with $\frac{x}{W(24e)}$ gives:

$$x = \frac{24}{W(24e)}$$

We initially substituted $\frac{pF}{w} = x$. Substituting back:

$$\frac{pF}{w} = \frac{24}{W(24e)}$$

With $p = 1$ and $F = 2000$,we get:

$$\frac{2000}{w} = \frac{24}{W(24e)}$$

And finally, transforming $LHS^{-1} = RHS^{-1}$ and multiplying by 2000 gives $w = w^*$:

$$w^* = \frac{2000}{24} W(24e) \tag{25}$$

With $W(24e) \approx 3.06$, we get

$$w^* \approx 255 \tag{26}$$

With w^* identified, we can also find the equilibrium labour demand and supply. Using $L(p,w) = \frac{pF}{w}$ and substituting $p = 1$, $F = 2000$, and $w \approx 255$, we get:

$$L^*(p = 1, w \approx= 255) \approx 7.84 \tag{27}$$

The optimal production can be found from $y(p,w) := \max \left\{ F \ln \left(\frac{pF}{w} \right); 0 \right\}$ and by substituting for p, F and W, which gives $y^*(p = 1, w \approx 255) = \max \left\{ 2000 \ln \left(\frac{2000}{255} \right); 0 \right\}$. This gives us an optimal production of:

$$y^*(p = 1, w \approx 255) \approx 4119 \tag{28}$$

With profit $\pi(p,w) := py - wL^d$, we can substitute and get $\pi^* \approx 1 \cdot 4119 - 255 \cdot 7.84$. Summing gives:

$$\pi^* \approx 2119 \tag{29}$$

Since non-labour income consist only of the firm's profit, we have:

$$m = \pi^* \approx 2119 \tag{30}$$

Working salary is $w^* \cdot L^{s*}(p,w,m) \approx 255 \cdot 7.84$, which sums to:

$$w^* \cdot L^{s*}(p,w,m) \approx 2000 \tag{31}$$

Please note that the working salary equals pF.

The consumer's total disposable income is therefore:

$$w^* \cdot L^{s*}(p,w,m) + m \approx 4119 \tag{32}$$

Please note that this equals the production value py^* of the firm, which is obvious considering that there is one owner, no other non-labour income, the consumer is the worker in its own firm, and there are no taxes.

Furthermore, we can calculate optimal leisure from (6) by substituting for β, m, w^*, and A. This gives $\ell^*(p = 1, w \approx 255, m \approx 4119) \approx \frac{1}{2}\frac{255 \cdot 24 + 2119}{255}$. Summing gives

$$\ell^*(p = 1, w \approx 255, m \approx 4119) = 16.16 \tag{33}$$

Please note that $16.16 + 7.84 = 24$. Also, please note that the cost of leisure is $16.16 \cdot 255 \approx 4119$.

And finally, we can calculate optimal consumption from (4) by substituting for β, m, w^*, and A. This gives $c^*(p=1, w \approx 255, m \approx 4119) \approx \frac{1}{2}\frac{255 \cdot 24 + 2119}{1}$. Summing gives:

$$c^*(p=1, w \approx 255, m \approx 4119) = 4119 \tag{34}$$

Please note that we spend an equal amount on consumption and leisure (because $\beta = \frac{1}{2}$ and $1-\beta = \frac{1}{2}$. But more importantly, please note that (34) equals the total income, and we have the cost of leisure in addition.

This is because the cost of leisure is an opportunity cost concept—for every hour you do not work, you lose one hour of income. Hence, the opportunity cost of one hour of leisure is w, and it is indirectly captured by working income we have available. It is also important to note that in this case, the opportunity cost is additionally present in the non-labour income because working less means less production, which in turn leads to less profit that is transferred as non-labour income. Hence, we use our total disposable income on consumption, and that is why optimal consumption approximately equals 4119—the same amount as total income.

9 Question 9

The manager prefers the mean-preserving spread because while the wage rate is in the denominator of a logarithmic function, the function will take a form similar to the one illustrated in figure 5.

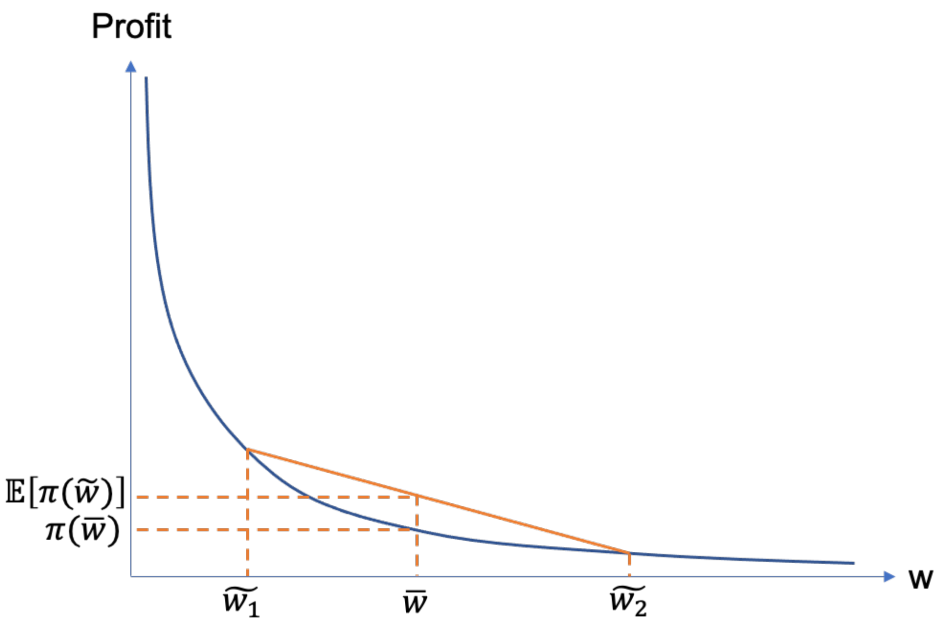

Figure 5: The manager prefers the mean-preserving spread.

However, the *employee* does not prefer the mean-preserving spread. Specifically, stochastic dominance lets us range risky alternatives by studying their probability distributions.

Whereas the first-order stochastic dominance studies better/poorer returns, second-order stochastic dominance (SOSD) studies the risk/spread of distributions with the same mean (i.e. a mean-preserving spread). For this question 9, we have one deterministic real wage rate $\bar{w}$ and we have one random real wage rate $\tilde{w}$ with expected value equal to $\bar{w}$. Hence, we can study the employee's preference with SOSD, as long as we implement a key assumption: that the employee is risk-averse—i.e. that the less risk/spread for a given return (i.e. we must study alternatives with a mean-preserving spread), the better.

We have that a distribution $F(\bullet)$ is preferred to a distribution $G(\bullet)$ for a risk-averse person when:

$$F(\bullet)\ SOSD\ G(\bullet) \iff \int_0^x F(t)dt \leq \int_0^x G(t)dt\ \forall x$$

For illustrative purposes, let us use the example numbers for question 10, in which $F(\bullet)$ represents the deterministic scenario ($\bar{w}$) and $G(\bullet)$ represents the stochastic scenario ($\tilde{w}$). We then have:

I. $\forall x \in [0;200\rangle$. $\int_0^x F(t)dt = 0,\ \int_0^x G(t)dt = 0$

II. $\forall x \in [200;250\rangle$. $\int_0^x F(t)dt = 0,\ \int_0^x G(t)dt > 0$

III. $\forall x \in [250;300\rangle$. $\int_0^x F(t)dt > 0,\ \int_0^x G(t)dt > \int_0^x F(t)dt$

IV. $\int_0^{300} F(t)dt = \int_0^{300} G(t)dt$ because of the mean-preserving spread.

Hence, the employee will dislike the mean-preserving spread and would rather want to have the deterministic wage $\bar{w}$.

Let us continue to use the numbers intended for question 10 and plug it into the following profit function from question 3:

$$\Pi(p,w) = pF\left(\ln\left(\frac{pF}{w}\right) - 1\right)\Bigg|_{\frac{pF}{w}>1}$$

Given $p = 1$ and $F = 2000$, the profit in the:

- deterministic case with $w = 250$ is approximately 2159.
- 1st random scenario with $w = 200$ is approximately 2605, with probability 0.5.
- 2nd random scenario with $w = 300$ is approximately 1794, with probability 0.5.

The expected profit from the random scenario is 2199, which is greater than 2159 from the deterministic case. This is what is illustrated in figure 5.

10 Question 10

The way I interpret this question, is that I am asked to calculate the certainty equivalent.

10.1 Exact calculation

We are interesting in finding the wage rate that would make the manager equally well off—in terms of expected profit—as in the uncertain case:

$$\Pi(p, w = CE) = \mathbb{E}\big[\Pi(p, w)\big]$$

Substituting for the profit function gives:

$$pF\left(\ln\left(\frac{pF}{CE}\right) - 1\right) = \mathbb{E}\big[\Pi(p, w)\big]$$

Dividing and moving over gives:

$$\ln\left(\frac{pF}{CE}\right) = \frac{\mathbb{E}\big[\Pi(p, w)\big]}{pF} + 1$$

Transforming into $e^{LHS} = e^{RHS}$ gives:

$$\frac{pF}{CE} = \exp\left(\frac{\mathbb{E}\big[\Pi(p, w)\big]}{pF} + 1\right)$$

Transforming $LHS^{-1} = RHS^{-1}$ and multiplying by pF gives:

$$CE = \frac{pF}{\exp\left(\frac{\mathbb{E}\big[\Pi(p,w)\big]}{pF} + 1\right)}$$

Substituting $p = 1$, $F = 2000$, and $\mathbb{E}\big[\Pi(p, w)\big] \approx 2199$ gives:

$$CE \approx \frac{2000}{\exp\left(\frac{2199}{2000}+1\right)}$$

Which ultimately leads gives us our certainty equivalent:

$$CE \approx 245.04 \tag{35}$$

In this context, the certainty equivalant means that the manager must be compensated with a difference of $250 - 245.04 = 4.96$ currency units (such as NOK) per hour to be indifferent between facing the certain real wage rate of 250 and the uncertain wage real wage rate in terms of profit levels.

10.2 Approximate calculation

Rao (2020) demonstrates that a Taylor-approximated utility function of the second degree can be rewritten as:

$$\mathbb{E}\left[\tilde{w}\right] - CE \approx \frac{1}{2}A(\tilde{w})\sigma^2_{\tilde{w}} \tag{36}$$

Since $A(\tilde{w}) = \frac{-\frac{\partial}{\partial \tilde{w}}[\Pi(p=1,\tilde{w})]}{-\frac{\partial^2}{\partial \tilde{w}^2}[\Pi(p=1,\tilde{w})]}$.

$$\frac{\partial}{\partial w}[\pi(p=1,w)] = \frac{\partial}{\partial w}\left[2000\left(\ln\left(\frac{2000}{w}\right)-1\right)\right]$$

However, this can be rewritten to:

$$\frac{\partial}{\partial w}[\Pi(p=1,w)] = \frac{\partial}{\partial w}[2000\ln 2000 - 2000\ln w - 2000]$$

Derivating then gives:

$$\frac{\partial}{\partial w}[\Pi(p=1,w)] = \frac{-2000}{w}$$

Which we can further use to find the double derivative of the profit function with respect to the real wage rate:

$$\frac{\partial^2}{\partial w^2}[\Pi(p=1,w)] = \frac{\partial}{\partial w}[\frac{-2000}{w}]$$

Which equals:

$$\frac{\partial^2}{\partial w^2}[\Pi(p=1,w)] = \frac{2000}{w^2}$$

Plugging into $A(\tilde{w}) = \frac{-\frac{\partial}{\partial \tilde{w}}[\Pi(p=1,\tilde{w})]}{-\frac{\partial^2}{\partial \tilde{w}^2}[\Pi(p=1,\tilde{w})]}$ gives:

$$A(\tilde{w}) = \frac{1}{\tilde{w}} \tag{37}$$

Substituting into (36), with $w = 250$, $\mathbb{E}\left[\tilde{w}\right] = 250$ and calculating the population variance to 2500 gives

$$250 - CE \approx \frac{1}{2}\frac{1}{250}2500$$

Which sums to

$$CE \approx 245 \tag{38}$$

This approximation is almost exactly equal to the exact calculation. However, this cannot always be expected since this approximation is good for low variance levels. With higher variance levels, this number will deviate more.

11 References

Rao, A. (2020). Understanding Risk-Aversion through Utility Theory [Power Point Slides]. Retrieved from `http://web.stanford.edu/class/cme241/lecture_slides/UtilityTheoryForRisk.pdf`