Summary of FOR10: pricing derivatives

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1 Useful formulas

$$x^- = max(-x; 0) \tag{1}$$

$$x^+ = max(x; 0) \tag{2}$$

$$R(\omega)^\varphi = \frac{V_1^\varphi(\omega) - V_0^\varphi}{V_0^\varphi} \tag{3}$$

$$S_0(t) = S_0(0)(1+r)^t \tag{4}$$

$$\forall t.\ \bar{X}_t = \frac{X_t}{(1+r)^t} \tag{5}$$

$$P_X(t,\omega) = E_Q\Big[\frac{X}{(1+r)^{T-t}}|\mathcal{F}_t\Big] = V^\varphi(t,\omega) \tag{6}$$

$$\text{Put-call parity: } P_t + S_t = C_t + \frac{K}{(1+r)^{T-t}} \tag{7}$$

$$\text{Fair forward price: } F_t = (1+r)^{T-t} S_t \tag{8}$$

$$E_Q[\bar{S}_1(1)|\mathcal{F}_0] = \bar{S}_1(0) \text{ is } E_Q\Big[\frac{S_1(1)}{(1+r)^1}\Big|\mathcal{F}_0\Big] = \frac{S_1(0)}{(1+r)^0} \text{ (two different discount rates.)} \tag{9}$$

$$E[aX + bY|\mathcal{F}_t = aE[X|\mathcal{F}_t] + bE[Y|\mathcal{F}_t] \tag{10}$$

$$E[X|\mathcal{F}_t] = X \text{ if } X \text{ is } \mathcal{F}_t\text{-measurable} \tag{11}$$

$$E[E[X|\mathcal{F}_t]|\mathcal{F}_s] = E[X|\mathcal{F}_s] \text{ if } \mathcal{F}_s \subset \mathcal{F}_t \tag{12}$$

$$E[X|\mathcal{F}_t] = E[X] \text{ if } X \text{ is independent of } \mathcal{F}_t \tag{13}$$

$$E[g(Y)|\mathcal{F}_t] \geq g(E[Y|\mathcal{F}_t]) \tag{14}$$

The risk-neutral price of any attainable derivate is the same for any choice of the equivalent martingale measure Q.

2 Key topics

I. A model is viable $\Leftrightarrow \nexists$ arbitrage opportunities.

II. $\nexists$ arbitrage opportunities $\Leftrightarrow \exists$ a risk-neutral measure $\mathbb{Q} = (q_1, \ldots, q_n)$.

III. $\exists!\mathbb{Q} = (q_1, \ldots, q_n) \Leftrightarrow$ The market is complete

IV. $\exists!\mathbb{Q} = (q_1, \ldots, q_n) \Leftrightarrow$ The market is replicable.

V. The market is replicable $\Leftrightarrow P_X(t, \omega) = V^{\varphi}(t, \omega) = E_{\mathbb{Q}}[\bar{X}(t, \omega)|\mathcal{F}_t]$

If a model is viable and complete, then every derivative in this model is attainable.

Note: the claim is replicable not because there exists a unique risk-neutral measure in the market, but because our specific claim's value is independent of our choice of λ. There may exist infinitely many risk-neutral measures in our market model, but as long as our claim does not rely on our choice of λ and, consequently, on our choice of $\mathbb{Q} = (q_1, \ldots, q_n)$, then the claim is replicable.

Properties of a risk-neutral measure:

$$\forall \omega \in \Omega, \forall t \in \mathbb{N}, (\forall j = 1, \ldots, d).\ E_{\mathbb{Q}}[\bar{S}_j(t, \omega)|\mathcal{F}_{t-1}] = \bar{S}_j(t-1) \tag{15}$$

$$\sum_{i=1}^{n} q_i = 1 \tag{16}$$

$$\forall i = i, ..., n.\ q_i > 0 \tag{17}$$

If we have multi-period models, work backwards to find a risk-neutral measure.

Summarized, here are our three possible states:

- $\nexists\mathbb{Q} = (q_1, \ldots, q_n) \Leftrightarrow$ arbitrage opportunities
- $\exists\mathbb{Q} = (q_1, \ldots, q_n) \Leftrightarrow$ no arbitrage opportunities, market is incomplete.
- $\exists!\mathbb{Q} = (q_1, \ldots, q_n) \Leftrightarrow$ no arbitrage opportunities, market is complete, replicable strategies exist.

Process to find a strategy for $t = 0, 1, 2$:

I. Find $V^{\varphi}(0)$ and $V^{\varphi}(1, \omega)$

II. Find $\vec{\varphi}(2, \omega)$ using prices from $t = 2$ and information from $t = 1$ because this strategy is made at $t = 1$. Then similarly for $t = 1$ (using information from $t = 0$), find $\vec{\varphi}(1, \omega)$ because this strategy is made at $t = 0$. Specifically, φ_t is held on the time interval $\langle t-1; t]$. Also, remember to write $\vec{\varphi}_t(\omega)$.

III. Control that $\forall t = 1, 2.\ \vec{S}^T(t-1)\vec{\varphi}(t, \omega) = V^{\varphi}(t-1)$

IV. Make decision map

3 Arbitrage

Arbitrage opportunity iff:

- $V(0)^{\varphi} = 0$
- $\forall \omega \in \Omega.\ V^{\varphi}(T, \omega) \geq 0$
- $\exists \omega \in \Omega.\ V^{\varphi}(T, \omega) > 0$

Arbitrage is risk-free profit by purchasing and selling simultaneously in both markets, and it is possible when one of three conditions is met:

I. The same asset does not trade at the same price on all markets

II. Two assets with identical cash flows do not trade at the same price.

III. An asset with a known price in the future does not today trade at its future price discountet at the risk-free interest rate.

Note: transations must occur simultaneously to avoid exposure to market risk.

4 Replicating a portfolio

It's possible to create a position $\vec{\varphi}$—in which the financing is borrowed at risk-free rate—which will produce identical cash flows to one option on the underlying share. The position created is known as a replicating portfolio since its cash flows replicate those of the option.

So *cash flow from position* $=$ *Cash flow from option* $\Rightarrow \vec{S}(\omega)^T\vec{\varphi} = D(\omega)\forall\omega = 1,\dots,n$.

Further, since the cash flows produced from the position and the cash flows produced from the option are identical, the price of the option today must equal the value of the position today—given no arbitrage opportunities. So $P_D(t) = V^{\varphi}(t)$. SJEKK AT DETTE STEMMER.

If $P_D(t) \neq V^{\varphi}(t)$, then arbitrage opportunities exist. Specifically and in the one-period case, if you go long on the option, you must finance the purchase and $\varphi_0' = -\varphi_0 - \frac{P_D - V_0^{\varphi}}{S_0(0)}$. However, if you go short on the option, you must place these money and $\varphi_0' = \varphi_0 + \frac{P_D - V_0^{\varphi}}{S_0(0)}$. Therefore, our extended portfolios become $\Psi = (\varphi_0 + \frac{P_D - V_0^{\varphi}}{S_0(0)}, y, -1)$ when you go short on the option and $\Psi = (-\varphi_0 - \frac{P_D - V_0^{\varphi}}{S_0(0)}, -y, 1)$ when you go long.

- The fair price is $P_X(t) = E_{\mathbb{Q}}[\bar{X}|\mathcal{F}_t]$
- A replicating strategy is $\forall t \in \mathbb{N}_0.\ \bar{V}^{\varphi}(t) = E_{\mathbb{Q}}[\bar{X}|\mathcal{F}_t]$, where φ is a self-financing strategy.

5 σ-algebras

- $\Omega \in \sigma(X)$
- Complements
- Unions
- $\mathcal{F}_t$ must also include $\mathcal{F}_s, s < t$.

Notes on measurability:

- X is measurable w.r.t a σ-algebra G iff $\sigma(X) \subseteq G$.
- X is an adapted process iff $\sigma(X_t) \subseteq \mathcal{F}_t$. Examples are $S_i, i \in \mathbb{N}$ and V^{φ}.
- X is a predictable process iff $\sigma(X_t) \subseteq \mathcal{F}_{t-1}$. Examples are S_0 and φ.

The trivial σ-algebra $\{\emptyset, \Omega\}$ is just a constant.

Note: when identifying $\mathbb{F}$, draw the nodes as a tree so I can see that equal values in time T are spanned from different nodes s.t. $\mathcal{F}_T = \mathcal{P}(\Omega)$

6 Conditional expectation

$$E[aX + bY|\mathcal{F}_t = aE[X|\mathcal{F}_t] + bE[Y|\mathcal{F}_t] \tag{18}$$

$$E[X|\mathcal{F}_t] = X \text{ if } X \text{ is } \mathcal{F}_t\text{-measurable} \tag{19}$$

$$E[E[X|\mathcal{F}_t]|\mathcal{F}_s] = E[X|\mathcal{F}_s] \text{ if } \mathcal{F}_s \subset \mathcal{F}_t \tag{20}$$

$$E[X|\mathcal{F}_t] = E[X] \text{ if } X \text{ is independent of } \mathcal{F}_t \tag{21}$$

$$E[g(Y)|\mathcal{F}_t] \geq g(E[Y|\mathcal{F}_t]) \tag{22}$$

Let

- X, Y, Z be random variables
- $a, b \in \mathbb{R}$
- $g : \mathbb{R} \to \mathbb{R}$

Then

I. $E[a|Y] = a$

II. $E[aX + bZ|Y] = aE[X|Y] + bE[Z|Y]$

III. $E[X|Y] \geq 0$ if $X \geq 0$

IV. $E[X|Y] = E[X]$ if X and Y are independent $\Rightarrow E[X|\{\emptyset, \Omega\}] = E[X]$

V. $E[E[X|Y]] = E[X]$

VI. $E[X \cdot g(Y)|Y] = E[X|Y] \cdot E[g(Y)|Y] = E[X|Y] \cdot g(Y)$. Also note that if Z is measurable w.r.t. G—in other words, that $Z(G)$, then $E[Z(G)|G] = Z(G)$.

VII. $E[X|Y, g(Y)] = E[X|Y]$

VIII. $E[E[X|Y, Z]|Y] = E[X|Y]$ This also implies that $E[E[Y|G_2]|G_1] = E[Y|G_1] if G_1 \subseteq G_2$.

IX. $E[g(Y)|G] \geq g(E[Y|G])$ for g convex s.t. $E[|g(X)|] < \infty$ (Jensen inequality)

X. $E[X|\mathcal{P}(X)] = X$

Note: when we use conditional expecation with σ-algebras and need to calculate the appropriate probability measures, remember:

$$\frac{\text{possible future scenarios}}{\text{possible scenarios w.r.t the information we have}} \tag{23}$$

This means that in $t = 2$, we get two different $E[X(2, \omega)|\mathcal{F}_1]$ since we are studying the node before $t = 2$: remember, we only have information up to the time $t = 1$ and must apply it.

7 Martingales

Interpretation: the expectation of the next value is equal to the present observed value no matter how many values we have observed.

A process X_t is a martingale w.r.t an equivalent probability measure $\mathbb{Q}$ if:

- $\forall t \in \mathbb{N}_0.\ \sigma(X_t) \subseteq \mathcal{F}_t$ (adapted)
- $\forall t \in \mathbb{N}_0.\ E_{\mathbb{Q}}[|X_t|] < \infty$

- $\forall s \leq t.\ E_{\mathbb{Q}}[X_t|\mathcal{F}_s] = X_s$

Supermartingale: $\forall s \leq t.\ E_{\mathbb{Q}}[X_t|\mathcal{F}_s] \leq X_s$

Sub-martingale: $\forall s \leq t.\ E_{\mathbb{Q}}[X_t|\mathcal{F}_s] \geq X_s$

The tower property is essential:

$$E_Q[E_Q[X|\mathcal{F}_t]|\mathcal{F}_s] = E_Q[X|\mathcal{F}_s], s < t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t). \tag{24}$$

Et bevis på Martingale:

$E_Q[M_{t+1}|\mathcal{F}_t] = M_t$

$E_Q[E_Q[M_{t+1}|\mathcal{F}_t]|\mathcal{F}_s] = E_Q[M_t|\mathcal{F}_s], s \leq t$

$E_Q[M_{t+1}|\mathcal{F}_s] = M_s$

Q is called an equivalent martingale measure.

1st Fundamental Theorem of Asset Pricing: Viable market model $\Leftrightarrow \exists Q$

2ndt Fundamental Theorem of Asset Pricing: Complete market model $\Leftrightarrow \exists! Q$

Proof of the less-than-infinity property: $E[|M_t|] = E[|E[Z|\mathcal{F}_t]|]$

$\Rightarrow E[|E[Z|\mathcal{F}_t]|] \leq E[E[|Z||\mathcal{F}_t]]$

$\Rightarrow E[|E[Z|\mathcal{F}_t]|] \leq E[|Z|] < \infty$

$\Rightarrow E[|E[Z|\mathcal{F}_t]|] < \infty$

$\Rightarrow E[|M_t|] < \infty$

Proof of the third property:

$E[M_t|\mathcal{F}_{t-1}]$. Inserting $M_t = E[Z|\mathcal{F}_t]$ gives

$E[E[Z|\mathcal{F}_t]|\mathcal{F}_{t-1}]$. The tower propert gives

$E[Z|\mathcal{F}_{t-1}]$. Inserting $M_{t-1} = E[Z|\mathcal{F}_{t-1}]$ gives

M_{t-1}.

Thus, we have proved that $E[M_t|\mathcal{F}_{t-1}] = M_{t-1}$

8 Optimal portfolio using the Martingale method

I. Solve $\max_W \{E[u(W)] - \lambda(E_{\mathbb{Q}}[\frac{W}{S_O(T)}] - v)\}$ to find $\hat{W}$. Note: $L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}$

II. Find $\hat{\lambda}$

III. Find $\hat{W}_{\hat{\lambda}}$

IV. Working backwards, find $\hat{\vec{\varphi}}$, using $\hat{W}_{\hat{\lambda}}$ as final values and controlling that $V^{\varphi}(0) = v$.

In other words, $W = V^{\varphi}(T, \omega)$ and $v = V^{\varphi}(0)$.

Note: $\gamma \in \langle \infty, 1 \rangle \backslash \{0\}$.

We want to find $\hat{\varphi}$ among all self-financing strategies with the same initial value v. $\hat{W}$ is the optimal wealth. $L(\omega)$ is the state-price density.

In other words, we want to find $\hat{\varphi}$ that replicates $\hat{W}$, i.e. $V^{\hat{\varphi}}(T) = \hat{W}\ (= \hat{W}_{\hat{\lambda}})$

9 Dynamic programming

$V_\varphi(T) = V_\varphi(0) + G_\varphi(T) = v + G_\varphi(T)$

At each time t, the x is then replaced by $V_\varphi(t)$

First find $u_1(V_1)$ from maximizing the value at time 1 plus the gain from time 1 to time 2. Then derive with respect to $\varphi_1(2,\omega)$, then find for time 0, and then find what you have to borrow in order to be self-financed..

10 The Put-Call Parity

$X_P = (4 - S_1(2))^+$ is a put option and $X_C = (S_1(2) - 4)^+$ is a call option. These have the same strike price $K = 4$ and the same underlying asset—S_1—and maturity $T = 2$. So we can use the put-call parity, which can be applied in markets with no arbitrage opportunities. The following properties hold.

I. X_P is replicable iff X_C is replicable.

II. $\forall t \in \mathbb{N}_0, \forall \omega \in \Omega.\ P_{X_P}(t,\omega) = P_{X_C}(t,\omega) + K \cdot E_{\mathbb{Q}}[\frac{1}{S_0(T)}] - S_1(t,\omega)$

The expectation requires an existing risk-neutral measure, and that explains why the put-call parity only works in markets with no arbitrage opportunities.

This parity is derived from the following equality:

$$(S_1(t) - K)^+ - (K - S_1(t))^+ = S_1(t) - K \tag{25}$$

Which gives

$$E_{\mathbb{Q}}\left[\frac{(S_1(T) - K)^+}{S_0(T)}\right] - E_{\mathbb{Q}}\left[\frac{(K - S_1(T))^+}{S_0(T)}\right] = E_{\mathbb{Q}}\left[\frac{S_1(T)}{S_0(T)}\right] - \frac{K}{S_0(T)} \tag{26}$$

Which gives the parity

$$\bar{P}_{X_C}(t,\omega) - \bar{P}_{X_P}(t,\omega) = \bar{S}_1(t,\omega) + \frac{K}{S_0(T)} \tag{27}$$

Note: the put-call parity can be extended and can also be

$$\bar{V}_P(t) = \bar{V}_C(t) + K \cdot E_{\mathbb{Q}}[\frac{1}{S_0(T)}|\mathcal{F}_s] - \bar{S}_1(t) \tag{28}$$

Examples of options include:

- European call: $(S_1(2) - 7)^+$
- European put: $(7 - S_1(2))^+$
- Asian call: $\left(\frac{S_1(0)+S_1(1)+S_1(2)}{3} - 7\right)^+$
- Look-back: $max(0; S_1(0) - 7; S_1(1) - 7; S_1(2) - 7)$