ECS504

1st Assignment

Candidate 27

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2 Homogeneity

Let $V(P,W,Y) := \frac{\exp\left(\beta_2 \frac{W}{P}\right)}{\beta_2}\left(\beta_2 \frac{Y}{P} + \beta_0 + \beta_1 \log\left(\frac{W}{P}\right)\right) - \frac{\beta_1}{\beta_2}\text{Ei}\left(\beta_2 \frac{W}{P}\right)$ be an indirect utility function.

$V(P,W,Y)$ is homogeneous of degree k in terms of (P,W,Y) if $V(\alpha P, \alpha W, \alpha Y) = \alpha^k V(P,W,Y)$. Hence, we must begin by multiplying all arguments with α:

$$V(\alpha P, \alpha W, \alpha Y) = \frac{\exp\left(\beta_2 \frac{\alpha W}{\alpha P}\right)}{\beta_2}\left(\beta_2 \frac{\alpha Y}{\alpha P} + \beta_0 + \beta_1 \log\left(\frac{\alpha W}{\alpha P}\right)\right) - \frac{\beta_1}{\beta_2}\text{Ei}\left(\beta_2 \frac{\alpha W}{\alpha P}\right)$$

All α values cancel each other out. Hence, $V(\alpha P, \alpha W, \alpha Y) = V(P,W,Y)$, or $V(\alpha P, \alpha W, \alpha Y) = \alpha^0 V(P,W,Y)$.

Therefore, $V(P,W,Y)$ is homogeneous of degree 0 in terms of (P,W,Y).

3 Minimal Expenditure Function

3.1 $e(w,u)$

We begin with the indirect utility function, having standardized prices P to 1.

$$v(w,y) = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 y + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)$$

The minimal expenditure function and the indirect utility function are inverses of each other. Hence, in optimum, we have $y = e(w,u)$ and $v(w,y) = u$.

$$u = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 e(w,u) + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)$$

Splitting up the parenthesis and moving terms over gives:

$$u + \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w) - \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_0 + \beta_1 \log(w)\right) = \exp\left(\beta_2 w\right) e(w,u)$$

Dividing by $\exp\left(\beta_2 w\right)$, splitting up the parenthesis, and switching sides gives:

$$e(w,u) = \frac{u}{\exp\left(\beta_2 w\right)} - \frac{\beta_0}{\beta_2} - \frac{\beta_1}{\beta_2}\log(w) + \frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}$$

3.2 $e^f(w,u)$

We begin with the indirect utility function, having standardized prices P to 1.

$$v(w,y) = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 y + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)$$

Let A be the total time endowment. Since we are interested in studying full income, we add two terms such that $\beta_2 y = \beta_2 y + \beta_2 wA - \beta_2 wA$. That gives:

$$v(w,y) = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 (y + wA) - \beta_2 wA + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)$$

The minimal expenditure function and the indirect utility function are inverses of each other. Hence, in optimum, we have $y + wA = e^f(w,u)$ and $v(w,y) = u$.

$$u = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 e^f(w,u) - \beta_2 wA + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\text{Ei}(\beta_2 w)$$

Splitting up the parenthesis and moving terms over gives:

$$u + \frac{\beta_1}{\beta_2}\text{Ei}(\beta_2 w) - \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_0 + \beta_1 \log(w) - \beta_2 wA\right) = \exp\left(\beta_2 w\right) e^f(w,u)$$

Dividing by $\exp\left(\beta_2 w\right)$, splitting up the parenthesis, and switching sides gives:

$$e^f(w,u) = \frac{u}{\exp\left(\beta_2 w\right)} - \frac{\beta_0}{\beta_2} - \frac{\beta_1}{\beta_2}\log(w) + \frac{\beta_1}{\beta_2}\frac{\text{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)} + wA$$

Since $e(w,u) = \frac{u}{\exp\left(\beta_2 w\right)} - \frac{\beta_0}{\beta_2} - \frac{\beta_1}{\beta_2}\log(w) + \frac{\beta_1}{\beta_2}\frac{\text{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}$, we can also write:

$$e^f(w,u) = e(w,u) + wA$$

4 Marshallian Labour Supply

Using the indirect utility function $v(w,y)$ and Roy's identity applied to labour supply, $L^u(w,y) = \frac{\partial v/\partial w}{\partial v/\partial y}$, we can find the Marshallian labour supply function. Roy's identity will be developed in steps.

4.1 Derivating Ei$(\beta_2 w)$

$$\text{Ei}(\beta_2 w) := \int_{-\infty}^{\beta_2 w} \frac{\exp(t)}{t} dt$$

Deriving with respect to w gives:

$$\frac{\partial}{\partial w}\left[\text{Ei}(\beta_2 w)\right] = \frac{\partial}{\partial w}\left[\int_{-\infty}^{\beta_2 w} \frac{\exp(t)}{t} dt\right]$$

Chain rule $\frac{\partial z}{\partial w} = \frac{\partial z}{\partial u}\frac{\partial u}{w}$ and setting $u = \beta_2 w$ gives:

$$\frac{\partial}{\partial w}\left[\text{Ei}(\beta_2 w)\right] = \frac{\partial}{\partial u}\left[\int_{-\infty}^{u} \frac{\exp(t)}{t} dt\right]\frac{\partial}{\partial w}[\beta_2 w]$$

$\int_{-\infty}^{u} \frac{\exp(t)}{t} dt =: \mathrm{Ei}(u)$ and $\frac{\partial}{\partial w}[\beta_2 w] = \beta_2$. This simplifies the right-hand side to:

$$\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\right] = \frac{\partial}{\partial u}\left[\mathrm{Ei}(u)\right]\beta_2$$

$\frac{\partial}{\partial u}[\mathrm{Ei}(u)] = \frac{e^u}{u}$. Since $u = \beta_2 w$, we get $\frac{\partial}{\partial u}[\mathrm{Ei}(u)] = \frac{e^{\beta_2 w}}{\beta_2 w}$ through substitution. This gives us:

$$\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\right] = \frac{e^{\beta_2 w}}{\beta_2 w}\beta_2$$

Which ultimately simplifies to:

$$\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\right] = \frac{e^{\beta_2 w}}{w}$$

4.2 $\partial v/\partial w$

We begin with the indirect utility function, having standardized prices P to 1.

$$v(w,y) = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 y + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)$$

Deriving with respect to w:

$$\frac{\partial}{\partial y}[v(w,y)] = \frac{\partial}{\partial w}\left[\frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 y + \beta_0 + \beta_1 \log(w)\right) - \frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)\right]$$

Splitting up the terms on the right-hand side and moving the coefficients out:

$$\frac{\partial}{\partial w}[v(w,y)] = \frac{1}{\beta_2}\frac{\partial}{\partial w}\left[\exp\left(\beta_2 w\right)\left(\beta_2 y + \beta_0 + \beta_1 \log(w)\right)\right] - \frac{\beta_1}{\beta_2}\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\right]$$

From section 4.1, we found that $\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\right] = \frac{\exp\left(\beta_2 w\right)}{w}$.

$$\frac{\partial}{\partial w}[v(w,y)] = \frac{1}{\beta_2}\frac{\partial}{\partial w}\left[\exp\left(\beta_2 w\right)\left(\beta_2 y + \beta_0 + \beta_1 \log(w)\right)\right] - \frac{\beta_1}{\beta_2}\frac{\exp\left(\beta_2 w\right)}{w}$$

Applying the product rule $D_x\left\{\prod_{i=1}^{n} f^{(i)}(x)\right\} = \sum_{i=1}^{n}\left(D_x\left\{f^{(i)}(x)\right\} \cdot \prod_{\substack{j=i,\\ j\neq i}}^{n} f^{(i)}(x)\right)$. gives us $\left(\frac{\partial}{\partial w}\left[\exp\left(\beta_2 w\right)\right]\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)+\exp\left(\beta_2 w\right)\frac{\partial}{\partial w}\left[\beta_2 y+\beta_0+\beta_1\log(w)\right]\right)$. Deriving gives us $\beta_2\exp\left(\beta_2 w\right)\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)+\frac{\beta_1\exp\left(\beta_2 w\right)}{w}$. Substitution gives us:

$$\frac{\partial}{\partial w}[v(w,y)] = \frac{1}{\beta_2}\left(\beta_2\exp\left(\beta_2 w\right)\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)+\frac{\beta_1\exp\left(\beta_2 w\right)}{w}\right)-\frac{\beta_1}{\beta_2}\frac{\exp\left(\beta_2 w\right)}{w}$$

The last two terms $\frac{\beta_1}{\beta_2}\frac{\exp\left(\beta_2 w\right)}{w}$ cancel each other out. Moreover, we have $\frac{1}{\beta_2}\beta_2 = 1$ for the first term. Hence, removing these terms and removing the first parenthesis gives us $\frac{\partial}{\partial w}[v(w,y)]$:

$$\frac{\partial}{\partial w}[v(w,y)] = \exp\left(\beta_2 w\right)\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)$$

4.3 $\partial v/\partial y$

We begin with the indirect utility function, having standardized prices P to 1.

$$v(w,y) = \frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)-\frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)$$

Deriving with respect to y:

$$\frac{\partial}{\partial y}[v(w,y)] = \frac{\partial}{\partial y}\left[\frac{\exp\left(\beta_2 w\right)}{\beta_2}\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)-\frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)\right]$$

Splitting up the terms on the right-hand side and moving the coefficients out:

$$\frac{\partial}{\partial y}[v(w,y)] = \frac{1}{\beta_2}\frac{\partial}{\partial y}\left[\exp\left(\beta_2 w\right)\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)\right]-\frac{\beta_1}{\beta_2}\frac{\partial}{\partial y}\left[\mathrm{Ei}(\beta_2 w)\right]$$

The second term on the right-hand side gives us $\frac{\partial}{\partial y}\left[\frac{\beta_1}{\beta_2}\mathrm{Ei}(\beta_2 w)\right] = 0$. For the first term with a derivate, we can use the product rule: $D_x\left\{\prod_{i=1}^{n} f^{(i)}(x)\right\} = \sum_{i=1}^{n}\left(D_x\left\{f^{(i)}(x)\right\}\cdot\prod_{\substack{j=i,\\ j\neq i}}^{n} f^{(i)}(x)\right)$. This gives:

$$\frac{\partial}{\partial y}[v(w,y)] = \frac{1}{\beta_2}\left(\frac{\partial}{\partial y}\left[\exp\left(\beta_2 w\right)\right]\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)+\exp\left(\beta_2 w\right)\frac{\partial}{\partial y}\left[\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)\right]\right)$$

Deriving gives us:

$$\frac{\partial}{\partial y}[v(w,y)] = \frac{1}{\beta_2}\left(0+\exp\left(\beta_2 w\right)\cdot\beta_2\right)$$

Which leaves us with:

$$\frac{\partial v(w,y)}{\partial y} = \exp\left(\beta_2 w\right)$$

4.4 Roy's Identity

Roy's identity can be applied to find the Marshallian labour supply (note the positive sign):

$$L^u(w,y) = \frac{\partial v/\partial w}{\partial v/\partial y}$$

From section 4.2 and 4.3 we calculated $\partial v/\partial w$ and $\partial v/\partial y$, respectively. Substituting gives:

$$L^u(w,y) = \frac{\exp\left(\beta_2 w\right)\left(\beta_2 y+\beta_0+\beta_1\log(w)\right)}{\exp\left(\beta_2 w\right)}$$

Which ultimately leads to a semi-log Marshallian labour supply function:

$$L^u(w,y) = \beta_0+\beta_1\log(w)+\beta_2 y$$

5 Hicksian Labour Supply

We can use Shephard's lemma to find the Hicksian labour supply:

$$L^c(w,u) = -\frac{\partial e(w,u)}{\partial w}$$

Substituting $e(w,u) = \frac{u}{\exp\left(\beta_2 w\right)} - \frac{\beta_0}{\beta_2} - \frac{\beta_1}{\beta_2}\log(w) + \frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}$ gives us:

$$L^c(w,u) = -\frac{\partial}{\partial w}\left[\frac{u}{\exp\left(\beta_2 w\right)} - \frac{\beta_0}{\beta_2} - \frac{\beta_1}{\beta_2}\log(w) + \frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}\right]$$

Splitting up the terms give us:

$$L^c(w,u) = -\left[\frac{\partial}{\partial w}\left[\frac{u}{\exp\left(\beta_2 w\right)}\right] - \frac{\partial}{\partial w}\left[\frac{\beta_0}{\beta_2}\right] - \frac{\partial}{\partial w}\left[\frac{\beta_1}{\beta_2}\log(w)\right] + \frac{\partial}{\partial w}\left[\frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}\right]\right]$$

We can rewrite the first term to $\frac{\partial}{\partial w}\left[\frac{u}{\exp\left(\beta_2 w\right)}\right] = \frac{\partial}{\partial w}\left[u \cdot \exp\left(\beta_2 w\right)^{-1}\right] = \frac{\partial}{\partial w}\left[u \cdot \exp\left(-\beta_2 w\right)\right]$. Hence, we must apply the product rule, which gives $\frac{\partial}{\partial w}\left[u \cdot \exp\left(-\beta_2 w\right)\right] = \frac{\partial}{\partial w}\left[u\right]\exp\left(-\beta_2 w\right) + u\frac{\partial}{\partial w}\left[\exp\left(-\beta_2 w\right)\right]$. This leads to $\frac{\partial}{\partial w}\left[u \cdot \exp\left(-\beta_2 w\right)\right] = u \cdot (-\beta_2)\exp\left(-\beta 2w\right)$, which can be rewritten to $\frac{\partial}{\partial w}\left[u \cdot \exp\left(-\beta_2 w\right)\right] = \frac{-u\beta_2}{\exp\left(\beta 2w\right)}$.

In a similar way can we rewrite the second term to $\frac{\partial}{\partial w}\left[\frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}\right] = \frac{\beta_1}{\beta_2}\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\exp\left(-\beta_2 w\right)\right]$. Applying the product rule gives $\frac{\partial}{\partial w}\left[\frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}\right] = \frac{\beta_1}{\beta_2}\left(\frac{\partial}{\partial w}\left[\mathrm{Ei}(\beta_2 w)\right]\exp\left(-\beta_2 w\right) + \mathrm{Ei}(\beta_2 w)\frac{\partial}{\partial w}\left[\exp\left(-\beta_2 w\right)\right]\right)$, which leads to $\frac{\beta_1}{\beta_2}\left[\frac{\exp\left(\beta_2 w\right)}{t}\exp\left(-\beta_2 w\right) + \mathrm{Ei}\left(\beta_2 w\right)(-\beta_2)\exp\left(-\beta_2 w\right)\right]$. Simplifying gives us $\frac{\partial}{\partial w}\left[\frac{\beta_1}{\beta_2}\frac{\mathrm{Ei}(\beta_2 w)}{\exp\left(\beta_2 w\right)}\right] = \frac{\beta_1}{\beta_2 w} - \frac{\beta_1 \mathrm{Ei}\left(\beta_2 w\right)}{\exp\left(\beta_2 w\right)}$.

Substituting gives us:

$$L^c(w,u) = -\left[\frac{-u\beta_2}{\exp\left(\beta_2 w\right)} + \frac{\beta_1}{\beta_2 w} - \frac{\beta_1}{\beta_2 w} - \frac{\beta_1 \mathrm{Ei}\left(\beta_2 w\right)}{\exp\left(\beta_2 w\right)}\right]$$

The second and third terms cancel each other out, and by also removing the brackets, we get the

Hicksian labour supply function.

$$L^c(w,u) = \frac{\beta_2 u + \beta_1 \mathrm{Ei}\big(\beta_2 w\big)}{\exp\left(\beta_2 w\right)}$$

6 Compensated Wage Effect on Labour Supply

6.1 $\frac{\partial L^c(w,u)}{\partial w}$

From section 5, we have:

$$L^c(w,u) = \frac{\beta_2 u + \beta_1 \mathrm{Ei}\big(\beta_2 w\big)}{\exp\left(\beta_2 w\right)}$$

We must derive with respect to w:

$$\frac{\partial L^c(w,u)}{\partial w} = \frac{\partial}{\partial w}\left[\frac{\beta_2 u + \beta_1 \mathrm{Ei}\big(\beta_2 w\big)}{\exp\left(\beta_2 w\right)}\right]$$

Splitting the terms and applying the product rule (also rewriting to $\frac{1}{\exp\left(\beta_2 w\right)} = \exp\left(-\beta_2 w\right)$:

$$\frac{\partial L^c(w,u)}{\partial w} = \left(\frac{\partial}{\partial w}\Big[\beta_2 u\Big]\exp\Big(-\beta_2 w\Big) + \beta_2 u \frac{\partial}{\partial w}\Big[\exp\Big(-\beta_2 w\Big)\Big]\right) + \left(\frac{\partial}{\partial w}\Big[\beta_1 \mathrm{Ei}\Big(\beta_2 w\Big)\Big]\exp\Big(-\beta_2 w\Big) + \beta_1 \mathrm{Ei}\Big(\beta_2 w\Big)\frac{\partial}{\partial w}\Big[\exp\Big(-\beta_2 w\Big)\Big]\right)$$

Solving, and setting $\exp\Big(-\beta_2 w\Big) = \frac{1}{\exp\left(\beta_2 w\right)}$ and $\frac{\partial}{\partial w}\Big[\mathrm{Ei}\Big(\beta_2 w\Big)\Big] = \frac{\exp\left(\beta_2 w\right)}{w}$, gives:

$$\frac{\partial L^c(w,u)}{\partial w} = 0 - \frac{\beta_2 u}{\exp\Big(\beta_2 w\Big)} + \beta_1 \frac{\exp\Big(\beta_2 w\Big)}{w}\frac{1}{\exp\Big(\beta_2 w\Big)} - \frac{\beta_1 \mathrm{Ei}\Big(\beta_2 w\Big)\beta_2}{\exp\Big(\beta_2 w\Big)}$$

Which simplifies to:

$$\frac{\partial L^c(w,u)}{\partial w} = \frac{\beta_1}{w} - \beta_2 \frac{\beta_2 u + \beta_1 \mathrm{Ei}\left(\beta_2 w\right)}{\exp\left(\beta_2 w\right)}$$

However, $L^c(w,u) = \frac{\beta_2 u + \beta_1 \mathrm{Ei}\left(\beta_2 w\right)}{\exp\left(\beta_2 w\right)}$. Substituting therefore gives us:

$$\frac{\partial L^c(w,u)}{\partial w} = \frac{\beta_1}{w} - \beta_2 L^c(w,u)$$

6.2 $\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(m,y)}$

In optimum, Marshallian demand equals Hicksian labour supply.

$$L^c(w,u) = L^u(w,y)$$

However, we also know that in optimum, $u = v(m,y)$ and $y = e(w,u)$:

$$L^c(w,u)|_{u=v(w,y)} = L^u(w, y = e(w,u))$$

Deriving with respect to w:

$$\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\partial L^u(w, y = e(w,u))}{\partial w}$$

Then totally differentiating with respect to wages w will expand the right-hand side to (please note that I substitute y with $e(w,u)$ in the last derivative, so the chain rule is still preserved):

$$\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\partial L^u(w,y)}{\partial w} + \frac{\partial L^u(w,y)}{\partial y}\frac{\partial e(w,u)}{\partial w}$$

Shephard's lemma gives us $L^c(w,u) = -\frac{\partial e(w,u)}{\partial w}$. Substituting into our optimum relation $L^c(w,u)|_{u=v(w,y)} = L^u(w, y = e(w,u))$ gives us $-\frac{\partial e(w,u)}{\partial w} = L^u(w, y = e(w,u)) \iff \frac{\partial e(w,u)}{\partial w} = -L^u(w,y)$. Substituting:

$$\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\partial L^u(w,y)}{\partial w} - \frac{\partial L^u(w,y)}{\partial y} L^u(w,y)$$

Please note that this is the Slutsky equation, although it's normally presented with the last term on the right-hand side moved over to the left-hand side to illustrate the substitution effect $\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)}$ and the income effect $\frac{\partial L^u(w,y)}{\partial y} L^u(w, y = e(w,u))$ together.

Moving on, we substitute $L^u(w,y) = \beta_0 + \beta_1 \log(w) + \beta_2 y$:

$$\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\partial}{\partial w}\Big[\beta_0 + \beta_1 \log(w) + \beta_2 y\Big] - \frac{\partial}{\partial y}\Big[\beta_0 + \beta_1 \log(w) + \beta_2 y\Big]\Big(\beta_0 + \beta_1 \log(w) + \beta_2 y\Big)$$

Deriving gives us:

$$\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\beta_1}{w} - \beta_2\Big(\beta_0 + \beta_1 \log(w) + \beta_2 y\Big)$$

Since $L^u(w,y) = \beta_0 + \beta_1 \log(w) + \beta_2 y$, we can substitute the last parenthesis, leaving us with:

$$\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\beta_1}{w} - \beta_2 L^u(w,y)$$

6.3 Comparing the Solutions

So we have both that $\frac{\partial L^c(w,u)}{\partial w} = \frac{\beta_1}{w} - \beta_2 L^c(w,u)$ and that $\frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} = \frac{\beta_1}{w} - \beta_2 L^u(w,y)$. Since $L^c(w,u) = L^u(w,y)$ in optimum, these answers are equivalent and will return the same values.

7 Income and Uncompensated Wage Elasticity

7.1 Income Elasticity

We have the following labour supply function:

$$L = -457.92 + 1156 \log\big(w\big) - 0.038y$$

Derivating both sides with respect to y gives us:

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}\Big[-457.92 + 1156 \log\big(w\big) - 0.038y\Big]$$

Multiplying both sides with $\frac{y}{L}$, and substituting for L on the right-hand side gives us:

$$\frac{\partial L}{\partial y}\frac{y}{L} = \frac{\partial}{\partial y}\left[-457.92 + 1156\log\left(w\right) - 0.038y\right]\frac{y}{-457.92 + 1156\log\left(w\right) - 0.038y}$$

$\frac{\partial L}{\partial y}\frac{y}{L} = \eta_{Ly}$, the income effect that measures by how many percent (approximately) labour supply will change when non-labour income increases by 1 %. Substituting and deriving the right-hand side gives us:

$$\eta_{Ly} = \frac{-0.038y}{-457.92 + 1156\log\left(w\right) - 0.038y}$$

Substituting the average male household values, $w = 18.05, y = 17462, L = 2223$, and calculating the right-hand side, we get

$$\eta_{Ly} \approx -0.298$$

Please note that substituting L directly in $\frac{\partial L}{\partial y}\frac{y}{L}$ would have yielded the same results.

7.2 Uncompensated Wage Elasticity

We have the following labour supply function:

$$L = -457.92 + 1156\log\left(w\right) - 0.038y$$

Derivating both sides with respect to w gives us

$$\frac{\partial L}{\partial w} = \frac{\partial}{\partial w}\left[-457.92 + 1156\log\left(w\right) - 0.038y\right]$$

Multiplying both sides with $\frac{w}{L}$, and substituting for L on the right-hand side gives us:

$$\frac{\partial L}{\partial w}\frac{w}{L} = \frac{\partial}{\partial w}\left[-457.92 + 1156\log\left(w\right) - 0.038y\right]\frac{w}{-457.92 + 1156\log\left(w\right) - 0.038y}$$

$\frac{\partial L}{\partial w}\frac{y}{L} = \varepsilon_{Lw}$, the uncompensated wage elasticity that measures by how many percent (approximately) labour supply will change when the net wage rate increases by 1 %. Substituting and deriving the right-hand side gives us:

$$\varepsilon_{Lw} = \frac{1156w}{w\Big(-457.92 + 1156\log\big(w\big) - 0.038y\Big)}$$

The two w cancel each other out, leaving us with:

$$\varepsilon_{Lw} = \frac{1156}{-457.92 + 1156\log\big(w\big) - 0.038y}$$

Substituting the average male household values, $w = 18.05, y = 17462, L = 2223$, and calculating the right-hand side, we get

$$\varepsilon_{Lw} \approx 0.520$$

7.3 The Compensated Wage Elasticity

We can also calculate the compensated wage elasticity. Let's start with the Slutsky equation:

$$\frac{\partial L^u(w,y)}{\partial w} = \frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)} + \frac{\partial L^u(w,y)}{\partial y} L^u(w,y)$$

Multiplying with $\frac{w}{L^u(w,y)}$ gives us:

$$\frac{\partial L^u(w,y)}{\partial w}\frac{w}{L^u(w,y)} = \frac{\partial L^c(w,u)}{\partial w}\Big|_{u=v(w,y)}\frac{w}{L^u(w,y)} + \frac{\partial L^u(w,y)}{\partial y} L^u(w,y)\frac{w}{L^u(w,y)}$$

$\frac{\partial L^u(w,y)}{\partial y} L^u(w,y)\frac{w}{L^u(w,y)}$ can be rewritten to $\frac{\partial L^u(w,y)}{\partial y} L^u(w,y)\frac{w}{L^u(w,y)}\frac{y}{y}$ (i.e. multiplied and divided with y), which can be rewritten to $\eta_{Ly}\frac{wL^u(w,y)}{y}$ since $\frac{\partial L}{\partial y}\frac{y}{L} = \eta_{Ly}$.

Moreover, $\varepsilon_{Lw} = \frac{\partial L^u(w,y)}{\partial w}\frac{w}{L^u(w,y)}$ and $\varepsilon^c_{Lw} = \frac{\partial L^c(w,y)}{\partial w}\frac{w}{L^u(w,y)}$.

Hence, substituting gives us:

$$\varepsilon_{Lw} = \varepsilon^c_{Lw} + \eta_{Ly}\frac{wL^u(w,y)}{y}$$

Moving over gives us the compensated wage elasticity:

$$\varepsilon_{Lw}^{c} = \varepsilon_{Lw} - \eta_{Ly}\frac{wL^{u}(w,y)}{y}$$

Substituting $\varepsilon_{Lw} = \frac{1156}{-457.92+1156\log\left(w\right)-0.038y}$, $\eta_{Ly} = \frac{-0.038y}{-457.92+1156\log\left(w\right)-0.038y}$, $w = 18.05$, $L^{u}(w,y) = 2223$, and $y = 17462$:

$$\varepsilon_{Lw}^{c} = \frac{1156}{-457.92+1156\log\left(18.05\right)-0.038\cdot 17462} - \frac{-0.038\cdot 17462}{-457.92+1156\log\left(18.05\right)-0.038\cdot 17462}\frac{18.05\cdot 2223}{17462}$$

Which ultimately leads to:

$$\varepsilon_{Lw}^{c} \approx 1.206$$

8 Marginal Tax Rate

8.1 Graphical Representation of the Labour Supply

First, to sketch the labour supply functions corresponding to the after tax situation, I interpret that I should sketch the functions with an intersection at $w = 18.05$, and then show the discrepancy after the tax rate is introduced.

Second, I must calculate $u = v(w = 18.05, y = 17462)$ to make the Hicksian labour supply actually compensated (that is, that the utility does not change when the net wage rate changes). Hence, u is substituted as a constant and not changing with w in the graph. Keeping w constant in $u = v(w,y)$ is the sole reason why the Marshallian and Hicksian labour supply curves are different.

Third, let us first have a look at the (w,L) space (that is, w is along the horizontal axis. The vertical lines show the gross and net wage rates.

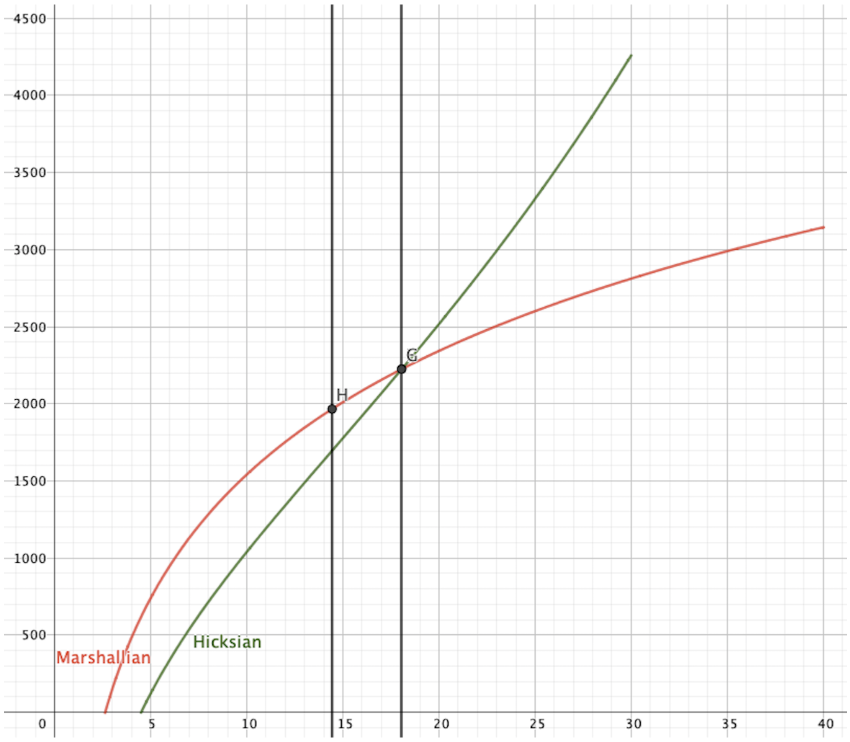

Figure 1: Labour supply functions in the (w, L) space.

Fourth, we must invert these functions. The Marshallian labour supply function is straightforward to invert, but the Hicksian is not because of the exponential integral. However, the Hicksian labour supply function is straightforward to approximate with a Taylor approximation, which in turn is easier to invert. The figure below shows the the Hicksian labour supply function and the 3rd order Taylor approximation around point A. Please notice in particular how similar these functions are between point A and B, the domain (or range, once we invert the function) that we are the most interested in.

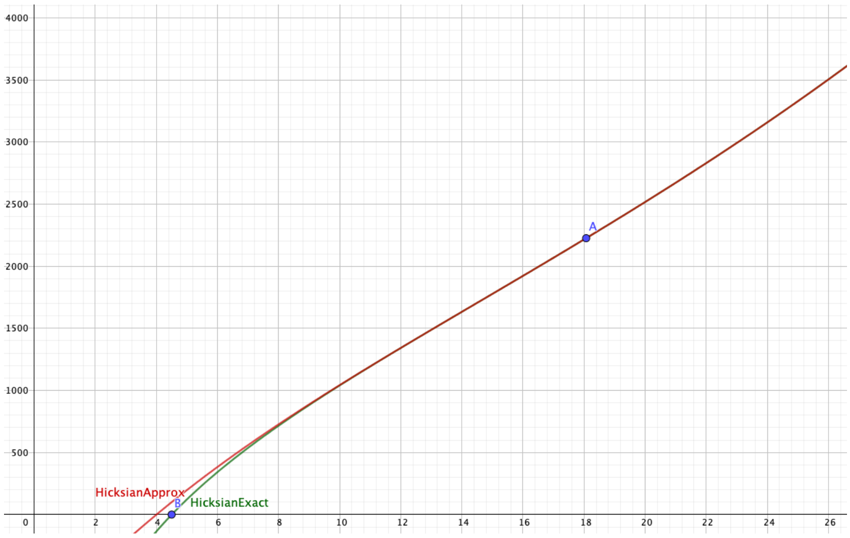

Figure 2: The exact and approximated Hicksian labour supply functions in the (w, L) space.

Fifth, we can then invert the Marshallian labour supply function and the approximated Hicksian labour supply function to show the labour supply functions in the (L, w) space. The horizontal lines show the gross and net wage rates.

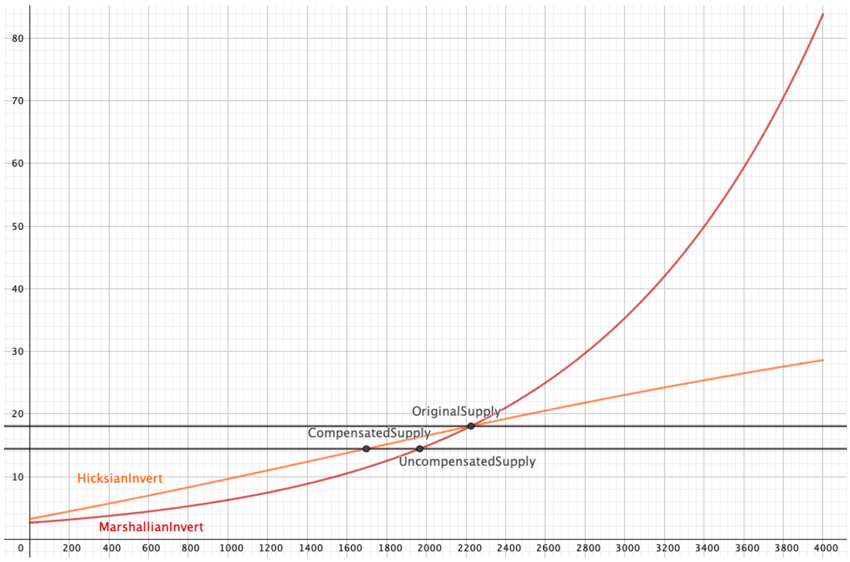

Figure 3: The labour supply functions in the (L, w) space.

8.2 Equivalent Variation

Equivalent Variation applied to labour supply is the compensation the worker must have to achieve the same utility level as before given that that the wage change (here, due to the marginal tax rate is introduced) has occurred. Since, in optimum, the budget will be used up, we can use the minimal expenditure function to calculate the equivalent variation. Let τ denote the marginal tax rate. Then:

$$EV = e(w(1-\tau, u) - e(w, u)$$

University of California, Berkeley (n.d.) details how $e(w, u)$ can be expanded into a second-order Taylor expansion, ultimately leaving us with the following equivalent variation:

$$EV = (\tau w) \cdot L^u\big(w(1-\tau)), y\big)\left(1 + \frac{1}{2}\tau\varepsilon^c\right)$$

With a marginal tax rate of 20 %, the average worker pays $0.2 \cdot 18.05 = 3.61$ dollars in taxes per hour worked. For a net wage rate of $18.05 - 3.61 = 14.44$, labour supply given $y = 17462$ will be $L^u(14.44, 17462) = -457.92 + 1156 \log\big(14.44\big) - 0.038 \cdot 17462 \approx 1965$ hours. Additionally, we have $\varepsilon^c_{Lw} \approx 1,206$. This leaves us with:

$$EV \approx 3.61 \cdot 1965\big(1 + \frac{1}{2}0.2 \cdot 1.206\big)$$

Which sums to:

$$EV \approx 7949$$

8.3 Tax Revenue

In the previous section, we found that tax revenues are approximately $3.61 \cdot 1965$. Hence:

$$Tax \approx 7094$$

8.4 Deadweight Loss

Deadweight loss as percent of tax revenue:

$$\frac{DWL}{Tax} = \frac{EV - Tax}{Tax}$$

Substituting $EV \approx 7949$ and $Tax \approx 7094$ gives us (I have used exact numbers in calculating the result below):

$$\frac{DWL}{Tax} \approx 0.1206$$

Please note the relation between $\frac{DWL}{Tax} \approx 0.1206$ and $\varepsilon^c_{Lw} \approx 1.206$, which is due to the fact that $\frac{DWL}{Tax} \approx \frac{1}{2}\tau\varepsilon^c_{Lw}$, with $\frac{1}{2}\tau = 0.1$.

9 Bibliography

- University of California, Berkeley (n.d.) *Economics 250a—Lecture 2*, University of California, Berkeley, `eml.berkeley.edu/~cle/e250a_f14/lecture2.pdf`.