Proof for CI of bridge model

~~Part~~ CI

~~$Var(\hat{\mu}_{y|x_0}) = Var(\sigma^2$~~

Future CI,

$$Var(y_{future}) = \sigma^2 + Var(x_{future}\hat{\beta}) \quad \text{where } \sigma^2 = Var(\varepsilon)$$

Consider law of total Variance

$$Var(x_{future}\hat{\beta}) = E(Var(x_{future}\hat{\beta} \mid x_{future})) + Var(E[x_{future}\hat{\beta} \mid x_{future}])$$

Consider the second term,

$$E[x_{future}\hat{\beta} \mid x_{future}] = x_{future}E[\hat{\beta}] \cancel{= x_{future}}$$

$$Var(x_{future}E[\beta]) = E[\beta]' Var(x_{future}) E[\beta]$$

Consider the first term,

$$Var(x_{future}\hat{\beta} \mid x_{future}) = x_{future}' \sigma^2 (X'X)^{-1} x_{future}$$

$$E(x_{future}' \sigma^2 (X'X)^{-1} x_{future}) = \sigma^2 tr((X'X)^{-1}\Sigma_{future}) + E[x_{future}]' \sigma^2 (X'X)^{-1} E[x_{future}]$$

$$Var(y_{future}) = \sigma^2 + E[\beta]' \Sigma_{x_{future}} E[\beta] + \sigma^2 tr((X'X)^{-1}\Sigma_{x_{future}}) + E[x_{future}]' \sigma^2 (X'X)^{-1} E[x_{future}]$$

Part CI

$$Var(\hat{\mu}_{y|x_0}) = Var(x_0\hat{\beta}) = x_0' Var(\hat{\beta}) x_0 = \sigma^2 x_0'(X'X)^{-1}x_0$$

For simplicity and since we use independent models to predict each component of x_{future}

$$x_{future} \sim N\left(\mu, \begin{pmatrix} Var(x_1) \\ \vdots \\ Var(x_n) \end{pmatrix}\right)$$, μ is the predicted value, $Var(x_n)$ = mse of predicted value

$$\Sigma_{x_{future}} = \begin{pmatrix} Var(x_1) & 0 & \cdots \\ 0 & & \vdots \\ \vdots & \cdots & \end{pmatrix}$$, should be a diagonal matrix.

Can say that $x_{future}\hat{\beta}$ follows a normal distribution by CLT, We assume lindenberg condition is true as we have many terms and all variances are similar

$x_{future}'\hat{\beta} = x_{future}(X'X)^{-1}X'y$ thus $= \sum_{i=1}\sum_{k=1} x_{future,i}\, y_k [(X'X)^{-1}X']_{ki}$. Thus the ~~product~~ sum of many independent variable, & by CLT

$$y_{future} \sim N\left(x_{future}'\hat{\beta},\ \sigma^2 + E[\beta]'\Sigma_{x_{future}}E[\beta] + \sigma^2 tr[(X'X)^{-1}\Sigma_{x_{future}}] + \sigma^2 E[x_{future}]'(X'X)^{-1}E[x_{future}]\right)$$