Analytical and Numerical Analysis of Fluid Flow In A Clyindrical Container Using The Navier-Stokes Equations

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1 Introduction

Here we will consider the flow of incompressible fluids with kinematic viscosity ν in a no-slip, circular cylinder of height H, radius R driven by the bottom endwall with a constant angular velocity Ω. This allows us to define on the length scale of R and time scale $1/\Omega$, we have the Reynolds number, $Re = \Omega R^2/\nu$. In the field of fluid dynamics the non-dimensional Navier-Stokes equations describe the flow of incompressible fluids and are given as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{Re} \nabla^2 \mathbf{u}, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2}$$

for the flow velocity of the fluid $\mathbf{u} = \left\langle \frac{\partial q_1}{\partial t}, \frac{\partial q_2}{\partial t}, \frac{\partial q_3}{\partial t} \right\rangle$ (in generalized coordinates). Here equation (1) represents the conservation of momentum of our system and equation (2) states that our fluid is incompressible. Our goal here is to get this into a form where we can apply methods for solving partial differential equations. Since our system has circular cylindrical symmetry we can map $\mathbf{u}$ to circular cylindrical coordinates as the vector $\langle \dot{r}, \dot{\theta}, \dot{z} \rangle = \langle u, v, w \rangle$. With this choice of coordinate system it is useful to note the gradient, Laplacian, divergence, and curl in cylidrical coordinates,[1]

$$\boldsymbol{\nabla} = \mathrm{r}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \mathrm{z}\frac{\partial}{\partial z} \tag{3}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{4}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r\ A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{5}$$

$$\boldsymbol{\nabla} \times \mathbf{A} = \left\langle \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right), \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right), \frac{1}{r}\left(\frac{\partial}{\partial r}(r\ A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \right\rangle. \tag{6}$$

Next we are going to want to define what $\nabla^2 \mathbf{u}$ and $\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}$ are in cylindrical coordinates. We can write

$$\nabla^2 \mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}).^2 \tag{7}$$

[1] John Taylor *Classical Mechanics* back cover
[2] ASU Physics Department *Tutoral: Vector Calculus*, equation 29.

When applying equations (3), (5), and (6) to $\nabla(\nabla\cdot\mathbf{u})-\nabla\times(\nabla\times\mathbf{u})$ we get (in component form),

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{u})-\nabla\times(\nabla\times\mathbf{u}) = \bigg\langle &\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u}{\partial\theta}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right),\\ &\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}+\frac{\partial^2 v}{\partial z^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{2}{r^2}\frac{\partial v}{\partial\theta}-\frac{v}{r^2}\right),\\ &\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial w}{\partial z}\right)\bigg\rangle.\end{aligned}\tag{8}$$

We know that $\mathbf{u}\cdot\nabla\mathbf{u}$ In cylindrical coordinates[3] is,

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = \bigg\langle &\left(u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)\\ &\left(u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+w\frac{\partial v}{\partial z}+\frac{v^2}{r}\right),\\ &\left(u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\theta}+w\frac{\partial w}{\partial z}\right)\bigg\rangle.\end{aligned}\tag{9}$$

For the following we will assume that the Reynolds number is sufficiently small ($Re \lesssim 10^3$) which implies that the flow is steady and axisymmetric (i.e. $\partial/\partial t = 0$ and $\partial/\partial\theta = 0$ respectively). Using the results from equations (3), (8), and (9) into equation (1) for low Reynolds numbers we will have the following components of the axisymmetric version of equation (1) with an aspect ration $H/R = \Gamma \sim \mathcal{O}(1)$,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{\partial p}{\partial r}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right]\tag{10}$$

$$\frac{\partial v}{\partial t}+u\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+w\frac{\partial v}{\partial z} = \frac{1}{Re}\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right]\tag{11}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial r}+\frac{1}{Re}\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right].\tag{12}$$

We can also get the axisymmetric version of equation (2) by applying the results from equation (5) to get,

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0.\tag{13}$$

Due to the axisymmetry, it is also convenient to define the streamfunction ψ such that,

$$\langle u, v, w\rangle = \left\langle -\frac{1}{r}\frac{\partial\psi}{\partial z}, v, \frac{1}{r}\frac{\partial\psi}{\partial r}\right\rangle.$$

With this we can show that this satisfies the incompressibiblity condition equation (13)

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} &= \frac{1}{r}\frac{\partial}{\partial r}\left(-r\frac{1}{r}\frac{\partial\psi}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right)\\ &= -\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial z}+\frac{1}{r}\frac{\partial^2\psi}{\partial z\partial r}\\ &= 0.\end{aligned}$$

[3] Weisstein, Eric W. "Convective Operator." From MathWorld–A Wolfram Web Resource.

Another useful quantity to define is the vorticity of the fluid. This can be defined as $\boldsymbol{\nabla} \times \mathbf{u}$. Applying the result of equation (6) we have the vorticity being,

$$\boldsymbol{\nabla} \times \mathbf{u} = \left\langle -\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rv) \right\rangle.$$

For notation sake we take

$$\begin{aligned} \eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ &= -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial r}, \end{aligned}$$

which we get by substituting the streamfunction into η. This leads to the implication that,

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial \psi}{\partial r} = -r\eta. \tag{14}$$

Looking at equations (10) and (12) we can eliminate the constraint on pressure p by doing,

$$\frac{\partial(10)}{\partial z} - \frac{\partial(12)}{\partial r},$$

resulting in the left hand side of the equation being

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right] + u\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) - \frac{2v}{r}\frac{\partial v}{\partial z},$$

which when applying the streamfunction becomes

$$\frac{\partial \eta}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \eta}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \eta}{\partial z} + \frac{\eta}{r^2}\frac{\partial \psi}{\partial z} - \frac{2v}{r}\frac{\partial v}{\partial z}.$$

Now examining the right hand side it becomes

$$\frac{1}{Re}\left[\frac{\partial^2}{\partial r^2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) - \frac{1}{r^2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right],$$

which after applying the streamfunction gives

$$\frac{1}{Re}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} - \frac{\eta}{r^2} + \frac{\partial^2 \eta}{\partial z^2}\right),$$

resulting in

$$\frac{\partial \eta}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \eta}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \eta}{\partial z} + \frac{\eta}{r^2}\frac{\partial \psi}{\partial z} - \frac{2v}{r}\frac{\partial v}{\partial z} = \frac{1}{Re}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} - \frac{\eta}{r^2} + \frac{\partial^2 \eta}{\partial z^2}\right). \tag{15}$$

. We also can write equation (11) in terms of the streamfunction which yields

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v}{\partial z} &= \frac{1}{Re}\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right] \\ Re\left[\frac{\partial v}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v}{\partial z}\right] &= \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}. \end{aligned} \tag{16}$$

Now we will consider the boundary conditions and other simplifications to obtain a solution. First since we will consider the inertia-less where the fluid is moving relatively slow meaning that the angular velocity is close to zero which in turn implies $Re \to 0$ which when applied to equation (16) gives,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial^2 v}{\partial z^2} = 0, \tag{17}$$

where $r \in [0,1], z \in [0,\Gamma]$. Now considering the no-slip condition of the cylinder (meaning that the velocity on any stationary wall is 0) along with the rotating bottom the following boundary conditions for equation (17) are,

$$\begin{aligned}
\text{Stationary sidewall at } r = 1 &: \langle u, v, w\rangle = \langle 0,0,0\rangle \\
\text{Stationary top at } z = \Gamma &: \langle u, v, w\rangle = \langle 0,0,0\rangle \\
\text{Rotating bottom at } z = 0 &: \langle u, v, w\rangle = \langle 0,r,0\rangle \\
\text{Axis } r = 0 &: \left\langle u, v, \frac{\partial w}{\partial t}\right\rangle = \langle 0,0,0\rangle
\end{aligned}$$

or if we describe v as a function of r and z we can write the conditions as,

$$v(0,z) = v(1,z) = v(r,\Gamma) = 0 \tag{18}$$

$$v(r,0) = r. \tag{19}$$

Note that the boundary conditions for ψ and η as

$$\begin{aligned}
&\psi(0,z,t) = \psi(1,z,t) = \psi(r,0,t) = \psi(r,\Gamma,t) = 0 \\
&\eta(0,z,t) = 0 \\
&\eta(1,z,t) = -\frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} \\
&\eta(r,0,t) = -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} \\
&\eta(r,\Gamma,t) = -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2}.
\end{aligned}$$

2 Analytic Solution for Low Reynolds Number Flow

We begin by noting we have non-homogeneous boundary conditions on the z conditions. We can apply a shift to our boundary conditions to make the z boundary conditions homogeneous and the r boundary conditions non-homogeneous. First we let $v(r,z) = f(r,z) + g(r,z)$ and assume that $f(r,z)$ is a bilinear function $f(r,z) = A + Br + Cz + Drz$ and rewrite the boundary conditions for $f(r,z)$ as

$$\begin{aligned}
f(r,\Gamma) &= 0 \\
f(r,0) &= r,
\end{aligned} \tag{20}$$

such that

$$g(r,\Gamma) = g(r,0) = 0. \tag{21}$$

When we apply the boundary conditions from equation (20) at $z = 0$ and then at $z = \Gamma$ we have

$$\begin{aligned} f(r,0) &= A + Br = r \implies A = 0, B = 1 \\ f(r,\Gamma) &= r + C\Gamma + Dr\Gamma = 0 \\ &= r(1 + D\Gamma) + C\Gamma = 0 \implies C = 0, D = \frac{1}{\Gamma}. \end{aligned}$$

Since $f(r,z) = r(1 + z/\Gamma)$ we can now solve for $g(r,z)$. To find the boundary conditions note that $g(r,z) = v(r,z) - f(r,z)$ meaning our boundary conditions for $g(r,z)$ are,

$$g(r,0) = v(r,0) - f(r,0) = r - r = 0 \tag{22}$$
$$g(r,\Gamma) = v(r,\Gamma) - f(r,\Gamma) = 0 - 0 = 0 \tag{23}$$
$$g(0,z) = v(0,z) - f(0,z) = 0 - 0 = 0 \tag{24}$$
$$g(1,z) = v(1,z) - f(1,z) = 0 - \left(1 - \frac{z}{\Gamma}\right) = \frac{z}{\Gamma} - 1 \tag{25}$$

From here we can use the method of separation of variables by assuming $g(r,z) = R(r)Z(z)$ and plugging it into equation (17) to get

$$\begin{aligned} \frac{R''Z}{RZ} + \frac{1}{r}\frac{R'Z}{RZ} - \frac{RZ}{r^2} &= -\frac{RZ''}{Z} \\ \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{R}{r^2} &= \lambda = -\frac{Z''}{Z} \end{aligned} \tag{26}$$

Here we will be looking at three cases for the eigenvalue λ.

2.1 Applying z Spacial Boundary Conditions

CASE I: $\lambda = 0$

Consider the z ODE,

$$Z'' = 0 \implies Z = Az + B.$$

When applying the boundary conditions $Z(0) = Z(\Gamma) = 0$ we have that,

$$\begin{aligned} Z(0) &= 0 = A(0) + B \implies B = 0 \\ Z(\Gamma) &= 0 = A\Gamma \implies A = 0, \end{aligned}$$

hence $\lambda = 0$ only yields trivial solutions and will not be considered.

CASE II: $\lambda = -\omega^2 < 0$

Considering the z ODE again we have

$$Z'' = \omega^2 Z \implies Z = Ae^{\omega Z} + Be^{-\omega Z}$$

Applying the same boundary conditions as in case I we have,

$$\begin{aligned} Z(0) &= 0 = A + B \implies A = -B \\ Z(\Gamma) &= 0 = A\left(e^{\omega Z} - e^{-\omega Z}\right) \implies A = 0. \end{aligned}$$

Since $\lambda = -\omega^2 < 0$ only yields trivial solutions it will not be considered.

CASE III: $\lambda = \omega^2 > 0$

Again consider the z ODE,

$$Z'' = -\omega^2 Z \implies Z = A\cos(\omega z) + B\sin(\omega z)\,.$$

Here when we apply the z spacial boundary conditions we get,

$$\begin{aligned} Z(0) &= 0 = A\cos(0) + B\sin(0) = A \implies A = 0 \\ Z(\Gamma) &= 0 = B\sin(\omega\Gamma) \implies 0 = B\sin(\omega\Gamma) \implies \omega\Gamma = n\pi \implies \omega = \frac{n\pi}{\Gamma}, n \in \mathbb{N}. \end{aligned}$$

Here we see that there is a non-trival contribution for the z ODE meaning,

$$g(r,z) = \sum_{n=1}^{\infty} B_n\ R(r)\sin\left(\frac{n\pi z}{\Gamma}\right). \tag{27}$$

2.2 Applying the r Boundary Conditions

Now we will consider the r ODE described in equation (26) we can rewrite it using the definition $\lambda = \omega^2 = n^2\pi^2/\Gamma^2$ as,

$$r^2R'' + rR' - \left[1 - \frac{n^2\pi^2}{\Gamma^2}r^2\right]R = 0. \tag{28}$$

This is a known ODE known as the modified Bessel's equation which has the solution of

$$R_n(r) = A\ I_1\left(\frac{n\pi}{\Gamma}r\right) + B\ K_1\left(\frac{n\pi}{\Gamma}r\right),$$

where $I_1(\frac{n\pi}{\Gamma}r)$ and $k_1(\frac{n\pi}{\Gamma}r)$ are the modified Bessel's functions of the first and second kind respectively of order one. Here we will apply the condition at $r = 0$ which gives,

$$R_n(0) = 0 = A\ I_1(0) + B\ K_1(0).$$

Since we need bounded solutions for our system $K_1(\frac{n\pi}{\Gamma}r)$ is ignored since as $r \to 0$, $K_1(r) \to \infty$ so we have now

$$g(r,z) = \sum_{n=1}^{\infty} C_n\ I_1\left(\frac{n\pi}{\Gamma}r\right)\sin\left(\frac{n\pi z}{\Gamma}\right), C_n = A \cdot B_n. \tag{29}$$

Now we need to solve for C_n. We can do this by applying the condition at $g(1,z)$. Thus we have,

$$g(1,z) = \frac{z}{\Gamma} - 1 = \sum_{n=1}^{\infty} C_n\ I_1\left(\frac{n\pi}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right). \tag{30}$$

We would like to apply orthogonality to this so we will multiply both sides of equation (30) by $\sin(m\pi z/\Gamma)$ to get,

$$\begin{aligned} \left(\frac{z}{\Gamma} - 1\right)\sin\left(\frac{m\pi z}{\Gamma}\right) &= \sum_{n=1}^{\infty} C_n\ I_1\left(\frac{n\pi}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right)\sin\left(\frac{m\pi z}{\Gamma}\right) \\ \int_0^{\Gamma}\left(\frac{z}{\Gamma} - 1\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz &= \sum_{n=1}^{\infty} C_n\ I_1\left(\frac{n\pi}{\Gamma}\right)\int_0^{\Gamma}\sin\left(\frac{n\pi z}{\Gamma}\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz. \end{aligned} \tag{31}$$

Using half wave orthogonality,

$$\int_{x_0}^{x_0+\lambda_1} \sin\left(\frac{m\pi z}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right)dz = \frac{\lambda_1}{4}\delta_{mn}, \tag{32}$$

and knowing $\lambda_1 = 2\Gamma$ we have

$$\int_0^{2\Gamma} \sin\left(\frac{m\pi z}{\Gamma}\right)\sin\left(\frac{n\pi z}{\Gamma}\right)dz = \frac{\Gamma}{2}\delta_{mn}, \tag{33}$$

for the integral on the right hand side of equation (32) leaving us with,

$$\begin{aligned}\int_0^{\Gamma}\left(\frac{z}{\Gamma}-1\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz &= \sum_{n=1}^{\infty} C_n\ I_1\left(\frac{n\pi}{\Gamma}\right)\frac{\Gamma}{2}\delta_{mn}\\ &= C_m I_1\left(\frac{m\pi}{\Gamma}\right)\frac{\Gamma}{2}.\end{aligned}$$

Now looking at the left hand side of equation (31) we can use Mathematica to solve this integral for us using the following code

```
Integrate[(z/gamma - 1) Sin[(m Pi z)/gamma], {z, 0, gamma}],
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which gives us that

$$\int_0^{\Gamma}\left(\frac{z}{\Gamma}-1\right)\sin\left(\frac{m\pi z}{\Gamma}\right)dz = -\frac{\Gamma}{m\pi}.$$

Thus equation (32) can be written as

$$\begin{aligned}-\frac{\Gamma}{m\pi} &= C_m I_1\left(\frac{m\pi}{\Gamma}\right)\frac{\Gamma}{2}\\ C_m &= -\frac{2}{m\pi I_1\left(\frac{m\pi}{\Gamma}\right)}.\end{aligned}$$

Thus we have the solution for $g(r,z)$ being,

$$g(r,z) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{I_1\left(n\pi r/\Gamma\right)}{nI_1\left(n\pi/\Gamma\right)}\sin\left(\frac{n\pi z}{\Gamma}\right)\right]. \tag{34}$$

2.3 The Solution

Using the principle of superposition we know that since $f(r,z)$ and $g(r,z)$ are both solutions to equation (17) the sum of the two is also a valid solution. Thus the solution to equation (17) given the boundary conditions described by (18) and (19) is

$$v(r,z) = r\left(1-\frac{z}{\Gamma}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{I_1\left(n\pi r/\Gamma\right)}{nI_1\left(n\pi/\Gamma\right)}\sin\left(\frac{n\pi z}{\Gamma}\right)\right]. \tag{35}$$

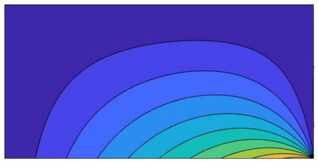

Figure 1: $v(r,z)$ with $\Gamma = 0.5$

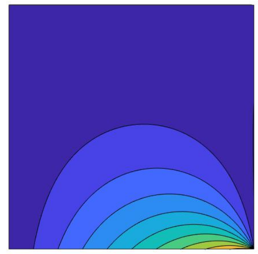

Figure 2: $v(r,z)$ with $\Gamma = 0.5$

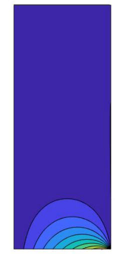

Figure 3: $v(r,z)$ with $\Gamma = 2.5$

From both the figures and from the the boundary conditions we can note that there is a singularity at $(r,z) = (1,0)$ since in the boundary conditions we define that point to be 0 with the condition $v(1,z) = 0$ and to be r with the condition on $v(r,0)$.

2.4 Additional notes

We can use an l_2 norm as a measure of stability for our numerical solution which can be given as,

$$||\mathbf{l_2}|| = \sqrt{\frac{1}{(n_r+1)(n_z+1)} \sum_{i=1}^{n_r} \sum_{j=1}^{n_z} (v_{i,j})^2}, \tag{36}$$

for a discrete array where $n_r, n_z \in \mathbb{N}$. Here we can see that as we add more terms to the series in equation (35) the l_2 norm converges in the following figures,

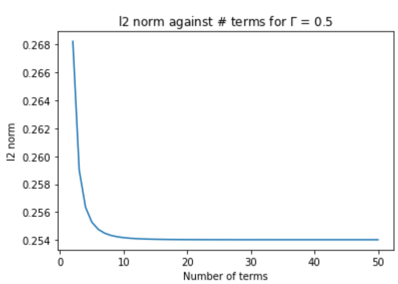

Figure 4: l_2 norm against #terms with $\Gamma = 0.5$

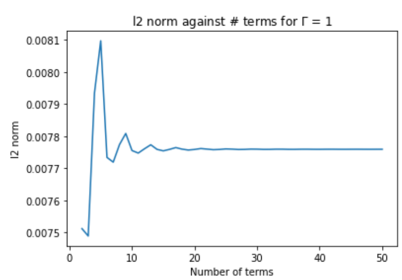

Figure 5: l_2 norm against #terms with $\Gamma = 0.5$

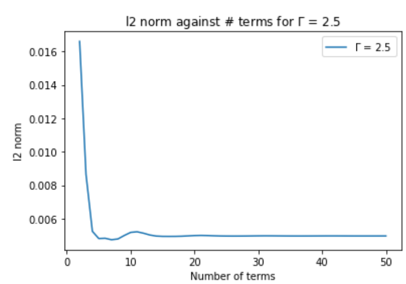

Figure 6: l_2 norm against #terms with $\Gamma = 2.5$

Here we see that for each value of Γ the l_2 norm converges as we increase n_r and n_z.

3 Numerical Analysis for Low Reynolds Number Flow

3.1 Steady State Calculations using Method of Finite Differences

Earlier we defined the partial differential equation to describe the flow of fluid in our cylindrical container

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial^2 v}{\partial z^2} = 0, \tag{37}$$

with boundary conditions,

$$v(0,z) = v(1,z) = v(r,\Gamma) = 0 \tag{38}$$

$$v(r,0) = r. \tag{39}$$

We made note that in our analytic solution there was a singularity at the point $(r,z) = (1,0)$ since our boundary conditions describe $v(1,0)$ being both 0 and 1. Since in nature we cannot have singularities we will apply numerical methods to get a more realistic solution. For this we will use the method of finite differences.

3.2 Brief Overview of the Finite Difference Method

Recall that for a smooth and continuous function $f(x)$ we have that the Taylor series approximation is

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + \frac{\delta_x^2}{2!}f''(x) + \cdots + \frac{\delta_x^n}{n!}f^{(n)}(x) + \cdots. \tag{40}$$

We will truncate the expansion after the second derivative to get the first order derivative's approximation which leaves us with,

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + \frac{\delta_x^2}{2!}f''(x) + \mathcal{O}(\delta_x^3) \tag{41}$$

$$f(x-\delta_x) = f(x) - \delta_x f'(x) + \frac{\delta_x^2}{2!}f''(x) - \mathcal{O}(\delta_x^3). \tag{42}$$

Subtracting equation (42) from (41) we have

$$f(x+\delta_x) - f(x-\delta_x) = 2\delta_x f'(x) + \mathcal{O}(\delta_x^2)$$

$$\implies f'(x) = \frac{f(x+\delta_x) - f(x-\delta_x)}{2\delta_x} + \mathcal{O}(\delta_x^2). \tag{43}$$

Next we'll expand equations (40) to the third order derivative to approximate the second order derivative as

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + \frac{\delta_x^2}{2!}f''(x) + \frac{\delta_x^3}{3!}f'''(x) + \mathcal{O}(\delta_x^4) \tag{44}$$

$$f(x-\delta_x) = f(x) - \delta_x f'(x) + \frac{\delta_x^2}{2!}f''(x) - \frac{\delta_x^3}{3!}f'''(x) + \mathcal{O}(\delta_x^4) \tag{45}$$

and adding equations (44) and (45) gives,

$$f(x+\delta_x) + f(x-\delta_x) = 2f(x) + \delta_x^2 f''(x) + \mathcal{O}(\delta_x^4)$$

$$\implies f''(x) = \frac{f(x+\delta_x) - 2f(x) + f(x-\delta_x)}{2\delta_x^2} + \mathcal{O}(\delta_x^2). \tag{46}$$

Next we will discretize the meridonal plane into a uniform grid by stating that $(r_i, z_j) = (i\delta_r, j\delta_z)$ for $i \in \{1, 2, \dots n_r\}$ and $j \in \{1, 2, \dots n_z\}$ where $\delta_r = 1/n_r$ and $\delta_z = \Gamma/n_z$. We'll use the notation $v_{i,j}$ for the function value, $v(r_i, z_j)$. With this we can rewrite equation (37) as

$$\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\delta_r^2} + \frac{v_{i+1,j} - v_{i-1,j}}{2i\delta_r^2} - \frac{v_{i,j}}{i^2\delta_r^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\delta_z^2} = 0$$

$$\frac{1}{\delta_r^2}\left[\left(1 - \frac{1}{2i}\right) v_{i-1,j} - \left(2 + \frac{1}{i^2}\right) v_{i,j} + \left(1 + \frac{1}{2i}\right) v_{i+1,j}\right] + \frac{1}{\delta_z^2}\left[v_{i,j-1} - 2v_{i,j} + v_{i,j+1}\right] = 0 \tag{47}$$

$$a_{n-1}v_{i-1,j} - a_{nn}v_{i,j} + a_{n+1}v_{i+1,j} + v_{i,j-1}\frac{1}{\delta_z^2} - v_{i,j}b_{mm} + v_{i,j+1}\frac{1}{\delta_z^2} = 0, \tag{48}$$

using our definitions above. From here since there is a discrete number of points we can represent our equation above as the matrix equation

$$A_{nn}V_{nm} + V_{nm}B_{mm} = F_{nm}, \tag{49}$$

where A_{nn} is the tridiagonal, $n \times n$, r-difference matrix, with the primary diagonal has elements a_{nn} for $i \in [1, n]$, sub-diagonal elements a_{n-1} for $i \in [2, n]$, and super-diagonal elements a_{n+1} for $i \in [1, n-1]$; B_{mm} is the tridiagonal $m \times m$, z-difference matrix, with main diagonal elements b_{mm}, and with sub and super diagonal elements being $1/\delta_z^2$; V_{nm} being our solution to equation (37) of size $n \times m$ with entries $v_{i,j}$; and F_{nm} is the right hand size with elements $f_{1,j}$ being $-i\delta_r/\delta_z^2$ for $i \in [1, n]$ as to account for the boundary condition $v(r, 0) = r$ and zero elsewhere. Solving matrix equation (49) directly is resource intensive so diagonalizing matrix A_{nn} by using a similarity transformation is used to speed up the calculations.
Using a similarity transform we get $Z_{nn}^{-1}A_{nn}Z_{nn} = E_{nn}$ where E_{nn} is the diagonal matrix containing the eigenvalues of A_{nn} and Z_{nn} is the matrix of the corresponding eigenvectors with Z_{nn}^{-1} being it's inverse matrix. When we Substitute $V_{nm} = Z_{nn}U_{nm}$ for matrix U_{nm} is to determined later into (49) we have,

$$\begin{aligned} A_{nn}Z_{nn}U_{nm} + Z_{nn}U_{nm}B_{mm} &= F_{nm} \\ Z_{nn}^{-1}A_{nn}Z_{nn}U_{nm} + Z_{nn}^{-1}Z_{nn}U_{nm}B_{mm} &= Z_{nn}^{-1}F_{nm} \\ E_{nn}U_{nm} + U_{nm}B_{mm} &= Z_{nn}^{-1}F_{nm}. \end{aligned}$$

Here to take advantage of the symmetries of E_{nn} and B_{mm}, we take the transpose of the matrix equation above resulting in

$$B_{mm}U_{nm}^T + U_{nm}^T E_{nn} = Z_{nn}^{-1}F_{nm} = H_{mn}. \tag{50}$$

From here we define a vector $\mathbf{u_i}$ being the rows of U_{nm}, $\mathbf{h_i}$ being the columns of H_{mn}, along with e_i being the eigenvalues of A_{nn}, equation (50) can be written as

$$\left(B_{mm} + e_i I_{mm}\right)\mathbf{u_i} = \mathbf{h_i}, i \in [i, n]. \tag{51}$$

This equation can be solved faster than (49) and upon solving these n equations for U_{nm} we get the numerical answer since $V_{nm} = Z_{nn}U_{nm}$.

3.3 Numerical Solutions

Here you can see when programming the method described above in Python 3 we have the following diagrams,

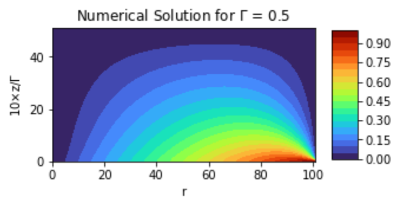

Figure 7: Numerical solution to $v(r, z)$ with $\Gamma = 0.5$

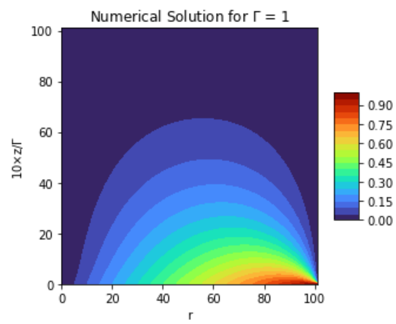

Figure 8: Numerical solution to $v(r, z)$ with $\Gamma = 1$

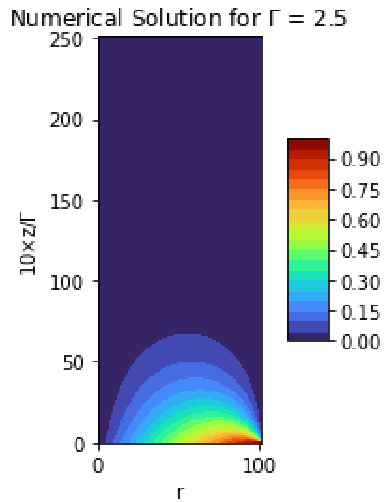

Figure 9: Numerical solution to $v(r, z)$ with $\Gamma = 2.5$

From here we note the accuracy of our numerical solution by taking the l_2 norm with the exact solution of $v(r, z)$. The relative error is defined as

$$\text{relative error} = \frac{l_2\left(v_{\text{exact}}\right) - l_2\left(v_{\text{numerical}}\right)}{l_2\left(v_{\text{exact}}\right)}. \tag{52}$$

First the l_2 norm was found to be for varying gammas,

Γ	l_2 norm
0.5	0.2448185051721005
1.0	0.18392053590295437
2.5	0.11755681692232807

Table 1: Values for l_2 norm for varying values of Γ.

with the relative error being calculated as,

Γ	relative error
0.5	0.011543499399148123
1.0	0.020582494476482086
2.5	0.024652470224409334

Table 2: Relative for varying values of Γ.

We can take note that if we take the limit as n_r and n_z approach infinity of equation (36) and knowing that summations in equation (36) converge as $n_r, n_z \to \infty$ to some value $v^2_{n_r,n_z}$ we have,

$$\begin{aligned}
\lim_{\substack{n_r\to\infty \\ n_z\to\infty}} \left[\sqrt{\frac{1}{(n_r+1)(n_z+1)}\sum_{i=1}^{n_r}\sum_{j=1}^{n_z}(v_{i,j})^2}\right] &= \lim_{\substack{n_r\to\infty \\ n_z\to\infty}} \left[\sqrt{\frac{1}{(n_r+1)(n_z+1)}}\sqrt{\sum_{i=1}^{n_r}\sum_{j=1}^{n_z}(v_{i,j})^2}\right] \\
&= v_{n_r,n_z} \lim_{\substack{n_r\to\infty \\ n_z\to\infty}} \sqrt{\frac{1}{(n_r+1)(n_z+1)}} \\
&= v_{n_r,n_z} \lim_{\substack{n_r\to\infty \\ n_z\to\infty}} \frac{1}{\sqrt{n_r+1}}\frac{1}{\sqrt{n_z+1}} \\
&= 0.
\end{aligned}$$

These results show two things. First is that the error of our numerical solution is at or near the order of 10^3. This high amount of error led to further investigation into why and where the error propagated to this degree. To visualize where the error was the highest a plot of of the relative error was made per value of Γ which can be seen below, and second that as we make our grid finer, the l_2 norm vanishes.

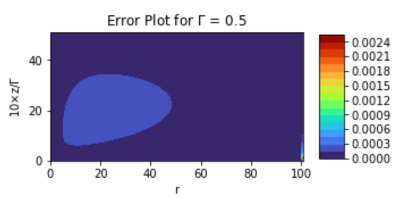

Figure 10: Numerical solution to $v(r, z)$ with $\Gamma = 0.5$

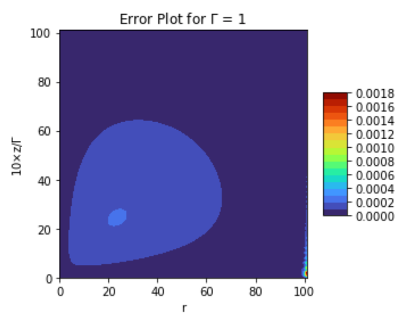

Figure 11: Numerical solution to $v(r, z)$ with $\Gamma = 1$

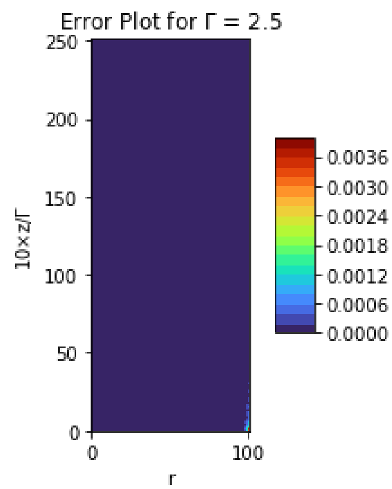

Figure 12: Numerical solution to $v(r, z)$ with $\Gamma = 2.5$

Here we see that a majority of the error is at or near the singularity so when summing over all the whole plane most of the error accumulation was near $(r, z) = (1, 0)$.

3.4 Transient Solution for Small Reynolds Numbers

To model the time evolution of our system, we used the method of lines. The method of lines (MOL for short), is the method of finding solutions to partial differential equations where we discretize all but one dimension (in our case the time dimension) and make a system of ODE's in which we can numerically integrate. We start by discretizing the PDE

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right], \tag{53}$$

giving us the equation for the interior gird points,

$$
\begin{aligned}
\frac{\partial v_{i,j}}{\partial t} &= \frac{1}{Re}\left[\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{\delta_r^2}+\frac{v_{i+1,j}-v_{i-1,j}}{2i\delta_r^2}-\frac{v_{i,j}}{i^2\delta_r^2}+\frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{\delta_z^2}\right] \\
&= \mathrm{RHS}(v_{i,j}).
\end{aligned} \tag{54}
$$

Here we make a function called $\mathrm{RHS}(v_{i,j})$ which will take in the array containing the exterior grid points and the interior grid points and will return another array. This here creates the system of linear ODE's we will numerically integrate. To do this Heun's method was used for the numerical integration. How Heun's method works is that it is a two step predictor-corrector method. First we predict where the next point of the function (time step $k+1$) by doing the following at time step k

$$v_{i,j}^p = v_{i,j}^k + \delta_t \cdot \mathrm{RHS}\left(v_{i,j}^k\right). \tag{55}$$

From here we use the result from equation (55) to correct our prediction by doing a similar process and averaging the previous step from the predicted step as seen below,

$$v_{i,j}^{k+1} = v_{i,j}^k + \frac{\delta_t}{2}\cdot\left(\mathrm{RHS}\left(v_{i,j}^p\right)+\mathrm{RHS}\left(v_{i,j}^k\right)\right). \tag{56}$$

A video of the time evolution of the system can be found HERE.
Upon testing it was noted that numerical stability happens on time steps

$$\mathcal{O}(\delta_t) = \max\left\{\mathcal{O}\left(\frac{\delta_r^2}{10\cdot Re}\right), \mathcal{O}\left(\frac{\delta_z^2}{10\cdot Re}\right)\right\}.$$

Calculating the relative error between the time evolution code and the steady state code was done by using the follwoing,

$$\text{relative error} = \frac{l_2\left(v_{\text{time evo}}\right)-l_2\left(v_{\text{steady state}}\right)}{l_2\left(v_{\text{time evo}}\right)}.$$

This result showed that the l_2 norm of the relative difference from the steady state solution and the time evolution code was 0.04820476770339076. This is a high error which means that the code used for time evolution does not have a fine enough time-step to to better approximate the value given by the steady state code. Another result to be noted is that when running different Reynolds numbers (specifically $Re = 10$ and $Re = 100$) the time to reach steady state took longer than when using $Re = 1$.

4 Numerical Analysis of Swirling Flow

Finally, we will analyze high Reynolds number flow of our system. Recall that equations (14), (15), and (16) which can be written as,

$$\frac{\partial v}{\partial t} = \frac{1}{r}\frac{\partial \psi}{\partial z}\Big(\frac{\partial v}{\partial r}+\frac{v}{r}\Big)-\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v}{\partial z}+\frac{1}{Re}\Big[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\Big] \tag{57}$$

$$\frac{\partial \eta}{\partial t} = \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \eta}{\partial r}-\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \eta}{\partial z}-\frac{\eta}{r^2}\frac{\partial \psi}{\partial z}+\frac{2v}{r}\frac{\partial v}{\partial z}+\frac{1}{Re}\Big[\frac{\partial^2 \eta}{\partial r^2}+\frac{1}{r}\frac{\partial \eta}{\partial r}-\frac{\eta}{r^2}+\frac{\partial^2 \eta}{\partial z^2}\Big] \tag{58}$$

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -r\eta, \tag{59}$$

subject to the boundary conditions

$$\begin{aligned}
v(0,z,t) &= v(1,z,t) = v(r,\Gamma,t) = 0\\
v(r,0,t) &= r\\
\psi(0,z,t) &= \psi(1,z,t) = \psi(r,0,t) = \psi(r,\Gamma,t) = 0\\
\eta(0,z,t) &= 0\\
\eta(1,z,t) &= -\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\\
\eta(r,0,t) &= -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\\
\eta(r,\Gamma,t) &= -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2},
\end{aligned}$$

govern this type of flow. Here we see that we cannot use method of lines on all of the equations in the system since the equation

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -r\eta,$$

does not explicitly depend on time. However since we know the values of $v(r,z,0), \psi(r,z,0)$, and $\eta(r,z,0)$ thus we can find the values of equations (57) and (58), then using the updated values of eta we can find the solution to equation (59). Discretizing equation (59) gives us

$$\frac{1}{\delta_r^2}\Big[\left(\psi^k_{i-1,j} - 2\psi^k_{i,j} + \psi^k_{i+1,j}\right) + \frac{1}{2i}\left(\psi^k_{i-1,j} - \psi^k_{i+1,j}\right)\Big] + \frac{1}{\delta_z^2}\left[\psi^k_{i,j-1} - 2\psi^k_{i,j} + \psi^k_{i,j+1}\right] = -i\delta_r\eta^k_{i,j}$$
$$a_{n-1}\psi_{i-1,j} + a_{nn}\psi_{i,j} + a_{n+1}\psi_{i+1,j} + b_{m-1}\psi_{i-1,j} + b_{mm}\psi_{i,j} + b_{m+1}\psi_{i+1,j} = -i\delta_r\eta^k_{i,j},$$

which we can write as the matrix equation

$$A_{nn}\Psi_{nm} + \Psi_{nm}B_{mm} = F_{nm}. \tag{60}$$

Here we can use the same method as in section 3 to solve for $\Psi_n m$. Once we have $\Psi_n m$ we can put use the values to update the boundary conditions for eta and use the method of lines for equations (57) and (58). Before we can use the method of lines we must resolve the eta boundary conditions we need to use finite differences. However unlike in section 3, we cannot use centered differences on any of the walls we only since values of $v, \eta,$ or ψ do not exist out side of the container. Thus we will use one sided differences. We begin by Taylor expanding the function ψ (where $' \equiv \partial r$) into

$$\psi_{n_r-1,j} = \cancelto{0}{\psi_{n_r,j}} - \delta_r\cancelto{0}{\psi'_{n_r,j}} + \frac{\delta_r^2}{2}\psi''_{n_r,j} - \frac{\delta_r^3}{3}\psi'''_{n_r,j} + O(\delta_r^4) \tag{61}$$

$$\psi_{n_r-2,j} = \cancelto{0}{\psi_{n_r,j}} - 2\delta_r\cancelto{0}{\psi'_{n_r,j}} + 2\delta_r^2\psi''_{n_r,j} - \frac{8\delta_r^3}{3}\psi'''_{n_r,j} + O(\delta_r^4) \tag{62}$$

which when doing (61) - (42) we get our second derivative as

$$\psi'' n_r, j = \frac{1}{2\delta_r^2}(8\psi_{n_r-1,j} - \psi_{n_r-2,j}), \tag{63}$$

and thus we get

$$\eta_{n_r,j} = -\frac{1}{r}\psi''_{n_r,j} = \frac{1}{2i\delta_r^3}(\psi_{n_r-2,j} - 8\psi_{n_r-1,j}). \tag{64}$$

Similar calculations can be done for the top and bottom walls (using $' \equiv \partial z$) we get for the top wall

$$\psi_{i,n_z-1}\psi_{i,n_z} - \delta_z \cancelto{0}{\psi'_{i,n_z}} + \tfrac{\delta_z^2}{2}\psi''_{i,n_z} - \tfrac{\delta_z^3}{3}\psi'''_{i,n_z} + O(\delta_z^4) = \tag{65}$$

$$\psi_{i,n_z-2} = \cancelto{0}{\psi_{i,n_z}} - 2\delta_z \cancelto{0}{\psi'_{i,n_z}} + 2\delta_z^2\psi''_{i,n_z} - \frac{8\delta_z^3}{3}\psi'''_{i,n_z} + O(\delta_z^4), \tag{66}$$

and doing (65) - (66) we get

$$\eta_{i,n_z} = -\frac{1}{r}\psi''_{i,n_z} = \frac{1}{2i\delta_r\delta_z^2}(\psi_{i,n_z-2} - 8\psi_{i,n_z-1}). \tag{67}$$

While for the bottom wall (using the same convention ' convention as the top wall) we have

$$\psi_{i,1} = \cancelto{0}{\psi_{i,0}} + \delta_z \cancelto{0}{\psi'_{i,0}} + \frac{\delta_z^2}{2}\psi''_{i,0} + \frac{\delta_z^3}{3}\psi'''_{i,0} + O(\delta_z^4) \tag{68}$$

$$\psi_{i,2} = \cancelto{0}{\psi_{i,0}} + 2\delta_z \cancelto{0}{\psi'_{i,0}} + 2\delta_z^2\psi''_{i,0} + \frac{8\delta_z^3}{3}\psi'''_{i,0} + O(\delta_z^4), \tag{69}$$

and subtracting (69) from (68) we get the boundary on the bottom wall to be

$$\eta_{i,0} = -\frac{1}{r}\psi''_{i,0} = \frac{1}{2i\delta_r\delta_z^2}(\psi_{i,2} - 8\psi_{i,1}). \tag{70}$$

After running the simulation for this system we note that for $Re \sim 1$ our solution roughly modeled the system described in section 3. However for $Re \sim 10$ we notice more distortion in the contours but have some resemblance to the numerical solutions described in section 3, and for $Re \sim 1000$ we have a completely different behavior that seems to behave sporadically compared to the solutions described in section 3.
Videos for the velocity ($v(r, z)$) for varying Reynolds numbers can be found by clicking on the following Re = 1, Re = 10, Re = 100, Re = 1000

5 Appendix

5.1 Documentation of Code

All code documentation can be found at the github link HERE.