MDS Assignment 6

1.

(a) We have: $A.\begin{bmatrix}1\\10\end{bmatrix} = \begin{bmatrix}x & y\\10 & 7\end{bmatrix}\begin{bmatrix}1\\10\end{bmatrix} = \begin{bmatrix}x+10y\\10+70\end{bmatrix} = \begin{bmatrix}x+10y\\80\end{bmatrix}$

Let the corresponding eigenvalue be λ, we have:

$$\begin{bmatrix}x+10y\\80\end{bmatrix} = \lambda \cdot \begin{bmatrix}1\\10\end{bmatrix} = \begin{bmatrix}\lambda\\10\lambda\end{bmatrix}$$

$$\Rightarrow \begin{cases}\lambda = x+10y & (1)\\ 10\lambda = 80 & (2)\end{cases}$$

From (2) $\Rightarrow$ $\lambda = 8$

(b) From the theorem about the sum of eigenvalues, we have:

$$\begin{aligned}\text{the sum of eigenvalues} &= \text{trace}(A)\\ &= x+7\\ \Rightarrow \quad \lambda + 2 &= x+7\end{aligned}$$

From (a), we have $\lambda = 8$

$$\Rightarrow \quad x+7 = 10$$

$$\Rightarrow \quad x = 3.$$

Also from equation (1) in (a), we have:

$$\lambda = x + 10y$$

$$\Rightarrow 8 = 3 + 10y$$

$$\Rightarrow y = \frac{1}{2}.$$

Therefore, we have $(x,y) = \left(3, \frac{1}{2}\right)$

2. In order to find the eigenvalues of B, we have to solve this equation: $|I\lambda - B| = 0$

$$\Leftrightarrow \left|\begin{bmatrix}\lambda & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}9 & 0 & 0\\0 & -7 & 4\\0 & -16 & 13\end{bmatrix}\right| = 0$$

$$\Leftrightarrow \begin{vmatrix} \lambda-9 & 0 & 0 \\ 0 & \lambda+7 & -4 \\ 0 & 16 & \lambda-13 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-9).\begin{vmatrix} \lambda+7 & -4 \\ 16 & \lambda-13 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-9).\left((\lambda+7)(\lambda-13)+64\right) = 0$$

$$\Leftrightarrow (\lambda-9).(\lambda^2-6\lambda-27) = 0$$

$$\Leftrightarrow (\lambda-9).(\lambda-9).(\lambda+3) = 0$$

$$\Leftrightarrow \left[\begin{array}{l} \lambda = 9 \\ \lambda = -3 \end{array}\right.$$

So the eigenvalues of B are $-3, 9$.

- Find corresponding eigenvectors for $\lambda = -3$:

$$\left[\begin{array}{ccc|c} -12 & 0 & 0 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 16 & -16 & 0 \end{array}\right] \quad \begin{array}{l} (R_1 = \frac{1}{-12} R_1) \\ \\ (R_3 = R_3 - 4R_2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (R_2 = \frac{1}{4} R_2)$$

Let $x_3 = t \Rightarrow x_2 = t$; and $x_1 = 0$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- Find corresponding eigenvectors for $\lambda = 9$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 16 & -4 & 0 \\ 0 & 16 & -4 & 0 \end{array}\right] \quad (R_2 = R_2 - R_3)$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right] \quad (R_3 = \frac{1}{4} R_3)$$

Let $x_1 = s$, $x_2 = t$ $\Rightarrow$ $x_3 = 4t$

$\Rightarrow$ The eigenvector corresponding to $\lambda = 9$ is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 4t \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 4t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

So therefore, we have:

- For $\lambda = -3$, $\mathbb{E}_{-3} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t \,\middle|\, t \in \mathbb{R} \right\}$

- For $\lambda = 9$, because $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ are linearly independent,

we have:

$$\mathbb{E}_9 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} t \,\middle|\, s,t \in \mathbb{R} \right\}$$

3. (a) Because $P.P^{-1} = I$, so we have:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 - R_1$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$R_3 \to R_3 - 2R_1$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 & 1 \end{array}\right)$$

$R_3 \to \frac{R_3}{2}$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right)$$

$R_3 \to R_3 + R_2$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1/2 \end{array}\right)$

$R_3 \to R_3/2$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & 1/4 \end{array}\right)$

$R_1 \to R_1 - R_3$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/4 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & 1/4 \end{array}\right)$

$R_2 \to R_2 - R_3$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/4 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & -1 & 1/2 & 1/4 \end{array}\right)$

$$\Rightarrow \quad P^{-1} = \begin{bmatrix} 2 & -1/2 & -1/4 \\ 0 & 1/2 & -1/4 \\ -1 & 1/2 & 1/4 \end{bmatrix}$$

(b) Let the diagonal matrix be $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

From the matrix factorisation theorem, we have:

$$C^n = (P \cdot D \cdot P^{-1})^n = \underbrace{(P.D.P^{-1}) \cdot (P.D.P^{-1}) \ldots (P.D.P^{-1})}_{n \text{ times}}$$

$$= P \cdot D \cdot I \cdot D \cdot I \ldots \ldots . D \cdot P^{-1}$$

$$= P \cdot D^n \cdot P^{-1} \qquad (1)$$

$$D^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 0^n & 0 & 0 \\ 0 & d^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d^n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substitute $D^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d^n & 0 \\ 0 & 0 & 1 \end{bmatrix}$ into (1), we have:

$$C^n = P. \begin{bmatrix} 0 & 0 & 0 \\ 0 & d^n & 0 \\ 0 & 0 & 1 \end{bmatrix} . P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & d^n & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & d^n & 2 \\ 0 & -2d^n & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1/2 & -1/4 \\ 0 & 1/2 & -1/4 \\ -1 & 1/2 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1/2 & 1/4 \\ -2 & (d^n+2)/2 & (-d^n+2)/4 \\ -4 & 2-d^n & (2+d^n)/2 \end{bmatrix}$$