MDS Assignment 4

1. According to the problem, we can define that matrix A has size $a\times 2$, B has size $b\times 5$, C has size $c\times 4$.

And because D^{-1} is defined $\Rightarrow$ D will have size $d\times d$.

+ $B.C$ is defined $\Rightarrow c = 5$ (1)

$\Rightarrow$ $B.C$ will have size $b\times 4$.

$\Rightarrow (B.C)^T$ will have size $4\times b$.

+ $(B.C)^T.D^{-1}$ is defined $\Rightarrow d = b$ (2)

$\Rightarrow (B.C)^T.D^{-1}$ has size $4\times d$.

+ $A + (B.C)^T.D^{-1}$ is defined $\Rightarrow \begin{cases} a = 4 & (3) \\ 2 = d & (4) \end{cases}$

With (1), (2), (3), (4) combined,

we have: $\boxed{A_{4\times 2}, B_{2\times 5}, C_{5\times 4}, D_{2\times 2}}$

2. Because the order of M^2 is $2\times 2 \Rightarrow$ The order of M is 2×2. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = M^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+cd & bc+d^2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a^2+bc = 1 & (1) \\ ab+bd = 2 & (2) \\ ca+cd = 0 & (3) \\ bc+d^2 = 4 & (4) \end{cases}$$

Subtract (1) from (4) we have: $(bc+d^2)-(a^2+bc) = 4-1$

$\Leftrightarrow d^2 - a^2 = 3$

$\Leftrightarrow (d+a)(d-a) = 3$.

$\Rightarrow d+a \neq 0$

From (3), we have: $c(a+d) = 0$

But $a+d \neq 0 \Rightarrow c=0$

Substitute $c=0$ into (1) & (4), we get: $\begin{cases} a^2=1 \\ d^2=4 \end{cases}$

$$\Rightarrow \begin{cases} (a,d)=(1,2) \Rightarrow a+d=3 \\ (a,d)=(1,-2) \Rightarrow a+d=-1 \\ (a,d)=(-1,2) \Rightarrow a+d=1 \\ (a,d)=(-1,-2) \Rightarrow a+d=-3 \end{cases}$$

From (2), we have: $b(a+d)=2$. (5)

Substitute each pair of (a,d) into (5), we get: all the solutions for $(a,d,b) = (1,2,\frac{2}{3})$; $(1,-2,-2)$; $(-1,2,2)$; $(-1,-2,\frac{-2}{3})$.

Combined with $c=0$, we have all the possible matrices M so that $M^2 = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ are:

$$\boxed{\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 2 \end{bmatrix}; \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix}; \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}; \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -2 \end{bmatrix}}.$$

3. (a) Let $\vec{q} = \vec{u} - \vec{p}$. Because $\vec{p}$ is the orthogonal projection of $\vec{u}$ on $\vec{v} \Rightarrow \vec{q}$ and $\vec{v}$ are orthogonal

$$\Rightarrow \vec{q} \cdot \vec{v} = 0$$

And, because $\vec{p}$ is the orthogondl projection of $\vec{u}$ onto $\vec{v}$, we have: $\vec{p} = \frac{\vec{u}.\vec{v}}{||v||^2} . \vec{v}$

Hence, We have the orthogonal pro ection of $\vec{p}$ onto $\vec{v}$ is

$$\frac{\vec{p}\cdot\vec{v}}{||v||^2}\cdot\vec{v} = \frac{(\vec{u}-\vec{q}).\vec{v}}{||v||^2}\cdot\vec{v} = \frac{\vec{u}\cdot\vec{v} - \vec{q}.\vec{v}}{||v||^2}\cdot\vec{v} = \frac{\vec{u}.\vec{v}}{||v||^2}\cdot\vec{v} = \vec{p} \text{ (Q.E.D.)}$$

(b) We have the orthogonal projection of $\vec{p}$ onto $\vec{u}$ is:

$$\frac{\vec{p}\cdot\vec{u}}{\|u\|^2}\cdot\vec{u} = \frac{\left(\frac{\vec{u}\cdot\vec{v}}{\|v\|^2}\cdot\vec{v}\right)\cdot\vec{u}}{\|u\|^2}\cdot\vec{u}$$

$$= \frac{\left(\frac{\vec{u}\cdot\vec{v}}{\|v\|^2}\right)\cdot(\vec{u}\cdot\vec{v})}{\|u\|^2}\cdot\vec{u}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{\|u\|.\|v\|}\right)^2\cdot\vec{u}$$

$$= \cos^2(\theta)\cdot u$$ (with θ is the angle between u & v) (Q.E.D.)

4.

(a) if $a=3$, we have : $\begin{cases} x+3y=12 & (1) \\ x-y=3 & (2) \end{cases}$

Let (1) − (2), we get: $(x+3y)-(x-y)=12-3$

$\Leftrightarrow \quad 4y = 9$

$\Leftrightarrow \quad y = \frac{9}{4}$.

Substitute $y=\frac{9}{4}$ into (2), we get $x-\frac{9}{4}=3$

$\Leftrightarrow \quad x = \frac{21}{4}$.

Hence, the solution (x,y) to the system if $a=3$ is $\boxed{\left(\frac{21}{4},\frac{9}{4}\right)}$.

(b) We have: $\begin{cases} x + ay = 12 & (3) \\ x - y = a & (4) \end{cases}$

let (3) − (4), we get: $(x+ay)-(x-y) = 12-a$

$\Leftrightarrow (a+1)y = 12-a$

Hence, if $a+1=0 \Leftrightarrow a=-1$, we would have: $0.y = 13$ (contradition)

and this system of equations will have no solutions.

But if $a+1 \neq 0 \Rightarrow y = \frac{12-a}{a+1}$, and this system will have a solution.

Therefore, for this system of equations to have no solutions, $\boxed{a=-1}$.

(c) If there a solution with $y=2$, we have: $\begin{cases} x+2a=12 \\ x-2=a \end{cases}$

$\Leftrightarrow \begin{cases} x+2a=12 & (5) \\ x = a+2 & (6) \end{cases}$

Substitute $x=a+2$ into (5), we get: $(a+2)+2a=12$

$\Leftrightarrow 3a = 10$

$\Leftrightarrow a = \frac{10}{3}$

* Retry, if $a=\frac{10}{3}$, we get: $\begin{cases} x+\frac{10}{3}y = 12 & (7) \\ x-y = \frac{10}{3}. & (8) \end{cases}$

Let (7) − (8) we get: $\frac{13}{3}y = \frac{26}{3} \Rightarrow y=2$

Substitute $y=2$ into (8), we get: $x-2=\frac{10}{3} \Leftrightarrow x=\frac{16}{3}$.

Hence, for there to exist a solution with $y=2$, we will have $\boxed{a=\frac{10}{3}}$

5. Using the properties of matrix determinants, we have:

$$\det\begin{bmatrix} b & e & -3h \\ a+3c & d+3f & -3g-9i \\ c & f & -3i \end{bmatrix} = \det\begin{bmatrix} b & e & -3h \\ a & d & -3g \\ c & f & -3i \end{bmatrix}$$

$$= -3 \cdot \det \begin{bmatrix} b & e & h \\ a & d & g \\ c & f & i \end{bmatrix} = (-3)(-1) \cdot \det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$= 3 \cdot \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 3 \cdot 6 = 18$$

Hence, $\det \begin{bmatrix} b & e & -3h \\ a+3c & d+3f & -3g-9i \\ c & f & -3i \end{bmatrix} = \boxed{18}$