MDS Assignment 5

1. We have the augmented matrix $[A|b]$ as follow:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 6 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 \to R_1 + R_2$
$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 8 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 \to R_1 - R_3$
$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Rows: $m = 3$, columns: $5 (n=4)$ and pivots: $p = 3$. So $m = p < n$.
There are $p = 3$ basic variables, $n - p = 1$ free variable, and an infinite number of solutions.

The equation from $R_3 \Rightarrow x_4 = 4$.

And from 2 other rows, we have:

$$\begin{cases} x_1 + 2x_3 = -4 \\ x_2 + 2x_3 = 2 \end{cases}$$

Let $x_3 = a \Rightarrow \begin{cases} x_1 = -4 - 2a \\ x_2 = 2 - 2a \end{cases}$

So the solution set is $(-4-2a, 2-2a, a, 4)$ for $a \in \mathbb{R}$.

2. We have the following augmented system:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1/3 & 0 \\ 0 & -3 & 0 & 4 & 0 \\ 0 & 0 & -3 & 5 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1/3 & 0 \\ 0 & 0 & -3 & 5 & 0 \\ 0 & 0 & -3 & 5 & 0 \end{array}\right]$

$R_4 \to R_4 - R_3$: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1/3 & 0 \\ 0 & 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$R_3 \to R_3 \times -1/3$: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1/3 & 0 \\ 0 & 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 + R_3$: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -4/3 & 0 \\ 0 & 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ We have the following system of equations:

$$\begin{cases} x_1 - 2x_4 = 0 \\ x_2 - \frac{4}{3}x_4 = 0 \\ x_3 - \frac{5}{3}x_4 = 0 \end{cases}$$

Let $x_4 = a$ $\Rightarrow$ $\begin{cases} x_1 = 2a \\ x_2 = 4a/3 \\ x_3 = 5a/3 \end{cases}$

So the solution set is $(2a, \frac{4a}{3}, \frac{5a}{3}, a)$ for $a \in R$

To find the smallest positive integer set, we should find the smallest a so that $2a, \frac{4a}{3}, \frac{5a}{3}, a$ are all integers.

Notice that $\frac{4a}{3}$ must be integers $\Rightarrow$ a is divisible by 3. $\Rightarrow$ let $a = 3$

$\Rightarrow$ The solution set is $(6, 4, 5, 3)$

For that reason, the rank in descending order is x_1, x_3, x_2, x_4.

3. We have: $\begin{cases} A = B + C & (1) \\ B = C^{-1} - A & (2) \end{cases}$

Substitute A from (1) into (2), we have:

$$B = C^{-1} - A$$
$$= C^{-1} - B - C$$
$$\Leftrightarrow 2B = C^{-1} - C \quad (3)$$

So we have:

$$2BC = (C^{-1} - C).C \quad \text{(from (3))}$$
$$= I - C.C$$
$$= C.C^{-1} - C.C$$
$$= C.(C^{-1} - C)$$
$$= C.\,2B \quad \text{(from (3))}$$
$$= 2.CB$$

$\Rightarrow BC = CB$ (Q.E.D)

4. Let a, b, c be scalars $(a, b, c \in \mathbb{R})$ so that

$$a(\vec{u} + \vec{v} + \vec{w}) + b(\vec{u} - \vec{v} + \vec{w}) + c(2\vec{u} - \vec{w}) = \vec{0} \quad (1)$$

In order to prove that $\{u+v+w, u-v+w, 2u-w\}$ is linearly independent, we need to prove that $(a, b, c) = (0, 0, 0)$ is the only solution set.

From equation (1) we get:

$$a.\vec{u} + a.\vec{v} + a.\vec{w} + b.\vec{u} - b.\vec{v} + b.\vec{w} + c.2\vec{u} - c.\vec{w} = \vec{0}$$

$$\Leftrightarrow (a+b+2c)\vec{u} + (a-b)\vec{v} + (a+b-c).\vec{w} = \vec{0} \quad (2)$$

Because $\{u, v, w\}$ is linearly independent, combined with (2) $\Rightarrow$ The only solution set of $(a+b+2c, a-b, a+b-c)$ is $(0,0,0)$.

From this we get this system of equations:

$$\begin{cases} a+b+2c=0 & (3) \\ a-b=0 & (4) \\ a+b-c=0 & (5) \end{cases}$$

From (4) $\Rightarrow$ $a=b$. Substitute into (3) & (5) we get:

$$\begin{cases} 2b+2c=0 \\ 2b-c=0 \end{cases} \Rightarrow \begin{cases} 2b+2c=0 \\ 2b=c \end{cases} \Rightarrow \begin{cases} c+2c=0 \\ 2b=c \end{cases}$$

$$\Rightarrow \begin{cases} 3c=0 \\ 2b=c \end{cases} \Rightarrow \begin{cases} c=0 \\ 2b=0 \end{cases} \Rightarrow \begin{cases} b=0 \\ c=0 \end{cases}$$

Combined with $a=b$ $\Rightarrow$ We have the solution to (a,b,c) is $(0,0,0)$, which is what we need to prove.

Therefore, we have $\{u+v+w, u-v+w, 2u-w\}$ is linearly independent (Q.E.D.)