MDS Assignment 3

1. We have:

$$g'(x) = \left(2^{f(x^2).x^n}\right)'$$
$$= \log 2 . 2^{f(x^2).x^n} . \left(f(x^2).x^n\right)'$$
$$= \log 2 . 2^{f(x^2).x^n} . \left((f(x^2))'.x^n + (x^n)'.f(x^2)\right)$$
$$= \log 2 . 2^{f(x^2).x^n} . \left(2x.f'(x^2).x^n + n.x^{n-1}.f(x^2)\right) (1)$$

Subtitute $x = 1$ into (1), we get:

$$g'(1) = \log 2 . 2^{f(1).1^n} . \left(2.1.f'(1^2).1^n + n.1^{n-1}.f(1)\right)$$

$$\Leftrightarrow -4 \log 2 = \log 2 . 2^{-2} . (6 - 2n)$$
$$\Leftrightarrow -4 = \frac{1}{2} . (3-n)$$
$$\Leftrightarrow -8 = 3 - n$$
$$\Leftrightarrow n = 11$$

$\boxed{\text{Ans: } 11}$

2.

(a) $Pr(2 < x \leq 3) = F(3) - F(2)$

$$= \left(-\frac{9}{5} + \frac{39}{10} - \frac{11}{10}\right) - \left(-\frac{4}{5} + \frac{26}{10} - \frac{11}{10}\right)$$
$$= \frac{-18 + 39 - 11 + 8 - 26 + 11}{10}$$
$$= \frac{3}{10}$$

$\boxed{\text{Ans: } \frac{3}{10}}$

(b) We will consider these 3 cases:

Case 1: $b < 1$

We have: $\Pr(X \le b) = F(b)$

$= 0$ (because $b < 1$)

$\Rightarrow \frac{1}{2} = 0$ (contradiction)

Case 2: $1 \le b \le 3$.

Then we have: $\Pr(x \le b) = F(b)$

$= -\frac{b^2}{5} + \frac{13b}{10} - \frac{11}{10}$

(because $1 \le b \le 3$)

$\Rightarrow \frac{1}{2} = -\frac{b^2}{5} + \frac{13b}{10} - \frac{11}{10}$

$\Leftrightarrow 5 = 13b - 2b^2 - 11$

$\Leftrightarrow 13b - 2b^2 - 16 = 0.$

$\Leftrightarrow 2b^2 - 13b + 16 = 0$

$\Leftrightarrow b^2 - \frac{13}{2} b + 8 = 0$

$\Leftrightarrow \left(b - \frac{13}{4}\right)^2 = \frac{13^2}{16} - 8$

$\Leftrightarrow \left(b - \frac{13}{4}\right)^2 = \frac{41}{16}.$

$\Leftrightarrow \left[\begin{array}{l} b = \frac{13 + \sqrt{41}}{4} \\ b = \frac{13 - \sqrt{41}}{4} \end{array}\right.$ (contradiction since $1 \le b \le 3$)

Case 3: $b > 3$

We have: $Pr(X \leq b) = F(b)$

$= 1$ (because $b > 3$)

$\Rightarrow \frac{1}{2} = 1$ (contradiction)

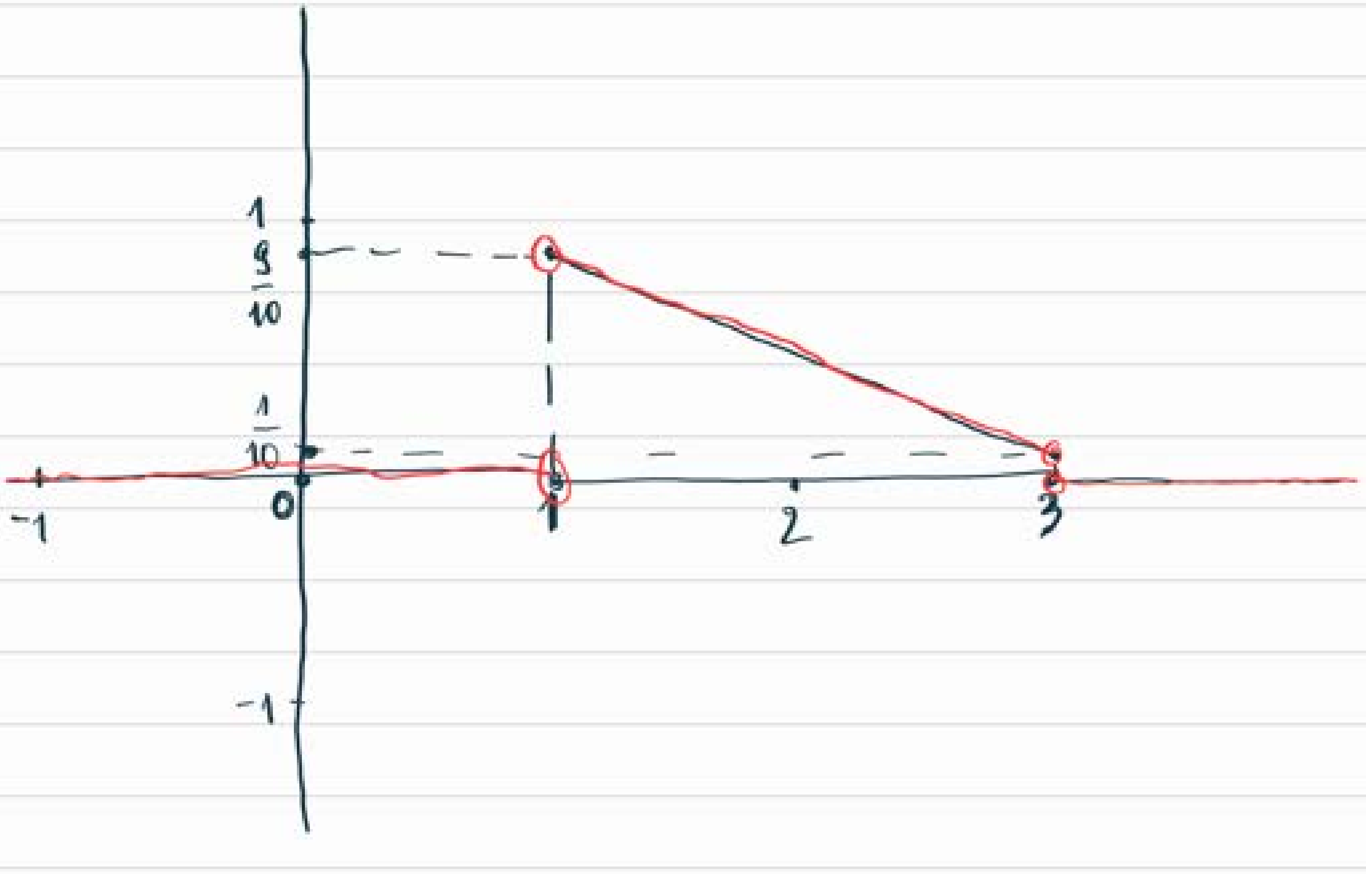
Hence, combining 3 cases, we have the only b that satisfies is $b = \frac{13+\sqrt{41}}{16}$.

$\boxed{\text{Ans: } \frac{13-\sqrt{41}}{4}}$

(c) Since $f(x) = F'(x)$, then from the definition of $F(x)$ we have:

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ -\frac{2}{5}x + \frac{13}{10} & \text{if } 1 \leq x \leq 3. \\ 0 & \text{if } x > 3. \end{cases}$$

So the graph of $f(x)$ is:



3.

(a) We have: $1 = \Pr(-1 \le X \le 5) \quad (c > 0)$.

$$= \int_{-1}^{5} f(x)\,dx$$

$$= \int_{-1}^{1} f(x)\,dx + \int_{4}^{5} f(x)\,dx$$

$$= \int_{-1}^{1} c\,dx + \int_{4}^{5} \frac{1}{5}\,dx$$

$$= c(1-(-1)) + \frac{1}{5}(5-4)$$

$$= 2c + \frac{1}{5}.$$

$$\Leftrightarrow \quad 2c = \frac{4}{5} \Leftrightarrow c = \frac{2}{5}.$$

$$\boxed{\text{Ans: } \frac{2}{5}}$$

(b) $$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$= \int_{-1}^{1} x f(x)\,dx + \int_{4}^{5} x f(x)\,dx$$

$$= \int_{-1}^{1} \frac{2}{5} x\,dx + \int_{4}^{5} \frac{1}{5} x\,dx$$

$$= \left(\frac{1}{5}\cdot 1^2 - \frac{1}{5}\cdot(-1)^2\right) + \frac{1}{10}\cdot 5^2 - \frac{1}{10}\cdot 4^2$$

$$= \frac{9}{10}$$

$$\boxed{\text{Ans}: \frac{9}{10}}$$

(c) We have:

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-1}^{1} x^2 f(x)\, dx + \int_{4}^{5} x^2 f(x)\, dx$$

$$= \int_{-1}^{1} \frac{2}{5} x^2 \cdot dx + \int_{4}^{5} \frac{1}{5} x^2\, dx$$

$$= \left(\frac{2}{15} \cdot 1^3 - \frac{2}{15} \cdot (-1)^3\right) + \left(\frac{1}{15} \cdot 5^3 - \frac{1}{15} \cdot 4^3\right)$$

$$= \frac{4}{15} + \frac{61}{15}$$

$$= \frac{13}{3}.$$

$$\Rightarrow Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{13}{3} - \left(\frac{9}{10}\right)^2 = 3.52\overline{3}$$

$$\boxed{\text{Ans}: 3.52\overline{3}}$$

4. In order to find the Maclaurin polynomial $P_6(x)$ for $f(x)$, we need to find the Maclaurin polynomial $P_6(x)$ for the function $g(x) = e^{3x^2}$.

According to the Maclaurin polynomial for e^x, we have:

$$e^x = \sum_{i=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow \quad e^{3x^2} = \sum_{i=0}^{\infty} \frac{(3x^2)^n}{n!}.$$

So the Maclaurin polynomial $P_6(x)$ for the function $g(x) = e^{3x^2}$ is:

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{27x^6}{6}.$$

$\Rightarrow$ The Maclaurin polynomial $P_7(x)$ for the function $f(x)$ is

$$(1+x)\left(1+3x^2+\frac{9x^4}{2}+\frac{27x^6}{6}\right)$$

$$= 1 + x + 3x^2 + 3x^3 + \frac{9x^4}{2} + \frac{9x^5}{2} + \frac{9x^6}{2} + \frac{9x^7}{2}.$$

$\Rightarrow$ The Maclaurin polynomial $P_6(x)$ for the function $f(x)$ is:

$$\boxed{P_6(x) = 1 + x + 3x^2 + 3x^3 + \frac{9x^4}{2} + \frac{9x^5}{2} + \frac{9x^6}{2}.} \quad (1)$$

(b) Since in a Maclaurin polynomial, the coefficent of x^6 is $\frac{f^{(6)}(0)}{6!}$

Combined with (1), we have:

$$\frac{f^{(6)}(0)}{6!} = \frac{9}{2} \Leftrightarrow f^{(6)}(0) = 3240$$

$\boxed{\text{Ans: } 3240}$