Activity: Fourier Coefficients

The following identities are useful when working with the *Fourier frequencies* $\omega_p = 2\pi p/N$:

(a) For each $p = 1, 2, \ldots, N/2$,

$$\sum_{t=1}^{N} \cos(\omega_p t) = \sum_{t=1}^{N} \sin(\omega_p t) = 0.$$

(b) For each $p, q = 0, 1, 2, \ldots, N/2$,

$$\sum_{t=1}^{N} \sin(\omega_p t) \cos(\omega_q t) = 0.$$

(c)

$$\sum_{t=1}^{N} \cos(\omega_p t) \cos(\omega_q t) = \begin{cases} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0, \frac{N}{2} \\ N & p = q = 0, \frac{N}{2} \end{cases}$$

(d)

$$\sum_{t=1}^{N} \sin(\omega_p t) \sin(\omega_q t) = \begin{cases} 0 & p \neq q \\ \frac{N}{2} & p = q \neq 0, \frac{N}{2} \\ 0 & p = q = 0, \frac{N}{2} \end{cases}$$

Let $\{x_1, \ldots, x_N\}$ be a sequence of N numbers. Assume N is even. Now by the above, for each $t = 1, \ldots, N$, it must be possible to write

$$x_t = \sum_{q=0}^{N/2} \left(a_q \cos(\omega_q t) + b_q \sin(\omega_q t)\right) \tag{1}$$

for some constants $\{a_q, b_q : q = 0, \ldots, N/2\}$ .We aim to find these constants, starting with the terms $\{a_q\}$.

1. Multiply (1) on both sides by $\cos(\omega_p t)$.

2. Then sum over $t = 1, \ldots, N$. Note the summation over q covers all the Fourier frequencies. Hence find the coefficients a_p, first for $p \neq 0, \frac{N}{2}$ and then when $p = 0, \frac{N}{2}$.

3. To find the coefficients $\{b_p\}$, multiply (1) on both sides by $\sin(\omega_p t)$, and then sum over $t = 1, \ldots, N$.