Activity: Examples of Spectral Densities

The *spectral density function* $f(\omega)$ of a stationary stochastic process $\{X_t\}$ is defined as

$$f(\omega) = \frac{1}{\pi}\left(\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)\cos(\omega k)\right)$$

for $\omega \in (0, \pi)$, where $\gamma(k)$ is the acvf of X_t. Basically, $f(\omega)$ is the FT of $\gamma(k)$. The normalized spectrum is

$$f^*(\omega) = \frac{f(\omega)}{\sigma_X^2},$$

where $\sigma_X^2 = \mathrm{Var}(X_t)$. So $f^*(\omega)$ is the FT of $\rho(k)$, the acf.

The aim here is to aid appreciation of the spectral density by consideration of some key special cases.

1. Consider first the MA(1) process

$$X_t = Z_t + \beta Z_{t-1}.$$

   (a) Remind yourself of, or derive if need be, the acvf and acf of X_t.

   (b) Write down the normalized spectrum and spectral density function here.

   (c) Plot the normalized spectral density when $\beta = 1$ and when $\beta = -1$. Briefly comment on what you observe and how you would relate the plots to realizations of these MA processes.

2. The AR(1) process

$$X_t = \alpha X_{t-1} + Z_t$$

   has a spectral density that can be written as

$$f(\omega) = \frac{\sigma^2}{\pi\left(1 - 2\alpha\cos(\omega) + \alpha^2\right)}.$$

   Plot the cases below, and in each case comment on the observed spectral density

   (a) $\alpha = -0.9$;

   (b) $\alpha = 0.9$;

   (c) $\alpha = 0.1$.