Activity Solution: Autoregressive Processes

Suppose $\{Z_t\}$ is white noise with mean zero and variance σ^2. We have seen that a process $\{X_t\}$ is said to be a moving average process of order q, denoted MA(q), if

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

for some constants $\beta_0, \beta_1, \ldots, \beta_q$, with usually $\beta_0 = 1$. This activity concerns a special case of the model above with $q \to \infty$, $\beta_0 = 1$ and $\beta_i = \alpha^i$ for $i \geq 1$, specifically

$$X_t = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots \tag{1}$$

1. What condition on α is necessary for the right–hand side of Equation (1) to be well-defined?
   $|\alpha| < 1$, *otherwise the sum diverges.*

2. Show that $X_t = \alpha X_{t-1} + Z_t$.
   By rearranging Equation (1) we obtain

$$\begin{aligned} Z_t =& X_t - \left(\alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots\right) \\ =& X_t - \alpha\left(Z_{t-1} + \alpha Z_{t-2} + \cdots\right) \\ =& X_t - \alpha X_{t-1}. \end{aligned}$$

3. We call $\{X_t\}$ an *autoregressive process* of order 1, denoted AR(1) . Each value in the series is modelled as the previous value plus noise. Find $\mathrm{E}(X_t)$.

$$\begin{aligned} E\left(X_t\right) =& E\left(Z_t + \alpha Z\left(t-1\right) + \alpha^2 Z_{t-2} + \cdots\right) \\ =& E\left(Z_t\right) + E\left(\alpha Z_{t-1}\right) + E\left(\alpha^2 Z_{t-2}\right) + \cdots \\ =& E\left(Z_t\right) + \alpha E\left(Z_{t-1}\right) + \alpha^2 E\left(Z_{t-2}\right) + \cdots \\ =& 0. \end{aligned}$$

4. Find the variance of X_t. (Remember if $|r| < 1$, $\sum_{k=0}^{\infty} ar^k = a/\left(1-r\right)$.) How does this relate to your answer to Question 1?
   So

$$\begin{aligned} Var\left(X_t\right) =& \ Var\left(Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots\right) \\ =& \ Var\left(Z_t\right) + Var\left(\alpha Z_{t-1}\right) + Var\left(\alpha^2 Z_{t-2}\right) + \cdots \\ =& \ Var\left(Z_t\right) + \alpha^2 Var\left(Z_{t-1}\right) + \alpha^4 Var\left(Z_{t-2}\right) + \cdots \\ =& \sigma^2\left(1 + \alpha^2 + \alpha^4 + \cdots\right) \end{aligned}$$

which if $|\alpha| < 1$ *(see Question 1) equals*

$$\frac{\sigma^2}{1-\alpha^2}.$$

5. If $E(X_t) = 0$ for all t, recall the acvf of X_t at lag h is

$$\gamma(h) = E\left(X_t X_{t+h}\right).$$

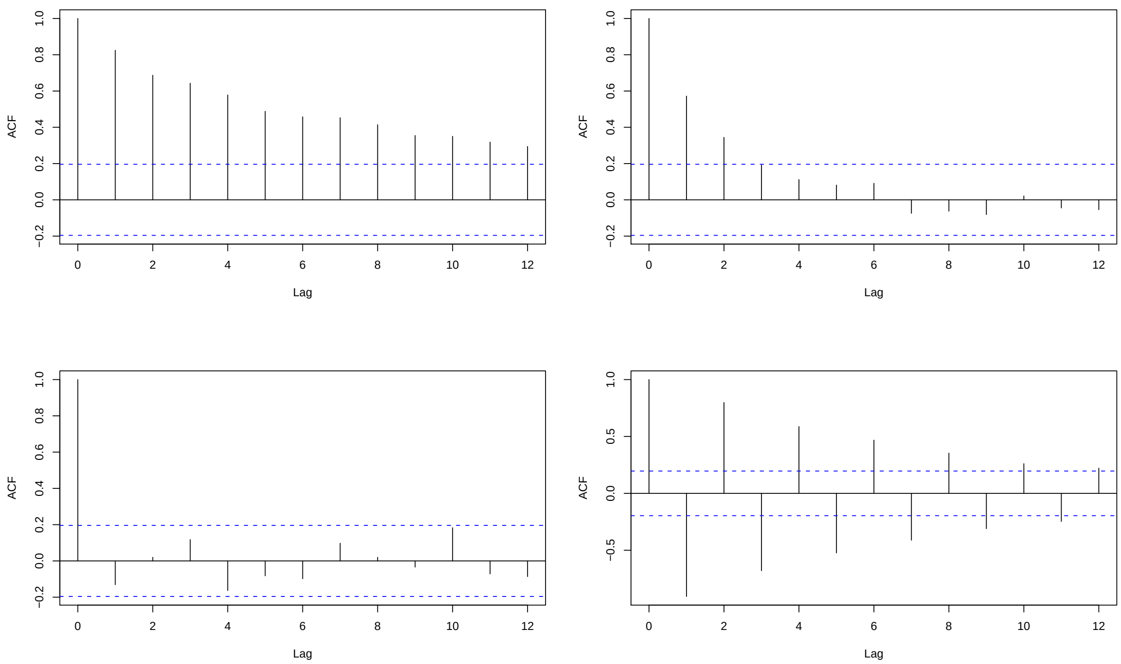
By substituting Equation (1) into the above, for $h \geq 0$ show that $\gamma(h) = \alpha^h \sigma_X^2$, where σ_X^2 is the variance found in Question 4. The identity given in Question 4 may again be useful.

$$\begin{aligned}
\gamma(h) =& E\left(X_t X_{t+h}\right) \\
=& E\left(\sum_{i=0}^{\infty} \alpha^i Z_{t-i} \sum_{j=0}^{\infty} \alpha^j Z_{t+h-j}\right) \\
=& \sigma^2 \sum_{i=0}^{\infty} \alpha^i \alpha^{h+i} \\
=& \frac{\alpha^h \sigma^2}{1-\alpha^2} \\
=& \alpha^h \sigma_X^2.
\end{aligned}$$

6. Is $\{X_t\}$ stationary?
Assuming $|\alpha| < 1$, *we have seen that neither the mean nor the acvf depend on* t. *Hence the process is stationary.*

7. Four AR(1) processes are generated with $\alpha = -0.9, 0, 0.5$ and 0.9. Identify which autocorrelation plot corresponds to each value of α.

Clockwise from top left: 0.9, 0.5, –0.9, 0.


ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.5
−0.5
0
2
4
6
8
10
12