Activity: Properties of the Periodogram

The *periodogram* of a time series $X_1, \ldots, X_N$ is a histogram over the frequencies $(0, \pi)$, the height of the histogram across $\omega_p \pm \frac{\pi}{N}$ being

$$I(\omega_p) = \frac{NR_p^2}{4\pi},$$

where

$$R_p = \left(a_p^2 + b_p^2\right)^{\frac{1}{2}}$$

with

$$\begin{aligned} a_p &= \frac{2}{N} \sum_{t=1}^{N} X_t \cos(\omega_p t), \\ b_p &= \frac{2}{N} \sum_{t=1}^{N} X_t \sin(\omega_p t), \end{aligned}$$

for $p = 1, \ldots, N/2 - 1$ (assuming N is even).

We consider a special case and show that the periodogram is not a consistent estimator of the underlying spectrum.

Assume $X_t \sim N(0, \sigma^2)$, independently for each $t = 1, \ldots, N$. It will be helpful to recall that if $X_i \sim N(0,1)$ for $i = 1, \ldots, n$ then

$$Q_n := \sum_{i=1}^{n} X_i^2 \sim \chi_n^2$$

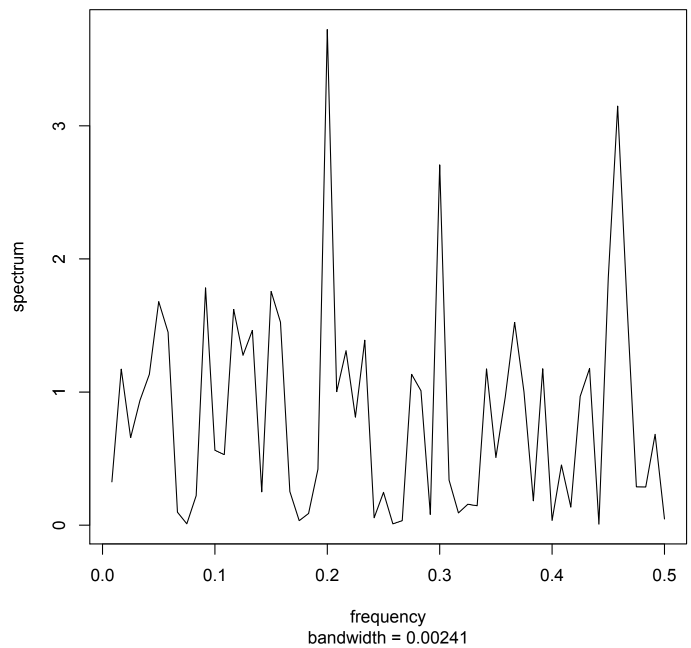
with $E(Q_n) = n$ and $\mathrm{Var}(Q_n) = 2n$.

1. Find $E(a_p)$ and $E(b_p)$, for $p = 1, \ldots, N/2 - 1$.
2. Find $\mathrm{Var}(a_p)$ and $\mathrm{Var}(b_p)$, for $p = 1, \ldots, N/2 - 1$.
3. Recalling that in our special case $X_t \sim N(0, \sigma^2)$ for each t, write down the probability distribution for a_p here.
4. Now show that a_p and b_q are uncorrelated, for all choices of $p, q = 1, \ldots, N/2 - 1$.
5. Find the expectation and variance of $I(\omega_p)$. Comment on these values.

6. What is the distribution of $2\pi I\left(\omega_p\right)/\sigma^2$? How does this behave as N grows large?

7. The periodogram for a sample of 120 observations simulated from $N(0,1)$ distribution is given in Figure 1.

   Figure 1: Periodogram for a white noise sample of size $N = 120$.

   How is R defining frequency in the above? In theory, what should the periodogram look like?

8. How might you define a "significant" value of the periodogram in the above example?