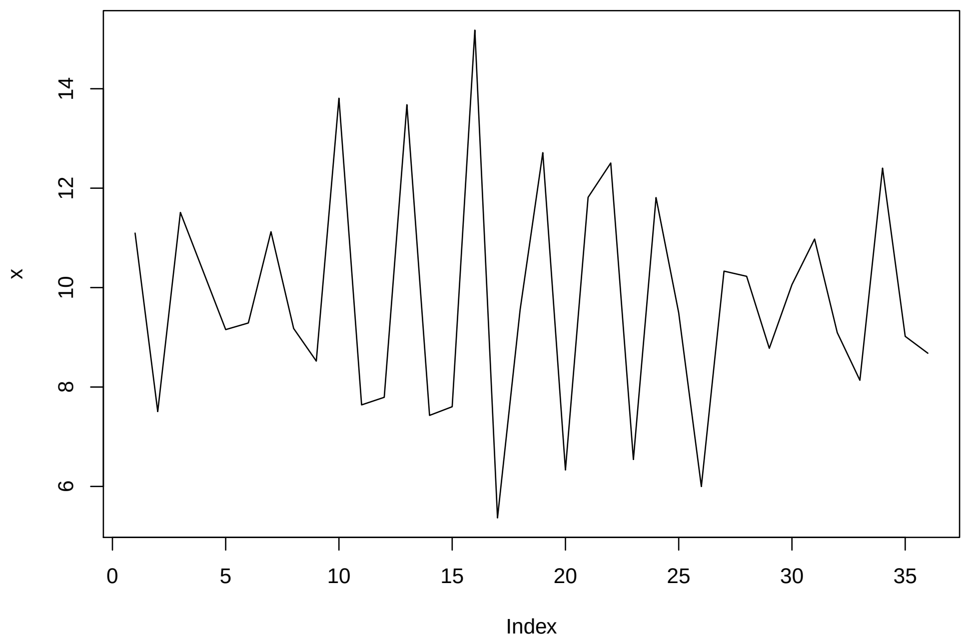
Activity: Modifying the Periodogram

As we have seen, the raw periodogram, given by

$$I(\omega_p) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{N-1} c_k \cos(\omega_p k)\right), \quad \omega_p = \frac{2\pi p}{N}, \quad p = 0, 1, \ldots, \frac{N}{2},$$

is not a good estimator of the underlying spectrum of a series as it is inconsistent. Here we explore the two common methods for modifying the periodogram to give a consistent estimator of the spectral density, $f(\omega)$.

We explore the methods as applied to the series $\{x_t\}_{t=1,\ldots,36}$ plotted below.



Note that in practice this series is rather too short for the methods to be robustly applied.

The sample autocovariance function (acvf) for this time series for lags up to 12 is:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
c_k	5.26	-2.70	-1.64	4.04	-2.75	-0.94	3.10	-2.22	-0.36	2.07	-1.54	-0.21	1.48

1. *Truncate and transform:* Since the sample acvf provides unreliable estimates of the true autocovariance function at large lag values, it is reasonable to truncate the raw periodogram by including c_k values only up to lag $k \leq M$ with $M < N$ as well as to modify it by attaching weights $\{\lambda_k\}_{k=1,\ldots,M}$ to the c_k values, where λ_k's decrease with k.

   This gives an estimator of the form

$$\hat{f}_T(\omega_p) = \frac{1}{\pi}\left(\lambda_0 c_0 + 2\sum_{k=1}^{M} \lambda_k c_k \cos(\omega_p k)\right). \tag{1}$$

   The weights $\{\lambda_k\}$ are called the *lag window*, and $M < N$ is the *truncation point.* A common choice of M is $M = 2\sqrt{N}$. Taking $M = 12$ as suggested, we will work out the estimator $\hat{f}_T(\omega_p)$, for $p = 0, 1, \ldots, N/2 = 18$, using the *Tukey window:*

$$\lambda_k = \frac{1}{2}\left(1 + \cos\left(\frac{\pi k}{12}\right)\right), \quad k = 0, 1 \ldots, 12.$$

   (a) To start, complete the table for the values of λ_k below:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
λ_k	1	0.98	...	0.85	0.75	0.63	...	0.37	0.25	0.15	0.07	0.02	...

   (b) The estimator $\hat{f}_T(\omega_p)$ involves values of $\cos(\omega_p k)$ for $k = 0, \ldots, 12$ and $p = 0, 1 \ldots, 18$. The table below gives values of this cosine function with p values across rows and k values across columns:

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.98	0.94	0.87	0.77	0.64	0.5	0.34	0.17	0	-0.17	-0.34	-0.5
2	0.94	0.77	0.50	0.17	-0.17	-0.5	-0.77	-0.94	-1	-0.94	-0.77	-0.5
3	0.87	0.50	0	-0.50	-0.87	-1	-0.87	-0.50	0	0.50	0.87	1
4	0.77	0.17	-0.50	-0.94	-0.94	-0.5	0.17	0.77	1	0.77	0.17	-0.5
5	0.64	-0.17	-0.87	-0.94	-0.34	0.5	0.98	0.77	0	-0.77	-0.98	-0.5
6	0.50	-0.50	-1.00	-0.50	0.50	1	0.50	-0.50	-1	-0.50	0.50	1
7	0.34	-0.77	-0.87	0.17	0.98	0.5	-0.64	-0.94	0	0.94	0.64	-0.5
8	0.17	-0.94	-0.50	0.77	0.77	-0.5	-0.94	0.17	1	0.17	-0.94	-0.5
9	0	-1	0	1	0	-1	0	1	0	-1	0	1
10	-0.17	-0.94	0.50	0.77	-0.77	-0.5	0.94	0.17	-1	0.17	0.94	-0.5
11	-0.34	-0.77	0.87	0.17	-0.98	0.5	0.64	-0.94	0	0.94	-0.64	-0.5
12	-0.50	-0.50	1.00	-0.50	-0.50	1	-0.50	-0.50	1	-0.50	-0.50	1
13	-0.64	-0.17	0.87	-0.94	0.34	0.5	-0.98	0.77	0	-0.77	0.98	-0.5
14	-0.77	0.17	0.50	-0.94	0.94	-0.5	-0.17	0.77	-1	0.77	-0.17	-0.5
15	-0.87	0.50	0	-0.50	0.87	-1	0.87	-0.50	0	0.50	-0.87	1
16	-0.94	0.77	-0.50	0.17	0.17	-0.5	0.77	-0.94	1	-0.94	0.77	-0.5
17	-0.98	0.94	-0.87	0.77	-0.64	0.5	-0.34	0.17	0	-0.17	0.34	-0.5
18	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1

Using these values as an intermediate step, the following table reports values of the sum $S_p = \sum_{k=1}^{M} \lambda_k c_k \cos(\omega_p k)$ for $p = 1, \ldots, 18$:

p	1	2	3	4	5	6	7	8	9
S_p	-2.524	-2.458	-2.368	-2.298	-2.314	-2.348	-2.372	-2.503	...
p	10	11	12	13	14	15	16	17	18
S_p	0.615	5.530	9.231	8.284	3.790	-0.214	-1.790	-1.961	...

Fill in the missing values in the above table.

(c) Now evaluate the modified estimator of the power spectrum $\hat{f}_T(\omega_p)$ in Eq. (1) by completing the following table:

p	1	2	3	4	5	6	7	8	9
$\hat{f}_T(\omega_p)$	0.067	0.109	...	...	0.201	0.179	0.164	0.081	0.362
p	10	11	12	13	14	15	16	17	18
$\hat{f}_T(\omega_p)$	2.065	5.195	7.551	...	...	1.538	0.535	0.426	0.448

(d) Comment on the shape of the estimate $\hat{f}_T$ here.

(e) If in fact asymptotically for $N \to \infty$

$$\frac{n\hat{f}_T(\omega)}{f(\omega)} \sim \chi^2_n \quad \text{with} \quad n = \frac{2N}{\sum_{k=-M}^{M} \lambda_k^2},$$

approximately, show how you could use this result here to construct 95% confidence intervals for $f(\omega)$. Evaluate these 95% confidence intervals for $f(\omega_5)$ and $f(\omega_{12})$.

2. *Smoothing:* Let

$$\hat{f}_S(\omega) = \frac{1}{m}\sum_j I(\omega_j)$$

where $\omega_j = 2\pi j/N$ and j ranges over m consecutive integers so that $\{\omega_j\}$ are symmetric about ω. We often take $m = 2\sqrt{N}$, which is 12 in this example. The formula needs to be adjusted near the end–points, though this has little impact if N is large.

(a) Assuming the asymptotic distribution results for the raw periodogram, explain why

$$\frac{2m\hat{f}_S(\omega_p)}{f(\omega_p)} \sim \chi^2_{2m}.$$

(b) Describe for our example how to use $\hat{f}_S(\omega_p)$ to produce 95% confidence intervals for $f(\omega_p)$.