Activity Solution: Moving Average Processes

The i.i.d sequence $\{Z_t\}$ has mean zero and variance σ^2. Suppose we define the stochastic process $\{X_t\}$ by

$$X_t = Z_t + 0.3Z_{t-1} + 0.2Z_{t-2} + 0.1Z_{t-3}.$$

1. The model X_t is an MA(q) for which value of q?
   3

2. Find $E(X_t)$.

$$\begin{aligned} E(X_t) &= E(Z_t + 0.3Z_{t-1} + 0.2Z_{t-2} + 0.1Z_{t-3}) \\ &= 0. \end{aligned}$$

3. Find $\text{Var}(X_t)$.

$$\begin{aligned} \text{Var}(X_t) &= \sigma^2\left(1 + 0.3^2 + 0.2^2 + 0.1^2\right) \\ &= 1.14\sigma^2. \end{aligned}$$

4. Find $\text{Cov}(X_t, X_{t+1})$.

$$\begin{aligned} \text{Cov}(X_t, X_{t+1}) &= \text{Cov}(Z_t + 0.3Z_{t-1} + 0.2Z_{t-2} + 0.1Z_{t-3}, \\ &\quad Z_{t+1} + 0.3Z_t + 0.2Z_{t-1} + 0.1Z_{t-2}) \\ &= \sigma^2(0.3 + 0.3 \times 0.2 + 0.2 \times 0.1) \\ &= 0.38\sigma^2. \end{aligned}$$

5. Find $\text{Cov}(X_t, X_{t+2})$.

$$\begin{aligned} \text{Cov}(X_t, X_{t+2}) &= \text{Cov}(Z_t + 0.3Z_{t-1} + 0.2Z_{t-2} + 0.1Z_{t-3}, \\ &\quad Z_{t+2} + 0.3Z_{t+1} + 0.2Z_t + 0.1Z_{t-1}) \\ &= \sigma^2(0.2 + 0.3 \times 0.1) \\ &= 0.23\sigma^2. \end{aligned}$$

6. Find $\text{Cov}(X_t, X_{t+3})$.

$$\begin{aligned} \text{Cov}(X_t, X_{t+3}) &= \text{Cov}(Z_t + 0.3Z_{t-1} + 0.2Z_{t-2} + 0.1Z_{t-3}, \\ &\quad Z_{t+3} + 0.3Z_{t+2} + 0.2Z_{t+1} + 0.1Z_t) \\ &= 0.1\sigma^2. \end{aligned}$$

7. What is $\mathrm{Cov}(X_t, X_{t+k})$ when $k > 3$?
   0

8. Is X_t stationary?
   Yes!

9. What is $\rho(k)$ for X_t?

$$\begin{aligned}
\rho(k) &= \frac{\gamma(k)}{\gamma(0)} \\
&= \begin{cases} 1 & k = 0 \\ 0.33 & k = \pm 1 \\ 0.20 & k = \pm 2 \\ 0.088 & k = \pm 3 \\ 0 & |k| > 3. \end{cases}
\end{aligned}$$

10. Suppose $E(Z_t) = 3$ in the above. What, if anything, would change?
    Only $E(X_t)$ would change, this being then

$$\begin{aligned}
E(X_t) &= 3 \times (1 + 0.3 + 0.2 + 0.1) \\
&= 4.8.
\end{aligned}$$