Activity Solution: Yule–Walker Equations

The i.i.d. sequence $\{Z_t\}$ has mean zero and variance σ^2. Suppose we define the stochastic process $\{X_t\}$ by

$$X_t = 1.30X_{t-1} - 0.22X_{t-2} - 0.10X_{t-3} + Z_t.$$

Assume that this process is stationary.

1. How many equations comprise the Yule–Walker equations here?
   3

2. Derive the Yule–Walker equations for $\{X_t\}$.
   Take the defining equation and multiply both sides by X_{t-k} for $k = 1, 2, 3$, giving:

$$\begin{aligned} X_t X_{t-1} &= 1.30X_{t-1}^2 - 0.22X_{t-2}X_{t-1} - 0.10X_{t-3}X_{t-1} \\ &\quad + Z_t X_{t-1} \\ X_t X_{t-2} &= 1.30X_{t-1}X_{t-2} - 0.22X_{t-2}^2 - 0.10X_{t-3}X_{t-2} \\ &\quad + Z_t X_{t-2} \\ X_t X_{t-3} &= 1.30X_{t-1}X_{t-3} - 0.22X_{t-2}X_{t-3} - 0.10X_{t-3}^2 \\ &\quad + Z_t X_{t-3}. \end{aligned}$$

   Now taking expectations on both sides and dividing by σ_X^2 we have

$$\begin{aligned} \rho(1) &= 1.30\rho(0) - 0.22\rho(1) - 0.10\rho(2) \\ \rho(2) &= 1.30\rho(1) - 0.22\rho(0) - 0.10\rho(1) \\ \rho(3) &= 1.30\rho(2) - 0.22\rho(1) - 0.10\rho(0) \end{aligned}$$

   since $E(X_t X_{t-k})/\sigma_X^2 = \rho(k)$ here. Since $\rho(0) = 1$ we have

$$\begin{aligned} \rho(1) &= 1.30 - 0.22\rho(1) - 0.10\rho(2) \\ \rho(2) &= 1.30\rho(1) - 0.22 - 0.10\rho(1) \\ \rho(3) &= 1.30\rho(2) - 0.22\rho(1) - 0.10, \end{aligned}$$

3. Find $\rho(1)$ and $\rho(2)$ for $\{X_t\}$. Comment on these values.
   Since

$$\begin{aligned} 1.22\rho(1) &= 1.30 - 0.1\rho(2), \\ \rho(2) &= 1.20\rho(1) - 0.22 \end{aligned}$$

we have

$$1.22\rho(1) = 1.3 - 0.1(1.20\rho(1) - 0.22)$$

giving that $\rho(1) = 0.987$ *and therefore* $\rho(2) = 0.964$. *Note these indicate very high correlations at lags 1 and 2, indicating strong short-term dependency. We have* $\rho(3) = 0.936$.

4. Given that the roots of

$$D^3 - 1.3D^2 + 0.22D + 0.1 = 0$$

are $d_1 = -0.195\,3$, $d_2 = 0.530\,97$ and $d_3 = 0.964\,33$ (you might like to check these), write down the general solution of the Yule–Walker equations for $\{X_t\}$. Working to 3 significant figures will suffice.

General solution must be of the form

$$\rho(k) = A_1(-0.195\,3)^{|k|} + A_2(0.530\,97)^{|k|} + A_3(0.964\,33)^{|k|}$$

where $A_1 + A_2 + A_3 = 1$ *since* $\rho(0) = 1$. *When* $k = 1$ and $k = 2$ *we have*

$$\begin{aligned} 0.987 &= A_1(-0.195) + A_2(0.531) + (1 - A_1 - A_2)(0.964) \\ 0.964 &= A_1(-0.195)^2 + A_2(0.531)^2 + (1 - A_1 - A_2)(0.964)^2 . \end{aligned}$$

Solving this linear system of equations and working up to 3 d.p. gives $A_1 = 0.000379$ and $A_2 = -0.05413255$. Then $A_3 = 1 - A_1 - A_2 = 1.0537534$, and the general solution is

$$\rho(k) = 0.00038(-0.195\,3)^{|k|} - 0.05413(0.530\,97)^{|k|} + 1.05375(0.964\,33)^{|k|} .$$

Remark*: if more than 3 s.f. are used in computations, the constants* A_i *are as follows:*

$$A_1 = 0.0008276096, \quad A_2 = -0.0535279793, \quad A_3 = 1.0527003697.$$

5. Find $\rho(1)$ and $\rho(2)$ using the general solution in Question 4. Check the values are what you found earlier.
We have

$$
\begin{aligned}
\rho(1) &= 0.00038\,(-0.195\,3) - 0.05413\,(0.530\,97) + 1.05375\,(0.964\,33) \\
&= 0.987
\end{aligned}
$$

as before. Further,

$$
\begin{aligned}
\rho(1) &= 0.00038\,(-0.195\,3)^2 - 0.05413\,(0.530\,97)^2 + 1.05375\,(0.964\,33)^2 \\
&= 0.964.
\end{aligned}
$$

6. Plot the acf of $\{X_t\}$ at lags $k = 0, 1, 2, \ldots, 10$. Comment on what you observe.
   We have $\rho(3) = 0.936$ *and so on. There is a slow decay in the acf here, indicating strong long-term dependency.*