1 Notation

- 1-second last mid price was observed for 1-month
- Daily volatility is being estimated
- Sampling period : K (60 * 10) (FIXME:10 minute; why?)
- Parameter estimation period : M (60 * 60 * 24 = 86400) (FIXME: 1 day; why?)
- Sequence length : n (86400 * 30 = 2592000)
- time : $0 < t < n$
- log-price : x_t
- log-return : r_t

2 Jump adjust

Removes jumps and stitch the log-price

2.1 Truncation

Truncation with 1σ was used.

$$\tilde{r}_t = \Psi_\tau(r_t) = \mathrm{sgn}(r_t)\min\left(|r_t|, \tau = \sigma_t\right)$$

where $\sigma^2 = \frac{1}{M}\sum_{i=t}^{t+M} r_i$.

$\tau = \sigma$ ensures the truncated process ensures sub-Gaussian condentration condition by Hoeffding's lemma

$$E[\exp(\lambda r_t)] \le \exp\left(\frac{\lambda^2\tau^2}{2}\right) = \exp\left(\frac{\lambda^2\sigma^2}{2}\right)$$

2.2 Truncation with MLE

It was assumed that the standard deviation of truncated return goes to the standard deviation of daily return.

- Raw return : r_t
- Truncated return with threshold τ : $\tilde{r}_t^\tau$
- Standard deviation : $\sigma_t^\tau = \sqrt{\sum_{i=1}^{M}(\tilde{r}_{t+i}^\tau)^2}$
- Daily return : $r_t^{\text{daily}} = \frac{1}{M}\sum_{i=0}^{M-1} r_{t+i}$

- Daily standard deviation : $\sigma_t^{\text{daily}} = \sqrt{\sum_{i=1}^{M}(r_{t+i}^{\text{daily}})^2}$
- Error : $\epsilon(\tau) = |\sigma_t^{\tau} - \sigma_t^{\text{daily}}|$
- Loss : $l^{\sigma}(\epsilon(\tau))$ where loss function l^{σ} is Huber loss with 1σ boundary.
- Optimal truncation threshold : $\hat{\tau} = \arg\min_{\tau}\left(l^{\sigma}(\epsilon(\tau))\right)$

2.3 Benchmark

Does not remove jump noise for benchmark purpose

3 Microstructure noise adjust

Removes high-frequency component

3.1 Pre-Averaging

Windowing

$$r_{PA} = r_t * w$$

was applied to the raw return r_t with window size = K and window function $w(s) = \min(s+1, (K-s))$

3.2 Fractional diffusion

Models price process as fractional diffusion, and removes drift.
Derivative with order ¡ 0.2 is considered as drift, because stationarity is achieved (ADF test with p=0.01) at 0.2-th derivative. Fractional differentiation is an analytic continuation of discrete differentiation.

$$\frac{\partial^n x_t}{\partial t^n} = x(t) * w(s) \quad \text{where} \quad w(s) = ({}_nC_s(-1)^s)$$

$$\frac{\partial^n x_t}{\partial t^n} = x(t) * w(s) \quad \text{where} \quad w(s) = \left(\frac{\Gamma(n+1)}{\Gamma(s+1)\Gamma(n-s+1)}(-1)^s\right)$$

See : Jiahao Jiang, Bing Miao; A study of anomalous stochastic processes via generalizing fractional calculus. Chaos 1 February 2025; 35 (2): 023156. https://doi.org/10.1063/5.0244009

3.3 Padé transform

Models jumps as Lorentz peak of oscillator resonance.

4 Volatility estimation

Estimate volatility, for given raw log-price and denoised log-price.

4.1 RV

Baseline. $\sigma_t = \frac{1}{K}\sum_{i=t}^{t+K} r_i^2$

4.2 TSRV

Two-scale realized volatility

$$(\sigma_t^{TRV})^2 = \frac{1}{K}\sum_{i=t}^{t+K}(r_t^{PA})^2 - \frac{1}{2}\sum_{i=t}^{t+K} r_i^2 \sum_{s=1}^{K-1}\left(w(s) - w(s-1)^2\right)$$

4.3 PRV

Pre-averaging realized volatility with noise adjustment

$$(\sigma_t^{PRV})^2 = \frac{1}{K}\sum_{i=t}^{t+K}(r_t^{PA})^2 - \frac{1}{2}\sum_{i=t}^{t+K} r_i^2 \sum_{s=1}^{K-1}\left(w(s) - w(s-1)^2\right)$$

where w is window function.

5 Evaluation

Compare methods based on information measure

5.1 KLD of Conditional distribution

Measures the amount of information in time Filtration $\mathcal{F}_t$.

$$\mathrm{KL}(P|P_0)$$

where $P = P(\sigma_t|\sigma_t{-}1) = P(\sigma_t|\mathcal{F}_t)P(\sigma_{t-1})$ and $P_0 = P(\sigma_t)P(\sigma_t) = P(\sigma_t)P(\sigma_{t-1})$. P is estimated using kernel density estimation.

5.2 Self-similarity dimension

. Fractal dimension is used to measure information amount, using box-counting method.

5.3 Minkowski Measure

For given fractal dimension, measures persistency.