

## 1 Notation

- 1-second last mid price was observed for 1-month
- Daily volatility is being estimated
- Sampling period : K ( $60 * 10$ ) (FIXME: 10 minute; why?)
- Parameter estimation period : M ( $60 * 60 * 24 = 86400$ ) (FIXME: 1 day; why?)
- Sequence length : n ( $86400 * 30 = 2592000$ )
- time :  $0 < t < n$
- log-price :  $x_t$
- log-return :  $r_t$

## 2 Jump adjust

Removes jumps and stitch the log-price

### 2.1 Truncation

Truncation with  $1\sigma$  was used.

$$\tilde{r}_t = \Psi_\tau(r_t) = \text{sgn}(r_t) \min(|r_t|, \tau = \sigma_t)$$

where  $\sigma^2 = \frac{1}{M} \sum_{i=t}^{t+M} r_i$ .

$\tau = \sigma$  ensures the truncated process ensures sub-Gaussian concentration condition by Hoeffding's lemma

$$E[\exp(\lambda r_t)] \leq \exp\left(\frac{\lambda^2 \tau^2}{2}\right) = \exp\left(\frac{\lambda^2 \sigma^2}{2}\right)$$

### 2.2 Truncation with MLE

It was assumed that the standard deviation of truncated return goes to the standard deviation of daily return.

- Raw return :  $r_t$
- Truncated return with threshold  $\tau$  :  $\tilde{r}_t^\tau$
- Standard deviation :  $\sigma_t^\tau = \sqrt{\sum_{i=1}^M (\tilde{r}_{t+i}^\tau)^2}$
- Daily return :  $r_t^{\text{daily}} = \frac{1}{M} \sum_{i=0}^{M-1} r_{t+i}$

- Daily standard deviation :  $\sigma_t^{\text{daily}} = \sqrt{\sum_{i=1}^M (r_{t+i}^{\text{daily}})^2}$
- Error :  $\epsilon(\tau) = |\sigma_t^\tau - \sigma_t^{\text{daily}}|$
- Loss :  $l^\sigma(\epsilon(\tau))$  where loss function  $l^\sigma$  is Huber loss with  $1\sigma$  boundary.
- Optimal truncation threshold :  $\hat{\tau} = \arg \min_\tau (l^\sigma(\epsilon(\tau)))$

### 2.3 Benchmark

Does not remove jump noise for benchmark purpose

## 3 Microstructure noise adjust

Removes high-frequency component

### 3.1 Pre-Averaging

Windowing

$$r_{PA} = r_t * w$$

was applied to the raw return  $r_t$  with window size = K and window function  $w(s) = \min(s+1, (K-s))$

### 3.2 Fractional diffusion

Models price process as fractional diffusion, and removes drift.

Derivative with order  $\downarrow 0.2$  is considered as drift, because stationarity is achieved (ADF test with p=0.01) at 0.2-th derivative. Fractional differentiation is an analytic continuation of discrete differentiation.

$$\frac{\partial^n x_t}{\partial t^n} = x(t) * w(s) \text{ where } w(s) = ({}_n C_s (-1)^s)$$

$$\frac{\partial^n x_t}{\partial t^n} = x(t) * w(s) \text{ where } w(s) = \left( \frac{\Gamma(n+1)}{\Gamma(s+1)\Gamma(n-s+1)} (-1)^s \right)$$

See : Jiahao Jiang, Bing Miao; A study of anomalous stochastic processes via generalizing fractional calculus. Chaos 1 February 2025; 35 (2): 023156. <https://doi.org/10.1063/5.0244009>

### 3.3 Padé transform

Models jumps as Lorentz peak of oscillator resonance.

## 4 Volatility estimation

Estimate volatility, for given raw log-price and denoised log-price.

### 4.1 RV

Baseline.  $\sigma_t = \frac{1}{K} \sum_{i=t}^{t+K} r_i^2$

### 4.2 TSRV

Two-scale realized volatility

$$(\sigma_t^{TRV})^2 = \frac{1}{K} \sum_{i=t}^{t+K} (r_t^{PA})^2 - \frac{1}{2} \sum_{i=t}^{t+K} r_i^2 \sum_{s=1}^{K-1} (w(s) - w(s-1))^2$$

```
logprc = self.logprc.copy() x1 = np.square(logprc[:-K] - logprc[K:]).mean()
x2 = np.square(np.diff(logprc)).mean() tsrv = x1 - x2
```

### 4.3 PRV

Pre-averaging realized volatility with noise adjustment

$$(\sigma_t^{PRV})^2 = \frac{1}{K} \sum_{i=t}^{t+K} (r_t^{PA})^2 - \frac{1}{2} \sum_{i=t}^{t+K} r_i^2 \sum_{s=1}^{K-1} (w(s) - w(s-1))^2$$

where  $w$  is window function.

## 5 Evaluation

Compare methods based on information measure

### 5.1 KLD of Conditional distribution

Measures the amount of information in time Filtration  $\mathcal{F}_t$ .

$$\text{KL}(P|P_0)$$

where  $P = P(\sigma_t|\sigma_{t-1}) = P(\sigma_t|\mathcal{F}_t)P(\sigma_{t-1})$  and  $P_0 = P(\sigma_t)P(\sigma_t) = P(\sigma_t)P(\sigma_{t-1})$ .  
P is estimated using kernel density estimation.

### 5.2 Self-similarity dimension

. Fractal dimension is used to measure information amount, using box-counting method.

### 5.3 Minkowski Measure

For given fractal dimension, measures persistency.