Differential Equations Notes

Based on Lectures and "An Introduction to ODEs"

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Contents

Basic Calculus

1 Differentiation

1.1 Definitions and methods

Definition 1.1 (Derivative). The derivative of a function $f(x)$ wrt its argument x is the function
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$
We define higher derivatives recursively by
$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} f}{\mathrm{d}x^{n-1}}\right).$$

For the derivative to exist, we need
$$\lim_{h\to 0-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}.$$

Rules for differentiation:

1. **Chain rule**: $(f(g(x)))' = f'(g(x))g'(x)$.
2. **Product rule**: $(u\cdot v)' = u\cdot v' + u'\cdot v$.
3. **Leibniz's rule**: generalisation of product rule.[1]
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(u\cdot v) = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}.$$

[1] There are multiple ways to prove, e.g. by induction.

1.2 Order of magnitude

The goal is to compare the sizes of functions, in the vicinity of specific points.

Definition 1.2 (Little and Big o). We say $f(x) = o(g(x))$ as $x\to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$. We say $f(x) = O(g(x))$ as $x \to x_0$ if $\exists M, \delta > 0, |x-x_0| < \delta \Rightarrow |f(x)| \leqslant M\,|g(x)|$. The infinite case is defined similarly.

To find the tangent line to f at x_0, note that
$$\begin{aligned} \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} &= \frac{f(x_0+h)-f(x_0)}{h} + \frac{o(h)}{h} \quad \text{when } h\to 0 \\ \Longrightarrow f(x_0+h) &= f(x_0) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} h + o(h) \quad \text{when } h\to 0 \end{aligned}$$

1.3 Taylor's Theorem and L'Hopital's Theorem

We want to approximate a function $f(x)$ with a polynomial of order n:
$$f(x) = \underbrace{a_0 + a_1 x + \cdots + a_n x^n}_{P_n(x)}.$$

Differentiating recursively we get

$$P_n(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0). \tag{1.1}$$

Alternatively, we can write $f(x) = P_n(x) + E_n$, where E_n is called the *remainder/error*.

By generalisation of $f(x + h) = f(x) + hf'(x) + o(h), h \to 0$, we get

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f'(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + o(h^n). \tag{1.2}$$

By refining the range of $o(h^n)$ we get

Theorem 1.1 (Taylor)**.** If the first $n + 1$ derivatives of $f(x)$ exist, then

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f'(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + O(h^{n+1})..$$

Using this we can prove

Theorem 1.2 (L'Hopital)**.** Let f and g be differentiable at $x = x_0$ and

$$\lim_{x \to x_0} f(x) = f(x_0) = 0, \quad \lim_{x \to x_0} g(x) = g(x_0) = 0.$$

Proof. (Not rigorous) As $x \to x_0$,

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)}{g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)} \\ &= \frac{(x - x_0)f'(x_0) + o(x - x_0)}{(x - x_0)g'(x_0) + o(x - x_0)} \\ &\to \frac{f'(x_0)}{g'(x_0)}. \end{aligned}$$

□

Note that it can be applied recursively.

2 Integration

2.1 Definition

All functions mentions are assumed to be well-hehaved.

We evaluate the area under the curve of $f(x)$ by considering

$$\sum_{n=0}^{N-1} f(x_n)\Delta x$$

where $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$.

Theorem 2.1 (MVT)**.** For a continuous function $f(x)$:

$$\int_{x_n}^{x_{n+1}} f(x)\,\mathrm{d}x = f(x_c)(x_{n+1} - x_n) \quad \text{for some } x_c \in (x_n, x_{n+1}).$$

Estimate $f(x_c)$ as follows:

$$f(x_c) = f(x_n) + O(x_c - x_n) = f(x_n) + O(x_{n+1} - x_n).$$

Hence

$$\begin{aligned}\int_{x_n}^{x_{n+1}} f(x)\,\mathrm{d}x &= f(x_c)(x_{n+1} - x_n) \\ &= [f(x_n) + O(x_{n+1} - x_n)](x_{n+1} - x_n) \\ &= \Delta x f(x_n) + O(\Delta x^2).\end{aligned}$$

Therefore the error $\epsilon = O(\Delta x^2)$. It follows that

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \left\{ \left[\sum_{n=0}^{N-1} f(x_n)\Delta x \right] + O(N\Delta x^2) \right\}.$$

Hence

Definition 2.1 (Definite integral). $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n)\Delta x$

2.2 Fundamental Theorem of Calculus

Theorem 2.2 (FTC). Let

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

then

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x).$$

Proof. From the definition of derivative:

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{h\to 0} \frac{1}{h} \left\{ \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \right\} \\ &= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t \\ &= \lim_{h\to 0} \frac{1}{h} \left(f(x)h + O(h^2) \right) \\ &= f(x).\end{aligned}$$

□

Corollary 2.3.

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_x^b -f(t)\,\mathrm{d}t.$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^{g(x)} f(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x} F(g(x)) = \frac{\mathrm{d}F}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} = f(g(x))\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Definition 2.2 (Indefinite integral).

$$\int f(x)\,\mathrm{d}x = \int_{x_0}^x f(t)\,\mathrm{d}t.$$

2.3 Techniques of Integration

skipped

3 Introduction to multivariable functions

Lecture 5.

3.1 Partial derivative

Definition 3.1. The *partial derivative* of $f(x, y)$ wrt x is

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}. \tag{3.1}$$

Similarly

$$\left.\frac{\partial f}{\partial y}\right|_x = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}.$$

We can take them in any order to form *cross derivatives*.

Note that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right). \tag{3.2}$$

3.2 Multivariable chain rule

Theorem 3.1. For well-behaved functions, we have

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \tag{3.3}$$

Proof. Note that

$$\begin{aligned}
\delta f &= f(x + \delta x, y + \delta y) - f(x + \delta x, y) + f(x + \delta x, y) - f(x, y) \\
&= f(x + \delta x, y) + \delta y \frac{\partial f}{\partial y}(x + \delta x, y) + o(\delta y) - f(x + \delta x, y) \\
&+ f(x, y) + \delta x \frac{\partial f}{\partial x}(x, y) + o(\delta x) - f(x, y) \\
&= \delta y \frac{\partial f}{\partial y}(x + \delta x, y) + \delta x \frac{\partial f}{\partial x}(x, y) + o(\delta x) + o(\delta y) \\
&= \delta y \left(\frac{\partial f}{\partial y}(x, y) + \delta x \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x, y)\right) + o(\delta x)\right) + \delta x \frac{\partial f}{\partial x}(x, y) + o(\delta x) + o(\delta y) \\
&= \delta y \frac{\partial f}{\partial y}(x, y) + \delta x \frac{\partial f}{\partial x}(x, y) + \delta x \delta y \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x, y)\right) + o(\delta x) + o(\delta y) + o(\delta x \delta y).
\end{aligned}$$

Taking limit gives the result. □

Remark. For $f(x(t), y(t))$, we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\delta x, \delta y, \delta t \to 0}\left[\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right] = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{3.4}$$

And integral form:

$$\int \mathrm{d}f = \int \frac{\partial f}{\partial x}\mathrm{d}x + \int \frac{\partial f}{\partial y}\mathrm{d}y \tag{3.5}$$

In this case we need to specify the *path* of integral as there might be some priority issues.

3.3 Applications of multivariable chain rule

3.3.1 Change of variables

It is often useful to write a DE in a different coordinate system before solving it. Need to transform the derivatives into the new coordinate system.

Example. Change from cartesian coordinates to polar coordinates: $x = r\cos\theta, y = r\sin\theta$. Firstly, write

$$f = f(x(r,\theta), y(r,\theta)).$$

We have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta.$$

By regarding $\frac{\partial f}{\partial r}$ as $\frac{\mathrm{d}f}{\mathrm{d}r}$ with θ fixed, we get this result.

Similar for other partial derivatives.

3.3.2 Implicit Differentiation

Consider $f(x,y,z) = c, c \in \mathbb{R}$. f describes a surface in 3d space. $f(x,y,z) = c$ implicitly defines $x(y,z), y(x,z), z(x,y)$. However, we can find $\frac{\partial z}{\partial x}$ here using implicit differentiation.

Consider $f(x,y,z(x,y)) = c$.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z.$$

Finding the partial derivative for x:

$$\begin{aligned}
\left.\frac{\partial f}{\partial x}\right|_y &= \left.\frac{\partial f}{\partial x}\right|_{yz}\left.\frac{\partial x}{\partial x}\right|_y + \left.\frac{\partial f}{\partial y}\right|_{xz}\left.\frac{\partial y}{\partial x}\right|_y + \left.\frac{\partial f}{\partial z}\right|_{xy}\left.\frac{\partial z}{\partial x}\right|_y \\
&= \left.\frac{\partial f}{\partial x}\right|_{yz} + \left.\frac{\partial f}{\partial y}\right|_{xz}\left.\frac{\partial y}{\partial x}\right|_y + \left.\frac{\partial f}{\partial z}\right|_{xy}\left.\frac{\partial z}{\partial x}\right|_y. \\
\iff \quad \left.\frac{\partial f}{\partial x}\right|_y &= \left.\frac{\partial f}{\partial x}\right|_{yz} + \left.\frac{\partial f}{\partial z}\right|_{xy}\left.\frac{\partial z}{\partial x}\right|_y \\
\iff \quad 0 &= \left.\frac{\partial f}{\partial x}\right|_{yz} + \left.\frac{\partial f}{\partial z}\right|_{xy}\left.\frac{\partial z}{\partial x}\right|_y \\
\iff \quad & \boxed{\left.\frac{\partial z}{\partial x}\right|_y = -\frac{\partial f/\partial x|_{yz}}{\partial f/\partial z|_{xy}}}
\end{aligned}$$

Notice the subscripts are very important since they discribes different functions

Since $\left.\frac{\partial y}{\partial x}\right|_y = 0$

Since $f = c$ along the surface $z(x,y)$.

Note that $\left.\frac{\partial f}{\partial x}\right|_{yz} \neq 0$ in general.

Remark. Reciprocal rule still holds as long as the same variable(s) are held fixed. e.g.

$$\left.\frac{\partial r}{\partial x}\right|_y = \frac{1}{\left.\frac{\partial x}{\partial r}\right|_y} \quad \text{but} \quad \left.\frac{\partial r}{\partial x}\right|_y \neq \frac{1}{\left.\frac{\partial x}{\partial r}\right|_\theta}.$$

3.3.3 Differentiation of an integral wrt its parameters

Consider a family of functions $f(x;\alpha)$, where α is the parameter. Define

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,\mathrm{d}x.$$

$$
\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= \lim_{\delta\alpha\to 0} \frac{I(\alpha+\delta\alpha)-I(\alpha)}{\delta\alpha} \\
&= \lim_{\delta\alpha\to 0} \frac{1}{\delta\alpha}\left[\int_{a(\alpha+\delta\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,\mathrm{d}x - \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,\mathrm{d}x\right] \\
&= \lim_{\delta\alpha\to 0} \frac{1}{\delta\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x;\alpha+\delta\alpha)-f(x;\alpha)\,\mathrm{d}x - \int_{a(\alpha)}^{a(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,\mathrm{d}x + \int_{b(\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,\mathrm{d}x\right] \\
&= \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial\alpha}\,\mathrm{d}x - f(a;\alpha)\lim_{\delta\alpha\to 0}\frac{a(\alpha+\delta\alpha)-a(\alpha)}{\delta\alpha} + f(b;\alpha)\lim_{\delta\alpha\to 0}\frac{b(\alpha+\delta\alpha)-b(\alpha)}{\delta\alpha}.
\end{aligned}
$$

Draw a graph to understand the steps.

When $\delta\alpha$ is very small, we can approximate the latter two integrals with the area of the rectangle of hight $f(a;\alpha)$ and width $a(\alpha+\delta\alpha)-a(\alpha)$.

Hence,

$$
\boxed{\frac{\mathrm{d}I}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,\mathrm{d}x = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial\alpha}\,\mathrm{d}x + f(b;\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a;\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha}}.
$$

First order linear ODEs

4 Terminology

Definition 4.1. An *ordinary differential equation* is a differential equation involving a function of one variable. A *partial differential equation* is (a) differential equation(s) involving a function of more than one variable.
nth order DE: the highest order of derivative is n.
Linear: dependent variable appears linearly.

5 Prelude: Exponential functions

Consider $f = a^x, a > 0$, we have

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} \\
&= \lambda a^x.
\end{aligned}
$$

Hence

Definition 5.1. Define $\exp(x) = e^x$ as the solution to the DE

$$
\frac{\mathrm{d}f}{\mathrm{d}x} = f(x), \quad f(0) = 1.
$$

Therefore e is the value of a such that $\lambda = 1$. i.e.,

$$
\lim_{h\to 0}\frac{e^h-1}{h} = 1.
$$

Define $\ln(x)$ as the inverse of e^x such that $e^{\ln(x)} = x$.
Consider $a^x = e^{\ln(a)x}$, so

$$
\frac{\mathrm{d}f}{\mathrm{d}x} = (\ln a)a^x, \lambda = \ln a.
$$

The exponential function is the *eigenfunction* of the differential operator.

The *eigenfunction* of an operator is unchanged by the action of the operator, except for a multiplicative scaling by the eigenvalue.

6 Rules for linear ODEs

1. Any linear homogeneous ODE with constant coefficients has solutions of form $e^{\lambda x}$, the eigenfunction. By *homogeneous* we mean that all terms involve the dependent variable or its derivatives.

This means that $y = 0$ is a trivial solution for all homogeneous ODEs.

Constant coefficients imply that the independent variable does not appear explicitly in DE.

2. For linear homogeneous ODEs, any constant multiple of a solution is also a solution.

3. An nth order ODE has n independent solutions.

For constant coefficient ODEs, this rule follows from the fundamental theorem of algebra.

4. An nth order ODE requires n initial/boundary conditions.

7 Inhomogeneous(forced) first order ODEs with constant coefficients

7.1 Constant forcing

Example. Consider the equation

$$5y' - 3y = 10.$$

Solution steps:

1. Write the general solution $y = y_p + y_c$ where y_p is a *particular integral* and y_c is a complementary function

2. Find y_p by simply setting $y' = 0$. In this case, $y = -10/3$.

3. Insert general solution into DE:

$$\begin{aligned} & 5(y_p + y_c)' - 3(y_p + y_c) && = 10 \\ \iff & 5y_c' + 10 - 3y_c && = 10 \\ \iff & 5y_c - 3y_c' = 0. \end{aligned}$$

   Note that y_c is a solution to corresponding homogeneous equation.

4. Solve for y_c. In this case, $y_c = Ae^{3x/5}$.

5. Combine y_p and y_c.

7.2 Eigenfunction forcing

Example problem: In a sample of rock, isotope A decays to isotope B at a rate proportional to a, the number of nuclei of A. B decays to C at a rate proportional to b, the number of nuclei of B. Find $b(t)$.

We have

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= -k_a a \implies a = a_0 e^{-k_a t}\\ \frac{\mathrm{d}b}{\mathrm{d}t} &= k_a a - k_b b,\end{aligned}$$

which means $\dot{b} + k_b b = k_a a_0 e^{-k_a t}$. RHS is called a *forcing term*, and it is an eigenfunction of differential operator.

We *guess* the form of the particular integral

$$b_p = ce^{-k_a t},$$

then the equation becomes

$$-k_a c + k_b c = k_a a_0 \iff c = \frac{k_a}{k_b - k_a} a_0, \quad \text{for } k_b \neq k_a.$$

Since the general solution for the DE is $b = b_p + b_c$,

$$\dot{b_c} + k_b b_c = 0 \iff b_c = De^{-k_b t}.$$

Hence

$$b = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + De^{-k_b t}.$$

If $b(0) = 0$, $D = -c$, then

$$b = \frac{k_a}{k_b - k_a} a_0 \left(e^{-k_a t} - e^{-k_b t}\right).$$

Taking the ratio of b and a:

$$\frac{b(t)}{a(t)} = \frac{k_a}{k_b - k_a} \left(1 - e^{(k_a - k_b)t}\right).$$

We can date the age without knowing a_0 is. This result allows rocks and other materials to be dated by measuring ratio of isotopes.

8 First order ODEs of non-constant coefficients

The general form is

$$a(x)y' + b(x) + y = c(x).$$

The standard form is

$$y' + p(x)y = f(x).$$

Solved using *integrating factors*, multiply by IF μ:

$$\mu y' + (\mu p)y = \mu f.$$

If $\mu p = \mu'$, LHS$=(\mu y)'$ by product rule. Hence we want $p = \mu'/\mu$.

$$\int p \, \mathrm{d}x = \int \frac{\mu'}{\mu} \, \mathrm{d}x = \ln \mu \Longrightarrow \boxed{\mu = e^{\int p(x) \, \mathrm{d}x}}.$$

Thus the DE becomes

$$(\mu y)' = \mu f \Longleftrightarrow y = \frac{1}{\mu} \int \mu f \, \mathrm{d}x.$$

9 Discrete equations

A *discrete equation* is an equation involving a function evaluated at a discrete set of points. Lecture 8

9.1 Numerical integration

Consider a discrete representation of $y(x)$, $y(x_1), \ldots, y(x_n)$. One approximation to y' is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_n} \approx \frac{y_{n+1} - y_n}{h}, \quad h = \frac{x_n}{n},$$

given that x_i are uniformly distributed. This is called the *Forward Euler* approximation, but it is not the best approximation of the derivative in most contexts.

Example. Consider $5y' - 3y = 0$. We can approximate the equation by

$$5\frac{y_{n+1} - y_n}{h} - 3y = 0,$$

which is called a *difference equation*, and deduce that

$$y_{n+1} = \left(1 + \frac{3h}{5}\right) y_n,$$

which is called a *recurrence relation*.
Apply recurrence relation repeatedly:

$$y_n = \left(1 + \frac{3h}{5}\right)^n y_0 = \left(1 + \frac{3x_n}{5n}\right)^n y_0.$$

Euler's definition of e^x is

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

It can be shown that this definition is equivalent to the previous definition. Hence

$$y(x) = \lim_{n \to \infty} y_n = y_0 e^{3x/5}.$$

Note for finite n, $y_n < y(x)$.

9.2 Series solutions

A powerful way to solve ODEs is to seek solutions in the form of an infinite power series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Plug into DE and find a solution.

Example. Consider $5y' - 3y = 0$. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Multiply both sides by x:

$$\begin{aligned} xy' &= \sum_{n=1}^{\infty} n a_n x^n, \\ xy &= \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n. \end{aligned}$$

Then the DE becomes

$$\begin{aligned} & 5 \sum_{n=1}^{\infty} n a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \\ \iff & \sum_{n=1}^{\infty} x^n \left(5 n a_n - 3 a_{n-1}\right) = 0. \end{aligned}$$

This holds for every $x \in \mathbb{R}$, so it holds if and only if

$$\forall x \in \mathbb{R}, 5 n a_n - 3 a_{n-1} = 0 \iff a_n = \frac{3}{5n} a_{n-1} \iff a_n = \left(\frac{3}{5}\right)^n \frac{a_0}{n!}.$$

Hence

$$y = a_0 \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n \frac{x^n}{n!} = a_0 e^{3x/5}.$$

This converges for all x, so $y(x) = a_0 e^{3x/5}$ is a solution.

First order nonlinear ODEs

General form is

$$Q(x, y) \frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0. \tag{9.1}$$

10 Separable equations

(10.1) is separable if and only if it can be written in the form

$$q(y) \mathrm{d}y = p(x) \mathrm{d}x,$$

and we simply solve x, y by integrating both sides.

11 Exact equations

(10.1) is an *exact equation* if and only if

$$Q(x, y) \mathrm{d}y + P(x, y) \mathrm{d}x \tag{*}$$

is an *exact differential* of function $f(x,y)$. i.e., $\mathrm{d}f = Q\mathrm{d}y + P\mathrm{d}x$. If this holds, then (10.1) implies that $\mathrm{d}f = 0$ and $f(x,y)$ is constant. We can use multivariable chain rule to check.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \Longrightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Comparing with (10.1), if $(*)$ is an exact differential, then $\exists f(x)$ such that

$$\frac{\partial f}{\partial x} = P(x,y), \quad \frac{\partial f}{\partial y} = Q(x,y). \tag{**}$$

Hence

$$\begin{aligned} &\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y} \wedge \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x} \\ \Longleftrightarrow &\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}. \end{aligned}$$

If it holds throughout a *simply connected* domain $\mathcal{D}$, then $P\mathrm{d}x + Q\mathrm{d}y$ is an exact differential of a single-valued function $f(x,y)$ in D. Henc we can use this to check exact equations.

$f(x,y)$ can be found by integrating $(**)$.

Example. Consider

$$6y(y-x)\frac{\mathrm{d}y}{\mathrm{d}x} + (2x - 3y^2) = 0.$$

Here $P = 2x - 3y^2, Q = 6y(y-x)$. We have

$$\frac{\partial P}{\partial y} = -6y = \frac{\partial Q}{\partial x},$$

so it is an exact equation. Note that

$$\begin{aligned} &\int \frac{\partial f}{\partial x}\,\mathrm{d}x = x^2 - 3xy^2 + h(y), \\ &\frac{\partial f}{\partial y} = (x^2 - 3xy^2 + h(y))'_y = -6xy + h'(y) = 6y(y-x), \\ \Longrightarrow &h' = 6y^2 \Longrightarrow h = 2y^3. \end{aligned}$$

Hence $f(x,y) = x^2 - 3xy^2 + 2y^3 + C$ and

$$x^2 - 3xy^2 + 2y^3 = C$$

is the general solution.

12 Isoclines and solution curves

Nonlinear equations are not guaranteed to have simple/closed form solutions. Nevertheless we can analyze the behaviour of the system without solving. Lecture 9.

Consider an ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t).$$

Each intial condition will give a different solution curve.

Example. Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t(1-y^2) = f(y,t). \tag{*}$$

It is separable:

$$\int \frac{\mathrm{d}y}{1-y^2}\,\mathrm{d}y = \int t\,\mathrm{d}t$$
$$\Longrightarrow y = \frac{A-e^{-t^2}}{A+e^{-t^2}}.$$

This general solution produces a family of solution curves, parameterised by by A.

Definition 12.1 (Isocline)**.** An *isocline* is the curve along which $f = \dot{y} = C$, where C is a constant.

Procedure of drawing a curve: draw isoclines, inspect the slope of y, draw a vector field, and plot the lines.

Remark. Since $f(y,t)$ is single-valued, any two solution curves do not cross.

13 Fixed(equilibrium) points

Definition 13.1. A fixed point is a point where

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t) = 0.$$

A fixed point is called *stable*(*unstable*) if solution curves in a small neighbourhood of the fixed point converge(diverge) to(away) the fixed point.

We can analyze the stability of fixed points using a *perturbation* analysis.

Let $y = a$ be a fixed point of $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t)$, i.e. $f(a,t) = 0$. Consider a small perturbation from the fixed point: $y = a + \epsilon(t)$ We have

$$\begin{aligned}\frac{\mathrm{d}\epsilon}{\mathrm{d}t} &= \frac{\mathrm{d}(y-a)}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t}\\ &= f(a+\epsilon,t)\\ &= f(a,t) + \epsilon\frac{\partial f}{\partial y}(a,t) + O(\epsilon^2).\end{aligned}$$

For small ϵ, we have

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} \approx \epsilon\frac{\partial f}{\partial y}(a,t).$$

Hence we've converted the non-linear ODE into a linear one wrt ϵ.

If $\lim_{t\to\infty}\epsilon = 0$, then a is a stable fixed point. Conversely if $\lim_{t\to\infty}\epsilon = \infty$, then a is an unstable fixed point. If $f'_y(a,t) = 0$, then we need higher order terms in Taylor series.

Example. Consider $f(y,t) = t(1-y^2)$. The fixed points are $y = \pm 1$. $f'_y = -2yt$. At $y = 1$, we have

$$\dot{\epsilon} \approx -2\epsilon t \Rightarrow \epsilon = \epsilon_0 e^{-t^2} \to 0.$$

Hence 1 is stable.

At $y = -1$, $\dot{\epsilon} = 2t\epsilon \Rightarrow \epsilon = \epsilon_0 e^{t^2} \to \infty$. Hence -1 is unstable.

14 Autonomous DEs

Definition 14.1. An *autonomous DE* is a special case when $\dot{y} = f(y)$.

In this case, near fixed points $y = a$, we have $\dot{\epsilon} = f'_y(a)\epsilon = \epsilon k$, where k is constant. Hence $\epsilon = \epsilon_0 e^{kt}$. Therefore for autonomous DEs we have

$$\text{if} \begin{cases} f'(a) < 0 \Rightarrow & \text{stable F.P.} \\ f'(a) > 0 \Rightarrow & \text{unstable F.P.} \end{cases}$$

15 Phase Portraits

Another way to analyze solutions to a DE is using a geometrical representation of the solution called a *phase portrait*. Lecture 10

Example (Chemical kinetics)**.** Consider the reaction

$$\mathrm{NaOH} + \mathrm{HCl} \longrightarrow \mathrm{NaCl} + \mathrm{H_2O}$$

with

	NaOH	+	HCl	→	H_2O	+	NaCl
Number of molecules	a		b		c		c
Initial number of molecules	a_0		b_0		0		0

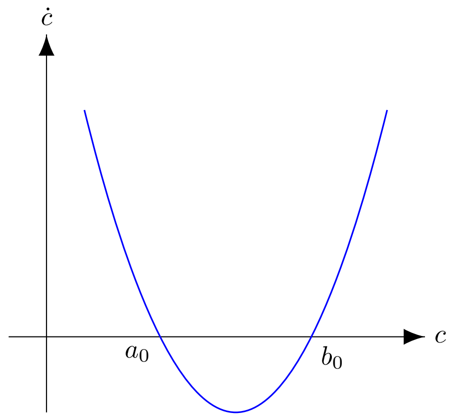
A model of reaction rate is

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \lambda ab,$$

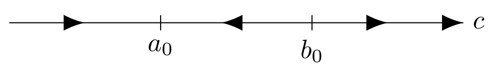
where λ is constant. Atoms are conserved: $a = a_0 - c, b = b_0 - c$. Then the equation can be written as

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \lambda(a_0 - c)(b_0 - c).$$

This is an example of nonlinear first order ODE.
Plot a 2D phase portrait for this DE. One way is to plot $\frac{\mathrm{d}c}{\mathrm{d}t}$ against t:



We can analyze the behaviour using 1D phase portrait:



arrows are drawn by sign of $\dot{c}$.

Example (Population dynamics)**.** Let $y(t)$ be population, αy be birth rate and βy be death rate.

(a) Linear model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y - \beta y \Rightarrow y = y_0 e^{(\alpha-\beta)t}.$$

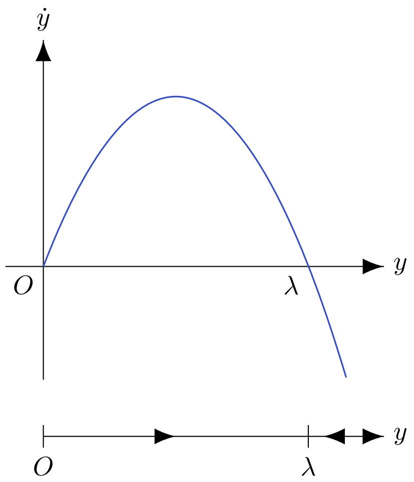
If $\alpha > \beta$, then $\lim_{t\to\infty} y = \infty$.

(b) Nonlinear model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (\alpha - \beta)y - \gamma y^2.$$

γy^2 is dominant when y is large. It models increased death rate at high population.

Equivalently we have $\dot{y} = ry(1 - \frac{y}{\lambda})$ where $r = \alpha - \beta, \lambda = \frac{\alpha-\beta}{\gamma}$. λ is called the *carrying capacity.*



16 Fixed points in discrete equations

16.1 Definitions

Consider a first order discrete(difference) equation of the form

$$x_{n+1} = f(x_n).$$

Define the *fixed point* as the value of x_n where $x_{n+1} = x_n$. That is, where

$$f(x_n) = x_n.$$

16.2 Stability

Use perturbation analysis to study its stability. Let x_f be a stable point of x_n, and perturb by a small ϵ:

$$\begin{aligned} f(x_f + \epsilon) &= f(x_f) + \epsilon \frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_{x_f} + O(\epsilon^2) \\ &= x_f + \epsilon \frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_{x_f} + O(\epsilon^2) \\ &\approx x_f + \epsilon \frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_{x_f}. \end{aligned}$$

Let $x_n = x_f + \epsilon$, then

$$x_{n+1} \approx x_f + \epsilon \frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_{x_f}.$$

Therefore,

$$x_f \text{ is} \begin{cases} \text{stable} & \text{if } \left| \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x_f} \right| < 1 \\ \text{unstable} & \text{if } \left| \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x_f} \right| > 1 \end{cases}$$

16.3 Example: Logistic map

Nonlinear discrete population model

$$\frac{x_{n+1} - x_n}{\Delta t} = \lambda x_n - \gamma x_n^2.$$

Formally,

$$x_{n+1} = (\lambda \Delta t + 1)x_n - \gamma \Delta t x_n^2.$$

A simpler version is

$$x_{n+1} = r x_n (1 - x_n) = f(x_n).$$

This equation is the Logistic map.

Fixed points: Let $f(x_n) = x_n$. We have $x_n(r-1-rx_n) = 0 \Rightarrow x_n = 0 \vee x_n = 1-1/r$.

Stability: Write $f(x) = rx(1-x)$, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = r(1-2x).$$

Hence for $x_n = 0$ we have $\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_0 = r$, so if $0 < r < 1$, it is stable. For $r > 1$, it is unstable.

For $x_n = 1 - \frac{1}{r}$, $\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x_n} = 2 - r$. Then for $0 < r < 1$ it is unphysical since $x_n < 0$. If $1 < r < 3$, it is stable. For $r > 3$ it is unstable.