

# ***Vectors and Matrices***

## ***Based on Lectures and "Intro to Linear Algebra"***

---

$\theta\omega\theta$

*Not in University of Cambridge  
skipped some talks irrelevant to contents*

*E-mail:* `not telling you`

---

## Contents

<b>I</b>	<b>Complex Numbers</b>	<b>1</b>
1	Definition	1
2	Basic Properties & Consequences	1
3	Exponential and Trigs in $\mathbb{C}$	2
4	Logarithms and Complex powers	2
5	Transformations, Lines, and Circles	3
<b>II</b>	<b>Vectors in 3 Dimensions</b>	<b>3</b>
6	Vector addition and scalar multiplication	3
7	Dot product	4
8	Vector cross product	4
9	Orthonormal Bases and Components	5
10	Triple products	5
10.1	Scalar triple product	5
10.2	Vector triple product	5
11	Lines, Planes, and Vector equations	6
11.1	Lines	6
11.2	Planes	6
11.3	Other vector equations	6
12	Index notation and the summation convention	7
12.1	Components, $\delta$ & $\epsilon$	7
12.2	Summation convention	8
12.3	Applications	8
12.4	$\epsilon$ $\epsilon$ identity	9
<b>III</b>	<b>Vectors in General</b>	<b>9</b>

<b>13 Vectors in <math>\mathbb{R}^n</math></b>	<b>9</b>
13.1 Definition and basic properties	9
13.2 Norm of a vector	10
13.3 Cauchy-Schwarz and Triangle inequalities	10
13.4 Inner Products and Cross products	11
<b>14 Vector Spaces</b>	<b>11</b>
14.1 Axioms, span, and subspaces	11
14.2 Linear dependence and independence	12
14.3 Inner products	12
<b>15 Bases and dimension</b>	<b>13</b>
<b>16 Vectors in <math>\mathbb{C}^n</math></b>	<b>15</b>
16.1 Introduction and definitions	15
16.2 Inner product	16

---

# Complex Numbers

## 1 Definition

**Definition 1.1.** Construct  $\mathbb{C}$  from  $\mathbb{R}$  by adding  $i$  that  $i^2 = -1$ . Any  $z \in \mathbb{C}$  is in the form

$$z = x + iy, x = \operatorname{Re} z, y = \operatorname{Im} z, x, y \in \mathbb{R}.$$

Addition and multiplication are defined by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2).$$

The *conjugate* is defined by

$$\bar{z} = z^* = x - iy.$$

The *modulus* is defined by

$$r = |z|, r \geq 0, r^2 = |z|^2 = z\bar{z} = x^2 + y^2.$$

The *argument* is defined by

$$z \neq 0 : \theta = \arg(z) \in \mathbb{R}, z = r(\cos \theta + i \sin \theta).$$

The values of  $\theta$  in  $(-\pi, \pi]$  are called the *principal values*.

Complex numbers can be plotted on an *Argand diagram*.

## 2 Basic Properties & Consequences

(1)  $+, \times$  are commutative and associative,

$\mathbb{C}$  under  $+$  is an abelian group,

$\mathbb{C}$  under  $\times$  is an abelian group,

$\mathbb{C}$  is a field.

(2) **Fundamental Theorem of Algebra:** A polynomial with  $\deg n$  with coefficients in  $\mathbb{C}$  can be written as a product of  $n$  linear factors, has at least one solution in  $\mathbb{C}$  and has  $n$  solutions connected with multiplicity.

(3) Parallelogram constructions.

(4)

$$|z_1| |z_2| = |z_1 z_2|, |z_1 + z_2| \leq |z_1| + |z_2|.$$

Alternative forms:

$$|z_2 - z_1| \geq |z_2| - |z_1|, |z_2 - z_1| \geq ||z_2| - |z_1||.$$

(5) **De Moivre's Theorem:**  $z^n = r^n(\cos n\theta + i \sin n\theta)$ .

### 3 Exponential and Trigs in $\mathbb{C}$

**Definition 3.1.** Define  $\exp, \cos, \sin$  on  $\mathbb{C}$  by

$$\begin{aligned}\exp(z) &= e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \\ \cos(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots, \\ \sin(z) &= \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots.\end{aligned}$$

These series converge for all  $z \in \mathbb{C}$ . Can be multiplied, rearranged, etc. Definitions reduce to familiar ones in the reals.

**Proposition 3.1.**  $\forall z, w \in \mathbb{C}, e^z e^w = e^{z+w}; e^z e^{-z} = 1, (e^z)^n = e^{nz}, n \in \mathbb{Z}$ .

**Lemma 3.1.** For  $z = x + iy$ :

- (1)  $e^z = e^x(\cos y + i \sin y)$ .
- (2)  $\exp(z) \in \mathbb{C} \setminus \{0\}$ .
- (3)  $e^z = 1 \Leftrightarrow z = 2\pi ni, n \in \mathbb{Z}$ .

**Definition 3.2** (Roots of unity).  $z$  is an  $N$ th root of unity if  $z^N = 1$ .

We have

$$z^N = r^N e^{iN\theta} = 1 \Leftrightarrow r = 1, N\theta = 2n\pi \Leftrightarrow \theta = \frac{2n\pi}{N},$$

which gives  $N$  distinct solutions

$$z = \frac{2n\pi}{N} = \omega^n, \quad n = 0, 1, \dots, N-1.$$

$\omega^n$  lie one the vertices of a regular  $n$ -gon on the unit circle.

### 4 Logarithms and Complex powers

**Definition 4.1.** Define  $w = \log z, z \in \mathbb{C} \wedge z \neq 0$  by  $e^w = e^{\log z} = z$ . Note that since  $\exp$  is many-to-one,  $\log$  is multi-valued.

$$\begin{aligned}z &= r e^{i\theta} = e^{\log r} e^{i\theta} = e^{\log r + i\theta} \\ \Rightarrow \log z &= \log r + i\theta = \log |z| + i \arg(z)\end{aligned}$$

To make it single-valued, simply take the principal value.

**Definition 4.2.** Define *complex power* by

$$z^\alpha = e^{\alpha \log z}, \quad z, \alpha \in \mathbb{C}, z \neq 0.$$

Note that since  $\arg z \rightarrow \arg z + 2n\pi \Rightarrow z^\alpha \rightarrow z^\alpha e^{2n\pi\alpha}$ , it is generally multi-valued. This also reduces to common powers when  $z, \alpha \in \mathbb{R}$ .

**Example.**

$$i^i = e^{i \log i} = e^{i(0+i(\frac{\pi}{2}+2n\pi))} = e^{-(\frac{\pi}{2}+2n\pi)}.$$

## 5 Transformations, Lines, and Circles

- We have five elementary transformations:

$$(1) z \mapsto z + a,$$

$$(2) z \mapsto \lambda z,$$

$$(3) z \mapsto e^{i\alpha} z,$$

$$(4) z \mapsto \bar{z},$$

$$(5) z \mapsto \frac{1}{z}.$$

- General point of a line in  $\mathbb{C}$  through  $z_0$  and parallel to  $w$ :

$$z = z_0 + \lambda w, \lambda \in \mathbb{R} \text{ or } \bar{w}z - w\bar{z} = \bar{w}z_0 - w\bar{z}_0.$$

- General point of a circle in  $\mathbb{C}$  with centre  $c$  and radius  $\rho$ :

$$z = c + \rho e^{i\theta} \text{ or } |z - c| = \rho \text{ or } |z|^2 - \bar{c}z - c\bar{z} = \rho^2 - |c|^2.$$

- Stereographic projection.

# Vectors in 3 Dimensions

## 6 Vector addition and scalar multiplication

**Definition 6.1** (scalar multiplication). Given  $\mathbf{a}$ , and scalar  $\lambda \in \mathbb{R}$ , define  $\lambda\mathbf{a}$  to be the position vector of  $A'$  on the line  $OA$  with length  $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$ . Direction depends on the sign of  $\lambda$ .

Define  $\text{span}\{\mathbf{a}\} = \{\lambda\mathbf{a} : \lambda \in \mathbb{R}\}$ . If  $\mathbf{a} \neq 0$ , then  $\text{span}\{\mathbf{a}\}$  is the entire line through  $O$  and  $A$ .

Define  $\mathbf{a} \parallel \mathbf{b}$  if and only if either  $\mathbf{a} = \lambda\mathbf{b}$  or  $\mathbf{b} = \lambda\mathbf{a}$ . Allow  $\lambda = 0$ , so  $\forall \mathbf{a}, \mathbf{0} \parallel \mathbf{a}$ . Also allow  $\lambda < 0$ .

**Definition 6.2** (vector addition). Give  $\mathbf{a}, \mathbf{b}$ , if  $\mathbf{a} \nparallel \mathbf{b}$ , construct a parallelogram  $OACB$  and define  $\mathbf{c} = \mathbf{a} + \mathbf{b}$ .

If  $\mathbf{a} \parallel \mathbf{b}$ , then  $\mathbf{a} = \alpha\mathbf{u}, \mathbf{b} = \beta\mathbf{u}$ , where  $\mathbf{u}$  is a unit vector and  $\mathbf{a} + \mathbf{b} = (\alpha + \beta)\mathbf{u}$ .

Given  $\mathbf{a}, \mathbf{b}, \dots, \mathbf{c}$ , we have a linear combination

$$\alpha\mathbf{a} + \beta\mathbf{b} + \dots + \gamma\mathbf{c}$$

for any  $\alpha, \beta, \dots, \gamma \in \mathbb{R}$ .

Define  $\text{span}\{\mathbf{a}, \mathbf{b}, \dots, \mathbf{c}\} = \{\alpha\mathbf{a} + \beta\mathbf{b} + \dots + \gamma\mathbf{c} : \alpha, \beta, \dots, \gamma \in \mathbb{R}\}$ . In 3d case, if  $\mathbf{a} \nparallel \mathbf{b}$ , then  $\text{span}\{\mathbf{a}, \mathbf{b}\}$  is a plane through  $O, A, B$ .

Here are some properties:

- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ , this says that  $\mathbf{0}$  is the identity for addition.
- $\exists -\mathbf{a}, \mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ . This says  $-\mathbf{a}$  is the inverse of  $\mathbf{a}$  under addition.
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ , this says that vector addition is commutative.
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c}, (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ , this says that vector addition is associative.

Hence, the set of vectors with addition form an abelian group.

Relation with scalars:

- $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$ .
- $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ .
- $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$ .

## 7 Dot product

**Definition 7.1** (dot product). Give  $\mathbf{a}, \mathbf{b}$ , let  $\theta$  be the angle between them, define  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ . Note that  $\theta$  is defined unless  $\mathbf{a} = \mathbf{0}$ , in which case we define  $\mathbf{a} \cdot \mathbf{b} = 0$ .  $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \theta = \frac{\pi}{2} \bmod \pi$  when  $\theta$  is defined. Allow  $\mathbf{a}$  or  $\mathbf{b} = \mathbf{0}$ , so  $\mathbf{a} \parallel \mathbf{0} \wedge \mathbf{a} \perp \mathbf{0}$ .

For  $\mathbf{a} \neq \mathbf{0}$ ,  $|\mathbf{b}| \cos \theta$  is the component of  $\mathbf{b}$  along  $\mathbf{a}$ .

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \mathbf{u} \cdot \mathbf{b}.$$

By resolving  $\mathbf{b}$  along and perpendicular to  $\mathbf{a}$ , we get

$$\mathbf{b} = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp}.$$

Properties:

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}, \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geq 0, = 0$  iff  $\mathbf{a} = \mathbf{0}$ .
- $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda\mathbf{b})$ .
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ .

## 8 Vector cross product

**Definition 8.1.** Given  $\mathbf{a}, \mathbf{b}$ , let  $\theta$  be the angle between them, wrt a unit vector  $\mathbf{n}$  normal to the plane they span. Define  $\mathbf{a} \wedge \mathbf{b}$  or  $\mathbf{a} \times \mathbf{b}$  as  $|\mathbf{a}||\mathbf{b}| \sin \theta \mathbf{n}$ .  $\mathbf{0}$  case is similar.

This is the *vector area* of the parallelogram generated by  $\mathbf{a}, \mathbf{b}$ . Note that  $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b}_{\perp}$ .

Properties:

- $\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$ .
- $(\lambda\mathbf{a}) \wedge \mathbf{b} = \lambda(\mathbf{a} \wedge \mathbf{b}) = \mathbf{a} \wedge (\lambda\mathbf{b})$ .
- $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$ .
- $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$  if and only if  $\mathbf{a} \parallel \mathbf{b}$ .
- $\mathbf{a} \wedge \mathbf{b} \perp \mathbf{a} \wedge \mathbf{b}_{\perp}$ .

## 9 Orthonormal Bases and Components

Choose  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  that are *orthonormal*. That is, they are of unit lengths and  $\mathbf{e}_i \cdot \mathbf{e}_j = 0, i \neq j \in \{1, 2, 3\}$ , which is equivalent to choose cartesian axes along the directions. Then  $\{\mathbf{e}_i\}$  is a basis and  $\forall \mathbf{a} \in \mathbb{R}^3$ ,

$$\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i.$$

By this spirite, we can write

$$\mathbf{a} = (a_1, a_2, a_3) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Scalar product in this form can be written as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3, \quad |\mathbf{a}| = a_1^2 + a_2^2 + a_3^2..$$

For vector products, choose this basis that it is also *right-handed*:

$$\mathbf{e}_i \times \mathbf{e}_{i+1} = \mathbf{e}_{i+2}.$$

Then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3)(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) \\ &= (a_3 b_2 - a_2 b_3) \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \mathbf{e}_2 + (a_1 b_2 - a_2 b_1) \mathbf{e}_3. \end{aligned}$$

## 10 Triple products

### 10.1 Scalar triple product

**Definition 10.1.** Define scalar triple product by

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

This is the volumn of the parallelepiped with bases  $\mathbf{b}, \mathbf{c}$  and side  $\mathbf{a}$ .

**Remark.**  $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$  is a "signed" volumn. If  $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) > 0$  then  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  is called a *right-handed set*.  $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$  if and only if  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are coplanar, e.g.,  $\mathbf{c} = \alpha \mathbf{a} + \beta \mathbf{b} \in \text{span}\{\mathbf{a}, \mathbf{b}\}$ .

In components,

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1 b_2 c_3 - a_1 b_3 c_2 \\ &+ a_2 b_3 c_1 - a_2 b_1 c_3 \\ &+ a_3 b_1 c_2 - a_3 b_2 c_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned}$$

### 10.2 Vector triple product

**Definition 10.2.** Define the vector triple product by  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ . Note that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  does not necessarily give the same result as  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ .

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}. \quad (10.1)$$

Lecture 4.

Each component  $a_i$  is uniquely determined by  $a_i = \mathbf{e}_i \cdot \mathbf{a}$ .



We have the following identities:

**Proposition 10.1.**

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{0} \\ (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d} \\ (\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{c} \times \mathbf{d}) \times (\mathbf{e} \times \mathbf{f})) &= [\mathbf{a}, \mathbf{b}, \mathbf{d}][\mathbf{c}, \mathbf{e}, \mathbf{f}] - [\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{d}, \mathbf{e}, \mathbf{f}] \\ (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= 0 \end{aligned}$$

## 11 Lines, Planes, and Vector equations

Vectors are defined as position vectors from  $O$ . But the definition of addition enables us to use them to describe displacements between points.

### 11.1 Lines

General point on a line through  $\mathbf{a}$  through  $\mathbf{u}$ :

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda \mathbf{u}, & \lambda \in \mathbb{R} & \text{The parametric form.} \\ \mathbf{u} \times \mathbf{r} &= \mathbf{u} \times \mathbf{a}, & & \text{Cross form.} \end{aligned}$$

**Proposition 11.1.** Any vector equation of the form  $\mathbf{u} \times \mathbf{r} = \mathbf{c}$  represents a line.

**Proof.**  $\mathbf{u} \times \mathbf{r} = \mathbf{c} \Rightarrow \mathbf{u} \cdot (\mathbf{u} \times \mathbf{r}) = \mathbf{u} \cdot \mathbf{c} \Leftrightarrow \mathbf{u} \cdot \mathbf{c} = 0$ . If  $\mathbf{u} \cdot \mathbf{c} \neq 0$  then the equation is inconsistent. If  $\mathbf{u} \cdot \mathbf{c} = 0$ , then note that

$$\mathbf{u} \times (\mathbf{u} \times \mathbf{c}) = (\mathbf{u} \cdot \mathbf{c})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{c} = -|\mathbf{u}|^2 \mathbf{c}.$$

Hence  $\mathbf{a} = -(\mathbf{u} \times \mathbf{c})/|\mathbf{u}|^2$  is a solution, and thus it represents a line.  $\square$

### 11.2 Planes

General point on a plane through  $\mathbf{a}$  with directions  $\mathbf{u}, \mathbf{v}$  in the plane ( $\mathbf{u} \nparallel \mathbf{v}$ ):

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda \mathbf{u} + \mu \mathbf{v}, & \lambda, \mu \in \mathbb{R} & \text{Parametric form,} \\ \mathbf{n} \cdot \mathbf{r} &= k = \mathbf{n} \cdot \mathbf{a}, & \mathbf{n} = \mathbf{u} \times \mathbf{v} & \text{Dot form.} \end{aligned}$$

The component of  $\mathbf{r}$  along  $\mathbf{n}$  is

$$\frac{\mathbf{n} \cdot \mathbf{r}}{|\mathbf{n}|} = \frac{k}{|\mathbf{n}|}.$$

### 11.3 Other vector equations

$$(1) |\mathbf{r}|^2 + \mathbf{r} \cdot \mathbf{a} = k \Leftrightarrow \left| \mathbf{r} + \frac{1}{2}\mathbf{a} \right|^2 = k + \frac{1}{4}|\mathbf{a}|^2, \text{ a sphere with centre } -\frac{1}{2}\mathbf{a} \text{ and radius } \sqrt{k + \frac{1}{4}|\mathbf{a}|^2}, \text{ provided } k > -\frac{1}{4}|\mathbf{a}|^2.$$

$$(2) \mathbf{r} + \mathbf{a} \times (\mathbf{b} \times \mathbf{r}) = \mathbf{c} \Leftrightarrow \mathbf{r} + (\mathbf{a} \cdot \mathbf{r})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{r} = \mathbf{c}. \text{ Dot with } \mathbf{a}:$$

$$\mathbf{a} \cdot \mathbf{r} = \mathbf{a} \cdot \mathbf{c} \implies (1 - \mathbf{a} \cdot \mathbf{b})\mathbf{r} = \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}.$$

If  $\mathbf{a} \cdot \mathbf{b} \neq 1$ , then there is a unique solution

$$\mathbf{r} = \frac{1}{1 - \mathbf{a} \cdot \mathbf{b}}(\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}),$$

which is a point.

If  $\mathbf{a} \cdot \mathbf{b} = 1$  and  $\text{RHS} \neq 0$ , then it is inconsistent.

If  $\mathbf{a} \cdot \mathbf{b}$  and  $\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \mathbf{0}$ , then

$$(\mathbf{a} \cdot \mathbf{r} - \mathbf{a} \cdot \mathbf{c})\mathbf{b} = \mathbf{0}.$$

Hence it is a plane.

## 12 Index notation and the summation convention

### 12.1 Components, $\delta$ & $\epsilon$

Write vectors  $\mathbf{a}, \mathbf{b}, \dots$  in terms of components  $a_i, b_i, \dots$  wrt an orthonormal right-handed basis  $\{\mathbf{e}_i\}$ . Indices  $i, j, \dots$  take values 1, 2, 3.

For example, if  $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ , then  $c_i = [\alpha\mathbf{a} + \beta\mathbf{b}]_i = \alpha a_i + \beta b_i$ , for  $i = 1, 2, 3$ .  $i$  is called a *free index*.

Hence

$$\bullet \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^3 a_i b_i.$$

$$\bullet \mathbf{x} = \mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\mathbf{d} \Leftrightarrow x_j = a_j + \left( \sum_{k=1}^3 b_k c_k \right) d_j.$$

**Definition 12.1** (Kronecker delta).

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We see that  $\delta_{ij} = \delta_{ji}$  and also

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using this,  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$

**Definition 12.2** (Levi-Civita epsilon).

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{else.} \end{cases}$$

We have  $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$ .  $\epsilon_{ijk}$  is totally anti-symmetric: exchanging any pair of indices produces a change in sign.

Then

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_k \epsilon_{ijk} \mathbf{e}_k$$

and

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \left( \sum_i a_i \mathbf{e}_i \right) \times \left( \sum_j b_j \mathbf{e}_j \right) \\ &= \sum_{ij} a_i b_j \mathbf{e}_i \times \mathbf{e}_j \\ &= \sum_{ij} a_i b_j \sum_k \epsilon_{ijk} \mathbf{e}_k = \sum_{ijk} a_i b_j \epsilon_{ijk} \mathbf{e}_k\end{aligned}$$

so

$$(\mathbf{a} \times \mathbf{b})_k = \sum_{ij} \epsilon_{ijk} a_i b_j.$$

## 12.2 Summation convention

With components and index notation, indices that appear twice in a given term are usually summed over. In the summation convention, we omit the sum signs for repeated indices. i.e., the sum is understood.

**Example.** (i) In  $a_i \delta_{ij} = a_1 \delta_{1j} + a_2 \delta_{2j} + a_3 \delta_{3j} = a_j$ , since  $\Sigma_i$  is understood.

(ii)  $\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j = a_i b_i$ .  $\Sigma_{ij}, \Sigma_i$  are understood.

(iii)  $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$ ,  $\Sigma_{jk}$  is understood.

(iv)  $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon_{ijk} a_i b_j c_k$ ,  $\Sigma_{ijk}$  is understood.

(v)  $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ .

(vi)

$$\begin{aligned}[(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}]_i &= (\mathbf{a} \cdot \mathbf{c})b_i - (\mathbf{a} \cdot \mathbf{b})c_i \\ &= a_j c_j b_i - a_j b_j c_i.\end{aligned}$$

$\Sigma_j$  is understood.

Here are the rules of summation convention.

- (1) An index occurring exactly once in any given term must appear once in every term in an equation, and it can take any value in 1, 2, 3, a *free* index.
- (2) An index occurring exactly twice in a given term is summed over. A *repeated*, *contracted*, or *dummy* index.
- (3) No index can occur more than twice in any given term.

## 12.3 Applications

We can use this to prove the vector triple product identity.

**Proof.** Write the huge sum in summation convention:

$$\begin{aligned}[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i &= \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ &= \epsilon_{ijk} a_j \epsilon_{kpq} b_p c_q \\ &= (\epsilon_{ijk} \epsilon_{kpq}) a_j b_p c_q.\end{aligned}$$

Notice that

$$\epsilon_{ijk} \epsilon_{kpq} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp} \quad (*)$$

see next subsection. So

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = \delta_{ip} \delta_{jq} a_j b_p c_q - \delta_{iq} \delta_{jp} a_j b_p c_q.$$

Notice also that  $a_i \delta_{ij} = a_j$ , so

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = a_q b_i c_q - a_j b_j c_i = (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i.$$

Hence the equation 10.1 is proved.  $\square$

## 12.4 $\epsilon \epsilon$ identity

**Proposition 12.1.**  $\epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp} = \epsilon_{kij} \epsilon_{kpq}$ .

**Proof.** Notice that LHS and RHS are both anti-symmetric, so both vanish when  $i, j$  or  $p, q$  take the same value. Inspection shows that<sup>1</sup> it suffices to show the cases  $i = p = 1 \wedge j = q = 2$  or  $i = q = 1, j = p = 2$  and all other index changings that give non-zero results.  $\square$

<sup>1</sup> Think carefully here.

**Proposition 12.2.**  $\epsilon_{ijk} \epsilon_{pjk} = 2\delta_{ip}$ .

**Proof.** Take  $q = j$  in the above equation:

$$\epsilon_{ijk} \epsilon_{pjk} = \delta_{ip} \delta_{jj} - \delta_{ij} \delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}.$$

$\square$

**Proposition 12.3.**  $\epsilon_{ijk} \epsilon_{ijk} = k$ .

**Proposition 12.4.**

$$\begin{aligned} \epsilon_{ijk} \epsilon_{pqr} &= \delta_{ip} \delta_{jq} \delta_{kr} - \delta_{jp} \delta_{iq} \delta_{kr} \\ &\quad + \delta_{jp} \delta_{kq} \delta_{ir} - \delta_{kp} \delta_{jq} \delta_{ir} \\ &\quad + \delta_{kp} \delta_{iq} \delta_{jr} - \delta_{ip} \delta_{kq} \delta_{jr}. \end{aligned}$$

**Proof.** Total anti-symmetry<sup>2</sup> in  $i, j, k$  and independently in  $p, q, r$  implies LHR, RHS agree up to an overall factor. To check the factor is 1, consider  $i = p = 1, j = q = 2, k = r = 3$ .  $\square$

<sup>2</sup> This simplifies most of the process and leaves only one case to check.

# Vectors in General

## 13 Vectors in $\mathbb{R}^n$

### 13.1 Definition and basic properties

Lecture 6

**Definition 13.1.** Regard vectors as sets of components, and let

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \mathbb{R}\}.$$

Define:

- Addition:  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$ ,
- Scalar multiplication:  $\lambda \mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$ .
- Linear combinations:  $\lambda \mathbf{x} + \mu \mathbf{y}$ ,
- Parallel:  $\mathbf{x} \parallel \mathbf{y} \Leftrightarrow \mathbf{x} = \lambda \mathbf{y} \vee \mathbf{y} = \lambda \mathbf{x}$ .
- Inner Product (Scalar product):  $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$ .

Properties of inner product:

(1). Symmetric:  $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ .

(2). Bilinear:

$$\begin{aligned} (\lambda \mathbf{x} + \lambda' \mathbf{x}') \cdot \mathbf{y} &= \lambda \mathbf{x} \cdot \mathbf{y} + \lambda' \mathbf{x}' \cdot \mathbf{y}, \\ \mathbf{x} \cdot (\mu \mathbf{y} + \mu' \mathbf{y}') &= \mu \mathbf{x} \cdot \mathbf{y} + \mu' \mathbf{x} \cdot \mathbf{y}'. \end{aligned}$$

(3). Positive definite:  $\mathbf{x} \cdot \mathbf{x} \geq 0$ , with  $=$  holds if and only if  $\mathbf{x} = \mathbf{0}$ .

## 13.2 Norm of a vector

**Definition 13.2.** The *norm* of a vector  $\mathbf{x}$  is denoted as  $|\mathbf{x}|$  with  $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$ .  $\mathbf{x}, \mathbf{y}$  are called *orthogonal* if  $\mathbf{x} \cdot \mathbf{y} = 0$ , denote as  $\mathbf{x} \perp \mathbf{y}$ .

The *standard basis* of  $\mathbb{R}^n$  is

$$\mathbf{e}_i = (0, \dots, 1, \dots, 0)$$

with 1 on the  $i$ th position. So that

$$\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$$

and  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ . i.e., standard basis is orthogonal.

## 13.3 Cauchy-Schwarz and Triangle inequalities

**Proposition 13.1** (Cauchy-Schwarz).

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \quad |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$$

with equality if and only if  $\mathbf{x} \parallel \mathbf{y}$ .

General deductions:

(i). Setting  $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos \theta$ , we can define angle  $\theta$  between  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ .

(ii). We have the *triangle inequality*:

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$

**Proof.** If  $\mathbf{y} = \mathbf{0}$ , then the result is immediate. If not, consider

$$\begin{aligned} |\mathbf{x} - \lambda\mathbf{y}|^2 &= (\mathbf{x} - \lambda\mathbf{y}) \cdot (\mathbf{x} - \lambda\mathbf{y}) \\ &= |\mathbf{x}|^2 - 2\lambda\mathbf{x} \cdot \mathbf{y} + \lambda^2|\mathbf{y}|^2 \geq 0. \end{aligned}$$

This is a real equation of  $\lambda$  with at most one root, so

$$(-2\mathbf{x} \cdot \mathbf{y})^2 - 4|\mathbf{x}|^2|\mathbf{y}|^2 \leq 0 \iff |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|.$$

Equality holds if and only if  $\mathbf{x} = \lambda\mathbf{y}$ .

Note also that for triangle inequality:

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2, \end{aligned}$$

as required. □

### 13.4 Inner Products and Cross products

Inner product in  $\mathbb{R}^n$  can be written as

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij}a_i b_j, \quad \text{by summation convention.}$$

For  $n = 3$ , it matches geometrical definition.

We can also define cross product in component definition. In 3d we have

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}a_j b_k,$$

and in  $n$  dimensions we have  $\epsilon_{ij\dots l}$  which is totally anti-symmetric. But there are only two  $a_i b_j$  so we cannot use this to define vector product in general.

However, in  $\mathbb{R}^2$  we have  $\epsilon_{ij}$  with  $\epsilon_{12} = -\epsilon_{21} = 1$ , so can use this to define a new scalar product

$$[\mathbf{a}, \mathbf{b}] = \epsilon_{ij}a_i b_j = a_1 b_2 - a_2 b_1.$$

Geometrically, this is the (signed) area of parallelogram formed by  $\mathbf{a}, \mathbf{b}$  and

$$|[\mathbf{a}, \mathbf{b}]| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$

Compare with  $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon_{ijk}a_i b_j c_k$ .

## 14 Vector Spaces

### 14.1 Axioms, span, and subspaces

**Definition 14.1.** Let  $V$  be a set of objects called *vectors* with operation

$$\begin{aligned} \mathbf{v} + \mathbf{w} &\in V \quad \forall \mathbf{v}, \mathbf{w} \in V \\ \lambda \mathbf{v} &\in V \quad \forall \mathbf{v} \in V, \lambda \in \mathbb{R}. \end{aligned}$$

Then  $V$  is called a *real vector space* if

- (i).  $V$  with  $+$  is an abelian group.

- (ii).  $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
- (iii).  $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$
- (iv).  $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$
- (v).  $1\mathbf{v} = \mathbf{v}$ .

**Example.** Let  $V = \{f : [0, 1] \rightarrow \mathbb{R} : f \wedge f(0) = f(1) = 0\}$ . By smooth we mean  $f$  is differentiable infinitely many times. Then  $V$  is a real vector space with  $+$  defined as  $(f + g)(x) = f(x) + g(x)$  and  $(\lambda f)(x) = \lambda(f(x))$ . Then all axioms apply.

**Definition 14.2.** A *subspace* of a real vector space  $V$  is a subset  $U \subseteq V$  that is also a vector space.

Lecture 7

**Remark.** A non-empty subset is a subspace if and only if  $\forall v, w \in U, \lambda v + \mu w \in U$ .

For any vectors  $v_1, \dots, v_r \in V$ , their *span*  $\text{span}\{v_1, \dots, v_r\} = \{\lambda_1 v_1 + \dots + \lambda_r v_r : v_i \in \mathbb{R}\}$  is a subspace.  $V$  and  $\{0\}$  are subspaces of  $V$ .

**Example.** A line or plane through  $O$  is a subspace in  $\mathbb{R}^3$ , but a line or plane that does not contain  $O$  is not a subspace.

## 14.2 Linear dependence and independence

For  $v_1, \dots, v_r \in V$ , a real vector space, consider a linear relation

$$\lambda_1 v_1 + \dots + \lambda_r v_r = 0. \quad (*)$$

If  $(*) \Rightarrow \lambda_i = 0$ , then the vectors form a *linearly independent set*. They obey only the trivial linear relation.

If  $(*)$  holds with at least  $\lambda_k \neq 0$ , then the vectors form a *linearly dependent set*. They obey a non-trivial linear relation.

**Example.** In  $\mathbb{R}^2$ ,  $\{(1, 0), (0, 1), (0, 2)\}$  is linearly dependent.

We cannot express  $(1, 0)$  in terms of the others.

Several facts:

- Any set containing  $0$  is linearly dependent.
- In  $\mathbb{R}^3$ ,  $\{\mathbf{a}\}$  is linearly independent if and only if  $\mathbf{a} \neq \mathbf{0}$ .
- $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  is linearly independent if  $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ . Since if

$$\alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c} = 0,$$

then dotting with  $\mathbf{b} \times \mathbf{c}$  we get  $\alpha[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0 \Rightarrow \alpha = 0$ . Similarly  $\beta = 0, \gamma = 0$ .

## 14.3 Inner products

This is an additional structure on a real vector space  $V$ , that can also be characterised by axioms or key properties.

For  $v, w \in V$ , denote inner product by

$$v \cdot w \text{ or } (v, w) \in \mathbb{R}.$$

Require this satisfies 1. it's symmetric, 2. it's bilinear, 3. it is positive definite.

Definition of length or norm and deductions such as Cauchy-Schwarz inequality depend just on these properties.

**Example.** Consider space of functions

$$V = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ smooth} \wedge f(0) = f(1) = 0\}.$$

Define an inner product by

$$(f, g) = \int_0^1 f(x)g(x) \, dx.$$

which has properties 123. Cauchy-Schwarz holds:

$$|(f, g)| \leq \|f\| \|g\|$$

with  $\|f\|^2 = (f, f)$ . i.e.

$$\left| \int_0^1 f(x)g(x) \, dx \right| \leq \left( \int_0^1 f(x) \, dx \right)^{1/2} \left( \int_0^1 g(x) \, dx \right)^{1/2}.$$

**Lemma 14.1.** In any real vector space  $V$  with an inner product, if  $v_1, v_2, \dots, v_r$  are non-zero and orthogonal vectors, then they are linearly independent.

**Proof.** If

$$\sum_i \alpha_i v_i = 0,$$

then

$$(v_j, \sum_i \alpha_i v_i) = 0 \iff \alpha_j = 0.$$

□

## 15 Bases and dimension

**Definition 15.1.** For a vector space  $V$ , a *basis* is a set

$$\mathfrak{B} = \{e_1, \dots, e_n\}$$

such that

(i)  $\mathfrak{B}$  spans  $V$ . i.e.,  $\forall v \in V$ ,

$$v = \sum_{i=1}^n v_i e_i.$$

(ii)  $\mathfrak{B}$  is linearly independent.

Given (ii), the coefficients  $v_i$  in (i) are unique, since

$$\sum_i v_i e_i = \sum_i v'_i e_i \iff v_i - v'_i = 0 \iff v_i = v'_i.$$



**Example.** Standard basis for  $\mathbb{R}^n$  consists of

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Many other bases can be chosen.

**Theorem 15.1.** If  $\{e_1, \dots, e_n\}, \{f_1, \dots, f_m\}$  are bases for a real vector space  $V$ , then  $m = n$ .

**Proof.** We have

$$f_a = \sum_i A_{ai} e_i,$$

$$e_i = \sum_a B_{ia} f_a$$

for  $A_{ai}, B_{ia} \in \mathbb{R}$ . Hence

$$f_a = \sum_i A_{ai} \sum_b B_{ib} f_b$$

$$= \sum_b \sum_i A_{ai} B_{ib} f_b$$

But the coefficients are unique, so

$$\sum_i A_{ai} B_{ib} = \delta_{ab}.$$

Similarly,

$$\sum_a B_{ia} A_{aj} = \delta_{ij}.$$

Now,

$$\sum_{i,a} A_{ai} B_{ia} = \sum_a \delta_{aa} = m = \sum_i \delta_{ii} = n.$$

□

**Definition 15.2.** The number of vectors in any basis is the *dimension* of the vector space.

**Remark.**  $\{0\}$  is called the *trivial* vector space and has dimension 0.

Lecture 8

The steps in the proof of basis theorem are within scope of this course, but the proof without prompts non-examinable. The same applies to the following:

**Proposition 15.1.** Let  $V$  be a vector space with finite subsets  $Y = \{w_1, \dots, w_m\}$  and  $X = \{u_1, \dots, u_k\}$ , with  $Y$  spans  $V$  and  $X$  linearly independent. Then

$$k \leq \dim V \leq m.$$

And

- (1) A basis can be found as a subset of  $Y$  by discarding vectors as necessary.
- (2)  $X$  can be extended to a basis by adding vectors from  $Y$  as necessary.

**Proof.** (1) If  $Y$  is linearly independent, then  $Y$  is a basis, and  $m = n = \dim V$ . If  $Y$  is not, then

$$\sum_{i=1}^m \lambda_i w_i = 0,$$

where  $\lambda_i$  are not all zero. wlog, can take  $\lambda_m \neq 0$ . Then

$$w_m = -\frac{1}{\lambda_m} \sum_{i=1}^{m-1} \lambda_i w_i,$$

so  $\text{span } Y = \text{span}(Y \setminus \{w_m\}) = \text{span } Y'$ . We repeat, until a basis is obtained.

(2) If  $X$  spans  $V$  then  $X$  is a basis  $k = n = \dim V$ . If not  $\exists u_{k+1}$  not in  $\text{span } X$ . Consider

$$\sum_{i=1}^{k+1} \mu_i u_i = 0$$

and thus  $\mu_i = 0, \forall i \in \{1, 2, \dots, k+1\}$ . Hence

$$X' = X \cup \{u_{k+1}\}$$

is linearly independent.

Furthermore, we can choose  $u_{k+1} \in Y$  since otherwise  $\text{span } X = V$ ,  $\#$ . Repeat this until a basis is achieved. The process stops since  $Y$  is finite.  $\square$

In this course, we will deal only with finite-dimensional spaces, except examples mentioned.

**Example.**  $V = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ smooth} \wedge f(0) = f(1) = 0\}$ . Note that

$$s_n(x) = \sqrt{2} \sin(n\pi x)$$

belong to  $V$  and

$$(s_n, s_m) = 2 \int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \delta_{nm},$$

so these functions are orthonormal and thus linearly independent. So  $V$  is infinite-dimensional.

## 16 Vectors in $\mathbb{C}^n$

### 16.1 Introduction and definitions

Let  $\mathbb{C}^n = \{\mathbf{z} = (z_1, \dots, z_n) : z_j \in \mathbb{C}\}$  and define

- Addition:  $\mathbf{z} + \mathbf{w} = (z_1 + w_1, \dots, z_n + w_n)$ ,
- Scalar multiplication:  $\lambda \mathbf{z} = (\lambda z_1, \dots, \lambda z_n)$ .

If scalars  $\lambda, \mu \in \mathbb{R}$ , then  $\mathbb{C}^n$  is a real vector space, and axioms apply.

If  $\lambda, \mu \in \mathbb{C}$ ,  $\mathbb{C}^n$  is a complex vector space. The same axioms hold, and definitions of linear combinations, linear dependence/independence, bases, dimension are generalised to  $\mathbb{C}$ .

The distinction between real and complex scalars is important.

**Example.**  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$  with  $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}$ . Then

$$\mathbf{z} = \sum_j x_j \mathbf{e}_j + \sum_j y_j \mathbf{f}_j,$$

a linear combination of  $\mathbf{e}$ , the usual standard basis in  $\mathbb{R}^n$ , and  $\mathbf{f}_j = (0, \dots, i, \dots, 0)$ .

We can see that  $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_n\}$  is a basis for  $\mathbb{C}^n$  as a *real* vector space, so it has dimension  $2n$ .

However,

$$\mathbf{z} = \sum_j z_j \mathbf{e}_j$$

is a *complex* linear combination, so the basis of  $\mathbb{C}^n$  as a *complex* vector space is simply  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  and the dimension is  $n$  over  $\mathbb{C}$ .

From now on we will view  $\mathbb{C}^n$  as a complex vector space unless mentioned otherwise.

## 16.2 Inner product

The inner product on  $\mathbb{C}^n$  is defined by

$$(\mathbf{z}, \mathbf{w}) = \sum_{j=1}^n \bar{z}_j w_j.$$

It has the following properties:

- (1) It is *hermitian*:  $(\mathbf{w}, \mathbf{z}) = \overline{(\mathbf{z}, \mathbf{w})}$ .
- (2) It is linear/anti-linear:  $(\mathbf{z}, \lambda \mathbf{w} + \lambda' \mathbf{w}') = \lambda(\mathbf{z}, \mathbf{w}) + \lambda'(\mathbf{z}, \mathbf{w}')$ . But  $(\lambda \mathbf{z} + \lambda' \mathbf{z}', \mathbf{w}) = \bar{\lambda}(\mathbf{z}, \mathbf{w}) + \bar{\lambda}'(\mathbf{z}', \mathbf{w})$ .
- (3) Positive definite:  $(\mathbf{z}, \mathbf{z}) \in \mathbb{R} \wedge \geq 0$ .  $= 0$  if and only if  $\mathbf{z} = \mathbf{0}$ .

Define the norm of  $\mathbf{z}$  to be

$$|\mathbf{z}| \geq 0, |\mathbf{z}|^2 = (\mathbf{z}, \mathbf{z}).$$

Also define  $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$  to be *orthogonal* if  $(\mathbf{z}, \mathbf{w}) = 0$ .

Note that the standard basis for  $\mathbb{C}^n$  is orthonormal. That is,

$$(\mathbf{e}_j, \mathbf{e}_k) = \delta_{jk}.$$

**Example.** Complex inner product of  $\mathbb{C}^1$  is

$$(z, w) = \bar{z}w.$$

Let  $z = a_1 + ia_2, w = b_1 + ib_2$ , and considers  $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ . Then

$$\bar{z}w = a_1b_1 + a_2b_2 + i(a_1b_2 - a_2b_1) = \mathbf{a} \cdot \mathbf{b} + i[\mathbf{a}, \mathbf{b}],$$

recover 2 scalar products in  $\mathbb{R}^2$ .