MIDTERM PRACTICE PROBLEMS SOLUTIONS.
CSC 262 Feb 2015

Q1

$E[X] = np = 30 \times \left(\frac{1}{365}\right) = 0.082$

a) $P(X=0) = p^n = 0.921$

$P(X=1) = np^{n-1}(1-p) = 0.0759$

$P(X=2) = \frac{n \cdot (n-1)}{2} p^{n-2}(1-p)^2 = .003$

b) $\lambda = 0.082$

$$\left.\begin{array}{l} P(X=0) = e^{-\lambda} = .921 \\ P(X=1) = \lambda e^{-\lambda} = .0757 \\ P(X=2) = \lambda^2/2 \, e^{-\lambda} = .003 \end{array}\right\} \text{Poisson.}$$

$\mu = .082$

$\sigma = \sqrt{30 \times \frac{1}{365} \cdot \frac{364}{365}} = 0.286$

$P(X=0) = P(-0.5 \leq X_{norm} \leq 0.5)$

$= P\left(\frac{-0.082-0.5}{0.286} \leq Z \leq \frac{-0.082+0.5}{0.286}\right)$

$= .907$

similarly

$P(X=1) = P(0.5 \leq X_{norm} \leq 1.5) = 0.072$

$P(X=2) = P(1.5 \leq X_{norm} \leq 2.5) = \text{ZERO}$

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Q2

a) $P(X \geq 1) = 1 - P(X=0) = 1 - (0.9)^N$

$X \sim bin(N, 0.1)$

To get $P(X \geq 1) = .95$ solve
$1 - 0.9^N \geq .95$

we find $1 - 0.9^{28} = 0.948$
and $1 - 0.9^{29} = 0.953$

so select $N = 29$.

[Note if $1 - .95 = 0.9^N$
then $\log(0.05) = N \log 0.9$].

b) sample without replacement.
X is binomial for any N.

Q3

~~4.~~ Outcomes: TTT TTH THT THH
HTT HTH HHT HHH.

a) 4/8
b) 1/8
c) 6/8
d) 4/8
e) 1/4

Q4

~~4~~ ~~6)~~. $\mu = 700$, $\sigma = 50$.

a) $Z_{.25} = -0.674$

$X_{.25} = \mu + \sigma Z_{.25} = 666.3$

b). $X_1 = N(750, 30^2)$

$P(X_1 \le 666.3) = P\left(Z \le \frac{666.3 - 750}{30}\right)$

$= 0.0026$

c) $X_{.95} = 750 + 30 * Z_{.95}$
$= 750 + 30 * 1.645$
$= 799.35$

~~[illegible]~~, feb 2014

Q5

~~1~~ The total number of hands (unordered) is $\binom{52}{5} = \frac{52\times\cdots\times 48}{5!} = 2,598,960$

a) To get 2 pairs we multiply.

$$\begin{aligned} &\binom{13}{2} && \{ \text{\# pair 'types'} \\ \times &\binom{4}{2} && \{ \text{\# ways to select} \\ \times &\binom{4}{2} && \{ \quad \text{2 from 4 ranks} \\ \times &\ 44 && \{ \text{\# ways to select remaining card} \end{aligned}$$

$$= \frac{13\times 12}{2!} \times \left(\frac{4\times 3}{2!}\right)^2 \times 44$$

$$= 123,552$$

so $P(\text{2 pair}) = \dfrac{123,552}{2,598,960} \approx 0.0475$

or, just under 1/20.

[illegible]

b) To get a Flush we multiply

$$4 \quad \{ \text{\# suits.}$$
$$\times \binom{13}{5} \quad \{ \text{\# ways to select 5 cards of the same suit.}$$

$$= 4 \times \frac{13 \times 12 \times 11 \times 10 \times 9}{5!} = 5,148.$$

So $P(\text{Flush}) = \dfrac{5,148}{2,598,960} \approx 0.00198$

or just under 1/500.

[Note that a flush, in an actual poker game might exclude a straight and royal flush, in which cards are also of consecutive rank. To make that adjustment, the probabilities of those hands would be calculated separately, and subtracted from the answer given above] .

Q6

a) A density satisfies $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$.

a) $$P(X \leq x) = \int_{-\infty}^{x} \phi(y)\,dy.$$

$$P(Z \leq z) = P\left(\frac{X-\alpha}{h} \leq z\right)$$

$$= P(X \leq hz + \alpha)$$

$$= \int_{-\infty}^{hz+\alpha} \phi(y)\,dy.$$

For $h > 0$, change of variable $\omega = \frac{y-\alpha}{h}$ gives

$$P(Z \leq z) = \int_{-\infty}^{z} \phi(h\omega + \alpha)\,h\,d\omega$$

$$= \int_{-\infty}^{z} \phi_Z(\omega)\,d\omega$$

so that $\phi_Z(\omega) = h\phi(h\omega+\alpha)$ is

the density of Z.

If $h < 0$ a similar argument gives

$$\phi_Z(\omega) = -h\,\phi(h\omega + \alpha)$$

so that the general solution is:

$$\phi_Z(\omega) = |h|\,\phi(h\omega + \alpha).$$

b) If $f_n(x) = \frac{1}{nh}\sum_{i=1}^{n} \phi\left(\frac{x-\alpha_i}{h}\right)$

then $f_n(x) \geq 0$ for all x, since ϕ is a density and $h > 0$.

Then $\int_{-\infty}^{\infty} \frac{1}{h}\phi\left(\frac{x-\alpha_i}{h}\right) dx =$

$= \int_{-\infty}^{\infty} \phi(u)\,du$ after change of variable $u = \frac{x-\alpha_i}{h}$

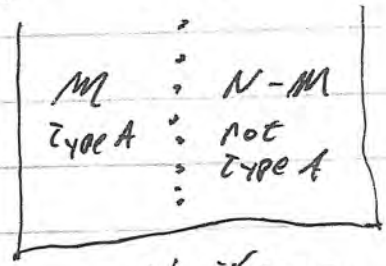
$= 1$

It follows that $f_n(x)$ is also a density.

Q7

~~Solutions, Sample Midterm 2, CSE 2?4, Feb 2014~~

1.



N items.

a). k items are selected.

- If $N-M < k$, at least $k-(N-M)$ Type A items must be selected, otherwise the minimum number of Type A items selected is 0.

- If $M < k$ at most M Type A items can be selected, otherwise k Type A items can be selected.

This can be summarized.

$$\max(0, k-(N-M)) \leq x \leq \min(M, k,$$

b) Assuming X is in the range given in part a), suppose all items are labelled uniquely (while keeping their Type A designations).

There are $\binom{N}{k}$ possible samples. Consider the event $\{X = x\}$. There are $\binom{m}{x}$ ways to select x Type A items and $\binom{N-m}{k-x}$ ways to select $k-x$ of the remaing items. This gives

$$P(X=x) = \frac{\binom{m}{x}\binom{N-m}{k-x}}{\binom{N}{k}}$$

That is, the Hypergeometric distribution. (see Example 2.35 from IPM-SR).

~~2~~5

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c) Then X is the sum of independent Bernoulli random variables, that is $X \sim \text{bin}(k, \frac{m}{N})$.

d). There are N fish in the lake, m have tags, k are caught,. X of those caught have tags.

so $\frac{X}{k} \approx \frac{m}{N}$. k, m are known,

$\therefore N \approx \frac{km}{X}$ is a good estimator.

The distribution can be obtained directly from the Hypergeometric distribution defined in part b). [or the binomial of part c) if the fish are released after capture].

~~3~~5

~~Ch See Assignt.~~ 3 .

~~7.31~~. $X \sim bin(n_1, p)$

Q8 $\quad Y \sim bin(n_2, p)$.

$X+Y$ is the sum of n_1+n_2 iid Bernoulli RV's, that is,

$$X+Y \sim bin(n_1+n_2, p).$$

a) $P(X=s \mid X+Y=t)$, s,t integers.

$$= \frac{P(X=s \text{ AND } X+Y=t)}{P(X+Y=t)}$$

$$= \frac{P(X=s)\,P(Y=t-s)}{P(X+Y=t)}$$

$$= \frac{\binom{n_1}{s} p^s (1-p)^{n_1-s} \binom{n_2}{t-s} p^{t-s} (1-p)^{n_2-(t-s)}}{\binom{n_1+n_2}{t} p^t (1-p)^{n_1+n_2-t}}$$

$$= \frac{\binom{n_1}{s}\binom{n_2}{t-s}}{\binom{n_1+n_2}{t}} \quad \text{for } 0 \le s \le n_1,\ 0 \le t-s \le n_2$$

[Hypergeometric] .

~~4/5~~

b) $P(X+Y=t \mid X=s)$

$$= \frac{P(X=s \text{ And } X+Y=t)}{P(X=s)}$$

$$= \frac{P(X=s)\,P(Y=t-s)}{P(X=s)}$$

$$= P(Y=t-s) = \binom{n_2}{t-s} p^{t-s}(1-p)^{n_2-(t-s)}$$

assume s,t are integers with $0 \le s \le n_1$, $0 \le t-s \le n_2$.

~~4/.2 See Section 5.2.3 in IPM-SR~~

49. ~~Pm~~

Theorem 7.2 of Lecture Notes

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