Practice Problems 1 - Chapters 2-3 - CSC/DSC 262/462

Q1. A coin is tossed twice, independently. Define the three events:

$$\begin{aligned} A_1 &= \{ \text{ first toss is Heads } \} \\ A_2 &= \{ \text{ second toss is Heads } \} \\ A_3 &= \{ \text{ two outcomes are the same } \}. \end{aligned}$$

Prove that these events are pairwise independent but not independent.

Q2. Three 6-sided dice are tossed independently. Label the dice red, green and blue. Suppose we define the following events:

$$\begin{aligned} A_1 &= \{\text{red dice} = \text{green dice}\} \\ A_2 &= \{\text{red dice} = \text{blue dice}\} \\ A_3 &= \{\text{green dice} = \text{blue dice}\}. \end{aligned}$$

Calculate the probabilities:

$$\begin{aligned} P(A_i), &\quad i = 1, 2, 3; \\ P(A_i \cap A_j), &\quad i \neq j; \\ P(A_1 \cap A_2 \cap A_3). & \end{aligned}$$

Are the events A_1, A_2, A_3 independent? Are they pairwise independent?

Q3. The *Monty Hall problem* is a good example of the often counterintuitive nature of probability. It is based on the television game show *Let's Make a Deal* (starring Monty Hall). There are three doors. Behind one is a car, and behind the other two are goats. The contestant picks one door. Then, one the other doors is opened, revealing a goat (this can always be done, since there are two goats). The contestant is offered the choice of staying with the original choice, or switching to the one remaining door. The contestant wins whatever is behind the selected door. Assume the contestant makes the first choice at random, and has decided in advance whether or not to switch. Determine the probability of winning the car if the contest doesn't switch, and if the contestant does switch.

Q4. A dice game is played in the following way. A player continues to toss a dice as long as the current outcome is strictly higher than the previous outcome. The score is the number of such outcomes. For example, for the sequence 1,3,5,2 the player stops at the fourth toss, and scores $X = 3$. What is the probability that the player scores at least $X = 3$?

Q5. The hour hand on a 12-point clock is positioned at 12. The hand moves backwards or forwards one position with equal probability N times. All moves are independent. Determine the probability that the hand rests at 3 if:

(a) $N = 9$,

(b) $N = 10$,

(c) $N = 19$.

Q6. Suppose any integer from 1 to 75 inclusive is chosen at random with equal probability, which will be denoted N. What is the probability that N is divisible by at least one of 5, 7 or 11?

Q7. A *random walk* can be described as follows. We have time points $i = 0, 1, 2, \ldots$. The random walk has value X_i at time point i, according to the following rules:

(1) $X_0 = 0$.

(2) At time point i, +1 or -1 is added to X_i with equal probability, resulting in $X_{i+1} = X_i - 1$ or $X_{i+1} = X_i + 1$. All increments are selected independently.

For example, we could have $X_0 = 0$, $X_1 = 1$, $X_2 = 0$, $X_3 = -1$, $X_4 = -2$, $X_5 = -1$ and so on.

Determine the following probabilities:

(a) $P(X_1 = 1,\ X_2 = 0,\ X_3 = -1,\ X_4 = 0)$,

(b) $P(X_1 = -1,\ X_2 = -2,\ X_3 = -3,\ X_4 = -2)$,

(c) $P(X_4 = 0)$,

(d) $P(X_i > 0 \text{ for } i = 1, 2, 3, 4)$.

Q8. A game is played in the following way. First a 6-sided dice is tossed. Suppose the dice shows N. Then a coin is tossed N times. The player wins if the coin shows the same face for each of the N tosses. What is the probability that the player wins? Use the law of total probability.

Q9. This question is adapted from *Introduction to Probability Models* (10th Edition), S.M. Ross. Three prisoners, labeled A, B and C, are informed by a guard that one of them has been chosen at random to be executed the following day. Prisoner A asks the guard, privately, to name one of the other prisoners who will be released. We then have the competing claims:

Claim 1: The guard argues that by eliminating one prisoner from the execution pool the probability that A is executed changes from 1/3 to 1/2.

Claim 2: Prisoner A argues that since it is already known that at least one of prisoners B or C will be released, the probability that A is executed remains 1/3.

Assume that if the guard names a prisoner to be released, and both B and C are to be released, the guard will name either one with equal probability. Otherwise, the guard names the only prisoner other than A being released. Define the following events.

$$\begin{aligned}
E_A &= \{\text{Prisoner } A \text{ chosen for execution}\} \\
E_B &= \{\text{Prisoner } B \text{ chosen for execution}\} \\
E_C &= \{\text{Prisoner } C \text{ chosen for execution}\} \\
F_B &= \{\text{Guard informs prisoner } A \text{ that prisoner } B \text{ is being released}\} \\
F_C &= \{\text{Guard informs prisoner } A \text{ that prisoner } C \text{ is being released}\}.
\end{aligned}$$

So, the event that B is to be released is equivalent to E_B^c, and the event that A is informed that B is to be released is F_B. Calculate the following probabilities:

(a) $P(E_B^c)$,

(b) $P(F_B)$,

(c) $P(E_A \mid E_B^c)$,

(d) $P(E_A \mid F_B)$.

Who is correct, the guard or prisoner A?

Q10. A bin contains 5 white and 5 black balls. A random selection of 2 balls is made. Let X be the number of white balls among the 2 selected. Determine $P(X = k)$ for $k = 0, 1, 2$.

Q11. A container contains 2 balls each of n colors (a total of $2n$ balls). The two balls of the same color are considered identical. Derive an expression for

$$\alpha_n = P(\text{All colors are adjacent in a random permutation of all balls}).$$

Q12. The letters in MISSISSIPPI are randomly permuted. What is the probability that there are no consecutive S's? What is the probability that the S's are consecutive (for example, IPSSSSIIMPI)?

Q13. Someone proposes playing a dice game, and kindly offers to provide the dice. You suspect that the dice may be *loaded*, that is, at least one outcome has a probably other than $1/6$. Suppose E is an event involving a die with the following special property. Let $P_f(E)$ be the probability of the event for a *fair* die (each outcome has probability $1/6$). Let $P_{uf}(E)$ be the probability of the event for some other die. If this special property holds, then if that die is loaded in any way, we have $P_{uf}(E) > P_f(E)$. Note that E can involve more than one toss *of the same* die. Can you think of an event with this property? If so, you can propose a bet that favors you if the dice is loaded, and is fair otherwise, without having to know how the dice is loaded.

Q14. A bin contains n balls labeled $1, \dots, n$. The balls are selected in order, at random. We say the ball labeled k was *selected correctly* if it is in position k of the selection order. For example, if $n = 5$, and the selection order was $5, 2, 1, 4, 3$ then balls 2 and 4 were selected correctly.

(a) Give an expression in terms of n for the probability that a specific ball was selected correctly.

(b) Suppose B is a specific subset of $m \leq n$ balls Give an expression in terms of n and m for the probability that all balls in B were selected correctly.

(c) Use the inclusion-exclusion principle to derive a formula for the probability that *no* ball is selected correctly (such a permutation is known as a *derangement*). Write an `R` program to calculate the probability of a derangement for $n = 1, 2, \dots, 25$. Comment briefly on the resulting sequence.

Q15. In some little known kingdom convicted prisoners are offered the possibility of a pardon according to a game of chance. The prisoner is given n red balls and n green balls. He/she then places the balls into two bins in any manner he/she chooses. The king then (i) selects a bin at random; (ii) selects one ball at random from that bin. If that ball is green the prisoner is pardoned. Note that the prisoner can leave one bin empty, and if that bin is selected by the king, then no green ball can be chosen, so no pardon is granted.

How should the balls be allocated to the bins in order to maximize the probability $P(G)$ of selecting green (and therefore winning a pardon)?

HINT: There are several ways to solve this. One way is to solve the following sub-problems, from which the optimal choice can be deduced:

P1 In a *simple allocation* all balls are in one bin. What is $P(G)$ for this case?

P2 In an *even allocation* each bin has the same number of balls, independent of their color. Show that $P(G)$ is the same for all even allocations, and derive this number.

P3 In an *uneven allocation* one bin (the *light bin*) has strictly fewer balls than the other (the *heavy bin*), but both have at least one. Show that for any uneven allocation the heavy bin has at least one green ball. Next, show that if the light bin has at least one red ball, and if a red ball from the light bin is exchanged with a green ball from the heavy bin, then $P(G)$ will strictly increase.

P4 Show that for an uneven allocation in which the light bin has no red balls and at least two green balls, if a green ball is moved from the light bin to the heavy bin, then $P(G)$ will strictly increase.

It also helps to assume $n > 1$. The case $n = 1$ can then be consider separately.

Q16. A standard 52 card playing deck assigns a unique combination of 13 ranks in the sequence (2,3,4,5,6,7,8,9,10,J,Q,K,A) and 4 suits (Clubs, Diamonds, Hearts, Spades) to each card ($13 \times 4 = 52$). Suppose a hand of 5 cards is selected at random. Using the *rule of product* calculate the probability that the cards form each of the hands listed below. Carefully list the *tasks* used in the application of the *rule of product*.

(a) **One Pair.** Exactly two cards of one rank, the remaining cards of distinct rank.

(b) **Two Pairs.** Two distinct ranks each represented by exactly two cards, the remaining card of distinct rank.

(c) **Three of a Kind.** Exactly three cards of one rank, the remaining cards of distinct rank.

(d) **Straight.** All ranks in consecutive sequence, but cards not all of the same suit.

(e) **Flush.** All cards of the same suit, but ranks not in consecutive sequence.

(f) **Full House.** Two cards of one rank, three cards of a different rank.

(g) **Straight Flush.** Ranks in consecutive sequence, all of the same suit, but not a royal flush.

(h) **Royal Flush.** Rank (10,J,Q,K,A) all of the same suit.

NOTE: There are many different rules for poker. Here, we will assume that in a consecutive sequence of ranks 'A' may preceed '2', that is (A,2,3,4,5) is a consecutive sequence. Other consecutive sequences are (3,4,5,6,7), (8,9,10,J,Q), (10,J,Q,K,A) and so on. However, (Q,K,A,2,3) is not a consecutive sequence.

Q17. In genetics, a genotype consists of two genes, each of which is one of (possibly) several types of alleles. When two organisms mate, each passes one allele, selected at random, to the offspring, forming that offspring's genotype. Suppose a gene of a type of flower has two alleles, `r` and `R`. A plant possessing genotype `rr`, `rR` or `RR` has white, pink or red petals, respectively. A trait like this, in which both alleles determine the trait, is called *codominant*.

(a) Suppose a white and pink flower produce offspring A. Give the color distribution of A (that is, the probability that A is a given color, for each color).

(b) Suppose that A mates with a pink flower to produce offspring B. Give the color distribution of B.

(c) Suppose that C is the offspring of two pink flowers, and that A mates with C to produce offspring D. Give the color distribution of D.