# ECE HW#4 Kefu Zhu

## Question 7

### (A)

**Is the process XN a MC?**

The process $X_n$ is a Markov chain because we can write $\begin{cases} X_{n+1} = 0, \ X_n = 0 \\ X_{n+1} = \sum_{i=1}^{X_n} D_i, \ X_n > 0 \end{cases}$

which indicate that the number of women in the $n+1$ generation only depends on the number of women in the $n$ generation ⇔ memory less property of markov chain.

**What are the transition probabilities $P(X_{n+1} = j|X_n = 0)$ and $P(X_{n+1} = j|X_n = 1)$?**

Because once the total number of women becomes zero at some point in time, the number will stay as zero and won't increase in the future. We have

$$P_{0j} = P(X_{n+1} = j|X_n = 0) = \begin{cases} 0, \ j \neq 0 \\ 1, \ j = 0 \end{cases}$$

As stated before, because $X_n$ is a Markov chain, we can write

$$P_{1j} = P(X_{n+1} = j|X_n = 1) = P(\sum_{i=1}^{X_n} D_i = j|X_n = 1) = P(D_i = j) = p_j$$

**What are the transition probabilities into state $Xn = 0$?**

The probability of the number of women becomes zero at some point given there are still $i$ women in the last generation is equivalent to the probability that every woman has no daughter, which can be represented as

$$P(X_{n+1} = 0|X_n = i) = \prod_{k=1}^{i} P(D_k = 0) = p_0^i$$

**Is the probability $P(X_{n+1} = i|X_n = i)$ of a state transitioning into itself strictly positive? Is this MC recurrent?**

As stated before, $P(X_{n+1} = i|X_n = i) = 1$, for $i = 0$

For $i > 0$, we know that one way of having the same number of women for two generations in a row is that every woman has exactly one daughter, with probability $p_1^i$. Therefore, we have

$P(X_{n+1} = i|X_n = i) \geq p_1^i > 0$, for $i > 0$

Since $P(X_{n+1} = 0|X_n = 0) = 1$, the state $0$ is recurrent. Because $\forall i \neq 0, p_0^i > 0$, all states that is not state $0$ are transient (Because there is a probability that state $i$ goest to state $0$ and then stays there forever → never come back to state $i$).

The MC is not recurrent.

## (B)

**Is $X_{rN}$ not a MC?**

For the special case, state $0$, which means that there are no women of type $r$ in the current generation, can happens because of two scenarios

1. Type $r$ has not been created so far → $P(X_{rm} > 0|X_{rn} = 0) = 0$
2. Type $r$ existed in the past but is now extinct → $P(X_{rm}) > 0$

Since the transition probabilities to other states starting from state $0$ depend on the past scenarios, this process must not be a Markov chain

**Given $X_{rn} > 0$, is the process $X_{r,n:\infty} = X_{rn}, X_{r,n+1}, \ldots$ a MC?**

Because we are conditioning on $X_{rn} > 0$, we eliminate the other scenario for the state $0$. The process now is a Markov chain. Now we have

$$P_{0j} = \begin{cases} 1, \; j = 0 \\ 0, \; j \neq 0 \end{cases}$$

$P_{1j} = (1 - q)p_j$, which indicate the probability of having exactly $j$ daughters without the mutation

**What is the value for $P_{i0}$?**

Consider one scenario of $P_{i0}$,

$P_{10} = p_0 + (1 - p_0)q$ because the number of women who has the mutation becomes extinct if one of the following two mutually exclusive scenarios happens:

1. woman has no daughters
2. woman has daughters but her daughters has type $r$

Because of independence, $P_{i0} = P_{10}^i = (p_0 + (1 - p_0)q)^i, \; i \geq 0$

**Is $P_{ii} > 0$? Is this MC recurrent**

Same logic as stated in part A, only state $0$ is recurrent. All other states are transient.

$$P_{ii} \geq ((1-q)p_i)^i > 0, \ i \geq 0$$

The MC is not recurrent

## (C)

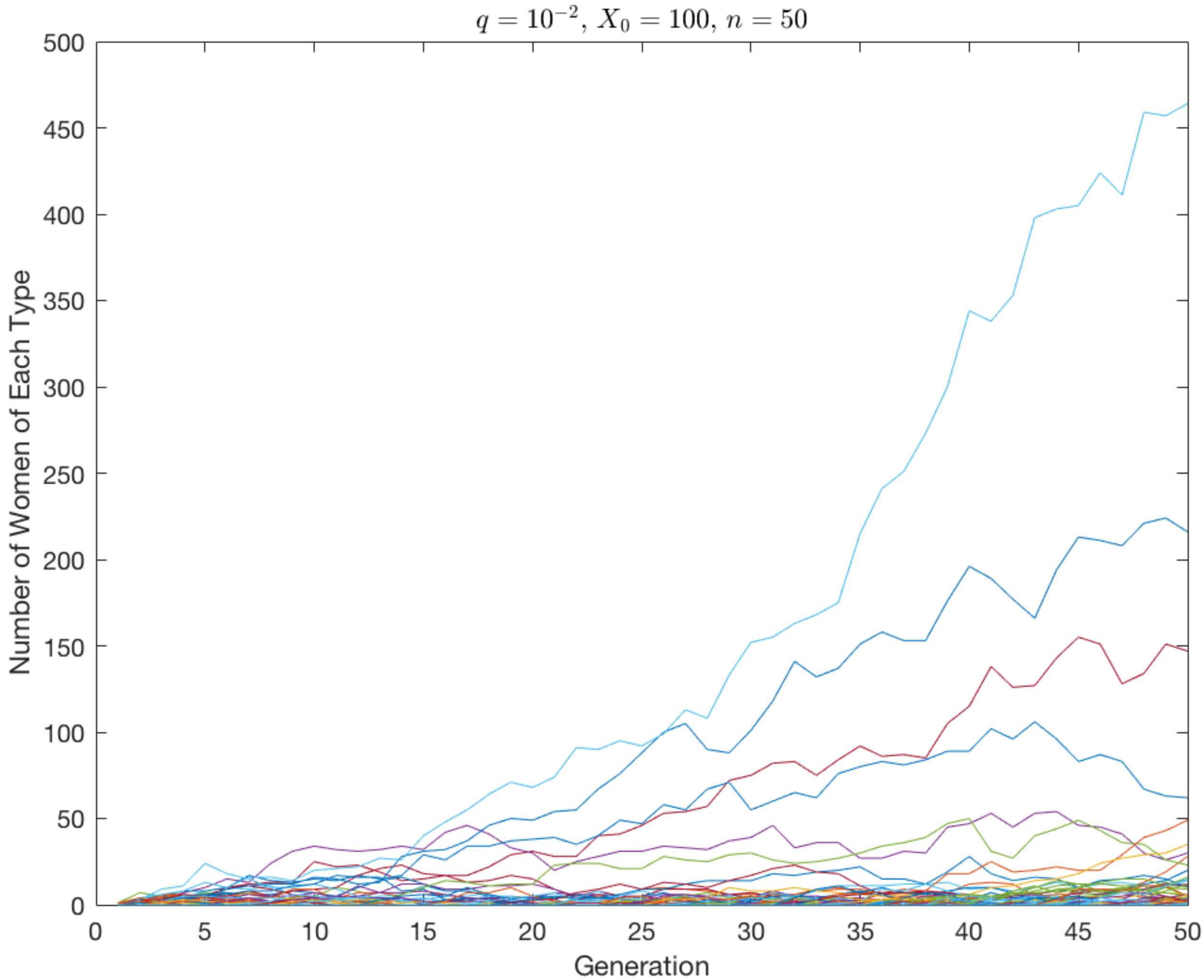
```
% Set simulation parameter
X_o = 100;
max_t = 50;
max_types = 1000;

% Set stochastic process parameters
mu = 1.05;
q = 10^-2;
% Initialize empty matrix with all zeros for storing population size
X=zeros(max_types, max_t);
% Initialize empty matrix with all zeros for storing number of types
number_of_types=zeros(1, max_t);
% Initialize population (1 people)
X(1:X_o,1) = 1;
% Initialize first generation
number_of_types(1)=X_o;
number_of_extinct_types=zeros(1,max_t);

% Start simulation
for n=2:max_t
    disp('n = '+string(n))
    number_of_types(n)=number_of_types(n-1);
    for type = 1:number_of_types(n-1);
        for i = 1:X(type,n-1)
            % Daughter/Not Daughter
            daughters = poissrnd(mu,1,1);
            % Mutation/No mutation
            mutation = binornd(1,q,1,1);

            % If there is a mutation
            if mutation
                number_of_types(n) = number_of_types(n)+1;
                X(number_of_types(n),n) = daughters;
            % Otherwise
```

```
            else
                X(type,n) = X(type,n) + daughters;
            end
        end

        % Add the extinct type
        if X(type,n)== 0
            number_of_extinct_types(n)=number_of_extinct_types(n)+1;
        end
    end
end
```

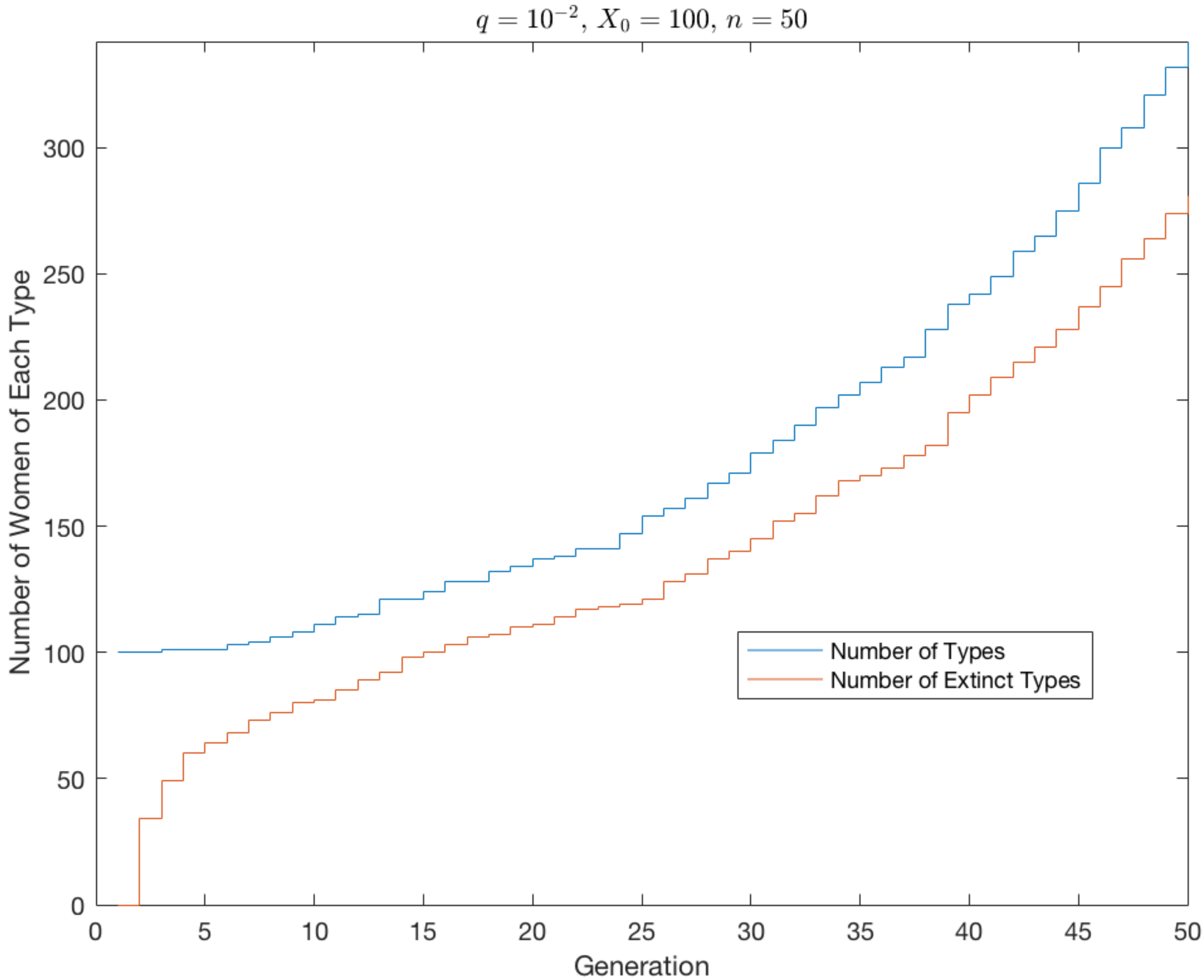
```
# The total number of mitochondrial DNA types by generation n
number_of_types

# The total number of extinct types by generation n
number_of_extinct
```

**(D)**

```
figure()
plot(1:max_t, X)
xlabel('Generation')
ylabel('Number of Women of Each Type')
title('$q=10^{-2},$ $X_{0}=100,$ $n=50$','Interpreter','latex')

figure()
stairs(1:max_t, [number_of_types;number_of_extinct_types]')
xlabel('Generation')
ylabel('Number of Women of Each Type')
title('$q=10^{-2},$ $X_{0}=100,$ $n=50$','Interpreter','latex')
axis([0,50,0,number_of_types(end)])
legend('Number of Types','Number of Extinct Types','Location','Best')

figure()
bar(1:number_of_types(end), X(1:number_of_types(end),max_t),'r')
xlabel('Types')
ylabel('Number of Women of Each Type')
title('Histogram of the Number of Women of Each Type, $n=50$','Interpreter','latex')
```



$q = 10^{-2}$, $X_0 = 100$, $n = 50$
Number of Women of Each Type
Generation
0
5
10
15
20
25
30
35
40
45
50
0
50
100
150
200
250
300
350
400
450
500

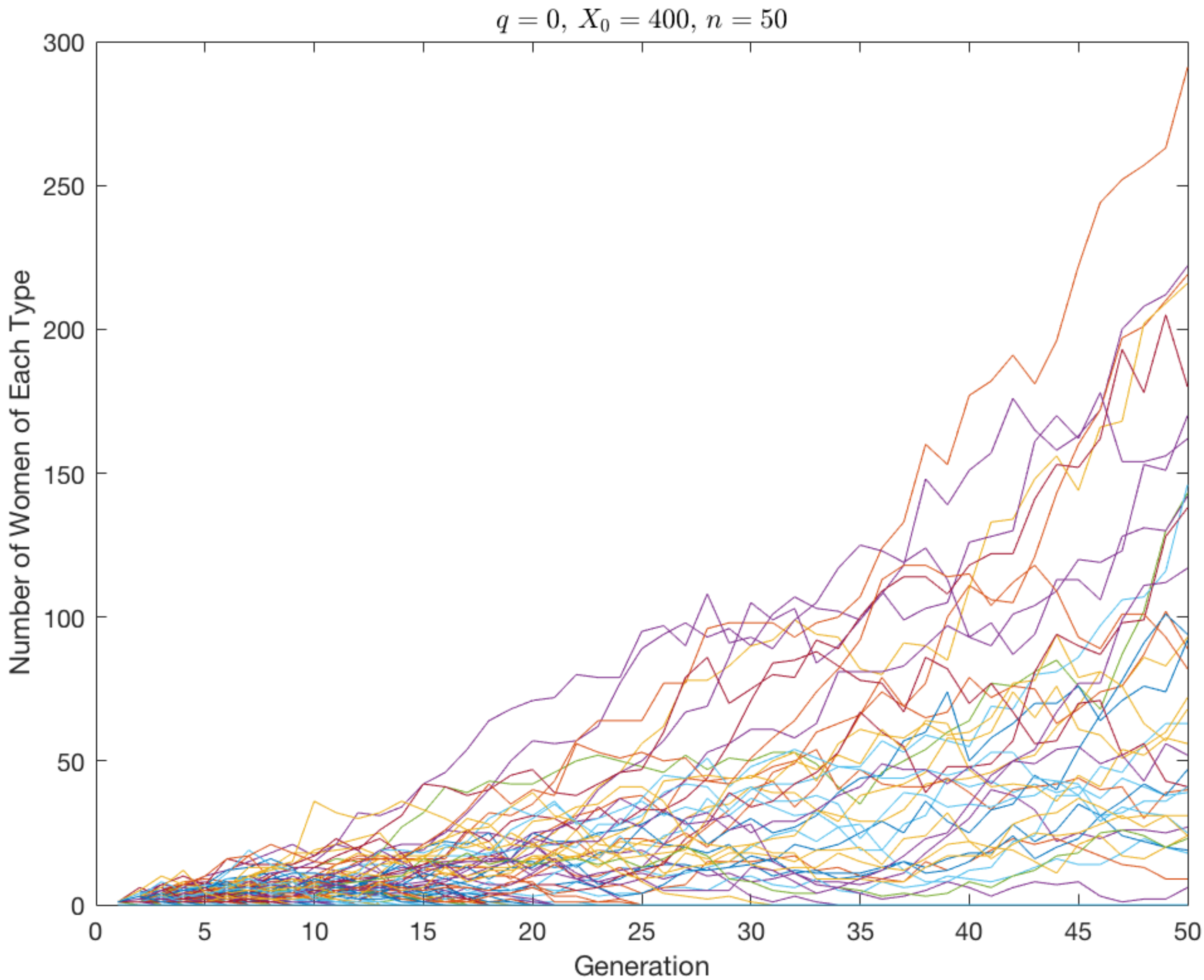


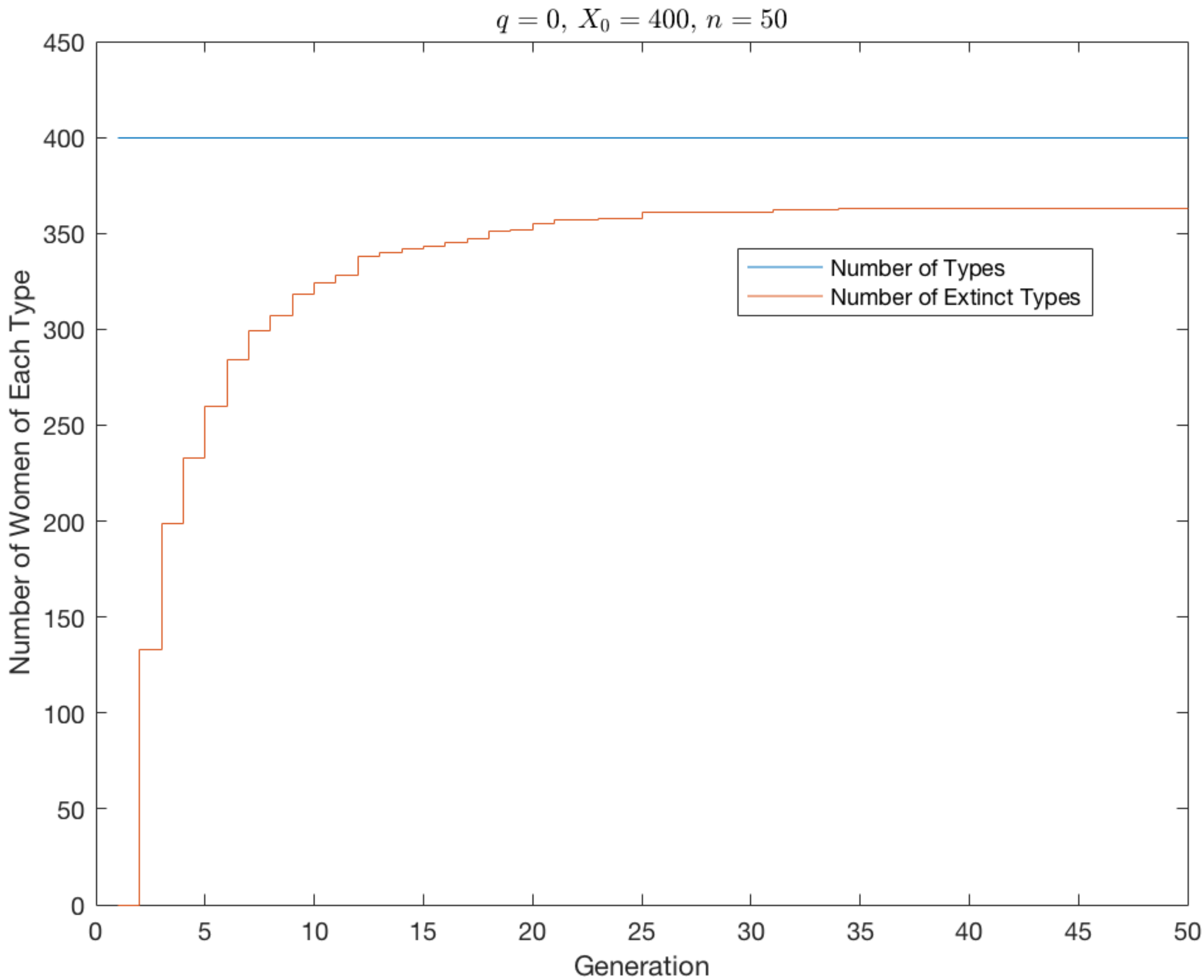
$q = 10^{-2}$, $X_0 = 100$, $n = 50$
Number of Women of Each Type
Generation
Number of Types
Number of Extinct Types
0
50
100
150
200
250
300
0
5
10
15
20
25
30
35
40
45
50


Histogram of the Number of Women of Each Type, $n = 50$
Number of Women of Each Type
Types
500
450
400
350
300
250
200
150
100
50
0
0
50
100
150
200
250
300

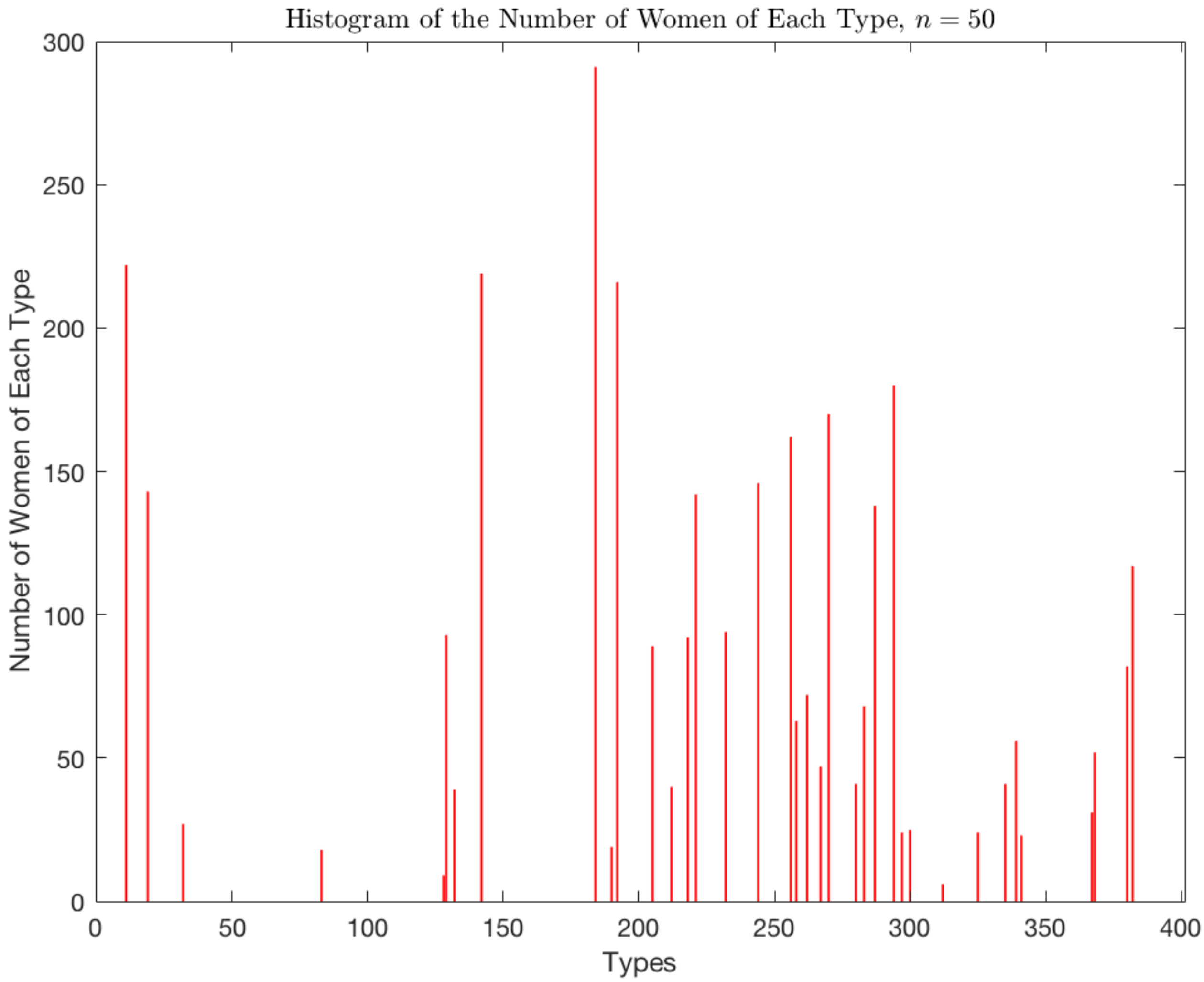
**(E)**

```
% Problem E
figure()
plot(1:max_t, X)
xlabel('Generation')
ylabel('Number of Women of Each Type')
title('$q=0,$ $X_{0}=400,$ $n=50$','Interpreter','latex')

figure()
stairs(1:max_t, [number_of_types;number_of_extinct_types]')
xlabel('Generation')
ylabel('Number of Women of Each Type')
title('$q=0,$ $X_{0}=400,$ $n=50$','Interpreter','latex')
axis([0,50,0,number_of_types(end)+50])
legend('Number of Types','Number of Extinct Types','Location','Best')

figure()
bar(1:number_of_types(end), X(1:number_of_types(end),max_t),'r')
xlabel('Types')
ylabel('Number of Women of Each Type')
title('Histogram of the Number of Women of Each Type, $n=50$','Interpreter','latex')
```


$q = 0,\ X_0 = 400,\ n = 50$
Number of Women of Each Type
Generation
0
50
100
150
200
250
300
0
5
10
15
20
25
30
35
40
45
50



$q = 0,\ X_0 = 400,\ n = 50$
Number of Women of Each Type
Generation
Number of Types
Number of Extinct Types
0
50
100
150
200
250
300
350
400
450
0
5
10
15
20
25
30
35
40
45
50




## (F)

$\because E(\sum_{i=1}^{X_{n-1}} D_i | X_{n-1} = k) = E(\sum_{i=1}^{k} D_i | X_{n-1} = k) = E(\sum_{i=1}^{k} D_i)$

As defined in the question, $E(D_i) = v$

$\therefore E(\sum_{i=1}^{X_{n-1}} D_i | X_{n-1} = k) = kv$

By iterated expectations

$E(X_n) = \sum_{k=0}^{\infty} E(X_n | X_{n-1} = k) P(X_{n-1} = k) = \sum_{k=0}^{\infty} kv \cdot P(X_{n-1} = k) = v \cdot E(X_{n-1})$

By iterations, $E(X_n) = v \cdot E(X_{n-1}) = v^2 \cdot E(X_{n-2}) = \ldots = v^n \cdot E(X_0) = v^n \cdot X_0$

For $v > 1$ ($v = \lambda$ in the part D and E), we expect to see an exponential increase in the expected value, which is similar to what we see in the simulation graph

Consider $X_0 = X_{r0}$, we then have $E(X_{rn}) = v_r^n \cdot E(X_{r0}) = v_r^n = (1-q)^n v^n$

## (G)

Based on Markov's inequality, $P(|X \geq a| \leq \frac{E(X)}{a})$, we have $P(|X_{rn} \geq a| \leq \frac{v_r^n}{a})$ for some value of $a$

If $v_r < 1$, then $lim_{n\to\infty} P(X_{rn} \geq a) \leq lim_{n\to\infty} \frac{v_r^n}{a} = 0$

$\Rightarrow lim_{n\to\infty} P(X_{rn} < a) = 1 \Leftrightarrow lim_{n\to\infty} P(X_{rn} = 0) = 1$

## (H)

$\because P_e(j) = \begin{cases} 1, \ v < 1 \ (Proved\ in\ part\ G) \\ < 1, \ v > 1 \ (Type\ r\ could\ extinct) \end{cases}$

$\therefore$ By law of total probability, consider conditioning on $X_{r1} = j$

$P(X_{r\infty} = 0|X_{r1} = j, X_{r0} = 1) = P(X_{r\infty} = 0|X_{r1} = j) = P(X_{r\infty} = 0|X_{r0} = 1)^j = (P_e(1))^j$

$\because P(X_{r\infty} = 0|X_{r1} = j) = (P_e(1))^j$

$\therefore$ Again, by law of total probability,
$P_e(1) = \sum_{j=1}^{\infty} P(X_{r\infty} = 0|X_{r1} = j, X_{r0} = 1)P(X_{r1} = j|X_{r0} = 1) = \sum_{j=1}^{\infty} p_j \cdot (P_e(1))^j$

If in general, the starting generation have $j$ individuals ($X_{r0} = j$), then by independence, we have $P_e(j) = (P_e(1))^j$