

# MIDTERM PRACTICE PROBLEMS SOLUTIONS.

CSC 262

Feb 2015

Q1

~~Q1~~.  $E[X] = np = 30 \times \left(\frac{1}{365}\right) = 0.082$

a)  $P(X=0) = p^n = 0.921$

$P(X=1) = np^{n-1}(1-p) = 0.0759$

$P(X=2) = \frac{n \cdot (n-1)}{2} p^{n-2} (1-p)^2 = .003$

b)  $\lambda = 0.082$

$$\left. \begin{aligned} P(X=0) &= e^{-\lambda} = .921 \\ P(X=1) &= \lambda e^{-\lambda} = .0759 \\ P(X=2) &= \frac{\lambda^2}{2} e^{-\lambda} = .003 \end{aligned} \right\} \text{Poisson.}$$

$\mu = .082$

$\sigma = \sqrt{30 \times \frac{1}{365} \times \frac{364}{365}} = 0.286$

$P(X=0) = P(-0.5 \leq X_{\text{norm}} \leq 0.5)$

$= P\left(\frac{-0.082 - 0.5}{0.286} \leq Z \leq \frac{-0.082 + 0.5}{0.286}\right)$   
 $= .907$

similarly

$P(X=1) = P(0.5 \leq X_{\text{norm}} \leq 1.5) = 0.072$

$P(X=2) = P(1.5 \leq X_{\text{norm}} \leq 2.5) = 0.010$

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Q2

~~Q1~~

$$a) P(X \geq 1) = 1 - P(X=0) = 1 - (0.9)^n$$

$$X \sim \text{bin}(N, 0.1)$$

$$\text{To get } P(X \geq 1) = .95 \text{ solve} \\ 1 - 0.9^n \geq .95$$

$$\text{We find } 1 - 0.9^{29} = 0.949 \\ \text{and } 1 - 0.9^{29} = 0.953$$

$$\text{so select } \underline{N=29}$$

$$[\text{Note if } 1 - .95 = 0.9^n \\ \text{then } \log(0.05) = N \log 0.9]$$

b) sample without replacement.  
X is binomial for any N.

Q3

~~Q3~~. Outcomes: TTT TTH THT THH  
HTT HTHT HHT HHH.

a)  $4/8$

b)  $1/8$

c)  $6/8$

d)  $4/8$

e)  $1/4$

Q4

~~Q3~~.  $\mu = 700, \sigma = 50.$

a)  $z_{.25} = -0.674$

$$X_{.25} = \mu + \sigma z_{.25} = 666.3$$

b).  $X_1 = N(750, 30^2)$

$$P(X_1 \leq 666.3) = P\left(Z \leq \frac{666.3 - 750}{30}\right)$$

$$= 0.0026$$

c) 
$$\begin{aligned} X_{.95} &= 750 + 30 * z_{.95} \\ &= 750 + 30 * 1.645 \\ &= 799.35 \end{aligned}$$

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~~Spoken to 5, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100, 105, 110, 115, 120, 125, 130, 135, 140, 145, 150, 155, 160, 165, 170, 175, 180, 185, 190, 195, 200, 205, 210, 215, 220, 225, 230, 235, 240, 245, 250, 255, 260, 265, 270, 275, 280, 285, 290, 295, 300, 305, 310, 315, 320, 325, 330, 335, 340, 345, 350, 355, 360, 365, 370, 375, 380, 385, 390, 395, 400, 405, 410, 415, 420, 425, 430, 435, 440, 445, 450, 455, 460, 465, 470, 475, 480, 485, 490, 495, 500, 505, 510, 515, 520, 525, 530, 535, 540, 545, 550, 555, 560, 565, 570, 575, 580, 585, 590, 595, 600, 605, 610, 615, 620, 625, 630, 635, 640, 645, 650, 655, 660, 665, 670, 675, 680, 685, 690, 695, 700, 705, 710, 715, 720, 725, 730, 735, 740, 745, 750, 755, 760, 765, 770, 775, 780, 785, 790, 795, 800, 805, 810, 815, 820, 825, 830, 835, 840, 845, 850, 855, 860, 865, 870, 875, 880, 885, 890, 895, 900, 905, 910, 915, 920, 925, 930, 935, 940, 945, 950, 955, 960, 965, 970, 975, 980, 985, 990, 995, 1000~~ ~~Feb 2014~~

Q5

The total number of hands (unordered) is  $\binom{52}{5} = \frac{52 \times \dots \times 48}{5!} = 2,598,960$

a) To get 2 pairs we multiply.

$$\begin{array}{ll} \binom{13}{2} & \{ \text{\# pair 'types'} \} \\ \times \binom{4}{2} & \{ \text{\# ways to select} \\ \times \binom{4}{2} & \{ \text{2 from 4 ranks} \\ \times 44 & \{ \text{\# ways to select remaining cards} \end{array}$$

$$\begin{aligned} &= \frac{13 \times 12}{2!} \times \left( \frac{4 \times 3}{2!} \right)^2 \times 44 \\ &= 123,552 \end{aligned}$$

$$\text{so } P(2 \text{ pair}) = \frac{123,552}{2,598,960} \approx 0.0475$$

or, just under  $1/20$ .

4/12 ~~4/12~~

b) To get a Flush we multiply

$$\begin{array}{ll} 4 & \{ \# \text{ suits} \\ \times \binom{13}{5} & \{ \# \text{ ways to select 5 cards} \\ & \text{of the same suit} \end{array}$$

$$= 4 \times \frac{13 \times 12 \times 11 \times 10 \times 9}{5!} = 5,148$$

$$\text{so } P(\text{Flush}) = \frac{5,148}{2,598,960} \approx 0.00198$$

as just under  $1/500$ .

[Note that a Flush, in an actual poker game might exclude a straight and royal flush, in which cards are also of consecutive rank. To make that adjustment, the probabilities of those hands would be calculated separately, and subtracted from the answer given above.]

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Q6

~~Th.~~ ~~Let~~ A density satisfies  $f(x) \geq 0$  and  $\int_{\mathbb{R}} f(x) dx = 1$ .

$$a) P(X \leq x) = \int_{-\infty}^x f(y) dy.$$

$$P(Z \leq z) = P\left(\frac{X - a}{h} \leq z\right)$$

$$= P(X \leq hz + a)$$

$$= \int_{-\infty}^{hz+a} f(y) dy.$$

For  $h > 0$ , change of variable  $w = \frac{y - a}{h}$  gives

$$\begin{aligned} P(Z \leq z) &= \int_{-\infty}^z f(hw + a) h dw \\ &= \int_{-\infty}^z \phi_z(w) dw \end{aligned}$$

so that  $\phi_z(w) = hf(hw + a)$  is

the density of  $Z$ .

~~Th.~~

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If  $h < 0$  a similar argument gives

$$\phi_z(\omega) = -h \phi(h\omega + d)$$

so that the general solution is

$$\phi_z(\omega) = |h| \phi(h\omega + d).$$

b) If  $f_n(x) = \frac{1}{nh} \sum_{i=1}^n \phi\left(\frac{x-d_i}{h}\right)$

then  $f_n(x) \geq 0$  for all  $x$ , since  $\phi$  is a density and  $h > 0$ .

$$\text{Then } \int_{-\infty}^{\infty} \frac{1}{h} \phi\left(\frac{x-d_i}{h}\right) dx =$$

$$= \int_{-\infty}^{\infty} \phi(u) du$$

after change of variable  
 $u = \frac{x-d_i}{h}$

$$= 1$$

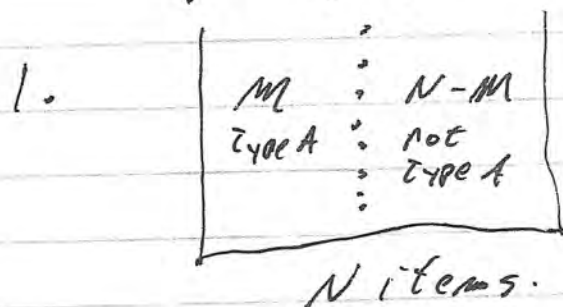
It follows that  $f_n(x)$  is  
also a density.

Q 7/12



Q7

~~8/12/14, Sample Question 2, Q5 2/14/14 Feb 2014.~~



a).  $k$  items are selected.

- If  $N-M < k$ , at least  $k - (N-M)$  Type A items must be selected, otherwise the minimum number of Type A items selected is 0.
- If  $M < k$  at most  $M$  Type A items can be selected, otherwise  $k$  Type A items can be selected.

This can be summarized.

$$\max(0, k - (N-M)) \leq x \leq \min(M, k)$$



b) Assuming  $X$  is in the range given in part a), suppose all items are labelled uniquely (while keeping their type A designations).

There are  $\binom{N}{12}$  possible samples. Consider the event  $\{X=x\}$ . There are  $\binom{m}{x}$  ways to select  $x$  type A items and  $\binom{N-m}{12-x}$  ways to select  $12-x$  of the remaining items. This gives

$$P(X=x) = \frac{\binom{m}{x} \binom{N-m}{12-x}}{\binom{N}{12}}$$

That is, the Hypergeometric distribution. (see Example 2.35 from IPM-SR).

~~25~~

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c) Then  $X$  is the sum of independent Bernoulli random Variables, that is  
$$X \sim \text{bin}(K, \frac{m}{N})$$
.

d). There are  $N$  Fish in the lake,  
 $M$  have tags 12 are caught,  
 $X$  of those caught have tags.

$$\text{so } \frac{X}{K} \approx \frac{M}{N} \quad . \quad 12, M \text{ are known,}$$

$$\therefore N \approx \frac{12M}{X} \quad \text{is a good estimator.}$$

The distribution can be obtained directly from the Hypergeometric distribution defined in part b). [or the binomial of part c) if the Fish are released after capture].

~~3/5~~

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See Assign. 3.

131.  $X \sim \text{bin}(n_1, p)$   
 $Y \sim \text{bin}(n_2, p)$ .

Q8

$X+Y$  is the sum of  $n_1+n_2$  iid Bernoulli RV's, that is,

$$X+Y \sim \text{bin}(n_1+n_2, p).$$

a)  $P(X=s \mid X+Y=t)$ ,  $s, t$  integers.

$$= \frac{P(X=s \text{ AND } X+Y=t)}{P(X+Y=t)}$$

$$= \frac{P(X=s) P(Y=t-s)}{P(X+Y=t)}$$

$$= \frac{\binom{n_1}{s} p^s (1-p)^{n_1-s} \binom{n_2}{t-s} p^{t-s} (1-p)^{n_2-(t-s)}}{\binom{n_1+n_2}{t} p^t (1-p)^{n_1+n_2-t}}$$

$$= \frac{\binom{n_1}{s} \binom{n_2}{t-s}}{\binom{n_1+n_2}{t}} \quad \begin{array}{l} \text{for } 0 \leq s \leq n_1 \\ 0 \leq t-s \leq n_2 \end{array}$$

[Hypergeometric]

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$$b). P(X+Y=t | X=s)$$

$$= \frac{P(X=s \text{ AND } X+Y=t)}{P(X=s)}$$

$$= \frac{P(X=s) P(Y=t-s)}{P(X=s)}$$

$$= P(Y=t-s) = \binom{n_2}{t-s} p^{t-s} (1-p)^{n_2-(t-s)}$$

assume  $s, t$  are integers with  
 $0 \leq s \leq n_1$ ,  $0 \leq t-s \leq n_2$ .

~~4.2 See section 5.2.3 in IPM-SIC~~

Q9. ~~From~~

Theorem 7.2 of Lecture notes

SPB

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