Assignment 2 - CSC/DSC 462 - Fall 2018 - Due October 9

All questions are worth equal marks.

Q1: A game involving a single dice is played in the following way. The dice is tossed repeatedly, and each outcome is recorded. Let X be the number of tosses needed to see an outcome already observed. For example, if the first four tosses are $3, 1, 6, 3$, then $X = 4$, and the game can stop. The largest value of X decides the winner.

(a) What is the support S_X of X?

(b) Using the rules of combinatorics, derive $P(X > i)$ for each $i \in S_X$.

(c) Use the answer of Part (b) to determine the PMF $p_i = P(X = i)$ for each $i \in S_X$.

Q2: A random variable X possesses a "V"-shaped density on the interval $[0, 4]$. This is constructed by taking the density to be $f_X(x) = cg(x)$ for some constant c, where $g(x)$ is the "V"-shaped function:

$$g(x) = \begin{cases} -(x-2) & ; \quad x \in (0,2) \\ x-2 & ; \quad x \in [2,4) \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

(a) Determine c.

(b) Determine the CDF $F(x) = P(X \leq x)$. Give this as a function of $x \in (-\infty, \infty)$. Sketch the CDF.

Q3: Recall the "hat" problem of Example 4.15 of the notes. At a party, N men bring identical hats. At the end of the party each man brings home one of the hats chosen at random. Let S_N be the number of men who bring home their own hats. It was shown that $E[S_N] = 1$, which, interestingly, does not depend on N.

For this problem, derive the variance of S_N.

HINT: Let $X_i = 1$ if the ith man brings home his own hat, and let $X_i = 0$ otherwise. Then $S_N = \sum_{i=1}^{n} X_i$. What is the covariance of X_i, X_j for any $i \neq j$?

Q4: Candidates for a position are first screened using an aptitude test. The scores are known to have a normal distribution with mean $\mu = 500$ and standard deviation $\sigma = 75$. Candidates who score at least $x = 625$ qualify for an interview.

(a) What is the probability that if 35 candidates take the aptitude test, at least 3 qualify for an interview?

(b) If we want the probability that at least 3 candidates qualify for an interview to be 80%, what should the cutoff score x be.

HINT: This problem can be solved by evaluating the root of an equation numerically using the `uniroot()` function. This calculates numerically the root u of a function f, that is, the solution to $f(u) = 0$. For example, suppose we wanted to find x which satisfies

$$5 = xe^x.$$

A quick sketch shows that the solution is in the interval $x \in [0, 2]$. Then we can use the following code to find that the solution is approximately $x \approx 1.326733$:

```
> f0 = function(x) {x*exp(x)-5}
> uniroot(f0,c(0,2))
$root
[1] 1.326733

$f.root
[1] 6.984409e-05

$iter
[1] 5

$init.it
[1] NA

$estim.prec
[1] 6.103516e-05

> 1.326733*exp(1.326733)
[1] 5.000073
>
```

Q5: Suppose $X_1, \ldots, X_n$ are independent observations from a uniform distribution on the interval $(0, c)$, for some constant $c > 0$. Define the maximum

$$M = \max_{i=1,\ldots,n} X_i.$$

Derive the expected value $E[M]$.

HINT: Note that $\{M \leq t\} = \cap_{i=1}^{n} \{X_i \leq t\}$, which is an interesection of n independent events.