Practice Problems 1 (with solutions) - Chapters 2-3 - CSC/DSC 262/462

Q1. A coin is tossed twice, independently. Define the three events:

$$\begin{aligned} A_1 &= \{ \text{ first toss is Heads } \} \\ A_2 &= \{ \text{ second toss is Heads } \} \\ A_3 &= \{ \text{ two outcomes are the same } \}. \end{aligned}$$

Prove that these events are pairwise independent but not independent.

SOLUTION
The sample space is $S = \{HH, HT, TH, TT\}$. We have

$$\begin{aligned} A_1 &= \{HH, HT\} \\ A_2 &= \{HH, TH\} \\ A_3 &= \{HH, TT\}. \end{aligned}$$

We have $P(A_i) = 1/2$ for $i = 1, 2, 3$.

To prove **pairwise independence** it suffices to show that $P(A_iA_j) = P(A_i)P(A_j) = 1/4$ for each pair $i \neq j$. We have

$$\begin{aligned} P(A_1A_2) &= P(\{HH\}) = 1/4, \\ P(A_1A_3) &= P(\{HH\}) = 1/4, \\ P(A_2A_3) &= P(\{HH\}) = 1/4. \end{aligned}$$

establishing pairwise independence.

To disprove **independence** we note

$$P(A_1 \cap A_2 \cap A_3) = P(\{HH\}) = 1/4 \neq P(A_1)P(A_2)P(A_3) = 1/8.$$

Q2. Three 6-sided dice are tossed independently. Label the dice red, green and blue. Suppose we define the following events:

$$\begin{aligned} A_1 &= \{\text{red dice} = \text{green dice}\} \\ A_2 &= \{\text{red dice} = \text{blue dice}\} \\ A_3 &= \{\text{green dice} = \text{blue dice}\}. \end{aligned}$$

Calculate the probabilities:

$$\begin{aligned} &P(A_i), \quad i = 1, 2, 3; \\ &P(A_i \cap A_j), \quad i \neq j; \\ &P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

Are the events A_1, A_2, A_3 independent? Are they pairwise independent?

SOLUTION
There are $6 \times 6 = 36$ outcomes involving two dice, and the two dice are equal for 6 of them:

$$P(A_i) = 6/36 = 1/6$$

for $i = 1, 2, 3$. Next, note that for any $i \neq j$

$$A_i \cap A_j = \{\text{all three dice are equal}\} = A_1 \cap A_2 \cap A_3.$$

There are $6 \times 6 \times 6 = 216$ outcomes involving three dice, and the three dice are equal for 6 of them:

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = P(A_1 \cap A_2 \cap A_3) = 6/216 = 1/36.$$

This means

$$(1/6)^3 = P(A_1)P(A_2)P(A_3) \neq P(A_1 \cap A_2 \cap A_3) = 1/36,$$

so that A_1, A_2, A_3 *are not independent*. However, for any $i \neq j$,

$$(1/6)^2 = P(A_i)P(A_j) = P(A_i \cap A_j) = 1/36,$$

so that A_1, A_2, A_3 *are pairwise independent*.

Q3. The *Monty Hall problem* is a good example of the often counterintuitive nature of probability. It is based on the television game show *Let's Make a Deal* (starring Monty Hall). There are three doors. Behind one is a car, and behind the other two are goats. The contestant picks one door. Then, one the other doors is opened, revealing a goat (this can always be done, since there are two goats). The contestant is offered the choice of staying with the original choice, or switching to the one remaining door. The contestant wins whatever is behind the selected door. Assume the contestant makes the first choice at random, and has decided in advance whether or not to switch. Determine the probability of winning the car if the contest doesn't switch, and if the contestant does switch.

SOLUTION
We can, without loss of generality, assume that the car is behind door 1 (the argument is identical wherever the car is). Define events

$$\begin{aligned} D_i &= \{\text{ Contestant initially picks door } i \text{ }\}, \ \ i = 1, 2, 3. \\ W &= \{\text{ Contestant wins car }\}. \end{aligned}$$

Then $P(D_i) = 1/3$.

If the **contestant doesn't switch**, we clearly have $P(W \mid D_1) = 1$ and $P(W \mid D_2) = P(W \mid D_3) = 0$. By conditioning on the initial selection (the Law of Total Probability) we have

$$P(W) = P(W \mid D_1)P(D_1) + P(W \mid D_2)P(D_2) + P(W \mid D_3)P(D_3) = 1 \times 1/3 + 0 \times 1/3 + 0 \times 1/3 = 1/3.$$

So the probability of winning the car is $1/3$ if the contestant doesn't switch.

If the **contestant does switch**, we have $P(W \mid D_1) = 0$, since the contestant necessarily switches to a goat. However, if the contestant initially picks door 2 (which has a goat) then it has to be door 3 which is opened. The contest would have to switch to door 1, thus winning the car. This means

$P(W \mid D_2) = 1$, and also $P(W \mid D_3) = 1$ by an identical argement. By conditioning on the initial selection (the Law of Total Probability) we have

$$P(W) = P(W \mid D_1)P(D_1) + P(W \mid D_2)P(D_2) + P(W \mid D_3)P(D_3) = 0 \times 1/3 + 1 \times 1/3 + 1 \times 1/3 = 2/3.$$

So the probability of winning the car is 2/3 if the contestant does switch.

Q4. A dice game is played in the following way. A player continues to toss a dice as long as the current outcome is strictly higher than the previous outcome. The score is the number of such outcomes. For example, for the sequence 1,3,5,2 the player stops at the fourth toss, and scores $X = 3$. What is the probability that the player scores at least $X = 3$?

SOLUTION

Noting that the player will always toss a dice at least twice, we define events

$$\begin{aligned} E &= \{ \text{ Player score at least } X = 3 \}, \\ A_{i,j} &= \{ \text{ First two tosses are } i, j \}. \end{aligned}$$

Consider event $A_{i,j}$. If $i \geq j$ then E cannot occur. If $i < j$, then E occurs with probability $(6 - j)/6$. Each event $A_{i,j}$ has probability $P(A_{i,j}) = 1/36$. This is expressed as conditional probabilities:

$$P(E \mid A_{i,j}) = \begin{cases} (6-j)/6 & ; \quad i < j \\ 0 & ; \quad i \geq j \end{cases} .$$

By conditioning on the events $A_{i,j}$ we have (including only those events for which $P(E \mid A_{i,j}) > 0$)

$$\begin{aligned} P(E) &= P(E \mid A_{1,5})P(A_{1,5}) + P(E \mid A_{2,5})P(A_{2,5}) + P(E \mid A_{3,5})P(A_{3,5}) + P(E \mid A_{4,5})P(A_{4,5}) \\ &\quad + P(E \mid A_{1,4})P(A_{1,4}) + P(E \mid A_{2,4})P(A_{2,4}) + P(E \mid A_{3,4})P(A_{3,4}) \\ &\quad + P(E \mid A_{1,3})P(A_{1,3}) + P(E \mid A_{2,3})P(A_{2,3}) \\ &\quad + P(E \mid A_{1,2})P(A_{1,2}) \\ &= \left(4 \times \frac{1}{6} \times \frac{1}{36}\right) + \left(3 \times \frac{2}{6} \times \frac{1}{36}\right) + \left(2 \times \frac{3}{6} \times \frac{1}{36}\right) + \left(1 \times \frac{4}{6} \times \frac{1}{36}\right) \\ &= \frac{4+6+6+4}{6 \times 36} = \frac{5}{54}. \end{aligned}$$

Q5. The hour hand on a 12-point clock is positioned at 12. The hand moves backwards or forwards one position with equal probability N times. All moves are independent. Determine the probability that the hand rests at 3 if:

(a) $N = 9$,

(b) $N = 10$,

(c) $N = 19$.

SOLUTION

An outcome consists of a sequence of N (F)orward or (B)ackwards directions, say, $BFF \dots BFB$. The problem is best approached by defining a random outcome X, defined as

$$X = \text{The number of } F\text{'s in the sequence.}$$

This is because the problem can be resolved by knowing X. By the rule of product, each outcome has the same probability $1/2^N$. The number of sequences of length N that have exactly k F's is $\binom{N}{k}$, since we are making an unordered selection of k positions for the F's from the N available. So,

$$P(X = k) = \binom{N}{k}(1/2)^N.$$

To determine if the final position of the hour is 3 we use the rule

$$\begin{aligned} (\#F - \#B) \mod 12 &= 3, \quad \text{or equivalently} \\ (2X - N) \mod 12 &= 3. \end{aligned}$$

Note that $x = y \mod n$ if $y - x$ is divisible by n. So $-9 \mod 12 = 3$. The order of the moves does not matter.

(a) There are two ways for the hour hand to rest on position 3 if $N = 9$. Either it moves forward 6 and backwards 3 positions, or it moves backwards 9 positions. The values of X for which which this occurs are 0 and 6, so

$$P(\text{ hand rests on 3 }) = P(X = 0) + P(X = 6) = \frac{\binom{9}{0} + \binom{9}{6}}{2^9} = \frac{85}{512} \approx 0.166.$$

(b) The hour hand cannot rest on 3 if $N = 10$, so

$$P(\text{ hand rests on 3 }) = 0.$$

(c) For $N = 19$ the hour hand rests on 3 if $X = 5, 11$ or 17. This means

$$\begin{aligned} P(\text{ hand rests on 3 }) &= P(X = 5) + P(X = 11) + P(X = 17) \\ &= \frac{\binom{19}{5} + \binom{19}{11} + \binom{19}{17}}{2^{19}} \\ &= \frac{87381}{524288} \approx 0.167. \end{aligned}$$

Note that the answers to (a) and (c) are both close to, but not exactly, 1/6.

Q6. Suppose any integer from 1 to 75 inclusive is chosen at random with equal probability, which will be denoted N. What is the probability that N is divisible by at least one of 5, 7 or 11?

SOLUTION

Let E_i be the event that N is divisible by i. So,

$$E_5 = \{N \in \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75\}\},$$

giving

$$P(E_5) = \frac{|E_5|}{73} = \frac{15}{75}.$$

Also, we must have (since 5, 7 and 11 are prime numbers)

$$E_i \cap E_j = E_{i\times j}, \;\; E_i \cap E_j \cap E_k = E_{i\times j\times k}.$$

So, by the inclusion-exclusion identity we have

$$\begin{aligned} P(E_5 \cup E_7 \cup E_{11}) &= P(E_5) + P(E_7) + P(E_{11}) - P(E_5E_7) - P(E_5E_{11}) - P(E_7E_{11}) + P(E_5E_7E_{11}) \\ &= P(E_5) + P(E_7) + P(E_{11}) - P(E_{35}) - P(E_{55}) - P(E_{77}) + P(E_{385}). \\ &= \frac{1}{75} \times (15 + 10 + 6 - 2 - 1 - 0 + 0) = \frac{28}{75}. \end{aligned}$$

Q7. A *random walk* can be described as follows. We have time points $i = 0, 1, 2, \ldots$. The random walk has value X_i at time point i, according to the following rules:

(1) $X_0 = 0$.

(2) At time point i, +1 or -1 is added to X_i with equal probability, resulting in $X_{i+1} = X_i - 1$ or $X_{i+1} = X_i + 1$. All increments are selected independently.

For example, we could have $X_0 = 0$, $X_1 = 1$, $X_2 = 0$, $X_3 = -1$, $X_4 = -2$, $X_5 = -1$ and so on.

Determine the following probabilities:

(a) $P(X_1 = 1, X_2 = 0, X_3 = -1, X_4 = 0)$,

(b) $P(X_1 = -1, X_2 = -2, X_3 = -3, X_4 = -2)$,

(c) $P(X_4 = 0)$,

(d) $P(X_i > 0 \text{ for } i = 1, 2, 3, 4)$.

SOLUTION

The key here is to recognize that there are 16 possible paths from X_0 to X_4, which can be enumerated with a bit of effort:

Path Index	X_0	X_1	X_2	X_3	X_4
1	0	1	2	3	4
2	0	1	2	3	2
3	0	1	2	1	2
4	0	1	2	1	0
5	0	1	0	1	2
6	0	1	0	1	0
7	0	1	0	-1	0
8	0	1	0	-1	-2
9	0	-1	0	1	2
10	0	-1	0	1	0
11	0	-1	0	-1	0
12	0	-1	0	-1	-2
13	0	-1	-2	-1	0
14	0	-1	-2	-1	-2
15	0	-1	-2	-3	-2
16	0	-1	-2	-3	-4

Then, for parts (a)-(b), we are simply being asked for the probability of a single path, which must be 1/16. For parts (c)-(d) we enumerate the paths which are in the event:

(a) $P(X_1 = 1,\ X_2 = 0,\ X_3 = -1,\ X_4 = 0) = P(\text{Path Index} = 7) = 1/16$,

(b) $P(X_1 = -1,\ X_2 = -2,\ X_3 = -3,\ X_4 = -2) = P(\text{Path Index} = 15) = 1/16$,

(c) $P(X_4 = 0) = P(\text{Path Index} \in \{4, 6, 7, 10, 11, 13\}) = 6/16$,

(d) $P(X_i > 0 \text{ for } i = 1, 2, 3, 4) = P(\text{Path Index} \in \{1, 2, 3\}) = 3/16$.

Q8. A game is played in the following way. First a 6-sided dice is tossed. Suppose the dice shows N. Then a coin is tossed N times. The player wins if the coin shows the same face for each of the N tosses. What is the probability that the player wins? Use the law of total probability.

SOLUTION
Let W be the event that the player wins. If $N = n$, then by independence

$$P(\text{All Heads}) = P(\text{All Tails}) = (1/2)^n,$$

so that

$$P(W \mid N = n) = (1/2)^n + (1/2)^n = (1/2)^{n-1}.$$

The sample space is partitioned by the 6 events $A_1 = \{N = 1\}, \ldots, A_6 = \{N = 6\}$. By the law of total probability

$$\begin{aligned}
P(W) &= \sum_{i=1}^{6} P(W \mid N = i) P(N = i) \\
&= \sum_{i=1}^{6} (1/2)^{i-1} \times \frac{1}{6} \\
&= \frac{1}{6} \times (1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32) \\
&= \frac{1}{6} \times 2 \times \frac{63}{64} \\
&= \frac{21}{64}.
\end{aligned}$$

Q9. This question is adapted from *Introduction to Probability Models* (10th Edition), S.M. Ross. Three prisoners, labeled A, B and C, are informed by a guard that one of them has been chosen at random to be executed the following day. Prisoner A asks the guard, privately, to name one of the other prisoners who will be released. We then have the competing claims:

Claim 1: The guard argues that by eliminating one prisoner from the execution pool the probability that A is executed changes from 1/3 to 1/2.

Claim 2: Prisoner A argues that since it is already known that at least one of prisoners B or C will be released, the probability that A is executed remains 1/3.

Assume that if the guard names a prisoner to be released, and both B and C are to be released, the guard will name either one with equal probability. Otherwise, the guard names the only prisoner other than A being released. Define the following events.

$$\begin{aligned}
E_A &= \{\text{Prisoner } A \text{ chosen for execution}\} \\
E_B &= \{\text{Prisoner } B \text{ chosen for execution}\} \\
E_C &= \{\text{Prisoner } C \text{ chosen for execution}\} \\
F_B &= \{\text{Guard informs prisoner } A \text{ that prisoner } B \text{ is being released}\} \\
F_C &= \{\text{Guard informs prisoner } A \text{ that prisoner } C \text{ is being released}\}.
\end{aligned}$$

So, the event that B is to be released is equivalent to E_B^c, and the event that A is informed that B is to be released is F_B. Calculate the following probabilities:

(a) $P(E_B^c)$,

(b) $P(F_B)$,

(c) $P(E_A \mid E_B^c)$,

(d) $P(E_A \mid F_B)$.

Who is correct, the guard or prisoner A?

SOLUTION

(a) We have $P(E_B) = 1/3$, so $P(E_B^c) = 1 - P(E_B) = 1 - 1/3 = 2/3$.

(b) We must have

$$\begin{aligned}
P(F_B \mid E_A) &= 1/2 \\
P(F_B \mid E_B) &= 0 \\
P(F_B \mid E_C) &= 1
\end{aligned}$$

so by the law of total probability:

$$\begin{aligned}
P(F_B) &= P(F_B \mid E_A)P(E_A) + P(F_B \mid E_B)P(E_B) + P(F_B \mid E_C)P(E_C) \\
&= (1/2)(1/3) + 0(1/3) + 1(1/3) \\
&= 1/2.
\end{aligned}$$

(c) Noting that $E_A \subset E_B^c$, and so $E_A E_B^c = E_A$ we have

$$P(E_A \mid E_B^c) = \frac{P(E_A E_B^c)}{P(E_B^c)} = \frac{P(E_A)}{P(E_B^c)} = \frac{1/3}{2/3} = 1/2.$$

(d)

$$P(E_A \mid F_B) = \frac{P(E_A F_B)}{P(F_B)} = \frac{P(F_B \mid E_A)P(E_A)}{P(F_B)} = \frac{(1/2)(1/3)}{1/2} = 1/3.$$

Prisoner A is correct. The key here is to recognize that E_B^c and F_B are not the same event.

Q10. A bin contains 5 white and 5 black balls. A random selection of 2 balls is made. Let X be the number of white balls among the 2 selected. Determine $P(X = k)$ for $k = 0, 1, 2$.

SOLUTION

We may temporarily label the balls within each color $1, \ldots, 5$, so that they are all distinct. Then the total number of selections is

$$D = \binom{10}{2} = 45.$$

To enumerate the selections for which $X = k$ use the *rule of product.* There are

$$N = n_1 \times n_2 = \binom{5}{k} \times \binom{5}{2-k}$$

such combinations. We then have the general expression:

$$P(X = k) = \frac{N}{D} = \frac{\binom{5}{k}\binom{5}{2-k}}{\binom{10}{2}}.$$

This gives

$$\begin{aligned} P(X = 0) &= \frac{\binom{5}{0}\binom{5}{2}}{\binom{10}{2}} = \frac{1 \times 10}{45} = \frac{2}{9} \\ P(X = 1) &= \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} = \frac{5 \times 5}{45} = \frac{5}{9} \\ P(X = 2) &= \frac{\binom{5}{2}\binom{5}{0}}{\binom{10}{2}} = \frac{10 \times 1}{45} = \frac{2}{9}. \end{aligned}$$

Q11. A container contains 2 balls each of n colors (a total of $2n$ balls). The two balls of the same color are considered identical. Derive an expression for

$$\alpha_n = P(\text{All colors are adjacent in a random permutation of all balls}).$$

SOLUTION

We can use the multinomial coefficient. There are n types of balls, with $n_i = 2$ of each type, $i = 1, \ldots, n$. The number of permutations is therefore

$$D = \binom{2n}{2, \ldots, 2} = \frac{(2n)!}{\prod_{i=1}^{n} 2!} = \frac{(2n)!}{2^n}.$$

The number of permutations for which all colors are adjacent is equal to the number of permutations of the n colors:

$$N = n!$$

So,

$$\alpha_n = \frac{N}{D} = \frac{2^n n!}{(2n)!}.$$

ALTERNATIVE SOLUTION

We can use the *rule of product*. Temporarily label the balls in each color pair 1 and 2. Then, to construct a permutation with adjacent colors use the following *tasks*:

Task 1: Select permutation of colors, $n_1 = n!$.

Task 2: Select ordering of temporary labels within each color pair, $n_2 = 2^n$.

There are

$$N = n_1 \times n_2 = n! \times 2^n$$

such permutations. There are a total of

$$D = (2n)!$$

(temporarily labelled) permutations, so

$$\alpha_n = \frac{N}{D} = \frac{2^n n!}{(2n)!}.$$

Q12. The letters in MISSISSIPPI are randomly permuted. What is the probability that there are no consecutive S's? What is the probability that the S's are consecutive (for example, IPSSSSIIMPI)?

SOLUTION

Use the *rule of product*.

Task 1: Permute the letters other than S, $n_1 = \binom{7}{1,2,4} = 105$.

Task 2: Once the remaining 7 letters have been permuted, we need to select for each S, uniquely, a position before, after or in between the letters. There are 8 such positions, so $n_2 = \binom{8}{4} = 70$.

There are

$$N = n_1 \times n_2 = 105 \times 70$$

such permutations. There are a total of

$$D = \binom{11}{1,2,4,4} = \frac{11!}{2! \times 4! \times 4!} = 34650$$

permutations, so

$$P(\text{No consecutive S's}) = \frac{N}{D} = \frac{105 \times 70}{34650} \approx 0.2121.$$

Next, permuting the letters of MISSISSIPPI such that the S's are consecutive is equivalent to replacing the 4 S's with 1 S, then counting the permutations. This gives

$$N = \binom{8}{1,1,2,4} = \frac{8!}{2! \times 4!} = 840,$$

so

$$P(\text{All S's consecutive}) = \frac{N}{D} = \frac{840}{34650} \approx 0.024.$$

Q13. Someone proposes playing a dice game, and kindly offers to provide the dice. You suspect that the dice may be *loaded*, that is, at least one outcome has a probably other than 1/6. Suppose E is an event involving a die with the following special property. Let $P_f(E)$ be the probability of the event for a *fair* die (each outcome has probability 1/6). Let $P_{uf}(E)$ be the probability of the event for some other die. If this special property holds, then if that die is loaded in any way, we have $P_{uf}(E) > P_f(E)$. Note that E can involve more than one toss *of the same* die. Can you think of an event with this property? If so, you can propose a bet that favors you if the dice is loaded, and is fair otherwise, without having to know how the dice is loaded.

SOLUTION

Toss a single die twice. Set $E = \{$ outcome is the same for both tosses $\}$. Then

$$P(E) = \sum_{i=1}^{6} p_i^2,$$

where p_i is the probability of tossing i. No matter what the loading, we must have $p_1 + \ldots + p_6 = 1$. It can then be shown that the sum is uniquely minimized by setting $p_i = 1/6$ for each i (using, for example, the Lagrange multiplier method).

Q14. A bin contains n balls labeled $1, \ldots, n$. The balls are selected in order, at random. We say the ball labeled k was *selected correctly* if it is in position k of the selection order. For example, if $n = 5$, and the selection order was $5, 2, 1, 4, 3$ then balls 2 and 4 were selected correctly.

(a) Give an expression in terms of n for the probability that a specific ball was selected correctly.

(b) Suppose B is a specific subset of $m \leq n$ balls Give an expression in terms of n and m for the probability that all balls in B were selected correctly.

(c) Use the inclusion-exclusion principle to derive a formula for the probability that *no* ball is selected correctly (such a permutation is known as a *derangement*). Write an `R` program to calculate the probability of a derangement for $n = 1, 2, \ldots, 25$. Comment briefly on the resulting sequence.

SOLUTION

(a) There are $D = n!$ possible selections. Let A_j be the set of ordered selections in which ball j is selected correctly. We can select from A_j using 2 tasks. First, put ball j in its correct place ($n_1 = 1$). Second, select positions for the remaining balls ($n_2 = (n-1)!$). Note that more than one ball can be selected correctly in A_j, as long as ball j is selected correctly. So $N = 1 \times (n-1)!$, giving

$$P(A_j) = \frac{N}{D} = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(b) Again, we do the selection in 2 tasks. First, place the balls in B in their correct place ($n_1 = 1$). Then permute the remaining balls ($n_2 = (n-m)!$). Let A_B be the set of ordered selections in which all balls $j \in B$ are selected correctly. This gives

$$P(A_B) = \frac{N}{D} = \frac{(n-m)!}{n!}. \tag{1}$$

(c) Let A_j be the event that ball j is selected correctly. Then

$$P(derangement) = 1 - P(\cup_{j=1}^{n} A_j).$$

Using the inclusion-exclusion principle we can write:

$$P(\cup_{j=1}^{n} A_j) = \sum_{j=1}^{n} (-1)^{j-1} S_j,$$

where S_j are the appropriate sums. Each term in S_j is equal to $(n-j)!/n!$ from (1). Furthermore, each sum S_j contains $\binom{n}{j}$ terms, giving

$$P(\cup_{i=1}^{n} A_i) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{(n-j)!}{n!} = \sum_{j=1}^{n} (-1)^{j-1} \frac{1}{j!}.$$

If we subtract from 1, we get

$$P(derangement) = \sum_{j=0}^{n} (-1)^{j} \frac{1}{j!}.$$

(d) The following code performs the required calculations:

```
> nmax = 25
> prob = rep(NA,nmax)
>
> for (n in 1:nmax) {
+   sm = 0
+   for (i in 1:n) {sm = sm + ((-1)^(i-1))/prod(1:i)}
+   prob[n] = 1 - sm
+ }
> prob
 [1] 0.0000000 0.5000000 0.3333333 0.3750000 0.3666667 0.3680556 0.3678571 0.3678819
 [9] 0.3678792 0.3678795 0.3678794 0.3678794 0.3678794 0.3678794 0.3678794 0.3678794
[17] 0.3678794 0.3678794 0.3678794 0.3678794 0.3678794 0.3678794 0.3678794 0.3678794
[25] 0.3678794
```

The derangement probability converges to a specific number as n increases (it can be shown that this number is exactly e^{-1}). More informally, for large enough n, the probablity of a derangement depends very little on n.

Q15. In some little known kingdom convicted prisoners are offered the possibility of a pardon according to a game of chance. The prisoner is given n red balls and n green balls. He/she then places the balls into two bins in any manner he/she chooses. The king then (i) selects a bin at random; (ii) selects one ball at random from that bin. If that ball is green the prisoner is pardoned. Note that the prisoner can leave one bin empty, and if that bin is selected by the king, then no green ball can be chosen, so no pardon is granted.

How should the balls be allocated to the bins in order to maximize the probability $P(G)$ of selecting green (and therefore winning a pardon)?

HINT: There are several ways to solve this. One way is to solve the following sub-problems, from which the optimal choice can be deduced:

P1 In a *simple allocation* all balls are in one bin. What is $P(G)$ for this case?

P2 In an *even allocation* each bin has the same number of balls, independent of their color. Show that $P(G)$ is the same for all even allocations, and derive this number.

P3 In an *uneven allocation* one bin (the *light bin*) has strictly fewer balls than the other (the *heavy bin*), but both have at least one. Show that for any uneven allocation the heavy bin has at least one green ball. Next, show that if the light bin has at least one red ball, and if a red ball from the light bin is exchanged with a green ball from the heavy bin, then $P(G)$ will strictly increase.

P4 Show that for an uneven allocation in which the light bin has no red balls and at least two green balls, if a green ball is moved from the light bin to the heavy bin, then $P(G)$ will strictly increase.

It also helps to assume $n > 1$. The case $n = 1$ can then be consider separately.

SOLUTION

It is possible to give an intuitive argument that the optimal allocation is to place one green ball only in one of the bins. Clearly, if bin 1, say, has fewer balls than bin 2, those balls have a higher probability of being selected. We would therefore like those balls to be green. Once we accept this, we then argue that it would be optimal to have only one ball in bin 1. The following two solution methods make this argument precise.

METHOD 1. Let $P_k(G)$ be the probability of selecting a green ball from bin k. Then

$$P(G) = [P_1(G) + P_2(G)]/2.$$

Suppose we put r and g red and green balls into bin 1. It will be convenient to set $m = r + g$. If bin k is empty we have $P_k(G) = 0$. Otherwise,

$$\begin{aligned} P_1(G) &= \frac{g}{m}, \;\; m \geq 1 \\ P_2(G) &= \frac{n-g}{2n-m}, \;\; m \leq 2n-1 \\ P(G) &= \frac{1}{2}\left[\frac{g}{m} + \frac{n-g}{2n-m}\right], \;\; 1 \leq m \leq 2n-1 \end{aligned} \tag{2}$$

We can, without loss of generality, assume that bin 1 is the light bin (or empty bin) for an uneven (or simple) allocation.

P1 We have $r = g = m = 0$, so $P(G) = 1/4$.

P2 In an *even allocation* we have $m = n$, but g may vary from 0 to n. This gives

$$P(B) = \frac{1}{2}\left[\frac{g}{n} + \frac{n-g}{n}\right] = \frac{1}{2}\left[\frac{n-g+g}{n}\right] = 1/2,$$

which does not depend on g.

P3 There are n red balls. Since the heavy bin contains more than n balls, at least one must be green. First, note that m is unchanged by the exchange. Let $P(G)$, $P'(G)$ be the probabilities of selecting green before and after the exchange. Then

$$\begin{aligned} P'(G) - P(G) &= \frac{1}{2}\left[\frac{g+1}{m} + \frac{n-g-1}{2n-m}\right] - \frac{1}{2}\left[\frac{g}{m} + \frac{n-g}{2n-m}\right] \\ &= \frac{1}{2}\left[\frac{1}{m} - \frac{1}{2n-m}\right] \\ &= > 0, \end{aligned}$$

where the inequality follows from the fact that $m < 2n - m$.

P4 First, note that $P_1(G) = 1$ before and after the move. Following the move, the number of green balls in the heavy bin increases and the number of red balls is unchanged, so $P_2(G)$, and therefore $P(G)$, must strictly increase.

The number of allocations is finite, so at least one allocation gives a maximum probability. We then proceed by elimination. From P1 and P2 we can eliminate the simple allocation, since any even allocation is strictly better. From P3 we can eliminate any uneven allocation with at least one red ball in the light bin. From P4 we can eliminate any uneven allocation in which the light bin contains no red balls and at least two green balls. This means that if an uneven allocation is optimal it must be $g = 1, r = 0, m = 1$ (or $g = n - 1, r = n, m = 2n - 1$). For this allocation we have

$$P(G) = \frac{1}{2}\left[\frac{1}{1} + \frac{n-1}{2n-1}\right] = \frac{3n-2}{4n-2}. \tag{3}$$

If we assume $n > 1$ then this number is greater than $1/2$, which eliminates the even allocation, exhausting all remaining possibilities. Finally, if $n = 1$, no uneven allocation is possible, so the even allocation is optimal, and equation (3) holds for this case also. Therefore, placing exactly one green ball in one of the bins yields the maximum value of $P(G)$, given in (3), for any $n \geq 1$. This value approaches $3/4$ from below as n increases (so larger n is more favorable to the prisoner).

METHOD 2. We have an optimization problem in two dimensions. Sometimes, such problems can be solved by holding one dimension fixed, solving the simpler one-dimensional problem under that constraint, then relaxing the constraint.
Let $q_m(g) = P(G)$ for m and g. First assume that $m \leq n$. In that case, this function is defined for $g = 0, 1, \ldots, m$ (see notaton for SOLUTION 1). Then let

$$q_m^* = \max_{g=0,1,\ldots,m} q_m(g).$$

We know (from METHOD 1) that $q_0(g) = 1/4$ and $q_n(g) = 1/2$ for all admissible g, so $q_0^* = 1/4$, $q_n^* = 1/2$.
Next, if $1 \leq m < n$ then it is easy to show that $q_m(g) = P(G) = ag + b$ for some constants a, b, for which $a > 0$ (other than that, the exact values of a, b don't matter). Therefore

$$q_m^* = q_m(m) = \begin{cases} \frac{1}{2}\left[1 + \frac{n-m}{2n-m}\right] & ; \quad m = 1, \ldots, n \\ 1/4 & ; \quad m = 0 \end{cases}.$$

This covers the cases $m = 0, 1, \ldots, n$. By symmetry (all we need to do is exchange the bin labels) we have $q_{2n-m}^* = q_m^*$, which exhausts all cases. The problem is then solved by maximizing q_m^*, achieved by setting $m = 1, g = 1$ or $m = 2n - 1, g = n - 1$, again yielding maximum probability (3).

A final note. If the two solutions are compared, it can be seen that they are very similar.

Q16. A standard 52 card playing deck assigns a unique combination of 13 ranks in the sequence (2,3,4,5,6,7,8,9,10,J,Q,K,A) and 4 suits (Clubs, Diamonds, Hearts, Spades) to each card ($13 \times 4 = 52$). Suppose a hand of 5 cards is selected at random. Using the *rule of product* calculate the probability that the cards form each of the hands listed below. Carefully list the *tasks* used in the application of the *rule of product*.

(a) **One Pair.** Exactly two cards of one rank, the remaining cards of distinct rank.

(b) **Two Pairs.** Two distinct ranks each represented by exactly two cards, the remaining card of distinct rank.

(c) **Three of a Kind.** Exactly three cards of one rank, the remaining cards of distinct rank.

(d) **Straight.** All ranks in consecutive sequence, but cards not all of the same suit.

(e) **Flush.** All cards of the same suit, but ranks not in consecutive sequence.

(f) **Full House.** Two cards of one rank, three cards of a different rank.

(g) **Straight Flush.** Ranks in consecutive sequence, all of the same suit, but not a royal flush.

(h) **Royal Flush.** Rank (10,J,Q,K,A) all of the same suit.

NOTE: There are many different rules for poker. Here, we will assume that in a consecutive sequence of ranks 'A' may preceed '2', that is (A,2,3,4,5) is a consecutive sequence. Other consecutive sequences are (3,4,5,6,7), (8,9,10,J,Q), (10,J,Q,K,A) and so on. However, (Q,K,A,2,3) is not a consecutive sequence.

SOLUTION
Recall that we can have

$$D = \binom{52}{5} = 2,598,960$$

possible hands. Use the rule of product for each problem:

(a) **One Pair**

Task 1: Select rank for pair, $n_1 = 13$.
Task 2: Select combination of 2 from 4 cards for pair rank, $n_2 = \binom{4}{2} = 6$.
Task 3: Select 3 distinct ranks for remaining cards, $n_3 = \binom{12}{3} = 220$.
Task 4: Select 1 of 4 suits for each of the remaining cards, $n_4 = 4^3 = 64$.
There are

$$N = n_1 \times n_2 \times n_3 \times n_4 = 13 \times 6 \times 220 \times 64 = 1,098,240$$

such selections, so

$$P(\text{ One Pair }) = \frac{1,098,240}{2,598,960} \approx 0.4226.$$

(b) **Two Pairs**

Task 1: Select combination of 2 from 13 ranks for the pairs, $n_1 = \binom{13}{2} = 78$.
Task 2: Select 2 from 4 cards for first pair rank, $n_2 = \binom{4}{2} = 6$.
Task 3: Select 2 from 4 cards for second pair rank, $n_3 = \binom{4}{2} = 6$.
Task 4: Select 1 of 44 = 52 - 8 remaining cards, $n_4 = 44$.
There are

$$N = n_1 \times n_2 \times n_3 \times n_4 = 78 \times 6 \times 6 \times 44 = 123,552$$

such selections, so

$$P(\text{ Two Pairs }) = \frac{123,552}{2,598,960} \approx 0.04754.$$

(c) **Three of a Kind**

Task 1: Select rank for three of a kind, $n_1 = 13$.
Task 2: Select combination of 3 from 4 cards for three of a kind, $n_2 = \binom{4}{3} = 4$.

Task 3: Select 2 distinct ranks for remaining cards, $n_3 = \binom{12}{2} = 66$.

Task 4: Select 1 of 4 suits for each of the remaining cards, $n_4 = 4^2 = 16$.

There are

$$N = n_1 \times n_2 \times n_3 \times n_4 = 13 \times 4 \times 66 \times 16 = 54,912$$

such selections, so

$$P(\text{ Three of a Kind }) = \frac{54,912}{2,598,960} \approx 0.02113.$$

(d) **Straight**

Task 1: Select sequence: $n_1 = 10$.

Task 2: Assign ordered selection of suits to ranks, excluding single suit assignments: $n_2 = 4^5 - 4 = 1,020$.

There are

$$N = n_1 \times n_2 = 10 \times 1,020 = 10,200$$

such selections, so

$$P(\text{ Straight }) = \frac{10,200}{2,598,960} \approx 0.003925.$$

(e) **Flush**

Task 1: Select suit: $n_1 = 4$.

Task 2: Select 5 ranks from 13, excluding sequences: $n_2 = \binom{13}{5} - 10 = 1,277$.

There are

$$N = n_1 \times n_2 = 4 \times 1,277 = 5,108$$

such selections, so

$$P(\text{ Flush }) = \frac{5,108}{2,598,960} \approx 0.001965.$$

(f) **Full House**

Task 1: Select 1 from 13 ranks for the three of a kind, $n_1 = 13$.

Task 2: Select 3 from 4 cards for the three of a kind, $n_2 = \binom{4}{3} = 4$.

Task 3: Select 1 from 12 remaining ranks for the two of a kind, $n_3 = 12$.

Task 4: Select 2 from 4 cards for the two of a kind, $n_4 = \binom{4}{2} = 6$.

There are

$$N = n_1 \times n_2 \times n_3 \times n_4 = 13 \times 4 \times 12 \times 6 = 3,744$$

such selections, so

$$P(\text{ Full House }) = \frac{3,744}{2,598,960} \approx 0.001441.$$

(g) **Four of a Kind**

Task 1: Select four cards of one rank, $n_1 = 13$.

Task 2: Select 1 of remaining card, $n_2 = 48$.

There are

$$N = n_1 \times n_2 = 13 \times 48 = 624$$

such selections, so

$$P(\text{ Four of a Kind }) = \frac{624}{2,598,960} \approx 0.0002401.$$

(h) **Straight Flush**

Task 1: Select suit: $n_1 = 4$.

Task 2: Select sequence, excluding 10,. . .,A: $n_2 = 9$.
There are
$$N = n_1 \times n_2 = 36$$
such selections, so
$$P(\text{ Straight Flush }) = \frac{36}{2,598,960} \approx 1.385 \times 10^{-5}.$$

(i) **Royal Flush**
Task 1: Select suit: $n_1 = 4$.
There are
$$N = n_1 = 4$$
such selections, so
$$P(\text{ Royal Flush }) = \frac{4}{2,598,960} \approx 1.539 \times 10^{-6}.$$

Q17. In genetics, a genotype consists of two genes, each of which is one of (possibly) several types of alleles. When two organisms mate, each passes one allele, selected at random, to the offspring, forming that offspring's genotype. Suppose a gene of a type of flower has two alleles, `r` and `R`. A plant possessing genotype `rr`, `rR` or `RR` has white, pink or red petals, respectively. A trait like this, in which both alleles determine the trait, is called *codominant*.

(a) Suppose a white and pink flower produce offspring A. Give the color distribution of A (that is, the probability that A is a given color, for each color).

(b) Suppose that A mates with a pink flower to produce offspring B. Give the color distribution of B.

(c) Suppose that C is the offspring of two pink flowers, and that A mates with C to produce offspring D. Give the color distribution of D.

SOLUTION
The following table gives the offspring genotype probability distribution for each combination of parents:

		Offspring Genotype		
Parent 1	Parent 2	`rr`	`rR`	`RR`
`rr`	`rr`	1	0	0
`rr`	`rR`	1/2	1/2	0
`rr`	`RR`	0	1	0
`rR`	`rR`	1/4	1/2	1/4
`rR`	`RR`	0	1/2	1/2
`RR`	`RR`	0	0	1

(a) The probabilities are given in row 2: $P(\text{white}) = 1/2$, $P(\text{pink}) = 1/2$.

(b) Let G_A, G_B be the genotypes of A and B. Using the law of total probability we have

$$\begin{aligned}
P(G_B = \texttt{rr}) &= P(G_B = \texttt{rr} \mid G_A = \texttt{rr})P(G_A = \texttt{rr}) + P(G_B = \texttt{rr} \mid G_A = \texttt{rR})P(G_A = \texttt{rR}) \\
&\quad + P(G_B = \texttt{rr} \mid G_A = \texttt{RR})P(G_A = \texttt{RR}) \\
&= \left[\frac{1}{2} \cdot \frac{1}{2}\right] + \left[\frac{1}{4} \cdot \frac{1}{2}\right] + 0 = \frac{3}{8} \\
P(G_B = \texttt{RR}) &= P(G_B = \texttt{RR} \mid G_A = \texttt{rr})P(G_A = \texttt{rr}) + P(G_B = \texttt{RR} \mid G_A = \texttt{rR})P(G_A = \texttt{rR}) \\
&\quad + P(G_B = \texttt{RR} \mid G_A = \texttt{RR})P(G_A = \texttt{RR}) \\
&= 0 + \left[\frac{1}{4} \cdot \frac{1}{2}\right] + 0 = \frac{1}{8}.
\end{aligned}$$

Then $P(G_B = \texttt{rR})$ is obtainable from the other probabilities, giving $P(\text{white}) = 3/8$, $P(\text{pink}) = 1/2$, $P(\text{red}) = 1/8$.

(c) It can be argued that the answers to (b) and (c) must be the same. In (b) the genetic contribution from A's mate comes from $\texttt{rR}$. In (c) it comes from one of C's parents, both of which are $\texttt{rR}$. So there is no difference, statistically. This is an acceptable answer.

The direct solution is as follows. Let G_C, G_D be the genotypes of C and D. Using the law of total probability we have (noting that $P(G_A = \texttt{RR}) = 0$):

$$\begin{aligned}
P(G_D = \texttt{rr}) &= P(G_D = \texttt{rr} \mid G_A = \texttt{rr}, G_C = \texttt{rr})P(G_A = \texttt{rr}, G_C = \texttt{rr}) \\
&\quad + P(G_D = \texttt{rr} \mid G_A = \texttt{rr}, G_C = \texttt{rR})P(G_A = \texttt{rr}, G_C = \texttt{rR}) \\
&\quad + P(G_D = \texttt{rr} \mid G_A = \texttt{rr}, G_C = \texttt{RR})P(G_A = \texttt{rr}, G_C = \texttt{RR}) \\
&\quad + P(G_D = \texttt{rr} \mid G_A = \texttt{rR}, G_C = \texttt{rr})P(G_A = \texttt{rR}, G_C = \texttt{rr}) \\
&\quad + P(G_D = \texttt{rr} \mid G_A = \texttt{rR}, G_C = \texttt{rR})P(G_A = \texttt{rR}, G_C = \texttt{rR}) \\
&\quad + P(G_D = \texttt{rr} \mid G_A = \texttt{rR}, G_C = \texttt{RR})P(G_A = \texttt{rR}, G_C = \texttt{RR}) \\
&= \left[1 \cdot \frac{1}{2} \cdot \frac{1}{4}\right] + \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right] + 0 + \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4}\right] + \left[\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2}\right] + 0 \\
&= \frac{3}{8}
\end{aligned}$$

and

$$\begin{aligned}
P(G_D = \texttt{RR}) &= P(G_D = \texttt{RR} \mid G_A = \texttt{rr}, G_C = \texttt{rr})P(G_A = \texttt{rr}, G_C = \texttt{rr}) \\
&\quad + P(G_D = \texttt{RR} \mid G_A = \texttt{rr}, G_C = \texttt{rR})P(G_A = \texttt{rr}, G_C = \texttt{rR}) \\
&\quad + P(G_D = \texttt{RR} \mid G_A = \texttt{rr}, G_C = \texttt{RR})P(G_A = \texttt{rr}, G_C = \texttt{RR}) \\
&\quad + P(G_D = \texttt{RR} \mid G_A = \texttt{rR}, G_C = \texttt{rr})P(G_A = \texttt{rR}, G_C = \texttt{rr}) \\
&\quad + P(G_D = \texttt{RR} \mid G_A = \texttt{rR}, G_C = \texttt{rR})P(G_A = \texttt{rR}, G_C = \texttt{rR}) \\
&\quad + P(G_D = \texttt{RR} \mid G_A = \texttt{rR}, G_C = \texttt{RR})P(G_A = \texttt{rR}, G_C = \texttt{RR}) \\
&= 0 + 0 + 0 + 0 + \left[\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2}\right] + \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4}\right] \\
&= \frac{1}{8}
\end{aligned}$$

Then $P(G_D = \texttt{rR})$ is obtainable from the other probabilities, giving $P(\text{white}) = 3/8$, $P(\text{pink}) = 1/2$, $P(\text{red}) = 1/8$.