DSC 462 HW#2, Kefu Zhu

Question 1

(a) $S_x = \{2, 3, 4, 5, 6, 7\}$

(b)(c)

Game Process:

(1) The 1st number (6 possible choices)

(2) The 2nd number

- The 2nd number is the same as the 1st number (1 possible choices) $\rightarrow X = 2, P(X = 2) = \frac{6 \cdot 1}{6^2}$
- The 2nd number is not the same as the 1st number (5 possible choices)

(3) The 3rd number

- The 3rd number is the same as one of the first 2 numbers (2 possible choices) $\rightarrow X = 3, P(X = 3) = \frac{(6 \cdot 5) \cdot 2}{6^3}$
- The 3rd number is not the same as any of the first 2 numbers (4 possible choices)

...

Therefore, we have the following calculation

$P(X = 2) = \frac{(6) \cdot 1}{6^2} = \frac{1}{6}$

$P(X = 3) = \frac{(6 \cdot 5) \cdot 2}{6^3} = \frac{5}{18}$

$P(X = 4) = \frac{(6 \cdot 5 \cdot 4) \cdot 3}{6^4} = \frac{5}{18}$

$P(X = 5) = \frac{(6 \cdot 5 \cdot 4 \cdot 3) \cdot 4}{6^5} = \frac{5}{27}$

$P(X = 6) = \frac{(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) \cdot 5}{6^6} = \frac{25}{324}$

$P(X = 7) = \frac{(6!) \cdot 6}{6^7} = \frac{5}{324}$

Sanity check: $P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)$
$=\frac{1}{6}+\frac{5}{18}+\frac{5}{18}+\frac{5}{27}+\frac{25}{324}+\frac{5}{324}=1$

Since we already have $P(X=i)$ for every $i \in S_x$, we can easily calculate every $P(X>i)$

$P(X>2)=1-P(X\leq 2)=1-P(X=2)=1-\frac{1}{6}\approx 0.83$

$P(X>3)=1-P(X=2)-P(X=3)=1-\frac{1}{6}-\frac{5}{18}\approx 0.56$

$P(X>4)=1-P(X=2)-P(X=3)-P(X=4)=1-\frac{1}{6}-\frac{5}{18}-\frac{5}{18}\approx 0.28$

$P(X>5)=P(X=6)+P(X=7)=\frac{25}{324}+\frac{5}{324}\approx 0.09$

$P(X>6)=P(X=7)\approx 0.02$

$P(X>7)=0$

Question 2

(a)

$$f(x)=\begin{cases}-c(x-2),\ x\in(0,2)\\ c(x-2),\ x\in[2,4)\\ 0,\ otherwise\end{cases}$$

$\sum p(x)=1=\int_0^2 -c(x-2)\mathrm{d}x+\int_2^4 c(x-2)\mathrm{d}x$

$=-c(\frac{x^2}{2}-2x\ |_0^2)+c(\frac{x^2}{2}-2x\ |_2^4)$

$=-c(2-4)+c[(8-8)-(2-4)]$

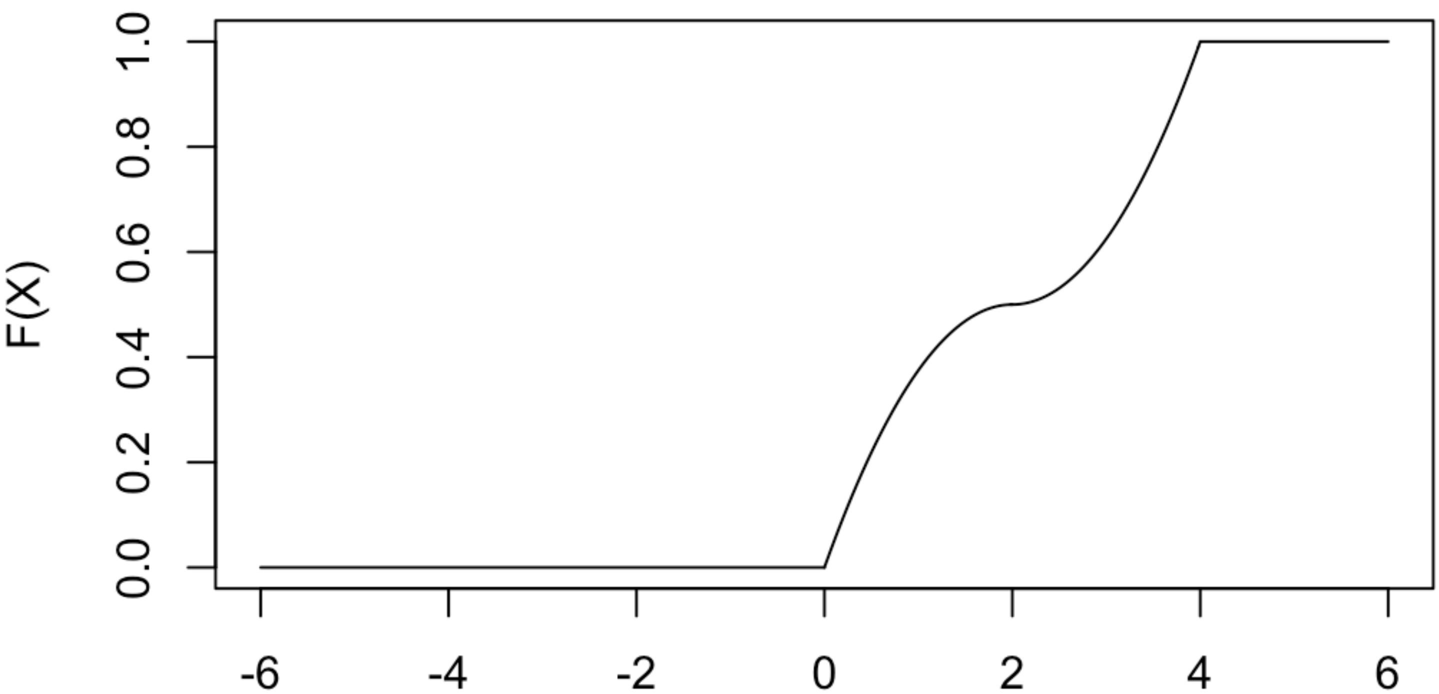
$=4c\Rightarrow c=\frac{1}{4}$

(b)

$$\because f(x)=\begin{cases}-\frac{1}{4}x+\frac{1}{2},\ x\in(0,2)\\ \frac{1}{4}x-\frac{1}{2},\ x\in[2,4)\\ 0,\ otherwise\end{cases}$$

$$\therefore F(x) = \begin{cases} 0,\ x \in (-\infty, 0] \\ \int_0^x -\frac{1}{4}x + \frac{1}{2}\mathrm{d}x = \frac{x}{2} - \frac{x^2}{8},\ x \in (0, 2) \\ (\int_0^2 -\frac{1}{4}x + \frac{1}{2}\mathrm{d}x) + (\int_2^x \frac{1}{4}x - \frac{1}{2}\mathrm{d}x) = 1 + \frac{x^2}{8} - \frac{x}{2},\ x \in [2, 4) \\ 1,\ x \in [4, +\infty) \end{cases}$$

```r
# F(x) for x in (0,2)
f1 = function(x) {x/2 - x^2/8}
# F(x) for x in (2,4)
f2 = function(x) {1 + x^2/8 - x/2}
# Plot F(x) in (0,2)
plot(f1, from = 0, to = 2, xlim = c(-6,6), ylim = c(0,1), xlab = 'X', ylab = 'F(X)')
# Plot F(x) in (2,4)
plot(f2, from = 2, to = 4, add = TRUE)
# Add F(x) = 0 from -infinity to 0
segments(-6,0,0,0)
# Add F(x) = 1 from 4 to +infinity
segments(4,1,6,1)
```



Question 3

$var(S_N) = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i<y} \sigma_{ij}$

$\because \sigma_i^2 = E[X_i^2] - E[X_i]^2 = \frac{1}{N} - (\frac{1}{N})^2 = \frac{1}{N} \cdot (1 - \frac{1}{N})$

$\because \sigma_{ij} = E[X_i X_j] - E[X_i]E[X_j] = 1 \cdot \frac{1}{N}\frac{1}{N-1} - \frac{1}{N}\frac{1}{N} = \frac{1}{N^2(N-1)}$

$\therefore var(S_N) = \sum_{i=1}^{n} \frac{1}{N} \cdot (1 - \frac{1}{N}) + 2\sum_{i,j=1,\, i<y}^{n} \frac{1}{N^2(N-1)} = 1 + 2 \cdot (\frac{N(N-1)}{2}) \cdot \frac{1}{N^2(N-1)} = 1$

Question 4

(a)

Define:

1. $X =$ the test score of an individual
2. $N =$ the number of candidates who passed the test

$P(X \geq 625) = P(Z \geq \frac{625-500}{75}) \approx 0.04779035$

```
# Calculate the P(Z >= (625-500)/75)
1 - pnorm(625, mean = 500, sd = 75) # 0.04779035
```

Define:

1. p = the probability that an individual passes the test = 0.04779035
2. q = the probability that an individual does not pass the test = $1 - p$ = 0.9522097

$P(N \geq 3) = 1 - P(N = 0) - P(N = 1) - P(N = 2)$

$= 1 - \binom{35}{0} \cdot q^{35} - \binom{35}{1} \cdot p \cdot q^{34} - \binom{35}{2} \cdot p^2 \cdot q^{33}$

$= 1 - 1 \cdot (0.9522097)^{35} - 35 \cdot 0.04779035 \cdot (0.9522097)^{34} - 595 \cdot (0.04779035)^2 \cdot (0.9522097)^{33}$

≈ 0.2333842

(b)

$P(N \geq 3) = 0.8 = 1 - \binom{35}{0} \cdot q^{35} - \binom{35}{1} \cdot p \cdot q^{34} - \binom{35}{2} \cdot p^2 \cdot q^{33}$

$\Rightarrow \binom{35}{0} \cdot q^{35} + \binom{35}{1} \cdot p \cdot q^{34} + \binom{35}{2} \cdot p^2 \cdot q^{33} - 0.2 = 0$

```
# Define function
f = function(x) {
choose(35,0)*(1-x)^35 +
choose(35,1)*x*(1-x)^34 +
choose(35,2)*(x^2)*(1-x)^33 - 0.2}
# Calculate the value of x
uniroot(f,c(0,1))
```

```
# Result
$root
[1] 0.1183283

$f.root
[1] -2.792853e-05

$iter
[1] 9

$init.it
[1] NA

$estim.prec
[1] 6.103516e-05
```

Therefore, $p \approx 0.1183283$. Based on the Z-score table, $Z \approx 1.18$

```
# Check the Z-score in R
qnorm(0.1183283, 0, 1, lower.tail = FALSE) # 1.183385
```

$\therefore Z \approx 1.18 = \frac{X-500}{75} \rightarrow X \approx 589$

Question 5

$F(M) = P(X_1 \leq M, X_2 \leq M, \ldots, X_n \leq M)$

$\because X_1, \ldots, X_n$ are i.i.d

$\therefore F_x(M) = P(X_1 \leq M)P(X_2 \leq M) \cdots P(X_n \leq M) = F(M)^k = (\frac{M-0}{c-0})^n = (\frac{M}{c})^n$

$\Rightarrow f(x) = \frac{\mathrm{d}}{\mathrm{d}M} F_x(M) = \frac{nM^{n-1}}{c^n}$

Then, $E(M) = \int_0^c M \cdot f(M) \mathrm{d}M$

$= \int_0^c M \cdot \frac{n}{c^n} \cdot M^{n-1} \mathrm{d}M$

$= \frac{n}{c^n \cdot (n+1)} \cdot M^{n+1} \mid_0^c$

$= \frac{n}{n+1} \cdot c$