ECE440 - Introduction to Random Processes

Final Exam

December 16, 2018

Instructions:

- This is an **individual** take-home exam, **collaborations are not allowed**.
- You will need to use Matlab to answer some of the questions in this exam.
- You should prepare a report in electronic format with your answers including all necessary derivations, discussions, Matlab code, and plots.
- Write your answers concisely and clearly. Show all your work.
- The report should be submitted via email to: gmateosb@ece.rochester.edu
- **The submission deadline is 10 pm ET, Wednesday December 19, 2018.**
- Late submissions will not be accepted.
- Perfect score: 100 (out of 105, extra points are bonus points).
- This exam has 4 numbered pages.

GOOD LUCK!

Determining the number of base stations to provide service in cellular communications. A cellular communication system like the one providing service to your cellphone divides the area of service into small units called cells. Each of these cells is serviced by a base station (for which we use the unsightly abbreviation BS). There are two reasons for this arrangement, the decay of electromagnetic power with distance and the scarcity of electromagnetic spectrum. Power decay is the reason why there are places in which your phone doesn't work and you get a no-service message, while spectrum scarcity is the reason why there are times at which you cannot place a call and get a busy signal. Power decay is the most important concern in sparsely populated areas where each BS has to cover a large area. Spectrum scarcity takes precedence in densely populated areas where a large number of customers are expected to require service in a small area.

The effect of having a limited amount of bandwidth is that there is a limit in the number of calls that a BS can handle. Exactly what this limit is and what is the system's behavior close to this limit, depends on the type of technology used. For our purposes suffices to say that there is a maximum number K of calls that the system can handle. A problem that telecommunication systems engineers have to solve is to determine the need to subdivide existing cells utilizing statistical traffic information collected by BSs.

To solve this problem assume that customers behave independently from where it follows that the time between call requests T_c is exponentially distributed with mean $1/\lambda$, i.e.,

$$T_c \sim \exp(\lambda). \tag{1}$$

If there are no channels available, something that happens when K calls are already established, service is denied and the customer's request to establish a call is denied. Otherwise a call is established and the channel is assigned to the customer for the duration of the call. The duration of calls, T_d, is random and modeled as exponentially distributed with parameter μ, i.e.,

$$T_d \sim \exp(\mu). \tag{2}$$

This exam is roughly divided in two sections. In the first section, which comprises parts *A-I* you are asked to build and analyze a stochastic model for the placement of calls in the service area of a BS. In the second section, which comprises parts *J* and *K*, you are asked to solve the problem of deciding when to add a new BS.

A) Departure process (5 points). We say that a departure occurs whenever a call is completed. Fix a given time t and let $1 \leq k \leq K$ be the number of calls in service at time t. Let $t + T_{di}$, with $1 \leq i \leq k$, be the random time at which customer i finishes her call. According to (2) T_{di} is exponentially distributed with parameter μ. Denote by T_k the random time until the next departure. The probability distribution of T_k is exponential with parameter $k\mu$. Write T_k as a function of the random times T_{di} and hence explain why the latter statement is true.

B) Four simple questions on the departure process (10 points). Given that there are k calls established, what is the probability that Customer 1 will be first to complete his call. Customer i has been talking for $s_i = 2$ minutes (mins.), while Customer j has been doing so for $s_j = 10$ mins. What is the probability of Customer i hanging up before Customer j? If $1/\mu = 3$ mins., what is $\mathrm{P}\left[T_{di} > 3 \text{ mins.}\right]$? What about $\mathrm{P}\left[T_{di} > 3 \text{ mins.} \,\middle|\, T_{di} > 2 \text{ mins.}\right]$?

C) Continuous-time Markov chain (CTMC) model (10 points). The number $X(t)$ of calls established at time t can be modeled as a CTMC with states $0 \leq k \leq K$. Explain why and specify the transition rates ν_k out of state k and the transition probabilities P_{ij}. Recall that the P_{ij} denote the probability of going from state i to state j given that the CTMC is transitioning out of state i. Notice that most of these transition probabilities are null.

D) Alternative CTMC representation (5 points). Give expressions for the transition rates q_{ij} from state i to state j. Notice that most of these transition rates are null. Draw a transition diagram.

E) Embedded Markov chain (MC) and ergodicity of the CMTC (5 points). Specify the embedded discrete-time MC associated with the CTMC $X(t)$. Explain why the CTMC $X(t)$ is ergodic.

F) System simulation (20 points). Write a function to simulate the placing of calls in the service area of a BS. Inputs to the function are the call rate λ, average call duration $1/\mu$, maximum number of channels K and a time $t_{\max}$ during which the system is simulated. The outputs are a vector of transition times $\mathbf{t}$ and a vector $\mathbf{X}$ with the corresponding values of the number of calls in service. Run your simulation for call rate $\lambda = 25$ calls/min., average call duration $1/\mu = 56$ seconds (s), number of available channels $K = 32$ and time $t_{\max} = 30$ mins $= 1800\ s$. Plot a realization of $X(t)$ for $0 \leq t \leq t_{\max}$.

G) Limit distribution (10 points). Define the limit distribution of the CTMC as the set of probabilities P_k of finding the CTMC in state k, for t sufficiently large, i.e.,

$$P_k := \lim_{t \to \infty} P_{ik}(t) \tag{3}$$

where $P_{ik}(t) = \text{P}\left[X(t+s) = k \,\middle|\, X(s) = i\right]$ is the transition probability function of the CTMC. Find the probabilities P_k for all $0 \leq k \leq K$. A good approach is to write down the balance equations, then express all probabilities in terms of P_0 and finally use the fact that $\sum_{k=0}^{K} P_k = 1$.

H) Ergodic limits (5 points). As is done with discrete-time MCs it is possible to relate the limit probabilities in (3) with the average amount of time spent visiting state k. Define then the ergodic limits

$$\bar{p}_k := \lim_{t\to\infty} \bar{p}_k(t) := \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{I}\left\{X(s) = k\right\} ds. \tag{4}$$

As is the case in discrete time, the average times in (4) are more important than the probabilities in (3) because they claim something for one realization of the process, whereas the probabilities in (3) are a metric across all realizations of the random process. In most cases, both of this quantities coincide, i.e,

$$\bar{p}_k := \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{I}\left\{X(s) = k\right\} ds = P_k, \qquad \text{almost surely.} \tag{5}$$

Explain the conditions under which (5) holds true and discuss if they are valid for the problem considered here. Write down the values of $\bar{p}_k$ for $0 \leq k \leq K$.

I) Approximating P_k using a simulation (10 points). For the same parameters of Part *F* (except for the value of $t_{\max}$ that you are asked to choose), compute and plot an approximation to the probabilities in (3) using your simulation code. You can choose to answer this question using a single run of the simulation, if possible at all, or using more than one run. What is the best option? Whatever your option you have to choose $t_{\max}$ and/or the number of runs to guarantee that your approximation has an accuracy of at least 10^{-3} for all probabilities larger than 10^{-2}. Present a plot of your choice to demonstrate that (5) holds.

J) Blocked-call probability (5 points). As a first step in determining the need to add a new cell or not, you are asked to compute the probability that a customer is denied service. This is referred to as the blocked-call probability P_b. Express P_b in terms of λ, μ and K.

K) Determining the need to add a new BS (20 points). Whenever a user makes a call request, the BS logs the attempt in a database. Typically, the BS reports an aggregate metric like "number of call requests" at periodic intervals, say every half hour. The information you are given as a system engineer is a large database containing the number of call requests at half hour intervals during the last year. I have done part of the work for you, which is the selection of the largest 10 values:

Date	**Time**	**Call attempts**	**Date**	**Time**	**Call attempts**
12/24	22:00 - 22:30	1,498	12/24	22:30 - 23:00	1,390
11/25	17:00 - 17:30	1,134	12/24	23:00 - 23:30	1,127
11/25	17:30 - 18:00	1,109	10/13	16:00 - 16:30	913
9/15	17:30 - 18:00	892	8/18	17:00 - 17:30	872
6/13	16:00 - 16:30	865	3/9	17:30 - 18:00	851

A typical design criteria is to discard the two busiest *days* of the year. Of the remaining values discard the two largest. The next largest value is your design target. The company plans a 5% increase in traffic for the upcoming year, which you add to your design target. For this value of number of calls during 30 minutes, you estimate the target arrival rate λ. For this value of λ the

BS is expected to provide service so that the blocked-call probability P_b is smaller than 0.02. If the average call duration is $1/\mu = 56$ s and the number of available channels is $K = 32$. Do you need to add a new BS? Justify your answer.

Addendum 1

I have used a cellular system as an example, but the problem you just solved appears in many different contexts. Staying close to communications this type of service dimensioning is also needed to determine the number of customer representatives in a call center. A minor variation would tell you about the shelving of products in a supermarket or of parts in a factory.

Addendum 2

On a different note, the fact that we discard the days with the largest number of attempted calls might let you understand why it is pretty much impossible to place a call on Christmas, Thanksgiving, or at the end of a football game. Still, the system is dimensioned for a very busy half hour of a very busy day. Most of the days and times, the BS is grossly over-dimensioned. This problem is common to all utilities, most notably to the production and distribution of electric energy. The power capacity installed has to be able to support the most demanding time of the most demanding day of the year – most likely a hot summer day. This is one of the most important limitations of renewable energies. Because their availability cannot be guaranteed – the sun might not be shining or the wind might not be blowing –, renewable energy requires conventional energy as backup, consequently duplicating investment. Here is another CTMC that you might want to study. A conclusion you will find is that you want to pool renewable energy from different sources and different geographical areas. This is why you hear about the need to develop a national "power superhighway."

Time estimate

To complete this exam I estimate that the total amount of time required is roughly 8 hours. A breakdown by parts is the following:

Problem part	**Time for execution**	**Problem part**	**Time for execution**
Problem comprehension	30 mins.	G	60 mins.
A	15 mins.	H	15 mins.
B	15 mins.	I	45 mins.
C	15 mins.	J	30 mins.
D	15 mins.	K	45 mins.
E	15 mins.	Report preparation	120 mins.
F	60 mins.		