Homework 1 - Introduction

1) Lower bounded random walk. Consider a game in which players bet \$1 to win \$1 with probability p and loose their bets with probability $q = 1 - p$. The wealth of a player as a function of time is a random process. If the player's wealth at time t is $w(t)$ (which denotes a realization of the random variable $W(t)$), the wealth at time $t+1$ is either $w(t)+1$ or $w(t)-1$. Moreover, the probability of the wealth increasing to $w(t)+1$ is p and the probability of the wealth decreasing to $w(t)-1$ is q. We write this as

$$\begin{aligned} \mathrm{P}\left[W(t+1) = w(t)+1 \,\middle|\, W(t) = w(t)\right] &= p, \\ \mathrm{P}\left[W(t+1) = w(t)-1 \,\middle|\, W(t) = w(t)\right] &= q. \end{aligned} \tag{1}$$

The first equation, e.g., is read as "the probability of $W(t+1)$ taking the value $w(t)+1$, given $W(t) = w(t)$ is p." The expression in (1) is true as long as $W(t) \neq 0$. When $W(t) = 0$ the gambler is ruined and $W(t+1) = 0$. A rather sophisticated, yet sometimes useful way of expressing this fact is

$$\mathrm{P}\left[W(t+1) = 0 \,\middle|\, W(t) = 0\right] = 1. \tag{2}$$

We saw in class that if $p > 1/2$ then it is likely that the sample paths $w(t)$ of the random process diverge making this a rather good game to play. In this exercise p can take any value. This process can be called a lower bounded random walk. Wealth can be reinterpreted as position on a line and wealth variations as steps taken randomly to left and right. The origin is home, in that if the walker reaches 0 it stays there. It is asked that:

A) Simulation of a process realization. Write a function that accepts as parameters the probability p, the initial wealth $W(0) = w_0$ and a maximum number of bets T. The function returns a vector of length at most $T+1$ containing the wealth's history $w(0), \ldots, w(T)$ randomly computed according to the probabilities in (1) and (2). If the wealth is depleted at time $t < T$, that is, if $w(t) = 0$ for some $t < T$, the function returns a vector of length $t+1$ with the wealth's history up to time t, i.e., $w(0), \ldots, w(t)$. Optionally, you can also return a boolean variable to distinguish between a run that resulted in a broken player and one that did not. This might be useful for parts B-E. Show plots with simulated processes for $w_0 = 20$, $T = 10^3$ and $p = 0.25$, $p = 0.5$ and $p = 0.75$.

B) Probability of reaching home. Fixing $p = 0.55$ and $w_0 = 10$ compute the probability $B(p, w_0)$ of eventually reaching home (going broke in the betting context), that is the probability of having $W(t) = 0$ for some t. Notice that because once $W(t) = 0$ wealth stays at 0 this probability can be written as the limit

$$B(p, w_0) = \lim_{t \to \infty} \mathrm{P}\left[W(t) = 0 \,\middle|\, W(0) = w_0\right]. \tag{3}$$

Strictly speaking, you would need to run the simulation forever to make sure the gambler does not run out of money. However, you can truncate simulations at time $T = 100$ for this exercise. With this approximation you would be aiming to compute the probability of reaching home between times 0 and T, which we assume approximates the probability of reaching home between times 0 and ∞ reasonably well. Put differently, we are assuming that $\mathrm{P}\left[W(T) = 0 \,\middle|\, W(0) = w_0\right]$ for $T = 100$ is a good approximation of the limit in (3). To estimate $\mathrm{P}\left[W(T) = 0 \,\middle|\, W(0) = w_0\right]$ we run the simulation code of part A multiple times. Each of these runs results in a wealth path $w_n(t)$, we then define the indicator function $\mathbb{I}\{w_n(T) = 0\}$ which equals 1 if wealth at time T is $w_n(T) = 0$ and 0 if not. The probability of reaching home is then estimated as (N is the number of simulations ran)

$$\hat{B}_N(p, w_0) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}\{w_n(T) = 0\}. \tag{4}$$

The expression in (4) is just the average number of times home was reached across all experiments. The function $\mathbb{I}\{w_n(T) = 0\}$ is called the indicator function of the event $w_n(T) = 0$ because it "indicates" the event by taking the value 1.

To compute $\hat{B}_N(p, w_0)$ you need to decide on a number of experiments N. The more experiments N you run the more accurate your estimation. Alas, the larger you need to wait. Report your probability estimate and the number of experiments N used. Explain your criteria for selecting N.

C) Probability of reaching home as a function of initial wealth. We want to study the probability of reaching home as a function of initial wealth. Fix $p = 0.55$ and vary initial wealth between $w_0 = 1$ and $w_0 = 20$. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of initial wealth. The number of experiments N run to compute probability estimates for different initial wealths need not be the same.

D) Probability of reaching home as a function of p. The goal is to understand the variation of the probability of reaching home for different values of the probability p. Fix $w_0 = 10$ and vary p between 0.3 and 0.7 – increments 0.02 should do. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of p. You should observe a fundamentally different behavior for $p < 1/2$ and $p > 1/2$. Comment on that.

E) Time to reach home. Fix $p = 0.4$. With this value of p it is possible to see that gamblers eventually deplete their wealth independently of their initial wealth w_0. This is something remarkable, despite the process being random it is possible to say that $W(t)$ eventually becomes 0. This needs to be qualified, though. Unlikely as it may be there is a chance of winning all hands. Of course, the probability of this happening becomes smaller as the gambler plays more hands. What we can say about a lower bounded random walk is that with probability 1, wealth $W(t)$ approaches 0 as t grows. More formally, the limit $\lim_{t\to\infty} W(t)$ satisfies

$$\mathrm{P}\left[\lim_{t\to\infty} W(t) = 0\right] = 1. \tag{5}$$

We say that $\lim_{t\to\infty} W(t) = 0$ almost surely. Different wealth paths are possible, but almost all of them result in a broken gambler. If we think of probabilities as measuring the likelihood of an event, the measure of the event $W(t) \neq 0$ is asymptotically null. An important quantity here is the time at which $W(t) = 0$ for the first time which we can write as

$$T_0 = \min_t \big(W(t) = 0\big). \tag{6}$$

Estimate the probability distribution of T_0 and its average value.