Midterm Practice Problems - CSC 262 - Feb 2015

1. A restaurant has 30 patrons. The probability that today is a patron's birthday is 1/365. Assume that birthdays occur independently.

   (a) What is the expected number of birthdays?

   (b) If X is the number of birthdays, calculate $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, directly from the binomial distribution.

   (c) Use both the Poisson and normal distributions to approximate the same probabilities (use the correction method for the normal distribution).

2. Suppose that if an infection exposure risk is present in a school, any given child is infected with a probability of 10%, and that infections occur independently. In order to detect the presence of an exposure risk, N children are selected at random for testing. If at least one child tests positive, an exposure risk is assumed to be present.

   (a) What is the smallest value for N which will ensure a probability of at least 95% that a true exposure risk will be detected?

   (b) We are assuming that the number of positive test results has a binomial distribution. Should the children be sampled with replacement or without replacement? If N represents the entire school, is the number of infections still binomially distributed?

3. A coin is tossed independently 3 times. Calculate the probabilities of the following events:

   (a) Getting an odd number of heads.

   (b) Getting exactly 3 heads.

   (c) Getting a exactly 1 head OR exactly 1 tail.

   (d) Getting more than 1 head.

   (e) Getting 3 heads GIVEN that there are more than 1 heads.

4. A college will not consider applicants scoring below the 25th percentile on a certain placement test. Nationally, this test has a mean score of 700 and a standard deviation of 50. The scores are normally distributed.

   (a) What should the minimum test score for consideration be?

   (b) Suppose the scores of applicants to this college are normally distributed with mean 750 and standard deviation 30. What proportion fail to meet this minimum score?

   (c) If the college finally admits 5% of it's applications, solely on the basis of the test score, what score was the effective cut-off?

5. A standard deck consists of 52 cards with all combinations of 13 ranks and 4 suits. Suppose 5 cards are selected at random from the deck.

(a) What is the probability of getting 2 pairs?

(b) What is the probability of getting a flush (all cards of the same suit)?

6. (a) Suppose ϕ is a true density function on $\Re$. If a random variable X has density ϕ what is the density of $Z = (X - \alpha)/h$?

   (b) Let $\alpha_1, \ldots, \alpha_n$ be any numbers, and let $h > 0$. Verify that f_n defined as

$$f_n(x) = \frac{1}{nh} \sum_{i=1}^{n} \phi\left(\frac{x - \alpha_i}{h}\right) \tag{1}$$

   is a true density.

7. An urn contains N items. Of these items M are of type A. A sample of K items are drawn from the urn without replacement. All selections of K have equal probability of being selected. Let X be the number of items in the sample of type A.

   (a) What range of values can X take?

   (b) Give the probability distribution of X.

   (c) How does the probability distribution change if the selection is made with replacement?

   (d) To estimate N, the number of fish in a lake, an initial sample of M are caught, tagged, then released back into the lake. After a suitable period of time K fish are caught. In this second sample X fish have tags. Give a sensible estimator of N. What is its distribution?

8. Suppose X has a binomial distribution with parameters (n_1, p) and Y has a binomial distribution with parameters (n_2, p). Suppose X and Y are independent.

   (a) Derive the conditional probability $P(X = s \mid X + Y = t)$.

   (b) Derive the conditional probability $P(X + Y = t \mid X = s)$.

9. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables, and that X_i has an exponential density with rate λ_i. Show that

$$Y = \min_i X_i$$

   is an exponentially distributed random variable with rate $\lambda_T = \sum_{i=1}^n \lambda_i$, and that

$$P(Y = X_i) = \lambda_i / \lambda_T.$$