ECE 440, HW#10, Kefu Zhu

Question 1 Arbitrages

(A) Arbitrage with single booker

Based on the investment strategy, we have

$$\begin{cases} 6.6x - c > 0 \rightarrow x > \frac{1}{6.6} \cdot c \\ 8.1y - c > 0 \rightarrow y > \frac{1}{8.1} \cdot c \\ 1.2z - c > 0 \rightarrow z > \frac{1}{1.2} \cdot c \end{cases}$$

$$\Rightarrow x + y + z > \frac{1}{6.6} \cdot c + \frac{1}{8.1} \cdot c + \frac{1}{1.2} \cdot c \Leftrightarrow c > 1.11c$$

Since $c > 0$, such condition can never be met, an arbitrage is not possible

(B) Arbitrage with many bookers

Based on the odds provided in the question, the best strategy is to bet on country from booker with the highest odds.

That means bet France win from Booker 1, bet Brazil win from Booker 3 and bet Other win from Booker 2. Similar to part (A), we have

$$\begin{cases} 6.6x - c > 0 \rightarrow x > \frac{1}{6.6} \cdot c \\ 8.4y - c > 0 \rightarrow y > \frac{1}{8.4} \cdot c \\ 1.3z - c > 0 \rightarrow z > \frac{1}{1.3} \cdot c \end{cases}$$

$$\Rightarrow x + y + z > \frac{1}{6.6} \cdot c + \frac{1}{8.4} \cdot c + \frac{1}{1.3} \cdot c \Leftrightarrow c > 1.04c$$

Since $c > 0$, such condition can never be met, an arbitrage is still not possible

Question 2 Option pricing

(A) Derivation

Because for different Y_n where $Y_n = Y_{(n-1)h}(h)$, they belong to disjoint intervals of length h, so Y_n are i.i.d normals with mean μh and variance $\sigma^2 h$

(B) Determination of drift and volatility

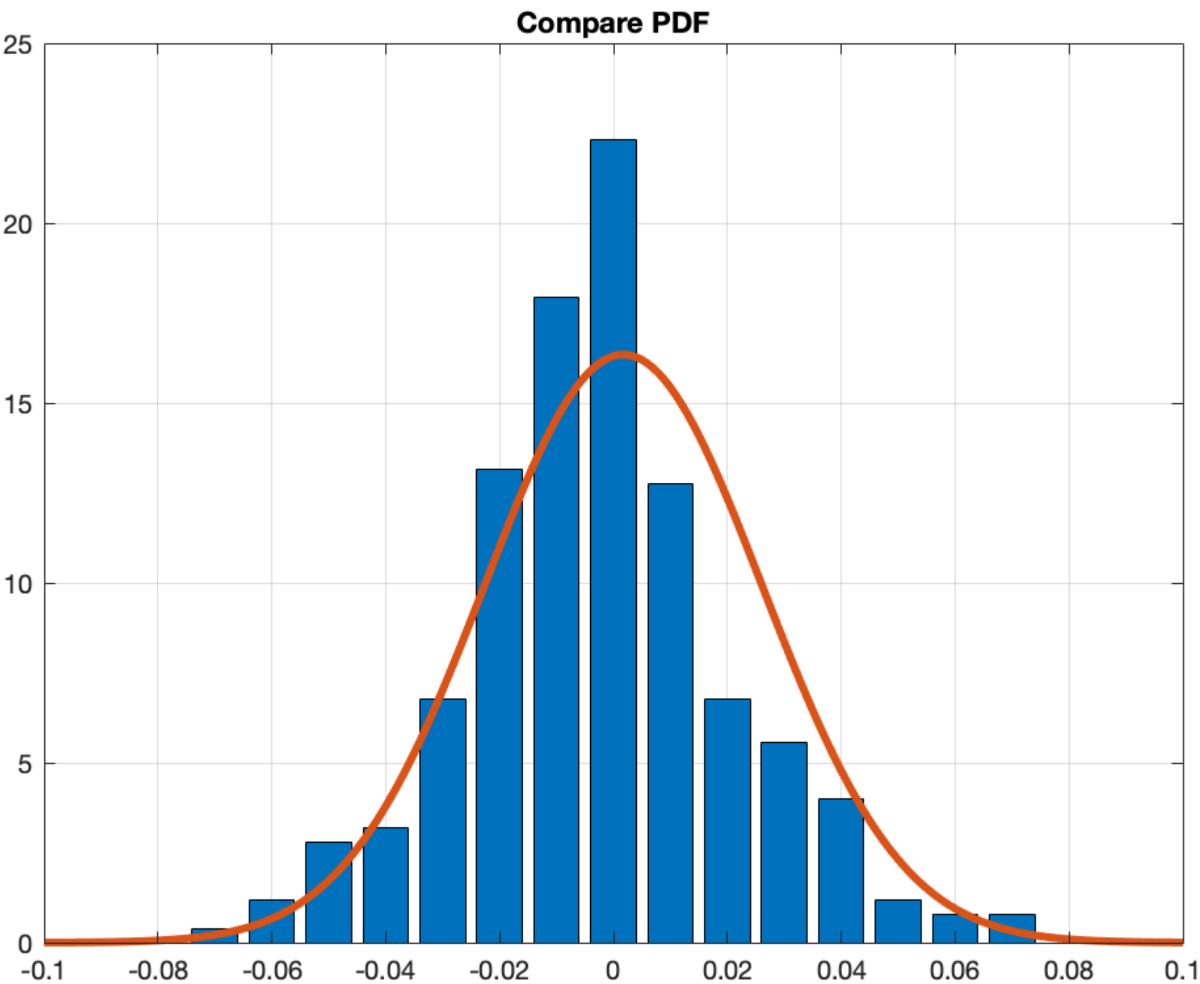
```
% Load data
cisco_stock_price
Z=log(close_price);
Y=Z(2:end)-Z(1:end-1);
N=length(Y);
h=1/365;

% Sample mean
mu_hat=sum(Y)/(N*h)
% Sample variance
sigma_sqr_hat=sum((Y-mu_hat*h).^2)/((N-1)*h)
```

The result from above matlab calculation is $\mu = 0.6275$ and $\hat{\sigma}^2 = 0.2174$

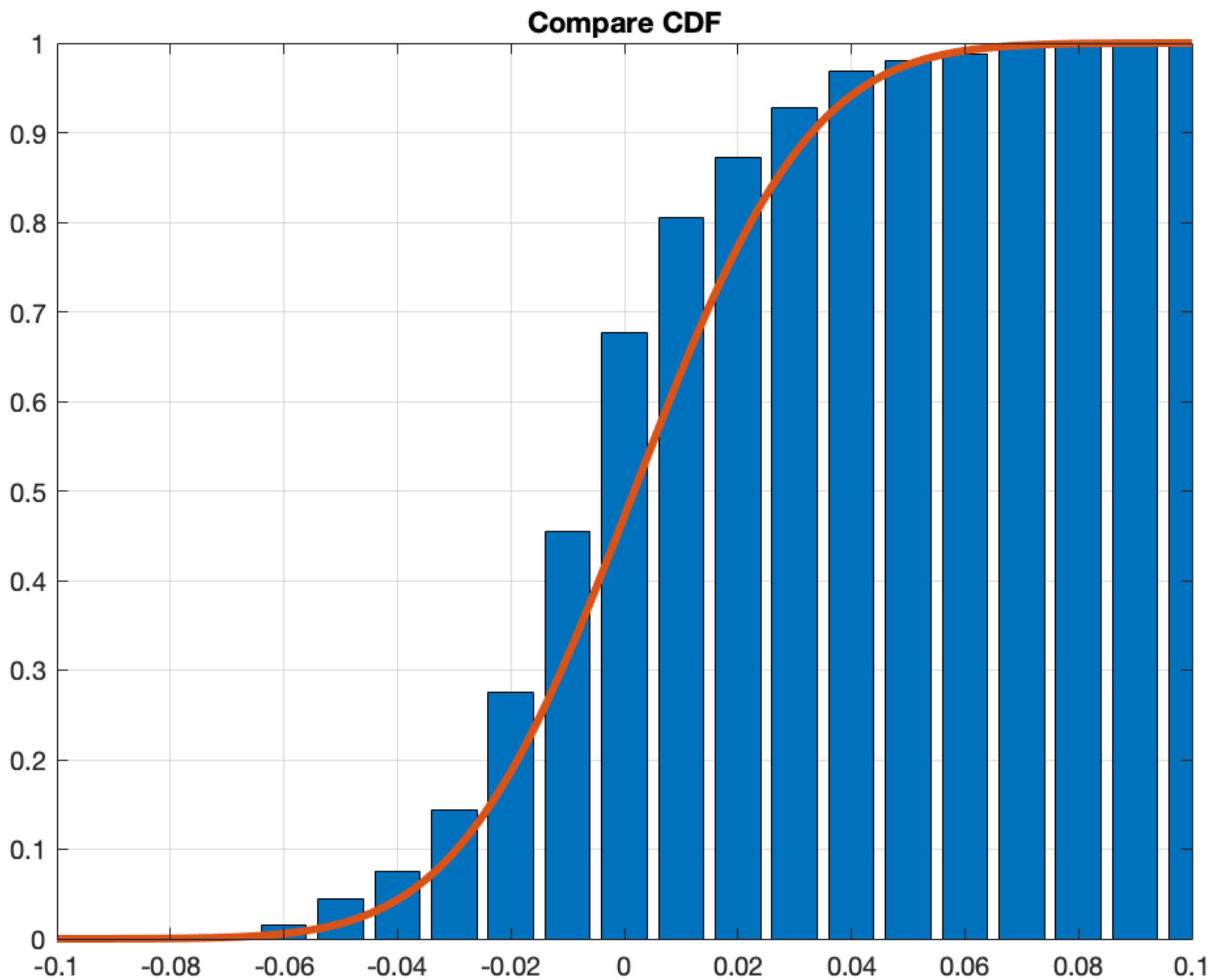
(C) Is geometric Brownian motion a good model

```
% Load CSCO data and set parameters
cisco_stock_price
Z = log(close_price);
Y = Z(2:end)-Z(1:end-1);
N = length(Y);
h = 1/365;
x = -0.1:0.01:0.1;
n_elements = histc(Y,x);
mu_hat = 0.6275;
sigma_sqr_hat = 0.2174;
```

```
% Compare pdf
figure(1)
bar(x,n_elements/N/0.01)
hold on
x_padded = -0.1:0.001:0.1;
plot(x_padded,normpdf(x_padded,mu_hat*h,sqrt(sigma_sqr_hat*h)), 'Linewidth', 3)
title('Compare PDF');
grid on
axis([-0.1,0.1,0,25])
```




```
% Compare cdf
figure(2)
c_elements = cumsum(n_elements)/N;
bar(x,c_elements)
hold on
x_padded = -0.1:0.001:0.1;
plot(x_padded,normcdf(x_padded,mu_hat*h,sqrt(sigma_sqr_hat*h)), 'Linewidth', 3)
title('Compare CDF');
grid on
axis([-0.1,0.1,0,1])
```



Answer: The geometric Brownian motion seems to model the evolution of CSCO stock price

(D) Expected return

Expected return = $E[\frac{e^{-\alpha t}X(t)}{X(0)} \mid X(0)] = e^{t\,(\hat{\mu}-\alpha+\frac{\hat{\sigma}^2}{2})}$

Given $\alpha = 3.75\%$ and $t = 1$, $E[\frac{e^{-\alpha t}X(t)}{X(0)} \mid X(0)] \approx 2.01$

$P[log(\frac{e^{-\alpha t}X(t)}{X(0)}) \geq 0.05 \mid X(0)] = P[Y(1) \geq 0.05 + 0.0375]$

$\because\ Y(1) \sim N(\hat{\mu}, \hat{\sigma}^2)\ \ \therefore P[Y(1) \geq 0.0875] \approx 0.88$

(E) Risk neutral measure

The risk neutral measure is a Brownian motion with $\mu = \alpha - \frac{\widehat{\alpha}^2}{2} \approx -0.07$ and $\sigma^2 = \widehat{\alpha}^2 \approx 0.22$

(F) Expected return for risk neutral measure

The expected discounted rate of return for an investment in CSCO is 0.

And the non-discounted rate of return is α

(G) Derive the Black-Scholes formula

By determining the price c, the Black-Scholes formula yields zero expected return with respect to the risk neutral measure so that there is no potential arbitrage/

The closed form expression is

$$c = X(0) \cdot \phi(\frac{log(K/X(0)) - \mu t}{\sqrt{\sigma^2 t}} - \sqrt{\sigma^2 t}) - e^{-\alpha t} \cdot K \cdot \phi(\frac{log(K/X(0)) - \mu t}{\sqrt{\sigma^2 t}})$$

(H) Determine option price

```
% Load CSCO data and set parameters
cisco_stock_price
Z=log(close_price);
Y=Z(2:end)-Z(1:end-1);
N=length(Y);
h=1/365;
mu_hat=sum(Y)/(N*h);
sigma_sqr_hat=sum((Y-mu_hat*h).^2)/((N-1)*h);
alpha=0.0375;

X_0=close_price(1,1);
EX=X_0*exp(mu_hat+sigma_sqr_hat/2);
K=[0.8,1,1.2]*EX;
a=(log(K/X_0)-(alpha-sigma_sqr_hat/2))/(sqrt(sigma_sqr_hat));
b=a-sqrt(sigma_sqr_hat);
Q_a=1-normcdf(a,0,1);
Q_b=1-normcdf(b,0,1);
c=X_0*Q_b-exp(-alpha)*K.*Q_a
```

By the calculation from code above, the price for $K = E[X(t)]$ is 0.2941, the price for $K = 1.2E[X(t)]$ is 0.1243, and the price for $K = 0.8E[X(t)]$ is 0.7190.

For $K = 0.8E[X(t)]$, since it has the least risk because it is more likely to see $K < X(t)$ and obtain a gain, it has the most expensive option price.