DSC 462 HW#1 Kefu Zhu

Q1

(a)

$$P(\bigcup_{i=1}^{3} A_i) = P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_2A_3)$$

(b)

Define:

- A_1 = The chosen integer from 1 to 105 inclusively is divisible by 2
- A_2 = The chosen integer from 1 to 105 inclusively is divisible by 9
- A_3 = The chosen integer from 1 to 105 inclusively is divisible by 13

$\because$

$$P(A_1) = \frac{105\ //\ 2}{105} = \frac{52}{105}, P(A_2) = \frac{105\ //\ 9}{105} = \frac{11}{105}, P(A_3) = \frac{105\ //\ 13}{105} = \frac{8}{105}$$

$$P(A_1A_2) = \frac{105\ //\ 18}{105} = \frac{5}{105}, P(A_1A_3) = \frac{105\ //\ 26}{105} = \frac{4}{105}, P(A_2A_3) = \frac{105\ //\ 117}{105} = \frac{0}{105}$$

$$\therefore P(\bigcup_{i=1}^{3} A_i) = P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_2A_3)$$

$$= \frac{52}{105} + \frac{11}{105} + \frac{8}{105} - \frac{5}{105} - \frac{4}{105} - \frac{0}{105} = \frac{62}{105}$$

Q2

(a)

Answer:

Consider four adjacent red birds as a whole, then the total number of possible arrangements equals to the value of A, where $A =$ number of blue birds sitting on the left side of red birds

$$A = \binom{7}{1} = 7$$

$\because B =$ Number of total possible arrangements $= \frac{10!}{6! \cdot 4!} = 210$

$$\therefore P\ (Four\ red\ birds\ are\ all\ adjacent) = \frac{A}{B} = \frac{1}{30}$$

(b)

In order to make sure no two red birds are adjacent, a subset of this line of birds must looks like: **red, blue, red, blue, red, blue, red**

Therefore, the scenario becomes similar to the previous question. Consider the formation above as a whole, then the total number of possible arrangements equals to the value of A, where $A =$ number of possible arrangements for the rest 3 blue birds

The rest 3 blue birds can be placed at any of the following 5 position

- Left side of the formation
- Right side of the formation
- Anywhere between two closest red birds
  - Between $1st$ red bird and the $2nd$ red bird
  - Between $2nd$ red bird and the $3rd$ red bird
  - Between $3rd$ red bird and the $4th$ red bird

$A = \binom{3+5-1}{5-1} = 35$

$\because B =$ Number of total possible arrangements $= \frac{10!}{6! \cdot 4!} = 210$

$\therefore P\,(no\ two\ red\ birds\ are\ adjacent) = \frac{A}{B} = \frac{35}{210} = \frac{1}{6}$

Q3

(a)

$P(All\ cards\ are\ face\ cards\ of\ a\ single\ color) = 2 \cdot (\frac{6}{52} \cdot \frac{5}{51} \cdot \frac{4}{50} \cdot \frac{3}{49} \cdot \frac{2}{48}) \approx 0.0000047$

(b)

- Pick one suit that appears twice: $\binom{4}{1} = 4$
- Pick two cards from that suit: $\binom{13}{2} = 78$
- Pick the rest 3 cards from the rest 3 suits: $\binom{13}{1}^3 = 2197$

$P(All\ suits\ are\ represented\ at\ least\ once) = \frac{4 \cdot 78 \cdot 2197}{\binom{52}{5}} \approx 0.2637$

(c)

- Pick five ranks: $\binom{13}{5} = 1287$
- Pick a color: $\binom{2}{1} = 2$
- Pick the suit of the card for every card in the five cards: 2^5

$P(All\ ranks\ are\ distinct,\ and\ of\ a\ single\ color) = \frac{1287 \cdot 2 \cdot 2^5}{\binom{52}{5}} \approx 0.03$

Q4

(a)

$P(R_1 \,|R_m) = \frac{1}{2} \cdot 2(1-q)q + q^2 = q$

(b)

$P(R_2)$ has four cases

- The mother is **rr** and father is **rR**: $\frac{1}{2} \cdot q^2 \cdot 2(1-q)q$
- The father is **rr** and mother is **rR**: $\frac{1}{2} \cdot q^2 \cdot 2(1-q)q$
- The mother is **rr** and father is **rr**: $1 \cdot q^2 \cdot q^2$
- The mother is **rR** and father is **rR**: $\frac{1}{2} \cdot \frac{1}{2} \cdot 2(1-q)q \cdot 2(1-q)q$

$P(R_2) = 2 \cdot [\frac{1}{2} \cdot q^2 \cdot 2(1-q)q] + [q^2 \cdot q^2] + [\frac{1}{2} \cdot \frac{1}{2} \cdot 2(1-q)q \cdot 2(1-q)q] = q^2$

$P(R_1 R_2)$ also has four cases

- The mother is **rr** and father is **rR**: $\frac{1}{2} \cdot \frac{1}{2} \cdot q^2 \cdot 2(1-q)q$
- The father is **rr** and mother is **rR**: $\frac{1}{2} \cdot \frac{1}{2} \cdot q^2 \cdot 2(1-q)q$
- The mother is **rr** and father is **rr**: $q^2 \cdot q^2$
- The mother is **rR** and father is **rR**: $\frac{1}{4} \cdot \frac{1}{4} \cdot 2(1-q)q \cdot 2(1-q)q$

$P(R_1 R_2) = 2 \cdot [\frac{1}{2} \cdot \frac{1}{2} \cdot q^2 \cdot 2(1-q)q] + [q^2 \cdot q^2] + [\frac{1}{4} \cdot \frac{1}{4} \cdot 2(1-q)q \cdot 2(1-q)q] = q^3 + \frac{1}{4}(1-q^2)q^2$

$P(R_1 | R_2) = \frac{P(R_1 R_2)}{P(R_2)} = \frac{q^3 + \frac{1}{4}(1-q^2)q^2}{q^2} = q + \frac{1}{4}(1-q^2)$

(c)

- $lim_{q \to 0} P(R_1 | R_m) = 0$
- $lim_{q \to 0} P(R_1 | R_2) = \frac{1}{4}$

Q5

(a)

(i) Proof of $P(\emptyset) = 0$

$\because P(S) = P(S \cup \{\cup_{i=1}^{\infty} \emptyset\})$, By Axiom 2, $P(S) = 1$, and the set $\emptyset$ is disjoint with all other sets, including $\emptyset$ itself

$\therefore$ By Axiom 3, $P(S) = P(S) + \sum_{i=1}^{\infty} \emptyset$

$\therefore 1 = 1 + \sum_{i=1}^{\infty} \emptyset$

$\therefore \sum_{i=1}^{\infty} \emptyset = 0 \to P(\emptyset) = 0$

(ii) Proof of $A \cap B = \emptyset$ implies $P(A \cap B) = P(A) + P(B)$

$\because A \cap B = \emptyset$

$\therefore A \text{ and } B \text{ are mutually exclusive}$

$\because$ the set $\emptyset$ is disjoint with all other sets, including $\emptyset$ itself

$\because P(\emptyset) = 0$

By Axiom 3, $P(A \cup B) = P(A \cup B \cup \{\cup_{i=1}^{\infty} \emptyset) = P(A) + P(B) + \sum_{i=1}^{\infty} \emptyset = P(A) + P(B)$

(iii) Proof of $P(A^c) = 1 - P(A)$

$\because P(S) = P(A) + P(A^c)$

By Axiom 2, $P(S) = 1$

$\therefore 1 = P(A) + P(A^c) \rightarrow P(A^c) = 1 - P(A)$

(iv) Proof of $A \subset B$ implies $P(A) \leq P(B)$

$P(B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$

$\because By\ Axiom\ 1,\ P(A^c \cap B) \geq 0$

$\therefore P(A) \leq P(B)$

(b)

(i)

$\because \overline{E_i} = E_i \cap E_{i-1}^c \cap E_{i-2}^c \cap \ldots \cap E_1^c$

$\therefore \forall A \in \overline{E_i}, A \in E_i \Rightarrow \overline{E_i} \subset E_i$

(ii)

$\forall m, n$ where $m < n$

We have $\overline{E_m} = E_m \cap E_{m-1}^c \cap \ldots E_1^c$ and $\overline{E_n} = E_n \cap E_{n-1}^c \cap \ldots E_1^c$

$\overline{E_n}$ can also be written as $\overline{E_n} = E_n \cap \ldots E_m^c \cap \ldots E_1^c$

$\forall A \in \overline{E_m}, A \in E_m$ and $\forall B \in \overline{E_n}, B \in E_m^c$

$\because E_m$ and E_m^c are mutually exclusive

$\therefore$ There is no pair of A and B such that $A = B$

$\therefore \forall m, n$ where $m < n$, $\overline{E_m}$ and $\overline{E_n}$ are mutually exclusive

$\Rightarrow$ The sets $\overline{E_1}, \overline{E_2}, \ldots$ are mutually exclusive

(iii)

$n = 1$

- $\cup_{i=1}^n E_i = E_1$
- $\cup_{i=1}^n \overline{E_i} = E_1$

$n = 2$

- $\cup_{i=1}^n E_i = E_1 \cup E_2$
- $\cup_{i=1}^n \overline{E_i} = (E_2 \cap E_1^c) \cup (\cup_{i=1}^1 \overline{E_i}) = (E_2 \cap E_1^c) \cup E_1 = E_2 \cup E_1$

$n = 3$

- $\cup_{i=1}^n E_i = E_1 \cup E_2 \cup E_3$
- $\cup_{i=1}^n \overline{E_i} = (E_3 \cap E_2^c \cap E_1^c) \cup (\cup_{i=1}^2 \overline{E_i}) = (E_3 \cap E_2^c \cap E_1^c) \cup (E_2 \cup E_1) = E_3 \cup E_2 \cup E_1$

...

$n = m$

- $\cup_{i=1}^{m} E_i = E_1 \cup E_2 \cup \ldots \cup E_m$
- $\cup_{i=1}^{m} \overline{E_i} = (E_m \cap E_{m-1}^c \cap E_{m-2}^c \cap \ldots \cap E_1^c) \cup (\cup_{i=1}^{m-1} \overline{E_i}) = (E_m \cap E_{m-1}^c \cap E_{m-2}^c \cap \ldots \cap E_1^c) \cup (E_{m-1} \cup E_{m-2} \cup \ldots \cup E_1) = E_m \cup E_{m-1} \cup \ldots \cup E_1$

$\therefore \forall m$, where $m \geq 1$, we have $\cup_{i=1}^{m} E_i = \cup_{i=1}^{m} \overline{E_i}$

As $m \to \infty$, $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} \overline{E_i}$

(c)

Proved in question (b)(iii), $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} \overline{E_i}$, therefore, $P(\cup_{i=1}^{\infty} E_i) = P(\cup_{i=1}^{\infty} \overline{E_i})$.

Since $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_i}$ are mutually exclusive (Proved in question (b)(ii)), by Axiom 3, $P(\cup_{i=1}^{\infty} \overline{E_i}) = \sum_{i=1}^{\infty} P(\overline{E_i})$

Proved in questions (b)(i), $\overline{E_i} \subset E_i, \forall i \geq 1$. Therefore, $P(\overline{E_i}) \leq P(E_i), \forall i \geq 1 \Rightarrow \sum_{i=1}^{\infty} P(\overline{E_i}) \leq \sum_{i=1}^{\infty} P(E_i)$

In conclusion, $P(\cup_{i=1}^{\infty} E_i) = P(\cup_{i=1}^{\infty} \overline{E_i}) = \sum_{i=1}^{\infty} P(\overline{E_i}) \leq \sum_{i=1}^{\infty} P(E_i)$

Simplify as $P(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} P(E_i)$

(d)

(1)

Based on the definition of $Q_i \to Q_i = P(\cap_{n=i}^{\infty} E_n^c)$

$\because P(\cap_{n=i}^{\infty} E_n^c) = 1 - P(\cup_{n=i}^{\infty} E_n)$

$\because P(\cup_{n=i}^{\infty} E_n) \leq \sum_{n=i}^{\infty} P(E_n)$ (Boole's Inequality proved in part c)

$\therefore Q_i = 1 - P(\cup_{n=i}^{\infty} E_n) \geq 1 - \sum_{n=i}^{\infty} P(E_n) \geq 1 - (\sum_{n=1}^{\infty} P(E_n) - \sum_{n=1}^{i} P(E_n))$

$\because \sum_{n=1}^{\infty} P(E_n) < \infty$

$\therefore \lim_{i \to \infty} Q_i = 1$

(2)

If $P(E_i) \leq \frac{c}{i^k}, c > 0, k > 1$, then

$\sum_{i=1}^{\infty} P(E_i) = c(1 + \frac{1}{2^k} + \frac{1}{3^k} + \ldots)$

$= c(1 + \int_1^{\infty} \frac{1}{x^k} \mathrm{d}x)$

$= c(1 + \frac{x^{-k+1}}{-k+1})|_i^{\infty}$

$= c + \frac{c}{1-k} = c \cdot \frac{k}{k-1} < \infty$

$\therefore \sum_{n=1}^{\infty} P(E_n) < \infty \to \lim_{i \to \infty} Q_i = 1$ (Proved in part 1)