ECE 440, HW8, Kefu Zhu

Question 7

(A)

Because all varialbes are integer based, and the range of $X(t)$ is between $X_{min} = 0$ and X_{max}, the states of the CTMC should be any integer between 0 and X_max, where X_{max} is nonnegative integer.

(B)

Because we have

- $T_p \sim exp(\lambda)$
- $T_c \sim exp(\alpha)$
- $T_d \sim exp(\beta)$

Follow the summary from the range table given in the question, it is obvious to show the foloowing v_x for different scenarios

Range	Possible Events	v_x
A	premium	λ
B	premium, claim paid at $X(t)$	$\lambda + \alpha$
C	preimum, claim	$\lambda + \alpha$
D	preimum, claim, dividend	$\lambda + \alpha + \beta$
E	claim, dividend	$\lambda + \beta$

(C)

(1) Possible transition states out of A:

The only possible transition out of this state is having a payment of premium $\rightarrow$ State 1

(2) Possible transition states out of B

- A payment of preimum: $x + 1$
- A claim paid at $X(t)$: $x - x = 0$

(3) Possible transition states out of C

- A payment of preimum: $x + 1$
- A payment of claim: $x - c$

(4) Possible transition states out of D

- A payment of preimum: $x + 1$
- A payment of claim: $x - c$
- A payment of divident: $x - d$

(5) Possible transition states out of E

- A payment of claim: $x - c$
- A payment of divident: $x - d$

(D)

Since we have alreadly listed all possible transition states for each range, it is easy to compute their transition probabilities given

- $T_p \sim exp(\lambda)$
- $T_c \sim exp(\alpha)$
- $T_d \sim exp(\beta)$

(1) Transition probabilites out of A:

- premium: $\frac{\lambda}{\lambda} = 1$

(2) Transition probabilites out of B

- premium: $\frac{\lambda}{\lambda+\alpha}$
- claim: $\frac{\alpha}{\lambda+\alpha}$

(3) Transition probabilites out of C

- premium: $\frac{\lambda}{\lambda+\alpha}$
- claim: $\frac{\alpha}{\lambda+\alpha}$

(4) Transition probabilites out of D

- premium: $\frac{\lambda}{\lambda+\alpha+\beta}$
- claim: $\frac{\alpha}{\lambda+\alpha+\beta}$
- divident: $\frac{\beta}{\lambda+\alpha+\beta}$

(5) Transition probabilites out of E

- claim: $\frac{\alpha}{\alpha+\beta}$
- divident: $\frac{\beta}{\alpha+\beta}$

(E)

cashflow.m

```
function [X,T]=cashflow(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
    index=1;
    X(index)=X_0;
    T(index)=0;
    while T(index)<T_max
        x=X(index);
        if x==0
            tau=exprnd(1/lambda);
            T(index+1)=T(index)+tau;
            X(index+1)=x+1;
        elseif 0<x && x<c
            tau=exprnd(1/(lambda+alpha));
            T(index+1)=T(index)+tau;
            u=rand;
            if u<(lambda/(lambda+alpha))
                X(index+1)=x+1;
            else
                X(index+1)=0;
            end
        elseif c<=x && x<X_r
            tau=exprnd(1/(lambda+alpha));
            T(index+1)=T(index)+tau;
            u=rand;
            if u<(lambda/(lambda+alpha))
                X(index+1)=x+1;
            else
                X(index+1)=x-c;
```

```
            end
        elseif X_r<=x && x<X_max
            tau=exprnd(1/(lambda+alpha+beta));
            T(index+1)=T(index)+tau;
            u=rand;
            if u<(lambda/(lambda+alpha+beta))
                X(index+1)=X(index)+1;
            elseif u<((lambda+alpha)/(lambda+alpha+beta))
                                X(index+1)=X(index)-c;
            else
                X(index+1)=X(index)-d;
            end
        elseif x==X_max
            tau=exprnd(1/(alpha+beta));
            T(index+1)=T(index)+tau;
            u=rand;
            if u<(alpha/(lambda+alpha))
                X(index+1)=x-c;
            else
                X(index+1)=x-d;
            end
        else
            disp('Out Of Range')
            break
        end
        index=index+1;
    end
end
```

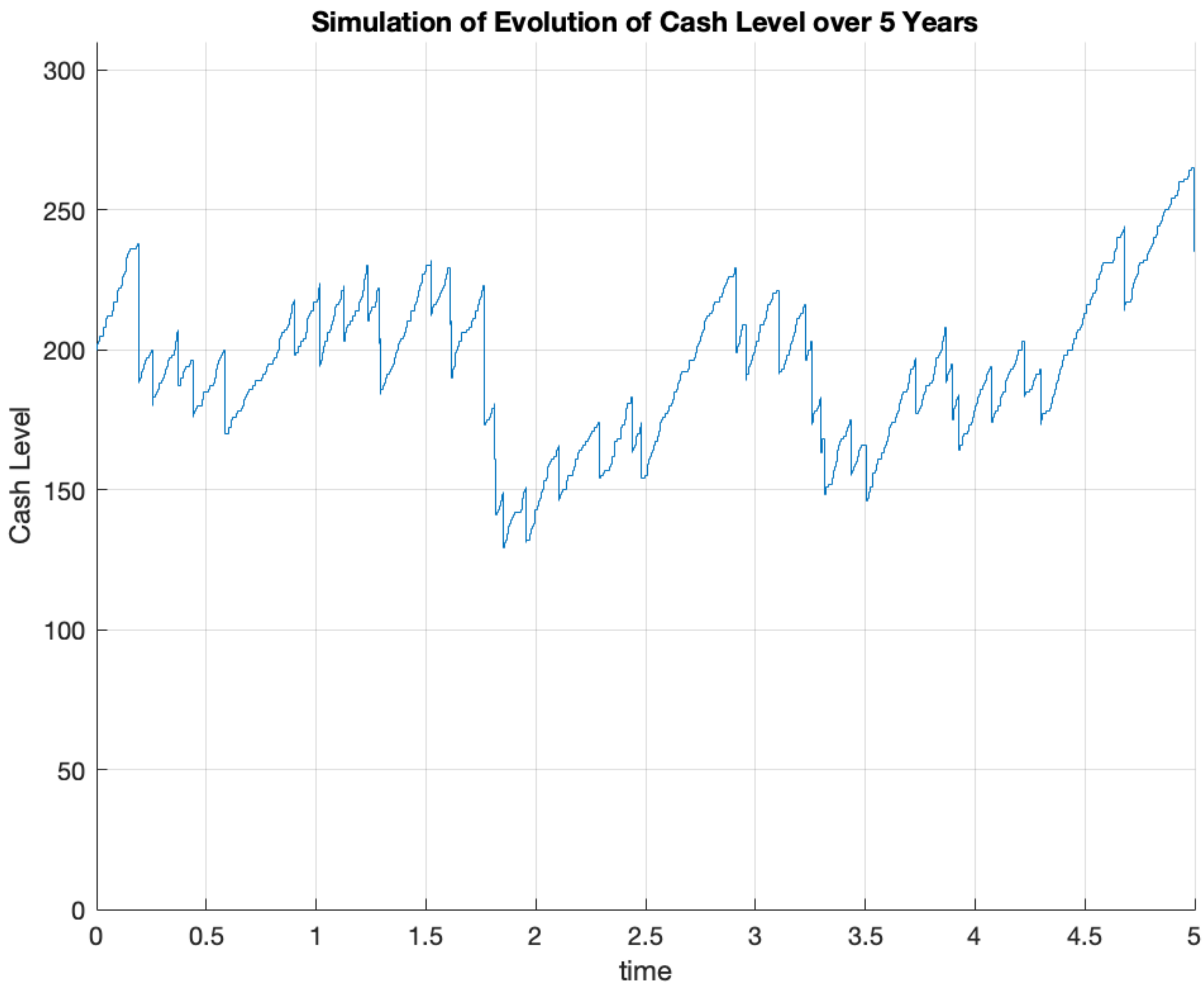
Plot

```
clc; clear all; close all;

X_0=200;
N=200;
r=0.04;
lambda=N;
alpha=r*N;
beta=4;
X_r=200;
X_max=300;
T_max=5;
d=30;
c=20;

[X,t]=cashflow(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max);

% Plot
hold on
grid on
xlabel('time')
ylabel('Cash Level')
title('Simulation of Evolution of Cash Level over 5 Years')
axis([0 5 0 310])
stairs(t,X);
```



(F)

$\because \frac{\mathrm{d}P_{xy}(t)}{\mathrm{d}t} = P'(xy) = \sum_{k=0,k\neq y}^{\infty} q_{ky} P_{xk}(t) - v_y P_{xy}(t)$

$\therefore$

Range A

$P'(xy) = \alpha \sum_{k=1}^{c} P_{x,k} - \lambda P_{x,y}$

Range B

$P'(xy) = \lambda P_{x,y-1} + \alpha P_{x,y+c} - (\lambda + \alpha) P_{x,y}$

Range C

- $y < x_r - d : P^{'}(xy) = \lambda P_{x,y-1} + \alpha P_{x,y+c} - (\lambda + \alpha)P_{x,y}$
- $y \geq x_r - d : P^{'}(xy) = \lambda P_{x,y-1} + \alpha P_{x,y+c} + \beta P_{x,y+d} - (\lambda + \alpha)P_{x,y}$

Range D

- $y \leq min(X_{max} - c, X_{max} - d) : P^{'}(xy) = \lambda P_{x,y-1} + \alpha P_{x,y+c} + \beta P_{x,y+d} - (\lambda + \alpha + \beta)P_{x,y}$
- $y > min(X_{max} - c, X_{max} - d) : P^{'}(xy) = \lambda P_{x,y-1} - (\lambda + \alpha + \beta)P_{x,y}$
- $c < d,\ min(X_{max} - d < y \leq X_{max} - c : P^{'}(xy) = \lambda P_{x,y-1} + \alpha P_{x,y+c} - (\lambda + \alpha + \beta)P_{x,y}$
- $d < c,\ min(X_{max} - c < y \leq X_{max} - d : P^{'}(xy) = \lambda P_{x,y-1} + \beta P_{x,y+d} - (\lambda + \alpha + \beta)P_{x,y}$

Range E

$$P^{'}(xy) = \lambda P_{x,y-1} - (\alpha + \beta)P_{x,y}$$

(G)

$$\because \frac{\mathrm{d}P_{xy}(t)}{\mathrm{d}t} = P^{'}(xy) = \sum_{k=0,k\neq x}^{\infty} q_{xk} P_{ky}(t) - v_x P_{xy}(t)$$

$\therefore$

Range A

$$P^{'}(xy) = \lambda P_{x+1,y} - P_{x,y}$$

Range B

$$P^{'}(xy) = \lambda P_{x+1,y} + \alpha P_{0,y} - (\lambda + \alpha)P_{x,y}$$

Range C

$$P^{'}(xy) = \lambda P_{x+1,y} + \alpha P_{x-c,y} - (\lambda + \alpha)P_{x,y}$$

Range D

$$P^{'}(xy) = \lambda P_{x+1,y} + \alpha P_{x-c,y} + \beta P_{x-d,y} - (\lambda + \alpha + \beta)P_{x,y}$$

Range E

$$P^{'}(xy) = \alpha P_{x-c,y} + \beta P_{x-d,y} - (\alpha + \beta)P_{x,y}$$

(H)

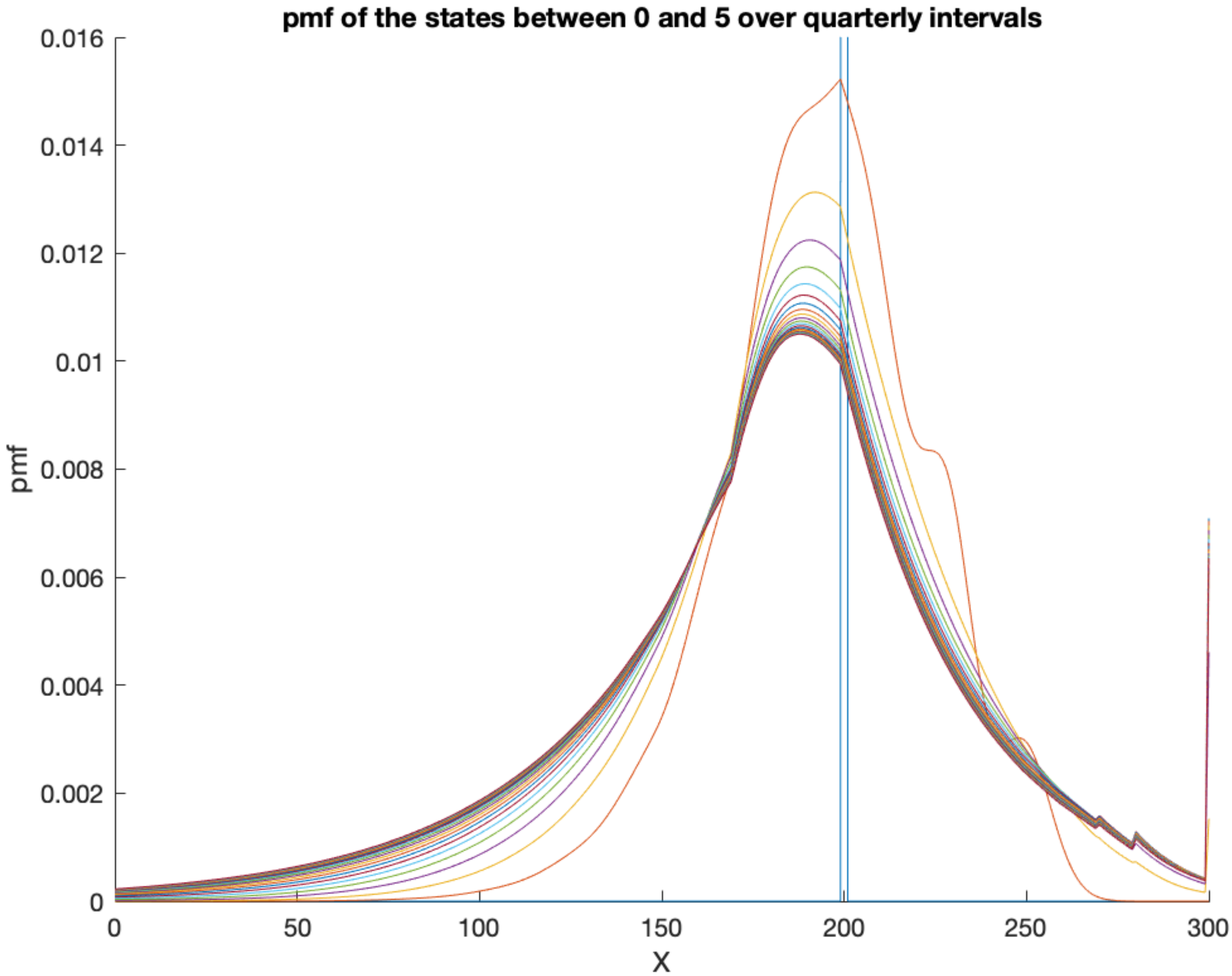
Kolmogrov_F

```matlab
function [R]=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max)
    % initialization
    R=zeros(X_max+1);
    % Range A:
    R(1,1)=-lambda;
    R(1,2:c+1)=alpha;
    % Range B:
    for i=2:c
        R(i,i-1)=lambda;
        R(i,i+c)=alpha;
        R(i,i)=-(lambda+alpha);
    end
    % Range C-1:
    for i=c+1:X_r-d
        R(i,i-1)=lambda;
        R(i,i+c)=alpha;
        R(i,i)=-(lambda+alpha);
    end
    % Range C_2:
    for i=X_r-d+1:X_r
        R(i,i-1)=lambda;
        R(i,i+c)=alpha;
        R(i,i+d)=beta;
        R(i,i)=-(lambda+alpha);
    end
    % Range D_1:
    for i=X_r+1:X_max-d+1
        R(i,i-1)=lambda;
        R(i,i+c)=alpha;
        R(i,i+d)=beta;
        R(i,i)=-(lambda+alpha+beta);
    end
    % Range D_2:
    for i=X_max-d+2:X_max-c+1
        R(i,i-1)=lambda;
        R(i,i+c)=alpha;
        R(i,i)=-(lambda+alpha+beta);
    end
    % Range D_3:
    for i=X_max-c+2:X_max
        R(i,i-1)=lambda;
        R(i,i)=-(lambda+alpha+beta);
    end
    % Range E:
    R(X_max+1, X_max)=lambda;
```

```
    R(X_max+1, X_max+1)=-(alpha+beta);
```

Plot

```
R=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max);
p0=zeros(X_max+1,1);
p0(X_0+1,1)=1;
T=0:0.25:5;
figure
hold on
xlabel('X')
ylabel('pmf')
title('pmf of the states between 0 and 5 over quarterly intervals')
axis([0 300 0 0.016])
for t=T
     pmf=expm(R.*t)*p0;
     plot(0:X_max,pmf)
end
```



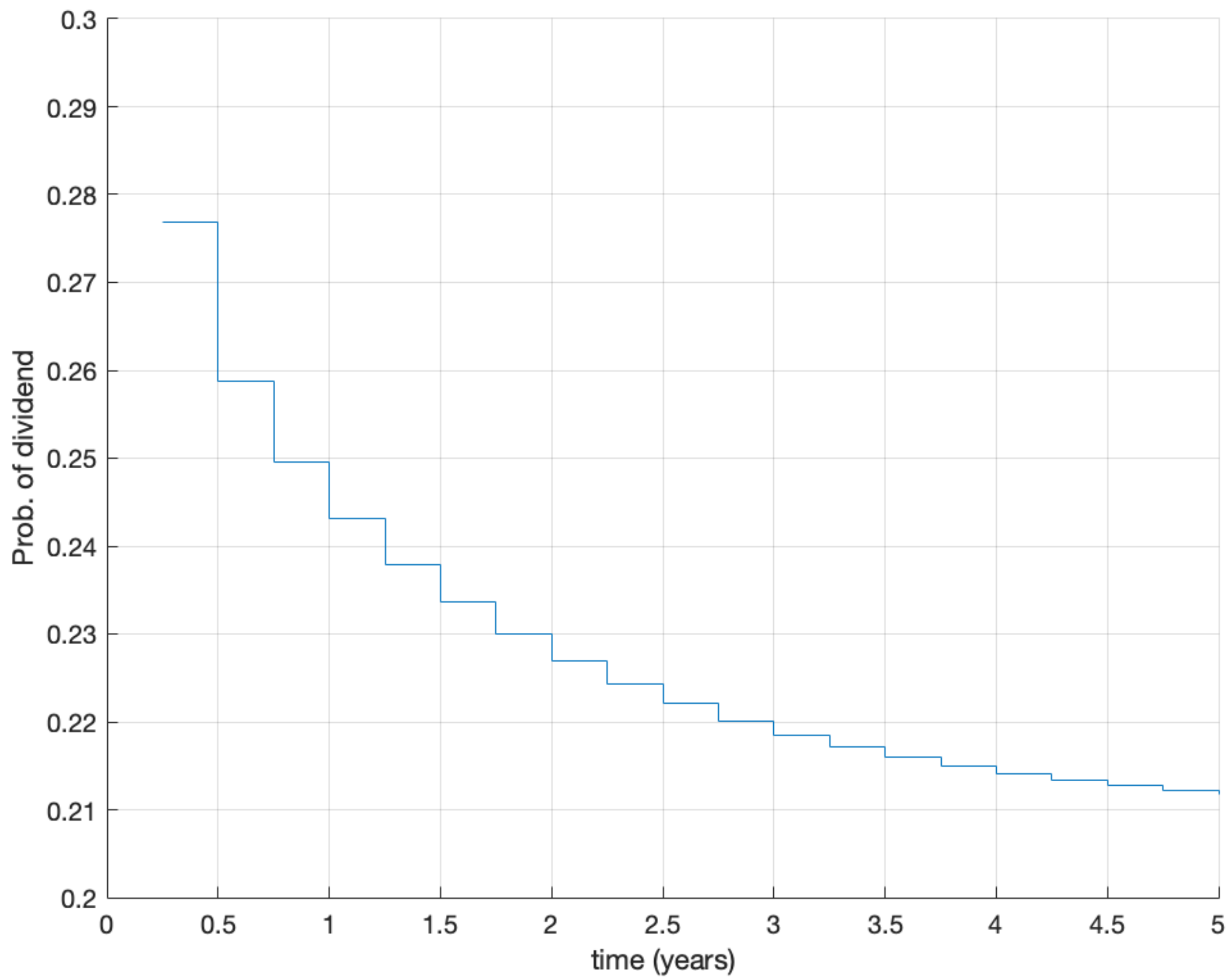
(I)

Because dividends can only be paid when x is in range (D) or (E), we can declare that

P(at least one dividend paid) $= 1 - P$(no dividends paid) ≈ 0.64

Then the probability of paying dividends in a given quarter can be estimated as 0.64 times the probability that $X(t) \geq X_r$. We repeat this procedure for every quarter and plot the result below

```
R=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max);
p0=zeros(X_max+1,1);
p0(X_0+1,1)=1;
T=0.25:0.25:5;
prob=zeros(20,1);
figure
hold on
xlabel('time (years)')
ylabel('Prob. of dividend')
axis([0 5 0.2 0.30])
for t=T
        pmf=expm(R.*t)*p0;
        prob(t/0.25) = sum(pmf(201:end))*0.64;
end
set(gca);
stairs(T,prob);
grid on
```



As shown on the plot, the probability of paying dividends will decrease as time increase. And eventually this probability tend to converge around 0.21.