ECE 440, Midterm Review

Probability

(1) Axioms of Probability

- **Non-negativity**: $P(E) \geq 0$
- **Probability of universe**: $P(S) = 1$
- **Additivity**: Given sequence of **disjoint** events $E_1, E_2, \ldots$

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

(2) Law of Large Numbers: Sequence of i.i.d. RVs $X_1, X_2, \ldots, X_n, \ldots$ with mean μ. Define sample average $\bar{X_N} := (1/N) \sum_{n=1}^{N} X_n$

- Weak version: Sample average $\bar{X_N}$ of i.i.d sequence **converges in prob** to $\mu = E[X_n]$

  $\lim_{N \to \infty} P(|\bar{X_N} - \mu| < \epsilon) = 1$, for all $\epsilon > 0$

- Strong version: Sample average $\bar{X_N}$ of i.i.d sequence **converges a.s.(almost surely)** to $\mu = E[X_n]$

  $\lim_{N \to \infty} P(|\bar{X_N}| = \mu) = 1$

(3) Bayes's Rule

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{\sum_i P(F|E_i)P(E_i)}$$

(4) Mutually Independent:

$P(\cap_i E_i) = \prod_i P(E_i)$ for **every finite** subset of i at least two integers (Every two pairs, three subset, four subset, and etc...)

(5) Pairwise Independent: $P(E_i \cap E_j) = P(E_i)P(E_j)$ for all (i, j)

(6) Bernoulli RV: X with parameter p indicate a random event E can succeed with $P(E) = p$

- $p(x) = p^x(1-p)^{1-x},\ E[X] = p,\ var[X] = p(1-p)$

(7) Geometric RV: X with parameter p counts the number of Bernoulli trials needed to register first success

- $p(x) = p(1-p)^{x-1},\ F(x) = 1-(1-p)^x,\ E[X] = \frac{1}{p}, var[X] = \frac{1-p}{p^2}$

(8) Binomial RV: X with parameters n and p counts the number of successes in n Bernoulli trials.

- $p(x) = \binom{n}{x} p^x (1-p)^{n-x},\ E[X] = np,\ var[X] = np(1-p)$
- For $B_1, B_2, \ldots, B_n$ i.i.d Bernoulli RVs with parameter p. Can write binomial X with parameters (n, p) as $X = \sum_{i=1}^{n} B_i$
- For binomials Y and Z with parameters (n_Y, p) and (n_Z, p), then $X = Y + Z \sim$ binomial$(n_Y + n_Z, p)$

(9) Poisson RV: X with parameter λ counts of rare events or "arrivals"

- $p(x) = e^{-\lambda} \frac{\lambda^x}{x!},\ E[X] = \lambda,\ var[X] = \lambda$
- For $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$, then $Y = X_1 + X_2 \sim Poisson(\lambda_1 + \lambda_2)$
- The law of rare events asserts that the distribution of $X \sim$ Binomial(n, p) converges to a Poisson(λ) as $n \to \infty$, provided $np = \lambda$

(10) Uniform RV: X with parameters a and b models problems with equal probability of landing on an interval $[a, b]$

- $f(x) = \frac{1}{b-a},\ F(x) = \frac{x-a}{b-a}$
- $E[X] = \frac{a+b}{2}$

(11) Exponential RV: X with parameter λ models duration of phone calls, lifetime of electronic components

- $f(x) = \lambda e^{-\lambda x}, x \geq 0$
- $F(x) = 1 - e^{-\lambda x}$
- $E[X] = \frac{1}{\lambda}$

(12) Gaussian/Normal RV: X with parameter μ and σ^2 models randomness arising from large number of random effects

- $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^3}$
- $E[X] = \mu,\ var[X] = \sigma^2$

(13) Markov's Inequality: $P(|X| \geq a) \leq \frac{E(|X|)}{a}$

(14) Chebyshev's Inequality: $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$

(15) Iterated Expectations: $E[X] = E_Y[E_X[X|Y]] = \sum_y E_X[X|Y=y] \cdot p_Y(y)$

- $var[X] = E_Y[var_X(X|Y)] + var_Y[E_X(X|Y)]$, using iterated expectation to compute the variance

Discrete Markov Chain

(1) Chapman-Kolmogorov Equation: $P_{ij}^{m+n} = \sum_{k=0}^{\infty} P_{kj}^{n} P_{ik}^{m} \rightarrow P^{(m+n)} = P^{(m)} P^{(n)}$

(2) n-step Transition Probabilities: $P^{(n)} = P^{n}$

(3) Communication: States i and j are said to **communicate ($i \leftrightarrow j$)** if $P_{ij}^{n} > 0$ and $P_{ji}^{m} > 0$ for some n and m

- **Reflexivity**: $i \leftrightarrow i$ (Because $P_{ii}^{0} = 1$ always holds)
- **Symmetry**: If $i \leftrightarrow j$ then $j \leftrightarrow i$
- **Transitivity**: If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$
- **Partitions set of states into disjoint classes**

(4) Irreducible MC

- All states communicate with each other
- If MC has finite number of states, the single class is recurrent
- If MC has infinite number of states, the single class is transient

(5) Recurrent

Define the return time to state i as $T_i = \min\{n > 0 : X_n = i \mid X_0 = i\}$

- **recurrent**: Probability of not returning is 0. $P(T_i = \infty \mid X_0 = i) = 0$
    - **positive recurrent**: Expected value of T_i is finite. $E[T_i \mid X_0 = i] = \sum_{n=1}^{\infty} nP(T_i = n \mid X_0 = i) < \infty$
    - **null recurrent**: Expected value of T_i is infinite. $E[T_i \mid X_0 = i] = \sum_{n=1}^{\infty} nP(T_i = n \mid X_0 = i) = \infty$

(6) Ergodic MC

- **irreducible**
- **positive recurrent**
- **aperiodic**

(7) Ensemble Average: across different realizations of the MC

$$E[f(X_n)] = \sum_{i=1}^{\infty} f(i)P(X_n = i) \rightarrow \sum_{i=1}^{\infty} f(i)\pi_i$$

(8) Ergodic Average: across time for a single realization of the MC

$$\bar{f}_n = \frac{1}{n} \sum_{m=1}^{n} f(X_m)$$

Note: Ensemble average equals to ergodic average almost surely, asymptotically in n

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(X_m) = \sum_{i=1}^{\infty} f(i)\pi_i$$