ECE 440, HW#9, Kefu Zhu

Question 7

(A)

Because $t_1 \neq t_2 \rightarrow t_1 - t_2 \neq 0$, and we have

$$\delta(t) = \begin{cases} \infty, \ t = 0 \\ 0, \ t \neq 0 \end{cases}$$

so $\delta(t_1 - t_2) = 0$. Then $R_W(t_1, t_2) = 0$, which tells us that $W(t_1)$ and $W(t_2)$ are uncorrelated. But since $W(t)$ is a Gaussian process, uncorrelation implies independence, so $W(t_1)$ and $W(t_2)$ are also independent.

(B)

(1) Show that the process $X(t)$ is Gaussian

Because $W(t)$ is a Gaussian process and $X(t)$ is a linear functional of $W(t)$, $X(t)$ is also a Gaussian process.

(2) Compute the mean and autocorrelation functions of $X(t)$

$\mu_X(t) = E[\int_0^t W(u)\ du] = \int_0^t E[W(u)]du = \int_0^t \mu_W(u)du = 0$

$R_X(t_1, t_2) = E[\int_0^{t_1} \int_0^{t_2} W(u)W(v)\ dv\ du]$

$= \int_0^{t_1} \int_0^{t_2} E[W(u)W(v)]\ dv\ du$

$= \int_0^{t_1} \int_0^{t_2} \sigma^2 \cdot \delta(u - v)\ dv\ du$

- If $t_1 < t_2$:

$= \int_0^{t_1} \int_0^{t_1} \sigma^2 \cdot \delta(u - v)\ d\ vdu + \int_0^{t_1} \int_{t_1}^{t_2} \sigma^2 \cdot \delta(u - v)\ dv\ du$

$= \int_0^{t_1} \int_0^{t_1} \sigma^2 \cdot \delta(u - v)\ dv\ du$

$= \int_0^{t_1} \sigma^2 du$

$= \sigma^2 \cdot t_1$

- If $t_1 > t_2$:

$= \int_0^{t_2} \int_0^{t_2} \sigma^2 \cdot \delta(u-v)\, dv\, du + \int_0^{t_2} \int_{t_2}^{t_1} \sigma^2 \cdot \delta(u-v)\, dv\, du$

$= \int_0^{t_2} \int_0^{t_2} \sigma^2 \cdot \delta(u-v)\, dv\, du$

$= \int_0^{t_2} \sigma^2 du$

$= \sigma^2 \cdot t_2$

To summarize, $R_X(t_1, t_2) = \sigma^2 \cdot \min(t_1, t_2)$

(3) Compute $P(X(t) > a)$ for arbitrary a and $t > 0$

$var(X(t)) = E[X^2(t)] - E^2[X(t)] = R_X(t,t) - \mu_X^2(t) = \sigma^2 t$

Based on Gaussian pdf,

$P(X(t) > a) = \int_0^{\infty} \frac{1}{\sqrt{(2\pi\sigma^2 t)}} \cdot \exp(-\frac{x^2}{2\sigma^2 t})\, dx$

(C)

$\mu_{W_h}(n) = E[W_h(n)] = E[\int_{nh}^{(n+1)h} W(u)\, du] = \int_{nh}^{(n+1)h} E[W(u)]\, du = \int_{nh}^{(n+1)h} \mu_W(u)\, du = 0$

$R_{W_h}(n_1, n_2) = E[\int_{n_1 h}^{(n_1+1)h} \int_{n_2 h}^{(n_2+1)h} W(u)W(v)\, du\, dv]$

$= \int_{n_1 h}^{(n_1+1)h} \int_{n_2 h}^{(n_2+1)h} E[W(u)W(v)]\, du\, dv$

$= \int_{n_1 h}^{(n_1+1)h} \int_{n_2 h}^{(n_2+1)h} \sigma^2 \delta(u-v)\, du\, dv$

- When $n_1 = n_2$, $R_{W_h}(n_1, n_2) = \sigma^2 h$
- When $n_1 \neq n_2$, $R_{W_h}(n_1, n_2) = 0$

By defining discrete-time $\delta_d(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$, we can express $R_{W_h}(n_1, n_2)$ as $\sigma^2 h \delta_d(n_1 - n_2)$

(D)

problem7_d

```
% Set parameters
h=0.01; sigma=1; t_MAX=10;

W_vector=normrnd(0,sigma*sqrt(h),1,t_MAX/h);
X_vector=cumsum(W_vector);

% Plot sample path
plot(h:h:t_MAX,X_vector);
xlabel('time');title('Weiner Process Simulated, h = ' + string(h));
grid on; axis([0 t_MAX -5 5])
```