Midterm - CSC/DSC 262/462 - October 12, 2017 - SOLUTIONS

NAME: ____

This exam is closed book. You are allowed one aid sheet on a standard 8.5×11 inch paper (both sides) and a calculator. Answer the questions in the space provided. Use the back of the sheet if needed (please indicate if you have done this). You have 1 hour and 10 minutes. Answer all four questions. All questions have equal weight. You are encouraged to read each question completely before starting.

1. A standard 52 card playing deck assigns a unique combination of 13 ranks in the sequence (2,3,4,5,6,7,8,9,10,J,Q,K,A) and 4 suits (Clubs, Diamonds, Hearts, Spades) to each card ($13 \times 4 = 52$). Suppose a hand of 5 cards is selected at random. Using the *rule of product* calculate the probability that the cards form a *Full House*, that is two cards of one rank, three cards of a different rank. Carefully list the *tasks* used in the application of the *rule of product*.

SOLUTION We can have

$$D = \binom{52}{5} = 2,598,960.$$

possible hands. **Full House**

Task 1: Select 1 from 13 ranks for the three of a kind, $n_1 = 13$.

Task 2: Select 3 from 4 cards for the three of a kind, $n_2 = \binom{4}{3} = 4$.

Task 3: Select 1 from 12 remaining ranks for the two of a kind, $n_3 = 12$.

Task 4: Select 2 from 4 cards for the two of a kind, $n_4 = \binom{4}{2} = 6$.

There are

$$N = n_1 \times n_2 \times n_3 \times n_4 = 13 \times 4 \times 12 \times 6 = 3,744$$

such selections, so

$$P(\text{ Full House }) = \frac{N}{D} = \frac{3,744}{2,598,960} \approx 0.001441.$$

2. A random variable X possesses the following density function for some constant c:

$$f_X(x) = \begin{cases} cx^4 & ; \quad x \in [0,2] \\ 0 & ; \quad otherwise \end{cases} .$$

(a) Determine c.

(b) Determine the expected value of X.

(a) The integral of a density evaluates to 1, so

$$1 = \int_0^2 cx^4 dx = cx^5/5\Big|_0^2 = c \times 32/5,$$

giving $c = 5/32$.

(b) Then $E[X]$ is

$$\begin{aligned} E[X] &= (5/32)\int_0^2 x \times x^4 dx \\ &= (5/32) \times x^6/6\Big|_0^2 \\ &= (5/32) \times (64/6) = 5/3. \end{aligned}$$

3. In a certain game, two dice are tossed independently. A player wins if at least one dice shows a 6. The game is played 10 times, and X is the number of times the player wins. Give the mean and variance of X.

Let $E_i = \{\text{Dice } i \text{ shows } 6\}$, $i = 1, 2$

$$\begin{aligned} P(\text{ At least one dice shows a 6 }) &= P(E_1 \cup E_2) \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= 1/6 + 1/6 - 1/36 \\ &= 11/36 \end{aligned}$$

Then $X \sim bin(n, p)$ where $n = 10$, $p = 11/36$. For any binomial random variable, the mean and variance is given by

$$\begin{aligned} E[X] &= np \\ &= 10 \times 11/36 \\ &= 55/18 \\ &\approx 3.06, \end{aligned}$$

and

$$\begin{aligned} var(X) &= np(1-p) \\ &= 10 \times (11/36) \times (25/36) \\ &= 1375/648 \\ &\approx 2.12. \end{aligned}$$

4. Assume that over a 25 year period the mean height of adult males increased from 175.5 cm to 179.1 cm, with the standard deviation remaining constant at $\sigma = 5.84$. Suppose the minimum height requirement to join the police force remained unchanged at 172 cm. Assume the heights are normally distributed.

   What proportion of adult males would not meet the minimum height requirement at the beginning and end of the 25 year period? [Use the probabilities $P(Z \leq -0.5993) \approx 0.274$, $P(Z \leq -1.2158) \approx 0.112$].

   SOLUTION At the beginning of the period $X \sim N(175.5, 5.84^2)$. The proportion not meeting the height requirement is

$$\begin{aligned} p &= P(X \leq 172) \\ &= P\left(\frac{X - 175.5}{5.84} \leq \frac{172 - 175.5}{5.84}\right) \\ &= P\left(Z \leq -0.5993\right) \\ &\approx 0.274, \end{aligned}$$

   where $Z \sim N(0, 1)$.

   At the beginning of the period $X \sim N(179.1, 5.84^2)$. The proportion not meeting the height requirement is

$$\begin{aligned} p &= P(X \leq 172) \\ &= P\left(\frac{X - 179.1}{5.84} \leq \frac{172 - 179.1}{5.84}\right) \\ &= P\left(Z \leq -1.2158\right) \\ &\approx 0.112, \end{aligned}$$

   where $Z \sim N(0, 1)$.

5. **[EXTRA QUESTION]** Suppose X is the sum of n independent Bernoulli random variables $U_1, \ldots, U_n$. Suppose the means are given by $E[U_i] = p_i$. What is the mean and variance of X?

SOLUTION The variance of U_i is $\sigma_i^2 = p_i(1 - p_i)$. Since the random variables are independent, we have

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} p_i, \text{ and} \\ var(X) &= \sum_{i=1}^{n} p_i(1 - p_i). \end{aligned}$$