Midterm - CSC/DSC 262/462 - October 13, 2016

NAME: ____

This exam is closed book. You are allowed one aid sheet on a standard 8.5×11 inch paper (both sides) and a calculator. Answer the questions in the space provided. Use the back of the sheet if needed (please indicate if you have done this). You have 1 hour. Answer all four questions. All questions have equal weight.

1. The letters ALABAMA are permuted at random. What is the probability that no two 'A's are next to each other?

SOLUTION The total number of permutations is

$$D = \binom{7}{1,1,1,4} = \frac{7!}{1 \times 1 \times 1 \times 4!} = 7 \times 6 \times 5 = 210.$$

Let N be the number of permutations for which no two 'A's are next to each other. To calculate N use the *rule of product*. First note that for such a permutation the 'A's must occupy position 1,3,5,7, and the letters 'L','B','M' must occupy positions 2,4,6.

1. There is exactly one way to arrange the 'A's, so $n_1 = 1$.
2. Then 'L','B','M' can be freely permuted among the remaining positions, so $n_2 = 3! = 6$.

There are

$$N = n_1 \times n_2 = 1 \times 6 = 6,$$

so

$$P(\text{No consecutive A's}) = \frac{N}{D} = \frac{6}{210} = \frac{1}{35}.$$

2. A random variable X possesses the following density function for some constant c:

$$f_X(x) = \begin{cases} c|x|^{1.5} & ; \quad x \in [-1, 1] \\ 0 & ; \quad otherwise \end{cases} .$$

(a) Determine c.

(b) Determine the 0.25-quantile for this density.

(a) The integral of a density evaluates to 1, so

$$\begin{aligned} 1 &= \int_{-1}^{1} c|x|^{1.5} dx \\ &= 2\int_{0}^{1} cx^{1.5} dx \\ &= 2 \times c \times x^{2.5}/2.5\Big|_0^1 \\ &= 2 \times c/2.5, \end{aligned}$$

giving $c = 5/4$ (the evaluation makes use of the symmetry of $f_X(x)$ about 0).

(b) By symmetry the 0.25-quantile is less than 0. For $x \in [-1, 0]$ the CDF is

$$\begin{aligned} F_X(x) &= (5/4)\int_{-1}^{x} (-x)^{1.5} dx \\ &= -(5/4)(-x)^{2.5}/2.5\Big|_{-1}^{x} \\ &= (1/2)\left[1 - (-x)^{2.5}\right]. \end{aligned}$$

The 0.25-quantile q is the solution to

$$0.25 = (1/2)\left[1 - (-x)^{2.5}\right],$$

or

$$x = -(1/2)^{1/2.5} \approx -0.7578583.$$

3. A certain hospital delivered 10 babies during the last year. Given that 6 of these were boys, what is the probability that the first six deliveries were all boys? Assume that a baby is equally (and independently) likely to be a boy or girl.

Let $X \sim bin(10, 1/2)$ be the number of boys. Let $A = \{\text{first 6 deliveries were boys}\}$. Then we need to evaluate

$$\begin{aligned}
P(A \mid X = 6) &= \frac{P(A \cap X = 6)}{P(X = 6)} \\
&= \frac{P(\text{first 6 are boys, last 4 are girls})}{\binom{10}{6}(1/2)^{10}} \\
&= \frac{(1/2)^{10}}{\binom{10}{6}(1/2)^{10}} \\
&= \frac{1}{\binom{10}{6}} = \frac{1 \times 2 \times 3 \times 4}{7 \times 8 \times 9 \times 10} = \frac{1}{210}.
\end{aligned}$$

4. The distribution of the height of a certain type of plant is normally distributed with mean $\mu = 39.8$ inches and standard deviation $\sigma = 2.05$ inches. What is the probability that of 20 randomly selected plants, exactly 5 have a height of at least 40 inches? Use the fact that $P(Z \leq 0.1) = 0.5398$ for a standard normal random variable $Z \sim N(0, 1)$.

The height of a single plant is distributed as $X \sim N(39.8, 2.05^2)$. The probability that a single plant has a height of at least 40 inches is

$$\begin{aligned} p &= P(X \geq 40) \\ &= P\left(\frac{X - 39.8}{2.05} \geq \frac{40 - 39.8}{2.05}\right) \\ &\approx P\left(Z \geq 0.1\right) \\ &\approx 1 - 0.5398 = 0.4602, \end{aligned}$$

where $Z \sim N(0, 1)$. The number of plants at least 40 inches high has binomial distribution $Y \sim bin(20, p)$, so

$$P(Y = 5) = \binom{20}{5} 0.4602^5 (1 - 0.4602)^{15} \approx 0.03.$$