The gaps between primes

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Abstract

It is proved that

· For any positive integer d, there are infinitely many prime gaps of size $2d$.

· For every integer greater than 2 is the sum of two prime numbers.

Contents

1. Introduction

In number theory, Polignac's conjecture was made by Alphonse de Polignac in 1849 and states:

For any positive even number n, there are infinitely many prime gaps of size n. In other words: There are infinitely many cases of two consecutive prime numbers with difference n. The case $n = 2$, it is the twin prime conjecture.

Although the conjecture has not yet been proven or disproven for any given value of n, in 2013 an important breakthrough was made by Zhang Yitang who proved that there are infinitely many prime gaps of size n for some value of $n < 70,000,000$. Later that year, James Maynard announced a related breakthrough which proved that there are infinitely many prime gaps of some size less than or equal to 600. As of April 14, 2014, one year after Zhang's announcement, according to the Polymath project wiki, n has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that n has been reduced to 12 and 6, respectively [6].

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It was proposed by the German mathematician Christian Goldbach in a letter to Leonhard Euler on 7 June 1742. It states that every even whole number greater than 2 is the sum of two prime numbers. The conjecture has been shown to hold for all integers less than 4×10^{18}, but remains unproven despite considerable effort [3].

In this paper, we prove the above two conjectures:

· **Theorem 1.** For any positive even number n, there are infinitely many prime gaps of size n. (This is Polignac's conjecture. The case n=2 is the twin prime conjecture.)

· **Theorem 2.** Every even whole number greater than 2 is the sum of two prime numbers. (This is Coldbach's conjecture.)

2. **Notation and definitions**

Notation.

$a,\ b,\ c,\ d,\ h,\ i,\ j,\ k,\ m,\ n,\ q,\ t,\ w,\ u$: integers.

p: a prime number.

p_t: the t-th odd prime number p_1=3, p_2=5, etc.

$a \mid d$ means a is a divisor of d.

$a \nmid d$ means a does not divide d.

x: variable.

$\lfloor x \rfloor$ means the largest integer which does not exceed x.

$\lceil x \rceil$ means the least integer not less than x.

$\binom{d}{a}$ means d choose a; the binomail coefficient $\frac{d!}{a!\,(d-a)!}$.

$\mathbb{A}$: an abstract field for function parameter.

$\mathbb{Z}$: the field of integers.

$\mathbb{M}^{\circ}$: the base set of $p_1, p_2, \cdots, p_n$.

$\mathbb{M}_i$: infinite set generated by elements of $\mathbb{M}^{\circ}$ with offset i.

$\mathbb{M}_{i\cup j}$ means $\mathbb{M}_i \cup \mathbb{M}_j$.

$\mathbb{A}\,[a,b)$ means $\mathbb{A} \cap [a,b)$.

$|\mathbb{A}|$ denote the cardinality of set $\mathbb{A}$.

$\lambda\,(\mathbb{A}, d)$: generate a new set by adding d to each element of set $\mathbb{A}$.

$T\,(a)$: product function.

$\chi\,(a, \mathbb{A})$: use 0 or 1 to indicate whether a belongs to $\mathbb{A}$.

$\Lambda\,(d)$: the von Mangoldt function.

$\theta\,(x)$: the first Chebyshev function.

$\psi\,(x)$: the second Chebyshev function.

$(a_1, a_2, a_3, \cdots)$, $(\cdots)$: ordered arrays.

$\rho\,((a_1, a_2, \cdots))$, $\vartheta\,((\cdots))$: custom functions for lemma declaration.

$\mu\,((a_1, a_2, \cdots), m)$: a custom function for proving lemma.

$\mathcal{J}(p)$, $\mathcal{K}(p)$, $\mathcal{S}(w)$: custom functions for proving lemma.
$\varrho(x)$: a custom function, we will prove that it is less than 1.
η: used to denote the boundary that satisfies some conditions.
$L_i(a,t)$: used to estimate a smaller upper bound of η .
$\mathcal{T}$, $\mathcal{H}$: custom sets.
$v(\mathcal{H}_1, \mathcal{H}_2, \cdots)$: defined to assist in estimating $L_i(a,t)$.
$(f(x))'$ means $f'(x)$, that is the derivative of $f(x)$.
$\exp\{\cdots\}$: exponential function.
$\inf\{\cdots\}$: greatest lower bound.
$\sup\{\cdots\}$: least upper bound.

Definition 1. For $n \geq 1$,

$$\mathbb{M}^{\circ} = \{p_1, p_2, \cdots, p_n\}.$$

Definition 2. For any i,

$$\mathbb{M}_i = \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^{\circ}}} \{km + i\}.$$

Definition 3. For any i and j,

$$\mathbb{M}_{i \cup j} = \mathbb{M}_i \cup \mathbb{M}_j.$$

Definition 4. Let λ be the function, defined by

$$\lambda(\mathbb{A}, d) = \{m : m = a + d \wedge a \in \mathbb{A}\}.$$

Definition 5. For any a,

$$T(a) = \prod_{m \in \mathbb{M}^{\circ}} (m - a).$$

Definition 6. Let the function χ be given by

$$\chi(a, \mathbb{A}) = \begin{cases} 1 & \text{if } a \in \mathbb{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 7. The von Mangoldt function Λ is defined by

$$\Lambda(d) = \begin{cases} \ln p & \text{if } d = p^k \wedge k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The unique factorization property of the natural numbers implies

$$\ln d = \sum_{a \mid d} \Lambda(a),$$

the sum is taken over all integers a that divide d [7].

Definition 8. The first Chebyshev function $\theta(x)$ is defined by

$$\theta(x) = \sum_{p \le x} \ln p,$$

where the sum is over primes $p \le x$ [2].

Definition 9. The second Chebyshev function $\psi(x)$ is defined similarly

$$\psi(x) = \sum_{k \in \mathbb{N}} \sum_{p^k \le x} \ln p = \sum_{d \le x} \Lambda(d),$$

with the sum extending over all prime powers not exceeding x [2].

3. **Lemmas**

In this section we introduce a number of prerequisite results, some of them given here may not be in the strongest forms, but they are adequate for the proofs of **Theorems 1** and **2**.

LEMMA 1. $(\forall i, j)\,(\ \mathbb{M}_j = \lambda(\mathbb{M}_i,\ j-i)\ )$.

Proof. By Definition 2 and Definition 4, we obtain

$$\begin{aligned}\mathbb{M}_j &= \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{km + j\} \\ &= \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{km + i + (j-i)\} \\ &= \lambda(\mathbb{M}_i,\ j-i).\end{aligned}$$

LEMMA 2. $(\forall i, h, a)\,(\ \chi(h,\ \mathbb{M}_i) = \chi(h+a,\ \lambda(\mathbb{M}_i,\ a)) = \chi(h+a,\ \mathbb{M}_{i+a})\ )$.

Proof. Let us suppose

$$\chi(h,\ \mathbb{M}_i) = 1,$$

then

$$(\exists k_0 \in \mathbb{Z} \wedge m_0 \in \mathbb{M}^\circ)\,(\ k_0 m_0 + i = h\ ).$$

And by Lemma 1,

$$\lambda(\mathbb{M}_i,\ a) = \mathbb{M}_{i+a}.$$

Hence,

$$\begin{aligned}\chi\left(h+a,\ \lambda\left(\mathbb{M}_i,\ a\right)\right) &= \chi\left(h+a,\ \mathbb{M}_{i+a}\right)\\ &= \chi\left(\left(k_0 m_0 + i\right)+a,\ \mathbb{M}_{i+a}\right)\\ &= \chi\left(k_0 m_0 + \left(i+a\right),\ \mathbb{M}_{i+a}\right)\\ &= 1.\end{aligned}$$

Otherwise,

$$\chi\left(h,\ \mathbb{M}_i\right) = 0,$$

then

$$\left(\forall k \in \mathbb{Z} \wedge m \in \mathbb{M}^{\circ}\right)\left(\ km + i \neq h\ \right).$$

Hence,

$$h + a \neq km + \left(i+a\right),$$

i.e.

$$\chi\left(h+a,\ \mathbb{M}_{i+a}\right) = \chi\left(h+a,\ \lambda\left(\mathbb{M}_i,\ a\right)\right) = 0.$$

So that

$$\left(\forall i, h, a\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right) = \chi\left(h+a,\ \lambda\left(\mathbb{M}_i,\ a\right)\right) = \chi\left(h+a, \mathbb{M}_{i+a}\right)\right).$$

Lemma 3. $\left(\forall i, h \wedge m \in \mathbb{M}_0\right)\left(\ \chi\left(m\left(h-i\right)+i,\ \mathbb{M}_i\right) = 1\ \right).$

Proof. Obviously,

$$\left(\exists k_0 \in \mathbb{Z} \wedge m_0 \in \mathbb{M}^{\circ}\right)\left(\ k_0 m_0 + 0 = m\ \right).$$

Let

$$k_1 = k_0\left(h-i\right),$$

then

$$m\left(h-i\right)+i\ = k_1 m_0 + i.$$

So that

$$\chi\left(m\left(h-i\right)+i,\ \mathbb{M}_i\right) = \ \chi\left(k_1 m_0 + i,\ \mathbb{M}_i\right) = 1.$$

Lemma 4. $\left(\forall i, h \wedge u \notin \mathbb{M}_0\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right) = \chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)\ \right).$

Proof. Suppose that

$$\chi\left(h,\ \mathbb{M}_i\right) = 1,$$

then

$$\left(\exists k_0 \in \mathbb{Z} \wedge m_0 \in \mathbb{M}^{\circ}\right)\left(\ k_0 m_0 + i = h\ \right).$$

Hence,

$$\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right) = \chi\left(\left(uk_0\right)m_0 + i,\ \mathbb{M}_i\right) = 1.$$

Otherwise,

$$\chi\left(h,\ \mathbb{M}_i\right)=0,$$

then

$$\left(\forall k_0\in\mathbb{Z}\wedge m_0\in\mathbb{M}^\circ\right)\left(\ k_0m_0+i\neq h\ \right).$$

Noting that

$$\left(\forall k_1\in\mathbb{Z}\wedge m_1\in\mathbb{M}^\circ\right)\left(\ k_1m_1+0\neq u\ \right).$$

Combining the both, we have

$$\left(\forall k_2\in\mathbb{Z}\wedge m_2\in\mathbb{M}^\circ\right)\left(\ k_2m_2\neq u\left(h-i\right)\ \right).$$

Thus,

$$u\left(h-i\right)+i\neq k_2m_2+i,$$

i.e.

$$\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)=0.$$

So that

$$\left(\forall i,h\wedge u\notin\mathbb{M}_0\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right)=\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)\ \right).$$

Remark 1. A stronger conclusion is that

$$\{m:\ 0\leq m<T\left(0\right)\wedge m\notin\mathbb{M}_0\}$$

is a multiplicative group of integers modulo $T\left(0\right)$. It will not be proved here because this conclusion is not used in the proofs of this paper.

LEMMA 5. $\left(\forall i,h,d\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h+dT\left(0\right),\ \mathbb{M}_i\right)\ \right).$

Proof. By Lemma 2,

$$\chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h+dT\left(0\right),\ \mathbb{M}_{i+dT(0)}\right),$$

and

$$\mathbb{M}_{i+dT(0)}=\lambda\left(\mathbb{M}_i,\ dT\left(0\right)\right)=\bigcup_{\substack{k\in\mathbb{Z}\\ m\in\mathbb{M}^\circ}}\{km+i+dT\left(0\right)\}.$$

By the Definition 5,

$$T\left(0\right)=\prod_{m\in\mathbb{M}^\circ}m.$$

This implies that

$$\left(\forall k\in\mathbb{Z}\wedge m\in\mathbb{M}^\circ\right)\left(\left(\exists k_0\in\mathbb{Z}\right)\left(\ km+dT\left(0\right)=k_0m\ \right)\right)$$

Combining this with above,

$$\mathbb{M}_{i+dT(0)} = \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^{\circ}}} \{km + i + dT(0)\} = \bigcup_{\substack{k_0 \in \mathbb{Z} \\ m \in \mathbb{M}^{\circ}}} \{k_0 m + i\} = \mathbb{M}_i. \tag{3.1}$$

Hence,

$$\chi(h,\ \mathbb{M}_i) = \chi\left(h + dT(0),\ \mathbb{M}_{i+dT(0)}\right) = \chi(h + dT(0),\ \mathbb{M}_i).$$

Remark 2. So we can see that $\mathbb{M}_i$ is periodic and its period is $T(0)$.

LEMMA 6. $(\forall i, j, h, d)\left(\ \chi(h,\ \mathbb{M}_{i\cup j}) = \chi(h + dT(0),\ \mathbb{M}_{i\cup j})\ \right).$

Proof. By Lemma 5 we have

$$\chi(h,\ \mathbb{M}_i) = \chi(h + dT(0),\ \mathbb{M}_i),$$

and

$$\chi(h,\ \mathbb{M}_j) = \chi(h + dT(0),\ \mathbb{M}_j).$$

It is easy to see that

$$\begin{aligned}\chi(h + dT(0),\ \mathbb{M}_{i\cup j}) &= \chi(h + dT(0),\ \mathbb{M}_i) \circledast \chi(h + dT(0),\ \mathbb{M}_j) \\ &= \chi(h,\ \mathbb{M}_i) \circledast \chi(h,\ \mathbb{M}_j) \\ &= \chi(h,\ \mathbb{M}_{i\cup j}),\end{aligned}$$

where we do not need to know exactly what operator $\circledast$ does.

Remark 3. We can also prove it by the truth table.

$\chi(h, \mathbb{M}_i)$	$\chi(h, \mathbb{M}_j)$	$\chi(h + dT(0), \mathbb{M}_i)$	$\chi(h + dT(0), \mathbb{M}_j)$	$\chi(h, \mathbb{M}_{i\cup j})$	$\chi(h + dT(0), \mathbb{M}_{i\cup j})$
0	0	0	0	0	0
0	1	0	1	1	1
1	0	1	0	1	1
1	1	1	1	1	1

So $\mathbb{M}_{i\cup j}$ and $\mathbb{M}_i$ have the same period.

LEMMA 7. $|\mathbb{M}_0[0, T(0))| = T(0) - T(1).$

Proof. It is easy to see that

$$
\begin{aligned}
&\left|\mathbb{M}_0\left[0,T\left(0\right)\right)\right| \\
&= \left|\bigcup_{m_0\in\mathbb{M}_0\left[0,T\left(0\right)\right)} m_0\right| \\
&= \left|\bigcup_{\left(\forall k\in\mathbb{Z}\wedge m\in\mathbb{M}^{\circ}\right)\left(m_0=km\wedge m_0\in\left[0,T\left(0\right)\right)\right)} m_0\right| \\
&= T\left(0\right)\left(\sum_{\substack{\{m_1\}\subseteq\mathbb{M}^{\circ}\\ \left|\{m_1\}\right|=1}}\left(\frac{1}{m_1}\right)-\sum_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^{\circ}\\ m_1<m_2}}\left(\frac{1}{m_1m_2}\right)+\sum_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^{\circ}\\ m_1<m_2<m_3}}\left(\frac{1}{m_1m_2m_3}\right)-\cdots+\left(-1\right)^{n-1}\sum_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^{\circ}\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{1}{m_1m_2m_3\cdots m_n}\right)\right).
\end{aligned}
$$

Then the alternating series can be reduced to showing that

$$\left|\mathbb{M}_0\left[0,T\left(0\right)\right)\right| = T\left(0\right)\left(1-\frac{T\left(1\right)}{T\left(0\right)}\right) = T\left(0\right)-T\left(1\right).$$

Lemma 8. $\left(\forall i,a\right)\left(\ \left|\mathbb{M}_i\left[a,a+T\left(0\right)\right)\right| = T\left(0\right)-T\left(1\right)\ \right).$

Proof. By Lemma 2,

$$
\begin{aligned}
\left|\mathbb{M}_i\left[a,a+T\left(0\right)\right)\right| &= \left|\lambda\left(\mathbb{M}_i,\ -a\right)\left[\mathrm{a}-\mathrm{a},\mathrm{a}+T\left(0\right)-\mathrm{a}\right)\right| \\
&= \left|\mathbb{M}_{i-a}\left[0,T\left(0\right)\right)\right| \\
&= \left|\mathbb{M}_0\left[0,T\left(0\right)\right)\right|.
\end{aligned}
$$

By Lemma 7,

$$\left|\mathbb{M}_i\left[a,a+T\left(0\right)\right)\right| = \left|\mathbb{M}_0\left[0,T\left(0\right)\right)\right| = T\left(0\right)-T\left(1\right).$$

Lemma 9. $\left(\forall i,j,a\right)\left(\ \left|\mathbb{M}_{i\cup j}\left[a,a+T\left(0\right)\right)\right| \le T\left(0\right)-\ T\left(2\right) < T\left(0\right)\ \right).$

Proof. If

$$(\exists k \in \mathbb{Z}) \left(\ j = i + kT(0) \ \right),$$

then (by (3.1))

$$\begin{aligned} |\mathbb{M}_{i \cup j}[a, a + T(0))| &= |\mathbb{M}_{i \cup i}[a, a + T(0))| \\ &= |\mathbb{M}_{i}[a, a + T(0))| \\ &= T(0) - T(1). \end{aligned}$$

Otherwise, let us suppose

$$(\forall m \in \mathbb{M}_0) \left(\ j - i \not\equiv 0 \, (mod \ m) \ \right).$$

It is similar to the proof of Lemma 7, we have

$$\begin{aligned} &|\mathbb{M}_{i \cup j}[0, T(0))| \\ &= T(0) \left(\sum_{\substack{\{m_1\} \subseteq \mathbb{M}^{\circ} \\ |\{m_1\}| = 1}} \left(\frac{2^1}{m_1} \right) - \sum_{\substack{\{m_1, m_2\} \subseteq \mathbb{M}^{\circ} \\ m_1 < m_2}} \left(\frac{2^2}{m_1 m_2} \right) + \right. \\ &\quad \sum_{\substack{\{m_1, m_2, m_3\} \subseteq \mathbb{M}^{\circ} \\ m_1 < m_2 < m_3}} \left(\frac{2^3}{m_1 m_2 m_3} \right) - \cdots + \\ &\quad \left. (-1)^{n-1} \sum_{\substack{\{m_1, m_2, m_3, \cdots, m_n\} \subseteq \mathbb{M}^{\circ} \\ m_1 < m_2 < m_3 < \cdots < m_n}} \left(\frac{2^n}{m_1 m_2 m_3 \cdots m_n} \right) \right). \end{aligned}$$

Then the alternating series can be reduced to showing that

$$|\mathbb{M}_{i \cup j}[0, T(0))| = T(0) \left(1 - \frac{T(2)}{T(0)} \right) = T(0) - T(2).$$

For the opposite case, there is at least one $m \in \mathbb{M}^{\circ}$ such that the coefficient of each term containing m in the above alternating series is divided by 2. The reason is that

$$(\exists m \in \mathbb{M}^{\circ}) \left(\ \{km + j : k \in \mathbb{Z}\} = \{km + i : k \in \mathbb{Z}\} \ \right).$$

Therefore,

$$|\mathbb{M}_{i \cup j}[0, T(0))| < T(0) \left(1 - \frac{T(2)}{T(0)} \right) = T(0) - T(2),$$

when
$$(\exists m \in \mathbb{M}_0)\,(\ j - i \equiv 0\,(mod\ m)\ )\,.$$
Obviously,
$$T\,(0) > T\,(1) > T\,(2) > 0.$$
Combining with the above, we have
$$|\mathbb{M}_{i\cup j}\,[0, T\,(0))| \leq T\,(0) - \ T\,(2) < T\,(0)\,.$$
By Lemma 6, $\mathbb{M}_{i\cup j}$ is periodic with $T\,(0)$, and considering Lemma 8, we can get
$$|\mathbb{M}_{i\cup j}\,[a, a + T\,(0))| = |\mathbb{M}_{i\cup j}\,[0, T\,(0))| \leq T\,(0) - T\,(2) < T\,(0)\,.$$

LEMMA 10. $(\exists \eta > 0)\,((\forall i, j, a)\,(\ |\mathbb{M}_{i\cup j}\,[a, a + \eta)| < \eta\ ))\,.$

Proof. By Lemma 9, there are at least $T\,(2)$ numbers in any range $T\,(0)$ that make
$$\chi\,(h,\ \mathbb{M}_{i\cup j}) = 0,$$
where
$$h \in [a, a + T\,(0))\,.$$
It can also be expressed as
$$(\forall i, j, a)\left(\left(\sum_{h\in[a,a+T(0))\wedge\chi(h,\,\mathbb{M}_{i\cup j})=0} 1\right) \geq T\,(2) > 0\right)$$
So that
$$0 < \eta \leq T\,(0)\,.$$

On the basis of Lemma 10 we have

LEMMA 11. $(\forall i, j, a)\,((\exists\, h \in [a, a + \eta))\,(\ \chi\,(h,\ \mathbb{M}_i) = \chi\,(h + j - i,\ \mathbb{M}_i) = 0\ ))\,.$

Proof. By Lemma 10,
$$(\forall i, j, \text{a})\,((\exists h_0 \in [a, a + \eta))\,(\ \chi\,(h_0,\ \mathbb{M}_{i\cup j}) = 0\ ))\,,$$
so that
$$\chi\,(h_0,\ \mathbb{M}_i) = \chi\,(h_0,\ \mathbb{M}_j) = 0.$$
By Lemma 2,
$$\chi\,(h_0,\ \mathbb{M}_j) = \chi\,(h_0 + j - i,\ \mathbb{M}_i)\,.$$
Therefore,
$$\chi\,(h_0,\ \mathbb{M}_i) = \chi\,(h_0 + j - i,\ \mathbb{M}_i) = 0.$$

Lemma 12. *For $t \geq 1$,*

$$(\forall m_1, m_2, m_3, \cdots, m_t \in \mathbb{M}^\circ)\left(\sum_{\delta \in \mathcal{T}} \rho(\delta) \leq \sum_{\delta \in \mathcal{T}} \vartheta(\delta)\right)$$

where

$$\rho((a_1, a_2, a_3, \cdots))$$

$$= \begin{cases} 0 & (\exists j > i \geq 1)(a_i = a_j \wedge a_i \nmid (j-i)), \\ \displaystyle\prod_{\substack{m \in \mathbb{M}^\circ \\ m \mid (a_1 a_2 a_3 \cdots)}} \frac{1}{m} & \text{otherwise.} \end{cases}$$

and

$$\vartheta((a_1, a_2, a_3, \cdots)) = \frac{1}{a_1 a_2 a_3 \cdots},$$

and

$$\mathcal{T} = \bigcup_{\{h_1, h_2, h_3, \cdots, h_t\} = \{1,2,3,\cdots,t\}} \{(m_{h_1}, m_{h_2}, m_{h_3}, \cdots, m_{h_t})\}.$$

Proof. Let

$$\mu((a_1, a_2, a_3, \cdots), m)$$

$$= \begin{cases} 0 & (\exists j > i \geq 1)(m = a_i \wedge a_i = a_j \wedge a_i \nmid (j-i)), \\ 1 & \text{otherwise.} \end{cases}$$

$$\mathcal{K}(p) = \sum_{\delta \in \mathcal{T}} \mu(\delta, p),$$

and

$$\mathcal{J}(p) = \sum_{d \in [1,t]} \begin{cases} 0 & m_d \neq p, \\ 1 & m_d = p. \end{cases}$$

$$\mathcal{S}(w)$$

$$= \frac{\binom{t - \mathcal{J}(p_w)}{\mathcal{J}(p_1) + \mathcal{J}(p_2) + \cdots + \mathcal{J}(p_{w-1})} \mathcal{K}(p_w)}{\binom{t}{\mathcal{J}(p_1) + \mathcal{J}(p_2) + \cdots + \mathcal{J}(p_{w-1})} \binom{t - \mathcal{J}(p_1) - \cdots - \mathcal{J}(p_{w-1})}{\mathcal{J}(p_w)}}.$$

Then

$$\mathcal{S}(1)=\frac{\binom{t-\mathcal{J}(p_1)}{0}\mathcal{K}(p_1)}{\binom{t}{0}\binom{t}{\mathcal{J}(p_1)}}=\frac{\mathcal{K}(p_1)}{\binom{t}{\mathcal{J}(p_1)}}$$

$$\leq\lim_{t\to\infty}\frac{\mathcal{K}(p_1)}{\binom{t}{\mathcal{J}(p_1)}}=\frac{\frac{\binom{t}{\mathcal{J}(p_1)}}{p_1^{\mathcal{J}(p_1)-1}}}{\binom{t}{\mathcal{J}(p_1)}}=\frac{1}{p_1^{\mathcal{J}(p_1)-1}},$$

$$\mathcal{S}(2)=\frac{\binom{t-\mathcal{J}(p_2)}{\mathcal{J}(p_1)}\mathcal{K}(p_2)}{\binom{t}{\mathcal{J}(p_1)}\binom{t-\mathcal{J}(p_1)}{\mathcal{J}(p_2)}}=\frac{\mathcal{K}(p_2)}{\binom{t}{\mathcal{J}(p_2)}}$$

$$\leq\frac{\frac{\binom{t}{\mathcal{J}(p_2)}}{p_2^{\mathcal{J}(p_2)-1}}}{\binom{t}{\mathcal{J}(p_2)}}=\frac{1}{p_2^{\mathcal{J}(p_2)-1}},$$

$$\mathcal{S}(3)=\frac{\binom{t-\mathcal{J}(p_3)}{\mathcal{J}(p_1)+\mathcal{J}(p_2)}\mathcal{K}(p_3)}{\binom{t}{\mathcal{J}(p_1)+\mathcal{J}(p_2)}\binom{t-\mathcal{J}(p_1)-\mathcal{J}(p_2)}{\mathcal{J}(p_3)}}$$

$$=\frac{\mathcal{K}(p_3)}{\binom{t}{\mathcal{J}(p_3)}}\leq\frac{\frac{\binom{t}{\mathcal{J}(p_3)}}{p_3^{\mathcal{J}(p_3)-1}}}{\binom{t}{\mathcal{J}(p_3)}}=\frac{1}{p_3^{\mathcal{J}(p_3)-1}},$$

$$\vdots$$

$$\mathcal{S}(n)=\frac{\mathcal{K}(p_n)}{\binom{t}{\mathcal{J}(p_n)}}\leq\frac{1}{p_n^{\mathcal{J}(p_n)-1}}.$$

So that

$$\begin{aligned}
\sum_{\delta\in\mathcal{T}}\rho(\delta) &\leq \frac{|\mathcal{T}|}{m_1m_2\cdots m_t}\prod_{\substack{w\in[1,n]\\ S(w)\neq 0}} S(w)\prod_{\substack{w\in[1,n]\\ S(w)\neq 0}} p_w^{\mathcal{J}(p_w)-1}\\
&\leq \frac{|\mathcal{T}|}{m_1m_2\cdots m_t}\prod_{w\in[1,n]}\frac{1}{p_w^{\mathcal{J}(p_w)-1}}\prod_{w\in[1,n]} p_w^{\mathcal{J}(p_w)-1}\\
&= \frac{|\mathcal{T}|}{m_1m_2\cdots m_t}\\
&= \sum_{\delta\in\mathcal{T}}\vartheta(\delta).
\end{aligned}$$

On the basis of Lemma 12 we have

Lemma 13. *For $t\geq 1$,*

$$\sum_{\delta\in\mathcal{T}}\rho(\delta)\leq\sum_{\delta\in\mathcal{T}}\vartheta(\delta)$$

where

$$\mathcal{T}=\bigcup_{\substack{m_1\in\mathbb{M}^\circ\\ m_2\in\mathbb{M}^\circ\\ \vdots\\ m_t\in\mathbb{M}^\circ}}\{(m_1,m_2,m_3,\cdots,m_t)\}.$$

Proof. Let

$$s=\sum_{\substack{h_1\geq 0\wedge h_2\geq 0\wedge\cdots\wedge h_n\geq 0\\ h_1+h_2+\cdots+h_n=t}} 1,$$

$$\{\mathcal{T}_1,\mathcal{T}_2,\cdots,\mathcal{T}_s\}$$

$$=\bigcup_{\substack{h_1\geq 0\wedge h_2\geq 0\wedge\cdots\wedge h_n\geq 0\\ h_1+h_2+\cdots+h_n=t}} ($$

$$\left\{\bigcup_{\substack{m_1\in\mathbb{M}^\circ\\ m_2\in\mathbb{M}^\circ\\ \vdots\\ m_t\in\mathbb{M}^\circ\\ (\forall d\in[1,n])\left(\mathcal{J}\left(p_d\right)=h_d\right)}}\{(m_1,m_2,m_3,\cdots,m_t)\}\right\}\Bigg]\Bigg).$$

It is easy to see that

$$\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_2\cup\cdots\cup\mathcal{T}_s,$$

and

$$(\forall s\geq j>i\geq 1)\left(\mathcal{T}_j\cap\mathcal{T}_i=\phi\right).$$

Combining this with Lemma 12, we have

$$\sum_{\delta\in\mathcal{T}}\rho\left(\delta\right)=\sum_{d\in[1,s]}\sum_{\delta_d\in\mathcal{T}_d}\rho\left(\delta_d\right)\leq\sum_{d\in[1,s]}\sum_{\delta_d\in\mathcal{T}_d}\vartheta\left(\delta_d\right)=\sum_{\delta\in\mathcal{T}}\vartheta\left(\delta\right).$$

Lemma 14. $(\forall x\geq 3)$

$$\left(\prod_{2<p\leq x}\left(1-2p^{-1}\right)\geq 0.4/\ln^2 x\right).$$

Proof. By Mertens' second theorem [5],

$$\sum_{p\leq x}\left(p^{-1}\right)=\ln\ln x+M+O\left(1/\ln x\right).$$

The value of M is approximately [4]

$$M\approx 0.2614972128476427 84\cdots.$$

For $p>2$,

$$\sum_{2<p\leq x}\left(p^{-1}\right)=\ln\ln x+M'+O\left(1/\ln x\right).$$

The value of M' is approximately

$$M'\approx -0.238502787152357217\cdots.$$

Since

$$\left|\ln\left(1-2p^{-1}\right)+2p^{-1}\right| = \left|\int_1^{1-2p^{-1}}\left(t^{-1}-1\right)dt\right|$$
$$= \left|\frac{2}{p}-\frac{2}{p}-\frac{2^2}{2p^2}-\frac{2^3}{3p^3}-\frac{2^4}{4p^4}-\cdots\right|$$
$$< \frac{2^2}{2p^2}+\frac{2^3}{2p^3}+\frac{2^4}{2p^4}+\cdots$$
$$= \frac{2}{p\left(p-2\right)},$$

and

$$\sum_{p>2}\frac{2}{p\left(p-2\right)}$$

is convergent, the series

$$\sum_{p>2}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)$$

must be convergent. Because the series

$$\sum_{p>2}\left(p^{-1}\right)$$

is divergent and so the product

$$\prod_{p>2}\left(1-2p^{-1}\right)$$

must diverge also. We can deduce that

$$\sum_{2<p\leq x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)$$

$$= \sum_{p}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)-\sum_{p>x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)$$
$$= \sum_{p}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)+O\left(\sum_{p>x}\left(\frac{1}{p\left(p-2\right)}\right)\right)$$
$$= \sum_{p}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)+O\left(x^{-1}\right),$$

$$\ln\left(\prod_{2<p\leq x}\left(1-2p^{-1}\right)\right)$$

$$
\begin{aligned}
&= -2\sum_{p\le x}\left(p^{-1}\right) + \sum_{2<p\le x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)\\
&= -2\ln\ln x - 2M' + \sum_{2<p\le x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right) + O\left(\ln^{-1}x\right).
\end{aligned}
$$

It's known from numerical calculation

$$\sum_{p>2}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right) \approx -0.660393386913.$$

Combining with the above, we can crudely estimate

$$\prod_{2<p\le x}\left(1-2p^{-1}\right) \ge \frac{0.4}{\ln^2 x}$$

through numerical analysis.

Lemma 15. $(\forall x\ge 1)$

$$\left(\frac{\lfloor x\rfloor!}{\left(\lfloor\frac{x}{2}\rfloor!\right)^2} < 6^{\frac{x}{2}}\right).$$

Proof. If $\lfloor x\rfloor = 2k$ is even, then

$$\frac{\lfloor x\rfloor!}{\left(\lfloor\frac{x}{2}\rfloor!\right)^2} = \begin{pmatrix} 2k \\ k\end{pmatrix} \le 2^{2k} = 4^{\frac{x}{2}} \le 2^{2k+1}\left(1+\frac{1}{2}\right)^k < 6^{\frac{x}{2}},$$

because it's the largest binomail coefficient in the binomail expansion of $(1+1)^{2k}$. Otherwise, $\lfloor x\rfloor = 2k+1$ is odd, then

$$\frac{\lfloor x\rfloor!}{\left(\lfloor\frac{x}{2}\rfloor!\right)^2} = \begin{pmatrix} 2k+1 \\ k\end{pmatrix}(k+1) \le 2^{2k}(k+1) \le 2^{2k+1}\left(1+\frac{1}{2}\right)^k < 6^{\frac{x}{2}}.$$

Lemma 16. *Upper bounds exist for both $\theta(x)$ and $\psi(x)$ that*

$$(\forall x\ge 1)\left(\ \theta(x)\le\psi(x) < x\ln 6\ \right).$$

Proof. By Definition 9, we have

$$\ln\left(\lfloor x\rfloor!\right) = \psi(x) + \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) + \psi\left(\frac{x}{4}\right) + \cdots.$$

Changing x to $\dfrac{x}{2}$, and inserting -2ln $\left(\lfloor\dfrac{x}{2}\rfloor!\right)$ into the above equation we obtain

$$\ln\left(\lfloor x\rfloor!\right) - 2\ln\left(\lfloor\frac{x}{2}\rfloor!\right) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) + \cdots.$$

It is obvious that

$$\psi(x) \ge \psi\left(\frac{x}{2}\right) \ge \psi\left(\frac{x}{3}\right) \ge \psi\left(\frac{x}{4}\right) \ge \cdots,$$

so that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \ln(\lfloor x \rfloor !) - 2\ln\left(\lfloor \frac{x}{2} \rfloor !\right) = \ln\left(\frac{\lfloor x \rfloor !}{\left(\lfloor \frac{x}{2} \rfloor !\right)^2}\right).$$

Combining this with Lemma 15, we can get

$$\psi(x) - \psi\left(\frac{x}{2}\right) < \left(\frac{x}{2}\right)\ln 6.$$

Changing x to $\frac{x}{2}$, $\frac{x}{4}$, $\frac{x}{8}$, $\cdots$, we have

$$\begin{aligned}
\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) &< \left(\frac{x}{4}\right)\ln 6,\\
\psi\left(\frac{x}{4}\right) - \psi\left(\frac{x}{8}\right) &< \left(\frac{x}{8}\right)\ln 6,\\
\psi\left(\frac{x}{8}\right) - \psi\left(\frac{x}{16}\right) &< \left(\frac{x}{16}\right)\ln 6,\\
&\vdots
\end{aligned}$$

Adding up them all, we have

$$\psi(x) < x\ln 6.$$

It is easy to see that the relationship between $\theta(x)$ and $\psi(x)$ is given by

$$\psi(x) = \sum_{d\geq 1}\theta\left(x^{\frac{1}{d}}\right).$$

There is the fact that

$$\theta(x) \leq \psi(x) < x\ln 6.$$

Lemma 17. *For $x \geq 3$, let*

$$\varrho(x) = \left(\prod_{2<p\leq x} p\right)\left(1 - \prod_{2<p\leq x}\left(1 - 2p^{-1}\right)\right)^{\left(\frac{\ln^2 x}{0.4}\right)x\ln 6},$$

then $\varrho(x) < 1$.

Proof. By Lemma 16,

$$\ln\left(\prod_{2<p\leq x} p\right) < \theta(x) < x\ln 6,$$

thus

$$\left(\prod_{2<p\leq x} p\right) < \exp\{x\ln 6\}.$$

By Lemma 14,

$$\left(1-\prod_{2<p\leq x}\left(1-2p^{-1}\right)\right)\leq\left(1-\frac{0.4}{\ln^2 x}\right).$$

Combining these results with numerical analysis we obtain

$$\begin{aligned}\varrho\left(x\right)&<\exp\left\{x\ln 6\right\}\left(1-\frac{0.4}{\ln^2 x}\right)^{\left(\frac{\ln^2 x}{0.4}\right)x\ln 6}\\&<\exp\left\{x\ln 6\right\}\exp\left\{-x\ln 6\right\}\\&=1.\end{aligned}$$

4. Estimation of $L_2(a,t)$ and η

In this section we estimate $L_2\left(a,t\right)$ and η.
First, for $t\geq 0$, let

$$L_1\left(a,t\right)=\left\{m:\ \left\{m,m+1,m+2,\cdots,m+t\right\}\subseteq\mathbb{M}_i\left[a,a+T\left(0\right)+t\right)\right\}.$$

We can see that for each element in $L_1\left(a,t\right)$, it denotes that there are $(t+1)$ consecutive elements in $\mathbb{M}_i\left[a,a+T\left(0\right)+t\right)$. We have

$$\left|L_1\left(a,t\right)\right|\leq T\left(0\right)\left(1-\frac{T\left(1\right)}{T\left(0\right)}\right)^{t+1}.$$

Proof. For $t=0$, obviously,

$$\left|L_1\left(a,0\right)\right|=T\left(0\right)-T\left(1\right)=T\left(0\right)\left(1-\frac{T\left(1\right)}{T\left(0\right)}\right).$$

Otherwise, let

$$s=\sum_{\substack{m_0\in\mathbb{M}^\circ\\ m_1\in\mathbb{M}^\circ\\ \vdots\\ m_t\in\mathbb{M}^\circ}}1,$$

$$\left\{\mathcal{H}_1,\mathcal{H}_2,\cdots,\mathcal{H}_s\right\}=\bigcup_{\substack{m_0\in\mathbb{M}^\circ\\ m_1\in\mathbb{M}^\circ\\ \vdots\\ m_t\in\mathbb{M}^\circ}}\left(\left\{\left\{m_0q^0,m_1q^1,\cdots,m_tq^t\right\}\right\}\right),$$

where $q>1$ is any pseudo prime number that

$$\left(\forall m\in\mathbb{M}^\circ\right)\left(q\not\equiv 0\left(mod\ m\right)\right).$$

And

$$v\left(\mathcal{H}_1, \mathcal{H}_2, \cdots\right) = \frac{q^{\alpha}}{\prod\limits_{\delta \in (\mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots)} \delta},$$

where α is the number of pseudo prime factor q contained in

$$\prod_{\delta \in (\mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots)} \delta.$$

Considering the proof of Lemma 7 and Lemma 8,

$$|L_1(a,t)| = \left| \bigcap_{w \in [0,t]} \mathbb{M}_{i+w}\left[a, a+T(0)\right) \right|$$

$$= \left| \bigcup_{\substack{m_i \in \mathbb{M}_i[a, a+T(0)) \\ m_{i+1} \in \mathbb{M}_{i+1}[a, a+T(0)) \\ \vdots \\ m_{i+t} \in \mathbb{M}_{i+t}[a, a+T(0))}} \left(\bigcap_{d \in [i, i+t]} m_d \right) \right|$$

$$= \left| \bigcup_{\substack{(\forall k_0 \in \mathbb{Z} \wedge m_0 \in \mathbb{M}^{\circ})\left(m_i = k_0 m_0 + i \wedge m_i \in [a, a+T(0))\right) \\ (\forall k_1 \in \mathbb{Z} \wedge m_1 \in \mathbb{M}^{\circ})\left(m_{i+1} = k_1 m_1 + i + 1 \wedge m_{i+1} \in [a, a+T(0))\right) \\ \vdots \\ (\forall k_t \in \mathbb{Z} \wedge m_t \in \mathbb{M}^{\circ})\left(m_{i+t} = k_t m_t + i + t \wedge m_{i+t} \in [a, a+T(0))\right)}} \left(\bigcap_{d \in [i, i+t]} m_d \right) \right|.$$

Combining this with Lemma 12 and Lemma 13,

$$|L_1(a,t)| \le$$

$$\mathrm{T}(0)\left(\begin{array}{c}\sum\limits_{\substack{\{d_1\}\subseteq\{1,2,\cdots,s\}\\ |\{d_1\}|=1}} v(\mathcal{H}_{d_1})-\sum\limits_{\substack{\{d_1,d_2\}\subseteq\{1,2,\cdots,s\}\\ d_1<d_2}} v(\mathcal{H}_{d_1},\mathcal{H}_{d_2})+\\ \sum\limits_{\substack{\{d_1,d_2,d_3\}\subseteq\{1,2,\cdots,s\}\\ d_1<d_2<d_3}} v(\mathcal{H}_{d_1},\mathcal{H}_{d_2},\mathcal{H}_{d_3})-\\ \cdots+(-1)^{s-1}\sum\limits_{\substack{\{d_1,d_2,\cdots,s\}\subseteq\{1,2,\cdots,s\}\\ d_1<d_2<\cdots<d_s}} v(\mathcal{H}_{d_1},\mathcal{H}_{d_2},\cdots,\mathcal{H}_{d_s})\end{array}\right)$$

$$=T(0)\left(\begin{array}{c}\sum\limits_{\substack{\{m_1\}\subseteq\mathbb{M}^\circ\\ |\{m_1\}|=1}}\left(\frac{1}{m_1}\right)-\sum\limits_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^\circ\\ m_1<m_2}}\left(\frac{1}{m_1m_2}\right)+\\ \sum\limits_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3}}\left(\frac{1}{m_1m_2m_3}\right)-\\ \cdots+(-1)^{n-1}\sum\limits_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{1}{m_1m_2m_3\cdots m_n}\right)\end{array}\right)^{t+1}$$

$$=T(0)\left(1-\frac{T(1)}{T(0)}\right)^{t+1}.$$

So that

$$(\forall t\geq 0)\left(|L_1(a,t)|\leq T(0)\left(1-\frac{T(1)}{T(0)}\right)^{t+1}\right).$$

Next, let us look at the case of

$$L_2(a,t)=\{m:\{m,m+1,m+2,\cdots,m+t\}\subseteq\mathbb{M}_{i\cup j}[a,a+T(0)+t)\}.$$

We can also see that for each element in $L_2(a,t)$, it denotes that there are $(t+1)$ consecutive elements in $\mathbb{M}_{i\cup j}[a,a+T(0)+t)$.
It is similar to the case of $L_1(a,t)$, combining this with Lemma 9, we have

$$|L_2(a,t+1)|\leq$$

$$T(0)\left(\begin{array}{c}\sum\limits_{\substack{\{m_1\}\subseteq\mathbb{M}^\circ\\ |\{m_1\}|=1}}\left(\frac{2^1}{m_1}\right)-\sum\limits_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^\circ\\ m_1<m_2}}\left(\frac{2^2}{m_1m_2}\right)+\\ \sum\limits_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3}}\left(\frac{2^3}{m_1m_2m_3}\right)-\\ \cdots+(-1)^{n-1}\sum\limits_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{2^n}{m_1m_2m_3\cdots m_n}\right)\end{array}\right)^{t+1}$$

$$= T(0)\left(1 - \frac{T(2)}{T(0)}\right)^{t+1}.$$

Now we can deduce that

$$(\forall t \geq 0)\left(\ |L_2(a,t)| \leq T(0)\left(1 - \frac{T(2)}{T(0)}\right)^{\mathrm{t+1}}\ \right). \tag{4.1}$$

Considering the relationship between $L_2(a,t)$ and η (in Lemma 10, Lemma 11), we have

$$\eta \geq \inf\{m+1 :\ m \geq 0 \wedge |L_2(a,m)| = 0\}, \tag{4.2}$$

according to the definition of $L_2(a,t)$.
For the next proof of theorems, we assume that η satisfies

$$\eta \leq \frac{p_n^2}{8}. \tag{4.3}$$

This requires

$$\left|L_2\left(a, \lfloor\frac{p_n^2}{8}\rfloor - 1\right)\right| = 0.$$

By (4.1), we have

$$\left|L_2\left(a, \lfloor\frac{p_n^2}{8}\rfloor - 1\right)\right| < T(0)\left(1 - \frac{T(2)}{T(0)}\right)^{\lfloor\frac{p_n^2}{8}\rfloor}$$

$$= \left(\prod_{p\in\mathbb{M}^\circ} p\right)\left(1 - \prod_{p\in\mathbb{M}^\circ}\left(1 - 2p^{-1}\right)\right)^{\lfloor\frac{p_n^2}{8}\rfloor}.$$

By Lemma 17, we know that we have

$$\left|L_2\left(a, \lfloor\frac{p_n^2}{8}\rfloor - 1\right)\right| < 1$$

when

$$\left(\frac{\ln^2 p_n}{0.4}\right) p_n \ln 6 < \lfloor\frac{p_n^2}{8}\rfloor. \tag{4.4}$$

Let

$$f(x) = \frac{x^2}{8} - \left(\frac{\ln^2 x}{0.4}\right) x \ln 6.$$

Then for $x \neq 0$,

$$\left(\frac{f(x)}{x}\right)' = \frac{1}{8} - \frac{2\ln 6 \ln x}{0.4x}.$$

We can easily get a crude result that

$$\left(\frac{f(x)}{x}\right)' > 0$$

when $x > 436$ through numerical analysis.
So that $f(x)$ is monotonically increasing when $x > 436$.
Next, the numerical analysis is continued, we can easily get another crude result that

$$f(x) > 0$$

when $x > 2096$.
Now we know that the condition (4.4) is satisfied when $p_n > 2096$.
Therefore, (4.3) holds for $p_n > 2096$.

5. **Proof of theorems**

We are now in the position to prove **Theorem 1** and **2**.
For n with $p_n \leq 2096$, we know that the theorems hold through computer verification.
Otherwise, we have

$$\eta \leq \frac{p_n^2}{8}.$$

Since

$$\eta \leq \frac{p_n^2}{8} < \frac{p_n^2 + 1}{2},$$

combining this with Lemma 11, we have

$$\left(\forall a, i, j \in \left[i+1, i+\frac{p_n+1}{2}\right)\right)$$
$$\left(\left(\exists h \in \left[a, a+\frac{p_n^2+1}{2}\right)\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right) = \chi\left(h+j-i,\ \mathbb{M}_i\right) = 0\ \right)\right).$$

And let

$$a = i + \frac{T(0) + p_n}{2},$$

we have

$$\left(\forall i, j \in \left[1, \frac{p_n+1}{2}\right)\right)\left(\left(\exists h \in \left[i+\frac{T(0)+p_n}{2},\right.\right.\right.$$
$$\left.\left.\left. i+\frac{T(0)+p_n^2+p_n+1}{2}\right)\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right) = \chi\left(h+j,\ \mathbb{M}_i\right) = 0\ \right)\right). \tag{5.1}$$

Then we can deduce that for every h in (5.1) satisfying the condition

$$\left(\ \chi\left(h,\ \mathbb{M}_i\right) = \chi\left(h+j,\ \mathbb{M}_i\right) = 0\ \right),$$

so we have q_1 and q_2 are both prime numbers, defined by

$$q_1 = 2(h-i) - T(0),$$
$$q_2 = q_1 + 2j = 2(h-i) - T(0) + 2j.$$

Proof. Since

$$\chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h+j,\ \mathbb{M}_i\right)=0,$$

we have

$$\chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h-i,\ \lambda\left(\mathbb{M}_i,\ 0-i\right)\right)=\chi\left(h-i,\ \mathbb{M}_0\right)=0.$$

Because the prime number 2 does not belong to $\mathbb{M}^{\circ}$,
by Lemma 4, we have

$$\chi\left(h,\ \mathbb{M}_i\right)=\chi\left(2\left(h-i\right)+i,\ \mathbb{M}_i\right)=0.$$

Combining this with Lemma 5 we have

$$\chi\left(h,\ \mathbb{M}_i\right)=\chi\left(\,2\left(h-i\right)+i-T\left(0\right),\ \mathbb{M}_i\right)=\chi\left(\,q_1,\ \mathbb{M}_0\right)=0.$$

i.e.

$$(\forall m\in\mathbb{M}^{\circ})\left(\ q_1\not\equiv 0\,(mod\ m)\ \right). \tag{5.2}$$

Similarly, we have

$$\chi\left(h+j,\ \mathbb{M}_i\right)=\chi\left(\,2\left(h-i\right)+i-T\left(0\right)+2j,\ \mathbb{M}_i\right)=\chi\left(q_2,\ \mathbb{M}_0\right)=0.$$

i.e.

$$(\forall m\in\mathbb{M}^{\circ})\left(\ q_2\not\equiv 0\,(mod\ m)\ \right). \tag{5.3}$$

Noting that the domain of h, we can deduce

$$\begin{aligned} q_1&\in\left[p_n,\ p_n\left(p_n+1\right)\right],\\ q_2&\in\left[p_n+2,\ p_n\left(p_n+2\right)\right].\end{aligned}$$

Obviously,

$$\begin{aligned} T\left(0\right)&\not\equiv 0\,(mod\ 2)\,,\\ q_1&\not\equiv 0\,(mod\ 2)\,,\\ q_2&\not\equiv 0\,(mod\ 2)\,.\end{aligned}$$

And $\mathbb{M}^{\circ}$ contains all odd primes not greater than p_n, so that

$$\forall w\in\left[p_n,\ p_n\left(p_n+2\right)\right],$$

if w is not a prime number, there must be

$$(\exists m\in(\mathbb{M}^{\circ}\cup\{2\}))\left(\ w\equiv 0\,(mod\ m)\ \right).$$

Thus, combined with (5.2) and (5.3), q_1 and q_2 must be prime numbers.
This implies that

for every $p_s>2096$, there must be primes p_a and p_b between p_s and $p_s^2+2p_s$,

$$\left(\forall d\in\left[1,\frac{p_s+1}{2}\right)\right)\left(\ p_a-p_b=2d\ \right).$$

i.e.

$$(\forall p_s > 2096)\left(\left(\exists p_a, p_b \in \left[p_s, p_s^2 + 2p_s\right]\right)\right.$$
$$\left.\left(\left(\forall d \in \left[1, \frac{p_s+1}{2}\right)\right)\left(\ p_a - p_b = 2d\ \right)\right)\right).$$

Since there are infinite primes, we can conclude that

for any positive integer d, there are infinitely many prime gaps of size $2d$.

This proves **Theorem 1**.

Next, let us transform the problem of gaps between primes into the problem of sums of primes.

Let

$$a = i + \frac{(T(0) + p_1)}{2}.$$

Since

$$\eta \leq \frac{p_n^2}{8},$$

combining this with Lemma 10, we have

$$\left(\forall i, j \in \left[i + p_1 + \lceil\frac{p_{\mathrm{n}}^2}{8}\rceil, i + p_1 + \lfloor\frac{p_n^2}{2}\rfloor\right)\right)\left(\left(\exists h \in \left[i + \frac{T(0) + p_1}{2},\right.\right.\right.$$
$$\left.\left.\left. i + \frac{T(0)+p_1}{2} + \lceil\frac{p_{\mathrm{n}}^2}{8}\rceil\right)\right)\ \left(\ \chi\left(h,\ \mathbb{M}_{i\cup j}\right) = 0\ \right)\right). \tag{5.4}$$

Then we can deduce that for every h in (5.4) satisfying the condition

$$\left(\ \chi\left(h,\ \mathbb{M}_{i\cup j}\right) = 0\ \right),$$

so we have q_1 and q_2 are both prime numbers, defined by

$$q_1 = 2(h-i) - T(0),$$
$$q_2 = 2(j-h) + T(0).$$

Proof. By the condition,

$$\chi\left(h,\ \mathbb{M}_i\right) = \chi\left(h,\ \mathbb{M}_{\mathrm{j}}\right) = 0.$$

Then it is similar to the proof of **Theorem 1**,

$$\chi\left(q_1,\ \mathbb{M}_0\right) = \chi\left(2(h-i),\ \mathbb{M}_0\right) = \chi\left(h-i,\ \mathbb{M}_0\right) = \chi\left(h,\ \mathbb{M}_i\right) = 0,$$
$$\chi\left(q_2,\ \mathbb{M}_0\right) = \chi\left(2(j-h),\ \mathbb{M}_0\right) = \chi\left(h-j,\ \mathbb{M}_0\right) = \chi\left(h,\ \mathbb{M}_j\right) = 0.$$

It is easy to see that

$$(\forall m \in (\mathbb{M}^\circ \cup \{2\}))\left(\ q_1 \not\equiv 0\ (mod\ m)\ \right),$$

$$(\forall m \in (\mathbb{M}^\circ \cup \{2\}))\left(\ q_2 \not\equiv 0\ (mod\ m)\ \right).$$

Noting that the domain of h, we can deduce

$$q_1 \in \left[p_1,\ 2\lceil\frac{p_n^2}{8}\rceil + p_1\right],$$

$$q_2 \in \left[p_1,\ p_n^2 + p_1\right].$$

So q_1 and q_2 are both prime numbers.
Now let us look at the domain of $(q_1 + q_2)$,

$$q_1 + q_2 = 2\,(j - i) \in \left[2p_1 + 2\lceil\frac{p_n^2}{8}\rceil,\ 2p_1 + 2\lfloor\frac{p_n^2}{2}\rfloor\right).$$

This implies that

for every $p_s > 2096$, there must be primes p_a and p_b between p_1 and $p_s^2 + p_1$,

$$\left(\forall d \in \left[p_1 + \lceil\frac{p_s^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\left(\ p_a + p_b = 2d\ \right).$$

i.e.

$$(\forall p_s > 2096)\left(\left(\exists p_a, p_b \in \left[p_1, p_s^2 + p_1\right]\right)\left(\left(\forall d \in \left[p_1 + \lceil\frac{p_s^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\left(\ p_a + p_b = 2d\ \right)\right)\right). \tag{5.5}$$

By Bertrand-Chebyshev theorem [1], we have

$$p_{s+1} < 2p_s,$$

then

$$\frac{p_{s+1}^2}{8} < \frac{p_s^2}{2},$$

so

$$(\forall s > 1)\left(\left[p_1 + \lceil\frac{p_s^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right) \cap \left[p_1 + \lceil\frac{p_{s+1}^2}{8}\rceil,\ p_1 + \lfloor\frac{p_{s+1}^2}{2}\rfloor\right) \neq \phi\right).$$

Combining this with (5.5), we can conclude that

$$(\forall p_s > 2096)\left((\exists p_a, p_b)\left(\left(\forall d \in \left[p_1 + \lceil\frac{p_u^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\left(\ p_a + p_b = 2d\ \right)\right)\right),$$

where p_u is the smallest prime greater than 2096, that is, 2099.
It is easy to get

$$p_1 + \lceil\frac{p_u^2}{8}\rceil = 3 + 550726 = 550729.$$

i.e.

$$(\forall p_s > 2096)\left((\exists p_a, p_b)\left(\left(\forall d \in \left[550729,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\left(\ p_a + p_b = 2d\ \right)\right)\right).$$

While the results of $d \in [1, 550729)$ can be obtained by computer-aided verification.
Since there are infinite primes, we can conclude that

for every even integer greater than 2 is the sum of two prime numbers.

This proves **Theorem 2**.

References

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