Álgebra Linear
Exercícios sobre Transformações Lineares

Prof. André Tiba

akot@cin.ufpe.br

Baia 65, ramais: 4765 ou 4338

Exercício 1

Qual a transformação linear T: $\mathbf{R}^2 \rightarrow \mathbf{R}^4$ tal que $T(2,1) = (1,0,1,-1)$ e $T(-1, 2) = (0,0,1,1)$?

<u>Solução 1:</u>

Deseja-se encontrar a transformação T tal que:
$T(\mathbf{v_1}) = \mathbf{w_1}$ e $T(\mathbf{v_2}) = \mathbf{w_2}$

$\mathbf{v_1} = (2,1)$, $\mathbf{v_2} = (-1,2)$, $\mathbf{w_1} = (1,0,1,-1)$, $\mathbf{w_2} = (0,0,1,1)$

$$(2,1) = 2.(1, 0) + 1.(0, 1)$$
$$T(2,1) = 2.T(1, 0) + 1.T(0,1) = (1, 0, 1, -1)$$

$$(-1,2) = -1.(1, 0) + 2.(0, 1)$$
$$T(-1, 2) = -1.T(1, 0) + 2.T(0, 1) = (0, 0, 1, 1)$$

Exercício 1 (continuação)

$$\begin{cases} T_{21} = 2T_{10} + T_{01} \\ T_{-12} = - T_{10} + 2T_{01} \end{cases}$$

onde $T_{ij} = T(i,j)$

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} T_{10} \\ T_{01} \end{bmatrix} = \begin{bmatrix} T_{21} \\ T_{-12} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & T_{21} \\ -1 & 2 & T_{-12} \end{bmatrix} \quad L_1 = L_1/2$$

$$\begin{bmatrix} 1 & 1/2 & T_{21}/2 \\ -1 & 2 & T_{-12} \end{bmatrix} L_2 = L_2 + L_1 \qquad \begin{bmatrix} 1 & 1/2 & T_{21}/2 \\ 0 & 5/2 & T_{21}/2 + T_{-12} \end{bmatrix} \quad L_2 = (2/5)\, L_2$$

$$\begin{bmatrix} 1 & 1/2 & T_{21}/2 \\ 0 & 1 & (T_{21} + 2T_{-12})/5 \end{bmatrix} L_1 = L_1 - L_2/2 \qquad \begin{bmatrix} 1 & 0 & (4T_{21} - 2T_{-12})/10 \\ 0 & 1 & (T_{21} + 2T_{-12})/5 \end{bmatrix}$$

$$\begin{bmatrix} T_{10} \\ T_{01} \end{bmatrix} = \begin{bmatrix} (4T_{21} - 2T_{-13})/10 \\ (T_{21} + 2T_{-13})/5 \end{bmatrix}$$

mas $T(2,1) = (1,0,1,-1)$ e $T(-1, 3) = (0,0,1,1)$

Exercício 1 (continuação)

$$\begin{bmatrix} T_{10} \\ T_{01} \end{bmatrix} = \begin{bmatrix} (4(1,0,1,-1) - 2(0,0,1,1))/10 \\ ((1,0,1,-1) + 2(0,0,1,1))/5 \end{bmatrix} = \begin{bmatrix} (2/5,0,1/5,-3/5) \\ (1/5,0,3/5,1/5) \end{bmatrix}$$

Seja um vetor (x,y) qualquer do $\mathbf{R}^2$:

$$(x,y) = x.(1,0) + y.(0,1)$$

$$\mathrm{T}(x,y) = x.\mathrm{T}(1,0) + y.\mathrm{T}(0,1)$$

$$\mathrm{T}(x,y) = x.\mathrm{T}_{10} + y.\mathrm{T}_{01}$$

$$\mathrm{T}(x,y) = x.(2/5,\ 0,\ 1/5,\ -3/5) + y\ .(1/5,\ 0,\ 3/5,\ 1/5)$$

$$\mathrm{T}(x,y) = ((2x+ y)/5,\ 0,\ (x + 3y)/5,\ (-3x+y)/5)$$

$$\mathrm{T}(2,1) = ((2.2+ 1)/5,\ 0,\ (2 + 3.1)/5,\ (-3.2 + 1)/5) = (1,\ 0,\ 1,\ -1)$$

$$\mathrm{T}(-1,2) = ((2.(-1)+ 2)/5,\ 0,\ (-1 + 3.2)/5,\ (-3.(-1) + 2)/5) = (0,\ 0,\ 1,\ 1)$$

Exercício 2

Qual a transformação linear T: $\mathbf{R}^3 \to \mathbf{R}^2$ tal que T(1, 0, 1) = (2, −1) T(2, −1, 0) = (0, 1) e T(1, 0, 3) = (1, 1) ?

Solução 1:

Deseja-se encontrar a transformação T tal que:

$T(\mathbf{v_1}) = \mathbf{w_1}$, $T(\mathbf{v_2}) = \mathbf{w_2}$ e $T(\mathbf{v_3}) = \mathbf{w_3}$

$\mathbf{v_1} = (1,0,1)$, $\mathbf{v_2} = (2,-1,0)$, $\mathbf{v_3} = (1,3,0)$, $\mathbf{w_1} = (2,-1)$, $\mathbf{w_2} = (0,1)$ e $\mathbf{w_3} = (1,1)$

$$(1, 0, 1) = (1, 0, 0) + (0, 0, 1)$$
$$T(1, 0, 1) = T(1, 0, 0) + T(0, 0, 1) = (2, -1)$$

$$(2, -1, 0) = 2(1, 0, 0) - 1(0, 1, 0)$$
$$T(2, -1, 0) = 2T(1, 0, 0) - T(0, 1, 0) = (0, 1)$$

$$(1, 0, 3) = (1, 0, 0) + 3(0, 1, 0)$$
$$T(1, 0, 3) = T(1, 0, 0) + 3T(0, 0, 1) = (1, 1)$$

Exercício 2 (continuação)

$$\begin{cases} T_{101} = T_{100} + T_{001} \\ T_{2\text{-}10} = 2T_{100} - T_{010} \\ T_{103} = T_{100} + 3T_{001} \end{cases}$$

onde $T_{ijz} = T(i,j,k)$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} T_{100} \\ T_{010} \\ T_{001} \end{bmatrix} = \begin{bmatrix} T_{101} \\ T_{2-10} \\ T_{130} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & T_{101} \\ 2 & -1 & 0 & T_{2-10} \\ 1 & 0 & 3 & T_{130} \end{bmatrix}$$

$L_2 = L_2 - 2L_1$

$L_3 = L_3 - L_1$

$$\begin{bmatrix} 1 & 0 & 1 & T_{101} \\ 0 & -1 & -2 & T_{2-10} - 2T_{101} \\ 0 & 0 & 2 & T_{130} - T_{101} \end{bmatrix}$$

$L_1 = L_1 - L_3/2$

$L_2 = L_2 + L_3$

$$\begin{bmatrix} 1 & 0 & 0 & (-T_{130} + 3T_{101})/2 \\ 0 & -1 & 0 & T_{2-10} + T_{130} - 3T_{101} \\ 0 & 0 & 2 & T_{130} - T_{101} \end{bmatrix}$$

$L_2 = -L_2$

$L_3 = L_3/2$

$$\Rightarrow \begin{bmatrix} T_{100} \\ T_{010} \\ T_{001} \end{bmatrix} = \begin{bmatrix} (-T_{130} + 3T_{101})/2 \\ -T_{130} + 3T_{101} - T_{2-10} \\ (T_{130} - T_{101})/2 \end{bmatrix}$$

Exercício 2 (continuação)

$$\begin{bmatrix} T_{100} \\ T_{010} \\ T_{001} \end{bmatrix} = \begin{bmatrix} (-T_{130} + 3T_{101})/2 \\ -T_{130} + 3T_{101} - T_{2-10} \\ (T_{130} - T_{101})/2 \end{bmatrix}$$

mas $T_{130} = (1,1)$, $T_{101} = (2,-1)$ e $T_{2\text{-}10} = (0,1)$

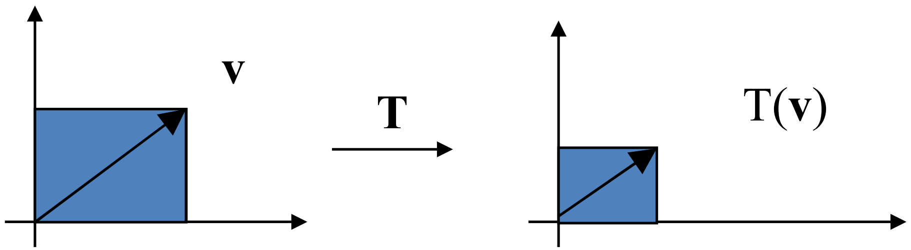
$$\begin{bmatrix} T_{100} \\ T_{010} \\ T_{001} \end{bmatrix} = \begin{bmatrix} (-(1,1) + 3(2,-1))/2 \\ -(1,1) + 3(2,-1) - (0,1) \\ ((1,1) - (2,-1))/2 \end{bmatrix} = \begin{bmatrix} (5/2,-2) \\ (5,-5) \\ (-1/2,1) \end{bmatrix}$$

Seja um vetor (x,y,z) qualquer do $\mathbf{R}^3$:

$$\begin{aligned}
(x,y,z) &= x.(1,0,0) + y.(0,1,0) + z.(0,0,1) \\
\mathrm{T}(x,y,z) &= x.\mathrm{T}(1,0,0) + y.\mathrm{T}(0,1,0) + z.\mathrm{T}(0,0,1) \\
\mathrm{T}(x,y,z) &= x.\mathrm{T}_{100} + y.\mathrm{T}_{010} + z.\ \mathrm{T}_{001} \\
\mathrm{T}(x,y,z) &= x.(5/2, -2) + y\ .(5, -5) + z.(-1/2, 1) \\
\mathrm{T}(x,y,z) &= (5x/2 + 5y - z\ /2, -2x - 5y + z)
\end{aligned}$$

Exercício 3

Ache a aplicação linear T, que representa uma contração de $2^{-1/2}$ seguida por uma rotação horária de 45º.



$$\alpha = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \mathrm{T}(x,y) = \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)$$

$$\begin{bmatrix} x/\sqrt{2} \\ y/\sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} \dfrac{x}{\sqrt{2}}\cos\theta - \dfrac{y}{\sqrt{2}}\sin\theta \\ \dfrac{x}{\sqrt{2}}\sin\theta + \dfrac{y}{\sqrt{2}}\cos\theta \end{bmatrix}$$

rotação horária de 45º
→ θ = -45º = - π/4

$$T(x,y) = \left(\frac{x}{\sqrt{2}}\frac{\sqrt{2}}{2} - \frac{y}{\sqrt{2}}\frac{-\sqrt{2}}{2}, \frac{x}{\sqrt{2}}\frac{-\sqrt{2}}{2} + \frac{y}{\sqrt{2}}\frac{\sqrt{2}}{2}\right) = \left(\frac{x+y}{2}, \frac{-x+y}{2}\right)$$

Exercício 4

Sejam: $\beta = \{(1,-1),(0,2)\}$ e $\beta' = \{(1,0,-1),(0,1,2),(1,2,0)\}$, bases de $\mathbf{R}^2$ e $\mathbf{R}^3$, e a matriz:

$$[\mathrm{T}]_{\beta'}^{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Então:

a) Ache T

b) Se $\mathrm{S}(x,y) = (2y, x-y, x)$, ache $[\mathrm{S}]_{\beta'}^{\beta}$

c) *ker* T, *Im* T, *ker* S e *Im* S

Exercício 4 (continuação)

<u>Letra a)</u>

$\beta = \{(1,-1),(0,2)\}$ e $\beta' = \{(1,0,-1),(0,1,2),(1,2,0)\}$ $\quad [T]^{\beta}_{\beta'} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$

$T(1,-1) = 1.(1, 0, -1) + 1.(0, 1, 2) + 0.(1, 2, 0) = (1, 1, 1)$

$T(0,2) = 0.(1, 0, -1) + 1.(0, 1, 2) - 1.(1, 2, 0) = (-1, -1, 2)$

Seja (x,y) um vetor do $\mathbf{R}^2$, podemos escrevê-lo em termos da base β como:

$(x,y) = a(1,-1) + b(0,2) = (a, -a + 2b) \rightarrow a = x$ e $b = (y + x)/2$

$(x,y) = x.(1,-1) + (½).(y + x).(0,1)$

Exercício 4 (continuação)

Aplicando T a equação,

$$T(x,y) = x.T(1,-1) + (½).(y + x).T(0,2)$$

$$T(x,y) = x.(1, 1, 1) + (½).(y + x).(-1, -1, 2)$$

$$T(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}, 2x+y\right)$$

Exercício 4 (continuação)

Letra b)

$$\beta = \{(1,-1),(0,2)\} \text{ e } \beta' = \{(1,0,-1),(0,1,2),(1,2,0)\}$$

$S(x,y) = (2y, x-y, x)$

$S(1,-1) = (-2, 2, 1)$ $\quad\quad$ $S(0,2) = (4,-2,0)$

Precisamos achar o vetor de coordenadas $[\mathbf{v}]_{\beta'}$ para um vetor qualquer $\mathbf{v} = (x,y,z)$ do $\mathbf{R}^3$.

$$(x,y,z) = a(1,0,-1) + b(0,1,2) + c(1,2,0)$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 2 & y \\ -1 & 2 & 0 & z \end{bmatrix} \quad L_3 = L_3 + L_1$$

Exercício 4 (continuação)

$$\begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 2 & y \\ 0 & 2 & 1 & x+z \end{bmatrix} \quad L_3 = L_3 - 2L_2 \quad \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 2 & y \\ 0 & 0 & -3 & x+z-2y \end{bmatrix} \quad L_3 = -L_3/3$$

$$\begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 2 & y \\ 0 & 0 & 1 & (-x+2y-z)/3 \end{bmatrix} \begin{matrix} L_1 = L_1 - L_3 \\ L_2 = L_2 - 2L_3 \\ \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 & (4x-2y+z)/3 \\ 0 & 1 & 0 & (2x-y+2z)/3 \\ 0 & 0 & 1 & (-x+2y-z)/3 \end{bmatrix}$$

$$\left[\mathbf{v} = (x,y,z)\right]_{\beta'} = \begin{bmatrix} (4x-2y+z)/3 \\ (2x-y+2z)/3 \\ (-x+2y-z)/3 \end{bmatrix}$$

Exercício 4 (continuação)

Vamos testar para verificar se $[\mathbf{v} = (x,y,z)]_{\beta'}$ foi encontrado corretamente. Para isso, basta jogar os vetores da base β'.

$$[\mathbf{v} = (x, y, z)]_{\beta'} = \begin{bmatrix} (4x - 2y + z)/3 \\ (2x - y + 2z)/3 \\ (-x + 2y - z)/3 \end{bmatrix}$$

$$[(1,0,-1)]_{\beta'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [(1,2,0)]_{\beta'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [(0,1,2)]_{\beta'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Teste ok ! Agora substitui os vetores: (–2, 2,1) e (4, –2, 0).

$$[(-2,2,1)]_{\beta'} = \begin{bmatrix} -11/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

$$[(4,-2,0)]_{\beta'} = \begin{bmatrix} 20/3 \\ 10/3 \\ -8/3 \end{bmatrix}$$

$$[s]_{\beta'}^{\beta} = \begin{bmatrix} -11/3 & 20/3 \\ -1/3 & 10/3 \\ 5/3 & -8/3 \end{bmatrix}$$

Exercício 4 (continuação)

<u>Letra c)</u>

Encontrar *ker* T, *Im* T, *ker* S e *Im* S

Seja T:$V \rightarrow W$

- $Im(\mathrm{T}) = \{\mathbf{w} \in W\ ;\ \mathrm{T}(\mathbf{v}) = \mathbf{w}$ para algum $\mathbf{v} \in V\}$
- $ker\ \mathrm{T} = \{\mathbf{v} \in V\ ;\ \mathrm{T}(\mathbf{v}) = \mathbf{0}\}$

$$T(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}, 2x + y \right)$$

$$\mathrm{Im}T = \left\{ \mathbf{w} \in R^3 : \mathbf{w} = \left(\frac{x+y}{2}, \frac{x+y}{2}, 2x + y \right); x, y \in \Re \right\}$$

$$\mathrm{Im}T = \left\{ x\left(\frac{1}{2}, \frac{1}{2}, 2 \right) + y\left(\frac{1}{2}, \frac{1}{2}, 1 \right); x, y \in \Re \right\} = \left\langle \left(\frac{1}{2}, \frac{1}{2}, 2 \right), \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \right\rangle$$

dim (*Im* T) = 2

Exercício 4 (continuação)

$$\ker T = \left\{ \mathbf{v} \in R^2 : T(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}, 2x+y \right) = (0,0,0); x, y \in \Re \right\}$$

$\ker T = \{(x,y) : (0,0)\} = \langle (0,0) \rangle$ dim (*ker* T) = 0

Observe o teorema: dim (*ker* T) + dim (*Im* T) = dim *V*

2 + *0* = *2*

S(*x*,*y*) = (2*y*, *x* – *y*, *x*)

$$\operatorname{Im} S = \left\{ \mathbf{w} \in R^3 : \mathbf{w} = (2y, x-y, x); x, y \in \Re \right\}$$

$$\operatorname{Im} S = \{ x(0,1,1) + y(2,-1,0); x, y \in \Re \} = \langle (0,1,1), (2,-1,0) \rangle$$

dim (*Im* T) = 2

Exercício 4 (continuação)

$$\ker S = \left\{ \mathbf{v} \in R^2 : T(x,y) = \left(2y, x-y, 2x\right) = (0,0,0); x, y \in \Re \right\}$$

$$\ker T = \{(x,y) : (0,0)\} = \langle (0,0) \rangle$$ dim (*ker* S) = 0

Observe o teorema: dim (*ker* S) + dim (*Im* S) = dim *V*

2 + *0* = *2*

Exercício 5

Seja a transformação linear T: $\mathbf{R}^3 \to \mathbf{R}^3$ tal que

$$T(x,y,z) = (z,\ x - y,\ -z)$$

Determine:

a) Uma base do núcleo de T
b) A dimensão da Imagem de T
c) T é sobrejetora ?

Exercício 5

Letra a):

Seja T:$V \rightarrow W$

- $Im(\mathrm{T}) = \{\mathbf{w} \in W \,;\, \mathrm{T}(\mathbf{v}) = \mathbf{w}$ para algum $\mathbf{v} \in V\}$
- $ker\ \mathrm{T} = \{\mathbf{v} \in V \,;\, \mathrm{T}(\mathbf{v}) = \mathbf{0}\}$

$$\mathrm{T}(x,y,z) = (z, x - y, -z) \rightarrow V = W = \mathbf{R}^3$$

$$kerT = \left\{\mathbf{v} \in R^3 : T(x,y,z) = \left(z, x - y, -z\right) = (0,0,0); x, y, z \in \Re\right\}$$

$$kerT = \left\{(1,1,0) : (z, x - y, -z) = (0,0,0)\right\} = \left\langle (1,1,0) \right\rangle$$

dim (ker T) = 1

Exercício 5

Letra b):

Seja T:$V \to W$

- $Im(\mathrm{T}) = \{\mathbf{w} \in W \,;\, \mathrm{T}(\mathbf{v}) = \mathbf{w}$ para algum $\mathbf{v} \in V\}$
- $ker\ \mathrm{T} = \{\mathbf{v} \in V \,;\, \mathrm{T}(\mathbf{v}) = \mathbf{0}\}$

$$\mathrm{T}(x,y,z) = (z,\ x - y,\ -z) \rightarrow V = W = \mathbf{R}^3$$

$$Im\ T = \left\{\mathbf{v} \in R^3 : T(x,y,z) = \left(z, x-y, -z\right); x,y,z \in \Re\right\}$$

$$ImT = \left\{(x,y,z) : x(0,1,0) + y(0,-1,0) + z(1,0,1)\right\} = \left\langle(0,1,0),(1,0,1)\right\rangle$$

dim (Im T) = 2

Exercício 5

<u>Letra c)</u>:

Seja T:$V \to W$

- $Im(\mathrm{T}) = \{\mathbf{w} \in W \,;\, \mathrm{T}(\mathbf{v}) = \mathbf{w}$ para algum $\mathbf{v} \in V\}$
- $ker\ \mathrm{T} = \{\mathbf{v} \in V \,;\, \mathrm{T}(\mathbf{v}) = \mathbf{0}\}$

$$\mathrm{T}(x,y,z) = (z,\ x - y,\ -z) \ \rightarrow\ V = W = \mathbf{R}^3$$

Uma transformação T:$V \to W$, dizemos que T é <u>*sobrejetora*</u> se a imagem de T coincidir com W, ou seja T(V) = W.

dim (Im T) = 2

dim W = 3

$\Rightarrow$

Não é sobrejetora pois dim (Im T) $\neq$ dim W