Livro: Álgebra Linear I
Autores: Boldrini - Costa - Figueiredo - Wetzler
3ª edição - 1980

Lista para primeira avaliação

Capítulo 6

① $T(1,0) = 0(x,0) + 0(0,y)$
$T(0,1) = 1(x,0) + 2(0,y)$

$[T] = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

Para verificar que $\lambda = 2$ $\quad T(v) = \lambda v$

$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{cases} y = 2x \\ \cancel{2y = 2y} \end{cases} \quad (x, 2x) = v_1$

$x(1,2)$

está verificado

② $T(1,0) = (0,1) = 0(1,0) + 1(0,1)$
$T(0,1) = (2,0) = 2(1,0) + 0(0,1)$

$[T] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

lembrando que $\det([T] - \lambda I) = 0$ * como $\lambda I = \lambda$
propriedade

$\begin{vmatrix} 0-\lambda & 2 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 2 = 0$ quando $\lambda_1 = -\sqrt{2}$ e $\lambda_2 = \sqrt{2}$

Exercícios do capítulo 6 do livro Álgebra Linear de Boldrini, Costa, Figueiredo e Wetzler.
Resolvidos por Djerly Simonetti

$T v = \lambda v$ * a matriz de transformação (T) multiplica o vetor (v)

* Para $\lambda_1 = -\sqrt{2}$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{2} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 2y = -\sqrt{2}x \\ x = -\sqrt{2}y \end{cases}$$

logo,

- a combinação que gera o espaço vetorial V

$$(-\sqrt{2}y, y) = y(-\sqrt{2}, 1)$$

- o autovetor

$$v_1 = (-\sqrt{2}, 1)$$

assim o V é gerado pelo autovetor v_1, ou seja,

$V_{-\sqrt{2}} = [(-\sqrt{2}, 1)]$ quando $\lambda = -\sqrt{2}$

• Para $\lambda_2 = \sqrt{2}$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 2y = \sqrt{2}x \\ x = \sqrt{2}y \end{cases}$$

logo,

- a combinação que gera o espaço vetorial V

$$(\sqrt{2}y, y) = y(\sqrt{2}, 1)$$

- o autovetor

$$v_2 = (\sqrt{2}, 1)$$

- $V_{\sqrt{2}} = [(\sqrt{2}, 1)]$ quando $\lambda = \sqrt{2}$

③ $T(1,0) = (1,2) = 1(1,0) + 2(0,1)$

$T(0,1) = (1,1) = 1(1,0) + 1(0,1)$

$$[T] = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Autovalores: $\det([T] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 2 = 0 \Rightarrow \lambda^2 - 2\lambda - 1 = 0$$

$$\frac{-b \pm \sqrt{\Delta}}{2a} = \frac{+2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$\lambda_1 = 1 + \sqrt{2}$ $\quad \lambda_2 = 1 - \sqrt{2}$ $\quad$ agora $T \cdot v = \lambda v$

- Para $\lambda_1 = 1 + \sqrt{2}$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 + \sqrt{2} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} x + y = (1+\sqrt{2})x \Rightarrow x + y = x + \sqrt{2}x \\ 2x + y = 1 + \sqrt{2}\, y \Rightarrow 2x + y = y + \sqrt{2}\, y \end{cases}$$

logo,

- $(x, \sqrt{2}x) = x(1, \sqrt{2})$
- $v_1 = (1, \sqrt{2})$
- $V_{1+\sqrt{2}} = [(1, \sqrt{2})]$

- Para $\lambda_2 = 1 - \sqrt{2}$

$$\begin{cases} x + y = (1 - \sqrt{2})x \\ 2x + y = (1 - \sqrt{2})y \end{cases} \quad \begin{cases} y = -\sqrt{2}x \\ 2x = -\sqrt{2}y \end{cases}$$

logo

- $(x, -\sqrt{2}x) = x(1, -\sqrt{2})$
- $v_1 = (1, -\sqrt{2})$
- $V_{1-\sqrt{2}} = [(1, -\sqrt{2})]$

(4) $T(1,0,0) = (1,1,2) = \underline{1}(1,0,0) + \underline{1}(0,1,0) + \underline{2}(0,0,1)$

$T(0,1,0) = (1,-1,1) = 1(1,0,0) - 1(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (0,2,-1) = 0(1,0,0) + 2(0,1,0) - 1(0,0,1)$

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

* $\det([T] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \text{usar Laplace}$$

$$(-1-\lambda)(-1)^{1+1} \begin{vmatrix} -1-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 1 & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-\lambda-1)^2 - 2\right] - \left[(-\lambda-1) - 4\right] = 0$$

$$(1-\lambda)\left[\lambda^2 + 2\lambda + 1 - 2\right] + (\lambda+5) = 0$$

$$(1-\lambda)\left[\lambda^2 + 2\lambda - 1\right] + (\lambda+5) = 0$$

$$\lambda^2 + 2\lambda - 1 - \lambda^3 - 2\lambda^2 + \lambda + \lambda + 5 = 0$$

$$-\lambda^3 - \lambda^2 + 4\lambda + 4 = 0$$

$$\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$$

- procure as possíveis raízes: divisores de -4 são $-1, 1, -4, 4, 2, -2$ agora testo e concluo:

$$\lambda_1 = -2 \qquad \lambda_2 = -1 \qquad \lambda_3 = 2$$

Para $\lambda_1 = -2$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x + y = -2x & \Rightarrow x = \frac{y}{3} \\ x - y + 2z = -2y & \Rightarrow 2z = -\frac{y}{3} - y \Rightarrow z = -\frac{2}{3}y \\ 2x + y - z = -2z \end{cases}$$

logo,

$* \ (\frac{y}{3}, y, -\frac{2}{3}y) = (y, 3y, -2y) = y(1, 3, -2)$

$* \ v_1 = (1, 3, -2)$

$* \ V_{-2} = [(1, 3, -2)]$

Para $\lambda_2 = -1$

$$\begin{cases} x + y = -x & \Rightarrow y = -2x \\ x - y + 2z = -y & \Rightarrow x - (-2x) + 2z = -(-2x) \Rightarrow z = \frac{x}{2} \\ 2x + y - z = -z \end{cases}$$

logo

$* \ (x, -2x, \frac{x}{2}) = (2x, -4x, x) = x(2, -4, 1)$

$* \ v_1 = (2, -4, 1)$

$* \ V_{-1} = [(2, -4, 1)]$

Para $\lambda_3 = 2$

$$\begin{cases} x + y = 2x & \Rightarrow y = x \\ x - y + 2z = 2y & \Rightarrow x - x + 2z = 2x \Rightarrow z = x \\ 2x + y - z = 2z \end{cases}$$

logo

$* \ (x, x, x) = x(1, 1, 1)$

$* \ v_1 = (1, 1, 1)$

$* \ V_2 = [(1, 1, 1)]$

(5) A base que gera qualquer polinômio de 2º grau é
$\beta = \{1, x, x^2\}$ polinômios

$T(0x^2+0x+1) = T(1) = 1x = 0(1) + 1(x) + 0 \cdot (x^2)$

$T(0x^2+x+0) = T(x) = 1 = 1(1) + 0(x) + 0(x^2)$

$T(x^2+0x+0) = T(x^2) = x^2 = 0(1) + 0(x) + 1(x^2)$

$T(ax^2+bx+c) = ax^2 + cx + b$

$0x^2 + 0x + 1 \qquad 1x$

$$[T] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$* \det([T] - \lambda I) = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \qquad (1-\lambda)[(\lambda^2-1)] = 0$$

$\lambda_1 = 1 \quad \lambda_2 = -1$

Para $\lambda_1 = 1$ → usar ordem da base $\{1, x, x^2\} = \{c, b, a\}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix} \qquad \begin{array}{l} b = c \\ c = b \\ a = a \end{array}$$

logo,

$* \; v_1 = ax^2 + bx + b$

$\mathbb{R}^3 \to \mathbb{R}^3$

$T(1,0,0) = \qquad = \underline{0}(1,0,0) + \underline{0}(0,1,0) + \underline{0}($

$T(0,1,0) =$

$T(0,0,1) =$

Para $\lambda_2 = -1$

$$\begin{cases} b = -c \Rightarrow c = -b \\ c = -b \\ a = -a \Rightarrow a = 0 \end{cases}$$

logo

$* \; v_1 = bx - b$

(6) $T\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$T\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+1\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$T\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+1\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$T\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$$[T]=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

* $\det([T]-\lambda I)=0$

$$\begin{vmatrix}1-\lambda&0&0&0\\0&-\lambda&1&0\\0&1&-\lambda&0\\0&0&0&1-\lambda\end{vmatrix}\qquad (1-\lambda)\begin{vmatrix}1-\lambda&0&0\\0&-\lambda&1\\0&1&-\lambda\end{vmatrix}$$

$$(1-\lambda)\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}$$

$(1-\lambda)(\lambda^2-1)=0$

$(1-\lambda)(1-\lambda)(\lambda^2-1)=0$

$\lambda_1=1 \quad \lambda_2=-1$

Para $\lambda_1=1$

$$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\Rightarrow \begin{array}{l}x=x\\z=y\\y=z\\w=w\end{array}$$

* $(x,z,z,w)=x(1,0,0,0)+z(0,1,1,0)+w(0,0,0,1)$
* $v_1=(1,0,0,0),(0,1,1,0),(0,0,0,1)$
* $V_1=[(1,0,0,0)\,(0,1,1,0),(0,0,0,1)]$

Para $\lambda_2 = -1$

$$\begin{cases} x = -x & 2x = 0 \\ z = -y & y = -z \\ y = -z & \\ w = -w & 2w = 0 \end{cases}$$

logo

* $(0, -z, z, 0) = z(0, -1, 1, 0)$

* $v_1 = (0, -1, 1, 0)$

* $V_{-1} = [(0, -1, 1, 0)]$

⑦ $T(\underset{e_1}{1, 0, 0, 0}) = (1, 1, 1, 1) = 1e_1 + 1e_2 + 1e_3 + 1e_4$

$T(\underset{e_2}{0, 1, 0, 0}) = (0, 1, 1, 1) = 0e_1 + 1e_2 + 1e_3 + 1e_4$

$T(\underset{e_3}{0, 0, 1, 0}) = (0, 0, 1, 1) = 0e_1 + 0e_2 + 1e_3 + 1e_4$

$T(\underset{e_4}{0, 0, 0, 1}) = (0, 0, 0, 1) = 0e_1 + 0e_2 + 0e_3 + 1e_4$

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

* $\det([T] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda_1 = 1$$

Para $\lambda_1 = 1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \Rightarrow \begin{cases} x = x \\ x + y = y & \Rightarrow x = 0 \\ x + y + z = z & \Rightarrow y = 0 \\ x + y + z + w = w & \Rightarrow z = 0 \end{cases}$$

Logo,

* $(0,0,0,w) = w(0,0,0,1)$

* $v_1 = (0,0,0,1)$

* $V_1 = [(0,0,0,1)]$

⑧ Usar definição $Tv = \lambda v$

seja $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ então aplicando a definição

para $\lambda = -2$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3y \\ y \end{pmatrix} = -2\begin{pmatrix} 3y \\ y \end{pmatrix}$ $\begin{cases} 3ay + by = -6y \\ 3cy + dy = -2y \end{cases}$ $\begin{cases} (3a+6+b)\,y = 0 \\ (3c+2+d)\,y = 0 \end{cases}$

para $\lambda = 3$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2y \\ y \end{pmatrix} = 3\begin{pmatrix} -2y \\ y \end{pmatrix}$ $\begin{cases} -2ay + by = -6y \\ -2cy + dy = 3y \end{cases}$ $\begin{cases} (-2a+6+b)\,y = 0 \\ (-2c-3+d)\,y = 0 \end{cases}$

$3a + \cancel{6} + \cancel{b} = -2a + \cancel{6} + \cancel{b} \Rightarrow a = 0$ $\quad b = -6$ $\quad [T] = \begin{bmatrix} 0 & -6 \\ -1 & 1 \end{bmatrix}$

$3c + 2 + \cancel{d} = -2c - 3 + \cancel{d} \Rightarrow c = -1$ $\quad d = 1$

Portanto $T(x,y) = (-6y, -x+y)$

⑨ $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

* $\det([A] - \lambda I) = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) = 0$

$\lambda_1 = -1$ e $\lambda_2 = 1$

Para $\lambda_1 = -1$

$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} x + 2y = -x \\ -y = -y \end{cases} \Rightarrow x = \frac{-2y}{2}$

* $(-y, y) = y(-1, 1)$

* $v_1 = (-1, 1)$

* $V_{-1} = [(-1, 1)]$

Para $\lambda_2 = 1$

$$\begin{cases} x + 2y = x \\ \quad -y = y \end{cases} \Rightarrow -2y = 0$$

logo

* $(x, 0) = x(1,0)$
* $v_1 = (1,0)$
* $V_1 = [(1,0)]$

(10) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\det([A] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \begin{aligned} (1-\lambda)(1-\lambda) - 1 &= 0 \\ 1 - \lambda - \lambda + \lambda^2 - 1 &= 0 \\ \lambda^2 - 2\lambda &= 0 \\ \lambda(\lambda - 2) &= 0 \end{aligned}$$

$\lambda_1 = 0$ e $\lambda_2 = 2$

Para $\lambda_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \Rightarrow \begin{cases} x + y = 0 \Rightarrow x = -y \\ x + y = 0 \end{cases}$$

logo

* $(-y, y) = y(-1,0)$
* $v_1 = (-1,0)$
* $V_0 = [(-1,0)]$

Para $\lambda_2 = 2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} x + y = 2x \Rightarrow y = x \\ x + y = 2y \end{cases}$$

logo

* $(y, y) = y(1,1)$
* $v_1 = (1,1)$
* $V_1 = [(1,1)]$

(11) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

* $\det([A] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda)(1-\lambda) = 0$$

$\lambda_1 = 1$

* $Tv = \lambda v$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x + 2y + 3z = x & \Rightarrow 2y = 0 \\ y + 2z = y & \Rightarrow 2z = 0 \\ z = z \end{cases}$$

logo

* $(x, 0, 0) = x(1, 0, 0)$
* $v_1 = (1, 0, 0)$
* $V_1 = [(1, 0, 0)]$

(12) $A = \begin{bmatrix} 3 & -3 & -4 \\ 0 & 3 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

* $\det([A] - \lambda I)$

$$\begin{vmatrix} 3-\lambda & -3 & -4 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$(3-\lambda)(3-\lambda)(-1-\lambda) = 0$

$\lambda_1 = 3 \quad \lambda_2 = -1$

$4x = \frac{15}{5}y - \frac{16}{5}y$

$x = -\frac{1}{20}y$

* Para $\lambda_1 = -1$

$$\begin{pmatrix} 3 & -3 & -4 \\ 0 & 3 & 5 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} 3x - 3y - 4z = -x & \Rightarrow 4x = 3y + 4z \\ 3y + 5z = -y & \Rightarrow z = -\frac{4}{5}y \\ -z = -z \end{cases}$$

$(-\frac{1}{20}y, y, -\frac{4}{5}y) = (-y, 20y, -16y) = y(-1, 20, -16)$

logo,

$* \ (-y, 20y, -16y) = y(-1, 20, -16)$

$* \ v_1 = (-1, 20, -16)$

$* \ V_{-1} = [(1, 20, -16)]$

Para $\lambda_2 = 3$

$$\begin{cases} 3x - 3y - 4z = 3x \\ 3y + 5z = 3y \\ -z = 3z \end{cases} \qquad \begin{cases} -3y = 0 \\ \\ 4z = 0 \end{cases}$$

logo

$* \ (x, 0, 0) = x(1, 0, 0)$

$* \ v_1 = (1, 0, 0)$

$* \ V_3 = [(1, 0, 0)]$

13) $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\det([A] - \lambda I)$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ -1 & -\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\{(1-\lambda)[(-\lambda)(2-\lambda) - 1] + 2(-1+\lambda)\} = 0$

$\{(1-\lambda)(\lambda^2 - 2\lambda - 1) + 2\lambda - 2\} = 0$

$\lambda^2 - \cancel{2\lambda} - 1 - \lambda^3 + 2\lambda^2 + \lambda + \cancel{2\lambda} - 2 = 0$

$-\lambda^3 + 3\lambda^2 + \lambda - 3 = 0$

$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$

divisores de 3 são: $-1, 1, -3, 3$

assim -1 é uma raiz

$$\begin{array}{l|l} \lambda^3 - 3\lambda^2 - \lambda + 3 & \lambda + 1 \\ \hline -\lambda^3 - \lambda^2 & \lambda^2 - 4\lambda + 3 \\ \overline{-4\lambda^2 - \lambda + 3} & \\ \quad 4\lambda^2 + 4\lambda & \\ \overline{\quad 3\lambda + 3} & \\ \quad -3\lambda - 3 & \\ \overline{\quad 0} & \end{array}$$

$\underline{1} + \underline{3} = 4$

$\underline{1} \times \underline{3} = 3$

$\therefore$ 1 e 3 são raízes

$\lambda_1 = -1$; $\lambda_2 = 1$; $\lambda_3 = 3$

Para $\lambda_1 = -1$

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x + 2z = -x \\ -x + z = -y \\ x + y + 2z = -z \end{cases} \begin{array}{l} \Rightarrow 2z = -2x \Rightarrow z = -x \\ \Rightarrow -x - x = -y \Rightarrow y = 2x \\ \end{array}$$

logo

$* \ (x, 2x, -x) = x(1, 2, -1)$

$* \ v_1 = (1, 2, -1)$

$* \ V_{-1} = [(1, 2, -1)]$

Para $\lambda_2 = 1$

$$\begin{cases} x + 2z = x \\ -x + z = y \\ x + y + 2z = z \end{cases} \begin{array}{l} \Rightarrow z = 0 \\ \Rightarrow y = -x \\ \end{array}$$

logo

$* \ (x, x, 0) = x(1, -1, 0)$

$* \ v_1 = (1, -1, 0)$

$* \ V_1 = [(1, -1, 0)]$

$x = -y + z$

$(-y + z) + y + 2z = z \Rightarrow z = 0$

logo

$* \ (-y, y, 0) = y(-1, 1, 0)$

$* \ v_1 = (-1, 1, 0)$

$* \ V_1 = [(-1, 1, 0)]$

Para $\lambda = 3$

$$\begin{cases} x + 2z = 3x \\ -x + z = 3y \\ x + y + 2z = 3z \end{cases} \begin{array}{l} \Rightarrow x = z \\ \Rightarrow y = 0 \\ \end{array}$$

logo

$* \ (x, 0, x) = x(1, 0, 1)$

$* \ v_1 = (1, 0, 1)$

$* \ V_3 = [(1, 0, 1)]$

(14) $[A] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

$\det([A] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = 0$$

$-1(1-\lambda-2) + (2-\lambda)\left[(1-\lambda)^2 - 4\right] - 1(1-\lambda-2) = 0$

$-1(-\lambda-1) + (2-\lambda)\left[1-2\lambda+\lambda^2-4\right] - 1(-\lambda-1) = 0$

$(\lambda+1) + (2-\lambda)(\lambda^2-2\lambda-3) + (\lambda+1) = 0$

$2\lambda + 2 + 2\lambda^2 - 4\lambda - 6 - \lambda^3 + 2\lambda^2 + 3\lambda = 0$

$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

divisores de 4 são: $-1, 1, -2, 2, -3, 3, -4, 4$.

como 1 é raiz, segue que

$$\begin{array}{l|l} \lambda^3 - 4\lambda^2 - \lambda + 4 & \lambda - 1 \\ \hline -\lambda^3 + \lambda^2 & \lambda^2 - 3\lambda - 4 \\ -3\lambda^2 - \lambda + 4 & \\ 3\lambda^2 - 3\lambda & \\ -4\lambda + 4 & \\ 4\lambda - 4 & \\ 0 & \end{array}$$

$\underline{-1} + \underline{4} = 3$

$\underline{-1} \times \underline{4} = -4$

portanto $\lambda_1 = -1$; $\lambda_2 = 1$; $\lambda_3 = 4$

Para $\lambda_1 = -1$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x + y + 2z = -x \\ x + 2y + z = -y \\ 2x + y + z = -z \end{cases}$$

$\Rightarrow y = -2z - 2x \Rightarrow y = -2(-x) - 2x \Rightarrow y = 0$

$\Rightarrow x + 3y + z = 0 \Rightarrow x - 6z - 6x + z = 0$

$z = -x$

logo $*\ (x, 0, -x) = x(1, 0, -1)$ $\quad *\ v_1 = (1, 0, -1)$ $\quad *\ V_{-1} = [(1, 0, -1)]$

Para $\lambda_2 = 1$

$$\begin{cases} x + y + 2z = x \Rightarrow y = -2z \\ x + 2y + z = y \Rightarrow x - 2z + z = 0 \Rightarrow x = z \\ 2x + y + z = z \end{cases}$$

logo

* $(z, -2z, z) = z(1, -2, 1)$
* $v_1 = (1, -2, 1)$
* $V_1 = [(1, -2, 1)]$

Para $\lambda_3 = 4$

$$\begin{cases} x + y + 2z = 4x \Rightarrow y = 3x - 2z \Rightarrow y = x \\ x + 2y + z = 4y \Rightarrow z = 2y - x \Rightarrow z = 6x - 4z - x \Rightarrow z = x \\ 2x + y + z = 4z \end{cases}$$

logo

* $(x, x, x) = x(1, 1, 1)$
* $v_1 = (1, 1, 1)$
* $V_4 = [(1, 1, 1)]$

15) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

* $\det([A] - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 0 & -\lambda \end{vmatrix} = 0 \qquad \begin{aligned} (-\lambda)(\lambda^2) - 1(1) &= 0 \\ -\lambda^3 - 1 &= 0 \\ \lambda^3 + 1 &= 0 \end{aligned}$$

$\lambda_1 = -1$

$* \; Tv = \lambda v$

Para $\lambda_1 = -1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{cases} y = -x \\ z = -y \\ -x = -z \end{cases}$$

logo

$* \; (x, -x, x) = x(1,-1,1)$

$* \; v_1 = (1,-1,1)$

$* \; V_{-1} = [(1,-1,1)]$

(16) $A = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 0 \\ -3 & 3 & 1 \end{bmatrix}$

$* \; \det([A] - \lambda I)$

$$\begin{vmatrix} 1-\lambda & 3 & -3 \\ 0 & 4-\lambda & 0 \\ -3 & 3 & 1-\lambda \end{vmatrix} = 0$$

$(4-\lambda)\left[(1-\lambda)^2 - 9\right] = 0$

$(4-\lambda)[1-2\lambda+\lambda^2-9] = 0$

$(4-\lambda)(\lambda^2-2\lambda-8) = 0$

$\underline{-2} + \underline{4} = 2$ e $\underline{-2} \times \underline{4} = -8$

$\lambda_1 = -2 \qquad \lambda_2 = 4$

$* \; Tv = \lambda v$

Para $\lambda_1 = -2$

$$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x + 3y - 3z = -2x & \Rightarrow -3z = -3x \Rightarrow z = x \\ 4y = -2y & \Rightarrow y = 0 \\ -3x + 3y + z = -2z \end{cases}$$

logo,

* $(x, 0, x) = x(1,0,1)$
* $v_1 = (1,0,1)$
* $V_{-2} = [(1,0,1)]$

Para $\lambda_2 = 4$

$$\begin{cases} x + 3y - 3z = 4x & \Rightarrow x = y - z \\ 4y = 4y & \\ -3x + 3y + z = 4z & \Rightarrow -3x + 3y = 3z \Rightarrow -3y + 3z + 3y = 3z \Rightarrow z = z \end{cases}$$

logo

* $(y - z, y, z) = y(1,1,0) + z(-1,0,1)$
* $v_1 = (1,1,0)\ (-1,0,1)$
* $V_4 = [(1,1,0)\ (-1,0,1)]$

Para conferir $Tv = \lambda v$

$$T\begin{pmatrix} y-z \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} y-z \\ y \\ z \end{pmatrix}$$

(19) $A = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

* $\det([A] - \lambda I)$

$$\begin{vmatrix} -1-\lambda & -2 & 0 \\ 0 & -1-\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-1-\lambda)^2 + (-2) = 0$$
$$(-\lambda)(1 + 2\lambda + \lambda^2) - 2 = 0$$
$$-\lambda^3 - 2\lambda^2 - \lambda - 2 = 0$$
$$\lambda^3 + 2\lambda^2 + \lambda + 2 = 0$$

- divisores de 2 são: $-1, 1, -2, 2$

como -2 é raiz, podemos aplicar

$$\begin{array}{l|l} \lambda^3 + 2\lambda^2 + \lambda + 2 & \lambda + 2 \\ \hline -\lambda^3 - 2\lambda^2 & \lambda^2 + 1 \\ \lambda + 2 & \\ -\lambda - 2 & \\ 0 & \end{array}$$

portanto $\lambda_1 = -2 \quad \lambda_2 = -i \quad \lambda_3 = -i$

a) $\lambda = -2$

$$\begin{pmatrix} -1 & -2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} -x - 2y = -2x \Rightarrow -2y = -x \Rightarrow y = \frac{x}{2} \\ -y + z = -2y \\ x = -2z \Rightarrow z = -\frac{x}{2} \end{cases}$$

logo

* $(x, x/2, -x/2) = (2x, x, -x) = x(2,1,-1)$
* $v_1 = (2,1,-1)$
* $V_{-2} = [(2,1,-1)]$

b) $\lambda = -i$

$$\begin{cases} -x - 2y = -ix \\ -y + z = -iy \\ x = -iz \end{cases} \quad \begin{matrix} \\ z = -iy + y \Rightarrow z = (1-i)y \\ x = -i(1-i)y \Rightarrow x = (-i-1)y \end{matrix} \quad \begin{matrix} i^2 = -1 \\ \\ (-i)(-i) = -1 \end{matrix}$$

logo

* $((-1-i)y, y, (1-i)y) = y((-1-i), 1, (1-i))$
* $v_1 = ((-1-i), 1, (1-i))$
* $V_{-i} = [(-1-i), 1, (1-i)]$

Para $\lambda = i$

$$\begin{cases} -x - 2y = ix \\ -y + z = iy \\ x = iz \end{cases} \quad \begin{matrix} \\ \Rightarrow z = (1+i)y \\ \Rightarrow x = i(1+i)y \Rightarrow x = (-1+i)y \end{matrix}$$

logo

* $((-1+i)y, y, (1+i)y) = y((-1+i), 1, (1+i))$
* $v_1 = ((-1+i), 1, (1+i))$
* $V_i = [(-1+1), 1, (1+i)]$

temos também para $\lambda = -2$

* $V_{-2} = [(2,1,-1)]$

(20) a) Se $Tv = \lambda v$ para $v \neq 0$, então para o autovetor Kv pode-se afirmar que $T(Kv) = KT(v) = K(\lambda v) = \lambda(Kv)$, portanto Kv é outro autovetor associado a λ se $K \neq 0$.

b) Lembre-se que: todo núcleo é um subespaço vetorial

Seja v os autovetores associados a λ e V o subespaço associado a λ, ou seja, $V_\lambda = \{v \in V : T(v) = \lambda v\}$.

Como $V_\lambda = Ker(T - \lambda I)$

temos que $v \in V_\lambda \Rightarrow T(v) = \lambda v \Rightarrow [T]v = \lambda v$

$\Rightarrow [T]v - \lambda I v = 0 \Rightarrow ([T] - \lambda I)v = 0$

logo, como tenho uma transformação que resulta em uma imagem nula, afirma-se que

$v \in Ker([T] - \lambda I) \Rightarrow V_\lambda = Ker(T - \lambda I)$.

(21) a) Lembre-se:

v_1, v_2 são l.i. se $a_1 v_1 + a_2 v_2 = 0$,

logo $a_1 = a_2 = 0$

assim

$a_1 v_1 + a_2 v_2 = 0$

$(T - \lambda_1 I)(a_1 v_1 + a_2 v_2) = 0$

$(T - \lambda_1 I)(a_1 v_1) + (T - \lambda_1 I)(a_2 v_2) = 0$

$a_1 T(v_1) - \lambda_1 a_1 I(v_1) + a_2 T(v_2) - \lambda_1 a_2 I(v_2) = 0$

$a_1 \lambda_1 v_1 - \lambda_1 a_1 v_1 + a_2 \lambda_2 v_2 - \lambda_1 a_2 v_2 = 0$

$a_2(\lambda_2 - \lambda_1) v_2 = 0$

$\therefore a_2 = 0$

Para a_1 é análogo.

22) a) $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$* \det([A] - \lambda I) = 0$

$\begin{vmatrix} 0-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = 0 \quad (-\lambda)(1-\lambda) - 2 = 0$

$\lambda^2 - \lambda - 2 = 0 \qquad \underline{-1} + \underline{2} = 1 \quad \underline{-1} \times \underline{2} = -2$

logo $\lambda_1 = -1$ e $\lambda_2 = 2$

$A^{-1} = \left[\begin{array}{cc|cc} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right] =$

$\left[\begin{array}{cc|cc} 2 & 0 & -1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 2 & 0 & -1 & 2 \\ 0 & 1 & 1/2 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & 1 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$

$\therefore A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

$* \det([A^{-1}] - \lambda I) = 0$

$\begin{vmatrix} -\frac{1}{2} - \frac{2}{2}\lambda & 1 \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \qquad \frac{\lambda}{2} + \lambda^2 - \frac{1}{2} = 0$

$\underline{-1} + \underline{1/2} = -1/2 \quad e \quad \underline{-1} \times \underline{1/2} = -1/2$

logo $\lambda_1 = -1$ e $\lambda_2 = \frac{1}{2}$

b) Para $[A]$ e $\lambda_1 = -1$

$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 2y = -x \\ x + y = -y \end{cases} \Rightarrow x = -2y$

logo

$* (-2y, y) = y(-2, 1) \quad * v_1 = (-2, 1) \quad * V_{-1} = [(-2, 1)]$

Para $\lambda_2 = 2$

$\begin{cases} 2y = 2x \\ x + y = 2y \end{cases} \Rightarrow y = x$

logo

$* (x, x) = x(1, 1) \quad * v_1 = (1, 1) \quad * V_1 = [(1, 1)]$

Para $[A^{-1}]$ com $\lambda_1 = -1$ temos que

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} -\frac{1}{2}x + y = -x \\ \frac{1}{2}x = -y \Rightarrow x = -2y \end{cases}$$

logo

$* \ (-2y, y) = y(-2,1) \quad * \ v_1 = (-2,1) \quad * \ V_{-1} = [(-2,1)]$

Para $\lambda_2 = \frac{1}{2}$

$$\begin{cases} -\frac{1}{2}x + y = \frac{1}{2}x \\ \frac{1}{2}x = \frac{1}{2}y \Rightarrow y = x \end{cases}$$

logo

$* \ (x, x) = x(1,1) \quad * \ v_1 = (1,1) \quad * \ V_{\frac{1}{2}} = [(1,1)]$

(23) Seja $Tv = \lambda v$

$\alpha T = \alpha(T(v)) = \alpha(\lambda v) = (\alpha\lambda)v$

logo $\alpha\lambda$ é autovalor com autovetor v de αT.

(24) a) $S + T$

$$\left.\begin{matrix} Sv = \lambda_1 v \\ Tv = \lambda_2 v \end{matrix}\right\} (S+T)v = (\lambda_1 + \lambda_2)v$$

logo $\lambda_1 + \lambda_2$ é autovalor e v autovetor

b) $S \circ T$

$(S \circ T)(v) \Rightarrow S(T(v)) = S(\lambda_2 v) \Rightarrow \lambda_2 S(v) =$

$\lambda_2 \lambda_1 v$

(25) Se $\lambda = 0$ é autovalor de T, temos que

$Tv = \lambda v \Rightarrow ([T] - \lambda I)v = 0$, assim tem-se uma aplicação com imagem nula, portanto $v \in Ker([T] - \lambda I)$, mas $\lambda = 0$, logo $Tv = 0$ e
conjunto infinitos
T não é injetora, pois para T ser injetora o $Ker T = \{0\}$

Como T não é injetora, então $Ker\, T \neq \{0\}$. Logo, existe um subconjunto de V, denominamos S tal que $v \in S$ implica $T(v) = 0$.
Portanto, $T(v) = 0 \cdot v$, assim $\lambda = 0$.

(26) a) $AB = \begin{bmatrix} 1 & 7 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & -3 \end{bmatrix}$ $BA = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

b) Para AB

$* \det([AB] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 7 & 4 \\ 0 & -2-\lambda & 3 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-2-\lambda)(-3-\lambda) = 0$

$\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$

Os autovalores de AB e BA são iguais

Para BA

$* \det([BA] - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 3 \\ 0 & -2-\lambda & 2 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-2-\lambda)(-3-\lambda) = 0$

$\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$

c) Para AB com $\lambda_1 = 1$

$$\begin{pmatrix} 1 & 7 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x + 7y + 4z = x \\ -2y + 3z = y \\ -3z = z \end{cases}$$

$z = 0 \quad y = 0 \quad x = x$

logo

$* \ (x, 0, 0) = x(1, 0, 0)$

Para $\lambda_2 = -2$

$$\begin{cases} x + 7y + 4z = -2x \\ -2y + 3z = -2y \\ -3z = -2z \end{cases}$$

$z = 0 \quad y = y \quad x = \frac{7}{3}y$

logo

$* \ (7y, 3y, 0) = y(7, 3, 0)$

Para $\lambda_3 = -3$

$$\begin{cases} x + 7y + 4z = -3x \\ -2y + 3z = -3y \\ -3z = -3z \end{cases}$$

$z = z \quad y = -3z \quad x = \frac{17}{4}z$

logo

$* \ (17z, -12z, z) = z(17, -12, 1)$

Para BA com $\lambda_1 = 1$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x - y + 3z = x \\ -2y + 2z = y \\ -3z = z \end{cases}$$

$z = 0 \quad y = 0 \quad x = x$

logo

$* \ (x, 0, 0) = x(1, 0, 0)$

Para $\lambda_2 = -2$

$$\begin{cases} x - y + 3z = -2x \\ -2y + 2z = -2y \\ -3z = -2z \end{cases}$$

$z = 0 \quad y = y \quad x = \frac{1}{3}y$

logo

$* \ (y, 3y, 0) = y(1, 3, 0)$

Para $\lambda_3 = -3$

$$\begin{cases} x - y + 3z = -3x \\ -2y + 2z = -3y \\ -3z = -3z \end{cases}$$

$z = z \quad y = -2z \quad x = \frac{5}{4}z$

logo

$* \ (5z, -8z, z) = z(5, -8, 1)$

Os autovetores são diferentes!