Transforms

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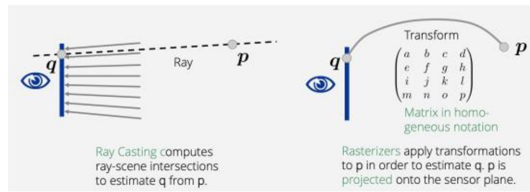
Rendering

- We can solve the visibility problem by ray casting or by applying transformations



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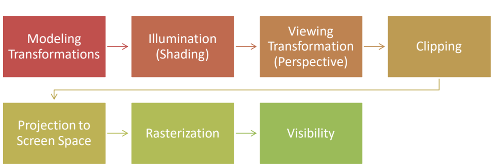
Rendering Pipeline



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Transformations

- Useful in modeling and rendering
  - Position, reshape, and animate objects, lights, cameras
  - Project 3D geometry onto the camera plane

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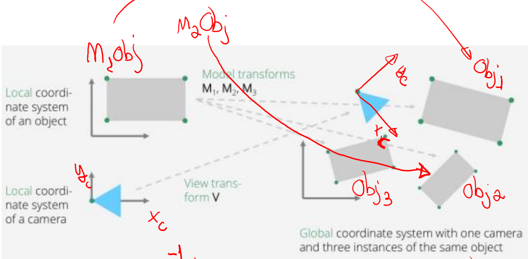
Coordinate Systems and Transformations



$V^{-1} Obj_1$

$Obj_2 = M_2 Obj$

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$V^{-1} M_1 Obj$

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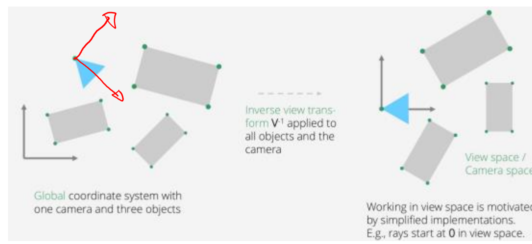
Coordinate Systems and Transformations



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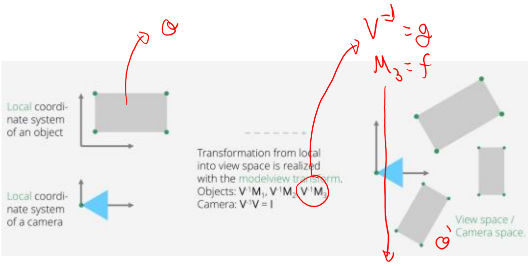
Coordinate Systems and Transformations



$O' = g(f(O))$

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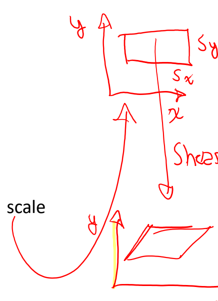
Type of transformations

- Euclidean transformations
  - Preserve shape and size
    - Translation, rotation, reflection
- Similarity transformations
  - Preserve shape
    - Translation, rotation, reflection, scale



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Type of transformations

- Affine transformations
  - Preserve **collinearity**: Points on a line are transformed to points on a line
  - Preserve **proportions**: Ratios of distances between points are preserved
  - Preserve **parallelism**: Parallel lines are transformed to parallel lines
    - Translation, rotation, reflection, scale, shear
  - **Do not preserve angles and lengths**

Shear

Shear

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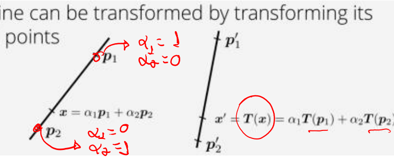
Affine transformations

$T\left(\begin{bmatrix}1.6\\1.8\end{bmatrix}\right) = \frac{4}{10}T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 0T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{6}{10}T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$

$\frac{4}{10}\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} + \frac{6}{10}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1.6\\1.8\end{bmatrix}$

- Preserve affine combinations
  - $T(\sum_i \alpha_i p_i) = \sum_i \alpha_i T(p_i), \sum_i \alpha_i = 1$
  - To transform a shape we only need to transform its defining vertices

E.g., a line can be transformed by transforming its control points



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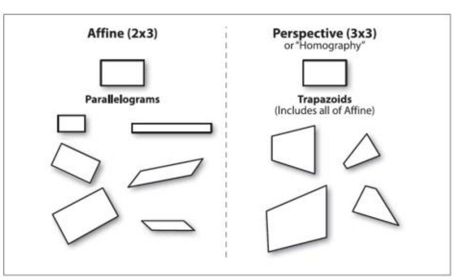
Affine transformations



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Points versus Vectors

- Points and vertices specify a location in space
- Vectors and normals specify a direction
- Operations
  - position - position = vector
  - position + vector = position
  - vector + vector = vector
  - position + position = not defined

$P_4 = \begin{bmatrix}20\\16\end{bmatrix}$

$P_3 = \begin{bmatrix}8\\6\end{bmatrix}$

$P_4 - P_3$

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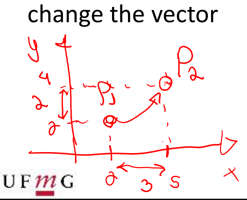
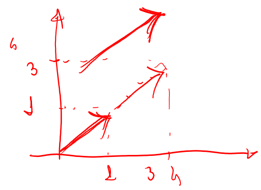
Points versus Vectors

- Transformations might have different effects on points and vectors
- E.g., translation of a point changes its position, but translation of a vector does not change the vector





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2D transformations

$(AB)^T = B^T A^T$ $\quad v \cdot p = \|v\|\|p\| \cos\theta$ $\quad p = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$\|v\| \sin\theta$ $\quad q = [10 \;\; 5] = p^T$ $\quad v = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

- 2D scale

$S_{s_x,s_y} p = (p^T S_{s_x,s_y}^T)^T$

$$\mathbf{p}' = \mathbf{S}_{s_x,s_y}\mathbf{p} \qquad \mathbf{S}_{s_x,s_y} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

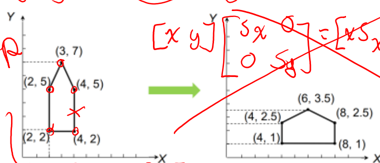
$\begin{bmatrix} S_x x \\ S_y y \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} S_x * x + 0 * y \\ 0 * x + S_y * y \end{bmatrix}$

$\|v\| \cos\theta$

$v^T p = v \cdot p$

$[x \;\; y] \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} = [x S_x \;\; y S_y]$

$[5 \;\; 6] \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 * 10 + 6 * 5$



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2D transformations

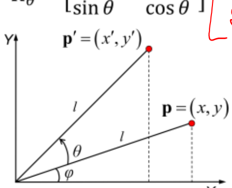
- 2D rotation

$x' = x\cos\theta - \sin\theta\, y$

$y' = \sin\theta + \cos\theta\, y$

$$\mathbf{p}' = \mathbf{R}_\theta \mathbf{p}$$

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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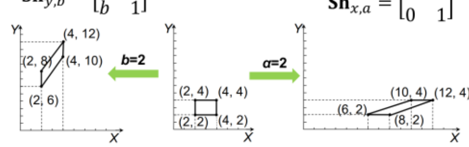
2D transformations

- 2D Shearing

$\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{p}' = \mathbf{Sh}_{y,b}\mathbf{p} \qquad \mathbf{Sh}_{y,b} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$\mathbf{p}' = \mathbf{Sh}_{x,a}\mathbf{p} \qquad \mathbf{Sh}_{x,a} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$



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Georgios Papaioannou

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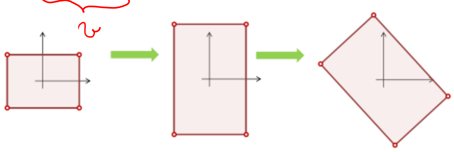
Compositive Linear Transformation

- Compositive transformation can be expressed using matrix multiplication

$$R_{45^o}\left(S_{1,2}(\mathbf{p})\right) = \mathbf{R}_{45^o}\mathbf{S}_{1,2}\mathbf{p}$$



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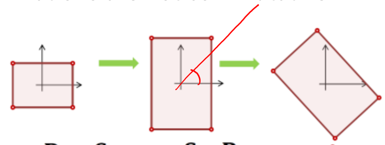
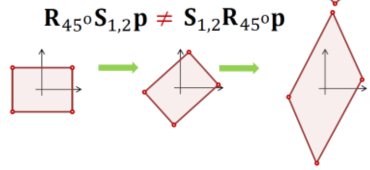
Compositive Linear Transformation

- Transformations are not commutative



$$\mathbf{R}_{45^o}\mathbf{S}_{1,2}\mathbf{p} \neq \mathbf{S}_{1,2}\mathbf{R}_{45^o}\mathbf{p}$$



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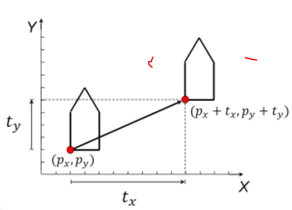
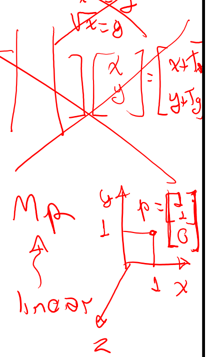
2D transformations

- 2D ~~scale~~ translation

ax+b=y

$$\mathbf{p}' = \mathbf{I}\mathbf{p} + \vec{\mathbf{t}} = \mathbf{p} + \vec{\mathbf{t}}$$





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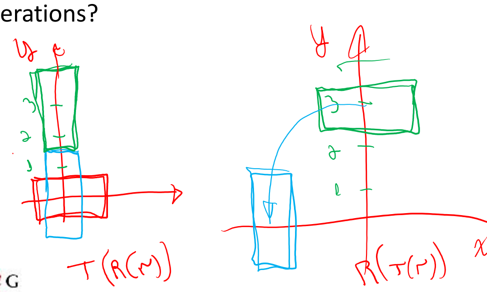
2D transformations

- Are rotations and translations commutative operations?



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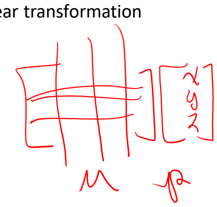
Homogeneous notation

- 3x3 matrix represents linear transformations
  - Scale, rotation, shear
- 3D vector represents translation
  - Translation is not a linear transformation



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Homogeneous notation

- Using the homogeneous notation, transformations of vectors and positions are handled in a unified way
- Using the homogeneous notation, all affine transformations are represented with one matrix-vector multiplication
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Homogeneous Coordinates

- Homogeneous coordinates
  – add the w coordinate (homogenization):

homogeneous image coordinates $(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ homogeneous scene coordinates $(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

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Homogeneous Coordinates

- Converting from homogeneous coordinates
  – Dehomogenization operation (**perspective division**)

$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$ $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$

- Ideal Points (points at infinity): w coordinate equal to zero $\omega \to 0$
- There is only one homogeneous coordinate vector forbidden:
  – $v = [0\ 0\ 0]^T$

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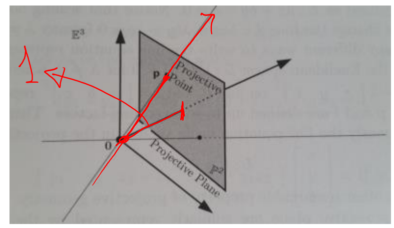
Homogeneous Coordinates

- The Euclidean space is said to be embedded into the projective space by *homogenization*
  - Points having the last w-coordinate equal to zero are called **ideal points** or **points at infinity**
- Duality: points and lines
  - Any point in the 2D projective plane can be interpreted as a 3D Euclidean line passing through the 3D origin and the projective plane point

Homogeneous Coordinates



Points at infinity

- Varying w, a point $(x, y, z, w)^T$ is scaled the points $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)^T$ represent a line in 3D
  - $(x, y, z)^T$ is the line direction
- $w \to 0$, the position $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)^T$ goes to infinity following the direction $(x, y, z)^T$
  - $(x, y, z, 0)^T$ represents the position at infinity in the direction of $(x, y, z)^T$
  - In other words, $(x, y, z, 0)^T$ is a vector in the direction of $(x, y, z)^T$

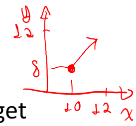
Positions and Vector

- When positions are normalized, we get position-vector relations well defined
  - vector + vector = vector

$$\begin{pmatrix} u_x \\ u_y \\ u_z \\ 0 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \begin{pmatrix} u_x + v_x \\ u_y + v_y \\ u_z + v_z \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \xrightarrow{\text{demo}} \begin{pmatrix} p_x/1 \\ p_y/1 \\ p_z/1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

Positions and Vector



- When positions are normalized, we get position-vector relations well defined
  - position + vector = position

$$\begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} + \begin{pmatrix} 12 \\ 12 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 24 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \begin{pmatrix} p_x + v_x \\ p_y + v_y \\ p_z + v_z \\ 1 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}}$$

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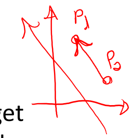
Positions and Vector



- When positions are normalized, we get position-vector relations well defined
  - position – position = vector

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} - \begin{pmatrix} r_x \\ r_y \\ r_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x - r_x \\ p_y - r_y \\ p_z - r_z \\ 0 \end{pmatrix}$$

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Transforms in Homogeneous notation

$v' = v + t$

- Linear transformations
  - Rotations, scale, shear

$m_{00}p_x + m_{01}p_y + m_{02}p_z + m_{03} \cdot 1 = p_x + t_x$

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \sim \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & 0 \\ m_{20} & m_{21} & m_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{pmatrix}$$

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Scale in Homogeneous Notation

- Scale

$(S^T S^{-1})^{T} = S^{-1} S = I^T$

$$\boldsymbol{S}(s_x, s_y, s_z)\boldsymbol{p} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x p_x \\ s_y p_y \\ s_z p_z \\ 1 \end{pmatrix}$$

- Inverse: $\boldsymbol{S}^{-1}(s_x, s_y, s_z) = \boldsymbol{S}(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z})$

$S^{-1} S = I$

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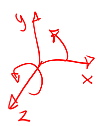
Rotation in Homogeneous Notation

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



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Rotation – inverse transform

$R_x^{-1}(\phi) = R_x(\phi)^T$

$R_x(-\phi)\, R_x(\phi) = I$

$R_x(-\phi) = R_x(\phi)^T$

$$R_x(-\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos -\phi & -\sin -\phi & 0 \\ 0 & \sin -\phi & \cos -\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$v \cdot u = \|v\|\,\|u\| \cos\theta$

$\sqrt{v^T v} = \sqrt{v_1^2 + v_2^2}$

Orthogonal Matrices

$\|v\|^2 = v^T v$

- An orthogonal matrix is a square matrix whose columns and rows are orthogonal unit vectors
  - $M^T M = M M^T = I$
  - $M^{-1} = M^T$
- Angles are preserved
  - $\langle Ru, Rv \rangle = \langle u, v \rangle$
- Length is preserved
  - $\|Ru\| = \|u\|$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ca+db & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Transforms in Homogeneous notation

- Translation?
  - Position

$$T(t)p = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{pmatrix}$$

$$T^{-1}(t) = T(-t) = \begin{pmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Transforms in Homogeneous notation

- Translation?
  - Vector

$$\boldsymbol{T}(\boldsymbol{t})\boldsymbol{v} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix}$$

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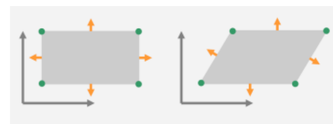
Transforms in Homogeneous notation

- Shear



$$\boldsymbol{H}_{xz}(s)\boldsymbol{p} = \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + sp_z \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

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$R^T p' - R^T t = p$

Rigid-Body Transform

$p' = Rp + t \rightsquigarrow p' - t = Rp + t - t$

$R^T(p' - t) = R^T R p$

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \boldsymbol{p} = \boldsymbol{T}(\boldsymbol{t})\boldsymbol{R}\boldsymbol{p} \qquad \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

$$\begin{pmatrix} r_1 & r_2 & r_3 & t_x \\ r_4 & r_5 & r_6 & t_y \\ r_7 & r_8 & r_9 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} r_1 & r_4 & r_7 & \\ r_2 & r_5 & r_8 & \\ r_3 & r_6 & r_9 & \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{pmatrix} \begin{pmatrix} -t_x \\ -t_y \\ -t_z \end{pmatrix}$$

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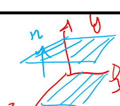
Planes and normals



- A plane can be represented by a normal n and a point r
  - All points p that $n.(p - r) = 0$ form a plane

$$n_x p_x + n_y p_y + n_z p_z + (-n_x r_x - n_y r_y - n_z r_z) = 0$$
$$\boxed{n_x p_x + n_y p_y + n_z p_z + d = 0}$$
$$(n_x\ n_y\ n_z\ d)(p_x\ p_y\ p_z\ 1)^T = 0$$

π $\quad p \quad \rightsquigarrow \pi p^T = 0$

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Planes and normals

- Suppose we transform all points that are on the plane using transformation A

$$\boldsymbol{A}(p_x\ p_y\ p_z\ 1)^{\mathsf{T}}$$

- Considering that the plane is represented by the $(n_x, n_y, n_z)^T$ normal

$$(n_x\ n_y\ n_z\ d)\boldsymbol{A}^{-1}\boldsymbol{A}(p_x\ p_y\ p_z\ 1)^{\mathsf{T}} = 0$$

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Planes and normals

- Since

$$(n_x\ n_y\ n_z\ d)\boldsymbol{A}^{-1} = ((\boldsymbol{A}^{-1})^{\mathsf{T}}(n_x\ n_y\ n_z\ d)^{\mathsf{T}})^{\mathsf{T}}$$

- The plane is transformed by
  $(\boldsymbol{A}^{-1})^T$

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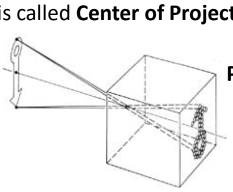
Pinhole camera model

- Pinhole model:
  - Captures pencil of rays – all rays through a single point
  - The point is called **Center of Projection (focal point)**
  - The image **Plane**



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Slide by Steve Seitz

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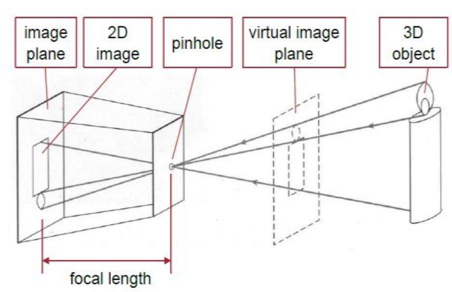
Pinhole Camera Model



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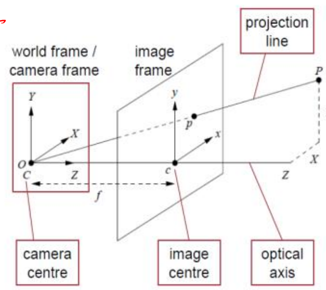
Pinhole Camera Model

- 3D point
  - $\mathbf{P} = (X, Y, Z)^T$
- 2D projection
  - $\mathbf{p} = (x, y)^T$



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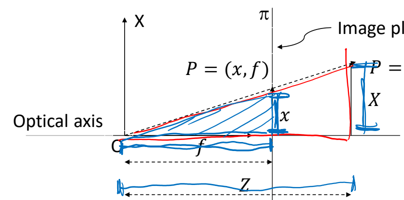
Pinhole Camera Model



$$\frac{x}{f} = \frac{X}{Z} \qquad x = f\,\frac{X}{Z}$$

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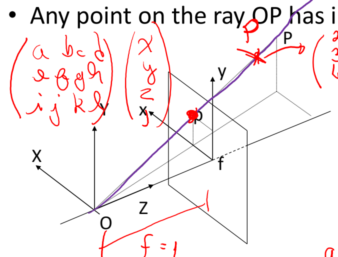
$Q = 2P$ $\quad Q = \lambda P = p$

Pinhole Camera Model

- Non-linear equations
- Any point on the ray OP has image p



$\begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$ $\quad \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \rightarrow p = \begin{pmatrix} 2/4 \cdot 1 \\ 3/4 \cdot 1 \end{pmatrix}$

$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$x = f\,\frac{X}{Z}$$

$$y = f\,\frac{Y}{Z}$$

$q = \begin{pmatrix} 4/8 \cdot 1 \\ 6/8 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2/4 \\ 3/4 \end{pmatrix}$

$f = 1$

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Perspective Matrix Equation

$p' = \Pi P$

$x' = fX$
$y' = fY$
$z' = Z$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$x = \frac{fX}{Z}$

$y = \frac{fY}{Z}$

$$x = \frac{x'}{z'} \qquad y = \frac{y'}{z'}$$

$x' = fX + 0Y + 0Z + 0 \cdot 1$
$y' = 0X + fY + 0Z + 0 \cdot 1$
$z' = 0X + 0Y + 1.Z + 0 \cdot 1$

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Perspective Matrix Equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$p = M_{\text{int}} \cdot P$$

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Transforms in Homogeneous notation

- And the projection?

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \sim \begin{pmatrix} m_{00} & m_{01} & m_{02} & \cancel{0}\ t_x \\ m_{10} & m_{11} & m_{12} & \cancel{0}\ t_y \\ m_{20} & m_{21} & m_{22} & \cancel{0}\ t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

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Projections in homogeneous notation

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{pmatrix} \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_z \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$$

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} & t_0 \\ m_{10} & m_{11} & m_{12} & t_1 \\ m_{20} & m_{21} & m_{22} & t_2 \\ p_0 & p_1 & p_2 & w \end{pmatrix}$$

- m_{ij} represent rotation, scale, shear
- t_i represent translation
- p_i are used in projections
- w is the homogeneous component

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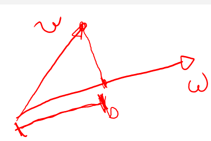
Projections in homogeneous notation

- We can use the last matrix row to realize divisions

$$p' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_0 & p_1 & p_2 & w \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ p_0 p_x + p_1 p_y + p_2 p_z + w \end{pmatrix} \sim \begin{pmatrix} \frac{p_x}{p_0 p_x + p_1 p_y + p_2 p_z + w} \\ \frac{p_y}{p_0 p_x + p_1 p_y + p_2 p_z + w} \\ \frac{p_z}{p_0 p_x + p_1 p_y + p_2 p_z + w} \end{pmatrix}$$



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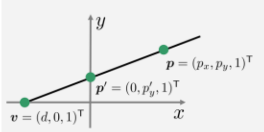
2D Example

$p = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ $x = \frac{fX}{Z}$

focal distance : f $y = \frac{fY}{Z}$



$p'_x = 0$

$\frac{p_y}{p_x - d} = \frac{p'_y}{-d} \Rightarrow p'_y = \frac{-dp_y}{p_x - d}$

$$p' = Mp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & 0 \\ 1 & 0 & -d \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_w \end{pmatrix} = \begin{pmatrix} 0 \\ -dp_y \\ p_x - d \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-dp_y}{p_x - d} \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{-dp_y}{p_x - d} \end{pmatrix}$$

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Line representation (review)

$ax + b = y$ $a' = -\frac{a}{b}$ $b' = -\frac{c}{b}$

$y = -\frac{a}{b}x - \frac{c}{b}$ $y = a'x + b'$

- Parametric equation: Consider the line through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the non-zero vector $v = (a, b, c)$
- A point $P = (x, y, z)$ is on the line if and only if
  - $(x - x_0, y - y_0, z - z_0) = t * v$
  - $x = x_0 + t * a$
  - $y = y_0 + t * b$
  - $z = z_0 + t * c$

$l: ax + by + c = 0$

$[x\ y\ 1] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

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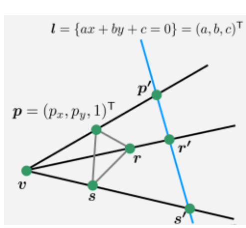
2D Example

- A 2D projection from $\boldsymbol{v}$ onto $\boldsymbol{l}$ maps a point $\boldsymbol{p}$ onto $\boldsymbol{p'}$
  - $\boldsymbol{p'}$ is the intersection of the line through $\boldsymbol{p}$ and $\boldsymbol{v}$ with line $\boldsymbol{l}$
  - $\boldsymbol{v}$ is the viewpoint, center of perspectivity
  - $\boldsymbol{l}$ is the viewline
  - The line through $\boldsymbol{p}$ and $\boldsymbol{v}$ is a projector



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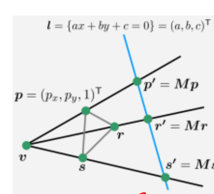
2D projection

- A 2D projection is represented by a matrix in homogeneous notation

$$M = vl^T - (l \cdot v)I_3$$

$$vl^T = \begin{pmatrix} v_x a & v_x b & v_x c \\ v_y a & v_y b & v_y c \\ v_w a & v_w b & v_w c \end{pmatrix}$$

$$(l \cdot v)I_3 = (av_x + bv_y + cv_w)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$v = \begin{bmatrix} a \\ b \end{bmatrix}$ $l = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix} [x\ y] = \begin{bmatrix} ax & ay \\ bx & by \end{bmatrix}$

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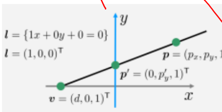
2D projection

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & 0 \\ 1 & 0 & -d \end{pmatrix} = \begin{pmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}$$



$$\boldsymbol{M} = \underbrace{\begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix} (1,0,0)} - \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix} \right) \boldsymbol{I}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & 0 \\ 1 & 0 & -d \end{pmatrix}$$

$$\boldsymbol{p}' = \boldsymbol{M}\boldsymbol{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & 0 \\ 1 & 0 & -d \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -dp_y \\ p_x - d \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-dp_y}{p_x - d} \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{-dp_y}{p_x - d} \end{pmatrix}$$

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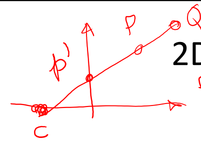
2D projection

$x = f\frac{X}{Z}$ $y = f\frac{Y}{Z}$



$P = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ $Q = \begin{pmatrix} \lambda X \\ \lambda Y \\ \lambda Z \end{pmatrix} = \lambda P$

– M and λM represent the same transformation ($\lambda \boldsymbol{M}\boldsymbol{p} = \lambda \boldsymbol{p}'$)

– $-\frac{1}{d}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & 0 \\ 1 & 0 & -d \end{pmatrix}$ ~~and~~ $=$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{d} & 0 & 1 \end{pmatrix}$ are the same transformation

– $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -d & 0 \\ 1 & 0 & -d \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_w \end{pmatrix} = \begin{pmatrix} 0 \\ -dp_y \\ p_x - dp_w \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-dp_y}{p_x - dp_w} \\ 1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{-dp_y}{p_x - dp_w} \end{pmatrix}$

$= \begin{pmatrix} 0 \\ \frac{-p_y}{\frac{p_x}{d} - p_w} \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{-p_y}{\frac{p_x}{d} - p_w} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ p_y \\ -\frac{p_x}{d} + p_w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{d} & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_w \end{pmatrix}$

$x_q = f\frac{\cancel{\lambda} X}{\cancel{\lambda} Z}$ $y_q = f\frac{\cancel{\lambda} Y}{\cancel{\lambda} Z}$

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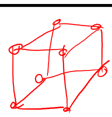
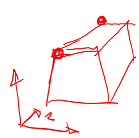
2D projection



$y = Y$
$x = X$



– Moving d to infinity results in parallel projection

$$\lim_{d \to \pm\infty} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{d} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ p_y \\ 1 \end{pmatrix}$$

– x-component is mapped to zero

– y- and w-component are unchanged

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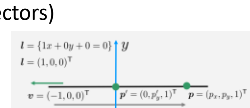
Parallel projection

- Parallel (viewpoint at infinity, parallel projectors)
  - Orthographic (viewline orthogonal to the projectors)



$\boldsymbol{M} = \boldsymbol{v}\boldsymbol{l}^{\mathsf{T}} - (\boldsymbol{l} \cdot \boldsymbol{v})\boldsymbol{I}_3$

$\boldsymbol{M} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} (1,0,0) - \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right) \boldsymbol{I}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

X-component is mapped to zero. Y-component is unchanged.

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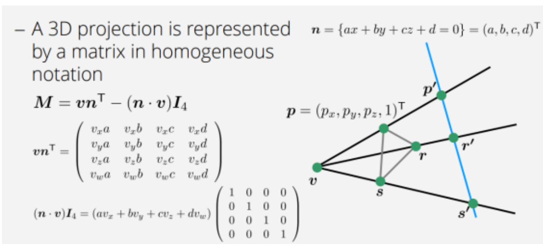
3D case



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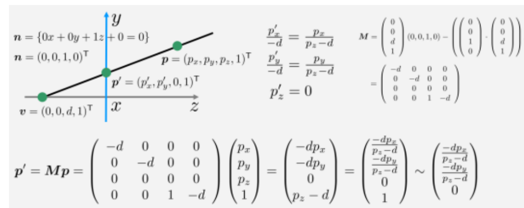
3D case



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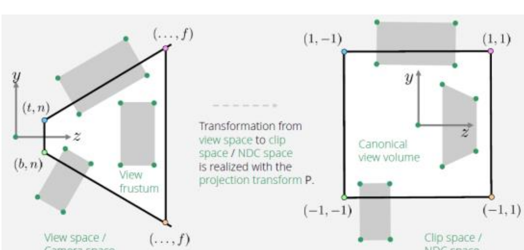
Projection Transform – Normalized Device Coordinates (NDC)



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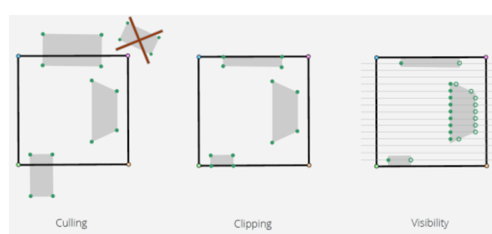
Clipping and Visibility



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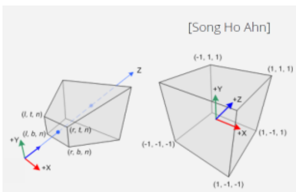
Projection Transform – Normalized Device Coordinates (NDC)

- Projection transform:
  - View volume/pyramidal frustum to a canonical view volume



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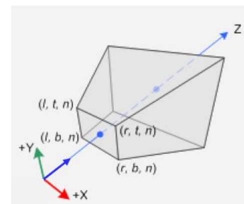
Perspective Projection

- The view volume is defined by its boundary
  - Left: l
  - Right: r
  - Bottom: b
  - Top: t
  - Near: n
  - Far: f



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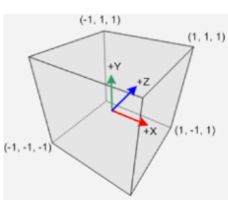
Perspective Projection

- The canonical view volume is a cube
  - From (-1,-1,-1)
  - To (1,1,1)



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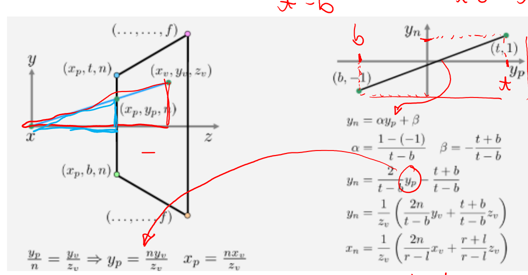
Perspective Projection

$1 = \left(\frac{1-(-1)}{t-b}\right)t + \beta$

$\alpha = \frac{1-(-1)}{t-b}$ $\beta = 1 - \frac{2t}{t-b} = \frac{t-b-2t}{t-b}$



$\beta = \frac{-b-t}{t-b} = -\frac{t+b}{t-b}$

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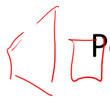

Perspective Projection

$x_c = \left(\frac{2n}{r-l}\right)x_v + \left(\frac{r+l}{r-l}\right)z_v$

– From

$$x_n = \frac{1}{z_v}\left(\frac{2n}{r-l}x_v + \frac{r+l}{r-l}z_v\right) \qquad y_n = \frac{1}{z_v}\left(\frac{2n}{t-b}y_v + \frac{t+b}{t-b}z_v\right)$$

we get

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \\ z_v \\ w_v \end{pmatrix}$$

Clip coordinates (clip space)

with

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ 1 \end{pmatrix} = \begin{pmatrix} x_c/w_c \\ y_c/w_c \\ z_c/w_c \\ w_c/w_c \end{pmatrix}$$

Normalized device coordinates (NDC space)

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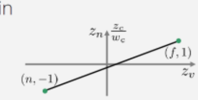
Perspective Projection

– z_v is mapped from (near, far) or (n, f) to (-1, 1)
– The transform does not depend on x_v and y_v
– So, we have to solve for A and B in

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & A & B \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \\ z_v \\ w_v \end{pmatrix}$$



$$z_n = \frac{z_c}{w_c} = \frac{Az_v + Bw_v}{z_v}$$

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Perspective Projection

- `glMatrixMode(GL_MODELVIEW)`
- `glFrustum (l, r, b, t, n, f)`

– $z_e = n$ with $w_v = 1$ is mapped to $z_n = -1$
– $z_e = f$ with $w_e = 1$ is mapped to $z_n = 1$

$\Rightarrow A = \frac{f+n}{f-n} \qquad \Rightarrow B = -\frac{2fn}{f-n}$

– The complete projection matrix is

$$\begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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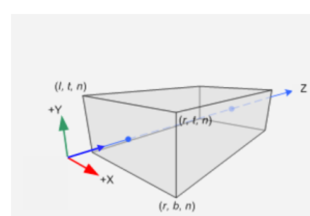
Orthographic Projection

- The view volume is defined by a cuboid and its boundary
  – Left: l
  – Right: r
  – Bottom: b
  – Top: t
  – Near: n
  – Far: f



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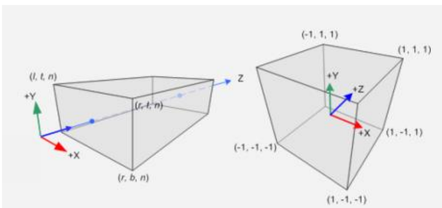
Orthographic Projection



[Song Ho Ahn]

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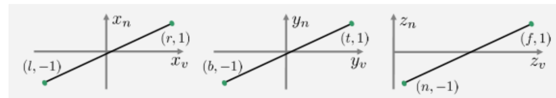
Orthographic Projection

- `glMatrixMode(GL_MODELVIEW)`
- `glOrtho(l, r, b, t, n, f)`



$x_n = \frac{2}{r-l} x_v - \frac{r+l}{r-l}$ $\quad y_n = \frac{2}{t-b} y_v - \frac{t+b}{t-b}$ $\quad z_n = \frac{2}{f-n} z_v - \frac{f+n}{f-n}$

$$\boldsymbol{P} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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