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Full paper

Kinematic modeling and singularity of wheeled mobile robots

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Abstract—This paper presents a global singularity analysis for wheeled mobile robots (WMRs). First, a kinematic model of a generic wheel is obtained using a recursive kinematics formulation. This novel and efficient approach is valid for all the common types of wheels: fixed, centered orientable, off-centered orientable (caster or castor) and Swedish or Mecanum. Then, a procedure for generating robot kinematic models is presented based on the set of wheel equations and the null space concept. Next, the singularity of kinematic models is discussed: first, the kinematic singularity condition in forward models is obtained, and then the singularity condition in inverse, or even mixed, models. A generic and practical geometric approach is established to characterize the singularity of any kinematic model of any WMR with the mentioned wheels. To illustrate the applications of the proposed approach, the singular configurations for many types of WMRs are depicted. Finally, the singularity characterization is extended to include other specialized wheels: dual-wheel, dual-wheel castor, ball-type and orthogonal.

Keywords: Model singularity; singular configurations; kinematic modeling; wheeled mobile robots.

1. INTRODUCTION

Wheeled mobile robots (WMRs) have been widely studied in the past 15 years. Due to kinematic constraints, WMRs are not integrable (non-holonomic). Therefore, standard techniques (either for kinematic/dynamic modeling) developed for robot manipulators are not directly applicable. This has led to abundant literature dealing with specific simplifications for kinematic models such as trailer-like, car-like, etc. Modeling and singularity characterization, which are often a prerequisite to motion planning and control design, are still, however, relevant issues. The aim of the present paper is to give a complete survey of WMR kinematic singularity.

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Examples of kinematic models for WMRs are available in the literature [1–9] among other relevant publications, although not all of them employ a systematic procedure. In this sense, Section 2 presents a systematic procedure for WMR kinematic modeling based upon the model of a generic wheel, obtained with a recursive kinematics formulation. This generic wheel includes all the common types: fixed, centered orientable (hereinafter orientable), off-centered orientable or castor and Swedish (also called Mecanum, Ilon or universal). It is worthwhile mentioning the fact that the generic wheel model makes the wheel velocity sliding vector explicit, which is useful for the subsequent stage of slip kinematic analysis and/or modeling. Moreover, Section 2 presents a systematic procedure, based upon the null space concept, for deriving forward and inverse kinematic models of WMRs under no-slip assumption.

Next, Section 3 discusses and characterizes the singularity of the WMR kinematic models. In particular, we formulate a general, practical and useful geometric approach for WMR singularity characterization. In order to illustrate the applications of the proposed approach, Section 5 derives the singularity for the kinematic models of the five types of WMRs classified according to Ref. [4]. Afterwards, Section 4 extends the singularity characterization to include other specialized wheels: dual-wheel, dual-wheel castor, ball-type and orthogonal. Finally, Section 6 points out the more outstanding contributions of this approach and suggests extensions to the present research.

2. KINEMATIC MODELING OF WMRs

First, some terminologies and relevant references are presented and commented.

Assuming horizontal movement, the position of the WMR body is completely specified by three scalar variables (e.g., x, y, θ), referred to in Ref. [4] as the WMR posture, $\mathbf{p}$ in vector form. Its first-order time derivative $\dot{\mathbf{p}}$ is called the WMR velocity vector, and separately (v_x, v_x, $\boldsymbol{\omega}$) WMR velocities [5]. Similarly, for each wheel, wheel velocity vector and wheel velocities [5] are defined.

Several publications have tackled the kinematic modeling of a wheel as a previous stage for modeling the whole WMR. One of the most outstanding kinematic modeling methodologies is presented in Ref. [5], where homogenous transformation matrices are used to relate coordinate systems. The result is a relationship, the Jacobian matrix, between the WMR velocities and the wheel velocities. Nevertheless, [5] presents the following drawbacks:

- Three equations per wheel are considered, while only two restrictions (no slip) should be derived for each wheel. Among other drawbacks, this produces an unnecessary computational cost. It would have been preferable to substitute one of the three scalar equations into the other two.
- Wheel rotation is included considering a fictitious planar pair between the wheel and the surface. This unnecessary *ad hoc* approach contrasts with the previous systematic method. It would have been preferable to use two additional frames.

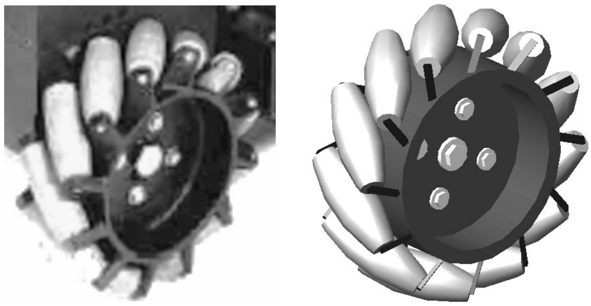


Figure 1. Swedish wheel (also called Mecanum or Ilon) with rollers at 45°.

- No slip is assumed at an unnecessarily early stage, complicating the identification of these magnitudes for subsequent stages.

The contribution of Ref. [6] follows the method of Ref. [5], expanding it to wheels with the orientation axle inclined and displaced.

References [4, 7] contribute to relevant kinematics: Ref. [7] uses a vector approach just for fixed and orientable wheel modeling, and Ref. [4] does not justify the wheel kinematics used, which is the key for the subsequent WMR classification and characterization. Another interesting study is Ref. [8], where the wheel kinematics are derived using a vector procedure. The wheel equation is defined with explicit sliding velocities (in both directions), which are occasionally used (augmented generalized coordinates) to make the Jacobian matrix square by adding trivial scalar equations. This modification facilitates forward and inverse kinematic relationships. Nevertheless, those trivial equations should be eliminated in a subsequent stage to avoid unnecessary computational cost. In addition, mobility analysis based on Grübler's formula, classical in mechanical systems, is also described. Finally, Ref. [9] is an example of a completely *ad hoc* geometric modeling with no systematic procedure.

Since the Swedish wheel may not be known by the reader, Fig. 1 shows an example with the usual roller orientation of 45°. Another classical orientation is 90°, used in Section 4.

2.1. Generic wheel equation

This work considers a modeling method based on recursive a kinematics formulation described in Ref. [10]:

$$\mathbf{v}_i = \frac{\mathrm{d}^*\mathbf{p}_i^*}{\mathrm{d}t} + (\boldsymbol{\omega}_{i-1} \times \mathbf{p}_i^*) + \mathbf{v}_{i-1}, \quad \boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_i^*, \tag{1}$$

with the following definitions (see Fig. 2):

$\frac{\mathrm{d}^*}{\mathrm{d}t}$ is the time derivative in the frame $i-1$, $\boldsymbol{\omega}_i^*$ is the angular velocity of the frame i with respect to frame $i-1$, in coordinate frame 0, $\boldsymbol{\omega}_i$ is the angular velocity of the frame i with respect to frame 0, $\mathbf{p}_i^*$ is the vector from the origin of the frame $i-1$

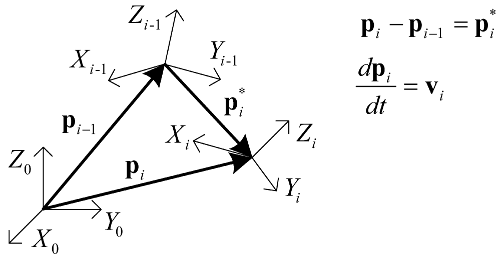


Figure 2. Frames and variables of the recursive a kinematics formulation described in Ref. [10].

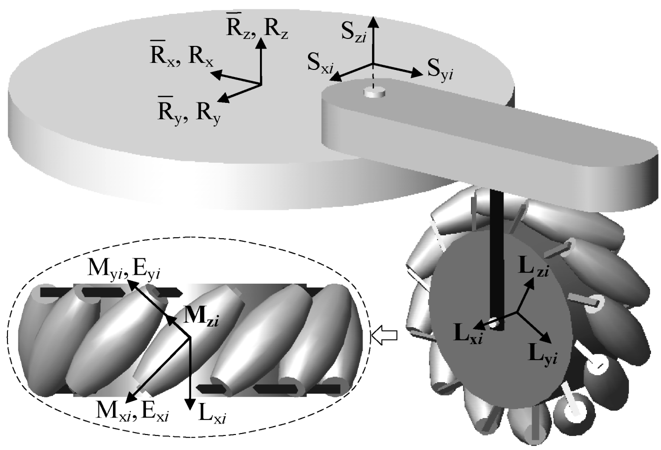


Figure 3. Assigned frames with a detailed bottom view of the wheel with rollers.

to the origin of the frame i, in coordinate frame 0 and $\mathbf{v}_i$ is the velocity of the origin of the frame i with respect to the origin of the frame 0, in coordinate frame 0.

Several methods for frame assignment in robotics have been developed [11, 12]. However, this paper considers one frame per mobile rigid link, two instantaneously coincident frames [5] and one additional frame for the roller in the contact point with the floor. Frames are described in Fig. 3 and Table 1.

In order to apply (1) recurrently (5 times), the notation of Table 2, taken from [5], is used:

$$\mathbf{v}_1 = {}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{R}} \qquad \boldsymbol{\omega}_1 = {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} \tag{2}$$

$$\mathbf{v}_2 = {}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{R}} + {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\mathbf{d}_{\mathrm{S}i} \qquad \boldsymbol{\omega}_2 = {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} \tag{3}$$

$$\mathbf{v}_3 = {}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{R}} + {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\mathbf{d}_{\mathrm{L}i} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} \times {}^{\mathrm{R,S}i}\mathbf{d}_{\mathrm{L}i} \quad \boldsymbol{\omega}_3 = {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} + {}^{\mathrm{R,S}i}\boldsymbol{\omega}_{\mathrm{L}i} \tag{4}$$

$$\begin{aligned} \mathbf{v}_4 &= {}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{R}} + {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\mathbf{d}_{\mathrm{M}i} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} \times {}^{\mathrm{R,S}i}\mathbf{d}_{\mathrm{M}i} + {}^{\mathrm{R,S}i}\boldsymbol{\omega}_{\mathrm{L}i} \times {}^{\mathrm{R,L}i}\mathbf{d}_{\mathrm{M}i} \\ \boldsymbol{\omega}_4 &= {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} + {}^{\mathrm{R,S}i}\boldsymbol{\omega}_{\mathrm{L}i} + {}^{\mathrm{R,L}i}\boldsymbol{\omega}_{\mathrm{M}i} \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbf{v}_5 &= {}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{R}} + {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} \times {}^{\mathrm{R}}\mathbf{d}_{\mathrm{E}i} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} \times {}^{\mathrm{R,S}i}\mathbf{d}_{\mathrm{E}i} \\ &\quad + {}^{\mathrm{R,S}i}\boldsymbol{\omega}_{\mathrm{L}i} \times {}^{\mathrm{R,L}i}\mathbf{d}_{\mathrm{E}i} + {}^{\mathrm{R,L}i}\boldsymbol{\omega}_{\mathrm{M}i} \times {}^{\mathrm{R,M}i}\mathbf{d}_{\mathrm{E}i} \\ \boldsymbol{\omega}_5 &= {}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}} + {}^{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{S}i} + {}^{\mathrm{R,S}i}\boldsymbol{\omega}_{\mathrm{L}i} + {}^{\mathrm{R,L}i}\boldsymbol{\omega}_{\mathrm{M}i} + {}^{\mathrm{R,M}i}\boldsymbol{\omega}_{\mathrm{E}i}. \end{aligned} \tag{6}$$

Table 1.
Assigned frames

No.	Symbol	Description
0	$\bar{\mathrm{R}}$	frame attached to the floor, coincident with the robot frame, with the Z-axis perpendicular to the floor surface
1	R	frame attached to the robot body with the Z-axis perpendicular to the floor surface
2	S_i	frame attached to the steering link of the wheel i, with the Z-axis coincident with the steering axle and the Y-axis parallel to the steering link
3	L_i	frame attached to the wheel i and with the X-axis coincident with the wheel rotation axle
4	M_i	frame attached to the roller of the wheel i and with the X-axis coincident with the roller rotation axle
5	E_i	frame attached to the roller but, in contrast with M_i, with the origin in the contact point between the wheel i roller and the floor, the Z-axis perpendicular to the floor surface and the X-axis parallel to M_{ix}
6	$\bar{\mathrm{E}}_i$	frame attached to the floor and coincident with E_i

Table 2.
Notation for variables and parameters

Symbol	Description
${}^{\mathrm{H,A}}\mathbf{d}_{\mathrm{B}}$	vector from the origin of the A frame to the frame B in coordinate frame H
${}^{\mathrm{A}}\theta_{\mathrm{B}}$	angle between the X-axis of the frame B and the X-axis of the frame A in coordinate Z of frame A
${}^{\mathrm{H,A}}\mathbf{v}_{\mathrm{B}}$	vector linear velocity of the origin of the frame B with respect to frame A in coordinate frame H
${}^{\mathrm{H,A}}\boldsymbol{\omega}_{\mathrm{B}}$	vector angular velocity of the frame B with respect to frame A in coordinate frame H

If H is not present in the expression, A takes its place.

In particular, $\mathbf{v}_5$ in (6) is ${}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{E}i}$, the sliding linear velocity between the rollers and the floor in coordinate frame $\bar{\mathrm{R}}$. Taking into account the frames and joints between the links depicted in Fig. 3, several terms of ${}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{E}i}$ are simplified:

$$\begin{aligned}\begin{pmatrix} {}^{\bar{\mathrm{R}}}v_{\mathrm{E}ix} \\ {}^{\bar{\mathrm{R}}}v_{\mathrm{E}iy} \\ 0 \end{pmatrix} &= \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\omega} \end{pmatrix} \times {}^{\mathrm{R}}\mathbf{d}_{\mathrm{E}i} + \begin{pmatrix} 0 \\ 0 \\ \dot{\beta}_i \end{pmatrix} \times {}^{\mathrm{R,S}i}\mathbf{d}_{\mathrm{E}i} \\ &\quad + \begin{pmatrix} \dot{\varphi}_i \cos {}^{\mathrm{R}}\theta_{\mathrm{L}i} \\ \dot{\varphi}_i \sin {}^{\mathrm{R}}\theta_{\mathrm{L}i} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -\mathrm{r}_i \end{pmatrix} + \begin{pmatrix} \dot{\varphi}_{\mathrm{r}i} \cos {}^{\mathrm{R}}\theta_{\mathrm{Ei}} \\ \dot{\varphi}_{\mathrm{r}i} \sin {}^{\mathrm{R}}\theta_{\mathrm{Ei}} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -\mathrm{r}_{\mathrm{r}i} \end{pmatrix}, \quad (7)\end{aligned}$$

with the meaning of variables and constants of Table 3.

Table 3.
New variables and constants in (7)

Symbol	Description
$\dot{\mathbf{p}} = (v_x \ \ v_y \ \ \boldsymbol{\omega})^{\mathrm{T}}$	WMR velocity vector in coordinate frame $\bar{\mathrm{R}}$, equivalent to $({}^{\bar{\mathrm{R}}}v_{\mathrm{R}x}{}^{\bar{\mathrm{R}}}v_{\mathrm{R}y}{}^{\bar{\mathrm{R}}}\boldsymbol{\omega}_{\mathrm{R}})^{\mathrm{T}}$
$\dot{\beta}_i$	angular velocity of the steering link with respect to the WMR
$(\dot{\varphi}_i, \dot{\varphi}_{\mathrm{r}i})$	rotation velocity of the wheel and the rollers in coordinate X of frame L_i and
$(\mathrm{r}_i, \mathrm{r}_{\mathrm{r}i})$	M_i wheel equivalent radius and roller radius

From (7) each wheel introduces two scalar equations, which can be expressed in a matrix form as:

$$
{}^{\bar{\mathrm{R}}}\mathbf{v}_{\mathrm{E}i} = \begin{pmatrix} 1 & 0 & -{}^{\mathrm{R}}d_{\mathrm{E}iy} & -{}^{\mathrm{R,S}i}d_{\mathrm{E}iy} & -r_i \sin {}^{\mathrm{R}}\theta_{\mathrm{L}i} & -r_{\mathrm{r}i} \sin {}^{\mathrm{R}}\theta_{\mathrm{Ei}} \\ 0 & 1 & {}^{\mathrm{R}}d_{\mathrm{E}ix} & {}^{\mathrm{R,S}i}d_{\mathrm{E}ix} & r_i \cos {}^{\mathrm{R}}\theta_{\mathrm{L}i} & r_{\mathrm{r}i} \cos {}^{\mathrm{R}}\theta_{\mathrm{Ei}} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\beta}_i \\ \dot{\varphi}_i \\ \dot{\varphi}_{\mathrm{r}i} \end{pmatrix}, \tag{8}
$$

where the Z velocity component is omitted and the distance vector subscript x/y indicates the corresponding component.

The sliding velocity in (8) can also be expressed with respect to the roller directions with a Z-axis rotation of $-{}^{\mathrm{R}}\theta_{\mathrm{E}i}$:

$$
{}^{\overline{\mathrm{E}i}}\mathbf{v}_{\mathrm{E}i} = \begin{pmatrix} \cos {}^{\mathrm{R}}\theta_{\mathrm{E}i} & \sin {}^{\mathrm{R}}\theta_{\mathrm{E}i} & {}^{\mathrm{E}i}d_{\mathrm{R}y} & {}^{\mathrm{E}i}d_{\mathrm{S}iy} & r_i \sin {}^{\mathrm{L}i}\theta_{\mathrm{E}i} & 0 \\ -\sin {}^{\mathrm{R}}\theta_{\mathrm{E}i} & \cos {}^{\mathrm{R}}\theta_{\mathrm{E}i} & -{}^{\mathrm{E}i}d_{\mathrm{R}x} & -{}^{\mathrm{E}i}d_{\mathrm{S}ix} & r_i \cos {}^{\mathrm{L}i}\theta_{\mathrm{E}i} & r_{\mathrm{r}i} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\beta}_i \\ \dot{\varphi}_i \\ \dot{\varphi}_{\mathrm{r}i} \end{pmatrix}. \tag{9}
$$

The following notation, depicted in Fig. 4, will be used:

$$
\begin{array}{lll} {}^{\mathrm{R}}d_{\mathrm{S}ix} = l_i \cos\alpha_i & {}^{\mathrm{R}}d_{\mathrm{S}iy} = l_i \sin\alpha_i & {}^{\mathrm{S}i}d_{\mathrm{E}iy} = d_i \\ {}^{\mathrm{R}}\theta_{\mathrm{Si}} = \beta_i & {}^{\mathrm{S}i}\theta_{\mathrm{L}i} = \delta_i & {}^{\mathrm{L}i}\theta_{\mathrm{E}i} = \gamma_i. \end{array} \tag{10}
$$

This notation is taken from Ref. [4], although exact matching for ${}^{\mathrm{R}}\theta_{\mathrm{S}i}$ should be $\alpha_i + \beta_i$ for fixed, orientable and Swedish wheels, and $\alpha_i + \beta_i + \pi$ for castor wheels. In both cases ${}^{\mathrm{L}i}\theta_{\mathrm{E}i}$ should be $\gamma_i - \pi/2$, while ${}^{\mathrm{S}i}\theta_{\mathrm{L}i}$ was not considered in Ref. [4].

The distance components of (9) with the notation (10) are:

$$
\begin{aligned} {}^{\mathrm{E}i}d_{\mathrm{R}y} &= l_i \sin(\beta_i + \delta_i + \gamma_i - \alpha_i) - d_i \cos(\delta_i + \gamma_i) \\ {}^{\mathrm{E}i}d_{\mathrm{R}x} &= -l_i \cos(\beta_i + \delta_i + \gamma_i - \alpha_i) - d_i \sin(\delta_i + \gamma_i) \\ {}^{\mathrm{E}i}d_{\mathrm{S}iy} &= -\mathrm{d}_i \cos(\delta_i + \gamma_i) \quad {}^{\mathrm{E}i}d_{\mathrm{S}ix} = -d_i \sin(\delta_i + \gamma_i). \end{aligned} \tag{11}
$$

The wheel model of Figs 3 and 4 is different from the models of other works because it makes it possible to include the Swedish wheel type and to consider a non-zero angle δ_i for castor wheels. Wheel equation (9) induces the minimum possible computational cost, i.e., just two scalar equations per wheel.

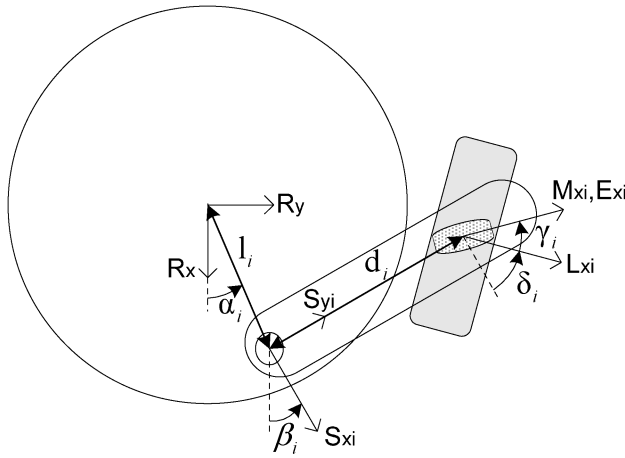


Figure 4. Wheel parameters l_i, d_i, α_i, β_i, δ_i and γ_i.

2.2. Particularization of the wheel equation

Here, (9) is particularized using ${}^{\overline{\mathrm{E}i}}\mathbf{v}_{\mathrm{E}i} \equiv \mathbf{v}_i$ and the subscripts f, o, c, s for fixed, orientable, castor and Swedish wheel.

2.2.1. Fixed wheels. In this case, parameters $r_{\mathrm{r}i}$, d_i, δ_i and γ_i are zero (wheel aligned with S_{iy}-axis), and β_i is constant:

$$\mathbf{v}_{\mathrm{f}i} = \begin{pmatrix} \cos\beta_{\mathrm{f}i} & \sin\beta_{\mathrm{f}i} & l_{\mathrm{f}i}\sin(\beta_{\mathrm{f}i}-\alpha_{\mathrm{f}i}) & 0 \\ -\sin\beta_{\mathrm{f}i} & \cos\beta_{\mathrm{f}i} & l_{\mathrm{f}i}\cos(\beta_{\mathrm{f}i}-\alpha_{\mathrm{f}i}) & r_{\mathrm{f}i} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{\mathrm{f}i} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{E}_{\mathrm{f}i} & 0 \\ \mathbf{F}_{\mathrm{f}i} & r_{\mathrm{f}i} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{\mathrm{f}i} \end{pmatrix}. \tag{12}$$

2.2.2. Orientable wheels. It is obtained the same equation (12) but with angle β_i variable:

$$\mathbf{v}_{\mathrm{o}i} = \begin{pmatrix} \cos\beta_{\mathrm{o}i} & \sin\beta_{\mathrm{o}i} & l_{\mathrm{o}i}\sin(\beta_{\mathrm{o}i}-\alpha_{\mathrm{o}i}) & 0 \\ -\sin\beta_{\mathrm{o}i} & \cos\beta_{\mathrm{o}i} & l_{\mathrm{o}i}\cos(\beta_{\mathrm{o}i}-\alpha_{\mathrm{o}i}) & r_{\mathrm{o}i} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{\mathrm{o}i} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{E}_{\mathrm{o}i} & 0 \\ \mathbf{F}_{\mathrm{o}i} & r_{\mathrm{o}i} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{\mathrm{o}i} \end{pmatrix}. \tag{13}$$

Note that, the steering velocity affects the wheel kinematics indirectly through its angle, but not instantaneously.

2.2.3. Castor wheels. Parameters $r_{\mathrm{r}i}$ and γ_i are zero:

$$\mathbf{v}_{\mathrm{c}i} = \begin{pmatrix} \cos(\beta_{\mathrm{c}i}+\delta_i) & \sin(\beta_{\mathrm{c}i}+\delta_i) & l_{\mathrm{c}i}\sin(\beta_{\mathrm{c}i}+\delta_i-\alpha_{\mathrm{c}i}) - d_i\cos\delta_i & -d_i\cos\delta_i & 0 \\ -\sin(\beta_{\mathrm{c}i}+\delta_i) & \cos(\beta_{\mathrm{c}i}+\delta_i) & l_{\mathrm{c}i}\cos(\beta_{\mathrm{c}i}+\delta_i-\alpha_{\mathrm{c}i}) + d_i\sin\delta_i & d_i\sin\delta_i & r_{\mathrm{c}i} \end{pmatrix}$$
$$\times \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\beta}_{\mathrm{c}i} \\ \dot{\varphi}_{\mathrm{c}i} \end{pmatrix} \tag{14}$$

$$\mathbf{v}_{ci} = \begin{pmatrix} \mathbf{F}_{\beta ci} & -d_i \cos\delta_i & 0 \\ \mathbf{F}_{\varphi ci} & d_i \sin\delta_i & r_{ci} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\beta}_{ci} \\ \dot{\varphi}_{ci} \end{pmatrix}.$$

2.2.4. Swedish wheels. Parameters d_i and δ_i are zero (wheel aligned with S_{iy}-axis), and β_i is constant:

$$\mathbf{v}_{si} = \begin{pmatrix} \cos(\beta_{si} + \gamma_i) & \sin(\beta_{si} + \gamma_i) & l_{si} \sin(\beta_{si} + \gamma_i - \alpha_{si}) & r_{si} \sin\gamma_{si} & 0 \\ -\sin(\beta_{si} + \gamma_i) & \cos(\beta_{si} + \gamma_i) & l_{si} \cos(\beta_{si} + \gamma_i - \alpha_{si}) & r_{si} \cos\gamma_{si} & r_{ri} \end{pmatrix} \times \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{si} \\ \dot{\varphi}_{rsi} \end{pmatrix}. \tag{15}$$

In the above expression, with the assumption that the frictional force always supplies the required acceleration, the rotational velocity $\dot{\varphi}_{ri}$ of the free roller guarantees no slip in the E_{iy} direction. Nevertheless, since that variable is not accessible, only the first slip component $\mathbf{v}_{six}$ has practical utility:

$$\begin{aligned} v_{six} &= \begin{pmatrix} \cos(\beta_{si} + \gamma_i) & \sin(\beta_{si} + \gamma_i) & l_{si} \sin(\beta_{si} + \gamma_i - \alpha_{si}) & r_{si} \sin\gamma_{si} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{si} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{F}_{si} & r_{si} \sin\gamma_{si} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{si} \end{pmatrix}. \end{aligned} \tag{16}$$

2.3. Kinematic equation of the WMR

Once the type of WMR wheels and their equations (through the corresponding wheel parameters: r_i, r_{ri}, l_i, d_i, α_i, β_i, δ_i and γ_i) are established, a compound global kinematic equation for the WMR may be defined. For that purpose, using (12)–(14) and (16), the compound WMR kinematic equation is derived:

$$\mathbf{v}_{\text{slip}} = \left(\begin{array}{c|ccccc} \mathbf{E}_{f} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{E}_{o} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{F}_{f} & \mathbf{r}_{f} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{o} & \mathbf{0} & \mathbf{r}_{o} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{s} & \mathbf{0} & \mathbf{0} & \mathbf{r}_{s} & \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{\beta c} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{d}_{\beta} & \mathbf{0} \\ \mathbf{F}_{\varphi c} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{d}_{\varphi} & \mathbf{r}_{c} \end{array}\right) \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\varphi}_{f} \\ \dot{\varphi}_{o} \\ \dot{\varphi}_{s} \\ \dot{\beta}_{c} \\ \dot{\varphi}_{c} \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{E} & \mathbf{0} \\ \hline \mathbf{F} & \mathbf{r} \end{array}\right) \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}}_{w} \end{pmatrix} = \mathbf{A}\dot{\mathbf{q}}, \tag{17}$$

where $\{\mathbf{E}_f, \mathbf{E}_o, \mathbf{F}_f, \mathbf{F}_o, \mathbf{F}_s, \mathbf{F}_{\beta c}, \mathbf{F}_{\phi c}\}$ are matrices obtained by the combination of row vectors $\{\mathbf{E}_{fi}, \mathbf{E}_{oi}, \mathbf{F}_{fi}, \mathbf{F}_{oi}, \mathbf{F}_{si}, \mathbf{F}_{\beta ci}, \mathbf{F}_{\phi ci}\}$, $\{\mathbf{r}_f, \mathbf{r}_o, \mathbf{r}_s, \mathbf{r}_c, \mathbf{d}_\beta, \mathbf{d}_\phi\}$ are diagonal matrices with the diagonal elements $\{r_{fi}, r_{oi}, r_{si} \sin\gamma_{si}, r_{ci}, -d_i \cos\delta_i, d_i \sin\delta_i\}$, $\{\dot{\varphi}_f, \dot{\varphi}_o, \dot{\varphi}_s, \dot{\beta}_c, \dot{\varphi}_c\}$ are the compound vectors of the elements $\{\dot{\varphi}_{fi}, \dot{\varphi}_{oi}, \dot{\varphi}_{si}, \dot{\beta}_{ci}, \dot{\varphi}_{ci}\}$, $\dot{\mathbf{q}}_w$ is the vector of all the wheel velocities, $\dot{\mathbf{q}}$ is the vector of all the velocities and $\mathbf{v}_{\text{slip}}$ is the vector of all the sliding velocities.

2.4. Kinematic modeling of WMRs with no slip

Under the no-slip condition, (17) becomes:

$$\mathbf{A}\dot{\mathbf{q}} = \mathbf{0}. \tag{18}$$

Therefore, the kinematic solution for velocity vector $\dot{\mathbf{q}}$ belongs to the null space of WMR matrix $\mathbf{A}$:

$$\dot{\mathbf{q}} \in \mathcal{N}(\mathbf{A}) \rightarrow \dot{\mathbf{q}} = \mathbf{B}\boldsymbol{\eta}, \tag{19}$$

where the matrix $\mathbf{B}$ forms a basis of $\mathcal{N}(\mathbf{A})$, $\boldsymbol{\eta}$ is an m-dimensional vector representing the WMR mobility and m is the WMR mobility degree given by the nullity of $\mathbf{A}$ (rank-nullity theorem):

$$m = \dim(\boldsymbol{\eta}) = \dim(\mathcal{N}(\mathbf{A})) = \dim(\dot{\mathbf{q}}) - \operatorname{rank}(\mathbf{A}) = k - g. \tag{20}$$

In order to use variables with physical meaning, the mobility vector $\boldsymbol{\eta}$ should be replaced with a set of freely assigned velocities. Depending on whether wheel velocities or WMR velocities are chosen, a forward or inverse kinematic model is obtained. If a mix of both types of velocities is chosen, a mixed solution is achieved. In order to check if an m-set of velocities $\dot{\mathbf{q}}_{\mathrm{a}}$ can be assigned, it must be verified that the determinant of the submatrix they define in (19) is non zero, i.e.:

$$\begin{pmatrix} \dot{\mathbf{q}}_{\mathrm{na}} \\ \dot{\mathbf{q}}_{\mathrm{a}} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{\mathrm{na}} \\ \mathbf{B}_{\mathrm{a}} \end{pmatrix} \boldsymbol{\eta} \tag{21}$$

$$\text{if } |\mathbf{B}_{\mathrm{a}}| \neq 0 \quad \rightarrow \quad \dot{\mathbf{q}}_{\mathrm{na}} = \mathbf{B}_{\mathrm{na}}\mathbf{B}_{\mathrm{a}}^{-1}\dot{\mathbf{q}}_{\mathrm{a}}, \tag{22}$$

where $\dot{\mathbf{q}}_{\mathrm{na}}$ are the remaining velocities of $\dot{\mathbf{q}}$.

Alternatively to the previous procedure, based upon the null space concept, it is possible to apply another method, based on separating the assigned velocities in (18):

$$\mathbf{A}_{\mathrm{na}}\dot{\mathbf{q}}_{\mathrm{na}} = -\mathbf{A}_{\mathrm{a}}\dot{\mathbf{q}}_{\mathrm{a}}. \tag{23}$$

To check if an m-set of velocities could be assigned $\dot{\mathbf{q}}_{\mathrm{a}}$, it must be verified that matrix $\mathbf{A}_{\mathrm{na}}$ is, in general, of full rank g:

$$\operatorname{rank}(\mathbf{A}_{\mathrm{na}}) = \operatorname{rank}(\mathbf{A}) = g. \tag{24}$$

3. SINGULARITY CHARACTERIZATION

3.1. Singularity of the kinematic model with no slip

In general, matrix $\mathbf{B}_{\mathrm{a}}$ (or alternatively matrix $\mathbf{A}_{\mathrm{na}}$) depends upon wheel parameters ($r_i, r_{\mathrm{r}i}, l_i, d_i, \alpha_i, \delta_i, \gamma_i, \beta_{\mathrm{f}i}, \beta_{\mathrm{s}i}$) and variable angles ($\beta_{\mathrm{o}i}, \beta_{\mathrm{c}i}$) of castor and orientable wheels. Therefore, the corresponding kinematic solution according to (22) also depends on these parameters and variables. So, for a given m-set of assigned velocities, there may be some particular values for parameters and variable angles where singularity arises, i.e., $\mathbf{B}_a$ becomes singular (or degenerate) or, alternatively, $\mathbf{A}_{na}$ losses rank (i.e., degenerates). For these situations, there are two possible approaches:

- Degrees of mobility loss: in order to avoid incompatibility the assigned velocities are coordinated properly, this implies a mobility degree loss.
- Kinematics incompatibility: no type of coordination for the assigned velocities is considered, so the kinematic incompatibility is not solved. If the assigned velocities are wheel velocities (forward kinematics), slip (due to the incompatibility, not because of accelerations) is inevitable. If they are WMR velocities (inverse kinematics), impossible (infinite) control action values are obtained.

Note that the WMR mobility degree m also depends upon the variable angles of castor and orientable wheels, since it depends on the rank of $\mathbf{A}$. For example, a WMR with two orientable wheels has in general $m = 1$, except for both wheels perpendicular to their joining line where $m = 2$. If less than m velocities are assigned, the model is indefinite.

The singularity characterization has been largely studied for robotic manipulators (see Refs [13–16] among many other relevant publications). In contrast, in WMR only a few authors have analyzed kinematic singularity. For instance, in Ref. [17] a partial singularity work is presented for an omnidirectional WMR with three castor wheels, while it is singularityless with three Swedish wheels [18]. In addition, study [9] focuses on the case of three orientable wheels.

3.2. Practical use of singularity

The previous subsection provides the following criterion: singularity has to be avoided (i.e., degrees of mobility loss, slip or impossible control actions). In particular, when the WMR has no orientable or castor wheels, this becomes a design criterion, since the conditions for singularity are permanent.

In contrast, if there are orientable or castor wheels, the singularity conditions arise for particular configurations, i.e., singular configurations. Then, the commented criterion becomes a planning and/or control criterion, i.e., the upper level planner (path generator) and/or controller have to develop paths and/or control actions that prevent singularities. This would be tackled analogously to the robotic manipulators case, where several planners and motion controllers have been developed considering singularity (see Refs [19–22] among many other relevant publications). In this case, an over-actuated WMR would be useful, since it allows change from one kinematic model to another, avoiding singularities.

3.3. Singularity of forward kinematic models

As it has been shown in Section 3.1, the singularity for a particular kinematic model (22) could come from the singularity of matrix $\mathbf{B}_a$. Nevertheless, it is possible to obtain a general singularity characterization, which is valid for any kinematic model of any WMR, through the analysis of the degeneration of matrix $\mathbf{A}_{\mathrm{na}}$. In the case of a forward kinematic model, the matrix $\mathbf{A}_{\mathrm{na}}$ is:

$$\mathbf{A}_{\mathrm{na}} = \left(\begin{array}{c|c} \mathbf{E} & \mathbf{0} \\ \mathbf{F}_{\mathrm{a}} & \mathbf{0} \\ \hline \mathbf{F}_{\mathrm{na}} & \mathbf{r}_{\mathrm{na}} \end{array}\right), \tag{25}$$

where the second matrix row is for the assigned velocities and the third for the remaining velocities.

First, matrix $\mathbf{A}_{\mathrm{na}}$ degenerates when:

$$|\mathbf{r}_{\mathrm{na}}| = 0 \quad \rightarrow \quad \exists i\colon \{\sin\gamma_i = 0 \text{ or } \cos\delta_i = 0\}, \tag{26}$$

where it has been assumed that $\mathrm{r}_i \neq 0$ and $\mathrm{d}_i \neq 0$.

Conditions (26) mean that there is some singular Swedish wheel (the roller orientation is parallel to the wheel plane) or some singular castor wheel (the wheel plane is perpendicular to the steering link). For this singular situation, velocities $\dot{\varphi}_{\mathrm{s}i}$ or $\dot{\beta}_{\mathrm{c}i}$, do not affect directly the kinematic model (while $\dot{\beta}_{\mathrm{c}i}$ has an indirect effect through $\beta_{\mathrm{c}i}$), so that its associated row vector would be included together with matrix $(\mathbf{E}^{\mathrm{T}}\mathbf{F}_{\mathrm{a}}^{\mathrm{T}})^{\mathrm{T}}$ in (25), i.e.:

$$\mathbf{G} = \begin{pmatrix} \mathbf{E} \\ \mathbf{F}_{\mathrm{a}} \\ \mathbf{F}_{\mathrm{na\_s}} \end{pmatrix} \tag{27}$$

$$\mathbf{A}_{\mathrm{na}} = \left(\begin{array}{c|c} \mathbf{G} & \mathbf{0} \\ \hline \mathbf{F}_{\mathrm{na\_ns}} & \mathbf{r}_{\mathrm{na\_ns}} \end{array}\right), \tag{28}$$

where $\mathbf{F}_{\mathrm{na\_s}}$ is a three-column matrix with the row vectors $\mathbf{F}_{\mathrm{s}i}$ and $\mathbf{F}_{\beta\mathrm{c}i}$ of the singular omnidirectional wheels, and $\{\mathbf{F}_{\mathrm{na\_ns}}, \mathbf{r}_{\mathrm{na\_ns}}\}$ are the same as $\{\mathbf{F}_{\mathrm{na}}, \mathbf{r}_{\mathrm{na}}\}$ but without those row vectors. Note that, $\mathbf{F}_{\mathrm{s}i}$ and/or $\mathbf{F}_{\beta\mathrm{c}i}$ are included in $\mathbf{G}$ either if the corresponding velocity is assigned or if the wheel i is singular.

The conditions (26) evidence why the classical roller orientation of 90° ($\gamma = 90°$) and the usual castor wheel orientation of 0° ($\delta = 0°$) are desirable, since those values are as far as possible from the omnidirectional wheel singularity.

Second, matrix $\mathbf{A}_{\mathrm{na}}$ degenerates with matrix $\mathbf{G}$:

$$\operatorname{rank}(\mathbf{G}) < 3. \tag{29}$$

In order to establish the conditions (29), the following row vector will be considered:

$$\boldsymbol{\lambda} = (\lambda_x \;\; \lambda_y \;\; \mathrm{d}_\lambda), \tag{30}$$

where $(\lambda_x \;\; \lambda_y)^{\mathrm{T}}$ is a two-dimensional unit vector in the direction of a straight line (see Fig. 5), and d_λ is the distance between the line and the origin, with positive sign if the cross product between the distance vector (from the origin to the line) and the direction vector is positive in coordinate Z and with negative sign otherwise.

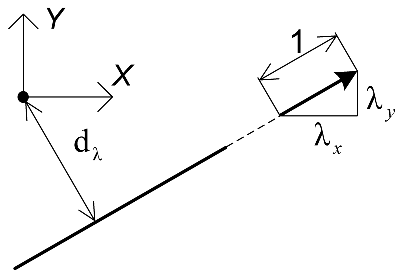


Figure 5. Line defined by the three element row vector $\boldsymbol{\lambda} = (\lambda_x \;\; \lambda_y \;\; d_\lambda)$.

The equation of the line is defined as

$$\lambda_x y - \lambda_y x + d_\lambda = (\lambda_x \;\; \lambda_y \;\; d_\lambda)(y \;\; -x \;\; 1)^{\mathrm{T}} = 0. \tag{31}$$

The linear dependency of three λ vectors representing three lines (a, b, c) in the plane is given by:

$$\begin{vmatrix} \lambda_{xa} & \lambda_{ya} & d_{\lambda a} \\ \lambda_{xb} & \lambda_{yb} & d_{\lambda b} \\ \lambda_{xc} & \lambda_{yc} & d_{\lambda c} \end{vmatrix} = 0. \tag{32}$$

The previous condition implies that all three lines (31) are parallel or intersect at a common point. As a straightforward extension, when more than three vectors/lines are considered and there is no set of three independent row vectors, all lines are parallel or intersect at a common point.

In our case, the row vectors ($\mathbf{E}_{\mathrm{f}i}, \mathbf{E}_{\mathrm{o}i}, \mathbf{F}_{\mathrm{f}i}, \mathbf{F}_{\mathrm{o}i}, \mathbf{F}_{\mathrm{s}i}, \mathbf{F}_{\beta\mathrm{c}i}, \mathbf{F}_{\phi\mathrm{c}i}$) (defined in (12)–(14) and (16)) represent the lines with respect to frame R:

- $\mathbf{E}_{\mathrm{f}i}/\mathbf{E}_{\mathrm{o}i}$: rotation axle of the fixed/orientable wheel i.
- $\mathbf{F}_{\mathrm{f}i}/\mathbf{F}_{\mathrm{o}i}/\mathbf{F}_{\phi\mathrm{c}i}$: line defined by the wheel plane of the fixed/orientable/castor wheel i.
- $\mathbf{F}_{\mathrm{s}i}$: rotation axle of the roller of the Swedish wheel i.
- $\mathbf{F}_{\beta\mathrm{c}i}$: rotation axle of the castor wheel i.

Thus, the degeneration of matrix $\mathbf{G}$ (i.e., the singularity of the kinematic model) is produced when all the lines associated with the row vectors are parallel or intersect at a common point. Therefore, the singularity characterization for the forward kinematic models is as follows:

For a kinematic model with an m-set of freely assigned wheel velocities, where m is the current WMR mobility degree (*computed from* (20)), *singularity arises when the following straight lines are parallel or intersect at a common point:*

(i) *The rotation axles of all the fixed, orientable and singular omnidirectional wheels.*

(ii) *The lines defined by the wheel planes of all the fixed, orientable and castor wheels with the rotation velocity* (*i.e.,* $\dot{\varphi}_{\mathrm{fi}}, \dot{\varphi}_{\mathrm{oi}}, \dot{\varphi}_{\mathrm{ci}}$) *as freely assigned.*

(iii) *The roller rotation axles of all the Swedish wheels with the rotation velocity* $\dot{\varphi}_{si}$ *as freely assigned.*

(iv) *The rotation axles of all the castor wheels with the steering velocity* $\dot{\beta}_{ci}$ *as freely assigned.*

3.4. Singularity of inverse or mixed kinematic models

The previous singularity characterization only considers assigned wheel velocities. This subsection extends the analysis also to assigned WMR velocities. For this purpose, the degeneration of matrix **G** has to be reconsidered.

In particular, if the WMR velocity v_x is considered assigned, the first column of matrix **G** must not be considered. Therefore, degeneration is produced by:

$$\begin{vmatrix} \lambda_{ya} & d_{\lambda a} \\ \lambda_{yb} & d_{\lambda b} \end{vmatrix} = \lambda_{ya} d_{\lambda b} - \lambda_{yb} d_{\lambda a} = 0, \tag{33}$$

which means that the lines associated with the row vectors of matrix **G** are parallel to the X-axis of frame R or intersect at this axis. Similarly, if the WMR velocity v_y is assigned, all the associated lines are parallel to the Y-axis of frame R or intersect at this axis.

Meanwhile, if both linear WMR velocities are assigned, only the third column of **G** is considered for degeneration:

$$|d_{\lambda a}| = d_{\lambda a} = 0, \tag{34}$$

which means that the lines associated with the row vectors of matrix **G** intersect at the origin of frame R.

Furthermore, if the angular WMR velocity $\boldsymbol{\omega}$ is assigned, the third column of **G** is not considered for degeneration:

$$\begin{vmatrix} \lambda_{xa} & \lambda_{ya} \\ \lambda_{xb} & \lambda_{yb} \end{vmatrix} = \lambda_{xa} \lambda_{yb} - \lambda_{xb} \lambda_{ya} = 0, \tag{35}$$

which means that the lines associated with the row vectors of matrix **G** are parallel. Also, if the angular and one linear WMR velocity ($\boldsymbol{\omega}$ and v_x or v_y) are assigned, the dependency between row vectors is:

$$|\lambda_{ya}| = \lambda_{ya} = 0 \quad \text{or} \quad |\lambda_{xa}| = \lambda_{xa} = 0, \tag{36}$$

which means that the associated lines are parallel to the X- or Y-axis of frame R.

Finally, if the three WMR velocities are assigned (if possible, i.e., $m = 3$) the inverse kinematic model is singularityless. Summarizing, the singularity characterization is extended, in order to include any kinematic model, with the following two points:

(v) *If linear WMR velocities* (*i.e.,* v_x *and/or* v_y) *are freely assigned, the line* (*or lines*) *defined by the corresponding* X- *and/or* Y*-axis of frame R are incorporated with the others.*

(vi) *If the angular WMR velocity $\boldsymbol{\omega}$ is freely assigned, all the indicated lines must be necessarily parallel for singularity.*

If more than m velocities are assigned, $\mathrm{rank}(\mathbf{A}_{\mathrm{na}}) < g$, the approach is valid to characterize the singularity ($\mathbf{A}_{\mathrm{na}}$ degeneration) when computing $\dot{\mathbf{q}}_{\mathrm{na}}$ in (23) with the left pseudo-inverse. The formulated characterization is absolutely generic and useful to establish the singularity of any kinematic model of any WMR with fixed, orientable, castor and/or Swedish wheels.

4. SINGULARITY OF THE FIVE TYPES OF VEHICLES

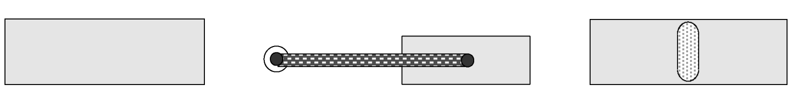
4.1. Introduction

The important contribution in Ref. [4] shows that WMRs can be classified into five generic types. Thus, in order to illustrate the applications of the geometric approach deduced in the previous section, the singularity of those five generic WMRs with three wheels will be characterized. The fact of considering WMRs with three wheels is for stability reasons and implies no loss of generality.

Therefore, their singularity for each set of assigned velocities is depicted in the next subsections for the case when all the corresponding lines intersect at a common point. The case of parallel lines can be considered a limit case where the intersection is at a 'point at infinity' (projective geometry). If the singularity of a WMR for a set of assigned velocities is not depicted and is not equivalent to another shown, it is because it is singularityless. The classical parameters values ($\gamma = 90°, \delta = 0°$) for Swedish and castor wheels will be considered. Figure 6 depicts the wheel representations that will be used in the next subsections; meanwhile the WMR body will be depicted with an equilateral triangle for the type-I WMRs and will be omitted for the other types.

4.2. Type I: omnidirectional WMR

This type of WMR has full mobility ($m = 3$) and is constructed with no fixed or orientable wheels, i.e., with either Swedish or castor wheels. In this work, the same kind of wheel, very common in practice, will be considered. Then, there are two options: three Swedish or three castor wheels. Their singularity, depending upon the set of assigned velocities $\dot{\mathbf{q}}_{\mathrm{a}}$, is depicted in Figs 7 and 8. The singular configurations of Fig. 8e and 8f have already been pointed out in Ref. [17]. Note that (assuming no singular omnidirectional wheels) the inverse kinematic model for this WMR type is singularityless.



(a) (b) (c)

Figure 6. Schematic representation of wheels: (a) fixed and orientable wheels; (b) castor wheel with $\delta = 0°$; (c) Swedish wheel with $\gamma = 90°$.

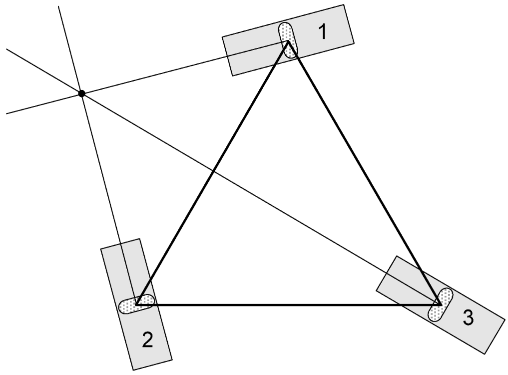
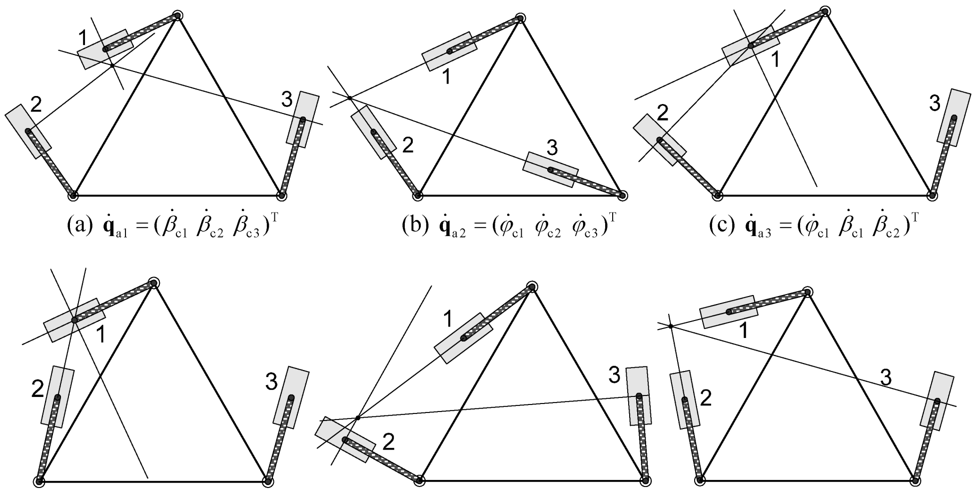


Figure 7. Singular condition for the omnidirectional WMR with Swedish wheels.



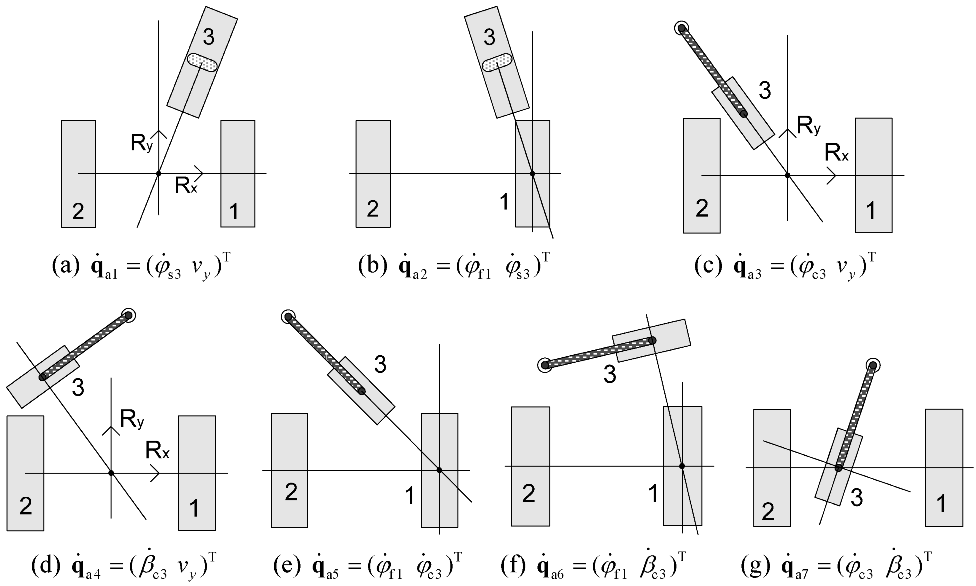
(a) $\dot{\mathbf{q}}_{a1} = (\dot{\beta}_{c1}\ \dot{\beta}_{c2}\ \dot{\beta}_{c3})^{T}$ (b) $\dot{\mathbf{q}}_{a2} = (\dot{\varphi}_{c1}\ \dot{\varphi}_{c2}\ \dot{\varphi}_{c3})^{T}$ (c) $\dot{\mathbf{q}}_{a3} = (\dot{\varphi}_{c1}\ \dot{\beta}_{c1}\ \dot{\beta}_{c2})^{T}$

(d) $\dot{\mathbf{q}}_{a4} = (\dot{\varphi}_{c1}\ \dot{\beta}_{c1}\ \dot{\varphi}_{c2})^{T}$ (e) $\dot{\mathbf{q}}_{a5} = (\dot{\varphi}_{c1}\ \dot{\beta}_{c2}\ \dot{\beta}_{c3})^{T}$ (f) $\dot{\mathbf{q}}_{a6} = (\dot{\varphi}_{c1}\ \dot{\varphi}_{c2}\ \dot{\beta}_{c3})^{T}$

Figure 8. Singular configurations for the omnidirectional WMR with castor wheels.

4.3. Type II: differential-drive WMR

This type of WMR has one independent fixed wheel and other possibly omnidirectional (Swedish or castor) wheels, resulting a mobility degree $m = 2$. In this work, it will be considered to have two fixed wheels with a common rotation axle (the classical differential-drive wheel mechanism) and one additional omnidirectional wheel, either Swedish or castor. The origin of frame R has been located (without loss of generality) at the middle point of the fixed wheels rotation axle, with its X-axis coincident with this rotation axle. The singularity of this WMR is depicted in Fig. 9, except for the cases where v_x is considered assigned, since they result singular for all the configuration/parameter space.



(a) $\dot{\mathbf{q}}_{\mathrm{a1}} = (\dot{\varphi}_{\mathrm{s3}}\ \ v_y)^{\mathrm{T}}$ (b) $\dot{\mathbf{q}}_{\mathrm{a2}} = (\dot{\varphi}_{\mathrm{f1}}\ \ \dot{\varphi}_{\mathrm{s3}})^{\mathrm{T}}$ (c) $\dot{\mathbf{q}}_{\mathrm{a3}} = (\dot{\varphi}_{\mathrm{c3}}\ \ v_y)^{\mathrm{T}}$

(d) $\dot{\mathbf{q}}_{\mathrm{a4}} = (\dot{\beta}_{\mathrm{c3}}\ \ v_y)^{\mathrm{T}}$ (e) $\dot{\mathbf{q}}_{\mathrm{a5}} = (\dot{\varphi}_{\mathrm{f1}}\ \ \dot{\varphi}_{\mathrm{c3}})^{\mathrm{T}}$ (f) $\dot{\mathbf{q}}_{\mathrm{a6}} = (\dot{\varphi}_{\mathrm{f1}}\ \ \dot{\beta}_{\mathrm{c3}})^{\mathrm{T}}$ (g) $\dot{\mathbf{q}}_{\mathrm{a7}} = (\dot{\varphi}_{\mathrm{c3}}\ \ \dot{\beta}_{\mathrm{c3}})^{\mathrm{T}}$

Figure 9. Singularity for the type-II WMRs with an extra Swedish/castor wheel.

4.4. Type III: WMR with one orientable wheel

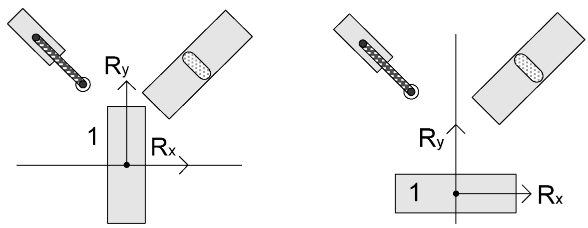
This type of WMR has one independent orientable wheel and other possibly omnidirectional (Swedish or castor) wheels, resulting in a mobility degree $m = 2$. Here, it will be considered to have one orientable wheel and two additional omnidirectional wheels of the same type, either Swedish or castor. For this WMR it will be considered (without loss of generality) that the origin of frame R is located at the orientable wheel.

The singularity for the type-III WMR with two additional Swedish or castor wheels is depicted in Figs 10–12. Note that Figs 11a, 12a and 12b are also valid if $\dot{\varphi}_{\mathrm{o1}}$ is replaced by v_x or v_y.

4.5. Type IV: tricycle and bicycle WMRs

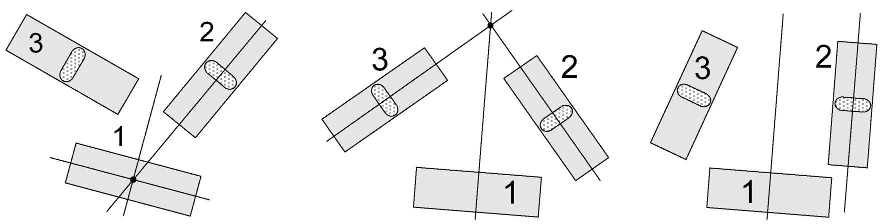
This type of WMR has one independent orientable wheel and another independent fixed wheel, resulting in a mobility $m = 1$. Here, it will be considered to have one orientable wheel, one fixed wheel and another wheel: fixed dependent, Swedish or castor.

4.5.1. Tricycle WMR. This classical WMR consists on one orientable wheel and other two symmetric fixed wheels. For this WMR, the origin of frame R will be located at the middle point of the fixed wheels rotation axle and with its X-axis coincident with this rotation axle. The singularity for this WMR is depicted in Fig. 13 except for the case $\dot{\mathbf{q}}_{\mathrm{a}} = v_x$, since it results singular for all the configuration space.



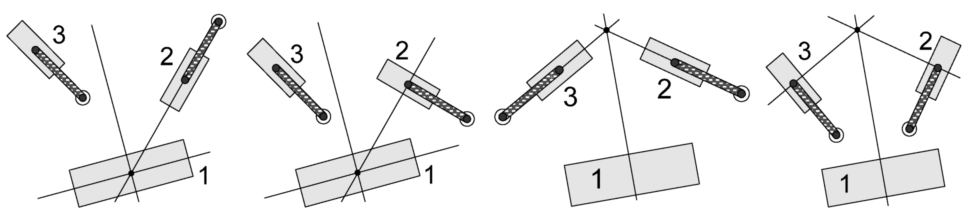
(a) $\dot{\mathbf{q}}_{a1} = (v_x \;\; \omega)^T$ (b) $\dot{\mathbf{q}}_{a2} = (v_y \;\; \omega)^T$

Figure 10. Common singular configurations for the type-III WMR.

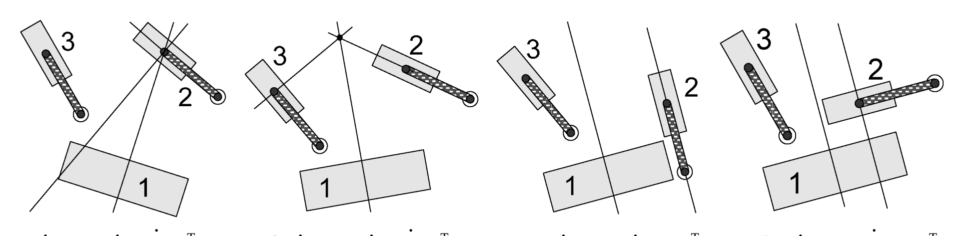


(a) $\dot{\mathbf{q}}_{a3} = (\dot{\varphi}_{o1} \;\; \dot{\varphi}_{s2})^T$ (b) $\dot{\mathbf{q}}_{a4} = (\dot{\varphi}_{s2} \;\; \dot{\varphi}_{s3})^T$ (c) $\dot{\mathbf{q}}_{a5} = (\dot{\varphi}_{s2} \;\; \omega)^T$

Figure 11. Singular configurations for the type-III WMR with Swedish wheels.



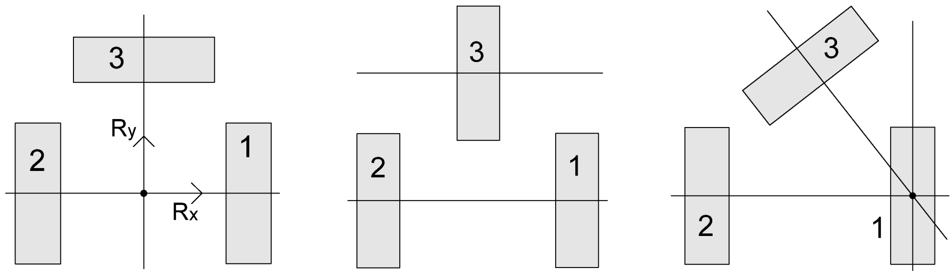
(a) $\dot{\mathbf{q}}_{a6} = (\dot{\varphi}_{o1} \;\; \dot{\varphi}_{c2})^T$ (b) $\dot{\mathbf{q}}_{a7} = (\dot{\varphi}_{o1} \;\; \dot{\beta}_{c2})^T$ (c) $\dot{\mathbf{q}}_{a8} = (\dot{\varphi}_{c2} \;\; \dot{\varphi}_{c3})^T$ (d) $\dot{\mathbf{q}}_{a9} = (\dot{\beta}_{c2} \;\; \dot{\beta}_{c3})^T$



(e) $\dot{\mathbf{q}}_{a10} = (\dot{\varphi}_{c2} \;\; \dot{\beta}_{c2})^T$ (f) $\dot{\mathbf{q}}_{a11} = (\dot{\varphi}_{c2} \;\; \dot{\beta}_{c3})^T$ (g) $\dot{\mathbf{q}}_{a12} = (\dot{\varphi}_{c2} \;\; \omega)^T$ (h) $\dot{\mathbf{q}}_{a13} = (\dot{\beta}_{c2} \;\; \omega)^T$

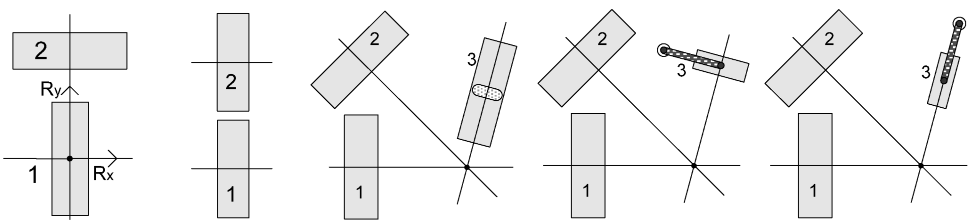
Figure 12. Singular configurations for the type-III WMR with castor wheels.

4.5.2. Bicycle WMR with an additional Swedish or castor wheel. This WMR has one fixed wheel aligned with one orientable wheel, and another Swedish or castor wheel for stability. For this WMR, the origin of frame R has been located at the center of the fixed wheel, with its X-axis coincident with the rotation axle of the



(a) $\dot{\mathbf{q}}_{a1} = v_y$ (b) $\dot{\mathbf{q}}_{a2} = \omega$ (c) $\dot{\mathbf{q}}_{a3} = \dot{\varphi}_{f1}$

Figure 13. Singular configurations for the tricycle WMR.



(a) $\dot{\mathbf{q}}_{a1} = v_y$ (b) $\dot{\mathbf{q}}_{a2} = \omega$ (c) $\dot{\mathbf{q}}_{a3} = \dot{\varphi}_{s3}$ (d) $\dot{\mathbf{q}}_{a4} = \dot{\varphi}_{c3}$ (e) $\dot{\mathbf{q}}_{a5} = \dot{\beta}_{c3}$

Figure 14. Singularity for the bicycle WMR with an extra Swedish or castor wheel.

fixed wheel. The singularity is depicted in Fig. 14 except for the case $\dot{\mathbf{q}}_a = v_x$, since it results singular for all the configuration space.

4.6. Type V: WMR with two orientable wheels

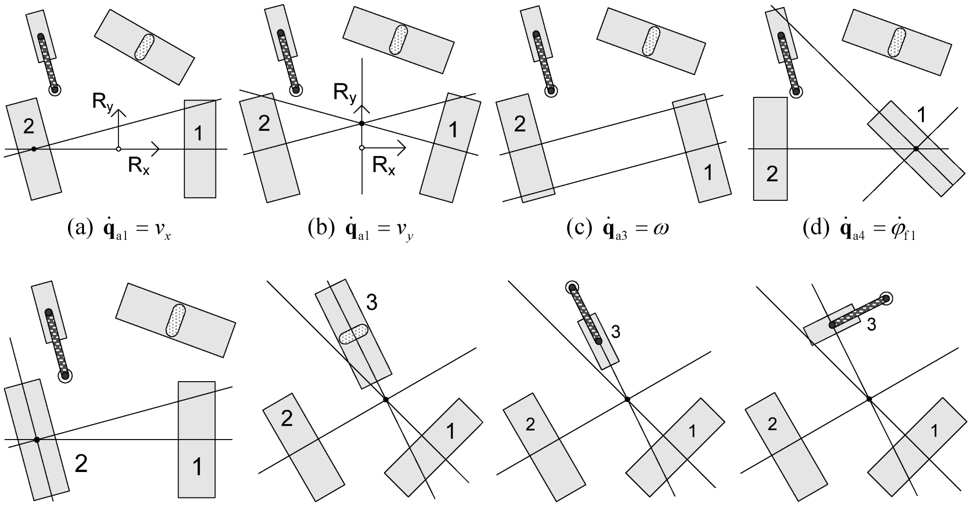
This type of WMR is characterized by two independent orientable wheels and has one degree of mobility ($m = 1$). In this work, it will be considered to have two orientable wheels and one additional Swedish or castor wheel. For this WMR, frame R has been located at the middle point of the joining line of both orientable wheels, with its X-axis coincident with this joining line. The singularity is depicted in Fig. 15.

5. EXTENSION OF THE SINGULARITY CHARACTERIZATION FOR SPECIALIZED WHEELS

Next, several specialized wheels are briefly described and discussed.

5.1. Dual-wheel

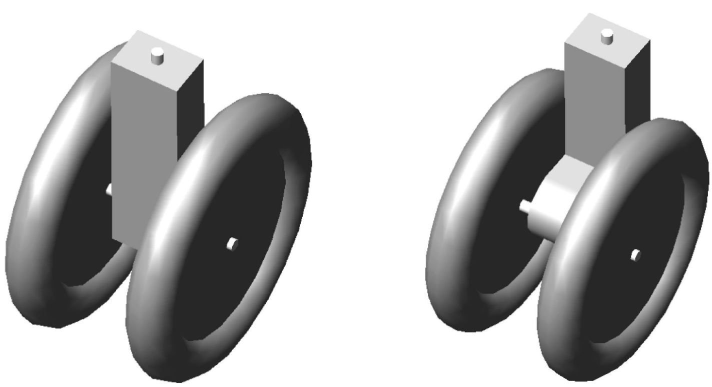
This type of wheel, shown in Fig. 16a, was presented in Ref. [23]. It is shown in Ref. [9] that this wheel is kinematically equivalent to the orientable wheel, although it has three practical advantages:



(a) $\dot{\mathbf{q}}_{a1} = v_x$ (b) $\dot{\mathbf{q}}_{a1} = v_y$ (c) $\dot{\mathbf{q}}_{a3} = \omega$ (d) $\dot{\mathbf{q}}_{a4} = \dot{\varphi}_{f1}$

(e) $\dot{\mathbf{q}}_{a5} = \dot{\varphi}_{f2}$ (f) $\dot{\mathbf{q}}_{a6} = \dot{\varphi}_{s3}$ (g) $\dot{\mathbf{q}}_{a7} = \dot{\varphi}_{c3}$ (h) $\dot{\mathbf{q}}_{a8} = \dot{\beta}_{c3}$

Figure 15. Singularity for the type-V WMR with Swedish or castor wheel.



(a) *Dual-wheel* (b) *Dual-wheel castor*

Figure 16. Dual-wheel and dual-wheel castor.

- The orientable wheel needs to overcome a dry-friction torque when reorienting the plane of the wheels with rolling disabled, meanwhile the dual-wheel must overcome only the rolling friction.
- The load is shared by two wheels, which means that the load-carrying capacity of the dual-wheel is twice as high as that of an orientable wheel using identical wheels and actuators.
- The orientable wheel needs two actuators to provide the steering and rolling motions; however, the two motors serve essentially different tasks, which makes it difficult to select an actuator off-the-shelf which is suitable for both operating characteristics. This problem does not exist for a dual-wheel.

The main disadvantage of this wheel is that it is bigger and with more elements than the orientable wheel.

5.2. Dual-wheel castor

This type of wheel, see Fig. 16b, was presented in Ref. [24]. As before, it is easy to show that this wheel is kinematically equivalent to the castor wheel. It has the same advantages and disadvantages as before.

5.3. Ball-type

This type of wheel consists of a spherical wheel and some kind of driving mechanism. Usually the driving mechanism utilizes a set of rollers: Ref. [25] uses two parallel and horizontal rollers, Ref. [26] uses three rollers forming a horizontal rectangle (see Fig. 17) and Ref. [27] uses several rollers with more complex locations. Another type of driving mechanism is proposed in Ref. [28] where one Swedish wheel rotates over the spherical wheel. In general, the driving mechanism of a ball-type wheel may constrain the movement of the spherical wheel in one or two directions. For example, the mechanism of Ref. [25] constrains the spherical wheel motion in the perpendicular direction of the parallel rollers when one roller (or both) is driven; the mechanism of Ref. [28] constrains the spherical wheel motion in the rotation axle of the Swedish wheel roller when the Swedish wheel is driven and the mechanism of Ref. [26] constrains the spherical wheel motion in the perpendicular directions of the driven rollers (one or two).

5.4. Orthogonal

This type of wheel was introduced in Ref. [29] and consists of two spheres of equal diameter that have been sliced to resemble wide, rounded-tire wheels that are placed with their axles in orthogonal directions. The wheels are able to rotate freely about their axles. A bracket holds the extremities of the wheel axle, which allows it to be driven to roll on its portion of spherical surface, while free-wheeling in the orthogonal direction. This gives the same effect as the universal wheels. With

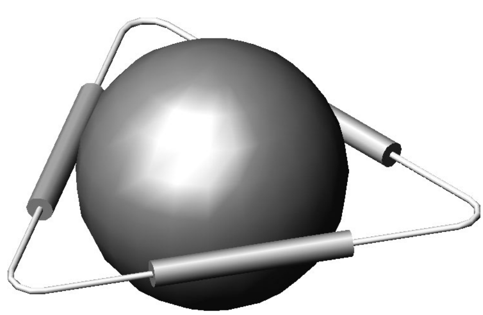
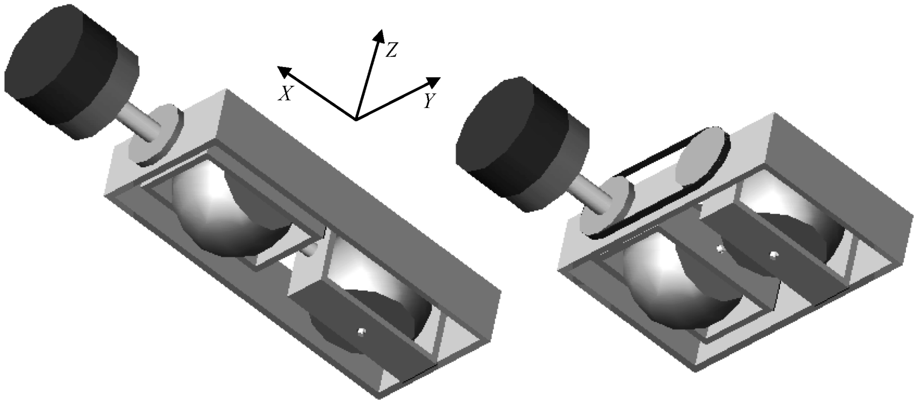


Figure 17. Ball-type wheel with three rollers forming a horizontal rectangle.



(a) (b)

Figure 18. Orthogonal wheel. (a) Longitudinal assembly, (b) lateral assembly.

correctly aligned wheels that are close to spherical and have synchronized motion, ground contact on one spherical surface or the other is assured at all times while leaving sufficient clearance for the bracket. Thus, an orthogonal pair can drive in one direction and remain unconstrained in the orthogonal direction.

They present two types of designs (Fig. 18): the longitudinal assembly and the lateral assembly. The latter case, although its construction is slightly more complicated since it requires a gear- or belt-type transmission, simplifies the robot kinematics and dynamics because all the wheels can be placed such that the point of contact is always a constant distance from the centre of the robot (for the longitudinal assembly this is not possible because kinematics singularity arises).

The main advantages of the orthogonal wheel with respect to the Swedish wheel are the continuous contact with the ground, fewer parts and a smaller wheel well. In order to overcome the discontinuous contact with the ground, some Swedish wheels use two rows of rollers, but add a complication for control and odometry in that the point of contact of the wheel moves between the inner and outer row of rollers. However, if the distance between the rollers is small in comparison with the radius of the transport system, the problem remains manageable.

5.5. Extension of the singularity characterization

The dual-wheel and the dual-wheel castor can be considered as two singular castor wheels and two castor wheels, respectively, with the same rotation axle L_{xi} and steering angle β_i. Therefore, using that equivalence the kinematic modeling and singularity characterization of previous sections are still valid.

On the other hand, although the ball-type and orthogonal wheels have not been modeled, they can be included in the singularity characterization of Section 3 by adding the following point:

(vii) *Each constrained direction of the spherical and orthogonal wheels must be considered as a freely assigned velocity of the m-set and the straight line defined by these constrained directions must be incorporated with the other lines of Sections 3.3 and 3.4.*

6. CONCLUSIONS AND FURTHER WORK

The main contribution of the present research is a novel and generic geometric approach for singularity characterization of WMR kinematics. For that purpose, a unified model has been formulated in order to include all common wheels: fixed, orientable, castor, and Swedish. An efficient recursive kinematics formulation was used to obtain wheel models. It is interesting to remark that, in contrast to Ref. [5], these models make explicit the wheel velocity sliding vector, which is useful for a subsequent stage of slip kinematic analysis and/or modeling.

In order to illustrate the proposed geometric approach, the singular configurations for all the kinematic models of the five types of WMRs classified in Ref. [4] have been characterized.

Furthermore, other specialized wheels were also considered for singularity characterization. In that case, some of them (dual-wheel and dual-wheel castor) were modeled based on the formulation of the common wheels and others (ball-type and orthogonal) were not required to be to modeled.

For further work, we will develop a planner and/or motion controller, for WMRs with orientable or castor wheels, that considers singular configurations for path and/or actuations generation. It would be analogous to the planners and motion controllers developed for robotic manipulators [19–22].

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