Advanced Robotics

ISSN: 0169-1864 (Print) 1568-5535 (Online) Journal homepage: https://www.tandfonline.com/loi/tadr20

Kinematic modeling, mobility analysis and design of wheeled mobile robots

K. H. Low & Y. P. Leow

Advanced Robotics, Vol. 19, No. 1, pp. 73–99 (2005)

Also available online - www.vsppub.com

Full paper

Kinematic modeling, mobility analysis and design of wheeled mobile robots

K. H. LOW [1,*] and Y. P. LEOW [2]

[1] *School of Mechanical and Production Engineering, Nanyang Technological University, Singapore 639798, Republic of Singapore*
[2] *Singapore Institute of Manufacturing Technology, Singapore 638075, Republic of Singapore*

Received 16 December 2003; accepted 10 February 2004

Abstract—Wheeled mobile robots (WMRs) consist of interconnections of many electromechanical systems. Their mechanical subsystem comprises primarily the platform and the wheel units. To formulate the kinematic model of this class of robots, we model the individual subsystems separately. The composite kinematic model of a WMR is then a coupling of the various kinematic submodels. We study WMRs with different wheels, i.e. offset wheels, centered wheels and dual-wheels. The study focuses on system mobility, which is derived using the functional matrix. We also identified the kinematic equivalence between the dual-wheel and the centered wheels, and some advantages of the dual-wheels over the centered wheels and offset wheels. Results suggest that WMRs with mobility less than 3 cannot track a trajectory with a discontinuous heading without incorporating a time delay, during which the wheel orientation should be changed. Moreover, the steering angles of WMRs equipped with steered wheels require proper coordination to avoid jamming of the drive subsystem. For design purposes, we aim at a kinetostatically robust WMR. The concept of kinetostatic isotropy is applied to find the location of the wheels with respect to the platform and their type in order to achieve isotropy. It is shown that WMRs with three conventional wheels can be made isotropic if the offset either vanishes or equals the radius of the wheel, and if the three wheels are mounted at the vertices of an equilateral triangle.

Keywords: Mobile robot; dual-wheel; kinetostatic isotropy; mobility; functional matrix.

NOMENCLATURE

$\mathcal{B}$	body coordinate frame
$\mathcal{F}$	fixed reference frame
$\mathcal{W}_i$	wheel coordinate frame
b	one-half the wheel separation of the dual-wheel drive

*To whom correspondence should be addressed. E-mail: mkhlow@ntu.edu.se

d	offset distance of the offset wheel
q	number of generalized coordinates
r	radius of the wheel
w	number of wheels
$\dot{x}$	velocity in the x-axis
$\dot{y}$	velocity in the y-axis
C	operation point of the platform
C_{f}	centroid of a triangular set of points
D_{ij}	intersection of two planes of any two wheels with the plane XY
D_{P}	intersection of planes of all three wheels with the plane XY
G_i	contact point between wheel and ground
H_i	center of the ith wheel
H_{i1}	center of wheel 1 of the ith dual-wheel
H_{i2}	center of wheel 2 of the ith dual-wheel
L	characteristic length
U	location of the instant center
$\dot{X}$	velocity in the X-axis
$\dot{Y}$	velocity in the Y-axis
$\mathbf{1}$	2×2 identity matrix
$\mathbf{c}$	position vector of point C
$\dot{\mathbf{c}}$	velocity of point C
$\mathbf{d}_i$	vector $\overrightarrow{CP_i}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors of *body* coordinate frame $\mathcal{B}$
$\mathbf{i}_{\mathcal{F}}, \mathbf{j}_{\mathcal{F}}$	unit vectors of *fixed* reference frame $\mathcal{F}$
$\mathbf{i}_i, \mathbf{j}_i$	unit vectors of *wheel* coordinate frame $\mathcal{W}_i$ of wheel unit i
$\dot{\mathbf{h}}_i$	velocity of point H_i
$\dot{\mathbf{h}}_{i1}$	velocity of point H_{i1}
$\dot{\mathbf{h}}_{i2}$	velocity of point H_{i2}
$\dot{\mathbf{p}}_i$	velocity of point P_i
$\dot{\mathbf{q}}$	$[\mathbf{t}^{\mathrm{T}}\ \dot{\boldsymbol{\theta}}^{\mathrm{T}}]^{\mathrm{T}}$
$\mathbf{t}$	planar twist of the platform
$\mathbf{t}_L$	planar twist of the platform associated with the characteristic length
$\mathbf{u}$	position vector of point U
$\mathbf{u}_{\mathrm{c}}$	vector $\overrightarrow{CU}$
$\mathbf{u}_i$	vector $\overrightarrow{P_iU}$

$\mathbf{A}$	arbitrary matrix
$\mathbf{E}$	2×2 skew symmetric matrix rotating vectors through 90° counter-clockwise
$\mathbf{F}$	functional matrix
$\mathbf{J}$	$\mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_2$
$\mathbf{J}_1$	$2w \times 3$ Jacobian matrix
$\mathbf{J}_{L1}$	Jacobian matrix $\mathbf{J}_1$ associated with characteristic length
$\mathbf{J}_2$	$2w \times 2w$ Jacobian matrix
$\mathbf{R}_{bwi}$	rotation matrix of ith wheel from frames w to b with angle β_i
β_i	angle of the ith intermediate body
$\dot{\beta}_i$	velocity of the ith intermediate body
$\dot{\boldsymbol{\theta}}$	actuated-joint rate vector
θ_i	angle of the ith wheel
θ	joint variable
$\dot{\theta}_i$	velocity of the ith wheel
$\dot{\theta}_{i1}$	velocity of wheel 1 of the ith dual-wheel
$\dot{\theta}_{i2}$	velocity of wheel 2 of the ith dual-wheel
κ	condition number
σ_1	largest singular value
σ_{s}	smallest singular value
ϕ_i	steering angle of the ith wheel
$\dot{\phi}_i$	steering velocity of the ith wheel
ψ	platform orientation
$\dot{\psi}$	platform angular velocity
$\dot{\psi}_{\mathrm{c}}$	platform angular velocity with instant center at the operation point
$\dot{\psi}_{\mathrm{u}}$	platform angular velocity with instant center at an arbitrary point
$\dim(\mathcal{N}(\mathbf{F}))$	dimension of nullspace of $\mathbf{F}$
$\mathcal{N}(\mathbf{F})$	nullspace of $\mathbf{F}$
$\mathrm{rank}(\mathbf{F})$	rank of matrix $\mathbf{F}$
$\nu(\mathbf{F})$	nullity of $\mathbf{F}$
$\det(\mathbf{J})$	determinant of $\mathbf{J}$

1. INTRODUCTION

The most common forms of locomotion for wheeled mobile robots (WMRs) are legged, treaded and wheeled system [1]. Wheeled locomotion is by far the most popular for several practical reasons. When compared to tracks and legs, wheels are simple in design, easy to fabricate, energy-efficient and offer a high load-carrying capability. The principal disadvantage of wheels, however, is their tendency to be bogged down in soft, uneven terrain. Nevertheless, wheeled locomotion has high potential for many applications on the manufacturing floor, which is typically smooth, flat and hard. Under the umbrella of wheeled locomotion, many types of wheels are available. They are defined as conventional, omni-directional and ball-wheels [1, 2]. Other types of wheel mechanisms proposed for WMR applications are also available, such as the dual-wheel [3], dual-wheel caster drive [4], orthogonal-wheel [5] and double-wheel drive [6]. A comprehensive overview of the development and classification of different kinds of WMR can be found elsewhere [7–10].

Conventional wheels are the most widely used among WMRs with wheeled locomotion. These wheels are simple to construct, require little maintenance, provide smooth motion, offer high load-carrying capacity and are cheap. In order to better understand the characteristics of WMRs with conventional wheels, we first formulate the kinematic models of this class of robots. The mathematical model of a WMR can be decomposed into many subsystems [1, 11], which can then be modeled and designed independently in parallel, thereby shortening the overall design cycle and time-to-market. In its simplest form, a WMR consists of a platform that is supported by w wheel units. These subsystems may be of the same type or a combination of various types. Thus, a WMR can be decomposed into the platform and the wheel units, as shown in Fig. 1.

The use of the wheel Jacobian matrix to relate the motion of each wheel to the motion of the robot and, subsequently, combining the individual wheel equations to obtain the composite equation of motion for robots was introduced in Refs [1, 2]. Along the same lines, a modular approach for modeling kinematically constrained multibody systems using Lagrange's equations was developed in Ref. [12]. In this approach, a procedure to derive the holonomy matrix of a multi-body system consisting of an arbitrary number of coupled subsystems is described in nine steps. This derivation not only verifies whether a coupled system is holonomic, non-holonomic or quasi-holonomic, but also derives the governing equations of the

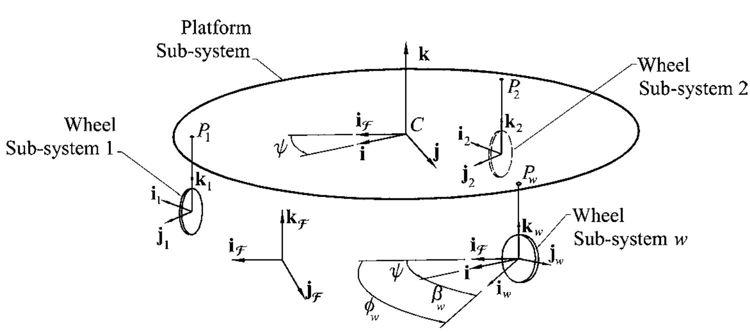

Figure 1. Subsystems of a WMR.

system. The second approach is, however, more systematic and simpler, not least because of the conciseness of the notation used.

In this paper, the kinematic models of three WMRs are derived. Each WMR model consists of a platform and an arbitrary number of wheel units. The three types of wheels formulated are offset wheels, centered wheels and dual-wheels. The dual-wheels are shown to be kinematically equivalent to centered wheels. However, WMRs with centered wheels need to overcome a dry-friction torque when reorienting the plane of the wheels with rolling disabled; WMRs with dual-wheels must overcome only rolling friction. Moreover, unlike WMRs with offset wheels and WMRs with centred wheels, the load is also shared by the two wheels, which means that the load-carrying capacity of dual-wheel drives is twice as high as that of offset wheels using identical wheels and actuators. The various subsystem models, derived separately, are subsequently assembled to form the overall WMR model.

2. GENERAL CONSIDERATIONS

The shape of WMRs can have a strong impact on the robustness of the system. We term a system robust when its performance is least sensitive to variations in the design environment parameters over which the designer has no control [13]. A non-cylindrical robot runs a greater risk of being trapped by an unfavorable arrangement of obstacles or failing to find its way through a narrow or cluttered space [14]. For this reason and for comparison, we will focus on circular platforms with three different types of wheels: offset, centred and dual-wheels. These wheels are selected as they are commonly available in industry. However, the same approach can be extended to platforms of other shapes and to other types of wheel.

To simplify our analysis, we assume that the WMR is a mechanical system composed of rigid bodies connected by ideal joints and operating on a regular horizontal surface, free of obstacles. Moreover, the translational friction at the point of contact between the wheel (modelled as a thin disk) and ground is assumed large enough to prevent slippage.

3. SUBSYSTEM MODELS

3.1. Platform kinematics

The platform is modeled as a round slab, which moves on a horizontal plane. Its pose can be described by an orientation and two position variables, and can be represented as $[\psi \quad \mathbf{c}^T]^T$, where ψ and $\mathbf{c}$ are depicted in Fig. 2.

In Fig. 2, ψ is the angle of the platform orientation with respect to the fixed reference frame $\mathcal{F}$ and $\mathbf{c}$ is the two-dimensional (2D) position vector of the platform center, which is taken as the operation point C [15]. Therefore, the necessary number q of generalized coordinates required to describe the configuration of the platform unambiguously is 3. Moreover, the fixed reference frame $\mathcal{F}$, which in dynamics is usually inertial, is selected so that the WMR moves on the XY plane, while $\mathbf{i}_{\mathcal{F}}$, $\mathbf{j}_{\mathcal{F}}$ and $\mathbf{k}$ are unit vectors, with $\mathbf{k}$ pointing upward in the vertical direction. Next, we define a body frame $\mathcal{B}$, whose orientation is fixed with respect to the platform, its associated unit vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, being shown in Figs 1 and 2. Furthermore, the body frame $\mathcal{B}$ rotates with respect to the reference frame $\mathcal{F}$ about a vertical axis with an angular velocity $\dot{\psi}$. The velocity of point C is, in turn:

$$\dot{\mathbf{c}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \dot{X}\mathbf{i}_{\mathcal{F}} + \dot{Y}\mathbf{j}_{\mathcal{F}}. \tag{1}$$

The absolute velocity of an arbitrary point P_i on the platform is then [16]:

$$\dot{\mathbf{p}}_i = \dot{\mathbf{c}}_w = \dot{\mathbf{c}} + \dot{\psi}\mathbf{E}\mathbf{d}_i \quad \text{as} \quad \dot{\boldsymbol{\psi}} \times \mathbf{d}_i = \dot{\psi}\mathbf{E}\mathbf{d}_i, \quad i = 1 \text{ to } w, \tag{2}$$

where:

$$\mathbf{E} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

is a 2×2 skew-symmetric matrix [16] rotating vectors in the plane through an angle of 90° counter-clockwise, while $\mathbf{d}_i$ is the 2D vector directed from C to P_i. The planar twist of the platform, denoted by $\mathbf{t}$, is defined as:

$$\mathbf{t} = \begin{bmatrix} \dot{\psi} & \dot{\mathbf{c}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{4}$$

This vector has three components, one for rotation, $\dot{\psi}$, and two for translation, the latter grouped in the 2D vector $\dot{\mathbf{c}}$. Equation (2) can be rearranged as:

$$[\dot{\mathbf{p}}_i] = [\mathbf{E}\mathbf{d}_i \quad \mathbf{1}] \begin{bmatrix} \dot{\psi} \\ \dot{\mathbf{c}} \end{bmatrix}, \tag{5}$$

where $\mathbf{1}$ is the 2×2 identity matrix, thereby completing the kinematic description of the platform subsystem.

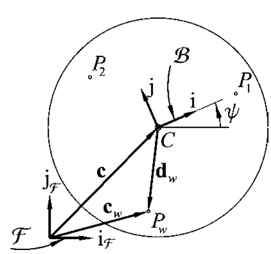

Figure 2. Notation for the platform.

3.2. Wheel kinematics

WMRs with offset wheels, centered wheels or dual-wheels all use the same type of conventional wheels [1–3]. The only difference between them is the intermediate body connecting the wheel to the platform. We start with the modeling of the wheel, which is common for all three types of wheels, followed by the intermediate body. The wheel is modeled as a thin disk of radius r rolling without sliding or skidding on a plane, as shown in Fig. 3. A wheel coordinate frame $\mathcal{W}_i$, whose unit vectors are $\mathbf{i}_i$, $\mathbf{j}_i$ and $\mathbf{k}_i$, is attached to the center of the ith wheel H_i, such that $\mathbf{i}_i$ is parallel to the axis of rolling, while $\mathbf{j}_i$ is normal to $\mathbf{i}_i$ in the disk plane. Finally, $\mathbf{k}_i$ is normal to both $\mathbf{i}_i$ in the disk plane as well as $\mathbf{j}_i$, thus forming an orthonormal triad. We shall restrict our analysis to the case where the plane of the wheel remains vertical; therefore, we can use a single unit vector $\mathbf{k}$ for all unit vectors pointing upward in the vertical direction. As illustrated in Fig. 3, θ_i and ϕ_i are the angles of rolling and steering, respectively. The non-slipping condition leads to the kinematic constraint:

$$\dot{\mathbf{h}}_i = -r\dot{\theta}_i\mathbf{j}_i, \tag{6}$$

where $\dot{\mathbf{h}}_i$ is the 2D velocity of the center of the wheel. Next, a model of the wheel mounted on the intermediate body, as shown in Fig. 4, is derived. In this architecture, the axis of rolling $\mathbf{i}_i$ is normal to the steering axis $\mathbf{k}$. The rates of rolling and steering of the ith wheel with respect to the ground are $\dot{\theta}_i$ and $\dot{\phi}_i$, respectively. The

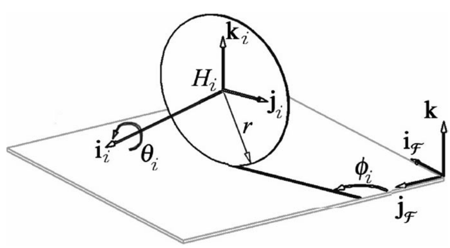

Figure 3. Notation for a wheel modeled as a rolling disk.

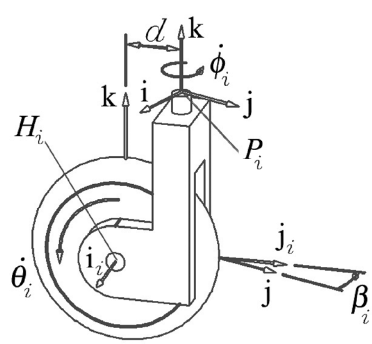

Figure 4. Architecture of an offset wheel.

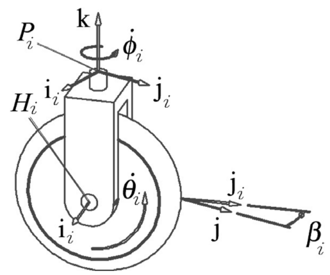

Figure 5. Architecture of a centered wheel.

angular velocity of the intermediate body with respect to the platform is $\dot{\beta}_i$ such that (see also Fig. 1):

$$\dot{\phi}_i = \dot{\psi} + \dot{\beta}_i, \tag{7}$$

which is integrable, i.e.:

$$\phi_i = \psi + \beta_i \tag{8}$$

in which ψ and β_i are both zero if $\phi_i = 0$. Furthermore, the center of the wheel is offset by a distance d from the axis of steering, while the velocity of P_i is:

$$\dot{\mathbf{p}}_i = -r\dot{\theta}_i\mathbf{j}_i - d\dot{\phi}_i\mathbf{i}_i. \tag{9}$$

When expressed in the body frame, (9) becomes:

$$[\dot{\mathbf{p}}_i]_{\mathcal{B}} = [\mathbf{R}_{bwi}][\dot{\mathbf{p}}_i] = [\mathbf{R}_{bwi}]\begin{bmatrix} -d\dot{\phi}_i \\ -r\dot{\theta}_i \end{bmatrix}, \tag{10}$$

where:

$$[\mathbf{R}_{bwi}] \equiv \begin{bmatrix} \cos\beta_i & -\sin\beta_i \\ \sin\beta_i & \cos\beta_i \end{bmatrix} \tag{11}$$

is the rotation matrix from the wheel frame to the body-attached frame. It is noteworthy that if the offset d is zero, (10) becomes:

$$[\dot{\mathbf{p}}_i]_{\mathcal{B}} = [\mathbf{R}_{bwi}][\dot{\mathbf{p}}_i] = [\mathbf{R}_{bwi}]\begin{bmatrix} 0 \\ -r\dot{\theta}_i \end{bmatrix}, \tag{12}$$

which is the velocity of P_i expressed in the body frame for the case with a centered wheel, as shown in Fig. 5. For both the offset wheel and the centered wheel, two actuators are need to provide the steering and rolling motions. Steering the wheel requires position control, while rolling motion requires velocity control. It is both economically and logistically sound that both actuators be identical; however, the two motors serve essentially different tasks. This makes it very difficult to select an actuator off-the-shelf which is suitable for both operating characteristics [15]. A wheel architecture known as 'dual-wheel', shown in Fig. 6, was proposed in

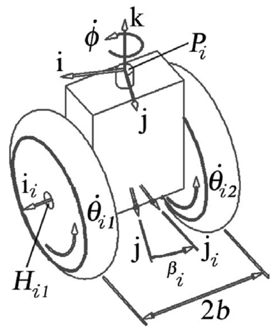

Figure 6. Architecture of a dual-wheel.

Ref. [3] to address this problem. In this architecture, two identical wheels of radius r are mounted symmetrically about the center of the intermediate body at a distance b. The two wheels, denoted as $i1$ and $i2$, are independently driven. The non-slipping condition leads to three kinematic constraints:

$$\dot{\mathbf{h}}_{i1} = -r\dot{\theta}_{i1}\mathbf{j}_i, \tag{13}$$

$$\dot{\mathbf{h}}_{i2} = -r\dot{\theta}_{i2}\mathbf{j}_i, \tag{14}$$

and:

$$\dot{\beta}_i = -\frac{r}{2b}\big(\dot{\theta}_{i1} - \dot{\theta}_{i2}\big), \tag{15}$$

which is integrable, i.e.:

$$\beta_i = -\frac{r}{2b}(\theta_{i1} - \theta_{i2}), \tag{16}$$

where θ_{i1} and θ_{i2} are both zero if $\beta_i = 0$. As illustrated in [3], the dual-wheel is kinematically equivalent to the centered wheel. This is possible with the introduction of the average velocities $\dot{\mathbf{h}}_i$ and $\dot{\theta}_i$, which are defined as:

$$\dot{\mathbf{h}}_i \equiv \frac{1}{2}\big(\dot{\mathbf{h}}_{i1} + \dot{\mathbf{h}}_{i2}\big) = -\frac{r}{2}\big(\dot{\theta}_{i1} + \dot{\theta}_{i2}\big)\mathbf{j}_i = -r\dot{\theta}_i\mathbf{j}_i, \tag{17}$$

and:

$$\dot{\theta}_i \equiv \frac{1}{2}\big(\dot{\theta}_{i1} + \dot{\theta}_{i2}\big), \tag{18}$$

Equation (18) is again integrable, i.e.:

$$\theta_i = \frac{1}{2}(\theta_{i1} + \theta_{i2}) \tag{19}$$

in which θ_{i1} and θ_{i2} are both zero if $\theta_i = 0$. Substituting (18) into (17) leads to (6), thereby showing that the velocity of P_i is the same as that for the centered wheels, as described in (12). This implies that the dual-wheel is kinematically equivalent to the centered wheel. However, both the centered wheel and offset wheel require two actuators to function as a powered wheel module. One of the actuators is for steering, while the other is for driving. The dual-wheel with its two wheels, however, does not have a steering actuator. It steers whenever there is a difference in the velocities of the two wheels. For this reason, we justify its different classification.

In Section 4, we assemble the wheels, the intermediate body and the platform to form the complete model of a WMR. We focus on WMRs with one single type of wheels, i.e. all offset wheels, all centered wheels or all dual-wheels. However, the same approach can be extended to hybrid wheel arrays.

4. WMR WITH OFFSET WHEELS

4.1. Kinematics of a WMR with offset wheels

We assume that the WMR has w active offset wheels mounted at various arbitrary points P_i, for $i = 1, \ldots, w$. Two actuators are needed to steer and power each wheel unit, which then requires a total of $2w$ actuators to steer and power a WMR with this type of wheel.

We begin the modeling of the WMR with offset wheels by equating the left-hand sides of (5) and (10), which leads to:

$$[\mathbf{Ed}_i \quad \mathbf{1}]\begin{bmatrix} \dot{\psi} \\ \dot{\mathbf{c}} \end{bmatrix} = [\dot{\mathbf{p}}_i]_{\mathcal{B}} = [\mathbf{R}_{bwi}][\dot{\mathbf{p}}_i] = [\mathbf{R}_{bwi}]\begin{bmatrix} -d\dot{\phi}_i \\ -r\dot{\theta}_i \end{bmatrix}. \tag{20}$$

For $i = 1, \ldots, w$, (20) can be written in compact form as:

$$\mathbf{J}_1\mathbf{t} = \mathbf{J}_2\dot{\boldsymbol{\theta}}. \tag{21}$$

The two Jacobian matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ of the WMR, the planar twist $\mathbf{t}$ of the platform, and the actuated-joint-rate vector $\dot{\boldsymbol{\theta}}$ are:

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{Ed}_1 & \mathbf{1} \\ \mathbf{Ed}_2 & \mathbf{1} \\ \vdots & \vdots \\ \mathbf{Ed}_w & \mathbf{1} \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} \dot{\psi} \\ \dot{\mathbf{c}} \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\phi}_1 & \dot{\theta}_1 & \dot{\phi}_2 & \dot{\theta}_2 & \ldots & \dot{\phi}_w & \dot{\theta}_w \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{J}_2 = \begin{bmatrix} \mathbf{R}_{bw1}\tilde{\mathbf{p}} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{bw2}\tilde{\mathbf{p}} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{R}_{bww}\tilde{\mathbf{p}} \end{bmatrix}, \quad \tilde{\mathbf{p}} = \begin{bmatrix} -d & 0 \\ 0 & -r \end{bmatrix},$$

where $\mathbf{J}_1$ is a $2w \times 3$ matrix, $\mathbf{J}_2$ is a $2w \times 2w$ matrix, while $\mathbf{t}$ and $\dot{\boldsymbol{\theta}}$ are 3D and $2w$-D vectors, respectively.

4.2. Mobility of a WMR with offset wheels

An unconstrained platform moving on a horizontal plane follows a planar rigid motion. Therefore, the platform has full mobility, i.e. a mobility of 3. If further constraints act on the platform, the mobility is reduced accordingly. With regard to our work on WMR, mobility refers to the number of independent generalized velocities that can be freely assigned without violating the kinematic constraints [12]. A minimum of three actuators will be required to generate motion for a WMR with three offset wheels. However, simply applying power to all three wheels does not give the platform full mobility. To provide the platform with full mobility, it is necessary to motorize the orientation of at least one wheel [11]. Moreover, a minimum of three wheels is sufficient to support and ensure stability of the platform. A system with more than three wheels is kinematically redundant, but not uncommon. However, to allow for any slight irregularities of the floor surface, redundant wheels have to be mounted on a suspension to ensure that all the wheels are in contact with the floor at all times. In our study, we consider a more general case whereby the WMR under consideration has a degree of freedom greater than its mobility. We examine below the mobility of a WMR with w offset wheels. Equation (21) can be rearranged as:

$$[\mathbf{J}_1 \quad -\mathbf{J}_2]\begin{bmatrix}\mathbf{t}\\ \dot{\boldsymbol{\theta}}\end{bmatrix} = \mathbf{0}, \tag{22}$$

where $\mathbf{0}$ denotes the $(3 + 2w)$-D zero vector. These relations are expressed in compact form as:

$$\mathbf{F}\dot{\mathbf{q}} = \mathbf{0}. \tag{23}$$

Note that matrix $\mathbf{F}$ is known as the $2w \times (3 + 2w)$ functional matrix [17], which is:

$$\mathbf{F} = \begin{bmatrix} \mathbf{Ed}_1 & \mathbf{1} & -\mathbf{R}_{bw1}\tilde{\mathbf{p}} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{Ed}_2 & \mathbf{1} & \mathbf{0} & -\mathbf{R}_{bw2}\tilde{\mathbf{p}} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{Ed}_w & \mathbf{1} & \mathbf{0} & \mathbf{0} & \dots & -\mathbf{R}_{bww}\tilde{\mathbf{p}} \end{bmatrix}, \tag{24}$$

while $\dot{\mathbf{q}}$ is defined as:

$$\dot{\mathbf{q}} \equiv \begin{bmatrix} \dot{\psi} & \dot{x} & \dot{y} & \dot{\phi}_1 & \dot{\theta}_1 & \dot{\phi}_2 & \dot{\theta}_2 & \dots & \dot{\phi}_w & \dot{\theta}_w \end{bmatrix}^{\mathrm{T}}. \tag{25}$$

Equation (23) implies that the vector $\dot{\mathbf{q}}$ lies in the nullspace of the matrix $\mathbf{F}$, i.e.:

$$\dot{\mathbf{q}} \in \mathcal{N}(\mathbf{F}). \tag{26}$$

The nullspace of matrix $\mathbf{F}$ reveals the number of constraints and the number of independent variables. The number of independent variables is the nullity ν of the matrix $\mathbf{F}$, which is defined as [18]:

$$\nu = \dim(\mathcal{N}(\mathbf{F})). \tag{27}$$

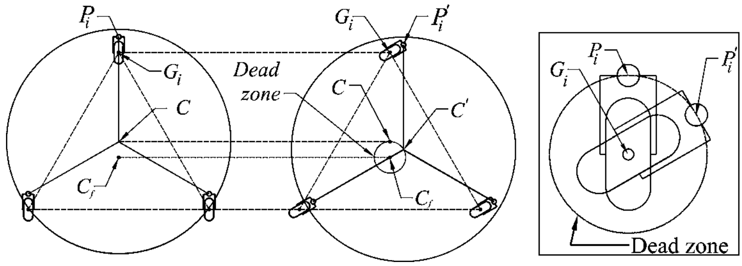

Figure 7. Dead zone of a WMR with offset wheels. Insert: Enlarged view of pivoting point of offset wheel.

The physical meaning of nullity ν in our case is the number of scalar velocities that we can freely specify, i.e. the mobility of the mechanical system.

More specifically, for a WMR with three active offset wheels, vector $\dot{\mathbf{q}}$ is 9D and the rank of matrix $\mathbf{F}$ is 6; thus $\nu(\mathbf{F}) = 3$. In other words, a WMR with three offset wheels has nine generalized coordinates, but a mobility of only 3. This is characteristic of non-holonomic systems [12, 19]. Therefore, the WMR at hand has full mobility and the WMR is omni-directional. However, if the driving actuators are locked, i.e. $\dot{\theta}_i = 0$ for $i = 1, 2, 3$, under a manoeuvre intended to reorient the wheels, then, the summation of (20) for $i = 1, 2, 3$ becomes:

$$\dot{\mathbf{c}} = \frac{d}{3}\sum_{i=1}^{3}\left([\mathbf{R}_{bwi}]\begin{bmatrix}-\dot{\phi}_i\\ 0\end{bmatrix}\right). \tag{28}$$

Upon multiplying both sides of (20) with $\dot{\mathbf{c}}^{\mathrm{T}}\mathbf{E}$ to eliminate $\dot{\theta}_i$, we obtain:

$$\dot{\psi} = \frac{d}{3}\sum_{i=1}^{3}\frac{\left([\mathbf{R}_{bwi}]\begin{bmatrix}-\dot{\phi}_i\\ 0\end{bmatrix}\right)}{\mathbf{d}_i}. \tag{29}$$

Equations (28) and (29) reveal a drift whenever the WMR is steered without driving, i.e. the platform will move simply by steering the three offset wheels. This motion, called drift, is undesirable. From Fig. 7, it is clear that when all the wheels are steered one full turn without driving, point C will trace a circle whose radius equals the distance between point C and point C_{f}, the centroid of the triangle whose vertices are the points of contact between the wheels and the ground; C_{f} remains fixed during steering. This distance d is the offset of the wheel and the circle traced by point C about point C_{f} is the dead zone of the WMR. It is noteworthy that the point of contact between the wheels and the ground, denoted as G_i, like C_{f}, remain stationary during steering when there is no driving. Furthermore, points C' and P' are the new locations of points C and P after steering, respectively. The dead zone makes it difficult for the WMR to execute small translations, especially if these are smaller than the radius of the dead zone. Therefore, accurate small displacements are difficult to achieve.

4.3. Kinetostatic isotropy of a WMR with three offset wheels

A fundamental problem in robotics lies in determining the twist when the joint rates are given or *vice versa*. The former is known as the forward kinematics problem, the latter as the inverse kinematics problem. However, the accuracy of the forward and inverse kinematics of WMRs depends on the condition number of the Jacobian matrices, whose inverses are required in the transformations from joint rates to platform twist and *vice versa*. The condition number of a matrix is a measure of the relative round-off error amplification of the computed results with respect to the relative round-off error of the input data, upon solving a linear system of equations associated with the matrix [20]. Hence, the accuracy of the kinematic results depends on the condition number of the matrices whose inverses are needed. Matrices with small condition numbers produce accurate results, whereas matrices with large condition numbers produce results with corresponding large round-off errors. In fact, a condition number equal to unity, which does not introduce any round-off error amplification in the solution, is the best that can be achieved. Thus, robustness of the kinematic control is ensured when inverting a matrix with a condition number of unity. Matrices with a unity condition number are called isotropic. The condition number, denoted by κ, of an $m \times n$ matrix $\mathbf{A}$ can be defined as:

$$\kappa \equiv \frac{\sigma_1}{\sigma_s}, \tag{30}$$

where σ_1 and σ_s are the largest and the smallest of the singular values of $\mathbf{A}$. Thus, rank-deficient matrices with $\sigma_s = 0$ have an infinitely large condition number, while isotropic matrices have all their singular values identical and different from zero. Hence, κ is bounded from below and unbounded from above, i.e.:

$$1 \leqslant \kappa \leqslant \infty. \tag{31}$$

From the above discussion, it follows that an $m \times n$ isotropic matrix $\mathbf{A}$ with $m < n$, when multiplied by its transpose, produces a matrix that is a multiple of the $m \times m$ identity matrix. Likewise, if $m > n$, then the product of $\mathbf{A}^T\mathbf{A}$ is a multiple of the $n \times n$ identity matrix.

4.3.1. Isotropy of $\mathbf{J}_1$. The immediate task at hand now is to verify whether the two Jacobian matrices can be rendered isotropic by design. In attempting to render $\mathbf{J}_1$ isotropic, we face a fundamental problem of incompatible units: the first column has units of length, while the last two are dimensionless. In order to overcome this problem, we introduce the characteristic length L [21]. Thus, we redefine the matrix $\mathbf{J}_1$ associated with L as:

$$\mathbf{J}_{L1} \equiv \begin{bmatrix} \frac{1}{L}\mathbf{E}\mathbf{d}_1 & \mathbf{1} \\ \frac{1}{L}\mathbf{E}\mathbf{d}_2 & \mathbf{1} \\ \vdots & \vdots \\ \frac{1}{L}\mathbf{E}\mathbf{d}_w & \mathbf{1} \end{bmatrix}. \tag{32}$$

Likewise we redefine the vector $\mathbf{t}$ as:

$$\mathbf{t}_L \equiv \begin{bmatrix} \dot{\psi}L & \dot{\mathbf{c}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{33}$$

in which L becomes an additional design variable, to be determined so as to either render the dimensionally homogeneous matrix $\mathbf{J}_{L1}$ isotropic or to minimize its condition number. If $\mathbf{J}_{L1}$ is isotropic, then the product $\mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_{L1}$ is proportional to the 3×3 identity matrix. This product, for a WMR with three active offset wheels, is:

$$\mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_{L1} = \begin{bmatrix} \frac{1}{L^2}\sum_{i=1}^{3}\mathbf{d}_i^{\mathrm{T}}\mathbf{d}_i & \frac{1}{L}\left(\sum_{i=1}^{3}\mathbf{d}_i^{\mathrm{T}}\right)\mathbf{E}^{\mathrm{T}} \\ \frac{1}{L}\mathbf{E}\left(\sum_{i=1}^{3}\mathbf{d}_i\right) & (3)\mathbf{1} \end{bmatrix}. \tag{34}$$

In order to render $\mathbf{J}_{L1}$ isotropic, it is apparent that:

$$\frac{1}{L}\left(\sum_{i=1}^{3}\mathbf{d}_i^{\mathrm{T}}\right)\mathbf{E}^{\mathrm{T}} = \mathbf{0} \quad \Rightarrow \quad \frac{1}{L}\mathbf{E}\left(\sum_{i=1}^{3}\mathbf{d}_i\right) = \mathbf{0}, \tag{35}$$

$$\frac{1}{L^2}\sum_{i=1}^{3}\mathbf{d}_i^{\mathrm{T}}\mathbf{d}_i = 3. \tag{36}$$

For (35) and (36) to be true, the three offset wheels must be mounted at the vertices of an equilateral triangle, such that:

$$\sum_{i=1}^{3}\mathbf{d}_i = \mathbf{0}, \tag{37}$$

and:

$$\mathbf{d}_i^{\mathrm{T}}\mathbf{d}_i = L^2, \quad i = 1, 2, 3, \tag{38}$$

which means that the centroid of the equilateral triangle must lie at point C, the operating point. The implication of the characteristic length L further emphasizes that the magnitudes of the vectors $\mathbf{d}_i$ must be identical. The value of L is naturally selected to be the identical magnitude of vectors $\mathbf{d}_i$, to render $\mathbf{J}_{L1}$ isotropic. The final form of the product of $\mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_{L1}$ is, thus:

$$\mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_{L1} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{39}$$

which is indeed proportional to the identity matrix.

4.3.2. Isotropy of $\mathbf{J}_2$. In attempting to render $\mathbf{J}_2$ isotropic, we need to verify that the product $\mathbf{J}_2\mathbf{J}_2^{\mathrm{T}}$ is proportional to the 6×6 identity matrix. This product for WMRs with three active offset wheels is:

$$\mathbf{J}_2\mathbf{J}_2^{\mathrm{T}} = \begin{bmatrix} \Upsilon_1 & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \Upsilon_2 & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \Upsilon_3 \end{bmatrix}, \tag{40}$$

where Υ_i, for $i = 1, 2, 3$, is defined as:

$$\Upsilon_i \equiv [\mathbf{R}_{bwi}]\left[\tilde{\mathbf{p}}^2\right][\mathbf{R}_{bwi}]^{\mathrm{T}}, \tag{41}$$

and:

$$[\tilde{\mathbf{p}}^2] = \begin{bmatrix} -d^2 & 0 \\ 0 & -r^2 \end{bmatrix}. \tag{42}$$

Upon expansion, Υ_i becomes:

$$\Upsilon_i = \begin{bmatrix} r^2\sin^2\beta_i + d^2\cos^2\beta_i & (d^2 - r^2)\sin\beta_i\cos\beta_i \\ (d^2 - r^2)\sin\beta_i\cos\beta_i & r^2\cos^2\beta_i + d^2\sin^2\beta_i \end{bmatrix}, \quad i = 1, 2, 3. \tag{43}$$

In order to render $\mathbf{J}_2$ isotropic, it is apparent that we should have all Υ_i matrices identical and proportional to the 2×2 identity matrix, i.e. we must have:

$$\left(d^2 - r^2\right)\sin\beta_i\cos\beta_i = 0, \tag{44}$$

and:

$$r^2\sin^2\beta_i + d^2\cos^2\beta_i = r^2\cos^2\beta_i + d^2\sin^2\beta_i. \tag{45}$$

Thus, for $\mathbf{J}_2$ to be isotropic, we then have:

$$d = r. \tag{46}$$

With this condition, Υ_i becomes:

$$\Upsilon_i = r^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{47}$$

The final form of the product $\mathbf{J}_2\mathbf{J}_2^{\mathrm{T}}$ is, thus:

$$\mathbf{J}_2\mathbf{J}_2^{\mathrm{T}} = r^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{48}$$

which is indeed proportional to the 6×6 identity matrix. It is noteworthy that the products $\mathbf{J}_2\mathbf{J}_2^{\mathrm{T}}$ and $\mathbf{J}_2^{\mathrm{T}}\mathbf{J}_2$ are identical if $d = r$. This is the condition when the steering axis is tangential to the circumference of the wheel. With these results, the computation of the planar twist reduces to:

$$\mathbf{t}_L = \frac{1}{3}\mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_2\dot{\boldsymbol{\theta}}, \tag{49}$$

computation of the joint rates reducing, correspondingly, to:

$$\dot{\boldsymbol{\theta}} = \frac{1}{r^2}\mathbf{J}_2^{\mathrm{T}}\mathbf{J}_{L1}\mathbf{t}_L. \tag{50}$$

In Ref. [22], a comprehensive treatment is given to identify the singular configurations of this class of WMR. We will not duplicate that effort here; instead, we will focus on WMR with centred wheels and dual-wheels, which will be discussed in Sections 5 and 6, respectively.

4.3.3. Remarks. It was shown in Section 4.2 that WMRs with three offset wheels have 3 d.o.f. It was argued in Section 4.3 that the accuracy of the kinematic results depends on the condition number of the matrices whose inverses are needed and, hence, our designs aim at isotropic Jacobian matrices, with a minimum condition number of unity. Robustness in the kinematic control can then be achieved if the wheel offset d is equal to the radius r of the wheel. However, (28) and (29), along with Fig. 7, reveal a drift and a dead zone in this type of WMR whenever the wheels need reorientation. Assuming that the maximum practical wheel offset is the wheel radius r (since most commercial offset wheels have $d = r$), isotropy leads to a maximum drift and a maximum dead zone. The conclusion is then that isotropy is achieved at the expense of a large drift, if offset wheels are used.

5. WMR WITH CENTERED WHEELS

5.1. Kinematics of a WMR with centered wheels

The architecture of a WMR with centered wheels is similar to that of a WMR with offset wheels, as shown in Fig. 5, except that the wheel offset d is set to zero. The WMR has w active centered wheels mounted at various arbitrary points P_i, for

$i = 1$ to w. Two actuators are needed to steer and drive each wheel unit, which then requires a total of $2w$ actuators to steer and drive a WMR with w wheels of this kind. We begin the modeling by equating the right-hand sides of (5) and (12), which leads to:

$$[\mathbf{Ed}_i \quad \mathbf{1}]\begin{bmatrix} \dot{\psi} \\ \dot{\mathbf{c}} \end{bmatrix} = [\mathbf{R}_{bwi}]\begin{bmatrix} 0 \\ -r\dot{\theta}_i \end{bmatrix}. \tag{51}$$

Similarly, (51) can be cast in the form of (21), with:

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{Ed}_1 & \mathbf{1} \\ \mathbf{Ed}_2 & \mathbf{1} \\ \vdots & \vdots \\ \mathbf{Ed}_w & \mathbf{1} \end{bmatrix},$$

$$\mathbf{J}_2 = \begin{bmatrix} [\mathbf{R}_{bw1}]\begin{bmatrix} 0 \\ -r \end{bmatrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & [\mathbf{R}_{bw2}]\begin{bmatrix} 0 \\ -r \end{bmatrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & [\mathbf{R}_{bww}]\begin{bmatrix} 0 \\ -r \end{bmatrix} \end{bmatrix},$$

$$\mathbf{t} = \begin{bmatrix} \dot{\psi} \\ \dot{\mathbf{c}} \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \ldots & \dot{\theta}_w \end{bmatrix}^{\mathrm{T}},$$

where $\mathbf{J}_1$ is a $2w \times 3$ matrix, $\mathbf{J}_2$ is a $2w \times w$ matrix, and $\mathbf{t}$ and $\dot{\boldsymbol{\theta}}$ are 3D and w-D vectors, respectively. It is noteworthy that the steering rates do not appear in (51); however, the steering angles do appear therein. This is a case of full decoupling of the steering rates from the driving rates.

5.2. Mobility analysis of a WMR with centered wheels

The functional matrix for a WMR with three centered wheels is given by (23) as:

$$\mathbf{F} \equiv \begin{bmatrix} \mathbf{Ed}_1 & \mathbf{1} & [\mathbf{R}_{bw1}]\begin{bmatrix} 0 \\ r \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{Ed}_2 & \mathbf{1} & \mathbf{0} & [\mathbf{R}_{bw2}]\begin{bmatrix} 0 \\ r \end{bmatrix} & \mathbf{0} \\ \mathbf{Ed}_3 & 1 & \mathbf{0} & \mathbf{0} & [\mathbf{R}_{bw3}]\begin{bmatrix} 0 \\ r \end{bmatrix} \end{bmatrix}, \tag{52}$$

while $\dot{\mathbf{q}}$ is defined as

$$\dot{\mathbf{q}} \equiv \begin{bmatrix} \dot{\psi} & \dot{x} & \dot{y} & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix}^{\mathrm{T}}. \tag{53}$$

The functional matrix $\mathbf{F}$ of a WMR with three centered wheels is of 6×6, while vector $\dot{\mathbf{q}}$ is 6D, meaning that if $\mathbf{F}$ is of full rank, then its nullity is zero. In such a situation, the WMR is unable to move. It has been demonstrated in [23, 24] that the rank of the functional matrix $\mathbf{F}$ is 5 when the WMR is in translation or in rotation. The resulting mobility of this WMR is unity, which leads to a unidirectional vehicle. In other words, the WMR can only move instantaneously in the direction prescribed by the steering angles, provided these angles satisfy the coordination condition. This coordination is equivalent to the fact that the wheel axes converge to a single point (possibly at infinity), which is the instantaneous center of rotation. Thus, by setting the offset d to zero, we have suffered in terms of mobility, but gained in terms of a zero dead zone. Moreover, the steering rates $\dot{\phi}_i$ are decoupled from the driving rates $\dot{\theta}_i$, such that the wheels are capable of continuously varying their orientation through 360°; as such, this WMR is omni-directional. The most basic requirement for a WMR maneuvering on a plane is its ability to execute the two fundamental motions of translation and rotation. Thus, if we can verify that the WMR at hand is capable of executing these two fundamental motions, then the WMR has full mobility.

5.2.1. WMR in translation. Translation is any rigid-body motion in which the body orientation remains unchanged. In rectilinear translation, all points in the platform move in parallel straight lines, while in curvilinear translation, all points move along congruent curves, as shown in Fig. 8.

Clearly, rectilinear translation is a special case of curvilinear translation. For this WMR in translation, the unit vectors $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ are identical, implying that the

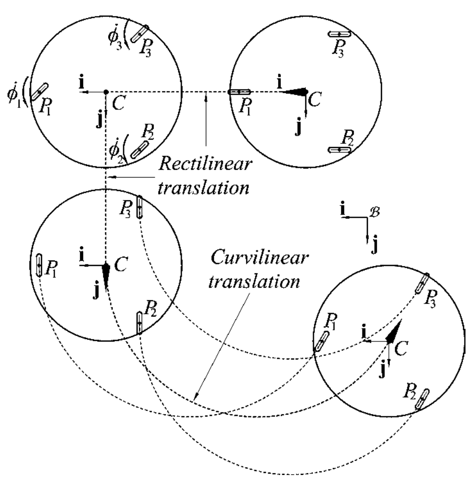

Figure 8. Rectilinear and curvilinear translations.

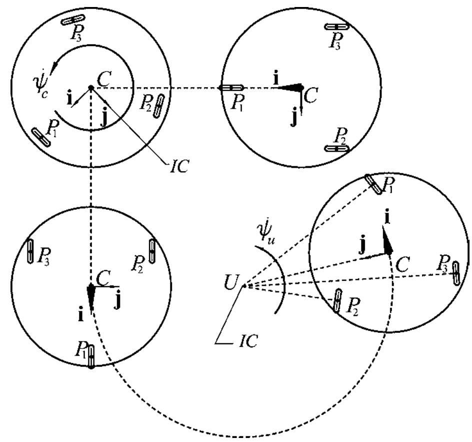

Figure 9. WMR in rotation with IC located at the operating point C and with IC located at an arbitrary point U.

instant center (IC) of the WMR undergoing translation is located at infinity. The rank of the functional matrix $\mathbf{F}$, in this case, is 5 and its nullity is 1. In other words, the WMR can only move instantaneously in the direction prescribed by the steering angles. However, the steering rates $\dot\phi_i$ are decoupled from the driving rates $\dot\theta_i$; therefore, the steering rates can be changed as the WMR is moving, thereby allowing the WMR to execute a curvilinear translation. In both cases, the motion of the platform is completely specified by the motion of any point in the body, since all points undergo the same motion.

5.2.2. WMR in rotation. For rotation, we will examine two cases. In the first case, as shown in Fig. 9, the IC is selected at the operation point C, while the WMR rotates with an angular velocity $\dot\psi_c$. In the second case, the IC is selected at an arbitrary point U, while the WMR rotates with an angular velocity $\dot\psi_u$.

One important difference between the two cases is the displacement of the operating point C during rotation, which for the former is fixed; for the latter, C is displaced during rotation. In both cases, the rank of the functional matrix $\mathbf{F}$ is 5 and its nullity is 1. The WMR can only rotate about the IC. However, the steering angles of the wheels must be compatible; i.e. the vectors $\mathbf{i}_i$ must all point towards the IC.

We have thus demonstrated that this WMR is fully capable of executing the two fundamental planar motions without the undesirable dead zone experienced by offset wheels.

5.3. Kinetostatic isotropy of a WMR with three centered wheels

5.3.1. Isotropy of $\mathbf{J}_1$. The platform design is essentially the same as the WMR with three offset wheels and the isotropy conditions of $\mathbf{J}_1$ are the same. In short, the three centered wheels must be mounted at the vertices of an equilateral triangle such that the three points P_i are equidistant from the operating point C.

5.3.2. Isotropy of $\mathbf{J}_2$. In attempting to render $\mathbf{J}_2$ isotropic, we need to verify that the product $\mathbf{J}_2^{\mathrm{T}}\mathbf{J}_2$ is proportional to the 3×3 identity matrix. The final form of this product for a WMR with three active centered wheels is:

$$\mathbf{J}_2^{\mathrm{T}}\mathbf{J}_2 = r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{54}$$

which is proportional to the identity matrix, $\mathbf{J}_2$ thus being isotropic. With both Jacobian matrices isotropic, robustness of the kinematic control is ensured. The computation of the planar twist reduces to:

$$\mathbf{t}_L = \frac{1}{3}\mathbf{J}\dot{\boldsymbol{\theta}}, \tag{55}$$

where $\mathbf{J}$ is the 3×3 matrix defined as:

$$\mathbf{J} \equiv \mathbf{J}_{L1}^{\mathrm{T}}\mathbf{J}_2, \tag{56}$$

while the computation of the joint rates reduces to:

$$\dot{\boldsymbol{\theta}} = \frac{1}{r^2}\mathbf{J}^{\mathrm{T}}\mathbf{t}_L. \tag{57}$$

Note that, in both cases, matrix $\mathbf{J}$ is singular if $\det(\mathbf{J}) = 0$. The singular configuration(s) of the WMR can be determined by solving for the angles between the intermediate body and the platform β_i, such that:

$$\det(\mathbf{J}) = 0. \tag{58}$$

For a WMR with three centered wheels, $\det(\mathbf{J})$ is computed using Maple [25], the result being:

$$\begin{aligned}\det(\mathbf{J}) = &-\frac{\sqrt{3}}{2}\cos(\beta_1 - \beta_2 - \beta_3) + \frac{3}{4}\sin(\beta_1 - \beta_2 + \beta_3) - \frac{3}{4}\sin(\beta_1 + \beta_2 - \beta_3) \\ &+ \frac{\sqrt{3}}{4}\cos(\beta_1 + \beta_2 - \beta_3) + \frac{\sqrt{3}}{4}\cos(\beta_1 - \beta_2 + \beta_3).\end{aligned} \tag{59}$$

Hence, $\det(\mathbf{J})$ vanishes for the following configurations:

(i) $\beta_1 = \beta_2 = \beta_3$
(ii) $\beta_1 = \beta_3 = 60^\circ$ or -120°
(iii) $\beta_1 = \beta_2 = -60^\circ$ or 120°
(iv) $\beta_2 = \beta_3 = 0^\circ$ or 180°; and
(v) planes of all three wheels intersect.

The singular configuration corresponding to $\beta_1 = \beta_2 = \beta_3$ is shown in Fig. 10. This configuration is encountered when the WMR is moving in translation. The IC is at infinity and the WMR can only move in a straight line.

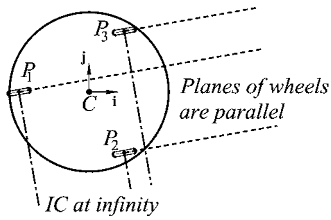

Figure 10. Singular configuration with IC at infinity ($\beta_1 = \beta_2 = \beta_3$).

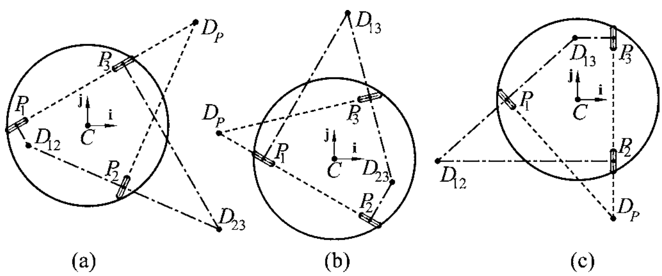

Figure 11. Singular configurations with two wheel planes coincident.

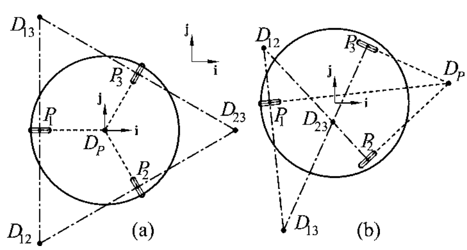

Figure 12. Singular configurations for the intersection of the planes of all the wheels: (a) at the operation point and (b) at an arbitrary point.

Shown in Fig. 11 are the singular configurations corresponding to: (a) $\beta_1 = \beta_3 = 60°$ or $-120°$, (b) $\beta_1 = \beta_2 = -60°$ or $120°$, and (c) $\beta_2 = \beta_3 = 0°$ or $180°$. These configurations happen whenever the planes of any two wheels are coincident, regardless of the plane of the third wheel. The point D_{ij}, for $i, j = 1, 2, 3$, denotes the intersection of two planes of any two wheels i and j with the XY plane. In these configurations, an IC is not possible. Instead, the axles of the three wheels intersect at two different locations. These configurations are unique to this WMR. When the number of wheels is changed or the wheels are rearranged, the singular configurations will change accordingly. However, the methodology developed here can be used to identify the singular configurations of other wheel arrangements.

The singular configurations corresponding to all the three planes of the wheels intersecting at a single point D_P are shown in Fig. 12. Point D_P denotes the intersection of the planes of the wheels with the XY plane. The axles of the three wheels intersect at three different locations. Under this condition, the WMR has lost all its mobility. The singular configuration shown in Fig. 12a can be exploited for braking when the WMR is at rest.

5.4. Steering of a WMR with three centered wheels

In Section 5.2 we demonstrated that this WMR can move in two modes. The first is translation, which includes rectilinear and curvilinear translations. The second mode is rotation. In both cases, the steering angles and the driving rates must be compatible to prevent singularities and skidding, which is undesirable.

5.4.1. WMR in translation. When this WMR operates in the translation mode, control of its orientation ψ is not possible, since only the wheels steer. Moreover, only one of the three steering angles can be specified, since all the planes of the wheel are parallel. The steering angles take the form:

$$\phi_i = \tan^{-1}\left(\frac{\dot{Y}}{\dot{X}}\right), \quad i = 1, 2, 3. \tag{60}$$

5.4.2. WMR in rotation. When the WMR operates in this mode, we have full control of its orientation ψ. However, all the steering angles must be compatible, such that vectors $\mathbf{u}_{\mathrm{c}}$ and $\mathbf{u}_i$ all point to the IC, denoted by U, as shown in Fig. 13. The vector $\overrightarrow{CU}$, denoted by $\mathbf{u}_{\mathrm{c}}$ is:

$$\mathbf{u} - \mathbf{c} = \mathbf{u}_{\mathrm{c}}, \tag{61}$$

while the velocity of U is:

$$\dot{\mathbf{u}} = \dot{\mathbf{c}} + \dot{\psi}\mathbf{E}\mathbf{u}_{\mathrm{c}}. \tag{62}$$

At the IC, the velocity is zero, i.e. $\dot{\mathbf{u}}$ is zero and (62) can be expressed as:

$$\mathbf{u}_{\mathrm{c}} = \frac{1}{\dot{\psi}}\mathbf{E}\dot{\mathbf{c}}. \tag{63}$$

It is apparent that if the WMR is in translation, $\dot{\psi}$ is zero and $\mathbf{u}_{\mathrm{c}}$ is infinity, meaning that the IC lies at infinity. The vector $\overrightarrow{P_iU}$, denoted by $\mathbf{u}_i$ for $\dot{\psi} \neq 0$, is then:

$$\mathbf{u}_i = \frac{1}{\dot{\psi}}\mathbf{E}\dot{\mathbf{c}} - \mathbf{d}_i. \tag{64}$$

The steering angle of the ith wheel can be calculated as:

$$\cos\beta_i = \frac{\mathbf{u}_{\mathrm{c}}^{\mathrm{T}}\mathbf{u}_i}{\|\mathbf{u}_{\mathrm{c}}\|\,\|\mathbf{u}_i\|}. \tag{65}$$

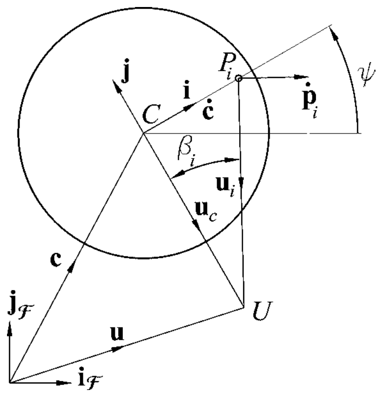

Figure 13. Steering angles of the wheels.

5.4.3. Remarks. We showed that a WMR with centered wheels can be omnidirectional. It was found that the WMR can be rendered isotropic, which leads to the robustness of the kinematic control of the WMR. Furthermore, this WMR does not exhibit the dead zone experienced by its counterpart with offset wheels. However, reorienting the plane of the wheel, with the rolling disabled, requires overcoming a dry-friction torque stemming from the unavoidable deformation of the wheel. Furthermore, like a WMR with offset wheels, a WMR with centered wheels requires two actuators, one for steering and one for driving. The two actuators perform two essentially different tasks. The steering actuator calls for position control while the driving actuator calls for velocity control. Therefore, two independent control systems are needed, that may end up operating in an uncoordinated fashion [15].

6. WMR WITH DUAL-WHEELS

We have shown in Section 3.2 that dual-wheels are kinematically equivalent to centered wheels, with the introduction of (17) and (18). Therefore, the analysis of WMRs with dual-wheels is similar to that of WMRs with centered wheels and will not be repeated.

A WMR with dual wheels has all the characteristics and merits of the WMR with centered wheels. An important difference lies in that, while WMRs with centered wheels need to overcome a dry-friction torque, stemming from the unavoidable deformation of the wheel when reorienting the plane of the wheels with rolling disabled, WMRs with dual-wheels must overcome only rolling friction. Moreover, unlike WMRs with offset wheels and WMRs with centered wheels, in which one actuator is used for steering and one for driving, WMRs with dual-wheels use their two actuators for driving, the difference in rates producing the steering. In this way, actuators of smaller capacity with the same characteristics can be used, since the load is shared by both actuators. Not only this, the load is also shared by the two wheels, which means that the load-carrying capacity of dual-wheel drives is twice as high as that of offset wheels using identical wheels and actuators. With all these merits, WMRs with dual-wheels still have to avoid singular configurations and the problem of skidding due to incompatible steering angles and driving rates due to errors in computation and control.

7. CONCLUSIONS

The work presented in this paper revolves around three main topics, i.e. kinematic modeling, mobility analysis and design of WMRs. In order to better understand the characteristics of the locomotion of WMRs, we derive the kinematic models of this class of robots. We first formulated the kinematic model of a generic platform that can be of any shape. Next, we formulated the kinematic model of the offset wheel, the centered wheel and the dual-wheel. Then, the composite kinematic models of WMRs with active offset wheels, active centered wheels and active dual-wheels were assembled. In the processes of analyzing the centered wheel and the dual-wheel, we exploited the kinematic equivalence between the dual-wheel and the centered wheel. In analyzing the offset wheel, we discovered a drift and a dead zone, which does not permit small displacement within the dead zone without making complex maneuvres. This drift and dead zone are proportional to the offset of the wheel. We studied the mobility of WMRs with (a) three active offset wheels, (b) three active centered wheels and (c) three active dual-wheels. The mobility of a given WMR was characterized by the functional matrix. The analysis shows that WMRs with three offset wheels have a mobility of 3, while WMRs with three centered wheels or three dual-wheels have a mobility of 1. We demonstrated that WMRs with three centered wheels or three dual-wheels are capable of executing the two fundamental motions of translation and rotation, thereby having full mobility. However, it is not possible for WMRs with three centered wheels or three dual-wheels to track a trajectory with discontinuous heading without incorporating a time delay, during which the wheel orientations can be changed. Furthermore, the steering angles of WMRs equipped with steered wheels should be properly coordinated in order to achieve a singularity-free operation. The singular configurations for a WMR with three centered wheels or three dual-wheels were identified by computing the determinant of the Jacobian matrix, and solving for the steering angles that make the determinant vanish. For design purposes, the objective of a kinematically robust WMR was the main focus. The concept of kinetostatic isotropy was applied to find the locations of the wheels with respect to the platform and to choose the types of wheel in order to achieve kinetostatic isotropy. We discovered that WMRs with conventional wheels are isotropic if the three wheels are mounted at the vertices of an equilateral triangle. However, the centroid of the equilateral triangle must coincide with the operation point. Additionally, WMRs with three offset wheels are isotropic if the offset is equal to the radius of the wheel.

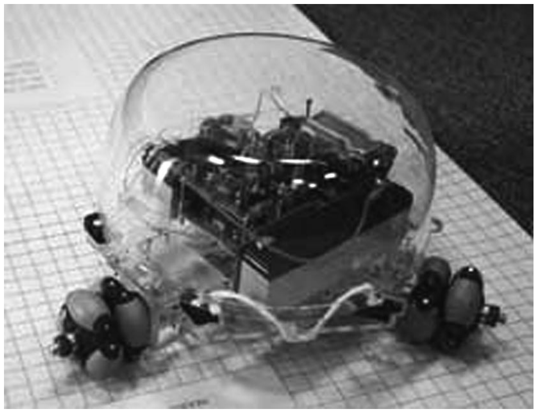

Figure 14. WMR system for future research.

In the future, we plan to combine the developed algorithm with a control scheme for an efficient locomotion, actuation and manipulation of specified objects to achieve certain tasks. It is interesting to compare the performance of our designed WMR [26], as shown in Fig. 14, with the mobile manipulators considered in references [22] and [27].

Acknowledgements

The authors would like to acknowledge the technical assistance given by the staff of Singapore Institute of Manufacturing Technology. We would also like to thank Professor Jorge Angeles, from McGill University, Canada, for his guidance while he was on sabbatical leave at the School of Mechanical and Production Engineering of NTU. Special thanks to Mr W. K. Loh, who helped us in the presentation of the manuscript. We would also like to thank the reviewers for their time and comments to help us improve on the quality of this paper.

REFERENCES

1. P. F. Muir and C. P. Neuman, Kinematic modeling of wheeled mobile robots, *J. Robotic Syst.* **4**, 281–340 (1987).
2. P. F. Muir, Modeling and control of wheeled mobile robots, PhD Thesis, Carnegie Mellon University, Pittsburgh, PA (1988).
3. Y. P. Leow, J. Angeles and K. H. Low, A comparative mobility study of three-wheeled mobile robots, in: *Proc. of 6th Int. Conf. on Control, Automation, Robotics and Vision*, Singapore (2000).
4. M. Wada, A. Takagi and S. Mori, Caster drive mechanisms for holonomic and omni-directional mobile platforms with no over constraint, in: *Proc. IEEE Int. Conf. on Robotics and Automation*, San Francisco, CA, pp. 1531–1538 (2000).
5. S. M. Killough and F. G. Pin, Design of an omni-directional and holonomic wheeled platform prototype, in: *Proc. IEEE Int. Conf. on Robotics and Automation*, Nice, pp. 84–90 (1992).
6. L. Ferrière and B. Raucent, Rollmobs, a new universal wheel concept, in: *Proc. Int. Conf. on Field and Service Robotics*, Leuven, pp. 1877–1882 (1998).

7. B. Carlisle, Robot mechanisms, in: *Proc. IEEE Int. Conf. on Robotics and Automation*, San Francisco, CA, pp. 701–708 (2000).
8. G. Dudek and M. Jenkin, *Computational Principles of Mobile Robotics*. Cambridge University Press, New York, NY (2000).
9. P. J. McKerrow, *Introduction to Robotics*. Addison-Wesley, Singapore (1991).
10. A. Meystel, *Autonomous Mobile Robots — Vehicles with Cognitive Control*. World Scientific, Singapore (1991).
11. G. Campion, G. Bastin and B. D'Andréa-Novel, Structural properties and classification of kinematic and dynamic models of wheeled mobile robots, *IEEE Trans. Robotics Automat.* **12**, 47-62 (1996).
12. S. Ostrovskaya, Dyanamics of quasiholonomic and nonholonomic reconfigurable rolling robots, PhD Thesis, McGill University, Montreal (2001).
13. J. Angeles, The robust design of parallel manipulators, in: *Proc. 1st Int. Coll. Robotic Systems for Handling and Assembly*, Braunschweig, pp. 9–30 (2002).
14. J. L. Jones and A. M. Flynn, *Mobile Robots: Inspiration to Implementation*. Peters, Wellesley, MA (1993).
15. J. Angeles, *Fundamentals of Robotic Mechanical Systems: Theory, Methods, and Algorithms*, 2nd edn. Springer-Verlag, New York, NY (2002).
16. K. H. Low, *Robotics: Principles and Systems Modeling*. Prentice-Hall, Singapore (2002).
17. F. Freudenstein, On the variety of motions generated by mechanisms, *ASME J. Engng Ind.* **84**, 156–160 (1962).
18. G. Strang, *Linear Algebra and Its Applications*, 3rd edn. Harcourt Brace Jovanorich, San Diego, CA (1988).
19. S. Ostrovskaya and J. Angeles, Nonholonomic systems revisited within the framework of analytical mechanics, *Appl. Mech. Rev.* **51**, 415–433 (1998).
20. G. H. Golub and C. F. Van Loan, *Matrix Computations*. The Johns Hopkins University Press, Batlimore, MD (1983).
21. S. K. Saha, J. Angeles and J. Darcovich, The design of kinematically isotropic rolling robots with omni-directional wheels, *Mechanism Machine Theory* **30**, 1127–1137 (1995).
22. B. J. Yi and W. K. Kim, The kinematics for redundantly actuated omnidirectional mobile robots, *J. Robotic Syst.* **19**, 255–267 (2002).
23. Y. P. Leow and K. H. Low, Mobile platform with centred wheels, in: *Proc. 2nd Asian Symp. on Industrial Automation and Robotics*, Bangkok, pp. 21–28 (2001).
24. Y. P. Leow, K. H. Low and S. Y. Lim, Kinematic modeling and analysis of mobile robots with centred wheels, in: *Proc. 5th Int. Conf. on Mechatronics Technology*, Singapore, pp. 200–204 (2001).
25. M. Kofler, *Maple: An Introduction and Reference*. Addison Wesley, Harlow (1997).
26. W. K. Loh, K. H. Low and Y. P. Leow, Mechatronics design and kinematic modelling of a singularityless omni-directional wheeled mobile robot, in: *Proc. IEEE Int. Conf. on Robotics and Automation*, Taipei, pp. 3237–3242 (2003).
27. M. T. Mason, *et al.*, Experiments with desktop mobile manipulators, in: *Proc. Int. Symp. of Experimental Robotics*, Sydney (1999).

ABOUT THE AUTHORS

K. H. Low obtained his MS and PhD degrees in Mechanical Engineering from the University of Waterloo, Canada, in 1983 and 1986, respectively. After spending 2 years as a Post-doctoral Fellow at the University of Waterloo, he joined the School of Mechanical and Production Engineering of the Nanyang Technological University. His teaching and research interests include robotics, vibrations, impacts, machines, mechanisms, exoskeleton systems and mechatronics design. He is the author of three books and more than 70 journal papers in the areas of robotics, impacts, power transmission systems, structural dynamics and vibrations. He is a member of IEEE, IASTED, SIAA and SiCToMM.

Y. P. Leow received both his BEng and MEng degrees in Mechanical Engineering from the Nanyang Technological University (NTU), Singapore, in 1996 and 2002, respectively. From 1999, he has been a Research Engineer with Singapore Institute of Manufacturing Technology, Singapore. His research interests include precision machine and mechanism design, and manufacturing automation. He is a member of SIAA and SiCToMM.